People's Democratic Republic of Algeria
Ministry of Higher Education and
Scientific Research

UNIVERSITY OF ECHAHID HAMMA LAKHDAR El OUED

FACULTY OF EXACTE SCIENCES

Final Master’s Thesis

MASTER ACADEMIQUE

Domain : Mathematics and informatics

Option : Mathematics

Speciality : Fundamental Mathematics and applications

Theme

Positive solutions of nonlinear multi-point boundary value problems

Presented by : **CHIKHA Yousra**
MESAI AOUN Messaouda

Discussed on **27-09-2025** by the jury :

President	**Zaiz Khaoula**	MCB	Univ. El Oued
Supervisor	**Ghendir Aoun Abdellatif**	MCA	Univ. El Oued
Examiner	**Beloul Said**	Prof	Univ. El Oued

Academic year : 2024/2025

Contents

Notations

We will use the following notations throughout this Memory:

General notations

- $\mathbb{N}$: The set of natural numbers.
- $\mathbb{R}$: The set of real numbers.
- $[a, b]$: The closed interval $a \leq x \leq b$.
- (a, b): The open interval $a < x < b$.
- K: a cone.
- ε: indicates a very small number.
- $\frac{d(.)}{dt}$: The ordinary derivative with respect to t.
- a.e.: Almost everywhere.
- i.e.: That is to say.

Spaces

- $\Omega \subset \mathbb{R}^n$: open set in $\mathbb{R}^n$.
- $\overline{\Omega}$: closure of Ω.
- $\partial\Omega$: boundary of Ω.
- $L^p(\Omega) = \{$u measurable on Ω and $\int_\Omega |u|^p dx < \infty\}$, $1 \leq p < \infty$..
- $C(\overline{\Omega})$: Space of continuous functions on Ω.
- $C^n(\overline{\Omega})$: Space of n times continuously differentiable functions on Ω.

  In this memory, we will specifically use the following spaces:

- $C([0,1],\mathbb{R})$: Banach space of continuous functions $u : [0,1] \to \mathbb{R}$, equipped with the norm
$$\|u\|_\infty = \sup\{|u(t)|,\, t \in [0,1]\}.$$
- $C^1([0,1],\mathbb{R})$: Space of functions $u : [0,1] \to \mathbb{R}$ such that u, u' are continuous functions on [0,1], equipped with the norm
$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}.$$
- $L^1([0,1],\mathbb{R})$: Space of Lebesgue integrable functions $u : [0,1] \to \mathbb{R}$ equipped with the norm
$$\|u\|_1 := \|u\|_{L^1} = \int_0^1 |u(t)| dt.$$

Introduction

Boundary value problems involving ordinary differential equations arise in physical sciences and applied mathematics. In some of these problems, additional conditions such that multi-point boundary conditions are important in practice since the measurements. It can also be extended to a n-point condition $u(t) = g(u)$, where g is a mapping defined on some space consisting of certain functions with $g(u) := g(\xi_1, \ldots, \xi_p, u(.)),\ p \in \mathbb{N}^*$, where $0 < \xi_1 < \ldots < \xi_p < +\infty$. The symbol $g(\xi_1, \ldots, \xi_p, u(.))$ is used in the sense that in place of $(.)$ we can put only elements of the set $\{\xi_1, \ldots, \xi_p\}$. For example g can be defined by the formula $g(u) = \sum_{i=1}^{p} \beta_i u(\xi_i)$, where $\beta_i,\ i = 1, 2, \ldots, p$ are given constants.
The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Marano [11]. Gupta [7] studied three-point boundary value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multi-point boundary value problems have been studied by several authors by using the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder, coincidence degree theory, and fixed point theorems in cones. We refer the reader to [1], [3], [16, 17, 18] and the references therein.
The existence and multiplicity of positive solutions for nonlinear ordinary differential equations have received much attention. To identify some of them, we refer the reader to [1], [3, 4], [6], [12, 13, 14, 15, 16, 17, 18, 19, 20, 21] and the references therein.
In this memory, we given some results on the existence of positive solutions for multi-point boundary value problems of second order ordinary differential equations on bounded intervals.
This memory is organized as follows.

The first chapter is devoted to presenting some preliminaries and general notions used throughout. Some basic tools from functional analysis and classical fixed point theorem of GuoKrasnoselskii.
In **the second chapter**, we present results on the existence of positive solutions to three point boundary value problems of second order ordinary differential equations. Let us consider one example from this chapter:

$$\begin{cases} x''(t) + \varphi(t) f(x) = 0, & t \in (0, 1), \\ x(0) = 0, \quad x(1) = \alpha\, x(\eta) \end{cases}$$

where $0 < \eta < 1$ and $0 < \alpha < \frac{1}{\eta}$.

We show the existence of at least one positive solution if f is either superlinear or sublinear by applying the fixed point theorem in cones.
In **the third chapter**, we discuss the of positive solutions of a nonlinear three-point

integral boundary value problem

$$\begin{cases} x''(t) + \varphi(t)f(x) = 0, & t \in (0,1), \\ x(0) = 0, \quad x(1) = \alpha \int_0^\eta x(s)ds \end{cases}$$

where $0 < \eta < 1,\ 0 < \alpha < \frac{2}{\eta^2}$ and f is either superlinear or sublinear.
In **the fourth chapter**, we present results on the existence of positive solutions to multi-point boundary value problem

$$\begin{cases} x''(t) + \varphi(t)f(x) = 0, & t \in (0,1), \\ x(0) = 0, \quad x(1) = \sum_{i=1}^{n-2} a_i x(\xi_i) \end{cases}$$

where $f \in C([0,\infty),[0,\infty))$, $\varphi \in C([0,1],[0,\infty))$,
$a_i > 0$, $i = 1,2,...,n-2$ and $0 < \xi_1 < \xi_2 < ... < \xi_{n-2} < 1$.
In all the phases of our work we give some examples which give applications to our results.

Chapter 1

Preliminaries

1.1 Some basic tools

Definition 1.1. Let E be a Banach space.
A closed nonempty convex subset K of E is a cone if it satisfies the following two conditions:

(i) $\forall x \in K$, $\forall \lambda \geq 0$ implies $\lambda x \in K$, $((\lambda K) \subset K)$

(ii) $(x \in K$ and $-x \in K)$ implies $(x = 0)$, $(K \cap (-K) = \{0\})$.

Definition 1.2. A real valued function f on an interval (or, more generally, on a convex set in a vector space) is said to be concave if for all x, y in the interval and for all $t \in [0, 1]$,

$$f\Big(tx + (1-t)y\Big) \;\geq\; t\, f(x) \;+\; (1-t)\, f(y).$$

Call a strictly concave function for all $t \in]0, 1[$ if :

$$f\Big(tx + (1-t)y\Big) \;>\; t\, f(x) \;+\; (1-t)\, f(y).$$

Remark 1.1.
(a) If f is concave on an interval, then for a, x, b in the interval with $a < x < b$, we have:

$$\frac{f(x) - f(a)}{x - a} \;\geq\; \frac{f(b) - f(a)}{b - a}.$$

(b) (Concavity criterion). Suppose $f : U \subset \mathbb{R} \to \mathbb{R}$ is of class C^2 (i.e.: f is continuous), then (f is concave), if and only if, ($f''(x) \leq 0$ for all $x \in U$).

Definition 1.3. Set

$$f_0 = \lim_{x \to 0^+} \frac{f(x)}{x}, \quad f_\infty = \lim_{x \to \infty} \frac{f(x)}{x}.$$

Then:

- $f_0 = 0$, $f_\infty = \infty$ $\;\Rightarrow$ correspond to superlinear case.
- $f_0 = \infty$, $f_\infty = 0$ $\;\Rightarrow$ correspond to sublinear case.

1.2 Ascoli-Arzela theorem

Let (X, d_X) be a compact metric space and (Y, d_Y) be a complete metric space. By $C(X, Y)$ we denote the vector space consisting of all continuous function $f : X \to Y$.

Definition 1.4. The family $F \subset C(X, Y)$ is called equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(t), f(s)) < \varepsilon$ for all $t, s \in X$ satisfying $d_X(t, s) < \delta$ and all $f \in F$.

Theorem 1.5. *([2]) The family $F \subset C(X, Y)$ is relatively compact if and only if:*
1. F is equicontinuous and
2. for all $t \in X$, the set

$$f(t) = \{x(t);\, x(.) \in F\}$$

is relatively compact in Y.

Remark 1.2.
1. If Y is a finite dimensional Banach space, the second condition of Ascoli-Arzela theorem is equivalent to F is uniformly bounded i.e., there exists $M > 0$ such that for all $x \in F$

$$\|x\|_\infty = \sup_{t \in X} \|x(t)\| \leq M.$$

2. If X is a compact metric space, then Ascoli-Arzela theorem can be expressed as follows: the family $F \subset C(X, Y)$ is relatively compact if and only if F is equicontinuous.

1.3 Lebesgue dominated convergence theorem

Theorem 1.6. *([2]) Let (f_n) be a sequence of functions in $L^1(\Omega)$ that satisfies*
(i) $f_n(t) \to f(t)$ for almost every $t \in \Omega$,
(ii) there is a function $g \in L^1(\Omega)$ such that $|f_n(t)| \leq g(t)$ for all $n \in \mathbb{N}$ and for almost every $t \in \Omega$.
Then $f \in L^1(\Omega)$ and $\|f_n - f\|_1 \to 0$.

1.4 Compact operators

Let X and Y be two normed vector spaces and Ω an open set of X.

Definition 1.7. A continuous mapping $T : \Omega \subset X \to Y$ is called compact if $T(\overline{\Omega})$ is relatively compact. If the image of any bounded subset B of Ω is relatively compact, then T is said to be completely continuous.

1.5 Fixed point theorem (Guo-Krasnoselskii)[5]

Let E be a Banach space, and let $K \subset E$ a cone. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$,

$$\bar{\Omega}_1 \subset \Omega_2 \quad \text{and let} \quad A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that:

(i) $\|Ax\| \leq \|x\|$, for $x \in K \cap \partial\Omega_1$ and $\|Ax\| \geq \|x\|$, for $x \in K \cap \partial\Omega_2$; or

(ii) $\|Ax\| \geq \|x\|$, for $x \in K \cap \partial\Omega_1$ and $\|Ax\| \leq \|x\|$, for $x \in K \cap \partial\Omega_2$

Then A has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Chapter 2

Existence results of positive solutions for a second order three-point boundary value problem

In this chapter, we study the existence of positive solutions for a second-order differential equation with a three-point boundary condition.

Consider the problem:

$$\begin{cases} x''(t) + \varphi(t) f(x) = 0, & t \in (0,1), \\ x(0) = 0, \quad x(1) = \alpha\, x(\eta) \end{cases} \tag{2.1}$$

where $0 < \eta < 1$ and $0 < \alpha < \frac{1}{\eta}$.
$\varphi \in C([0,1],[0,\infty))$, We show the existence of at least one positive solution if f is either superlinear or sublinear by applying the fixed point theorem in cones.

2.1 Auxiliary lemmas

We first give some preliminary lemmas.

Lemma 2.1. *Let $h \in C([0,1])$, $\alpha\eta \neq 1$. Then the problem*

$$\begin{cases} x''(t) + h(t) = 0, & t \in (0,1), \\ x(0) = 0, \quad x(1) = \alpha\, x(\eta) \end{cases} \tag{2.2}$$

has a unique solution given by

$$x(t) = -\int_0^t (t-s)h(s)\,ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)h(s)\,ds + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)h(s)\,ds.$$

Proof.
From $x''(t) = -h(t)$ with integrating we obtain

$$x'(t) = x'(0) - \int_0^t h(s)\,ds,$$

and integrating again

$$x(t) = x'(0)t - \int_0^t (t-s)h(s)\,ds.$$

Then we have

$$x(1) = x'(0) - \int_0^1 (1-s)h(s)\,ds,$$
$$x(\eta) = x'(0)\eta - \int_0^\eta (\eta-s)h(s)\,ds.$$

By using the condition $x(1) = \alpha x(\eta)$ we obtain

$$(1-\alpha\eta)\,x'(0) = -\alpha \int_0^\eta (\eta-s)h(s)\,ds + \int_0^1 (1-s)h(s)\,ds.$$

Thus,

$$x(t) = -\int_0^t (t-s)h(s)\,ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)h(s)\,ds + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)h(s)\,ds.$$

□

Lemma 2.2. *Let $0 < \alpha < \frac{1}{\eta}$. If $h \in C([0,1],[0,+\infty))$, then the solution x of the problem (2.2) satisfies $x \geq 0$, $t \in [0,1]$.*

Proof.
From $x''(t) = -h(t) \leq 0$, we know that the graph of $x : t \mapsto x(t)$ is concave downwards on $(0,\,1)$. Which implies that for all $t \in (0,\,1)$,

$$\frac{x(t)-x(0)}{t} \geq \frac{x(1)-x(0)}{1}$$

i.e. $(x(t) \geq tx(1))$.
Then, if $x(1) \geq 0$, then the concavity of x and the boundary condition $x(0) = 0$ implies that $x(t) \geq 0$ for $t \in [0,1]$.
If $x(1) < 0$, so we have that $x(\eta) < 0$ and $x(1) = \alpha x(\eta) > \frac{1}{\eta}x(\eta)$, which contradicts the concavity of x, $\left(\frac{x(\eta)-x(0)}{\eta-0} \geq \frac{x(1)-x(0)}{1-0}\right)$. □

Lemma 2.3. *Let $\alpha > \frac{1}{\eta}$. If $h \in C([0,1],[0,+\infty))$, then the problem (2.2) has no positive solution.*

Proof.
We use proof by contradiction.
Suppose that problem (2.2) has a positive solution x.
If $x(1) > 0$, then $x(\eta) > 0$ and $\frac{x(1)}{1} = \frac{\alpha x(\eta)}{1} > \frac{x(\eta)}{\eta}$ which contradicts the concavity of x.
If $x(1) = 0$ and $x(\tau) > 0$ for some $\tau \in (0,1)$, then $x(\eta) = \frac{x(1)}{\alpha} = 0$, $\tau \neq \eta$.
If $\tau \in (0,\,\eta)$ i.e. $\tau < \eta < 1$, then by the concavity of x we have

$$\frac{x(\eta)-x(\tau)}{\eta-\tau} \geqslant \frac{x(1)-x(\tau)}{1-\tau}$$

which gives

$$\frac{-x(\tau)}{\eta-\tau} \geq \frac{-x(\tau)}{1-\tau}$$

i.e. $\eta \geq 1$, which is contradicted at $\eta \in (0,\,1)$.
If $\tau \in (\eta,\,1)$ i.e. $0 < \eta < \tau$, then by the concavity of x we get

$$\frac{x(\eta)-x(0)}{\eta-0} \geq \frac{x(\tau)-x(0)}{\tau-0}$$

i.e. $x(\tau) \leq 0$, which is contradicted at $x(\tau) > 0$. □

Lemma 2.4. *Let $0 < \alpha < \frac{1}{\eta}$. If $h \in C([0,1],[0,+\infty))$, then the solution x of the problem (2.2) satisfies*

$$\inf_{t\in[\eta,1]} x(t) \geq \gamma\|x\|, \quad \textit{where } \gamma = \min\left\{\alpha\eta, \frac{\alpha(1-\eta)}{1-\alpha\eta}, \eta\right\}.$$

Proof.
We divide the demonstration into two steps.

Step 1 : We treat the case $0 < \alpha < 1$. In the case, we have $x(\eta) \geq x(1)$.
We pose $x(\bar{t}) = \|x\|$.
If $\bar{t} \leq \eta < 1$, thus

$$\min_{t\in[\eta,1]} x(t) = x(1)$$

because, by the concavity of x and for all $t \in]\eta, 1]$ we get

$$\frac{x(t)-x(\eta)}{t-\eta} \geq \frac{x(1)-x(\eta)}{1-\eta} \geq \frac{x(1)-x(\eta)}{t-\eta}$$

which implies that for all $t \in]\eta, 1]$, $x(t) \geq x(1)$ and with $x(\eta) \geq x(1)$ so for all $t \in [\eta, 1]$, $x(t) \geq x(1)$.
In addition,

$$\frac{x(1)-x(\eta)}{1-\eta} \leq \frac{x(\bar{t})-x(1)}{\bar{t}-1} \leq \frac{x(\bar{t})-x(1)}{0-1}$$

which gives

$$x(\bar{t}) \leq x(1) + \frac{x(1)-x(\eta)}{1-\eta}(0-1) = x(1)\frac{1-\alpha\eta}{\alpha(1-\eta)}.$$

Then

$$\min_{t\in[\eta,1]} x(t) \geq \frac{\alpha(1-\eta)}{1-\alpha\eta}\|x\|.$$

If $0 < \eta < \bar{t} < 1$, so

$$\min_{t\in[\eta,1]} x(t) = x(1).$$

According to the concavity of x, we have $\frac{x(\eta)}{\eta} \geq \frac{x(\bar{t})}{\bar{t}}$ which combined with boundary conditions $x(1) = \alpha x(\eta)$ we conclude that

$$\frac{x(1)}{\alpha\eta} \geq \frac{x(\bar{t})}{\bar{t}} \geq x(\bar{t}) = \|x\|.$$

Then

$$\min_{t\in[\eta,1]} x(t) \geq \alpha\eta\|x\|.$$

Step 2 : We treat the case $1 < \alpha < \frac{1}{\eta}$. In the case, we get $x(\eta) \leq x(1)$ thus

$$\min_{t\in[\eta,1]} x(t) = x(\eta)$$

because, by the concavity of x and with $x(\eta) \leq x(1)$, we have

$$\frac{x(t)-x(\eta)}{t-\eta} \geq \frac{x(1)-x(\eta)}{1-\eta} \geq 0$$

thus, for all $t \in [\eta, 1]$ we have
$$x(t) \geq x(\eta).$$
We pose $x(\bar{t}) = \|x\|$.
From $x(\eta) \leq x(1)$ and the concavity of x, we have that
$$\min_{t\in[\eta,1]} x(t) = x(\eta).$$
Use of the concavity of x and the Lemma 2.2, we have that $\frac{x(\eta)}{\eta} \geq \frac{x(\bar{t})}{\bar{t}}$ which implies that
$$\min_{t\in[\eta,1]} x(t) \geq \eta\|x\|.$$

□

2.2 Result of existence

Theorem 2.5. *Assume that:*

(H1) $f \in C([0, +\infty), [0, +\infty))$,

(H2) $\varphi \in C([0, 1], [0, +\infty))$, *and there exists* $t_0 \in (\eta, 1)$ *such that* $\varphi(t_0) > 0$.

Then the problem (2.1) *has at least one positive solution in the case:*

(i) $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear), or*

(ii) $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear).*

Proof.

(i) Let's suppose that $f_0 = 0$ and $f_\infty = \infty$. We will show the existence of a positive solution to the problem (2.1). We have
(the problem (2.1) has a solution $x = x(t)$), if and only if, (x is a solution of the following equation of the operator
$$\begin{aligned} x(t) &= -\int_0^t (t-s)\varphi(s)f(x(s))ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)\varphi(s)f(x(s))ds \\ &\quad + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)\varphi(s)f(x(s))ds \\ &:= Ax(t)). \end{aligned}$$
Denote that
$$K = \left\{x : x \in C[0,1], x \geq 0, \min_{t\in[\eta,1]} x(t) \geq \gamma\|x\|\right\}.$$
It is obvious that K is a cone in $C[0, 1]$. Moreover, by the Lemma 2.3, $AK \subset K$. we prove that $A : K \longrightarrow K$ is completely continuous.
Indeed, on the one hand, according to *(H1)*, *(H2)* the operator A is continuous.

On the other hand, let us prove that the operator A is compact.
We have for all $t \in C[0,1]$,

$$\begin{aligned}
|Ax(t)| &\leq \int_0^1 |\varphi(s)|\,|f(x(s))|\,ds + \frac{\alpha\eta}{1-\alpha\eta}\int_0^1 |\varphi(s)|\,|f(x(s))|\,ds \\
&\quad + \frac{1}{1-\alpha\eta}\int_0^1 |\varphi(s)|\,|f(x(s))|\,ds \\
&\leq \left(1 + \frac{\alpha\eta}{1-\alpha\eta} + \frac{1}{1-\alpha\eta}\right)\int_0^1 |\varphi(s)|\,|f(x(s))|\,ds \\
&\leq \frac{2}{1-\alpha\eta}\int_0^1 |\varphi(s)|\,|f(x(s))|\,ds
\end{aligned}$$

which proves that A is bounded.
Now show that (Ax) is equicontinuous:
for $t_1, t_2 \in [0,1], (t_2 \leq t_1)$ with $|t_1 - t_2| \longrightarrow 0$, we have

$$\begin{aligned}
|Ax(t_1) - Ax(t_2)| &= \Bigg| -\int_0^{t_1}(t_1-s)\varphi(s)f(x(s))ds - \frac{\alpha t_1}{1-\alpha\eta}\int_0^{\eta}(\eta-s)\varphi(s)f(x(s))ds \\
&\quad + \frac{t_1}{1-\alpha\eta}\int_0^1 (1-s)\varphi(s)f(x(s))ds + \int_0^{t_2}(t_2-s)\varphi(s)f(x(s))ds \\
&\quad + \frac{\alpha t_2}{1-\alpha\eta}\int_0^{\eta}(\eta-s)\varphi(s)f(x(s))ds - \frac{t_2}{1-\alpha\eta}\int_0^1(1-s)\varphi(s)f(x(s))ds \Bigg| \\
&= \Bigg| \frac{\alpha(t_1-t_2)}{1-\alpha\eta}\int_0^{\eta}(\eta-s)\varphi(s)f(x(s))ds + \frac{t_1-t_2}{1-\alpha\eta}\int_0^1(1-s)\varphi(s)f(x(s))ds \\
&\quad - \int_0^{t_1}(t_1-s)\varphi(s)f(x(s))ds - \int_0^{t_2}(t_1-s)\varphi(s)f(x(s))ds \\
&\quad + \int_0^{t_2}(t_1-s)\varphi(s)f(x(s))ds + \int_0^{t_2}(t_2-s)\varphi(s)f(x(s))ds \Bigg| \\
&= \Bigg| \frac{\alpha(t_1-t_2)}{1-\alpha\eta}\int_0^{\eta}(\eta-s)\varphi(s)f(x(s))ds + \frac{t_1-t_2}{1-\alpha\eta}\int_0^1(1-s)\varphi(s)f(x(s))ds \\
&\quad + \int_0^{t_2}(t_2-t_1)\varphi(s)f(x(s))ds + \int_{t_1}^{t_2}(t_1-s)\varphi(s)f(x(s))ds \Bigg| \\
&\leq \frac{\alpha|t_1-t_2|}{1-\alpha\eta}\int_0^{\eta}(\eta-s)\varphi(s)f(x(s))ds + \frac{|t_1-t_2|}{1-\alpha\eta}\int_0^1(1-s)\varphi(s)f(x(s))ds \\
&\quad + |t_1-t_2|\int_0^{t_2}\varphi(s)f(x(s))ds + \int_{t_1}^{t_2}(t_1-s)\varphi(s)f(x(s))ds,
\end{aligned}$$

which implies, $|Ax(t_1) - Ax(t_2)| \to 0$ when $|t_1 - t_2| \to 0$.
Since $f_0 = 0$, we can choose $H_1 > 0$ so that $f(x) \leq \epsilon x$, for $0 < x < H_1$, such that $\epsilon > 0$ satisfied

$$\frac{\epsilon}{1-\alpha\eta}\int_0^1 (1-s)\varphi(s)ds \leq 1.$$

Thus, if $x \in K$ and $\|x\| = H_1$ we get

$$\begin{aligned}
Ax(t) &\leq \frac{t}{1-\alpha\eta}\int_0^1 (1-s)\varphi(s)f(x(s))ds \\
&\leq \frac{t}{1-\alpha\eta}\int_0^1 (1-s)\varphi(s)\epsilon x(s)ds \\
&\leq \frac{\epsilon}{1-\alpha\eta}\int_0^1 (1-s)\varphi(s)ds\|x\| \\
&\leq \frac{\epsilon}{1-\alpha\eta}\int_0^1 (1-s)\varphi(s)dsH_1.
\end{aligned}$$

Now, let

$$\Omega_1 = \{x \in C[0,1] : \|x\| < H_1\},$$

then $\|Ax\| \leq \|x\|$, for $x \in K \cap \partial\Omega_1$.
From $f_\infty = \infty$, it exists $\widehat{H_2} > 0$ such that $f(x) \geq \rho x$, for $x \geq \widehat{H_2}$, with $\rho > 0$ is chosen so that

$$\rho\frac{\eta\gamma}{1-\alpha\eta}\int_\eta^1 (1-s)\varphi(s)ds > 1.$$

Let $H_2 = \max\left\{2H_1, \frac{\widehat{H_2}}{\gamma}\right\}$ and $\Omega_2 = \{x \in C[0,1] : \|x\| < H_2\}$,
then $x \in K$ and $\|x\| = H_2$ implies that

$$\min_{t\in[\eta,1]} x(t) \geq \gamma\|x\| \geq \widehat{H_2}.$$

Thus

$$\begin{aligned}
Ax(\eta) &= -\int_0^\eta (\eta-s)\varphi(s)f(x(s))ds - \frac{\alpha\eta}{1-\alpha\eta}\int_0^\eta (\eta-s)\varphi(s)f(x(s))ds \\
&\quad +\frac{\eta}{1-\alpha\eta}\int_0^1 (1-s)\varphi(s)f(x(s))ds \\
&= -\frac{1}{1-\alpha\eta}\int_0^\eta (\eta-s)\varphi(s)f(x(s))ds + \frac{\eta}{1-\alpha\eta}\int_0^1 (1-s)\varphi(s)f(x(s))ds \\
&= -\frac{1}{1-\alpha\eta}\int_0^\eta \eta\varphi(s)f(x(s))ds + \frac{1}{1-\alpha\eta}\int_0^\eta s\varphi(s)f(x(s))ds \\
&\quad +\frac{\eta}{1-\alpha\eta}\int_0^1 \varphi(s)f(x(s))ds - \frac{\eta}{1-\alpha\eta}\int_0^1 s\varphi(s)f(x(s))ds \\
&= \frac{\eta}{1-\alpha\eta}\int_\eta^1 \varphi(s)f(x(s))ds + \frac{1}{1-\alpha\eta}\int_0^\eta s\varphi(s)f(x(s))ds \\
&\quad -\frac{\eta}{1-\alpha\eta}\int_0^1 s\varphi(s)f(x(s))ds \\
&= \frac{\eta}{1-\alpha\eta}\int_\eta^1 \varphi(s)f(x(s))ds + \frac{1-\eta}{1-\alpha\eta}\int_0^\eta s\varphi(s)f(x(s))ds \\
&\quad -\frac{\eta}{1-\alpha\eta}\int_\eta^1 s\varphi(s)f(x(s))ds \\
&\geq \frac{\eta}{1-\alpha\eta}\int_\eta^1 \varphi(s)f(x(s))ds - \frac{\eta}{1-\alpha\eta}\int_\eta^1 s\varphi(s)f(x(s))ds \\
&= \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)\varphi(s)f(x(s))ds.
\end{aligned}$$

Then, for $x \in K \cap \partial\Omega_2$,

$$\|Ax\| \geq \rho\frac{\eta\gamma}{1-\alpha\eta}\int_\eta^1 (1-s)\varphi(s)ds\|x\| \geq \|x\|.$$

Therefore, the first part of the theorem of Guo-Krasnosel'skii satisfied, it follows that A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, such that $H_1 \leq \|x\| \leq H_2$.

(ii) Let's suppose that $f_0 = \infty$ and $f_\infty = 0$. We first have to choose $H_3 > 0$ such that $f(x) \geq Mx$ for $0 < x < H_3$ with

$$M\gamma\left(\frac{\eta}{1-\alpha\eta}\right)\int_\eta^1 (1-s)\varphi(s)ds \geq 1.$$

Then

$$\begin{aligned}
Ax(\eta) &= -\int_0^\eta (\eta - s)\varphi(s)f(x(s))ds - \frac{\alpha\eta}{1-\alpha\eta}\int_0^\eta (\eta-s)\varphi(s)f(x(s))ds \\
&\quad + \frac{\eta}{1-\alpha\eta}\int_0^1 (1-s)\varphi(s)f(x(s))ds \\
&\geq \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)\varphi(s)f(x(s))ds \\
&\geq \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)\varphi(s)Mx(s)ds \\
&\geq \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)\varphi(s)M\gamma ds\|x\| \\
&\geq \|x\|.
\end{aligned}$$

Thus, we can choose $\Omega_3 = \{x \in C[0,1] : \|x\| < H_3\}$, such that

$$\|Ax\| \geq \|x\|, \quad x \in K \cap \partial\Omega_3.$$

On the other hand, since $f_\infty = 0$, it exists $\widehat{H_4} > 0$ such that $f(x) \leq \epsilon x$, for $x \geq \widehat{H_4}$, with $\epsilon > 0$ satisfied

$$\frac{\epsilon}{1-\alpha\eta}\int_0^1 (1-s)\varphi(s)ds \leq 1.$$

Let $H_4 = \max\left\{2H_3, \frac{\widehat{H_4}}{\gamma}\right\}$ and $\Omega_4 = \{x \in C[0,1] : \|x\| < H_4\}$,
then $x \in K$ and $\|x\| = H_4$ implies that

$$\min_{t\in[\eta,1]} x(t) \geq \gamma\|x\| \geq \widehat{H_4}.$$

Thus, if $x \in K$ and $\|x\| = H_4$ we get

$$\begin{aligned}
Ax(t) &\leq \frac{t}{1-\alpha\eta}\int_0^1 (1-s)\varphi(s)f(x(s))ds \\
&\leq \frac{t}{1-\alpha\eta}\int_0^1 (1-s)\varphi(s)\epsilon x(s)ds \\
&\leq \frac{\epsilon}{1-\alpha\eta}\int_0^1 (1-s)\varphi(s)ds\|x\| \\
&\leq \frac{\epsilon}{1-\alpha\eta}\int_0^1 (1-s)\varphi(s)dsH_4 \\
&\leq H_4,
\end{aligned}$$

then

$$\|Ax\| \leq \|x\|.$$

So, for $x \in K \cap \partial\Omega_4$ we can have $\|Ax\| \leq \|x\|$. Therefore, the second part of the theorem of Guo-Krasnosel'skii satisfied, it follows that the problem (2.1) admits a positive solution.

□

2.3 Some examples

To illustrate our result, we consider ... examples.

Example 2.1.
the boundary value problem

$$\begin{cases} x'' + (1-t)^3 x^2 e^x = 0, & t \in (0,\,1), \\ x(0) = 0, \quad x(1) = \frac{1}{4} x(\frac{1}{2}), \end{cases} \tag{2.3}$$

admits at least one positive solution.
Indeed, when we assume

$$\varphi(t) = (1-t)^3,\ f(x) = x^2 e^x$$

and in this case we have

$$\alpha = \frac{1}{4},\ \eta = \frac{1}{2}.$$

Then we obtain $\varphi \in C([0,\,1], [0,\infty))$ and it exists $t_0 \in [\eta,\,1] = [\frac{1}{2},\,1]$ such that $\varphi(t_0) > 0$, we choose $t_0 = \frac{1}{5}$, $\varphi(\frac{1}{5}) > 0$.
Also $f \in C([0,\infty),[0,\infty))$ and $f_0 = 0,\ f_\infty = \infty$.
Then, the conditions *(H1)*, *(H2)* and case (i) are satisfied.
So, by the theorem 2.5, the conclusion is satisfied.

Example 2.2.
Consider the boundary value problem

$$\begin{cases} x'' + t^2 \cdot x^2 \sin\left(\dfrac{1}{1+x}\right) = 0, & t \in (0,\,1), \\ x(0) = 0, \quad x(1) = \dfrac{1}{6} x\left(\dfrac{3}{4}\right), \end{cases} \tag{2.4}$$

admits at least one positive solution.
Let us define

$$\varphi(t) = t^2, \quad f(x) = x^2 \sin\left(\frac{1}{1+x}\right), \quad \alpha = \frac{1}{6}, \quad \eta = \frac{3}{4}.$$

Clearly, $\varphi \in C([0,1],[0,\infty))$, and there exists $t_0 \in [\eta, 1] = \left[\frac{3}{4}, 1\right]$ such that $\varphi(t_0) > 0$. For instance, $t_0 = \frac{5}{6}$, then $\varphi\left(\frac{5}{6}\right) = \frac{25}{36} > 0$.
Also, $f \in C([0,\infty),[0,\infty))$, and:

$$f_0 = \lim_{x\to 0^+} x^2 \sin\left(\frac{1}{1+x}\right) = 0, \quad f_\infty = \lim_{x\to\infty} x^2 \sin\left(\frac{1}{1+x}\right) = \infty.$$

Hence, assumptions *(H1)*, *(H2)*, and case (i) are satisfied.
Therefore, by Theorem 2.5, the boundary value problem has at least one positive solution.

Chapter 3

Positive solutions of three-point integral boundary value problem

In this chapter, we study a boundary value problem involving a second-order differential equation with a three-point integral boundary condition.

We consider the problem:

$$\begin{cases} x''(t) + \varphi(t)f(x) = 0, & t \in (0,1) \\ x(0) = 0, \quad x(1) = \alpha \int_0^\eta x(s)\,ds \end{cases} \tag{3.1}$$

where $0 < \eta < 1$.

The aim of this chapter is to give results for existence of positive solutions to (3.1). Assuming:

- $0 < \alpha < \frac{2}{\eta^2}$,
- f is either superlinear or sublinear.

Throughout this chpter, we suppose the following conditions hold:

(C1) $f \in C([0,\infty),[0,\infty))$,

(C2) $\varphi \in C([0,1],[0,\infty))$ and there exists $t_0 \in [\eta, 1]$ such that $\varphi(t_0) > 0$.

3.1 Auxiliary Lemmas

Lemma 3.1. *If $\alpha\eta^2 \neq 2$, then for $h \in C([0,1])$, the problem*

$$\begin{cases} x''(t) + h(t) = 0, & t \in (0,1) \\ x(0) = 0, \quad x(1) = \alpha \int_0^\eta x(s)\,ds \end{cases} \tag{3.2}$$

has a unique solution

$$x(t) = \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)h(s)\,ds - \frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 h(s)\,ds - \int_0^t (t-s)h(s)\,ds.$$

Proof. From $x''(t) = -h(t)$ with integration from 0 to t, we get

$$x'(t) = x'(0) - \int_0^t h(s)\,ds,$$

and with integration again, we obtain

$$x(t) = x'(0)t - \int_0^t (t-s)h(s)\,ds.$$

So, we have

$$x(1) = x'(0) - \int_0^1 (1-s)h(s)\,ds,$$

and

$$\begin{aligned}\int_0^\eta x(s)\,ds &= x'(0)\frac{\eta^2}{2} - \int_0^\eta \left(\int_0^\tau (\tau-s)h(s)\,ds\right) d\tau \\ &= x'(0)\frac{\eta^2}{2} - \frac{1}{2}\int_0^\eta (\eta-s)^2 h(s)\,ds.\end{aligned}$$

From the condition $x(1) = \alpha \int_0^\eta x(s)\,ds$, we obtain that

$$x'(0) - \int_0^1 (1-s)h(s)\,ds = x'(0)\alpha\frac{\eta^2}{2} - \frac{\alpha}{2}\int_0^\eta (\eta-s)^2 h(s)\,ds.$$

Thus,

$$x'(0) = \frac{2}{2-\alpha\eta^2}\int_0^1 (1-s)h(s)\,ds - \frac{\alpha}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 h(s)\,ds.$$

So,

$$x(t) = \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)h(s)\,ds - \frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 h(s)\,ds - \int_0^t (t-s)h(s)\,ds.$$

□

Lemma 3.2. *Let $0 < \alpha < \frac{2}{\eta^2}$. If $h \in C([0,1],[0,+\infty))$, then the unique solution x of (3.2) is a positive.*

Proof. From the fact that $x''(t) = -h(t) \le 0$, x is concave on $[0,1]$.

- If $x(1) \ge 0$, then by the concavity of x and $x(0) = 0$, we get

  $$\frac{x(t)}{t} \ge x(1), \quad \text{for } t \in (0,1],$$

  thus $x(t) \ge 0$ for $t \in [0,1]$.

- If $x(1) < 0$, then by the concavity of x and $x(0) = 0$, we get

  $$\frac{x(t)}{t} \ge \frac{x(\eta)}{\eta}, \quad \text{for } t \in (0,\eta],$$

  it implies that

  $$\int_0^\eta x(s)\,ds \ge \frac{1}{2}\eta x(\eta).$$

  From $x(1) = \alpha \int_0^\eta x(s)\,ds$, and $x(1) < 0$, we have $\int_0^\eta x(s)\,ds < 0$. Then, $x(\eta) < 0$. Hence,

  $$x(1) = \alpha\int_0^\eta x(s)\,ds \ge \alpha\frac{\eta}{2}x(\eta) > \frac{x(\eta)}{\eta}$$

  which contradicts the concavity of x.

□

Lemma 3.3. *Let* $\alpha\eta^2 > 2$. *If* $h \in C([0,1],[0,\infty))$, *then* (3.2) *has no positive solution.*

Proof. Assume (3.2) has a positive solution x.

- If $x(1) > 0$, we have on one hand by the concavity of x, $\frac{x(\eta)}{\eta} \geq \frac{x(1)}{1}$ which implies that $x(\eta) > 0$, and on the other hand,

$$\frac{x(1)}{1} = \alpha \int_0^\eta x(s)\,ds \geq \alpha \frac{\eta}{2} x(\eta) = \frac{\alpha\eta^2}{2} \cdot \frac{x(\eta)}{\eta} > \frac{x(\eta)}{\eta},$$

  which contradicts the concavity of x.

- If $x(1) = 0$, then $\int_0^\eta x(s)\,ds = 0$, this is $x(t) = 0$ for all $t \in [0,\eta]$.
  If there exists $\rho \in (\eta, 1)$ such that $x(\rho) > 0$, then by concavity of x, we have

$$\frac{x(\eta) - x(0)}{\eta} \geq \frac{x(\rho) - x(0)}{\rho},$$

  it implies that $x(\rho) \leq 0$, which is a contradiction.

□

Lemma 3.4. *Let* $0 < \alpha < \frac{2}{\eta^2}$. *If* $h \in C([0,1],[0,\infty))$, *then the unique solution* x *of the problem* (3.2) *satisfies*

$$\inf_{t\in[\eta,1]} x(t) \geq \delta \|x\|,$$

where

$$\delta = \min\left\{\eta, \frac{\alpha\eta^2}{2}, \frac{\alpha\eta(1-\eta)}{2-\alpha\eta^2}\right\}.$$

Proof. Set $x(T) = \|x\|$. We divide the proof into three cases.

Case 1. If $\eta \leq T \leq 1$ and $\inf_{t\in[\eta,1]} x(t) = x(\eta)$, then the concavity of x implies that

$$\frac{x(\eta)}{\eta} \geq \frac{x(T)}{T} \geq x(T).$$

Thus,

$$\inf_{t\in[\eta,1]} x(t) \geq \eta \|x\|.$$

Case 2. If $\eta \leq T \leq 1$ and $\inf_{t\in[\eta,1]} x(t) = x(1)$, then the concavity of x implies

$$x(1) = \alpha \int_0^\eta x(s)\,ds \geq \frac{\alpha\eta^2}{2}\frac{x(\eta)}{\eta} \geq \frac{\alpha\eta^2}{2}\frac{x(T)}{T} \geq \frac{\alpha\eta^2}{2} x(T).$$

Thus,

$$\inf_{t\in[\eta,1]} x(t) > \frac{\alpha\eta^2}{2} \|x\|.$$

Case 3. If $T \leq \eta \leq 1$, then $\inf_{t\in[\eta,1]} x(t) = x(1)$. Using the concavity of x and from

$$\int_0^\eta x(s)\,ds \geq \frac{\eta}{2}x(\eta), \left(x(1) \geq \frac{\alpha\eta}{2}x(\eta)\right)$$

we have

$$\frac{x(1)-x(\eta)}{1-\eta} \leq \frac{x(T)-x(1)}{T-1} \leq \frac{x(T)-x(1)}{0-1}.$$

then

$$\begin{aligned} x(T) &\leq x(1) - \frac{x(1)-x(\eta)}{1-\eta} \\ &\leq x(1)\left(1 - \frac{\alpha\eta-2}{\alpha\eta(1-\eta)}\right) \\ &= x(1)\frac{2-\alpha\eta^2}{\alpha\eta(1-\eta)}. \end{aligned}$$

This implies that

$$\inf_{t\in[\eta,1]} x(t) \geq \frac{\alpha\eta(1-\eta)}{2-\alpha\eta^2}\|x\|.$$

□

3.2 Main Result

Theorem 3.5. *Assume that:*

(H1) $f \in C([0,+\infty),[0,+\infty))$,

(H2) $\varphi \in C([0,1],[0,+\infty))$, *and there exists* $t_0 \in (\eta,1)$ *such that* $\varphi(t_0) > 0$.

Then the problem (3.1) *has at least one positive solution in the case:*

(i) $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear), or*

(ii) $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear).*

Proof.
It is known that $0 < \alpha < \frac{2}{\eta^2}$. From Lemma 3.1. , x is a solution to the boundary value problem ((3.2)) if and only if x is a fixed point of operator A, where A is defined by

$$\begin{aligned} Ax(t) =& \frac{2t}{2-\alpha\eta^2}\left(\int_0^1 (1-s)\varphi(s)f(x(s))ds\right) - \frac{\alpha t}{2-\alpha\eta^2}\left(\int_0^\eta (\eta-s)^2\varphi(s)f(x(s))ds\right) \\ &- \int_0^t (t-s)\varphi(s)f(x(s))ds. \end{aligned}$$

Denote that

$$K = \left\{x/x \in C([0,1]), x \geq 0, \inf_{t\in[\eta,1]} x(t) \geq \delta\|x\|\right\},$$

where

$$\delta = \min\left\{\eta, \frac{\alpha\eta^2}{2}, \frac{\alpha\eta(1-\eta)}{2-\alpha\eta^2}\right\}.$$

It is obvious that K is a cone in $C([0,1])$.

Moreover, by Lemma 3.2. and Lemma 3.4. , $A(K) \sqsubset K$. It can be confirmed that $A : K \to K$ is completely continuous.

• **Superlinear Case**

Since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(x) \leq \varepsilon x$ for $0 < x \leq H_1$, where $\varepsilon > 0$ satisfies

$$\frac{2\varepsilon}{2-\alpha\eta^2}\int_0^1 (1-s)\varphi(s)ds \leq 1.$$

Thus, if we let

$$\Omega_1 = \{x \in C([0,1]) : \|x\| < H_1\},$$

then, for $x \in K \cap \partial\Omega_1$, we get

$$\begin{aligned} Ax(t) &\leq \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)\varphi(s)f(x(s))ds \\ &\leq \frac{2t\varepsilon}{2-\alpha\eta^2}\int_0^1 (1-s)\varphi(s)f(x(s)) \\ &\leq \frac{2\varepsilon}{2-\alpha\eta^2}\int_0^1 (1-s)\varphi(s)ds\|x\| \\ &\leq \|x\|. \end{aligned}$$

Thus $\|Ax\| \leq \|x\|$, for $x \in K \cap \partial\Omega_1$.

On the other hand, since $f_\infty = \infty$, there exists $\widetilde{H}_2 > 0$ such that $f(x) \geq \rho x$ for $x \geq \widetilde{H}_2$, where $\rho > 0$ is chosen so that

$$\rho\delta\frac{2\eta}{2-\alpha\eta^2}\int_0^1 (1-s)\varphi(s)ds \geq 1.$$

Let

$$H_2 = \max\left\{2H_1, \frac{\widetilde{H}_2}{\delta}\right\}, \quad \Omega_2 = \{x \in C([0,1]) : \|x\| < H_2\}.$$

Then $x \in K \cap \partial\Omega_2$ implies that

$$\inf_{t\in[\eta,1]} x(t) \geq \delta\|x\| = \delta H_2 \geq \widetilde{H}_2,$$

and so

$$\begin{aligned}
Ax(\eta) &= \frac{2\eta}{2-\alpha\eta^2}\int_0^1 (1-s)\varphi(s)f(x(s))ds - \frac{\alpha\eta}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2\varphi(s)f(x(s))ds \\
&\quad - \int_0^\eta (\eta-s)\varphi(s)f(x(s))ds \\
&= \frac{2\eta}{2-\alpha\eta^2}\int_0^1 (1-s)\varphi(s)f(x(s))ds - \frac{\alpha\eta}{2-\alpha\eta^2}\int_0^\eta (\eta^2-2\eta s+s^2)\varphi(s)f(x(s))ds \\
&\quad - \int_0^\eta (\eta-s)\varphi(s)f(x(s))ds \\
&= \int_0^1 \frac{2\eta-2\eta s}{2-\alpha\eta^2}\varphi(s)f(x(s))ds + \int_0^\eta \frac{\alpha\eta^2 s-\alpha\eta s^2-2\eta+2s}{2-\alpha\eta^2}\varphi(s)f(x(s))ds \\
&= \int_\eta^1 \frac{2\eta-2\eta s}{2-\alpha\eta^2}\varphi(s)f(x(s))ds + \int_0^\eta \frac{-2\eta s+\alpha\eta^2 s-\alpha\eta s^2+2s}{2-\alpha\eta^2}\varphi(s)f(x(s))ds \\
&= \int_\eta^1 \frac{2\eta(1-s)}{2-\alpha\eta^2}\varphi(s)f(x(s))ds + \int_0^\eta \frac{\alpha\eta s(\eta-s)+2s(1-\eta)}{2-\alpha\eta^2}\varphi(s)f(x(s))ds. \\
&\geq \frac{2\eta}{2-\alpha\eta^2}\int_\eta^1 (1-s)\,\varphi(s)\,f(x(s))\,ds \\
&\geq \frac{2\eta\rho}{2-\alpha\eta^2}\int_\eta^1 (1-s)\,\varphi(s)\,x(s)\,ds \\
&\geq \frac{2\eta\rho\delta}{2-\alpha\eta^2}\int_\eta^1 (1-s)\,\varphi(s)\,ds\,\|x\| \geq \|x\|.
\end{aligned}$$

Hence, $\|Ax\| \geq \|x\|, \quad x \in K \cap \partial\Omega_2$.

By the first part of Theorem of Guo-Krasnoselskii, A has a fixed point in $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that $H_1 \leq \|x\| \leq H_2$.

• **Sublinear Case**

Since $f_0 = \infty$, choose $H_3 > 0$ such that $f(x) \geq Mx$ for $0 < x \leq H_3$, where $M > 0$ satisfies

$$\frac{2\eta\delta M}{2-\alpha\eta^2}\int_\eta^1 (1-s)\,\varphi(s)\,ds \geq 1.$$

Let

$$\Omega_3 = \{x \in C([0,1]),\ \|x\| < H_3\},$$

then for $x \in K \cap \partial\Omega_3$, we get

$$\begin{aligned}
Ax(\eta) &= \frac{2\eta}{2-\alpha\eta^2}\int_\eta^1 (1-s)\,\varphi(s)\,f(x(s))\,ds - \frac{\alpha\eta}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2\varphi(s)f(x(s))ds \\
&\quad - \int_0^\eta (\eta-s)\,\varphi(s)\,f(x(s))\,ds \\
&\geq \frac{2\eta}{2-\alpha\eta^2}\int_\eta^1 (1-s)\,\varphi(s)\,f(x(s))\,ds \\
&\geq \frac{2\eta\delta M}{2-\alpha\eta^2}\int_\eta^1 (1-s)\,\varphi(s)\,ds\,\|x\| \geq \|x\|.
\end{aligned}$$

Thus, $\|Ax\| \geq \|x\|, \quad x \in K \cap \partial\Omega_3$.

On the other hand, since $f_\infty = 0$, there exists $\widetilde{H}_4 > 0$ so that $f(x) \geq \lambda x$ for $x \geq \widetilde{H}_4$, where $\lambda > 0$ satisfies

$$\frac{2\lambda}{2-\alpha\eta^2}\int_0^1 (1-s)\,\varphi(s)\,ds \leq 1.$$

Choose $H_4 = \max\left\{2H_3,\ \frac{\widetilde{H}_4}{\delta}\right\}$. Let

$$\Omega_4 = \{x \in C([0,1]),\ \|x\| < H_4\},$$

then $x \in K \cap \partial\Omega_4$ implies that

$$\inf_{t\in[\eta,1]} x(t) \geq \delta\|x\| = \delta H_4 \geq \widetilde{H}_4.$$

Therefore,

$$\begin{aligned} Ax(t) = &\frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)\,\varphi(s)\,f(x(s))\,ds - \frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2\,\varphi(s)\,f(x(s))\,ds \\ &- \int_0^t (t-s)\,\varphi(s)\,f(x(s))\,ds \\ \leq &\frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)\,\varphi(s)\,f(x(s))\,ds \\ \leq &\frac{2\lambda\|x\|}{2-\alpha\eta^2}\int_0^1 (1-s)\,\varphi(s)\,ds \leq \|x\|. \end{aligned}$$

Thus, $\|Ax\| \leq \|x\|, \quad x \in K \cap \partial\Omega_4$.

By the second part of the theorem of Guo-Krasnoselskii, A has a fixed point $x \in K \cap (\bar{\Omega}_4 \backslash \Omega_3)$ such that $H_3 \leq \|x\| \leq H_4$. This completes the sublinear part and the problem (3.1) has at least one positive solution.

□

3.3 Some examples

Example 3.1.
The boundary value problem

$$\begin{cases} x'' + (\sin t)e^{-x} = 0, & t \in (0,1) \\ x(0) = 0, \quad x(1) = 2\int_0^{1/2} x(s)\,ds \end{cases}$$

has at least one positive solution. In fact, assume that

$$\varphi(t) = \sin t, \quad f(x) = e^{-x}$$

In this case, we have $\alpha = 2$, $\eta = \frac{1}{2}$. Then, we get:

- $0 < \alpha < \frac{2}{\eta^2} \quad (0 < 2 < 8)$
- $\varphi \in C([0,1],[0,\infty))$ and there exists $t_0 \in [\eta,1] = [\frac{1}{2},1]$ such that $\varphi(t_0) > 0$. We choose $t_0 = \frac{3}{4}$, so
$$\varphi\left(\frac{3}{4}\right) = \sin\left(\frac{3}{4}\right) > 0$$
- $f \in C([0,1],[0,\infty))$ and $f_0 = \infty,\ f_\infty = 0$ (sublinear)

Then, all the conditions of the theorem 3.5. hold. So the conclusion is satisfied.

Example 3.2.
The boundary value problem

$$\begin{cases} x'' + (t^2+1)\ln(x+1) = 0, & t \in (0,1) \\ x(0) = 0, \quad x(1) = \int_0^{2/3} x(s)\,ds \end{cases}$$

has at least one positive solution.
Assume that:

$$\varphi(t) = t^2 + 1, \quad f(x) = \ln(x+1)$$

In this case, we have $\alpha = 1$, $\eta = \frac{2}{3}$. Then:

- $0 < \alpha < \frac{2}{\eta^2} \quad \left(0 < 1 < \frac{2}{(2/3)^2} = \frac{2}{4/9} = 4.5\right)$
- $\varphi \in C([0,1],[0,\infty))$ and there exists $t_0 \in [\eta, 1] = [\frac{2}{3}, 1]$ such that $\varphi(t_0) > 0$. We choose $t_0 = \frac{3}{4}$, so
$$\varphi\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^2 + 1 = \frac{9}{16} + 1 = \frac{25}{16} > 0$$
- $f \in C([0,\infty),[0,\infty))$, with $f_0 = 0$, $f_\infty = \infty$ (superlinear)

Then, all the conditions of Theorem 3.5 hold. So the conclusion is satisfied.

Example 3.3.
The boundary value problem

$$\begin{cases} x''(t) + \dfrac{\cos^2(t)}{1+x(t)} = 0, & t \in (0,1), \\ x(0) = 0, \quad x(1) = \dfrac{2}{3}\int_0^{\frac{3}{4}} x(s)\,ds \end{cases}$$

has at least one positive solution.
Assume that:

$$\varphi(t) = \cos^2(t), \quad f(x) = \frac{1}{1+x}$$

In this case, we have $\alpha = \dfrac{2}{3}$, $\eta = \dfrac{3}{4}$. Then:

- $0 < \alpha < \frac{2}{\eta^2} \quad \left(0 < \dfrac{2}{3} < \dfrac{2}{(3/4)^2} = \dfrac{32}{9} \approx 3.56\right)$
- $\varphi \in C([0,1],[0,\infty))$, and $\varphi(t_0) > 0$ for any $t_0 \in [\dfrac{3}{4}, 1]$, for example, $\varphi(0.8) = \cos^2(0.8) > 0$
- $f \in C([0,\infty),[0,\infty))$, with $f_0 = \infty$, $f_\infty = 0$ (sublinear)

Then, all the conditions of Theorem 3.5 hold. So the conclusion is satisfied.

Chapter 4

Positive solutions of n-point boundary value problem

In this chapter, we study the existence of positive solutions for a nonlinear second-order differential equation with an n-point boundary condition.

We consider the problem:

$$\begin{cases} x''(t) + \varphi(t)f(x) = 0, & t \in (0,\, 1) \\ x(0) = 0, & x(1) = \sum\limits_{i=1}^{n-2} a_i x(\xi_i) \end{cases} \tag{4.1}$$

where $f \in C([0,\infty),[0,\infty))$, $\varphi \in C([0,1],[0,\infty))$,
$a_i > 0$, $i = 1, 2, ..., n-2$ and $0 < \xi_1 < \xi_2 < ... < \xi_{n-2} < 1$.

4.1 Auxiliary Lemmas

We first give some auxiliary Lemmas.

Lemma 4.1. *Let $h \in C([0,1])$, $a_i > 0$ for $i = 1, 2, ..., n-2$ and $\sum\limits_{i=1}^{n-2} a_i\xi_i \neq 1$. Then the problem*

$$\begin{cases} x''(t) + h(t) = 0, & t \in (0,\, 1) \\ x(0) = 0, & x(1) = \sum\limits_{i=1}^{n-2} a_i x(\xi_i) \end{cases} \tag{4.2}$$

has a unique solution,

$$\begin{aligned} x(t) &= -\int_0^t (t-s)h(s)ds - t\frac{\sum_{i=1}^{n-2} a_i \int_0^{\xi_i}(\xi_i - s)h(s)ds}{1 - \sum\limits_{i=1}^{n-2} a_i\xi_i} \\ &\quad + t\frac{\int_0^1 (1-s)h(s)ds}{1 - \sum\limits_{i=1}^{n-2} a_i\xi_i}. \end{aligned}$$

Proof. We start from the differential equation:

$$x''(t) = -h(t).$$

Integrating twice, we obtain:

$$x'(t) = -\int_0^t h(s)\,\mathrm{d}s + C_1, \qquad x(t) = -\int_0^t (t-s)h(s)\,\mathrm{d}s + C_1 t + C_2.$$

Using the boundary condition $x(0) = 0$, we find $C_2 = 0$. Thus,

$$x(t) = -\int_0^t (t-s)h(s)\,\mathrm{d}s + C_1 t.$$

Now applying the nonlocal boundary condition:

$$x(1) = \sum_{i=1}^{n-2} a_i x(\xi_i),$$

we compute both sides.

Left-hand side:

$$x(1) = -\int_0^1 (1-s)h(s)\,\mathrm{d}s + C_1.$$

Right-hand side:

$$x(\xi_i) = -\int_0^{\xi_i} (\xi_i - s)h(s)\,\mathrm{d}s + C_1\xi_i,$$

so

$$\sum_{i=1}^{n-2} a_i x(\xi_i) = -\sum_{i=1}^{n-2} a_i \int_0^{\xi_i} (\xi_i - s)h(s)\,\mathrm{d}s + C_1 \sum_{i=1}^{n-2} a_i\xi_i.$$

Equating both sides:

$$-\int_0^1 (1-s)h(s)\,\mathrm{d}s + C_1 = -\sum_{i=1}^{n-2} a_i \int_0^{\xi_i} (\xi_i - s)h(s)\,\mathrm{d}s + C_1 \sum_{i=1}^{n-2} a_i\xi_i.$$

Solving for C_1, we obtain:

$$C_1\left(1 - \sum_{i=1}^{n-2} a_i\xi_i\right) = \int_0^1 (1-s)h(s)\,\mathrm{d}s - \sum_{i=1}^{n-2} a_i \int_0^{\xi_i} (\xi_i - s)h(s)\,\mathrm{d}s.$$

Since $\sum_{i=1}^{n-2} a_i\xi_i \neq 1$, the denominator is nonzero, and we find:

$$C_1 = \frac{\int_0^1 (1-s)h(s)\,\mathrm{d}s - \sum_{i=1}^{n-2} a_i \int_0^{\xi_i} (\xi_i - s)h(s)\,\mathrm{d}s}{1 - \sum_{i=1}^{n-2} a_i\xi_i}.$$

Substituting back into the expression for $x(t)$, we obtain the unique solution:

$$x(t) = -\int_0^t (t-s)h(s)\,\mathrm{d}s + t \cdot \frac{\int_0^1 (1-s)h(s)\,\mathrm{d}s - \sum_{i=1}^{n-2} a_i \int_0^{\xi_i} (\xi_i - s)h(s)\,\mathrm{d}s}{1 - \sum_{i=1}^{n-2} a_i\xi_i}.$$

□

Lemma 4.2. *Let $a_i \geq 0$ for $i = 1, \ldots, n-2$ and $\sum_{i=1}^{n-2} a_i \xi_i < 1$.*
If $h \in C([0,1])$ and $h \geq 0$, then the unique solution x of problem (4.2) satisfies

$$x \geq 0, \quad t \in [0,1].$$

Proof. From the fact that $x''(t) = -h(t) \leq 0$, we know the graph of $x(t)$ is concave down on $(0,1)$.
So, we have two cases:

- If $x(1) \geq 0$, then the concavity of x together with the boundary condition $x(0) = 0$:

  $$\frac{x(t) - x(0)}{t} \geq \frac{x(1) - x(0)}{1} \quad \text{for } t \in [0,1]$$

  implies that $x(t) \geq 0$ for $t \in [0,1]$.

- If $x(1) < 0$, then from the concavity of x, we know that

  $$\frac{x(\xi_i) - x(0)}{\xi_i} \geq \frac{x(1) - x(0)}{1} \quad \text{for } \xi_i \in [0,1], \text{ with } i = 1, \ldots, n-2.$$

  i.e.

  $$\frac{x(\xi_i)}{\xi_i} \geq x(1).$$

  This implies:

  $$x(1) = \sum_{i=1}^{n-2} a_i x(\xi_i) \geq \sum_{i=1}^{n-2} a_i \xi_i x(1).$$

  This contradicts the fact that $\sum_{i=1}^{n-2} a_i \xi_i < 1$.

□

Lemma 4.3. *Let $a_i \geq 0$ for $i = 1, \ldots, n-3$, $a_{n-2} > 0$ and $\sum_{i=1}^{n-2} a_i \xi_i > 1$.*
If $h \in C([0,1],[0,+\infty))$, then the problem (4.2) has no positive solution.

Proof. We use proof by contradiction.
Assume that (4.2) has a positive solution x, then

$$x(\xi_i) > 0 \quad \text{for } i = 1, \ldots, n-2$$

and

$$\begin{aligned} x(1) &= \sum_{i=1}^{n-2} a_i x(\xi_i) \\ &= \sum_{i=1}^{n-2} a_i \xi_i \frac{x(\xi_i)}{\xi_i} \\ &\geq \sum_{i=1}^{n-2} a_i \xi_i \frac{x(\bar{\xi})}{\bar{\xi}} > \frac{x(\bar{\xi})}{\bar{\xi}} \end{aligned}$$

(where $\bar{\xi} \in \{\xi_1, \ldots, \xi_{n-2}\}$ satisfies $\frac{x(\bar{\xi})}{\bar{\xi}} = \min\left\{\frac{x(\xi_i)}{\xi_i}; i = 1, \ldots, n-2\right\}$)

So, we can written in the form

$$\frac{x(\bar{\xi}) - x(0)}{\bar{\xi}} < \frac{x(1) - x(0)}{1},$$

This contradicts the concavity of x.

If $x(1) = 0$, then applying $a_{n-2} > 0$, we know that

$$x(\xi_{n-2}) = 0.$$

Then, from the concavity of x when $0 < \xi_{n-2} < t$,

$$\frac{x(\xi_{n-2}) - x(0)}{\xi_{n-2}} \geq \frac{x(t) - x(0)}{t}$$

it's who gives $x(t) \leq 0$.

This contradicts that $x(t) > 0$ for $t \in [0, 1]$. □

Lemma 4.4. *Let* $a_i \geq 0$ *for* $i = 1, \ldots, n-2$ *and* $\sum_{i=1}^{n-2} a_i \xi_i < 1$.
If $h \in C([0,1], [0, +\infty))$, *then the unique solution* x *of problem (4.2) satisfies*

$$\min_{t \in [\xi_{n-2}, 1]} x(t) \geq \gamma \|x\|$$

where

$$\gamma = \min\left\{\frac{a_{n-2}(1 - \xi_{n-2})}{1 - a_{n-2}\xi_{n-2}},\ a_{n-2}\xi_{n-2}), \xi_1\right\}.$$

Proof. We divide the proof into two steps.

Step 1 : We deal with the case that $\sum_{i=1}^{n-2} a_i < 1$.

Set $x(\bar{t}) = \|x\|$.

If $\bar{t} \leq \xi_{n-2} < 1$, then

$$\min_{t \in [\xi_{n-2}, 1]} x(t) = x(1).$$

Indeed, by the concavity of x, and for all $t \in]\xi_{n-2}, 1]$, we have

$$\frac{x(t) - x(\xi_{n-2})}{t - \xi_{n-2}} \geq \frac{x(1) - x(\xi_{n-2})}{1 - \xi_{n-2}} \geq \frac{x(1) - x(\xi_{n-2})}{t - \xi_{n-2}}$$

which implies that for all $t \in]\xi_{n-2}, 1]$, $x(t) \geq x(1)$
and with $x(\xi_{n-2}) \geq x(1)$, so for all $t \in [\xi_{n-2}, 1]$, we get $x(t) \geq x(1)$.

From the fact that

$$x(1) = \sum_{i=1}^{n-2} a_i x(\xi_i) \geq a_{n-2} x(\xi_{n-2}),$$

and

$$\frac{x(1) - x(\xi_{n-2})}{1 - \xi_{n-2}} \leq \frac{x(\bar{t}) - x(1)}{\bar{t} - 1} \leq \frac{x(\bar{t}) - x(1)}{0 - 1},$$

we get:

$$\begin{aligned}
x(\bar{t}) &\le x(1) + \frac{x(1) - x(\xi_{n-2})}{1 - \xi_{n-2}}(0 - 1) \\
&= x(1) - \frac{x(1)}{1 - \xi_{n-2}} + \frac{x(\xi_{n-2})}{1 - \xi_{n-2}} \\
&\le x(1) - \frac{x(1)}{1 - \xi_{n-2}} + \frac{x(1)}{a_{n-2}(1 - \xi_{n-2})} \\
&= x(1)\frac{1 - a_{n-2}\xi_{n-2}}{a_{n-2}(1 - \xi_{n-2})}.
\end{aligned}$$

This implies

$$\min_{t\in[\xi_{n-2},1]} x(t) \ge \|x\| \frac{a_{n-2}(1 - \xi_{n-2})}{1 - a_{n-2}\xi_{n-2}}.$$

If $\xi_{n-2} < \bar{t} < 1$, then we claim that

$$\min_{t\in[\xi_{n-2},1]} x(t) = x(1).$$

In fact, if $\min_{t\in[\xi_{n-2},1]} x(t) = x(\xi_{n-2})$, then we have that $\bar{t} \in [\xi_{n-2}, 1]$ and

$$x(\xi_{n-2}) \ge \cdots \ge x(\xi_2) \ge x(\xi_1).$$

Then we have

$$x(1) = \sum_{i=1}^{n-2} a_i x(\xi_i) \le \sum_{i=1}^{n-2} a_i x(\xi_{n-2}) < x(\xi_{n-2}) \le x(1), \quad (\sum a_i < 1)$$

a contradiction.

From the concavity of x, we know that

$$\frac{x(\xi_{n-1})}{\xi_{n-2}} \ge \frac{x(\bar{t})}{\bar{t}} \ge x(\bar{t})$$

and from $x(1) \ge a_{n-2}x(\xi_{n-2})$, we conclude that

$$\frac{x(1)}{a_{n-2}\xi_{n-2}} \ge x(\bar{t}).$$

Thus

$$\min_{t\in[\xi_{n-2},1]} x(t) \ge a_{n-2}\xi_{n-2}\|x\|.$$

Step 2 : We deal with the case that

$$\sum_{i=1}^{n-2} a_i \ge 1.$$

Set $x(\bar{t}) = \|x\|$.

- If $x(\xi_{n-2}) \le x(1)$, then

$$\min_{t\in[\xi_{n-2},1]} x(t) = x(\xi_{n-2}).$$

From the concavity of x, we know that $\bar{t} \in [\xi_{n-2}, 1]$. This implies

$$\frac{x(\xi_{n-2})}{\xi_{n-2}} \ge \frac{x(\bar{t})}{\bar{t}} \ge x(\bar{t}).$$

Then,

$$\min_{t\in[\xi_{n-2},1]} x(t) \ge \xi_{n-2}\|x\|.$$

- If $x(\xi_{n-2}) > x(1)$, then

$$\min_{t\in[\xi_{n-2},1]} x(t) = x(1).$$

Furthermore, we have $\bar{t} \in [\xi_1, 1]$. In fact, assume to the contrary that $\bar{t} \in [0, \xi_1)$, then

$$x(\xi_1) \ge x(\xi_2) \ge \cdots \ge x(\xi_{n-2}) > x(1).$$

This implies

$$x(1) = \sum_{i=1}^{n-2} a_i x(\xi_i) \ge \sum_{i=1}^{n-2} a_i x(\xi_{n-2}) > \sum a_i x(1) \ge x(1),$$

a contradiction, so $\bar{t} \in [\xi_1, 1]$.
Since $\sum_{i=1}^{n-2} a_i \ge 1$, we know that there exists $\bar{\xi} \in \{\xi_1, \ldots, \xi_{n-2}\}$ such that $x(\bar{\xi}) \le x(1)$. This implies that

$$x(\xi_1) \le x(\xi_2) \le \cdots \le x(\bar{\xi}) \le x(1).$$

From $\bar{t} \in [\xi_1, 1]$ and with concavity of x, we can conclude that

$$\frac{x(1)}{\xi_1} \ge \frac{x(\xi_1)}{\xi_1} \ge \frac{x(\bar{t})}{\bar{t}} \ge x(\bar{t}).$$

This implies that

$$\min_{t\in[\xi_{n-2},1]} x(t) \ge \xi_1\|x\|.$$

We conclude:

$$\min_{t\in[\xi_{n-2},1]} x(t) \ge \delta\|x\|, \quad \text{where}$$

$$\begin{aligned}\delta &= \min\left\{\frac{a_{n-2}(1-\xi_{n-2})}{1-a_{n-2}\xi_{n-2}},\ a_{n-2}\xi_{n-2},\ \xi_{n-2},\ \xi_1\right\} \\ &= \min\left\{\frac{a_{n-2}(1-\xi_{n-2})}{1-a_{n-2}\xi_{n-2}},\ a_{n-2}\xi_{n-2},\ \xi_1\right\}.\end{aligned}$$

□

4.2 Result of existence

Theorem 4.5. *Assume that :*

(C1) $f \in C([0,\infty),[0,\infty))$ *and the following limits are exist:*

$$f_0 = 0 \quad and \quad f_\infty = \infty \quad (superlinear\ case)$$

or

$$f_0 = \infty \quad and \quad f_\infty = 0 \quad (sublinear\ case)$$

(C2) $\varphi \in C([0,1],[0,\infty))$ *and there exists* $t_0 \in [\xi_{n-2}, 1]$ *such that* $\varphi(t_0) > 0$.

(C3) For $i = 1, \ldots, n-2$, $a_i \geq 0$ *and* $\sum_{i=1}^{n-2} a_i\xi_i < 1$.

Then, the problem (4.1) has at least one positive solution.

Proof.
• **Superlinear case:** Suppose that $f_0 = 0$ and $f_\infty = \infty$.
We know, problem (4.1) has a solution $x : t \mapsto x(t)$ if and only if x solves the operator equation:

$$\begin{aligned} x(t) = &- \int_0^t (t-s)a(s)f(x(s))\,ds - t\frac{\sum_{i=1}^{n-2} a_i \int_0^{\xi_i}(\xi_i - s)a(s)f(x(s))\,ds}{1-\sum_{i=1}^{n-2} a_i\xi_i} \\ &+ t\frac{\int_0^1 (1-s)a(s)f(x(s))\,ds}{1-\sum_{i=1}^{n-2} a_i\xi_i} \\ :=& Tx(t). \end{aligned}$$

Denote

$$K = \left\{ x/x \in C([0,1],[0,\infty)) : \min_{t\in[\xi_{n-2},1]} x(t) \geq \delta\|x\| \right\},$$

where δ is defined in Lemma 3.4.

It is obvious that K is a cone in $C([0,1])$.

Moreover, by Lemma 3.4., $TK \sqsubset K$.

In addition, $T : K \to K$ is completely continuous. In fact, according to (C2), (C3) the operator T is continuous.

On the other hand, we show that the operator T is compact.

We have for all $t \in [0,1]$:

$$\begin{aligned} |Tx(t)| &\leq \int_0^1 |\varphi(s)||f(x(s))|\,ds + \frac{\sum a_i\xi_i}{1-\sum a_i\xi_i}\int_0^1 |\varphi(s)||f(x(s))|\,ds \\ &+ \frac{1}{1-\sum a_i\xi_i}\int_0^1 |\varphi(s)||f(x(s))|\,ds \\ &= \frac{2}{1-\sum a_i\xi_i}\int_0^1 |\varphi(s)||f(x(s))|\,ds \end{aligned}$$

which proves that T is bounded.
Now, we show that T is equicontinuous:

For $t_1, t_2 \in [0,1]$ with $(t_2 \leq t_1)$ and $|t_1 - t_2| \to 0$, we have:

$$\begin{aligned}
|Tx(t_1) - Tx(t_2)| = \Bigg| &- \int_0^{t_1} (t_1 - s)\varphi(s) f(x(s))\, ds - t_1 \frac{\sum a_i \int_0^{\xi_i} (\xi_i - s)\varphi(s) f(x(s))\, ds}{1 - \sum a_i \xi_i} \\
&+ t_1 \frac{\int_0^1 (1 - s)\varphi(s) f(x(s))\, ds}{1 - \sum a_i \xi_i} \\
&+ \int_0^{t_2} (t_2 - s)\varphi(s) f(x(s))\, ds + t_2 \frac{\sum a_i \int_0^{\xi_i} (\xi_i - s)\varphi(s) f(x(s))\, ds}{1 - \sum a_i \xi_i} \\
&- t_2 \frac{\int_0^1 (1 - s)\varphi(s) f(x(s))\, ds}{1 - \sum a_i \xi_i} \Bigg| \\
= \Bigg| &(t_2 - t_1) \frac{\sum a_i \int_0^{\xi_i} (\xi_i - s)\varphi(s) f(x(s))\, ds}{1 - \sum a_i \xi_i} + (t_1 - t_2) \frac{\int_0^1 (1 - s)\varphi(s) f(x(s))\, ds}{1 - \sum a_i \xi_i} \\
&- \int_0^{t_1} (t_1 - s)\varphi(s) f(x(s))\, ds - \int_0^{t_2} (t_1 - s)\varphi(s) f(x(s))\, ds \\
&+ \int_0^{t_2} (t_1 - s)\varphi(s) f(x(s))\, ds + \int_0^{t_2} (t_2 - s)\varphi(s) f(x(s))\, ds \Bigg| \\
= \Bigg| &(t_2 - t_1) \frac{\sum a_i \int_0^{\xi_i} (\xi_i - s)\varphi(s) f(x(s))\, ds}{1 - \sum a_i \xi_i} + (t_1 - t_2) \frac{\int_0^1 (1 - s)\varphi(s) f(x(s))\, ds}{1 - \sum a_i \xi_i} \\
&+ (t_2 - t_1) \int_0^{t_2} \varphi(s) f(x(s))\, ds + \int_{t_1}^{t_2} (t_1 - s)\varphi(s) f(x(s))\, ds \Bigg| \\
\le |t_2 - t_1| &\frac{\sum a_i \int_0^{\xi_i} (\xi_i - s)\varphi(s) f(x(s))\, ds}{1 - \sum a_i \xi_i} + |t_1 - t_2| \frac{\int_0^1 (1 - s)\varphi(s) f(x(s))\, ds}{1 - \sum a_i \xi_i} \\
&+ |t_1 - t_2| \int_0^{t_2} \varphi(s) |f(x(s))|\, ds + \int_{t_1}^{t_2} (t_1 - s)\varphi(s) f(x(s))\, ds
\end{aligned}$$

Which implies:

$$|Tx(t_1) - Tx(t_2)| \to 0 \quad \text{as} \quad |t_1 - t_2| \to 0.$$

Now since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(x) \le \varepsilon x$, for $0 < x < H_1$, where $\varepsilon > 0$ satisfies:

$$\frac{\varepsilon \int_0^1 (1 - s)\varphi(s)\, ds}{1 - \sum_{i=1}^{n-2} a_i \xi_i} \le 1.$$

Thus, if $x \in K$ and $\|x\| = H_1$, then we get:

$$\begin{aligned}
Tx(t) &\le \frac{t \int_0^1 (1 - s)\varphi(s) f(x(s))\, ds}{1 - \sum a_i \xi_i} \\
&\le \frac{\int_0^1 (1 - s)\varphi(s) \varepsilon x(s)\, ds}{1 - \sum a_i \xi_i} \\
&\le \frac{\int_0^1 (1 - s)\varphi(s) \varepsilon\, ds \|x\|}{1 - \sum a_i \xi_i} \\
&\le \frac{\int_0^1 (1 - s)\varphi(s) \varepsilon\, ds H_1}{1 - \sum a_i \xi_i}.
\end{aligned}$$

If we let

$$\Omega_1 = \{x \in C([0, 1]) : \|x\| < H_1\},$$

then we get $\|Tx\| \leq \|x\|$, for $x \in K \cap \partial\Omega_1$.

Furthermore, since $f_\infty = \infty$, there exists $\widehat{H}_2 > 0$ such that $f(x) \geq \rho x$ for $x \geq \widehat{H}_2$, where $\rho > 0$ is chosen so that:

$$\rho\delta \frac{1}{1-\sum a_i\xi_i} \sum_{i=1}^{n-2} a_i \int_{\xi_{n-2}}^{1} \xi_1(1-s)a(s)\,ds \geq 1.$$

Let

$$H_2 = \max\left\{2H_1, \frac{\widehat{H}_2}{\delta}\right\}, \quad \Omega_2 = \{x \in C([0,1]) : \|x\| < H_2\},$$

then $x \in K$ and $\|x\| = H_2$ implies

$$\min_{t\in[\xi_{n-2},1]} x(t) \geq \delta\|x\| \geq \widehat{H}_2,$$

and so

$$\begin{aligned} Tx(\xi_i) = &-\int_0^{\xi_i} (\xi_i - s)\varphi(s)f(x(s))\,ds - \xi_i \frac{\sum a_i \int_0^{\xi_i} (\xi_i - s)\varphi(s)f(x(s))\,ds}{1-\sum a_i\xi_i} \\ &+\xi_i \frac{\int_0^1 (1-s)\varphi(s)f(x(s))\,ds}{1-\sum a_i\xi_i}. \end{aligned}$$

This implies:

$$\begin{aligned} x(1) &= \sum_{i=1}^{n-2} a_i\, x(\xi_i) \\ &= -\sum a_i \int_0^{\xi_i} (\xi_i - s)\,\varphi(s)\,f(x(s))\,ds \quad -\sum a_i\xi_i \frac{\sum a_i \int_0^{\xi_i} (\xi_i - s)\,\varphi(s)\,f(x(s))\,ds}{1-\sum a_i\xi_i} \\ &\quad + \sum a_i\xi_i \frac{\int_0^1 (1-s)\,\varphi(s)\,f(x(s))\,ds}{1-\sum a_i\xi_i} \\ &= -\frac{1}{1-\sum a_i\xi_i} \sum a_i \int_0^{\xi_i} (\xi_i - s)\,\varphi(s)\,f(x(s))\,dx \\ &\quad + \sum a_i\xi_i \frac{\int_0^1 (1-s)\,\varphi(s)\,f(x(s))\,ds}{1-\sum a_i\xi_i} \\ &= \frac{1}{1-\sum a_i\xi_i} \sum a_i \Bigg(-\int_0^{\xi_i} \xi_i\varphi(s)\,f(x(s))\,ds + \int_0^{\xi_i} s\,\varphi(s)\,f(x(s))\,ds \\ &\quad\; + \int_0^1 \xi_i\varphi(s)\,f(x(s))\,ds - \int_0^1 \xi_i s\,\varphi(s)\,f(x(s))\,ds \Bigg) \\ &\geq \frac{1}{1-\sum a_i\xi_i} \sum a_i \left(\int_{\xi_i}^1 \xi_i\varphi(s)\,f(x(s))\,ds - \xi_i \int_{\xi_i}^1 s\,\varphi(s)\,f(x(s))\,ds \right) \\ &= \frac{1}{1-\sum a_i\xi_i} \sum_{i=1}^{n-2} a_i \int_{\xi_i}^1 \xi_i(1-s)\,\varphi(s)\,f(x(s))\,ds \\ &\geq \frac{1}{1-\sum a_i\xi_i} \sum_{i=1}^{n-2} a_i \int_{\xi_i}^1 \xi_1(1-s)\,\varphi(s)\,f(x(s))\,ds \\ &\geq \frac{1}{1-\sum a_i\xi_i} \sum a_i \int_{\xi_{n-2}}^1 \xi_1(1-s)\,\varphi(s)\,f(x(s))\,ds \end{aligned}$$

Hence, for $x \in K \cap \partial\Omega_2$,

$$\|Tx\| \geq x(1) \geq \rho\delta \frac{1}{1-\sum a_i\xi_i} \sum a_i \int_{\xi_{n-2}}^{1} \xi_1(1-s)\varphi(s)f(x(s))\,ds\|x\| \geq \|x\|.$$

Therefore, by the first part of the fixed-point theorem, it follows that T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, such that

$$H_1 \leq \|x\| \leq H_2.$$

• **Superlinear case:** Now suppose that $f_0 = \infty$ and $f_\infty = 0$.
We first choose $H_3 > 0$ such that $f(x) \geq Mx$ for $0 < x < H_3$, where

$$M\delta \frac{1}{1-\sum a_i\xi_i} \sum a_i \int_{\xi_{n-2}}^{1} \xi_1(1-s)\varphi(s)\,ds \geq 1.$$

For $x \in K$ and $\|x\| = H_3$, using the same method as above, we get:

$$\begin{aligned}
Tx(1) &= \sum_{i=1}^{n-2} a_i Tx(\xi_i) \\
&\geq \frac{1}{1-\sum a_i\xi_i} \sum a_i \int_{\xi_i}^{1} \xi_i(1-s)\varphi(s)f(x(s))\,ds \\
&\geq \frac{1}{1-\sum a_i\xi_i} \sum a_i \int_{\xi_{n-2}}^{1} \xi_1(1-s)\varphi(s)Mx(s)\,ds \\
&\geq \frac{1}{1-\sum a_i\xi_i} \sum a_i \int_{\xi_{n-2}}^{1} \xi_1(1-s)\varphi(s)M\delta\,ds\,\|x\| \\
&\geq M\delta \frac{1}{1-\sum a_i\xi_i} \sum a_i \int_{\xi_{n-2}}^{1} \xi_1(1-s)\varphi(s)\,ds\,H_3 \\
&\geq H_3.
\end{aligned}$$

Thus, we may let $\Omega_3 = \{x \in C([0,1]) : \|x\| < H_3\}$ so that

$$\|Tx\| \geq \|x\|, \quad x \in K \cap \partial\Omega_3$$

Now, since $f_\infty = 0$, there exists $\hat{H}_4 > 0$ so that $f(x) \leq \lambda x$ for $x \geq \hat{H}_4$ where $\lambda > 0$ satisfies

$$\frac{\lambda \int_0^1 (1-s)\varphi(s)ds}{1-\sum a_i\xi_i} \leq 1$$

We consider two cases:
Case 1: Suppose f is bounded, say $f(x) \leq N$ for all $x \in [0,\infty)$. In this case, choose

$$H_4 = \max\left\{2H_3, \frac{N\int_0^1(1-s)\varphi(s)ds}{1-\sum a_i\xi_i}\right\}$$

so that, for $x \in K$ with $\|x\| = H_4$, we have

$$\begin{aligned}
Tx(t) &= -\int_0^t (t-s)\varphi(s)f(x(s))\,ds - t\frac{\sum a_i \int_0^{\xi_i}(\xi_i-s)\varphi(s)f(x(s))\,ds}{1-\sum a_i\xi_i} \\
&\quad + t\frac{\int_0^1(1-s)\varphi(s)f(x(s))\,ds}{1-\sum a_i\xi_i} \\
&\leq \frac{\int_0^1(1-s)\varphi(s)f(x(s))\,ds}{1-\sum a_i\xi_i} \\
&\leq \frac{\int_0^1(1-s)\varphi(s)N\,ds}{1-\sum a_i\xi_i} \\
&\leq H_4,
\end{aligned}$$

and therefore, $\|Tx\| \leq \|x\|$.

Case 2: If f is unbounded, then we know from (C1) that there exists H_4 such that

$$H_4 > \max\left\{2H_3, \frac{1}{\delta}\hat{H}_4\right\}$$

and

$$f(x) \leq f(H_4), \quad \text{for } 0 < x \leq H_4.$$

Then, for $x \in K$ and $\|x\| = H_4$, we have

$$\begin{aligned}
Tx(t) &= -\int_0^t (t-s)\varphi(s)f(x(s))\,ds - t\frac{\sum a_i \int_0^{\xi_i}(\xi_i - s)\varphi(s)f(x(s))\,ds}{1-\sum a_i\xi_i} \\
&\quad + t\frac{\int_0^1 (1-s)\varphi(s)f(x(s))\,ds}{1-\sum a_i\xi_i} \\
&\leq t\frac{\int_0^1 (1-s)\varphi(s)f(x(s))\,ds}{1-\sum a_i\xi_i} \\
&\leq \frac{\int_0^1 (1-s)\varphi(s)f(H_4)\,ds}{1-\sum a_i\xi_i} \\
&\leq \frac{\int_0^1 (1-s)\varphi(s)\lambda H_4\,ds}{1-\sum a_i\xi_i} \\
&\leq H_4.
\end{aligned}$$

Therefore, in either case, we may put

$$\Omega_4 = \{x \in C([0,1]), \|x\| < H_4\}$$

and for $x \in K \cap \partial\Omega_4$, we may have $\|Tx\| \leq \|x\|$.

By the second part of the fixed-point theorem of GuoKrasnosel'skii, the problem (4.1) has a positive solution.

□

4.3 Some examples

Example 4.1.
The boundary value problem

$$\begin{cases} x'' + \frac{1}{2-t}e^{-x} = 0, & t \in (0,1) \\ x(0) = 0, \quad x(1) = \frac{1}{3}x\left(\frac{1}{4}\right) + \frac{1}{6}x\left(\frac{1}{2}\right) \end{cases}$$

has at least one positive solution.

Indeed, assume that

$$\varphi(t) = \frac{1}{2-t}, \quad f(x) = e^{-x}$$

In this case, we have:

$$a_1 = \frac{1}{3}, \quad a_2 = \frac{1}{6}, \quad \xi_1 = \frac{1}{4}, \quad \xi_2 = \frac{1}{2}$$

Then, we get:

- $\varphi \in C([0,1],[0,\infty))$ and there exists $t_0 \in [\xi_2, 1] = [\frac{1}{2}, 1]$ such that $\varphi(t_0) > 0$, we choose $t_0 = \frac{3}{4}$, so that $\varphi(\frac{3}{4}) = \frac{4}{5} > 0$.
- $f \in C([0,1],[0,\infty))$ and $f_0 = \infty$, $f_\infty = 0$ (sublinear case)
- $\sum_{i=1}^{2} a_i\xi_i = a_1\xi_1 + a_2\xi_2 = \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{6}\cdot\frac{1}{2} = \frac{1}{12} + \frac{1}{12} = \frac{1}{6} < 1$

Then, all the conditions of Theorem 4.1 hold. So the conclusion is satisfied.

Example 4.2.
Consider the following boundary value problem:

$$\begin{cases} x'' + \sin(t+1)\, x^3 = 0, & t \in (0,1) \\ x(0) = 0, \quad x(1) = \dfrac{1}{4}x\left(\dfrac{1}{4}\right) + \dfrac{1}{5}x\left(\dfrac{3}{4}\right) \end{cases}$$

We define:

$$\varphi(t) = \sin(t+1), \quad f(x) = x^3$$

Given:

$$a_1 = \frac{1}{4}, \quad a_2 = \frac{1}{5}, \quad \xi_1 = \frac{1}{4}, \quad \xi_2 = \frac{3}{4}$$

Now we verify the conditions of the theorem:

- $\varphi \in C([0,1],[0,\infty))$, and there exists $t_0 \in [\xi_2, 1] = [\frac{3}{4}, 1]$ such that $\varphi(t_0) > 0$. We choose $t_0 = 0.8$, so that $\varphi(0.8) = \sin(1.8) \approx 0.0314 > 0$.
- $f(x) = x^3$ satisfies:

$$f_0 = \lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} x^2 = 0, \quad f_\infty = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} x^2 = \infty$$

  So the function is superlinear.

- For the boundary condition:

$$\sum_{i=1}^{2} a_i\xi_i = \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{5}\cdot\frac{3}{4} = \frac{1}{16} + \frac{3}{20} = \frac{5}{80} + \frac{12}{80} = \frac{17}{80} < 1$$

Hence, all the hypotheses of the second existence theorem are satisfied, and we conclude that the problem admits at least one positive solution.

Example 4.3.
Consider the boundary value problem:

$$\begin{cases} x'' + \dfrac{2}{t+2}\cdot\sqrt{x+1} = 0, & t \in (0,1) \\ x(0) = 0, \quad x(1) = \dfrac{1}{6}x\left(\dfrac{1}{5}\right) + \dfrac{1}{5}x\left(\dfrac{1}{2}\right) + \dfrac{1}{8}x\left(\dfrac{4}{5}\right) \end{cases}$$

Define:

$$\varphi(t) = \frac{2}{t+2}, \quad f(x) = \sqrt{x+1}$$

Given:

$$a_1 = \frac{1}{6}, \quad a_2 = \frac{1}{5}, \quad a_3 = \frac{1}{8}, \quad \xi_1 = \frac{1}{5}, \quad \xi_2 = \frac{1}{2}, \quad \xi_3 = \frac{4}{5}$$

We verify the conditions of the existence theorem:

- $\varphi(t) = \dfrac{2}{t+2}$ is continuous and positive on $[0,1]$, and there exists $t_0 \in [\xi_2, 1] = [\frac{1}{2}, 1]$ such that $\varphi(t_0) > 0$. We choose $t_0 = 0.9$ we have $\varphi(0.9) = \dfrac{2}{2.9} \approx 0.689 > 0$.

- $f(x) = \sqrt{x+1}$ satisfies:

$$f_0 = \lim_{x\to 0^+} \frac{\sqrt{x+1}}{x} = \infty, \quad f_\infty = \lim_{x\to\infty} \frac{\sqrt{x+1}}{x} = 0$$

  So f is superlinear.

- For the boundary condition:

$$\sum_{i=1}^{3} a_i \xi_i = \frac{1}{6}\cdot\frac{1}{5} + \frac{1}{5}\cdot\frac{1}{2} + \frac{1}{8}\cdot\frac{4}{5} = \frac{1}{30} + \frac{1}{10} + \frac{1}{10} = \frac{7}{30} < 1$$

Therefore, all the hypotheses of the sublinear existence theorem are satisfied, and the problem admits at least one positive solution.

Conclusion

In this memory, we have presented existence results of positive solutions for some types of second-order boundary value problems associated with differential equations posed on bounded intervals and where boundary conditions are multi-point.
We have used several methods to search for the existence of positive solutions to these problems, these methods are based on the corresponding function associated with the proposed problems in the superlinear or sublinear case, we need a compactness criterion to prove that the proposed operators are compact. We also devote in all phases of our work to construct and choose the necessary and sufficient conditions on the elements of the studied problems.
This study is divided into four chapters. The first chapter provides a preliminary overview of general concepts used in the various chapters of the work, and introduces some basic tools. We began our work in the second chapter with a study of the existence of positive solutions to the second-order three-point problem.
Dans le troisième chapitre,nous avons étudié quelques résultats d'existence des solutions positives du problème du second ordre à trois points où les conditions aux bords sont intégral.
In the fourth chapter, we studied an existence result for positive solutions to the second-order multi-point problem.
The methodology for proving existence theories depends on converting the proposed problems into operational equations, in addition to using the compactness criterion to apply the Ascoli-Arzéla theorem and inserting the Guo-Krasnosel'skii fixed-point theorem to obtain the appropriate results.
As for us, we hope that this memort opens the way for us to embark on the mathematical study of some boundary problems.

Bibliography

[1] D. Cao, R. Ma, Positive solutions to a second order multi-point boundary value problem, Electronic Journal of Differential Equations, Vol. 2000(2000), No. 65, pp. 1-8.

[2] K.Deimling, Nonlinear Functional Analysis, Springer Verlag, Berlin Heidelberg New York Tokyo, 1985.

[3] M. Feng, W. Ge, Positive solutions for a class of m-point singular boundary value problems, Math. Comput. Modelling, 46(2007), 375-383.

[4] C.S. Goodrich, Positive solutions to boundary value problems with nonlinear boundary conditions, Nonlinear Analysis 75 (2012) 417-432.

[5] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones. Academic Press, San Diego, CA, 1988.

[6] Y. Guo, W. Ge, Positive solutions for three-point boundary value problems with dependence on the first order derivative, J. Math. Anal. Appl. 290 (2004), 291-301.

[7] C. Gupta, Solvability of a three-point nonlinear boundary value problem for second order ordinary differential equations, J. Math. Anal. Appl. 168 (1992), 540-551.

[8] L-G. Hu, Positive solutions for singular boundary value problems involving integral conditions, Boundary Value Problems 2012, 2012:72.

[9] M. Krasnosel'skii, Positive Solutions of Operator Equations, Noordhoff, Groningen (1964).

[10] B. Liu, Positive solutions of a nonlinear three-point boundary value problem, Appl. Math. Comput. 132 (2002), 11-28.

[11] S. Marano, A remark on a second order three-point boundary value problem, J. Math. Anal. Appl. 183 (1994), 518-522.

[12] C. Miao, W. Ge, Existence and uniqueness of positive solutions for singular three-point boundary value problems, J. Appl. Math. and Informatics Vol. 27(2009), No. 3-4, pp. 895-902.

[13] R. Ma, Positive solutions of a nonlinear three-point boundary value problem, Electron. J. Differential Equations 34 (1999), 8 pp. (electronic).

[14] R. Ma, N. Castaneda, Positive solutions for second-order three-point boundary value problems, Appl. Math. Lett. 14 (2001), 1-5.

[15] R. Ma, H. Wang, Positive solutions of nonlinear three-point boundary-value problems, J. Math. Anal. Appl. 279 (2003), 216-227.

[16] A.L.M. Martinez, C.A.P. Martinez, E.V. Castelani, T. de Souza Pinto, Remarks on Positive Solutions for an m-Point Boundary Value Problem, Int. J. of Pure and App. Math., Vol. 25, 2012, no. 2, 281 -297.

[17] J. Tariboon, W. Sudsutad, Positive Solutions of a Nonlinear m-Point Integral Boundary Value Problem, Applied Mathematical Sciences, Vol. 6, 2012, no. 15, 711 - 729.

[18] P. Thiramanus, J. Tariboon, Positive Solutions to second order m-Point Integral Boundary Value Problem, Int. J. of Pure and App. Math., Vol. 74, 2012, no. 1, 125 -146.

[19] J. R. L. Webb, Positive solutions of some three-point boundary value problems via fixed point index theory, Nonlinear Analysis, 47(2001), 4319-4332.

[20] Z. Yang, Positive solutions of a second order integral boundary value problem, J. Math. Anal. Appl., 321(2006), 51-765.

[21] Z. Zhang, J. Wang, Positive solutions to a second order three-point boundary value problem, J. Math. Anal. Appl. 285 (2003), 237-249.

Abstract

In this memory, we have studied the existence solutions of some boundary value problems associated with differential equations of second order set on bounded intervals where the boundary conditions are at multipoint.
We have used some methods (fixed point theory and the compactness criterion) to demonstrate our results.
In order to study these problems, we transform the given boundary value problem into an operational equation, in this case we use the criterion of compactness
with applying the Guo-Krasnosel'skii fixed point theorem to prove the given results.
In all the phases of our work we give some examples which give applications to our results.
Keywords: Cone, concave, multipoint, fixed point theorem, super-linear, sub-linear.

Résumé

Dans ce mémoire, nous avons étudié l'existence des solutions positives de quelques problèmes aux limites du second ordre associés à des équations différentielles posées sur des intervalles bornés et où les conditions aux bords à multipoints.
Nous avons utilisé quelques méthodes (théorie du point fixe et le critère de compacité) pour démontrer nos résultats.
Afin d'étudier ces problèmes, nous transformons le problème aux limites donner en une équation opérationnelle, dans ce cas on utilise le critère de compacité avec appliquer le théorème de points fixe de Guo-Krasnosel'skii pour prouver les résultats fournis.
Dans toutes les phases de notre travail nous donnons quelques exemples qui donnent des applications à nos résultats.
Mots clés : Cône, concave, multipoints, théorème de point fixe, super-linéaire, sous-linéaire.

الملخص

في هذه المذكرة درسنا وجود وتعدد الحلول الموجبة لبعض الجمل الحدية المرتبطة بالمعادلات التفاضلية من الرتبة الثانية على المجالات المحدودة أين تكون الشروط الحدية متعدّدة النقط.
لقد استخدمنا بعض الطرق (نظرية النقطة الصامدة ومعيار التراص) لاثبات نتائجنا.
من أجل دراسة هذه الجمل، نقوم بتحويل الجملة الحدية المعطاة إلى معادلة بالمؤثر فى هذه الحالة تستخدم معيار التراص مع نظرية النقطة الصامدة Guo−Krasnosel'skiiتطبيق لإثبات النتائج المعطاة.
في جمع مراحل عملنا نقدم بعض الأمثلة التي تعطى تطبيقات لنتائجنا.
الكلمات المفتاحية : مخروط ، مقعر ، متعدّدة النقط، نظرية النقطة الصامدة ، فوق الخطية، تحت الخطية.