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Some results for the almost and weakly star Lindelof

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Abstract

We discuss some of the selected results obtained in the past fifteen years on the principles of star selection in topology, which is an important sub-field in the field of theory of selection principles. The main theory of selections mainly describes. overlap properties and local properties in topological spaces. we find that they are used in many domains, including probability games ....
So in this paper we introduce and study some new types of star-selection principles: almost star Lindelof and weakly star Lindelof and we give some properties of these selection principles are proved.

Keywords:Selection principles, star Rothberger, star Lindelof, star-Menger space, almost , weakly

1. Introduction

Classical selection principles, based on the diagonalization arguments, have a long history going back to the works by Borel, Menger, Hurewicz, Rothberger, and others Let A and B be collections of families of subsets of an infinite set X. then we have following definition ;
$S_1(A, B)$**(Rothberger in 1938)**: For each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of A, there is a sequence $\{B_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, we have $B_n \in A_n$ and $\{B_n : n \in \mathbb{N}\} \in B$.
$S_{fin}(A, B)$**(Menger)**: For each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of A, there is a sequence $\{B_n : n \in \mathbb{N}\}$ of finite sets such that for each $n \in \mathbb{N}$, $B_n \subset A_n$ and $\bigcup\{B_n : n \in \mathbb{N}\} \in B$.
In 1999, Kocinac (see [5],[6] and [7]) introduced the following selection principles in connection with the star operator.
$S_1^*(A, B)$**(star Rothberger)**: For each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of elements of A, there exists a sequence $\{V_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $V_n \in \mathcal{U}_n$ and $\{St(V_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in B$.
$S_{fin}^*(A, B)$ **(star Menger)**: denotes the following selection hypothesis: For each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of elements of A, there exists a sequence $\{V_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}, V_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}}\{St(V, \mathcal{U}_n) : V \in V_n\} \in B$.
$SS_{\mathcal{K}}^*(A, B)$**(Strongly star ..)** We say that X belongs to the class $SS_{\mathcal{K}}^*(A, B)$ if X satisfies the following selection hypothesis: for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of A there exists a sequence $(K_n : n \in \mathbb{N})$ of elements of $\mathcal{K}$ such that $\{St(K_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in B$. When K is the collection of all one-point [resp., finite, compact] subspaces of X we write $SS_1^*(A, B)$ [resp., $SS_{fin}^*(A, B)$, $SS_{comp}^*(A, B)$] instead of $SS_K^*(A, B)$
For a space X we use the following notation:

1. $\mathcal{O}$ denotes the collection of all open covers of X.
2. Ω denotes the collection of all $\omega-covers$ of X; an open cover $\mathcal{U}$ of X is an $\omega-cover$ [1] if every finite subset of X is contained in a member of $\mathcal{U}$.

3. Γ denotes the collection of all $\gamma - covers$ of X; an open cover $\mathcal{U}$ of X is a $\gamma - cover$ [1] if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$

4. D : the collection of $\mathcal{U} \subset \mathcal{T}$ whose union is dense in X

Definition 1 *[5]*
A space X has the star-Menger (resp. star-Rothberger) property if it satisfies the selection hypothesis $S^*_{fin}(\mathcal{O}, \mathcal{O})$ (resp. $S^*_1(\mathcal{O}, \mathcal{O})$)

Definition 2 *[3] A space X is Lidelof if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $\{V_n : n \in \mathbb{N}\}$ with V_n is a countable subset of $\mathcal{U}_n$, for each n, and $\bigcup\{V_n : n \in \mathbb{N}\} = X$.*

2. New Selection Principles

In this section we introduce an selection principles in connection with the star operator : star Lindelof, Almost star Lindelof and weakly star Lindelof then we have.

Definition 3 (star Lindelof) *: denotes the following selection principle: Let a sequence $\mathcal{U}_n$ of open cover of X, there exists a countable subset F_n of X such that: $\bigcup_{n\in\mathbb{N}} St(F_n, \mathcal{U}_n) = X$*

Definition 4 (Almost star Lindelof) *: denotes the following selection principle: Let a sequence $\mathcal{U}_n$ of open cover of$X$, there exists a countable subset F_n of X such that: $\bigcup_{n\in\mathbb{N}} \overline{St(F_n, \mathcal{U}_n)} = X$*

Definition 5 (Weakly star Lindelof) *: denotes the following selection principle: Let a sequence $\mathcal{U}_n$ of open cover of X, there exists a countable subset F_n of X such that: $\overline{\bigcup_{n\in\mathbb{N}} St(F_n, \mathcal{U}_n)} = X$*

3. Results and theories

In the partial we give some proposition and theorem:

Proposition 6 *It is clear we have star-Lindelof $\Rightarrow$ almost star-Lindelof . and we have almost star-Lindelof $\Rightarrow$ weakly star-Lindelof*

Proposition 7 *Every almost star-Menger space X is almost star-Lindelof*

Proof. Let $\mathcal{U}$ be an open cover of X. Then, by definition, there is a sequence $(V_n : n \in \mathbb{N})$ such that for every n, V_n is a finite subset of $\mathcal{U}$ and $\bigcup_{n\in\mathcal{N}} \overline{St(\cup V_n, \mathcal{U})} = X$. Then $V = \bigcup_{n\in\mathbb{N}} V_n$ is a countable subfamily of $\mathcal{U}$ satisfying $\bigcup \overline{St(\cup V, \mathcal{U})} = X$, **i.e.** X is a almost star-Lindelof space. ■

From the definitions above, we have the following diagram.

star Rothberger$\Rightarrow$ almost $S^*_1(\mathcal{O}, \mathcal{O}) \Rightarrow$ weakly $S^*_1(\mathcal{O}, \mathcal{O})$

$\Downarrow$ $\Downarrow$ $\Downarrow$

star Menger$\Rightarrow$ almost $S^*_{fin}(\mathcal{O}, \mathcal{O}) \Rightarrow$ weakly $S^*_{fin}(\mathcal{O}, \mathcal{O})$

$\Downarrow$ $\Downarrow$ $\Downarrow$

star Lindelof$\Rightarrow$ almost star Lindelof $\Rightarrow$ weakly star Lindelof

Proof. Each point of X can be considered as a finite and since every finite is countable than proof is evident ■

Theorem 8 *For a paracompact (Hausdorff) space X the following are equivalent:*

a) *X is almost star-Lindelof space;*

b) *X is almost strongly star-Lindelof space;*

c) *X is almost Lindelof space.*

Proof. We have to prove only that (a) implies (c). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of a paracompact almost star-Lindelof space X. By the well known Stone characterization of paracompactness for every $n \in \mathbb{N}$ let V_n be an open star-refinement of $\mathcal{U}_n$. Since X is almost star-Lindelof there exists a sequence $\{\mathcal{W}_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a countable subfamily of V_n and $\bigcup_{n\in\mathbb{N}} \overline{St(\cup\mathcal{W}_n, V_n)} = X$. For every $W \in \mathcal{W}_n$ let U_W be a member of $\mathcal{U}$ n such that $St(W, V_n) \subset U_W$. Then $\mathcal{U}_n^{'} = \{U_W : W \in \mathcal{W}_n\}$ is a countable subfamily of $\mathcal{U}_n$ for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \overline{\bigcup \mathcal{U}_n^{'}} = X$ which means that X is a almost Lindelof space. ■

Theorem 9 *For a paracompact (Hausdorff) space X the following are equivalent:*

I) *X is weakly star-Lindelof space;*

II) *X is weakly strongly star-Lindelof space;*

III) *X is weakly Lindelof space.*

Definition 10 *[2] We say that a topological space X is d-paracompact if every dense family of subsets of X has a locally finite refinement.*

Theorem 11 *If a topological space X is weakly star Lindelof and d-paracompact, then X is almost star Lindelof.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Since X is weakly star Lindelof, there exists a sequence $(V_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, V_n is a countable subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \cup V_n$ is dense in X. By the assumption, $V_n : n \in \mathbb{N}$ has a locally finite refinement$\mathcal{W}$. Then $\bigcup \mathcal{W} = \bigcup_{n\in\mathbb{N}} \bigcup V_n$ and therefore $\overline{\cup\mathcal{W}} = \overline{\bigcup_{n\in\mathbb{N}} \bigcup V_n}$. As $\mathcal{W}$ is a locally finite family, we have that $\overline{\bigcup \mathcal{W}} = \bigcup_{W\in\mathcal{W}} \overline{W}$. Since for every $W \in \mathcal{W}$ there exists $n \in \mathbb{N}$ and $V_W \in V_n$ so that $W \subset V_W$, we have that $\bigcup\{\overline{\mathcal{V}} : \mathcal{V} \in V_n\} = X$, so we showed that X is almost star Lindelof. ■

Definition 12 *Recall that a topological space X is P-space if every intersection of countably many open subsets of X is open.*

Proposition 13 *If a topological space $(X,\mathcal{T})$ is weakly star Menger P-space, then $(X,\mathcal{T})$ is almost star Menger*

Theorem 14 *If a topological space $(X,\mathcal{T})$ is weakly star Lindelof P-space, then $(X,\mathcal{T})$ is almost star Lindelof.*

$*$ We will also see that in P-spaces the properties of almost star Lindelof and almost star Menger are equivalent.

Theorem 15 *Every almost star-Lindelof P-space is almost star-Menger*

Proposition 16 *Let X be a regular P-space. Then the following statements are equivalent:*

1. *X is almost Menger;*
2. *X is weakly Menger;*
3. *X is almost star-Menger;*
4. *X is weakly star-Menger;*
5. *X is almost Lindelof;*
6. *X is weakly Lindelod;*
7. *X is almost star-Lindelof;*
8. *X is weakly star-Lindelof*

Theorem 17 *Let X be an almost star-Lidelof topological space and let Y be a topological space. If $f : X \to Y$ is an almost continuous surjection, then Y is an almost star-Lindelof space.*

Proof. . Let $(\mathcal{U}_n : n\mathbb{N})$ be a sequence of covers of Y. Let $\mathcal{U}_n^{'} = \{f^{-1}(U) : U \in \mathcal{U}_n\}$. Then each $\mathcal{U}_n^{'}$ is an open cover of X since f is almost continuous and each $U \in \mathcal{U}_n$. Since X is almost star-Lindelof, there is a sequence $(V_n^{'} : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}, V_n^{'}$ is a countable subset of $\mathcal{U}_n^{'}$ and $\bigcup \overline{St(V_n^{'},\mathcal{U}_n^{'})} : n \in \mathbb{N}$ is a cover for X. Let $V_n = \{U : f^{-1}(U) \in V_n^{'}\}$ and $x \in X$. Then $\cup f^{-1}(V_n) = \cup V_n^{'}$ and there is $n \in \mathbb{N}$ such that $x \in \overline{St(f^{-1}(V_n),\mathcal{U}_n^{'})}$. If $y = f(x) \in Y$, then $y \in \bigcup \overline{f(St(f^{-1}(V_n),\mathcal{U}_n^{'}))} \subseteq \overline{f(St(f^{-1}(\cup V_n),\mathcal{U}_n^{'}))} \subseteq \overline{St(V_n,\mathcal{U}_n)}$. We will prove the last inclusion: Suppose that $f^{-1}(V_n) \cap f^{-1}(U) \neq \emptyset$. Then also $f(f^{-1}(V_n)) \cap f(f^{-1}(U)) \neq \emptyset$, so $V_n \cap U \neq \emptyset$. So, the sequence $(V_n : n \in \mathbb{N})$ witnesses that X is an almost star-Lindelof space. ■

Definition 18 *Let us recall that a subset U of a space X is regular open if $U = Int(\overline{U})$. Let us recall that a subset U of a space X is regular closed if $U = \overline{Int(U)}$.*

Theorem 19 *A topological space X is almost star-Lindelof if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by regular open sets there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a countable subset of $\mathcal{U}_n$ and $\overline{St(\bigcup \mathcal{V}_n,\mathcal{U}_n)} : n \in \mathbb{N}$ is a cover of X.*

Theorem 20 *A space X is weakly star-Lindelof if and only if for each sequence $(\mathcal{U}_n : n\mathbb{N})$ of covers of X by regular open sets there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that, for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a countable subset of $\mathcal{U}_n$ and $\overline{\bigcup_n St(\bigcup \mathcal{V}_n, \mathcal{U}_n)} = X$.*

Lemma 21 *A space X having a dense almost Lindelof subspace is almost star-Lindelof*

Proof. Let X have a dense almost Lindelof subspace D. We show that X is almost star-Lindelof. Let $\mathcal{U}$ be an open cover of X. Since D is a dense almost Lindelof subset of X. Then there exists a countable subset V of $\mathcal{U}$ such that $D \subseteq \bigcup \overline{V}$. Hence $\bigcup \overline{St(V, \mathcal{U})} = X$, which shows that X is almost star-Lindelof. ■

Theorem 22 *Let X be a weakly star-Lindelof space and Y be a space. If $f : X \longrightarrow Y$ is an almost continuous mapping, then Y is a weakly star-Lindelof space*

4. conclusion

The main theory of selections mainly describes. overlap properties and local properties in topological spaces, especially function spaces. His fist sources come from the theory of measurements and the theory of bases in metric spaces. In mathematics, the main theory of selections describes mainly the properties of combination overlays of topological spaces.

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