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Existence results of n-point boundary value problems for second order differential equations on bounded domain

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الإهداء

بسم الله الرحمن الرحيم

الحمد لله الذي علّم بالقلم، علّم الإنسان ما لم يعلم، والصلاة والسلام على معلم البشرية، سيدنا محمد، وعلى آله وصحبه أجمعين

إلى والدي الحبيب، الذي غرس في نفسي قيمة الاجتهاد، وثقافة الإصرار، فكنت دائمًا قدوتي في الصبر والتحمّل، والمثل الأعلى في السعي والمثابرة...
علمتني أن الأحلام لا تتحقق إلا بالجد والتعب،

إلى والدتي الغالية، نبع الحنان وصوت الدعاء الذي يسبقني في كل خطوة...
يا من علمتني أن بالحب والإيمان نصنع المستحيل، وأن رضاكِ مفتاح كل توفيق
أسأل الله أن يحفظكما لي.

إلى من رافقوني في رحلة العلم الطويلة...
إلى إخي وأخواتي، الذين كانوا لي سندًا وعونًا في أوقات الضيق، ومصدر فرح وسعادة في أوقات الفرح...

إلى أصدقائي وزملائي الأعزاء، رفاق الدرب والمواقف الصادقة، الذين تقاسمنا معًا لحظات التعب والاجتهاد، وتشاركنا الآمال والتطلعات...
شكرًا لابتساماتكم، لشجاعتكم، وللكلمات التي كانت بلسمًا في لحظات الإنهاك

الى أساتذتي الأفاضل أتقدم بجزيل الشكر والامتنان على ما قدموه من علم وتوجيه ودعم طوال مسيرتي الدراسية.
والى استاذتي الفاضلة التي كان لها دور كبير في نجاحي
كحيلي هدى

الى مشرفي الفاضل،اتوجه بجزيل الشكر والامتنان على توجيهاته السديدة وجهوده القيّمة التي كان لها الأثر الكبير في إنجاز هذا العمل

إلى شريكتي ناريمان التي وقفت بجانبي تشاركنا لحضات التعب والفرح طيلة مشوار هاذا العمل

أهدي هذه المذكرة المتواضعة، خلاصة سنوات من السعي والبحث المستمر، لكل من كان له أثر في مسيرتي العلمية والإنسانية.

بديده إكرام

الإهداء

من قال أنا لها.....نالها
لم تكن الرحلة قصيرة ولا ينبغي لها ان تكون ولم يكن الحلم قريبا ولا الطريق كان محفوفا بالتسهيلات ...... لكنني فعلتها ونلتها

إلى والدي الحبيب، الذي غرس في نفسي قيمة الاجتهاد، وثقافة الإصرار، فكنت دائمًا قدوتي في الصبر والتحمّل، والمثل الأعلى في السعي والمثابرة...
إلى والدتي الغالية، منبع الحب والطمأنينة، التي لولا دعاؤها وصبرها ما وصلت إلى هذا الإنجاز..أهديك ثمرة جهدي، عرفانًا بجميلكِ الذي لا يُرد.
أسأل الله أن يحفظكما لي.

إلى من رافقوني في رحلة العلم الطويلة...
إلى إخوتي وأختي، الذين كانوا لي سندًا وعونًا في أوقات الضيق، ومصدر فرح وسعادة في أوقات الفرح...
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الى مشرفي الفاضل،اتوجه بجزيل الشكر والامتنان على توجيهاته السديدة وجهوده القيّمة التي كان لها الأثر الكبير في إنجاز هذا العمل
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سليماني ناريمان

Contents

Notations

We will use the following notations throughout this thesis:

General notations

- L: A continuous linear operator.
- $D(L)$: Domain of the operator L.
- L^{-1}: The inverse application of L.
- $\mathbb{N}$: The set of natural numbers.
- $\mathbb{Z}$: The set of whole numbers.
- $\mathbb{R}$: The set of real numbers.
- $[a, b]$: The closed interval $a \leq x \leq b$.
- (a, b): The open interval $a < x < b$.
- $\frac{d(.)}{dt}$: The ordinary derivative with respect to t.
- a.e.: Almost everywhere.
- i.e.: That is to say.

Spaces

- $\Omega \subset \mathbb{R}^n$: open set in $\mathbb{R}^n$.
- $\overline{\Omega}$: closure of Ω.
- $\partial\Omega$: boundary of Ω.
- $L^p(\Omega) = \{$u measurable on Ω and $\int_\Omega |u|^p dx < \infty\}$, $1 \leq p < \infty$..
- $C(\overline{\Omega})$: Space of continuous functions on Ω.
- $C^n(\overline{\Omega})$: Space of n times continuously differentiable functions on Ω.
- $W^{m,p}$: Sobolev space of functions in L^p with derivatives up to the order m in L^p.

  In this memory, we will specifically use the following spaces:

- $C([0,1],\mathbb{R})$: Banach space of continuous functions $u : [0,1] \to \mathbb{R}$, equipped with the norm
$$\|u\|_\infty = \sup\{|u(t)|,\, t \in [0,1]\}.$$
- $C^1([0,1],\mathbb{R})$: Space of functions $u : [0,1] \to \mathbb{R}$ such that u, u' are continuous functions on [0,1], equipped with the norm
$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}.$$
- $L^1([0,1],\mathbb{R})$: Space of Lebesgue integrable functions $u : [0,1] \to \mathbb{R}$ equipped with the norm
$$\|u\|_1 := \|u\|_{L^1} = \int_0^1 |u(t)|dt.$$
- $W^{2,1}(0,1) = \{u : [0,1] \to \mathbb{R};\, u,\, u'$ are absolutely continuous in $[0,1]$, $u'' \in L^1(0,1)\}$, equipped with the norm
$$\|u\|_{2,1} = \int_0^1 (|u(t)| + |u'(t)| + |u''(t)|)dt.$$

Introduction

Boundary value problems involving ordinary differential equations arise in physical sciences and applied mathematics. In some of these problems, additional conditions such that n-point boundary conditions are important in practice since the measurements.For example, the classical Robin problem with local conditions is given by

$$\begin{cases} u''(t) + f(t, u(t), u'(t)) = 0, & t \in (0, 1) \\ u(0) = 0, \qquad u'(1) = 0. \end{cases}$$

If the local condition $u'(1) = 0$ is replaced by the condition $u(1) = u(\eta)$ (where $\eta \in (0, 1)$), then the problem

$$\begin{cases} u''(t) + f(t, u(t), u'(t)) = 0, & t \in (0, 1) \\ u(0) = 0, \qquad u(1) = u(\eta), \end{cases}$$

is a three point problem. On the interval $(0, 1)$, by the Rolle theorem, local problem can be thought as the limiting case of n-point problem as $\eta \to 1^-$. In the process of scientific experiment and numerical computation, it is more difficult to determine the value of $u'(1)$ than that of $\frac{u(\eta)-u(1)}{\eta-1}$. So the nonlocal problem gives better effect than the local problem.

It can also be extended to a n-point condition $u(t) = g(u)$, where g is a mapping defined on some space consisting of certain functions with $g(u) := g(\xi_1, \dots, \xi_p, u(.)),\ p \in \mathbb{N}^*$, where $0 < \xi_1 < \dots < \xi_p < +\infty$. The symbol $g(\xi_1, \dots, \xi_p, u(.))$ is used in the sense that in place of (.) we can put only elements of the set $\{\xi_1, \dots, \xi_p\}$. For example g can be defined by the formula $g(u) = \sum_{i=1}^{p} \beta_i u(\xi_i)$, where $\beta_i,\ i = 1, 2, \dots, p$ are given constants.

Three-point bvps and more generally n-point boundary value problems were initiated in the early 1960s by Bitsadze [2] and later by Il'in and Moiseev [11]. In the simplest case of a three-point boundary value problems, Gupta [7], [8] has first applied functional analytical methods to prove existence of solutions, he was then followed by Ma [13], Marano [16], and others. Since then, the existence of solutions for nonlinear three-point, n-point, integral boundary value problems has been studied by several authors by using the nonlinear alternatives theorem of Leray-Schauder. We refer the reader to [2], [8, 10, 13, 14, 15, 16, 17] and the references therein. However, most of those papers have studied the existence and multiplicity of solutions to bvps with boundary conditions specified as solution values or the slope of solution values at given points.

n-point boundary value problems have been well studied especially on compact intervals; in this context, we refer to [15] for a survey of some of the related research works and for references to many further contributions.

For instance, integral boundary conditions appear in blood flow models as well as in chemical engineering.

In this memory, we given some results on the existence and multiplicity of solutions for n-point boundary value problems of second order ordinary differential equations on bounded intervals.

This memory is organized as follows.

The first chapter is devoted to presenting some preliminaries and general notions used throughout. Some basic tools from functional analysis, and Leray-Schauder's nonlinear alternative fixed point.

In **the second chapter**, we discuss the three-point boundary value problems:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \in (0,1), \\ x(0) = 0, \qquad x(1) = x(\eta). \end{cases}$$

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0, & t \in (0,1), \\ x(0) = 0, \qquad x(1) = \alpha x(\eta), & (\alpha \neq 1) \end{cases}$$

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0, & t \in (0,1). \\ x'(0) = 0, \quad x(1) = \alpha x(\eta) \end{cases}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a given function $\eta \in (0,1)$ and $\alpha \in \mathbb{R}$ with $1 < \alpha < \frac{1}{\eta}$.
In **the third chapter**, we studyed the existence solutions of the second order differential equation with multi-point boundary conditions in general case which represented in the folowing problem:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \in (0,\ 1) \\ x(0) = \sum_{i=1}^{m-2} c_i x(\xi_i), & x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j), \end{cases}$$

such that $f : [0,\ 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and $c_i,\ a_j \in \mathbb{R},\ \xi_i,\ \tau_j \in (0,\ 1)$, $i = 1, 2, ..., m-2,\ j = 1, 2, ..., n-2$,
$0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1,\ 0 < \tau_1 < \tau_2 < ... < \tau_{n-2} < 1$.
Every time when studying any problem, we transform each given differential equation into an equation with the operator and by means of the Leray-Schauder nonlinear alternative

theorem, sufficient conditions are obtained that guarantee the existence of solutions to the above problems. As applications, some examples are included to illustrate the existence results.

Chapter 1

Preliminaries

1.1 Some basic tools

Definition 1.1. *We say that* $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ *is a* L^1*-Carathéodory function if*

(i) $t \mapsto f(t,\, u)$ *is measurable on* $[0,\ 1]$ *for each* $u \in \mathbb{R}^n$

(ii) $u \mapsto f(t,\, u)$ *is continuous on* $\mathbb{R}^n$ *for a.e.* $t \in [0,1]$

(iii) *for each* $r \in (0,\, +\infty)$*, there exists an* $g_r \in L^1[0,1]$ *such that*

$$|f(t,\, u)| \leq g_r(t) \text{ for a.e. } t \in [0,1] \text{ and all } u \in \mathbb{R}^n \text{ with } \|u\| \leq r.$$

Theorem 1.1. *(Rolle's theorem)*
Let f *an application of the interval* $[a,b]$ *on* $\mathbb{R}$ *verifying the following conditions*

1. f *is continuous on* $[a,b]$.
2. f *is derivable on* $]a,b[$.
3. $f(a) = f(b)$.

So, it exists $c \in\ ]a,b[$ *such that* $f'(c) = 0$.

Theorem 1.2. *(finite increments)*
Let f *an application of the interval* $[a,b]$ *on* $\mathbb{R}$ *verifying the following conditions*

1. f *is continuous on* $[a,b]$.

2. f is derivable on $]a, b[$.

So, it exists $c \in\ ]a, b[$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

Wirtinger Inequalities:

Let a, b be two real numbers such that $a < b$, and let u be a continuously differentiable function C^1 on $[a, b]$ to real values. We have

$$\int_a^b |u(t) - u(a)|^2 \, dt \leq 4 \left(\frac{b-a}{\pi} \right)^2 \int_a^b |u'(t)|^2 \, dt.$$

$$\int_a^b |u(t) - u(b)|^2 \, dt \leq 4 \left(\frac{b-a}{\pi} \right)^2 \int_a^b |u'(t)|^2 \, dt.$$

1.1.1 Important theorems

Ascoli-Arzela theorem

Let (X, d_X) be a compact metric space and (Y, d_Y) be a complete metric space. By $C(X, Y)$ we denote the vector space consisting of all continuous function $f : X \to Y$.

Definition 1.2. *The family $F \subset C(X, Y)$ is called equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(t), f(s)) < \varepsilon$ for all $t, s \in X$ satisfying $d_X(t, s) < \delta$ and all $f \in F$.*

Theorem 1.3. *([4]) The family $F \subset C(X, Y)$ is relatively compact if and only if*
1. F is equicontinuous
2. for all $t \in X$, the set

$$M(t) = \{x(t);\, x(.) \in F\}$$

is relatively compact in Y.

Remark 1.1. *1. If Y is a finite dimensional Banach space, the second condition of Ascoli-Arzela theorem is equivalent to F is uniformly bounded i.e., there exists $M > 0$ such that for all $x \in F$*

$$\|x\|_\infty = \sup_{t \in X} \|x(t)\| \leq M.$$

2. If X is a compact metric space, then Ascoli-Arzela theorem can be expressed as follows: the family $F \subset C(X, Y)$ is relatively compact if and only if F is equicontinuous.

1.1.2 Lebesgue dominated convergence theorem

Theorem 1.4. *([3]) Let (f_n) be a sequence of functions in $L^1(\Omega)$ that satisfies*
(i) $f_n(t) \to f(t)$ for almost every $t \in \Omega$
(ii) There is a function $g \in L^1(\Omega)$ such that $|f_n(t)| \leq g(t)$ for all $n \in \mathbb{N}$ and for almost every $t \in \Omega$.
Then $f \in L^1(\Omega)$ and $\|f_n - f\|_1 \to 0$.

1.1.3 Operators

Let X and Y be two normed vector spaces and Ω an open set of X.

Definition 1.3. *A continuous mapping $T : \Omega \subset X \to Y$ is called compact if $T(\overline{\Omega})$ is relatively compact. If the image of any bounded subset B of Ω is relatively compact, then T is said to be completely continuous.*

Definition 1.4. *The operator $T : \Omega \subset X \to Y$ is said to be of finite rank, if there exists a finite dimensional subspace F such that $Im(T) \subseteq F \subseteq Y$.*

Remark 1.2. (a) *Any compact mapping is completely continuous. (Because for every bounded $B \subset \Omega$, we have $T(B) \subset T(\overline{\Omega})$). The converse is true if Ω is bounded.*

(b) *Let X, Y be two Banach spaces, $T : X \to Y$ be a linear mapping. If at least one of spaces X or Y has finite dimension, then T is compact if and only if T is continuous.*

1.1.4 The Green Function

Let p, q, $f \in C([a,b])$ where $p \in C^1([a,b])$, $(a < b)$ and $(\alpha_i, \beta_i) \in \mathbb{R}^2$ such that $\forall\, i = 1, 2$, $|\alpha_1| + |\alpha_2|$, $|\beta_1| + |\beta_2| \neq 0$. Consider the ordinary differential equations

$$\begin{aligned} (H) &\quad (py')' + qy = 0 \\ (NH) &\quad (py')' + qy = f \end{aligned}$$

and the associated edge conditions

$$(CB)_h \quad \begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \end{cases}$$

$$(CB)_{nh}\begin{cases}\alpha_1 y(a)+\alpha_2 y'(a)=\gamma\\ \beta_1 y(b)+\beta_2 y'(b)=\delta.\end{cases}$$

Definition 1.5. *We call the Green's function associated with the homogeneous problem* $(H)-(CB)_h$ *a function* $G\colon\ [a,b]\times[a,b]\to\mathbb{R}$ *satisfied The following properties*

(a) *G is continuous on* $[a,b]\times[a,b]$.

(b) *G is symmetric:* $G(x,y)=G(y,x)$, $\forall\,(x,y)\in[a,b]^2$.

(c) $\dfrac{\partial G}{\partial x}(x,y)$ *is continuous for all* $x\neq y$.

(d) $\dfrac{\partial G}{\partial x}(y^+,y)-\dfrac{\partial G}{\partial x}(y^-,y)=\dfrac{1}{p(y)}$ *for all* $y\in[a,b]$.

(e) *Partial Function* $x\mapsto G(x,y)$ *is the solution of the equation* (H) *for all* $x\neq y$.

(f) *Partial Function* $x\mapsto G(x,y)$ *satisfied the conditions* $(CB)_h$ *for all* $y\in[a,b]$.

Theorem 1.5. *(***(Existence and uniqueness of the Green function)***) Suppose that the homogeneous problem* $(H)-(CB)_h$ *does not admit of a non-trivial solution. Then there exists a (and only one) function G that does not depend on f, and is called Green's function, such that, for any function f, the solution y of the non-homogeneous problem* $(NH)-(CB)_h$ *is uniquely written as*

$$y(x)=\int_a^b G(x,s)f(s)\,ds.$$

Proof

(i) Existence of the function G: *Let ϕ_1 and ϕ_2 the respective solutions of the problems with initial conditions.*

$$(H)+\begin{cases}\phi_1(a)=\alpha_2\\ \phi_1'(a)=-\alpha_1.\end{cases}\quad \textit{and}\quad (H)+\begin{cases}\phi_2(b)=\beta_2\\ \phi_2'(b)=-\beta_1.\end{cases}$$

Then, ϕ_1, $\phi_2\neq 0$ *are linearly independent because otherwise ϕ_1 (and also ϕ_2) would be the solution to the problem* $(P_0):(H)+(CB)_h$ *a contradiction hypotheses . So be it* $W=\phi_1\phi_2'-\phi_1'\phi_2\neq 0$ *their Vronskian and G the Green function defined by*

$$G(x,s)=\begin{cases}\dfrac{\phi_1(s)\phi_2(x)}{p(s)W(s)}, & a\leqslant s\leqslant x,\\[2mm] \dfrac{\phi_1(x)\phi_2(s)}{p(s)W(s)}, & x\leqslant s\leqslant b.\end{cases}$$

Because, for any solution y of the non-homogeneous problem $(NH)-(CB)_h$ is written in the form

$$\begin{aligned} y(x) &= \int_a^x \frac{\phi_1(s)\phi_2(x)}{p(s)W(s)} f(s)\,ds + \int_x^b \frac{\phi_1(x)\phi_2(s)}{p(s)W(s)} f(s)\,ds \\ &= \int_a^b G(x,s)f(s)\,ds. \end{aligned}$$

Note that the product pW is constant because $p(x)W(x) = p(s)W(s) = p(a)W(a) = p(b)W(b) \neq 0$, $\forall\, x, s \in [a,b]$. Indeed, according to Lagrange's formula, we have

$$\begin{aligned} \phi_2(p\phi_1')' - \phi_1(p\phi_1')' = 0 \;&\Leftrightarrow\; (p\phi_2\phi_2')' - (p\phi_1\phi_2')' = 0 \\ &\Leftrightarrow\; [p(\phi_2\phi_1' - \phi_1\phi_2')]' = 0 \\ &\Leftrightarrow\; pW = cte. \end{aligned}$$

Finally, G satisfies the hypotheses of the Green function.

(ii) Uniqueness of the function G: *Let G, H two Green functions then $\int_a^b [G(x,s) - H(x,s)]f(s)ds = 0$, $\forall\, x \in [a,b]$ and $\forall\, f\colon [a,b] \to \mathbb{R}$ continuous function. For x fixed, let's put down $f(s) = G(x,s) - H(x,s)$; We have $\int_a^b [G(x,s) - H(x,s)]^2 ds = 0$, $\forall\, x \in [a,b]$. As G and H are continuous , $G \equiv H$, that is $(G(x,.) = H(x,.), \forall\, x \in [a,b])$.*

(iii) Existence and uniqueness of a solution: *The function F defined by*

$$F(x) = \int_a^b G(x,s)f(s)ds = \frac{\phi_2(x)}{pW}\int_a^x \phi_1(s)f(s)ds + \frac{\phi_1(x)}{pW}\int_x^b \phi_2(s)f(s)ds$$

is the solution to the problem $(NH)+(CB)_h$.Indeed

$$\begin{aligned} F'(x) &= \int_a^b \frac{\partial G}{\partial x}(x,s)f(s)\,ds \\ \text{and } \; F''(x) &= \int_a^b \frac{\partial^2 G}{\partial x^2}(x,s)f(s)\,ds + f(x)[\frac{\partial G}{\partial x}(x,x^-) - \frac{\partial G}{\partial x}(x,x^+)]. \end{aligned}$$

Or

$$\begin{array}{rcl}\dfrac{\partial G}{\partial x}(x^+,x) & = & \dfrac{\partial G}{\partial x}(x,x^-)\\ \dfrac{\partial G}{\partial x}(x^-,x) & = & \dfrac{\partial G}{\partial x}(x,x^+)\\ \dfrac{\partial G}{\partial x}(x^+,x) & - & \dfrac{\partial G}{\partial x}(x^-,x) = \dfrac{1}{p}\cdot\end{array}$$

From this we deduce the expression

$$F''(x) = \int_a^b \frac{\partial^2 G}{\partial x^2}(x,s)f(s)\,ds + \frac{f(x)}{p(x)}$$

As well as

$$(pF')' = \int_a^b \left(p\frac{\partial G}{\partial x}\right)'(x,s)f(s)\,ds + f(x).$$

As a result

$$\begin{array}{rcl}(pF')' + qF & = & \displaystyle\int_a^b \left[\left(p\frac{\partial G}{\partial x}\right)' + qG\right] f(s)\,ds + f(x)\\ (\textit{by definition of } G) & = & \displaystyle -\int_a^b 0 f(s)\,ds + f(x)\\ & = & f(x).\end{array}$$

The uniqueness of the solution y results from the hypothesis on the homogeneous problem as well as from Fredholm's Alternative.

Example 1.1. *Consider the two-point problem posed over an interval $[a,b]$.*

$$\left\{\begin{array}{l} y'' = f(x), \quad a < x < b\\ y(a) = 0, \quad y(b) = 0.\end{array}\right. \tag{1.1}$$

Let's build the functions φ_1 et φ_2 solution of Cauchy problems

$$\left\{\begin{array}{l}\varphi_1'' = 0\\ \varphi_1(a) = 0\\ \varphi_1'(a) = -1\end{array}\right. \qquad \left\{\begin{array}{l}\varphi_2'' = 0\\ \varphi_2(b) = 0\\ \varphi_2'(b) = -1.\end{array}\right.$$

Then, $\varphi_1(x) = (a-x)$, $\varphi_2(x) = (b-x)$ et $W(\varphi_1,\varphi_2) = b-a$.

Hence the function of Green

$$G(x,s)=\begin{cases}\dfrac{(x-b)(s-a)}{(b-a)}, & si\ a\leqslant s\leqslant x,\\ \dfrac{(x-a)(s-b)}{(b-a)}, & si\ x\leqslant s\leqslant b.\end{cases}$$

The unique solution of the problem (1.1) *is therefore given by*

$$y(x)=\int_a^b G(x,s)f(s)ds=\frac{x-b}{b-a}\int_a^x (s-a)f(s)\,ds+\frac{x-a}{b-a}\int_x^b (s-b)f(s)\,ds.$$

1.2 fixed point theorems

We recall the following fixed point theorems: nonlinear alternative for Leray-Schauder.

Definition 1.6. *Let E be a Banach space equipped with the norm $\|.\|$ and $\mathrm{T}:E\longrightarrow E$ a mapping. An element x of E is said to be a fixed point of T if $\mathrm{T}x=x$.*

Theorem 1.6. *([1])(Leray-Schauder nonlinear alternative theorem). Let C be a convex subset of a Banach space and Ω an open subset of C with $0\in\Omega$. Then every completely continuous map $N:\overline{\Omega}\to C$ satisfies at least one of the following two properties:*
$(A1)$ N has a fixed point in $\overline{\Omega}$, or
$(A2)$ There is an $x\in\partial\Omega$ and $\lambda\in(0,\ 1)$ with $x=\lambda Nx$.

Remark 1.3. *Hypothesis $(A2)$ indicates that the set $S=\{x\in\Omega: x=\lambda\mathrm{T}x;\lambda\in[0,1]\}$ is not bounded. So to apply the nonlinear alternative of Leray-Schauder, one can just show that T is completely continuous by checking the following a priori estimate*

$$\exists R>0,\forall x\in E,\forall\lambda\in[0,1]:(x=\lambda\mathrm{T}x\Longrightarrow\|x\|<R)$$

to assert that T admits at least one fixed point in $B(0,\ R)$.

Chapter 2

Second-order three-point boundary value problems

In this chapter, we consider the following two three-point boundary value problems

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \in (0,1), \\ x(0) = 0, \qquad x(1) = x(\eta). \end{cases} \tag{2.1}$$

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0, & t \in (0,1), \\ x(0) = 0, \qquad x(1) = \alpha x(\eta), & (\alpha \neq 1). \end{cases} \tag{2.2}$$

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0, & t \in (0,1) \\ x'(0) = 0, \quad x(1) = \alpha x(\eta). \end{cases} \tag{2.3}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *is a given function* $\eta \in (0,1)$ *and* $\alpha \in \mathbb{R}$ *with* $1 < \alpha < \frac{1}{\eta}$.

2.1 Results for the Boundary Value Problem (2.1)

We can formulate the reference equation in the form of an operator equation

$$Lx = Nx$$

where $L : D(L) \subset X \to Y$ *is a linear operator with*

$$D(L) = \{x \in W^{2,1}(0,1), x(0) = 0, \qquad x(1) = x(\eta)\}$$

and for

$$x \in D(L), \qquad L(x) = x''$$

$N : X \to Y$ *be a nonlinear operator such that*

$$(Nx)(t) = f(t, x(t), x'(t)), \qquad t \in [0,1]$$

with
$X = \mathbb{C}^1([0,1]), ||x||_X = max\,\{||x||_\infty, ||x'||_\infty\}$
$Y = \mathbb{L}^1([0,1]), ||x||_1 = \int_0^1 |x(t)|dt$
and the Sobolev space
$W^{2,1}(0,1) = \{x : [0,1] \to \mathbb{R}, x, x'$ *are absolutely continuous in* $[0,1]$ *and* $x'' \in \mathbb{L}^1([0,1])\}$

$$||x||_{2,1} = \int_0^1 (|x(t)| + |x'(t)| + |x''(t)|)dt$$

Lemma 2.1. *Let* $h \in Y$ *and* $x \in X$ *In such a way that is*

$$\begin{cases} x''(t) = h(t), & t \in (0,1), \\ x(0) = 0, & x(1) = x(\eta). \end{cases} \tag{2.2}$$

So

$$x(t) = \int_0^t (t-s)h(s)ds - \frac{t}{1-\eta}\int_0^1 (1-s)h(s)ds + \frac{t}{1-\eta}\int_0^\eta (\eta - s)h(s)ds$$

Proof

From $x''(t) = h(t)$,*we have* $x'(t) = x'(0) + \int_0^t h(s)ds$
which implies that

$$x(t) = x'(0)t + \int_0^t (t-s)h(s)ds$$

and of $x(1) = x(\eta)$,*we have*

$$x'(0) = -\frac{1}{1-\eta}\int_0^1 (1-s)h(s)ds + \frac{1}{1-\eta}\int_0^\eta (\eta-s)h(s)ds$$

so

$$x(t) = \int_0^t (t-s)h(s)ds - \frac{t}{1-\eta}\int_0^1 (1-s)h(s)ds + \frac{t}{1-\eta}\int_0^\eta (\eta-s)h(s)ds$$

Lemma 2.2. *Under the assumptions of Lemma (2.1), the solution of problem (2.1) is written in the form*

$$x(t) = \int_0^1 G(t,s)h(s)ds$$

Such that $G(t,s)$ (the associated Green's function) is given as follows

$$G(t,s) = \begin{cases} -s, & s \leq min\{\eta, t\} \\ \frac{s(t-1)+\eta(s-t)}{1-\eta}, & \eta \leq s \leq t \\ -t, & t \leq s \leq \eta \\ -\frac{t(1-s)}{1-\eta}, & s \geq max\{\eta, t\}. \end{cases}$$

Proof

1. *For $t \leq \eta$ the solution to the problem in Lemma (2.1) can be written as*

$$\begin{aligned} x(t) &= \int_0^t (t-s)h(s)ds - \frac{t}{1-\eta}\left(\int_0^t (1-s)h(s)ds + \int_t^\eta (1-s)h(s)ds + \int_\eta^1 (1-s)h(s)ds\right) \\ &\quad + \frac{t}{1-\eta}\left(\int_0^t (\eta-s)h(s)ds + \int_t^\eta (\eta-s)h(s)ds\right) \\ &= \frac{1}{1-\eta}\int_0^t \left((1-\eta)(t-s) - t(1-s) + t(\eta-s)\right) h(s)ds \\ &\quad + \frac{t}{1-\eta}\int_t^\eta \left((1-s)+(\eta-s)\right) h(s)ds - \frac{t}{1-\eta}\int_\eta^1 (1-s)h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds. \end{aligned}$$

2. For $\eta \le t$ the solution to the problem in Lemma (2.1) can be written as

$$
\begin{aligned}
x(t) &= \int_0^\eta (t-s)h(s)ds + \int_\eta^t (t-s)h(s)ds \\
&\quad - \frac{t}{1-\eta}\left(\int_0^\eta (1-s)h(s)ds + \int_\eta^t (1-s)h(s)ds + \int_t^1 (1-s)h(s)ds\right) \\
&\quad + \frac{t}{1-\eta}\int_0^\eta (\eta-s)h(s)ds \\
&= \frac{1}{1-\eta}\int_0^\eta \left((1-\eta)(t-s) - t(\eta-s)\right)h(s)ds \\
&\quad + \frac{1}{1-\eta}\int_\eta^t \left((1-\eta)(t-s) - t(1-s)\right)h(s)ds - \frac{t}{1-\eta}\int_t^1 (1-s)h(s)ds \\
&= \int_0^1 G(t,s)h(s)ds.
\end{aligned}
$$

From 1 and 2 we obtain

$$
G(t,s) = \begin{cases} -s, & s \le \min\{\eta, s\}, \\ \frac{s(t-1)+\eta(s-t)}{1-\eta}, & \eta \le s \le t, \\ -t, & t \le s \le \eta, \\ \frac{-t(1-s)}{1-\eta}, & s \ge \max\{\eta, t\}. \end{cases}
$$

Lemma 2.3. *For everything $x \in W^{2,1}(0,1)$ with $x(0) = 0$, $x(1) = x(\eta)$. There is $\xi \in (0,1)$, $\eta < \xi < 1$ such that $x'(\xi) = 0$ and*

$$\|x\|\infty \le \|x'\|\infty \le \|x''\|_1.$$

Proof

Let $x \in W^{2,1}(0,1)$ and according to Rolle's theorem, there are $\xi \in (\eta,1)$ such as $x'(\xi) = 0$.

From

$$\int_0^t x'(s)ds = x(t) - x(0) = x(t)$$

so

$$\forall t \in [0,1], \quad x(t) = \int_0^t x'(s)ds \le \int_0^1 |x'(s)|ds.$$

Which implies that

$$\|x\|_\infty \le \|x'\|_1.$$

On the other hand, we have

$$t \in [0,1], \quad \|x''\|_1 = \int_0^1 |x''(s)|ds \ge \int_\xi^t x''(s)ds = x'(t).$$

Which implies that

$$\|x'\|_\infty \le \|x''\|_1.$$

Theorem 2.1. *That is $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ a function L^1- Caratheodory. Suppose that there are functions p, q, r in $L^1(0,1)$ such as*

$$|f(t,u,v)| \le p(t)|u| + q(t)|v| + r(t), \quad \textit{for everything } t \in [0,1], \quad (u,v) \in \mathbb{R}^2 \textit{ et } \eta \in (0,1).$$

So, the problem (2.1) admits at least one solution in $C^1[0,1]$ provided that

$$\|p\|_1 + \|q\|_1 < 1.$$

proof

Let $X = C^1([0,1])$, $Y = L^1([0,1])$.
We define the linear operator $L : D(L) \subset X \to Y$ as

$$D(L) = \{x \in W^{2,1}(0,1) \mid x(0) = 0,\ x(1) = x(\eta)\}$$

and for $x \in D(L)$,

$$L(x) = x''.$$

Also, we define the nonlinear operator $N : X \to Y$ as

$$(Nx)(t) = f(t, x(t), x'(t)), \quad t \in [0,1].$$

Next, we assume the linear operator $K : Y \to X$ is defined for $y \in Y$ by

$$(Ky)(t) = \int_0^t (t-s)y(s)ds - \frac{1}{\eta}\int_0^\eta (1-s)y(s)ds + \frac{t}{1-\eta}\int_0^\eta (\eta - s)y(s)ds.$$

For $y \in Y$, we have $Ky \in D(L)$ and $LKy = y$, and for $x \in D(L)$, $KLx = x$, which implies that $L^{-1} = K$. We have
($x \in C^1[0,1]$ is a solution of (2.1)) $\Leftrightarrow$ (x is a solution of the operator equation $Lx = Nx$) $\Leftrightarrow x = KNx$. We apply the Leray-Schauder alternative theorem to obtain the existence of a solution to the equation $x = KNx$.
To do this, it is sufficient to verify that the set of all possible solutions of the family of equations

$$\begin{cases} x''(t) = \lambda f(t, x(t), x'(t)), & t \in (0,1) \\ x(0) = 0, \quad x(1) = x(\eta) \end{cases} \tag{2.3}$$

is bounded in X by a constant independent of $\lambda \in [0,1]$. First, we show that KN is completely continuous.

- *KN is continuous*

$$\begin{aligned} \|KNx\| &\leq \int_0^1 |G(t,s)f(s,x(s),x'(s))|ds \\ &\leq \max_{(t,s)\in[0,1]^2} G(t,s) \int_0^1 |f(t,x(s),x'(s))|ds \\ &\leq \max_{(t,s)\in[0,1]^2} G(t,s) \int_0^1 h_r(s)ds \\ &\leq \max_{(t,s)\in[0,1]^2} G(t,s)\|h_r\|_1. \end{aligned}$$

This proves that KN is bounded, hence continuous.

- *Moreover, using the ArzelàAscoli theorem, we conclude that KN is compact.*

Let $\Omega = \{x \in X, \|x\| \leq r\}$, where $r > 0$ is a bounded set of X.
Since $\|KNx\| \leq \max_{(t,s)\in[0,1]^2} G(t,s)\|h_r\|_1$ for all $x \in \Omega$, for all $t \in [0,1]$,so the functions in Ω are uniformly bounded.
Furthermore, for any $\varepsilon > 0$, since $G(t,s)$ is continuous on $[0,1] \times [0,1]$ (which is compact), so $G(t,s)$ is uniformly continuous.
Thus, there exists $\delta > 0$ such that $|G(t_1,s) - G(t_2,s)| \leq \frac{\varepsilon}{\|h_r\|_1}$ for $|t_1 - t_2| < \delta$, and for all s fixed in $[0,1]$, we have for $x \in \Omega$

$$|KNx(t_1) - KNx(t_2)| \leq \int_0^1 |G(t_1, s) - G(t_2, s)| \cdot |f(s, x(s), x'(s))| ds$$
$$\leq \frac{\varepsilon}{\|h_r\|_1} \int_0^1 \|h_r\|_1 ds = \varepsilon$$

which proves that the functions in Ω are equicontinuous.
By the ArzelàAscoli theorem, $KN(\Omega)$ is relatively compact, so KN transforms every bounded set into a relatively compact set, proving that KN is compact.
Since KN is continuous, we conclude that it is completely continuous.
Now, let x be a solution of (2.2) for some $\lambda \in [0, 1]$:

$$\begin{aligned}\|x\|_1 &= \lambda \|f(t, x(t), x'(t))\| \\ &\leq \|p\|_1 \|x\|_\infty + \|q\|_1 \|x'\|_\infty + ||r||_1 \\ &\leq (\|p\|_1 + \|q\|_1) \|x\|_1\end{aligned}$$

Thus

$$(1 - \|p\|_1 + \|q\|_1) \|x\|_1 \leq \|r\|_1$$

Hence

$$||x\|_1 \leq \frac{1}{1 - (\|p\|_1 + \|q\|_1)} \|r\|_1 = C \text{ (constant)}$$

Which implies that $\|x\|_\infty \leq C$ and $\|x'\|_\infty \leq C$, therefore $\|x\|_X \leq C$.

We now give the uniqueness result for the problem (2.1).

Example 2.1: *We consider the problem*

$$\begin{cases} x''(t) = \frac{t}{3}x + \frac{1}{5}x' + \frac{1}{\sqrt{1+x^2}}, & t \in (0, 1), \\ x(0) = 0, \qquad x(1) = x(\frac{1}{2}). \end{cases}$$

In this case, $\eta = \frac{1}{2}$ and we chose

$$f(t, u, v) = \frac{t}{3}u + \frac{1}{5}v + \frac{1}{\sqrt{1 + u^2}}.$$

Then, we have f is $L^1 - Caratheodory$ and

$$|f(t, u, v)| \leq \frac{t}{3}|u| + \frac{1}{5}|v| + 1$$

Then we chose , $p(t) = \frac{t}{3}, q(t) = \frac{1}{5}, r(t) = 1$ *these functions are on* $L^1([0,1])$.
We have

$$\|p\|_1 + \|q\|_1 = \frac{1}{6} + \frac{1}{5} = \frac{11}{30}$$

then we have

$$\|p\|_1 + \|q\|_1 < 1.$$

Thus all condition of theorem (2.1) *are satisfied ,then the problem admits a solution on* $C^1([0,1])$.

Theorem 2.2. *Let* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *be a function of* L^1 *Caratheodory. Assume that* $a > 0, b > 0$ *such that*

$$(***) \quad |f(t,u_1,v_1) - f(t,u_2,v_2)| \leq a|u_1 - u_2| + b|v_1 - v_2|$$

almost everywhere $t \in [0,1]$*, for all* $(u_1,v_1),(u_2,v_2) \in \mathbb{R}^2$*. Then the problem* (2.1) *admits a unique solution in* $C^1[0,1]$ *provided that*

$$\frac{4a\pi\sqrt{\eta} + 2\pi^2\sqrt{\eta}b + 4a + 4ab^2\pi + 2\pi^2 b^3}{\pi^2} < 1.$$

Proof

We have from $(***)$,

$$|f(t,u,v)| \leq a|u| + b|v| + |f(t,0,0)|$$

then, the problem (2.1) *admits at least one solution by Theorem* (2.2)*. Now, let* x_1, x_2 *are two solutions of problem* (2.1) *on* $C^1[0,1]$ *and put* $z = x_1 - x_2$*, we get*

$$\begin{cases} z''(t) = f(t,x_1(t),x_1'(t)) - f(t,x_2(t),x_2'(t)), & t \in (0,1) \\ z(0) = 0, \quad z(1) = z'(\eta). \end{cases}$$

From

$$z''(t) = f(t,x_1(t),x_1'(t)) - f(t,x_2(t),x_2'(t))$$

and multiplying this equation by z*, after integrating from 0 to 1, we get*

$$\begin{aligned}
0 &= -\int_0^1 z''(t)z(t)dt + \int_0^1 (f(t,x_1(t),x_1'(t)) - f(t,x_2(t),x_2'(t))z(t)dt. \\
&\geq -z(1)z'(1) + \int_0^1 |z'(t)|^2 dt - \int_0^1 |f(t,x_1(t),x_1'(t)) - f(t,x_2(t),x_2'(t)))||z(t)|dt. \\
&\geq -|z(1)||z'(1)| + ||z'||_2^2 - \int_0^1 (a|x_1(t)-x_2(t)|^2 + b|x_1'(t)-x_2'(t)||x_1(t)-x_2(t)|)dt. \\
&\geq -\sqrt{\eta}||z'||_2 2||Z''||_1 + ||z'||_2^2 - \int_0^1 (a|z(t)|^2 + b|z'(t)||z(t)|)dt. \\
&\geq -2\sqrt{\eta}||z'||_2||Z''||_1 + ||z'||_2^2 - a||z(t)||_2^2 + b||z'||_\infty||z||_\infty.
\end{aligned}$$

But

$$a\|z\|_2^2 \leq a\frac{4}{\pi^2}\|z'\|_2^2$$

because, by Wirtinger's inequality

$$\int_0^1 |z(t)-z(0)|^2 dt \leq 4\left(\frac{1-0}{\pi}\right)^2 \int_0^1 |z'(t)|^2 dt.$$

i,e

$$\|z\|_2^2 \leq \frac{4}{\pi^2}\|z'\|_2^2.$$

And

$$||z||_\infty \leq ||z'||_2$$

$$||z'||_\infty \leq 2||z''||_1.$$

Then

$$b||z'||_\infty||z||_\infty \leq 2b||z'||_2||z''||_1$$

Also

$$\begin{aligned}
\|z''\|_1 &= \|f(t,x_1(t),x_1'(t)) - f(t,x_2(t),x_2'(t))\|_1 \\
&\leq a\|z\|_1 + b\|z'\|_1 \\
&\leq a\frac{2}{\pi}\|z'\|_2 + b\|z'\|_2 \\
&\leq \left(\frac{2a}{\pi} + b\right)\|z'\|_2.
\end{aligned}$$

Then

$$\begin{aligned}
0 &\geq -2\sqrt{\eta}\|z'\|_2\|z''\|_1 + \|z'\|_2^2 - a\|z\|_2^2 - b\|z'\|_\infty\|z\|_\infty \\
&\geq -2\sqrt{\eta}\|z'\|_2\left(\frac{2a}{\pi}+b\right)\|z'\|_2 + \|z'\|_2^2 - \frac{4a}{\pi^2}\|z'\|_2^2 - 2b\left(\frac{2a}{\pi}+b\right)\|z'\|_2^2. \\
&\geq \left(-2\sqrt{\eta}\left(\frac{2a}{\pi}+b\right) + 1 - \frac{4a}{\pi^2} - 2b\left(\frac{2a}{\pi}+b\right)\right)\|z'\|_2^2. \\
&\left(1 - \frac{4a\pi\sqrt{\eta} + 2\pi^2\sqrt{\eta}b + 4a + 4a\pi b + 2\pi^2 b^2}{\pi^2}\right)\|z'\|_2^2.
\end{aligned}$$

This implies that

$$\|z'\|_2 = 0.$$

From the condition $z(0) = 0$*, we obtain*

$$z(t) = \int_0^t z'(s)\,ds.$$

Since we also have

$$\|z\|_\infty \leq \|z'\|_2 = 0,$$

it follows that

$$z = 0.$$

Thus, we conclude that

$$x_1 = x_2.$$

Example 2.2: *We consider the problem*

$$\begin{cases} x''(t) = \frac{1}{8}x + \frac{1}{2}x' + cos(x), & t \in (0,1), \\ x(0) = 0, \qquad x(1) = x(\frac{1}{4}). \end{cases}$$

In this case, $\eta = \frac{1}{4}$ *and we chose*

$$f(t,u,v) = \frac{1}{8}u + \frac{1}{2}v + cos(u).$$

Then, we have f *is* $L^1 - Caratheodory$ *and*

$$|f(t,u,v)| \leq \frac{1}{8}|u| + \frac{1}{2}|v| + 1$$

Then we chose $,p(t) = \frac{1}{8}, q(t) = \frac{1}{2}, r(t) = 1$ *these functions are on* $L^1([0,1])$.
We have

$$\|p\|_1 + \|q\|_1 = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$$

Then we have $\|p\|_1 + \|q\|_1 < 1$.
Then the problem admits a solution on $C^1([0,1])$.
We have

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq a|u_1 - u_2| + b|v_1 - v_2|$$

We chose $a = \frac{1}{8}, b = \frac{1}{2}, \eta = \frac{1}{4}$

$$\frac{4a\pi\sqrt{\eta} + 2\pi^2\sqrt{\eta}b + 4a + 4ab^2\pi + 2\pi^2b^3}{\pi^2} = \frac{\frac{1}{2}\pi\sqrt{\frac{1}{4}} + 2\pi^2\sqrt{\frac{1}{4}} \times \frac{1}{2} + 4 \times \frac{1}{8} + 4 \times \frac{1}{8} \times \frac{1}{4}\pi + 2\pi^2 \times \frac{1}{8}}{\pi^2} < 1$$

Thus all condition of theorem (2.2) *are satisfied , then the problem admits a unique solution on* $C^1([0,1])$.

2.2 Results for the Boundary Value Problem (2.2)

Lemma 2.4. *Let* $h \in \mathbb{L}^1([0,1])$, $\alpha \in \mathbb{R}$ *with* $\eta(0,1)$ *and* $x \in \mathbb{C}([0,1])$.
If

$$\begin{cases} x''(t) + h(t) = 0, & t \in (0,1), \\ x(0) = 0, \qquad x(1) = \alpha x(\eta), & (\alpha \neq 1) \end{cases} \tag{2.4}$$

Then

$$x(t) = \int_0^t (t-s)h(s)ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)h(s)ds + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)h(s)ds$$

Proof

From $x''(t) = -h(t)$ *with integration we find*

$$x'(t) - x'(0) = -\int_0^t h(s)ds.$$

and with integration again

$$x(t) = x'(0)t - \int_0^t (t-s)h(s)ds.$$

Now, using the condition $x(1) = \alpha x(\eta)$, with

$$x(1) = x'(0) - \int_0^1 (1-s)h(s)ds$$
$$x(\eta) = x'(0)\eta - \int_0^\eta (\eta - s)h(s)ds$$

We get

$$(1-\alpha\eta)x'(0) = -\alpha \int_0^\eta (\eta - s)h(s)ds + \int_0^1 (1-s)h(s)ds$$

Then

$$x(t) = -\int_0^t (t-s)h(s)ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)h(s)ds + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)h(s)ds$$

Lemma 2.5. *Under the assumption of Lemma (2.4),the solution to problem can be writen as*

$$x(t) = \int_0^1 G(t,s)h(s)ds$$

Where $G(t,s)$ is the Green's function defined by

$$G(t,s) = \frac{1}{1-\alpha\eta}\begin{cases} s - \alpha\eta s + \alpha t s - ts, & s \leq \min\{t,\eta\}, \\ -\alpha t\eta + \alpha t s + t - ts, & t \leq s \leq \eta, \\ \alpha\eta t - \alpha\eta s, & \eta \leq s \leq t, \\ t - ts, & s \geq \max\{t,\eta\}. \end{cases}$$

proof

We have two cases

(a) *If $t \leq \eta$,the solution of problem (2.2) can be expressed as*

$$
\begin{aligned}
x(t) &= \int_0^t (t-s)h(s)ds - \frac{\alpha t}{1-\alpha\eta}\left(\int_0^t (\eta-s)h(s)ds + \int_t^\eta (\eta-s)h(s)ds\right) \\
&\quad + \frac{t}{1-\alpha\eta}\left(\int_0^t (1-s)h(s)ds + \int_t^\eta (1-s)h(s)ds + \int_\eta^1 (1-s)h(s)ds\right) \\
&= \frac{1}{1-\alpha\eta}\int_0^t \left(-(1-\alpha\eta)(t-s) + \alpha t(\eta-s) + t(1-s)\right)h(s)ds \\
&\quad + \frac{1}{1-\alpha\eta}\int_t^\eta \left(-\alpha t(\eta-s) + t(1-s)\right)h(s)ds + \frac{t}{1-\alpha\eta}\int_\eta^1 (1-s)h(s)ds \\
&= \frac{1}{1-\alpha\eta}\int_0^t \left(s - \alpha\eta s + \alpha ts - ts\right)h(s)ds \\
&\quad + \frac{1}{1-\alpha\eta}\int_t^\eta \left(-\alpha t\eta + \alpha ts + t - ts\right)h(s)ds + \frac{t}{1-\alpha\eta}\int_\eta^1 (1-s)h(s)ds \\
&= \int_0^1 G(t,s)h(s)ds.
\end{aligned}
$$

(b) *If $t \geq \eta$,the solution of problem (2.2) can be expressed as*

$$
\begin{aligned}
x(t) &= \int_0^\eta (t-s)h(s)ds - \int_\eta^t (t-s)h(s)ds \\
&\quad + \frac{t}{1-\alpha\eta}\left(\int_0^\eta (1-s)h(s)ds + \int_\eta^t (1-s)h(s)ds + \int_t^1 (1-s)h(s)ds\right) \\
&\quad - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)h(s)ds \\
&= \frac{1}{1-\alpha\eta}\int_0^\eta \left(-(1-\alpha\eta)(t-s) - \alpha t(\eta-s) + t(1-s)\right)h(s)ds \\
&\quad + \frac{1}{1-\alpha\eta}\int_\eta^t \left(-(1-\alpha\eta)(t-s) + t(1-s)\right)h(s)ds + \frac{t}{1-\alpha\eta}\int_t^1 (1-s)h(s)ds \\
&= \int_0^\eta \frac{s-\alpha\eta s + \alpha ts - ts}{1-\alpha\eta}h(s)ds + \int_\eta^t \frac{\alpha\eta t - \alpha\eta s}{1-\alpha\eta}h(s)ds + \int_t^1 \frac{t-ts}{1-\alpha\eta}h(s)ds \\
&= \int_0^1 G(t,s)h(s)ds.
\end{aligned}
$$

Lemma 2.6. *If $1 < \alpha < \frac{1}{\eta}$ with $\eta \in (0,1)$, then the Green's function to problem (2.2) satisfies*

$$
0 \leq G(t,s) \leq \frac{1}{8}\left(\frac{\alpha\eta^2-1}{\alpha\eta-1}\right)^2 \qquad t,s \in (0,1)
$$

Proof

step1: *We show that* $G(t,s) \geq 0$.

- *For* $0 \leq s \leq min\{t, \eta\}$ *we have*

$$G(t,s) = \frac{s - \alpha\eta s + \alpha ts - ts}{1 - \alpha\eta}$$

$$= \frac{(1-\alpha\eta)s + (\alpha - 1)ts}{1 - \alpha\eta} \geq 0$$

- *For* $0 \leq t \leq s \leq \eta$, *we get*

$$G(t,s) = \frac{-\alpha t\eta + \alpha ts + t - ts}{1 - \alpha\eta}$$

$$= \frac{(1-\alpha\eta)t + (\alpha - 1)ts}{1 - \alpha\eta} \geq 0$$

- *For* $0 \leq \eta \leq s \leq t$, *we have*

$$G(t,s) = \frac{\alpha\eta t - \alpha\eta s}{1 - \alpha\eta}$$

$$= \frac{(t-s)\alpha\eta}{1 - \alpha\eta} \geq 0$$

- *For* $s \geq max\{t, \eta\} \geq 0$

$$G(t,s) = \frac{t - ts}{1 - \alpha\eta}$$

$$= \frac{t(1-s)}{1 - \alpha\eta)} \geq 0$$

step2: *We show that* $G(t,s) \leq \frac{1}{8}(\frac{\alpha\eta^2 - 1}{\alpha\eta - 1})^2$, *for* $t, s \in (0,1)$:
In fact

$$\varphi(t) = \int_0^1 G(t,s)ds$$

then

$$\|\varphi\|_\infty = \|G\|_\infty$$

and we have that φ is a solution to problem

$$\begin{cases} -\varphi''(t) = 1 \\ \varphi(0) = 0, \qquad \varphi(1) = \alpha\varphi(\eta). \end{cases}$$

This is implies,
$\varphi'(t) = \varphi'(0) - t$ *with* $\qquad \varphi(0) = 0, \qquad \varphi(1) = \alpha\varphi(\eta)$
and $\varphi(t) = \varphi'(0)t - \frac{1}{2}t^2$ *with* $\qquad \varphi(0) = 0, \qquad \varphi(1) = \alpha\varphi(\eta)$
then

$$\varphi(t) = -\frac{1}{2}t^2 + \frac{1}{2}\frac{\alpha\eta^2 - 1}{\alpha\eta - 1}t$$

it is who gives

$$\begin{aligned} \|\varphi\|_\infty = \sup_{0\le t\le 1} |\varphi(t)| &= |\varphi(\frac{1}{2}\frac{\alpha\eta^2 - 1}{\alpha\eta - 1})| \\ &= \frac{1}{8}(\frac{\alpha\eta^2 - 1}{\alpha\eta - 1})^2 = \|G\|_\infty \end{aligned}$$

Lemma 2.7. *If* $1 < \alpha < \frac{1}{\eta}$*, then*

$$\|x\|_\infty \le \|x'\|_\infty \le \frac{\alpha(1-\eta)}{1-\alpha\eta}\|x''\|_1$$

Proof

Let $x \in C^1([0,1])$ *with* $x(0) = 0$*,* $x(1) < x(\eta)$*, and* $1 < \alpha < \frac{1}{\eta}$*. From the finite increment theorem, there exists* $\xi \in (\eta, 1)$ *such that*

$$x(1) - x(\eta) = x'(\xi)(1-\eta)$$

and from $x(1) - x(\eta) = (\alpha - 1)x(\eta)$*, then*

$$x(\eta) = \frac{1-\eta}{\alpha - 1}x'(\xi).$$

We have

$$
\begin{aligned}
\|x''\|_1 &\geq \int_\xi^t |x''(s)|ds \quad \forall t \in [0,1] \\
&\geq \left| \int_\xi^t x''(s)ds \right| \quad \forall t \in [0,1] \\
&\geq |x'(t) - x'(\xi)| \quad \forall t \in [0,1] \\
\Rightarrow \|x''\|_1 &\geq \sup_{t\in[0,1]} |x'(t)| - \frac{\alpha-1}{1-\eta} x(\eta) \\
\Rightarrow \|x''\|_1 &\geq \|x'\|_\infty - \frac{\alpha-1}{1-\eta} x(\eta)
\end{aligned}
$$

And from $\alpha|x(\eta)| \leq \|x'\|_1$, *and* $\|x''\|_1 \leq \|x'\|_\infty$, *we get*

$$
\begin{aligned}
\|x''\|_1 &\geq \|x'\|_\infty - \frac{\alpha-1}{1-\eta} \cdot \frac{1}{\alpha} \|x'\|_\infty \\
&\geq \|x'\|_\infty - \frac{\alpha-1}{1-\eta} \cdot \frac{1}{\alpha} \|x'\|_1 \\
&= \frac{1-\alpha\eta}{\alpha(1-\eta)} \|x'\|_\infty \\
\Rightarrow \|x'\|_\infty &\leq \frac{1-\alpha\eta}{\alpha(1-\eta)} \|x'\|_1
\end{aligned}
$$

and from $\|x\|_\infty \leq \|x'\|_\infty$ *then*

$$
\|x\|_\infty \leq \|x'\|_\infty \leq \frac{\alpha(1-\eta)}{1-\alpha\eta} \|x''\|_1.
$$

Theorem 2.3. *Let* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *is* L^1*-Caratheodory function.*
Assume that it exists the function p, q, r *on* $L^1[0,1]$ *such that,*

$$
|f(t,u,v)| \leq p(t)|u| + q(t)|v| + r(t)
$$

a,e $t \in [0,1]$ *and for* $(u,v) \in \mathbb{R}^2$, *(almost everywhere)*
And let $\alpha \in \mathbb{R}$, $n \in [0,1]$ *such that* $1 < \alpha < \frac{1}{\eta}$, *are given.*
Then, the problem (2.2) *admits a solution on* $C^1[0,1]$ *provided that*

$$
\|p\|_1 + \|q\|_1 < \frac{1-\alpha\eta}{\alpha(1-\eta)}.
$$

Proof

Let $X = C^1([0,1])$, $Y = L^1([0,1])$.*Note that* $Lx = x''$ *and* $N : X \to Y$ *with* $(Nx)(t) = f(t, x(t), x'(t)), \quad t \in [0,1]$

We have that $x \in X$ *is a solution of problem* (2.2) *if and only if* x *is a solution of equation*

$$Lx + Nx = 0$$

To do this, just check that the set of all possible solutions of the family of equations

$$\begin{cases} x''(t) = \lambda f(t, x(t), x'(t)), \quad t \in (0,1) \\ x(0) = 0, \quad x(1) = x(\eta) \end{cases} \tag{2.5}$$

is bounded on $C^1([0,1])$ *by a constant independent of* $\lambda \in [0,1]$. *If* x *is a solution of problem* (2.2) *, we have*

$$\begin{aligned} \|x''\|_1 =& \lambda \|f(t, x(t), x'(t))\|_1 \\ &\leq \|P\|1\|x\|_\infty + \|Q\|1\|x'\|_\infty + \|r\|_1 \\ &\leq (\|P\|_1 + \|Q\|_1)\frac{\alpha(1-\eta)}{1-\alpha\eta}\|x''\|_1 + \|r\|_1 \quad \textit{(by lemma (2.7))} \end{aligned}$$

$$\Rightarrow \left(\frac{1-\alpha\eta}{\alpha(1-\eta)} - (C\|P\|_1 + \|Q\|_1)\right)\frac{\alpha(1-\eta)}{1-\alpha\eta}\|x''\|_1 \leq \|r\|_1$$

$$\Rightarrow \|x''\|_1 \leq \frac{1}{\frac{1-\alpha\eta}{\alpha(1-\eta)} - C(\|P\|_1 + \|Q\|_1)} = C.$$

Then

$$\|x\|_\infty \leq C \quad \textit{and} \quad \|x'\|_\infty \leq C.$$

This implies

$$\|u\| \leq C.$$

We conclude by the theorem of LeraySchauder the problem (2.2) *admits at one solution on* $C^1([0,1])$ *provided that*

$$\|p\| + \|q\| < \frac{1-\alpha x}{\alpha(1-x)}.$$

Example2.3: *We consider the problem*

$$\begin{cases} x''(t) = \frac{t}{50}x + \frac{1}{100}x' + sin(x), & t \in (0,1), \\ x(0) = 0, & x(1) = 3x(\frac{1}{4}). \end{cases}$$

In this case, $\alpha = 3, \eta = \frac{1}{4}$ *and we chose*

$$f(t,u,v) = \frac{t}{50}u + \frac{1}{100}v + sin(u).$$

Then, we have f *is* $L^1 - Caratheodory$ *and*

$$|f(t,u,v)| \leq \frac{t}{50}|u| + \frac{1}{100}|v| + 1$$

Then we chose ,$p(t) = \frac{t}{50}, q(t) = \frac{1}{100}, r(t) = 1$ *these functions are on* $L^1([0,1])$.

We have

$$\|p\|_1 + \|q\|_1 = \frac{1}{100} + \frac{1}{100} = \frac{1}{50}$$

and

$$\frac{1-\alpha\eta}{\alpha(1-\eta)} = \frac{1}{9}$$

then we have

$$\|p\|_1 + \|q\|_1 < \frac{1-\alpha\eta}{\alpha(1-\eta)}.$$

Thus all condition of theorem (2.3) are satisfied ,then the problem admits a solution on $C^1([0,1])$.

2.3 Results for the Boundary Value Problem (2.3)

Now we consider the problem

$$\begin{cases} x''(t) + f(t,x(t),x'(t)) = 0, & t \in (0,1) \\ x'(0) = 0, & x(1) = \alpha x(\eta) \end{cases}$$

Lemma 2.8. *Let $h \in L^1[0,1]$, $\alpha \in \mathbb{R}$ ($\alpha \neq 1$), $\eta \in (0,1)$ and $x \in C^1[0,1]$.*
If

$$\begin{cases} x''(t) + h(t) = 0, & t \in (0,1) \\ x'(0) = 0, \quad x(1) = \alpha x(\eta) \end{cases}$$

Then

$$x(t) = -\int_0^t (t-s)h(s)\,ds - \frac{\alpha}{1-\alpha}\int_0^\eta (\eta - s)h(s)\,ds + \frac{1}{1-\alpha}\int_0^1 (1-s)h(s)\,ds$$

proof

From $x''(t) = -h(t)$,

$$x'(t) - x'(0) = -\int_0^t h(s)\,ds \Rightarrow x'(t) = -\int_0^t h(s)\,ds$$

and

$$x(t) - x(0) = -\int_0^t (t-s)h(s)\,ds$$

$$x(1) = x(0) - \int_0^1 (t-s)h(s)\,ds$$

By the condition $x(1) = \alpha x(\eta)$, we have

$$x(1) = x(0) - \int_0^1 (1-s)h(s)\,ds$$

$$x(\eta) = x(0) - \int_0^\eta (\eta - s)h(s)ds$$

We get

$$(1-\alpha)x(0) = -\alpha\int_0^\eta (\eta - s)h(s)\,ds + \int_0^1 (1-s)h(s)\,ds$$

i.e

$$x(0) = -\frac{\alpha}{1-\alpha}\int_0^\eta (\eta - s)h(s)\,ds + \frac{1}{1-\alpha}\int_0^1 (1-s)h(s)ds$$

Then

$$x(t) = -\int_0^t (t-s)h(s)\,ds - \frac{\alpha}{1-\alpha}\int_0^\eta (\eta - s)h(s)\,ds + \frac{1}{1-\alpha}\int_0^1 (1-s)h(s)\,ds$$

Lemma 2.9. *Under the assumption of Lemma (2.8), the solution to problem can be written* *as*

$$x(t) = \int_0^1 G(t,s)h(s)ds$$

Where $G(t,s)$ is the Green's function defined by

$$G(t,s) = \frac{1}{1-\alpha}\begin{cases} -t+\alpha t-\alpha\eta+1, & s \leq \min\{t,\eta\}, \\ -\alpha\eta+\alpha s+1-s, & t \leq s \leq \eta, \\ \alpha t-\alpha s-t+1, & \eta \leq s \leq t, \\ 1-s, & s \geq \max\{t,\eta\}. \end{cases}$$

Proof

We have two cases

(a) *If $t \leq \eta$, the solution of problem (2.3) can be expressed as*

$$\begin{aligned} x(t) &= -\int_0^t (t-s)h(s)ds - \frac{\alpha}{1-\alpha}\left(\int_0^t (\eta-s)h(s)ds + \int_t^\eta (\eta-s)h(s)ds\right) \\ &\quad + \frac{1}{1-\alpha}\left(\int_0^t (1-s)h(s)ds + \int_t^\eta (1-s)h(s)ds + \int_\eta^1 (1-s)h(s)ds\right) \\ &= \frac{1}{1-\alpha}\int_0^t \left(-(1-\alpha)(t-s) - \alpha(\eta-s) + (1-s)\right)h(s)ds \\ &\quad + \frac{1}{1-\alpha}\int_t^\eta \left(-\alpha(\eta-s) + (1-s)\right)h(s)ds + \frac{1}{1-\alpha}\int_\eta^1 (1-s)h(s)ds \\ &= \frac{1}{1-\alpha}\int_0^t \left(-t+\alpha t-\alpha\eta+1\right)h(s)ds \\ &\quad + \frac{1}{1-\alpha}\int_t^\eta \left(-\alpha\eta+\alpha s+1-s\right)h(s)ds + \frac{1}{1-\alpha}\int_\eta^1 (1-s)h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds. \end{aligned}$$

(b) *If $t \geq \eta$, the solution of problem (2.3) can be expressed as*

$$\begin{aligned}
x(t) &= -\int_0^{\eta}(t-s)h(s)ds - \int_{\eta}^{t}(t-s)h(s)ds \\
&\quad + \frac{1}{1-\alpha}\left(\int_0^{\eta}(1-s)h(s)ds + \int_{\eta}^{t}(1-s)h(s)ds + \int_t^1(1-s)h(s)ds\right) \\
&\quad - \frac{\alpha}{1-\alpha}\int_0^{\eta}(\eta-s)h(s)ds \\
&= \frac{1}{1-\alpha}\int_0^{\eta}\left(-(1-\alpha)(t-s)-\alpha(\eta-s)+(1-s)\right)h(s)ds \\
&\quad + \frac{1}{1-\alpha}\int_{\eta}^{t}\left(-(1-\alpha)(t-s)+(1-s)\right)h(s)ds + \frac{1}{1-\alpha}\int_t^1(1-s)h(s)ds \\
&= \int_0^{\eta}\frac{-t+\alpha t-\alpha\eta+1}{1-\alpha}h(s)ds + \int_{\eta}^{t}\frac{\alpha t-\alpha s-t+1}{1-\alpha}h(s)ds + \int_t^1\frac{1-s}{1-\alpha}h(s)ds \\
&= \int_0^1 G(t,s)h(s)ds.
\end{aligned}$$

Lemma 2.10. *Let $x \in (0,1), \alpha \in \mathbb{R}$ and $x \in C^1([0,1])$ with $x(1) = \alpha x(\eta)$, Then*

- *If $\alpha \leq 0$,it exist $\xi \in [\eta, 1]$ such that $x(\xi) = 0$.*
- *If $0 < \alpha \neq 0$,it exist $\xi \in [\eta, 1]$ such that $x(\eta) = \frac{1-\eta}{\alpha-1}x'(\xi)$.*

Proof

- *If $\alpha = 0$ then $x(1) = 0$ i.e $\xi = 1$.*
- *If $\alpha < 0$ with $x(\eta) \neq 0$ then $x(1) \times x(\xi) < 0$*
  and by the intermediate value theory, it exists $\xi \in (\eta, 1)$ such that $x(\xi) = 0$
- *if $\alpha > 0, \alpha \neq 1$ by the theory of finite increments, it exist $\xi \in (\eta, 1)$ such that $x(1) - x(\eta) = (1-\eta)x'(\xi)$*
  it's who gives
  $x(\eta) = \frac{1-\eta}{\alpha-1}x'(\xi)$

Theorem 2.4. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be an L^1-Caratheodory function.*
Assume that there exist functions $p, q, r \in L^1[0,1]$ such that:

$$|f(t,u,v)| \leq p(t)|u| + q(t)|v| + r(t)$$

a.e. $t \in [0,1]$ *and for all* $(u,v) \in \mathbb{R}^2$, *let* $\alpha \in \mathbb{R}$ *and* $\eta \in (0,1)$. *Then the problem* (2.3) *admits a solution on* $C^1[0,1]$ *provided that*

$$\begin{cases} \|p\|_1 + \|q\|_1 < 1 \quad \text{if } \alpha \leq 0, \\ (1 + \max\left\{\frac{\alpha(1-\eta)}{|\alpha-1|}, \frac{1-\eta}{|\alpha-1|}\right\})\|p\|_1 + \|q\|_1 < 1 \quad \text{if } 0 < \alpha \neq 1. \end{cases}$$

Proof

Let $X = C^1[0,1]$, $Y = L^1[0,1]$. *We note that the linear operator*

$$L : D(L) \subset X \to Y \quad \text{with}$$

$$D(L) = \left\{x \in W^{2,1}(0,1) : x'(0) = 0,\ x(1) = \alpha x(\eta)\right\}$$

and for all $x \in D(L)$, *we have* $Lx = x''$. *We define the nonlinear application*

$$N : X \to Y \quad \text{by}$$

$$(Nx)(t) = f(t, x(t), x'(t)), \quad t \in [0,1].$$

and define the linear application

$$K : Y \to X \quad \text{by}$$

$$(ky)(t) = -\int_0^t (t-s)y(s)ds - \frac{\alpha}{1-\alpha}\int_0^\eta (\eta - s)y(s)ds + \frac{1}{1-\alpha}\int_0^\eta (1-s)y(s)ds.$$

According to above,it can be proven that $KN : X \to X$ *is completely continuous. By theorem of Leray-Schauder*

$$(4_\lambda)\begin{cases} x'' = \lambda f(t, x(t), x'(t)), \\ x'(0) = 0, \qquad x(1) = \alpha x(\eta). \end{cases}$$

Then we have tow cases :

case 1: *If* $\alpha \leq 0$, *from lemma* (2.10)*,it exist* $\xi \in (\eta, 1)$ *such that* $x(\xi) = 0$ *,and form* $x(t) = \int_\xi^t x'(s)ds$, $0 < t < 1$ *which implies that*

$$|x(t)| \leq \int_\xi^t |x'(s)|ds, \qquad 0 < t < 1$$

Then, $\|x\|_\infty \leq \|x'\|_\infty$ *and from* $|x'(t)| \leq \int_0^1 x''(s)ds$, $0 < t < 1$
we get $\|x'\|_\infty \leq \|x''\|_1$,*then*

$$\|x\|_\infty \leq \|x'\|_\infty \leq \|x''\|_1$$

If x *is a solution of problem* (4_λ) *we have*

$$\begin{aligned}\|x''\|_1 &\leq \|p\|_1\|x\|_\infty + \|q\|_1\|x''\|_\infty + \|r\|_1 \\ &\leq (\|p\|_1 + \|q\|_1)\|x''\|_1 + \|r\|_1\end{aligned}$$

then

$$\|x''\|_1 \leq \frac{1}{1-(\|p\|_1 + \|q\|_1)}\|r\|_1 = C,$$

Thus $\|x\|_\infty \leq C$,$\|x'\|_\infty \leq C$ *and* $\|x\|_\infty \leq C$ *on* $C^1([0,1])$.

case 2: *If* $0 < \alpha \neq 1$, *from lemma* (2.10), *it exist* $\xi \in [\eta, 1]$ *such that* $x(\xi) = \frac{1-\eta}{\alpha-1}x'(\xi)$, *and by* $x(t) = x(\eta) + \int_\eta^t x'(s)ds$, $0 < t < 1$
then $x(t) = \frac{1-\eta}{\alpha-1}x'(\xi) + \int_\eta^t x'(s)ds$ *which implies that*

$$\|x\|_\infty \leq \left(\frac{1-\eta}{|\alpha-1|} + 1\right)\|x'\|_\infty$$

And form $\|x'\|_\infty \leq \|x''\|_1$ *we get*

$$\|x\|_\infty \leq \left(\frac{1-\eta}{|\alpha-1|} + 1\right)\|x'\|_1.$$

On the other hand

$$x(t) = x(1) - \int_t^1 x'(s)ds, \qquad 0 < t < 1$$

i.e.

$$x(t) = \alpha x(\eta) - \int_t^1 x'(s)ds, \qquad 0 < t < 1$$

Then

$$x(t) = \frac{\alpha(1-\eta)}{\alpha-1}x'(\xi) - \int_t^1 x'(s)ds, \qquad 0 < t < 1$$

wich implies that

$$\begin{aligned}\|x\|_\infty &\leq \left(\frac{\alpha(1-\eta)}{|\alpha-1|}+1\right)\|x'\|_\infty \\ &\leq \left(\frac{\alpha(1-\eta)}{|\alpha-1|}+1\right)\|x'\|_1\end{aligned}$$

then we have

$$\|x\|_\infty \leq \left(1+max(\frac{\alpha(1-\eta)}{|\alpha-1|},\frac{(1-\eta)}{|\alpha-1|}\right)\|x'\|_1$$

If x is a solution of problem (4_λ) we have

$$\|x''\|_1 \leq (\|p\|_1\left(1+max(\frac{\alpha(1-\eta)}{|\alpha-1|},\frac{(1-\eta)}{|\alpha-1|}+\|q\|_1)\right)\|x''\|_1+\|r'\|_1$$

Then

$$\|x''\|_1 \leq \frac{1}{1-\left(\|p\|_1(1+max(\frac{\alpha(1-\eta)}{|\alpha-1|},\frac{(1-\eta)}{|\alpha-1|+})+\|q\|_1\right)}\|r'\|_1 = C'$$

Thus $\|x\|_\infty \leq C'$, $\quad \|x''\|_\infty \leq C'$
and $\|x\|_\infty \leq C'$ *on* $C^1([0,1])$.

Exemple2.4: *We consider the problem*

$$\begin{cases} x''(t)=\frac{t}{4}x+\frac{1}{2}x'+\frac{2025}{\sqrt{1+x^2}}, & t\in(0,1),\\ x(0)=0, \qquad x(1)=5x(\frac{1}{3}). \end{cases}$$

In this case, $\alpha=5,\eta=\frac{1}{3}$ *and we chose*

$$f(t,u,v)=\frac{t}{4}u+\frac{1}{2}v+\frac{2025}{\sqrt{1+u^2}}.$$

Then, we have f is $L^1-Caratheodory$ and

$$|f(t,u,v)|\leq \frac{t}{4}|u|+\frac{1}{2}|v|+2025$$

Then we chose , $p(t) = \frac{t}{4}, q(t) = \frac{1}{2}, r(t) = 2025$ *these functions are on* $L^1([0,1])$.
We have

$$(1 + \max\left\{\frac{\alpha(1-\eta)}{|\alpha-1|}, \frac{1-\eta}{|\alpha-1|}\right\})\|p\|_1 + \|q\|_1 = (1 + \frac{5}{6})\frac{1}{8} + \frac{1}{2} = \frac{35}{48} < 1$$

Then we have

$$(1 + \max\left\{\frac{\alpha(1-\eta)}{|\alpha-1|}, \frac{1-\eta}{|\alpha-1|}\right\})\|p\|_1 + \|q\|_1 < 1$$

Thus all condition of theorem (2.4) *are satisfied* , *then the problem admits a solution on* $C^1([0,1])$.

Chapter 3

Solvability of Second-Order Boundary Value Problems at N-Points

3.1 Introduction

This chapter deals with the results concerning the existence of a solution for boundary value problems at n-points. For example, we consider the boundary value problem at multi-point:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \in (0,\, 1) \\ x(0) = \sum_{i=1}^{m-2} c_i x(\xi_i), & x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j), \end{cases}$$

or $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ *is a continuous function or a Caratheodory function and* c_i, $a_j \in \mathbb{R}$, ξ_i, $\tau_j \in (0,\, 1)$, $i = 1, 2, ..., m-2$, $j = 1, 2, ..., n-2$,
$0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < ... < \tau_{n-2} < 1$.
In one case, we assume that each of the $c_i's$*, and each of the* $a_j's$*, have the same sign, and in another case, we assume that the* $c_i's$ *and the* $a_j's$ *do not have the same sign.*
In certain cases, or in the general case, we study the existence of solutions to this problem. In several cases, in order to study the existence of solutions for the multi-point boundary value problem as described above, we can use (as we will study in the third section of this chapter) a priori estimates obtained for the corresponding four-point boundary value problem given below.

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \in (0,\, 1) \\ x(0) = \gamma x(\zeta), & x(1) = \alpha x(\eta), \end{cases}$$

with $\gamma = \sum_{i=1}^{m-2} c_i$, $\alpha = \sum_{j=1}^{n-2} a_j$, *and* $\zeta \in [\xi_1, \xi_{m-2}]$, $\eta \in [\tau_1, \tau_{n-2}]$.
In the case where $\gamma = \sum_{i=1}^{m-2} c_i = 0$, *the first problem is called an n-point boundary value problem. It can be studied directly or reduced to a three-point problem.*

The study of n-point boundary value problems was initiated by Il'in and Moiseev [12]. Motivated by the work of Il'in and Moiseev [12], Gupta [8] studied nonlinear three-point boundary value problems for nonlinear ordinary differential equations. Since then, more general nonlinear n-point boundary value problems have been studied by several authors using the nonlinear alternative theorem of Leray-Schauder. Some recent results for nonlinear n-point boundary value problems can be found in ([7, 8, 9, 10, 15, 16]).

3.2 N-point boundary value problems: Reduction to three points

Let $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ *be a continuous function or an* L^1- *Caratheodory function,* $a_i \in \mathbb{R}$, *with all the* a_i'*s having the same sign,*
$\xi_i \in (0, 1), i = 1, 2, ..., n-2; \quad 0 < \xi_1 < \xi_2 < ... < \xi_{n-2} < 1.$
We consider the following second-order differential equation

$$x''(t) = f(t, x(t), x'(t)), \qquad t \in (0, 1) \tag{3.1}$$

subject to one of the following boundary conditions

$$x(0) = 0, \qquad x(1) = \sum_{i=1}^{n-2} a_i x(\xi_i), \tag{3.2}$$

$$x'(0) = 0, \qquad x(1) = \sum_{i=1}^{n-2} a_i x(\xi_i), \tag{3.3}$$

$$x(0) = 0, \qquad x'(1) = \sum_{i=1}^{n-2} a_i x'(\xi_i). \tag{3.4}$$

It is well known that the existence of a solution to these problems can be studied through the existence of a solution to equation (3.1) , *subject to one of the following three-point* *boundary conditions*

$$x(0) = 0, \qquad x(1) = \alpha x(\eta), \tag{3.5}$$

$$x'(0) = 0, \qquad x(1) = \alpha x(\eta), \tag{3.6}$$

$$x(0) = 0, \qquad x'(1) = \alpha x'(\eta). \tag{3.7}$$

where $\alpha = \sum_{i=1}^{n-2} a_i$ *and* $\eta \in [\xi_1, \xi_{n-2}]$.
Indeed, for each solution $x(t)$ *of problem* (3.1) , (3.2), *let us denote*

$$m = \min_{t\in[\xi_1,\xi_{n-2}]} x(t), \quad M = \max_{t\in[\xi_1,\xi_{n-2}]} x(t).$$

- *If* $a_i \in [0, +\infty)$, $i \in \{1, 2, ..., n-2\}$ *so*

$$\begin{aligned} a_i m \leqslant a_i x(\xi_i) &\leqslant a_i M, \\ \sum_{i=1}^{n-2} a_i m \leqslant \sum_{i=1}^{n-2} a_i x(\xi_i) &\leqslant \sum_{i=1}^{n-2} M, \\ m \leqslant \frac{\sum_{i=1}^{n-2} a_i x(\xi_i)}{\sum_{i=1}^{n-2} a_i} &\leqslant M, \\ m \leqslant \frac{x(1)}{\alpha} &\leqslant M. \end{aligned}$$

- *If* $a_i \in (-\infty, 0]$, $i \in \{1, 2, ..., n-2\}$ *so*

$$\begin{aligned} a_i m \geqslant a_i x(\xi_i) &\geqslant a_i M, \\ \sum_{i=1}^{n-2} a_i m \geqslant \sum_{i=1}^{n-2} a_i x(\xi_i) &\geqslant \sum_{i=1}^{n-2} M, \\ m \leqslant \frac{\sum_{i=1}^{n-2} a_i x(\xi_i)}{\sum_{i=1}^{n-2} a_i} &\leqslant M, \\ m \leqslant \frac{x(1)}{\alpha} &\leqslant M. \end{aligned}$$

It follows that there exists $\eta \in [\xi_1, \xi_{m-2}]$ *such that* $x(\eta) = \frac{x(1)}{\alpha}$, *which implies that* $x(t)$ *is also a solution of the boundary value problem* (3.1) , (3.5) .

Existence results for the problem (3.1) , (3.2)

Now we are studying the n-point problem (3.1) , (3.2) with $\xi_i \in (0,\, 1)$ *for* $i = 1, 2, ..., n-2$ *satisfies* $0 < \xi_1 < \xi_2 < ... < \xi_{n-2} < 1$ *and* $a_i \in \mathbb{R}$; $i = 1, 2, ..., n-2$, *that satisfy the conditions and have the same sign* $\alpha = \sum_{i=1}^{n-2} a_i \neq 0$.
We are now studying the n-point problem (3.1) , (3.2), with solutions that satisfy the conditions and have the same sign.

Lemma 3.1. *Let* $h \in L^1[0,\, 1]$, *and* $\xi_i \in (0,\, 1)$ *for* $i = 1, 2, ..., n-2$ *such that* $0 < \xi_1 < \xi_2 < ... < \xi_{n-2} < 1$ *and* $a_i \in \mathbb{R}$, $i = 1, 2, ..., n-2$, *have the same sign with* $\sum_{i=1}^{n-2} a_i \neq 0$ *and* $\sum_{i=1}^{n-2} a_i\xi_i \neq 1$ *for any*

$$\begin{cases} x''(t) = h(t), & t \in (0,\, 1) \\ x(0) = 0, & x(1) = \sum_{i=1}^{n2} a_i x(\xi_i). \end{cases}$$

Then

$$x(t) = \int_0^t (t-s)h(s)ds + At,$$

where A is given by

$$A\left(1 - \sum_{i=1}^{n-2} a_i\xi_i\right) = \sum_{i=1}^{n-2} a_i \int_0^{\xi_i} (\xi_i - s)h(s)ds - \int_0^1 (1-s)h(s)ds.$$

Proof

We have $(x''(t) = h(t)) \Longrightarrow (x(t) = x'(0)t + \int_0^t (t-s)h(s)ds)$, *from which we obtain*

$$x(1) = x'(0) + \int_0^1 (1-s)h(s)ds$$

and

$$\begin{aligned} \sum_{i=1}^{n-2} a_i x(\xi_i) &= \sum_{i=1}^{n-2} a_i \left[x'(0)\xi_i + \int_0^{\xi_i} (\xi_i - s)h(s)ds\right] \\ &= x'(0)\sum_{i=1}^{n-2} a_i\xi_i + \sum_{i=1}^{n-2} a_i \int_0^{\xi_i} (\xi_i - s)h(s)ds, \end{aligned}$$

And from $x(1) = \sum_{i=1}^{n-2} a_i x(\xi_i)$ *we obtain,*

$$x'(0)\left(1 - \sum_{i=1}^{n-2} a_i\xi_i\right) = \sum_{i=1}^{n-2} a_i \int_0^{\xi_i} (\xi_i - s)h(s)ds - \int_0^1 (1-s)h(s)ds,$$

Then

$$x(t) = \int_0^t (t-s)h(s)ds + At,$$

where A *is given by,*

$$A\left(1 - \sum_{i=1}^{n-2} a_i\xi_i\right) = \sum_{i=1}^{n-2} a_i \int_0^{\xi_i} (\xi_i - s)h(s)ds - \int_0^1 (1-s)h(s)ds.$$

Theorem 3.1. *Let* $f : [0,\, 1]\mathbb{R}^2 \to \mathbb{R}$ *be an* L^1*-caratheodory function.*
Assume that there exist functions $p,\, q,\, r$ *in* $L^1[0,\, 1]$ *such that ,*

$$|f(t,u,v)| \leqslant p(t)|u| + q(t)|v| + r(t),$$

p.p. $t \in [0,\, 1]$ *and for all* $(u,v) \in \mathbb{R}^2$*, and let* $\alpha = \sum_{i=1}^{n-2} a_i$*, and* $\eta \in (0,\, 1)$ *are given .*
Then, the problem (3.1) ,(3.2) *admits a solution in* $C^1[0,\, 1]$ *provided that*

$$\begin{cases} \|p\|_1 + \|q\|_1 < 1, & \text{if } \alpha \leqslant 1, \\ \|p\|_1 + \|q\|_1 < \frac{1-\alpha\xi_{n-2}}{\alpha(1-\xi_1)}, & \text{if } 1 < \alpha < \frac{1}{\xi_{n-2}}. \end{cases}$$

Proof

Let $\alpha = \sum_{i=1}^{n-2} a_i \leqslant 1.$
Let X *Banach space* $C^1[0,\, 1]$ *and* Y *represent the Banach space* $L^1(0,\, 1)$ *with their usual norms.*
We note that the linear operator $L : D(L) \subset X \longrightarrow Y$*, with*

$$D(L) = \left\{x \in W^{2,1}(0,1) : x(0) = 0,\, x(1) = \sum_{i=1}^{n-2} a_i x(\xi_i)\right\},$$

and for $x \in D(L)$*,*

$$Lx = x''.$$

We have also defined a nonlinear mapping $N : X \longrightarrow Y$ par

$$(Nx)(t) = f(t, x(t), x'(t)), \qquad t \in [0,\, 1]$$

N is a bounded operator from X to Y, and L is an injective operator .
Then we defined the linear operator $K : Y \longrightarrow X$, for $y \in Y$ by

$$(Ky)(t) = \int_0^t (t-s)y(s)ds + At,$$

where A is given by,

$$A\left(1 - \sum_{i=1}^{n-2} a_i\xi_i\right) = \sum_{i=1}^{n-2} a_i \int_0^{\xi_i} (\xi_i - s)y(s)ds - \int_0^1 (1-s)y(s)ds.$$

We have for $y \in Y, Ky \in D(L)$ and $LKy = y$; and for $u \in D(L), KLu = u$.
Moreover, the application $KN : X \longrightarrow X$ is compact. We have that,
($x \in C^1[0,\, 1]$ is a solution of problem (3.1) ,(3.2) , if and only if, (x is a solution of the operator equation: $Lx = Nx$), and ($Lx = Nx$) is equivalent to ($x = KNx$).
We apply the Leray-Schauder theorem to obtain the existence of a solution to the equation $x = KNx$.
To do so, it is sufficient to verify that the set of all possible solutions of the family of equations

$$\begin{cases} x''(t) = \lambda f(t, x(t), x'(t)), & t \in (0,\, 1) \\ x(0) = 0, \qquad x(1) = \sum_{i=1}^{n-2} a_i x(\xi_i), & \end{cases} \tag{3.8}$$

is, a priori, bounded in $X = C^1[0,\, 1]$ by a constant that is independent of $\lambda \in [0,\, 1]$.
Let $x(t)$ be a solution of problem (3.8)for some $\lambda \in [0,\, 1]$, such that $x \in W^{2,1}(0,\, 1)$ with $x(0) = 0$, $x(1) = \sum_{i=1}^{n-2} a_i x(\xi_i)$, There exists $\eta \in [\xi_1,\, \xi_{n-2}]$ such that $x(t)$ is a solution of problem (3.8) which is bounded in $X = C^1[0,\, 1]$ by a constant C that is independent of $\lambda \in [0,\, 1]$, such that

$$\|x\|_\infty \leqslant \|x'\|_\infty \leqslant \|x''\|_1 \leqslant C.$$

Then the set of all possible solutions of the family of equations (3.8) is, a priori, bounded in $X = C^1[0,\, 1]$ by a constant that is independent of $\lambda \in [0,\, 1]$.

Exemple3.1: *We consider the problem*

$$\begin{cases} x''(t) = \frac{t}{5}x + \frac{1}{3}x' + cos(x), & t \in (0,1), \\ x(0) = 0, \qquad x(1) = 4x(\frac{1}{6}) + 3x(\frac{1}{5}). \end{cases}$$

and we chose

$$f(t,u,v) = \frac{t}{5}u + \frac{1}{3}v + cos(u).$$

Then,we have f is $L^1 - Caratheodory$ and

$$|f(t,u,v)| \leq \frac{t}{5}|u| + \frac{1}{3}|v| + 1$$

Then we chose ,$p(t) = \frac{t}{5}, q(t) = \frac{1}{3}, r(t) = 1$ these functions are on $L^1([0,1])$.
We have

$$\begin{cases} \|p\|_1 + \|q\|_1 < \frac{1}{10} + \frac{1}{3} = \frac{13}{30} < 1 & if\alpha = 1 \\ \|p\|_1 + \|q\|_1 < \frac{1-4(\frac{1}{5})}{4-(1-\frac{1}{6})} = 0.06 & if 1 < \alpha < 5. \end{cases}$$

Thus all condition of theorem (3.1) are satisfied ,then the problem admits a solution on $C^1([0,1])$.

3.3 N-point boundary value problems: Reduction to four points

Let $f : [0,\, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function or a Carathéodory function, with , $c_i,\, a_j \in \mathbb{R}$, and each of the $c_i's$, and each of the $a_j's$, having the same sign, $\xi_i,\, \tau_j \in (0,\, 1)$, $i = 1, 2, ..., m-2,\;\; j = 1, 2, ..., n-2$,
$0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1,\;\; 0 < \tau_1 < \tau_2 < ... < \tau_{n-2} < 1.$
Let us consider the following second-order ordinary differential equation (3.1) subject to one

of the following boundary conditions:

$$x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j), \tag{3.9}$$

$$x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x'(1) = \sum_{j=1}^{n-2} a_j x'(\tau_j), \tag{3.10}$$

$$x(0) = \sum_{i=1}^{m-2} c_i x(\xi_i), \qquad x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j), \tag{3.11}$$

$$x(0) = \sum_{i=1}^{m-2} c_i x(\xi_i), \qquad x'(1) = \sum_{j=1}^{n-2} a_j x'(\tau_j), \tag{3.12}$$

$$x'(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j), \tag{3.13}$$

$$x'(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x'(1) = \sum_{j=1}^{n-2} a_j x'(\tau_j). \tag{3.14}$$

It is well known that if a function $x \in C^1$ satisfies one of the boundary conditions (3.14) -(3.19) and c_i, a_j, $i = 1, 2, ..., m-2$, $j = 1, 2, ..., n-2$ are as above, then there exist $\zeta \in [\xi_1, \xi_{m-2}]$, $\eta \in [\tau_1, \tau_{n-2}]$ such that

$$x(0) = \gamma x'(\zeta), \qquad x(1) = \alpha x(\eta), \tag{3.15}$$

$$x(0) = \gamma x'(\zeta), \qquad x'(1) = \alpha x'(\eta), \tag{3.16}$$

$$x(0) = \gamma x(\zeta), \qquad x(1) = \alpha x(\eta), \tag{3.17}$$

$$x(0) = \gamma x(\zeta), \qquad x'(1) = \alpha x'(\eta), \tag{3.18}$$

$$x'(0) = \gamma x'(\zeta), \qquad x(1) = \alpha x(\eta), \tag{3.19}$$

$$x'(0) = \gamma x'(\zeta), \qquad x'(1) = \alpha x'(\eta), \tag{3.20}$$

respectively with $\gamma = \sum_{i=1}^{m-2} c_i$, $\alpha = \sum_{j=1}^{n-2} a_j$.
As in the case of the m-point problem which is reduced to the three-point problem, we will prove an existence result for the problem (3.1) ,(3.14) for example, which first establishes an existence result for the problem (3.1),(3.20) ,and by using the a priori bounds obtained for this problem, for the problem (3.1),(3.14).

Existence results for the problem (3.1),(3.20)

For each solution x of the problem (3.1),(3.14) *there exist* $\zeta \in [\xi_1, \xi_{m-2}]$, $\eta \in [\tau_1, \tau_{n-2}]$ *such that x is a solution* (3.1),(3.20) *with* $\alpha = 1$. *We will prove that all the solutions of the problem*

$$\begin{cases} x''(t) = \lambda f(t, x(t), x'(t)), & t \in (0, 1) \\ x(0) = \gamma x'(\zeta), \quad x(1) = x(\eta), \end{cases}$$

is, a priori, bounded in $X = C^1[0,1]$ *by a constant independent of* $\lambda \in [0, 1]$.
Lately, we have studied the above problem when $\gamma = 0$ *(for some recent results on the three-point problem).*
Here, we extend the results to the general case ($\gamma \neq 0$).
In the following lemma, we obtain the necessary a priori estimates which are independent of ζ and η, for the problem. (3.1),(3.20).

Lemma 3.2. *Let* $\zeta, \eta \in (0, 1)$ *are given and* $x(t) \in W^{2,1}(0, 1)$ *such that* $x(0) = \gamma x'(\zeta)$, $x(1) = x(\eta)$. *Then*

$$\|x\|_\infty \leqslant (1 + |\gamma|)\|x'\|_\infty, \qquad \|x'\|_\infty \leqslant \|x''\|_1$$

Proof

Since $x(1) = x(\eta)$ *there exists* $\omega \in (\eta, 1)$ *with* $x'(\omega) = 0$.
Therefore $\|x'\|_\infty \leqslant \|x''\|_1$.
Also, since $x(t) = x(0) + \int_0^t x'(s)ds = \gamma x'(\zeta) + \int_0^t x'(s)ds$, *Then* $\|x\|_\infty \leqslant (1 + |\gamma|)\|x'\|_\infty$.

Lemma 3.3. *Let* $y \in L^1[0,1]$ *and* $\gamma \in \mathbb{R}$; $\eta, \zeta \in (0,1)$, *and* $x \in C^1[0,1]$, *in such a way that if*

$$\begin{cases} x''(t) = y(t), & t \in (0,1) \\ x(0) = \gamma x'(\zeta), \quad x(1) = x(\eta). \end{cases}$$

Then,

$$\begin{aligned} x(t) &= \int_0^t (t-s)y(s)ds + \gamma \int_0^\zeta y(s)ds \\ &\quad + \frac{\gamma + t}{1-\eta}[\int_0^\eta (\eta - s)y(s)ds \\ &\quad - \int_0^1 (1-s)y(s)ds], \qquad t \in [0,1] \end{aligned}$$

Proof

From $x''(t) = y(t)$ *we have*
$x'(t) = x'(0) + \int_0^t y(s)ds$ *which implies that* $x(t) - x(0) = x'(0)t + \int_0^t (t-s)y(s)ds,$
Since $x(0) = \gamma x'(\zeta)$ *and* $x'(\zeta) = \int_0^\zeta x''(s)ds + x'(0) = \int_0^\zeta y(s)ds + x'(0)$ *then*

$$x(t) = x'(0)(t+\gamma) + \gamma \int_0^\zeta y(s)ds + \int_0^t (t-s)y(s)ds,$$

We have, $x(1) = \alpha x(\eta)$ *which implies that*

$$x'(0)(1+\gamma) + \gamma \int_0^\zeta y(s)ds + \int_0^1 (1-s)y(s)ds =$$

$$x'(0)(\eta+\gamma) + \gamma \int_0^\zeta y(s)ds + \int_0^\eta (\eta-s)y(s)ds,$$

So we obtain

$$x'(0) = \frac{1}{1-\eta}[\int_0^\eta (\eta-s)y(s)ds - \int_0^1 (1-s)y(s)ds+],$$

Then

$$\begin{aligned} x(t) &= \int_0^t (t-s)y(s)ds + \gamma \int_0^\zeta y(s)ds \\ &+ \frac{\gamma+t}{1-\eta}[\int_0^\eta (\eta-s)y(s)ds \\ &- \int_0^1 (1-s)y(s)ds]. \end{aligned}$$

Theorem 3.2. *Let* $f : [0,\, 1]\mathbb{R}^2 \to \mathbb{R}$ *It is an* L^1*-Carathéodory function. Suppose that there exist functions* p, q, r *in* $L^1[0,1]$ *such that*

$$|f(t,u,v)| \leqslant p(t)|u| + q(t)|v| + r(t),$$

p.p. $t \in [0,\, 1]$ *and For all* $(u,v) \in \mathbb{R}^2$, *and let* $\zeta, \eta \in (0,\, 1)$, $\alpha, \gamma \in \mathbb{R}$ *with* $1+\gamma \neq \alpha(\gamma+\eta)$. *Then the problem* (3.1),(3.20) *admits a solution in* $C^1[0,1]$ *provided that*

$$(1+|\gamma|)\|p\|_1 + \|q\|_1 < 1.$$

Proof

Let $X = C^1[0,1]$, $Y = L^1[0,1]$.
We note that the linear operator $L : D(L) \subset X \longrightarrow Y$, *with*

$$D(L) = \left\{ x \in W^{2,1}(0,1) : x(0) = \gamma x'(\zeta), x(1) = x(\eta) \right\},$$

and for $x \in D(L)$,

$$Lx = x''.$$

We also define a nonlinear application $N : X \longrightarrow Y$ *by*

$$(Nx)(t) = f(t, x(t), x'(t)), \qquad t \in [0,\, 1]$$

N *is a bounded application from* X *to* Y, *and* L *is an injective application.*
Then we define the linear application $K : Y \longrightarrow X$, *for* $y \in Y$ *by*

$$\begin{aligned}(Ky)(t) &= \int_0^t (t-s)y(s)ds + \gamma \int_0^\zeta y(s)ds \\ &\quad + \frac{\gamma + t}{1-\eta}\left[\int_0^\eta (\eta - s)y(s)ds \right. \\ &\quad \left. - \int_0^1 (1-s)y(s)ds\right], \qquad t \in [0,\, 1]\end{aligned}$$

For $y \in Y, Ky \in D(L)$ *and* $LKy = y$; *and for* $u \in D(L), KLu = u$.
Moreover, the application $KN : X \longrightarrow X$ *is compact, We have*
($x \in C^1[0,1]$ *is a solution to the problem* (3.1),(3.20) $\Longleftrightarrow$ x *is a solution of the operator equation:* $Lx = Nx$), *which is equivalent to the equation* $x = KNx$.
We apply the Leray-Schauder theorem to obtain the existence of a solution to the equation $x = KNx$.
To do this, it is sufficient to verify that the set of all possible solutions of the family of equations

$$\begin{cases} x''(t) = \lambda f(t, x(t), x'(t)), & t \in (0,\, 1) \\ x(0) = \gamma x'(\zeta), \qquad x(1) = x(\eta), \end{cases} \tag{3.21}$$

It is, a priori, bounded in $X = C^1[0,1]$ *by a constant independent of* $\lambda \in [0,\, 1]$.

- *Suppose that* $\alpha \leqslant 0,\ \gamma = 0,$ *According to Lemma* (3.2) *we have*

$$||x||_\infty \leqslant ||x'||_\infty \leqslant ||x''||_1.$$

Now let $x(t)$ *be a solution of problem* (3.21) *for some* $\lambda \in [0,\ 1]$
such that $x(0) = \gamma x'(\zeta),\ x(1) = \alpha x(\eta),$ *then we have*

$$\begin{aligned} \|x''\|_1 &= \lambda \|f(t, x(t), x'(t))\|_1 \\ &\leqslant \|p\|_1 \|x\|_\infty + \|q\|_1 \|x'\|_\infty + \|r\|_1 \\ &\leqslant (\|p\|_1 + \|q\|_1)\|x''\|_1 + \|r\|_1. \end{aligned}$$

Since $\|p\|_1 + \|q\|_1 < 1,$ *there exists a constant* C *independent of* $\lambda \in [0, 1]$ *such that*

$$\|x''\|_1 \leq C.$$

According to Lemma (3.2) *we have*

$$\|x\|_\infty \leq (1 + |\gamma|)\|x'\|_\infty, \quad \|x'\|_\infty \leq \|x''\|_1,$$

therefore we have

$$\begin{aligned} \|x''\|_1 &= \lambda \|f(t, x(t), x'(t))\| \\ &\leq \|p\|_1 \|x\|_\infty + \|q\|_1 \|x'\|_\infty + \|r\|_1 \\ &\leq [(1 + |\gamma|)\|p\|_1 + \|q\|_1]\|x''\|_1 + \|r\|_1. \end{aligned}$$

Since $(1 + |\gamma|)\|p\|_1 + \|q\|_1 < 1,$ *there exists a constant* C *independent of* $\lambda \in [0, 1]$ *such that*

$$\|x''\|_1 \leq C.$$

Exemple3.2: *We consider the problem*

$$\begin{cases} x''(t) = \frac{t}{4}x + \frac{1}{7}x' + \frac{1}{\sqrt{1+x^4}}, & t \in (0, 1), \\ x'(0) = x'(\frac{1}{2}), \qquad x'(1) = 2x'(\frac{1}{4}). \end{cases}$$

In this case, $\alpha = 2, \gamma = 1, \eta = \frac{1}{4}, \zeta = \frac{1}{2}$ *with this condition* $1 + \gamma \neq \alpha(\gamma + \eta)$ *satisfied*
We chose

$$f(t, u, v) = \frac{t}{4}u + \frac{1}{7}v + \frac{1}{\sqrt{1 + u^4}}.$$

Then, we have f *is* $L^1-Caratheodory$ *and*

$$|f(t,u,v)| \leq \frac{t}{4}|u| + \frac{1}{7}|v| + 1$$

Then we chose ,$p(t) = \frac{t}{4}, q(t) = \frac{1}{7}, r(t) = 1$ *these functions are on* $L^1([0,1])$.
We have $(1+|\gamma|)(\|p\|_1 + \|q\|_1) < 1$
Then we have $(1+|1|)\frac{5}{8} + \frac{1}{7} = \frac{11}{28} < 1$
Thus all condition of theorem (3.2) *are satisfied , then the problem admits a solution on* $C^1([0,1])$.

3.4 Existence Results for Multi-Point Boundary Value Problems

We have already studied the problem (3.1),(3.11) *when* $\gamma = 0$, $\alpha = 1$, *now we present an existence result for the problem* (3.1),(3.11).

First, we have the following lemmas

Lemma 3.4. *Let* $c_i, a_j \in \mathbb{R}$, $\xi_i, \tau_j \in (0,1)$, *for* $i = 1,2,...,m-2$, $j = 1,2,...,n-2$, *with*

$$0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1, \quad 0 < \tau_1 < \tau_2 < ... < \tau_{n-2} < 1,$$

such that

$$\left(\sum_{i=1}^{m-2} c_i\xi_i\right)\left(1 - \sum_{j=1}^{n-2} a_j\right) \neq \left(1 - \sum_{i=1}^{m-2} c_i\right)\left(\sum_{j=1}^{n-2} a_j\tau_j - 1\right).$$

Let $x(t) \in W^{2,1}(0,1)$ *such that* $x(0) = \sum_{i=1}^{m-2} c_i x(\xi_i)$, $x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j)$. *Then*

$$\|x\|_\infty \leq M\|x''\|_\infty.$$

where

$$M = \min\left\{ \frac{1}{|\sum_{i=1}^{m-2} c_i|}\left(\sum_{i=1}^{m-2}|c_i|\lambda_i + \frac{\sum_{i=1}^{m-2}|c_i\xi_i|}{|1-\sum_{i=1}^{m-2}c_i|}\right), \frac{1}{\sum_{j=1}^{n-2} a_j}\left(\sum_{j=1}^{n-2}|a_j|\mu_j + \frac{\sum_{j=1}^{n-2}|a_j(1-\tau_j)|}{|1-\sum_{j=1}^{n-2}a_j|}\right), 1+\frac{\sum_{i=1}^{m-2}|c_i\xi_i|}{|1-\sum_{i=1}^{m-2}c_i|} + 1 + \sum_{j=1}^{n-2}\frac{|a_j(1-\tau_j)|}{|1-\sum_{j=1}^{n-2}a_j|}\right\},$$

with $\lambda_i = \max\{\xi_i, 1-\xi_i\}$, $i=1,2,\ldots,m-2$, $\mu_j = \max\{\tau_j, 1-\tau_j\}$, $j=1,2,\ldots,n-2$.

Proof *From the hypothesis*

$$\left(\sum_{i=1}^{m-2}c_i\xi_i\right)\left(1-\sum_{j=1}^{n-2}a_j\right) \neq \left(1-\sum_{i=1}^{m-2}c_i\right)\left(\sum_{j=1}^{n-2}a_j\tau_j - 1\right),$$

we observe that each of $\left(1-\sum_{i=1}^{m-2}c_i\right)$ *and* $\left(1-\sum_{j=1}^{n-2}a_j\right)$ *is nonzero, hence* $M < +\infty$. *Also, since* $x(\xi_i) - x(0) = \int_0^{\xi_i} x'(s)ds$ *for* $i=1,2,\ldots,m-2$, *and since* $x(0) = \sum_{i=1}^{m-2}c_i x(\xi_i)$, *we obtain* $\left(1-\sum_{i=1}^{m-2}c_i\right)x(0) = \sum_{i=1}^{m-2}c_i\int_0^{\xi_i}x'(s)ds$, *thus*

$$|x(0)| \le \left(\frac{\sum_{i=1}^{m-2}|c_i\xi_i|}{|1-\sum_{i=1}^{m-2}c_i|}\right)\|x'\|_\infty.$$

Also, since $x(t) = x(\xi_i) + \int_{\xi_i}^t x'(s)ds$, *we have*

$$\sum_{i=1}^{m-2}c_i x(t) = \sum_{i=1}^{m-2}c_i x(\xi_i) + \sum_{i=1}^{m-2}c_i\int_{\xi_i}^t x'(s)ds = x(0) + \sum_{i=1}^{m-2}c_i\int_{\xi_i}^t x'(s)ds,$$

thus

$$\left|\sum_{i=1}^{m-2}c_i x(t)\right| \le |x(0)| + \sum_{i=1}^{m-2}|c_i|\left|\int_{\xi_i}^t x'(s)ds\right|,$$

leading to

$$\left|\sum_{i=1}^{m-2}c_i x(t)\right| \le \left(\frac{\sum_{i=1}^{m-2}|c_i\xi_i|}{|1-\sum_{i=1}^{m-2}c_i|} + \sum_{i=1}^{m-2}|c_i|\lambda_i\right)\|x'\|_\infty.$$

which implies

$$\|x\|_\infty \le \frac{1}{\left|\sum_{i=1}^{m-2} c_i\right|}\left(\left(\frac{\sum_{i=1}^{m-2}|c_i\xi_i|}{1-\sum_{i=1}^{m-2}c_i}+\sum_{i=1}^{m-2}|c_i|\lambda_i\right)\|x'\|_\infty\right)$$

Similarly, starting from $x(1)=x(\tau_j)+\int_{\tau_j}^1 x'(s)\,ds$ *for* $j=1,2,\ldots,n-2$, *and since* $x(1)=\sum_{j=1}^{n-2}a_jx(\tau_j)$, *we obtain* $(1-\sum_{j=1}^{n-2}a_j)x(1)=\sum_{j=1}^{n-2}a_j\int_{\tau_j}^1 x'(s)\,ds$, *thus*

$$|x(1)|\le\frac{\sum_{j=1}^{n-2}|a_j(1-\tau_j)|}{|1-\sum_{j=1}^{n-2}a_j|}\|x'\|_\infty$$

Also, since $x(t)=x(\tau_j)+\int_{\tau_j}^t x'(s)\,ds$, *we have* $\left(\sum_{j=1}^{n-2}a_j\right)x(t)=\sum_{j=1}^{n-2}a_jx(\tau_j)+\sum_{j=1}^{n-2}a_j\int_{\tau_j}^t x'(s)\,ds$ *or* $\left(\sum_{j=1}^{n-2}a_j\right)x(t)=x(1)+\sum_{j=1}^{n-2}\int_{\tau_j}^t a_jx'(s)\,ds$ *thus*

$$\begin{aligned}\left|\sum_{j=1}^{n-2}a_j\right||x(t)| &\le |x(1)|+\sum_{j=1}^{n-2}|a_j|\left|\int_{\tau_j}^t x'(s)ds\right|,\\ &\le\left(\frac{\sum_{j=1}^{n-2}|a_j(1-\tau_j)|}{|1-\sum_{j=1}^{n-2}a_j|}+\sum_{j=1}^{n-2}a_j\mu_j\right)\|x'\|_\infty.\end{aligned}$$

which implies

$$\|x\|_\infty\le\frac{1}{\left|\sum_{j=1}^{n-2}a_j\right|}\left(\frac{\sum_{j=1}^{n-2}|a_j(1-\tau_j)|}{|1-\sum_{j=1}^{n-2}a_j|}+\sum_{j=1}^{n-2}a_j\mu_j\right)\|x'\|_\infty.$$

Since $x(t)=x(0)+\int_0^t x'(s)\,ds$, *we have*

$$\|x\|_\infty\le\left(\frac{\sum_{i=1}^{m-2}|c_i\xi_i|}{|1-\sum_{i=1}^{m-2}c_i|}+1\right)\|x'\|_\infty.$$

Also, starting from $x(t)=x(1)-\int_t^1 x'(s)\,ds$, *we obtain*

$$\|x\|_\infty\le\left(\frac{\sum_{j=1}^{n-2}|a_j|(1-\tau_j)}{|1-\sum_{j=1}^{n-2}a_j|}+1\right)\|x'\|_\infty.$$

Conclusion

$$\|x\|_\infty \leq M\|x'\|_\infty,$$

where

$$M = min\left\{\frac{1}{|\sum_{i=1}^{m-2} c_i|}\left(\sum_{i=1}^{m-2} \mid c_i \mid \lambda_i + \frac{\sum_{i=1}^{m-2}|c_i\xi_i|}{|1-\sum_{i=1}^{m-2} c_i|}\right),\right.$$
$$\frac{1}{|\sum_{i=1}^{n-2} a_j|}\left(\sum_{j=1}^{n-2} \mid a_j \mid \mu_j + \frac{\sum_{j=1}^{n-2}|a_j(1-\tau_j)|}{|1-\sum_{j=1}^{n-2} a_j|}\right),$$
$$\left. 1+\frac{\sum_{i=1}^{m-2}|c_i\xi_i|}{|1-\sum_{i=1}^{m-2} c_i|}, 1+\frac{\sum_{j=1}^{n-2}|a_j(1-\tau_j)|}{|1-\sum_{j=1}^{n-2} a_j|}\right\}.$$

Lemma 3.5. *Let* $c_i, a_j \in \mathbb{R}$, $\xi_i, \tau_j \in (0,1)$, $i = 1,2,\ldots,m-2$, $j = 1,2,\ldots,n-2$, *with* $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ *and* $0 < \tau_1 < \tau_2 < \cdots < \tau_{n-2} < 1$, *and*

$$\left(\sum_{i=1}^{m-2} c_i\xi_i\right)\left(1-\sum_{j=1}^{n-2} a_j\right) \neq \left(1-\sum_{i=1}^{m-2} c_i\right)\left(\sum_{j=1}^{n-2} a_j\tau_j - 1\right).$$

Let $x(t) \in W^{2,1}(0,1)$ *such that* $x(0) = \sum_{i=1}^{m-2} c_i x(\xi_i)$, $x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j)$. *Suppose that* M *is as in the previous lemma such that*

$$\left|\sum_{i=1}^{m-2} c_i\xi_i\right| > M\left|1-\sum_{i=1}^{m-2} c_i\right|, \quad \left|\sum_{j=1}^{n-2} a_j(1-\tau_j)\right| > M\left|\sum_{j=1}^{n-2} a_j - 1\right|.$$

Then

$$\|x'\|_\infty \leq Q\|x''\|_1,$$

where

$$Q = min\left\{\frac{\sum_{i=1}^{m-2}|c_i\xi_i|}{|\sum_{i=1}^{m-2} c_i\xi_i| - M|1-\sum_{i=1}^{m-2} c_i|},\right.$$
$$\left.\frac{\sum_{j=1}^{n-2}|a_j(1-\tau_j)|}{|\sum_{j=1}^{n-2} a_j(1-\tau_j)| - M|\sum_{j=1}^{n-2} a_j - 1|}\right\}.$$

Proof

First, note that for each $i = 1,2,\ldots,m-2$, *there exists* $\eta_i \in (0,\xi_i)$ *such that*

$x(\xi_i) - x(0) = \xi_i x'(\eta_i)$, *which implies*

$$\sum_{i=1}^{m-2} c_i x(\xi_i) - \sum_{i=1}^{m-2} c_i x(0) = \sum_{i=1}^{m-2} c_i \xi_i x'(\eta_i),$$

thus

$$\sum_{i=1}^{m-2} c_i \xi_i x'(\eta_i) = \left(1 - \sum_{i=1}^{m-2} c_i\right) x(0).$$

We also observe from the equation $x'(t) = x'(\eta_i) + \int_{\eta_i}^{t} x''(s)\,ds$ *that*

$$\begin{aligned}\left(\sum_{i=1}^{m-2} c_i\xi_i\right) x'(t) &= \sum_{i=1}^{m-2} c_i \xi_i x'(\eta_i) + \sum_{i=1}^{m-2} c_i\xi_i \int_{\eta_i}^{t} x''(s)ds \\ &= \left(1 - \sum_{i=1}^{m-2} c_i\right) x(0) + \sum_{i=1}^{m-2} c_i\xi_i \int_{\eta_i}^{t} x''(s)ds.\end{aligned}$$

As a consequence

$$\begin{aligned}\left|\sum_{i=1}^{m-2} c_i\right| \|x'\|_\infty &\le \left|1 - \sum_{i=1}^{m-2} c_i\right| \left(|x(0)| + \sum_{i=1}^{m-2} |c_i\xi_i| \left|\int_0^1 x''(s)ds\right|\right) \\ &\le \left|1 - \sum_{i=1}^{m-2} c_i\right| \|x\|_\infty + \sum_{i=1}^{m-2} |c_i\xi_i| \|x''\|_1 \\ &\le M\left|1 - \sum_{i=1}^{m-2} c_i\right| \|x'\|_\infty + \sum_{i=1}^{m-2} |c_i\xi_i| \|x''\|_1\end{aligned}$$

which implies that

$$\|x'\|_\infty \le \frac{\sum_{i=1}^{m-2} |c_i\xi_i|}{\left|\sum_{i=1}^{m-2} c_i\xi_i\right| - M\left|1 - \sum_{i=1}^{m-2} c_i\right|} \|x''\|_1.$$

Similarly, for each $j = 1, 2, ..., n-2$, *there exists* $\zeta_j \in (\tau_j, 1)$ *such that* $x(1) - x(\tau_j) = (1-\tau_j)x'(\zeta_j)$, *which implies*

$$\sum_{j=1}^{n-2} a_j x(1) - \sum_{j=1}^{n-2} a_j x(\tau_j) = \sum_{j=1}^{n-2} a_j (1-\tau_j) x'(\zeta_j),$$

thus

$$\sum_{j=1}^{n-2} a_j(1-\tau_j)x'(\zeta_j) = \left(\sum_{j=1}^{n-2} a_j - 1\right) x(1).$$

We observe in the equation, $x'(t) = x'(\zeta_j) + \int_{\zeta_j}^{t} x''(s)ds$, *that*

$$(1-\tau_j)x'(t) = (1-\tau_j)x'(\zeta_j) + (1-\tau_j)\int_{\zeta_j}^{t} x''(s)ds$$

$$\begin{aligned}\sum_{j=1}^{n-2} a_j(1-\tau_j)x'(t) &= \sum_{j=1}^{n-2} a_j(1-\tau_j)x'(\zeta_j) + \sum_{j=1}^{n-2} a_j(1-\tau_j)\int_{\zeta_j}^{t} x''(s)ds \\ &= \left(\sum_{j=1}^{n-2} a_j - 1\right) x(1) + \sum_{j=1}^{n-2} a_j(1-\tau_j)\int_{\zeta_j}^{t} x''(s)ds.\end{aligned}$$

As a consequence

$$\begin{aligned}\left|\sum_{j=1}^{n-2} a_j(1-\tau_j)\right| \|x'\|_\infty &\leq \left|\sum j = 1^{n-2} a_j - 1\right| \|x\|_\infty + \sum_{j=1}^{n-2} |a_j(1-\tau_j)| \left|\int_{\zeta_j}^{t} x''(s)ds\right| \\ &\leq M\left|\sum_{j=1}^{n-2} a_j - 1\right| \|x\|_\infty + \sum_{j=1}^{n-2} |a_j(1-\tau_j)| \|x''\|_1\end{aligned}$$

which implies that

$$\|x'\|_\infty \leq \frac{\sum_{j=1}^{n-2} |a_j(1-\tau_j)|}{\left|\sum_{j=1}^{n-2} a_j(1-\tau_j)\right| - M\left|\sum_{j=1}^{n-2} a_j - 1\right|}\|x''\|_1.$$

Conclusion

$$\|x'\|_\infty \leq Q\|x''\|_1,$$

where

$$\begin{aligned}Q = min\Bigg\{ &\frac{\sum_{i=1}^{m-2} |c_i\xi_i|}{\left|\sum_{i=1}^{m-2} c_i\xi_i\right| - M\left|1 - \sum_{i=1}^{m-2} c_i\right|}, \\ &\frac{\sum_{j=1}^{n-2} |a_j(1-\tau_j)|}{\left|\sum_{j=1}^{n-2} a_j(1-\tau_j)\right| - M\left|\sum_{j=1}^{n-2} a_j - 1\right|}\Bigg\}.\end{aligned}$$

Lemma 3.6. *Let* $y \in L^1[0,1]$ *and let* $c_i, a_j \in \mathbb{R}$, $\xi_i, \tau_j \in (0,1)$, $i = 1, 2, \ldots, m-2$, $j =$

$1, 2, \ldots, n-2$, *with* $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \ldots < \tau_{n-2} < 1$. *We assume that*

$$\left(\sum_{i=1}^{m-2} c_i\xi_i\right)\left(1-\sum_{j=1}^{n-2} a_j\right) \neq \left(1-\sum_{i=1}^{m-2} c_i\right)\left(\sum_{j=1}^{n-2} a_j\tau_j - 1\right),$$

and $x \in C^1[0,1]$, *such that if*

$$\begin{cases} x''(t) = y(t), & t \in (0,1), \\ x(0) = \sum_{i=1}^{m-2} c_i x(\xi_i), & x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j), \end{cases}$$

then

$$x(t) = \int_0^t (t-s)y(s)ds + At + B,$$

where

$$\begin{aligned} A&\left(\left(\sum_{i=1}^{m-2} c_i\xi_i\right)\left(1-\sum_{j=1}^{n-2} a_j\right) - \left(1-\sum_{i=1}^{m-2} c_i\right)\left(\sum_{i=1}^{n-2} a_j\tau_j - 1\right)\right) \\ &= -\left(1-\sum_{j=1}^{n-2} a_j\right)\left(\sum_{i=1}^{m-2} c_i \int_0^{\xi_i} (\xi_i - s)y(s)ds\right) \\ &\quad + \left(1-\sum_{i=1}^{m-2} c_i\right)\left(\sum_{j=1}^{n-2} a_j \int_0^{\tau_j} (\tau_j - s)y(s)ds - \int_0^1 (1-s)y(s)ds\right), \\ B&\left(\left(\sum_{i=1}^{m-2} c_i\xi_i\right)\left(1-\sum_{j=1}^{n-2} a_j\right) - \left(1-\sum_{i=1}^{m-2} c_i\right)\left(\sum_{i=1}^{n-2} a_j\tau_j - 1\right)\right) \\ &= \left(1-\sum_{j=1}^{n-2} a_j\tau_j\right)\left(\sum_{i=1}^{m-2} c_i \int_0^{\xi_i} (\xi_i - s)y(s)ds\right) \\ &\quad + \left(\sum_{i=1}^{m-2} c_i\xi_i\right)\left(\sum_{j=1}^{n-2} a_j \int_0^{\tau_j} (\tau_j - s)y(s)ds - \int_0^1 (1-s)y(s)ds\right). \end{aligned}$$

Proof *From* $x''(t) = y(t)$ *we have:*

$$x'(t) = x'(0) + \int_0^t y(s)ds,$$

which implies

$$x(t) = x(0) + x'(0)t + \int_0^t (t-s)y(s)ds.$$

Thus, we have:

$$x(\xi_i) = x(0) + x'(0)\xi_i + \int_0^{\xi_i} (\xi_i - s)y(s)ds,$$

$$\sum_{i=1}^{m-2} c_i x(\xi_i) = x(0)\sum_{i=1}^{m-2} c_i + x'(0)\sum_{i=1}^{m-2} c_i\xi_i + \sum_{i=1}^{m-2} c_i \int_0^{\xi_i} (\xi_i - s)y(s)ds. \tag{R1}$$

Since $x(0) = \sum_{i=1}^{m-2} c_i x(\xi_i)$, *we get*

$$\left(1 - \sum_{i=1}^{m-2} c_i\right) x(0) = x'(0)\sum_{i=1}^{m-2} c_i\xi_i + \sum_{i=1}^{m-2} c_i \int_0^{\xi_i} (\xi_i - s)y(s)ds.$$

On the other hand:

$$x(\tau_j) = x(0) + x'(0)\tau_j + \int_0^{\tau_j} (\tau_j - s)y(s)ds,$$

$$x(1) = x(0) + x'(0) + \int_0^1 (1-s)y(s)ds.$$

Since $x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j)$, *then*

$$\left(1 - \sum_{j=1}^{n-2} a_j\right) x(0) = x'(0)\left(\sum_{j=1}^{n-2} a_j\tau_j - 1\right) - \int_0^1 (1-s)y(s)ds + \sum_{j=1}^{n-2} a_j \int_0^{\tau_j} (\tau_j - s)y(s)ds. \tag{R2}$$

As a consequence, we get from $(1 - \sum_{j=1}^{n-2} a_j) \times$ *(R1) and* $(1 - \sum_{i=1}^{m-2} c_i) \times$ *(R2) that*

$$\begin{aligned} x'(0) &\left(\left(\sum_{i=1}^{m-2} c_i\xi_i\right)\left(1 - \sum_{j=1}^{n-2} a_j\right) - \left(1 - \sum_{i=1}^{m-2} c_i\right)\left(\sum_{i=1}^{n-2} a_j\tau_j - 1\right)\right) \\ &= -\left(1 - \sum_{j=1}^{n-2} a_j\right)\left(\sum_{i=1}^{m-2} c_i \int_0^{\xi_i} (\xi_i - s)y(s)ds\right) \\ &\quad + \left(1 - \sum_{i=1}^{m-2} c_i\right)\left(\sum_{j=1}^{n-2} a_j \int_0^{\tau_j} (\tau_j - s)y(s)ds - \int_0^1 (1-s)y(s)ds\right), \end{aligned}$$

and also of $(\sum_{i=1}^{n-2} a_j\tau_j - 1) \times (R1)$ *and* $(\sum_{i=1}^{m-2} c_i\xi_i) \times (R2)$, *we have*

$$x(0)\left(\left(\sum_{i=1}^{m-2} c_i\xi_i\right)\left(1-\sum_{j=1}^{n-2} a_j\right) - \left(1-\sum_{i=1}^{m-2} c_i\right)\left(\sum_{i=1}^{n-2} a_j\tau_j - 1\right)\right)$$

$$= \left(1-\sum_{j=1}^{n-2} a_j\tau_j\right)\left(\sum_{i=1}^{m-2} c_i \int_0^{\xi_i} (\xi_i - s)y(s)ds\right)$$

$$+ \left(\sum_{i=1}^{m-2} c_i\xi_i\right)\left(\sum_{j=1}^{n-2} a_j \int_0^{\tau_j} (\tau_j - s)y(s)ds - \int_0^1 (1-s)y(s)ds\right).$$

We note that $x'(0) = A$ *and* $x(0) = B$.

Theorem 3.3. *Let* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *be an* L^1*-Carathéodory function. Suppose there exist functions* $p, q, r \in L^1[0,1]$ *such that*

$$|f(t,u,v)| \leq p(t)|u| + q(t)|v| + r(t),$$

for almost every $t \in [0,1]$ *and for all* $(u,v) \in \mathbb{R}^2$, *and let* $c_i, a_j \in \mathbb{R}$, *with all the* $c_i\xi_i$ *(respectively,* $a_j\tau_j$*) having the same sign,* $\xi_i, \tau_j \in (0,1)$, $i = 1,2,\ldots,m-2$, $j = 1,2,\ldots,n-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \cdots < \tau_{n-2} < 1$. *We suppose that*

$$\left(\sum_{i=1}^{m-2} c_i\xi_i\right)\left(1-\sum_{j=1}^{n-2} a_j\right) \neq \left(1-\sum_{i=1}^{m-2} c_i\right)\left(\sum_{i=1}^{n-2} a_j\tau_j - 1\right).$$

Then problem (3.1)-(3.11) *admits a solution in* $C^1[0,1]$ *under the condition*

$$Q(M\|p\|_1 + \|q\|_1) < 1,$$

where

$$M = min\left\{\frac{1}{\left|\sum_{i=1}^{m-2} c_i\right|}\left(\sum_{i=1}^{m-2} \mid c_i \mid \lambda_i + \frac{\sum_{i=1}^{m-2}|c_i\xi_i|}{\left|1-\sum_{i=1}^{m-2} c_i\right|}\right),\right.$$

$$\frac{1}{\left|\sum_{i=1}^{n-2} a_j\right|}\left(\sum_{j=1}^{n-2} \mid a_j \mid \mu_j + \frac{\sum_{j=1}^{n-2}|a_j(1-\tau_j)|}{\left|1-\sum_{j=1}^{n-2} a_j\right|}\right),$$

$$\left. 1 + \frac{\sum_{i=1}^{m-2}|c_i\xi_i|}{\left|1-\sum_{i=1}^{m-2} c_i\right|}, 1 + \frac{\sum_{j=1}^{n-2}|a_j(1-\tau_j)|}{\left|1-\sum_{j=1}^{n-2} a_j\right|}\right\},$$

$\lambda_i = max\{\xi_i,\, 1-\xi_i\}$, $i = 1, 2, ..., m-2$, $\mu_j = max\{\tau_j,\, 1-\tau_j\}$, $j = 1, 2, ..., n-2$ *and*

$$Q = min\left\{\frac{\sum_{i=1}^{m-2}|c_i\xi_i|}{\left|\sum_{i=1}^{m-2}c_i\xi_i\right| - M\left|1-\sum_{i=1}^{m-2}c_i\right|},\right.$$

$$\left.\frac{\sum_{j=1}^{n-2}|a_j(1-\tau_j)|}{\left|\sum_{j=1}^{n-2}a_j(1-\tau_j)\right| - M\left|\sum_{j=1}^{n-2}a_j - 1\right|}\right\}.$$

Proof

Let $X = C^1[0,1]$, $Y = L^1[0,1]$. *We assume the linear operator* $L : D(L) \subset X \to Y$, *with*

$$D(L) = \left\{x \in W^{2,1}(0,1) : x(0) = \sum_{i=1}^{m-2} c_i x(\xi_i), \quad x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j)\right\},$$

and for $x \in D(L)$, $Lx = x''$. *We also define a nonlinear mapping* $N : X \to Y$ *by*

$$(Nx)(t) = f(t, x(t), x'(t)), \quad t \in [0,1],$$

N *is a bounded mapping from* X *to* Y, *and* L *is an injective operator. Then we define the linear operator* $K : Y \to X$, *for* $y \in Y$, *by*

$$(Ky)(t) = \int_0^t (t-s)y(s)ds + At + B,$$

Where

$$A\left(\left(\sum_{i=1}^{m-2} c_i\xi_i\right)\left(1-\sum_{j=1}^{n-2} a_j\right) - \left(1-\sum_{i=1}^{m-2} c_i\right)\left(\sum_{i=1}^{n-2} a_j\tau_j - 1\right)\right)$$
$$= -\left(1-\sum_{j=1}^{n-2} a_j\right)\left(\sum_{i=1}^{m-2} c_i\int_0^{\xi_i}(\xi_i - s)y(s)ds\right)$$
$$+\left(1-\sum_{i=1}^{m-2} c_i\right)\left(\sum_{j=1}^{n-2} a_j\int_0^{\tau_j}(\tau_j - s)y(s)ds - \int_0^1(1-s)y(s)ds\right),$$

$$B\left(\left(\sum_{i=1}^{m-2} c_i\xi_i\right)\left(1-\sum_{j=1}^{n-2} a_j\right) - \left(1-\sum_{i=1}^{m-2} c_i\right)\left(\sum_{i=1}^{n-2} a_j\tau_j - 1\right)\right)$$
$$= \left(1-\sum_{j=1}^{n-2} a_j\tau_j\right)\left(\sum_{i=1}^{m-2} c_i\int_0^{\xi_i}(\xi_i - s)y(s)ds\right)$$

$$+\left(\sum_{i=1}^{m-2} c_i\xi_i\right)\left(\sum_{j=1}^{n-2} a_j \int_0^{\tau_j} (\tau_j - s)y(s)ds - \int_0^1 (1-s)y(s)ds\right).$$

Let $y \in Y$, then $Ky \in D(L)$ and $LKy = y$; and for $u \in D(L)$, $KLu = u$.
Moreover, the application $KN : X \to X$ is compact.
We have $x \in C^1[0,1]$ is a solution to problem (3.1),(3.11) if and only if (x is a solution to the operator equation $Lx = Nx$), which is equivalent to the equation $x = KNx$.
We apply the LeraySchauder theorem to obtain
the existence of a solution to the equation $x = KNx$.
To do this, it is sufficient to verify that the set of all possible solutions of the family of equations

$$\begin{cases} x''(t) = \lambda f(t, x(t), x'(t)), & t \in (0,1), \\ x(0) = \sum_{i=1}^{m-2} c_i x(\xi_i), & x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j). \end{cases}$$

is, a priori, bounded in $X = C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$. We observe that if $x \in W^{2,1}(0,1)$, with $x(0) = \sum_{i=1}^{m-2} c_i x(\xi_i)$, $\quad x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j)$,
then we have the inequalities

$$\|x\|_\infty \leq M\|x'\|_\infty, \qquad \|x'\|_\infty \leq Q\|x''\|_1,$$

(according to the previous lemmas).
Let now $x(t)$ be a solution of (3.4) for some $\lambda \in [0,1]$, so that $x \in W^{2,1}(0,1)$, with $x(0) = \sum_{i=1}^{m-2} c_i x(\xi_i)$, $\quad x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j)$.
We then obtain

$$\begin{aligned} \|x''\|_1 &= \lambda\|f(t, x(t), x'(t)) \\ &\leqslant \|p\|_1\|x\|_\infty + \|q\|_1\|x'\|_\infty + \|r\|_1 \\ &\leqslant Q\left(M\|p\|_1 + \|q\|_1\right)\|x''\|_1 + \|r\|_1. \end{aligned}$$

As $Q\left(M\|p\|_1 + \|q\|_1\right) < 1$, there exists is a constant C independent of $\lambda \in [0,\, 1]$, such that

$$\|x''\|_1 \leqslant C.$$

Conclusion: the set of solutions of the family of equations is, a prior, limited in $X = C^1[0,1]$ by a constant independent of $\lambda \in [0,\, 1]$.

Exemple3.2: *We consider the problem*

$$\begin{cases} x''(t) = \frac{t}{2}x + \frac{1}{8}x' + sinx, & t \in (0,1), \\ x(0) = 3x(\frac{1}{4}) + 4x(\frac{1}{2}), \qquad x(1) = 2x(\frac{1}{6}) + 4x(\frac{1}{3}). \end{cases}$$

In this case, $\tau_1 = \frac{1}{6}, \tau_2 = \frac{1}{3}c_1 = 3, c_2 = 4, a_1 = 2, a_2 =, \zeta_1 = \frac{1}{4}, \zeta_2 = \frac{1}{2}$
We chose

$$f(t,u,v) = \frac{t}{4}u + \frac{1}{8}v + sinu.$$

Then, we have f *is* $L^1 - Caratheodory$ *and*

$$|f(t,u,v)| \leq \frac{t}{4}|u| + \frac{1}{8}|v| + 1$$

Then we chose ,$p(t) = \frac{t}{4}, q(t) = \frac{1}{8}, r(t) = 1$ *these functions are on* $L^1([0,1])$.
We have

$$\left(\sum_{i=1}^{m-2} c_i\xi_i\right)\left(1 - \sum_{j=1}^{n-2} a_j\right) \neq \left(1 - \sum_{i=1}^{m-2} c_i\right)\left(\sum_{i=1}^{n-2} a_j\tau_j - 1\right).$$

By substitution,we find

$$(\frac{11}{4})(-5) \neq (-6)(\frac{2}{3})$$

$$\frac{-55}{4} \neq -4$$

Then we have $Q(M\|p\|_1 + \|q\|_1) < 1$,
Such that $\|p\|_1 = \frac{1}{8}, \|q\|_1 = \frac{1}{8}, M = \frac{113}{168}, Q = \frac{728}{163}$

Thus all condition of theorem (3.2) *are satisfied , then the problem admits a solution on* $C^1([0,1])$.

Conclusion and perspectives

In the present memory, we have considered some boundary value problems of second order differential equations posed on bounded intervals and associated with n-point boundary conditions. Most of these problems stem from some physical phenomena.

We studied at the beginning three point at nonlinear boundary value problems of second-order over bounded intervals.

Some methods have been employed to prove existence of solutions. These are methods based on the study of the corresponding Green's function, some of its properties and fixed point theorems. For this, some sufficient conditions on the nonlinear term are assumed throughout.

All of our existence results are supported by examples of applications.

Through the results obtained in this memory, we hope that we have contributed to the study of some classes of n-point boundary value problems for second order differential equations.

In the future, we intend to investigate some questions related to the properties of solutions for such classes of problems.

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Abstract

In this memoire, we studied the existence and uniqueness of solutions to some boundary value problems of second order differential equations posed on bounded intervals and associated with n-point boundary conditions. Some methods have been employed to prove existence of solutions. These are methods based on the study of the corresponding Green's function, some of its properties and and we transformed differential equation into an operational equation and applied the fixed point theory to find the solutions. On the other hand, we chose sufficient conditions to demonstrate our results. In all phases of our work we have submitted some examples that give applications for the results obtained.

key words: Differential equation, existence, uniqueness, boundary value problem, bounded interval, fixed point theorem.

Résumé

Dans ce mémoire, nous avons étudié l'existence et l'unicité des solutions à certains problèmes aux limites d'équations différentielles du second ordre posées sur des intervalles bornés et associées à des conditions aux limites à n points. Des méthodes ont été employées pour prouver l'existence de solutions. Ces méthodes reposent sur l'étude de la fonction de Green correspondante et de certaines de ses propriétés. Nous avons transformé l'équation différentielle en équation opérationnelle et appliqué la théorie du point fixe pour trouver les solutions. D'autre part, nous avons choisi des conditions suffisantes pour démontrer nos résultats. À chaque étape de notre travail, nous avons présenté des exemples illustrant les applications des résultats obtenus.

Mots clés : Équation différentielle, existence, unicité, problème aux limites, intervalle borné, théorème du point fixe.

ملخص

درسنا في هذا البحث وجود ووحدانية حلول بعض الجمل الحدية لمعادلات تفاضلية من الرتبة الثانية، المطروحة على مجالات محدودة، والمرتبطة بشروط حدية لعدة نقط. استخدمنا بعض الطرق لإثبات وجود الحلول. تعتمد هذه الطرق على دراسة دالة جرين المرتبطة لها، وبعض خصائصها، وقد حولنا المعادلة التفاضلية إلى معادلة بالمؤثر، وطبقنا نظرية النقطة الصامدة لإيجاد الحلول. من ناحية أخرى، اخترنا شروطاً كافية لإثبات نتائجنا. في جميع مراحل عملنا، قدمنا بعض الأمثلة كتطبيقات للنتائج التي حصلنا عليها.

الكلمات المفتاحية: المعادلة التفاضلية، الوجود، الوحدانية، الجملة الحدية، المجال المحدود، نظرية النقطة الصامدة.