People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

Echahide Hamma Lakhdar University-El-Oued

Faculty of Exact Sciences

Department of Mathematics

Thesis

To obtain the 3*rd* cycle LMD Doctorate degree in mathematics

Specialty: APPLIED MATHEMATICS

Theme

RECENT GENERALIZED METRIC SPACES AND RELATED FIXED POINT THEOREMS

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Publicly supported the: 26/01/2025

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University year: 2024-2025

Acknowledgement

*Praise to **ALLAH THE ALMIGHTY** who gave me the strength and will to complete this work!*

*I want to express my gratitude to my supervisor, **Dr. KHALED HABITA**, assistant professor at the university of El Oued, for having agreed to be my thesis supervisor as well as for his availability, his patience and his encouragement throughout the completion of this Doctorate thesis.*

*Also, I would like to thank honorable teacher for his effort with us, **Prof. SAID BELOUL**. My thanks also go to all the members of the Jury, for the honor they did me by agreeing to be on the Jury for my thesis.*

I also extend my thanks to my family, my teachers, my friends and my colleagues for their support.

Dedication

I *dedicate this fruit of my long years of study first of all:*

T*o my **father's pure soul SALEM and My mother MABROUKA**, who are the light of my life, who suffered and sacrificed so much for me to be happy, for their advice, their affection and their encouragement.*

A *unique sense of appreciation for the persons who supported me throughout of this project: my husband **MOSTEFA MEBARKI**.* and my children **ABDERAZZAK, BILEL and SOUDJOUD.**

And i dedicate it to :

A*ll the members of my family,*

My brothers,

My sisters especially **SALIMA and HER HUSBAND.**

A*ll my friends and colleagues with whom I have shared very good times throughout these years.*

Abstract

In this thesis, some fixed point theorems are studied on different spaces, using a class of functions, new contractions are introduced on dislocated quasi b-metric spaces and on dislocated b-metric spaces, and the fixed point theorem is presented for a self mapping satisfying a generalized contraction condition. Finally, two applications are presented to prove the existence of solutions to: integral equations of the Volterra type and the boundary value problem of fractional differential equations.

Keywords:

Dislocated quasi b-metric space, dislocated b-metric spaces, fixed point theorem, contractives mappings.

Résumé

Dans cette thèse, certains théorèmes des points fixes sont étudiés dans différents espaces, en utilisant une classe de fonctions, de nouvelles contractions introduites sur les espaces quasi b-métriques disloqués et sur les espaces b-métriques disloqués, et présenté théorème du point fixe pour une auto-application satisfaisant une condition contractive généralisée. Enfin, présentées deux applications pour l'existence de solutions: d'équation intégrale de type Volterra et d'un problème de valeur aux limites des équations différentiels fractionnaires.

Mots clés:

Espace quasi b-métrique disloqué, espaces b-métriques disloqués, théorème du point fixe, applications contractions.

ملخص

في هذه الاطروحة، تمت دراسة بعض نظريات النقطة الثابتة على فضاءات مختلفة باستخدام فئة من الدوال، تم تقديم تقلصات جديدة في فضاءات من نوع ب-شبه المترية المفككة، و فضاءات من نوع ب-المترية المفككة، و عرض نظرية النقطة الثابتة لتطبيق ذاتي يلبي شرط تقلصي معمم. اخيرا، قدمنا تطبيقين لاثبات وجود حلول لكل من: معادلة تكاملية من نوع فولتيرا، ومسألة القيمة الحدية للمعادلات التفاضلية الكسيرية.

الكلمات المفتاحية :

الفضاءات من نوع ب-شبه المترية المفككة، الفضاءات من نوع ب-المترية المفككة، نظرية النقطة الثابتة، تطبيقات انكماشية.

Published paper

N. Hala, K. Habita and S. Beloul, Fixed point results for generalized contractions via simulation functions in dislocated quasi b-metric space, *Journal of Analysis. Vol(32) (2024), 2833-2846.*

Submitted article

N. Hala, K. Habita and S. Beloul, Some fixed point results for js-contractions in b-metric like spaces with an application.

Contents

GENERAL INTRODUCTION

In 1906, Frechet introduced the notion of metric space, which is one of the cornerstones of not only mathematics but also several quantitative sciences. Due to its importance and application potential, this notion has been extended, improved and generalized in many different ways.

In 1989, Bakhtin [11] introduced the very interesting concept of a b-metric space as an analog of metric space. He proved the contraction mapping principle in b-metric space that generalizes the famous Banach contraction principle in metric spaces. Since then many mathematicians have done work involving fixed points for single-valued and multivalued operators in b-metric spaces.

Hitzler and Seda [29] are the first who considered the concept of metric-like (or dislocated metric) spaces. Later Amini-Harandi [8] established some fixed point results in the class of metric-like space. Very recently many fixed point results on metric-like spaces have been provided.

In [63], Wilson initiated the concept of quasi-metric space (also known as asymmetric metric space) as an extension of the metric space.

For another thing, from spaces, there are too many generalizations of metric spaces. For instance, we have the quasi-b-metric space, dislocated b-metric space (or b-metric-like space), dislocated quasi-metric space (or quasi-metric-like space), and the dislocated quasi-b-metric space (or quasi-b-metric-like space) (see [65] [62]).

In the field of fixed point theory, fixed point theory has become the focus of many

researchers in the last couple decades and that is due to its applications in the existence and uniqueness of solutions of differential and integral equations, engineering, mathematical economics,dynamical systems, neural networks and many other fields.
In 1922, a very strong result in the history of mathematics was given by a Polish mathematician, Stefan Banach, [12] known as "Banach Contraction Principle". According to this "every contraction has a unique fixed point in a complete metric space". Mathematically we can say:
If $(X,\ d)$ is a metric space and $T\ :\ X \to X$ be a self map, which satisfies the following inequality for $\alpha\ \in\ [0,\ 1)$,

$$d(Tx,\ Ty) \leq \alpha d(x,\ y) \qquad \forall x,\ y \in X;$$

then T has unique fixed point.
The Banach contraction principle is one of the simplest and most applicable results of metric fixed point theory. It is a popular tool for proving the existence of solution of problems in different fields of mathematics. There are several generalizations of the Banach contraction principle in literature on metric fixed point theory [5, 40].
Banach's fixed theorem has been generalized by expanding the underlying metric space or by changing the contraction condition.
In 1973, Hardy and Rogers [28] presented a fixed points theorem using a more wider contractive expression which improved the fixed point result.
Berinde [15] introduced the concept of almost as a generalization to weak contraction in the context of single valued mappings, later some results have been obtained using this concept.
Samet and al.[58] furnished a new concept called as α- admissible and obtained some fixed point results for $\alpha - \psi$-contraction maps, later some results were established in this direction, see [[9], [32]].
Recently, Jleli and Samet [34] introduced a new contractions type called θ-contraction to prove the existence of fixed points for such contractions. It is worth mentioning here, that a contraction in the sense of Banach is a particular of θ contraction, example see [13, 47].
Chandok [23] who introduced the concept of (α, β)-admissible Geraghty type contractive mapping, which sufficient condition for the existence of a fixed point for such class

of generalized non-linear contractive mapping in metric space proved some fixed point results

Khojasteh, Shukla and Radenovic[41] presented the notion of Z-contraction with respect to Γ , that is a non-linear contraction involving a new class of mapping namely simulation function. The Z-contraction generalized the Banach contraction and unify several known type of contractions involving the combination of $d(Tx,\ Ty)$ and $d(x,\ y)$ and satisfies some particular conditions in complete metric space. They studied the existence and uniqueness of fixed point for Z-contraction type operators.

This thesis is divided into four chapters divided as follows:

Chapter 1: This chapter includes contains the definition and examples and topology all of metric space and generalized spaces, In addition to the fixed point theorem for every space.

Chapter 2: We will present the fixed point theorem for almost JS-contractions of Hardy-Rogers type in dislocated b-metric spaces and their proof, in addition to practical examples of theorem, and corollary of fixed point on partially ordered dislocated b-metric spaces.

Chapter 3: we present fixed point theorem in dislocated quasi b-metric space for generalized $(\alpha, \beta, \epsilon, \psi, \phi, \Gamma)$ contraction mapping, in addition to practical examples of theorem, and some corollarys.

Chapter 4: This chapter will be devoted to some applications concerning the study of some problems such as:

1. The existence of solutions for a Volterra integral equations.

2. The existence of solutions of boundary value problems of fractional differential equations.

CHAPTER

1

METRIC SPACES AND THEIR GENERALIZATIONS

The concept of metric space has been expanded, improved, and generalized in several different ways.

This chapter is reserved to present some definitions, concepts and properties to metric spaces with their recent generalizations, where we give some.

1.1 Metric Space

A metric space is a set in which we can talk of the distance between any two of its elements. The definition below imposes certain natural conditions on the distance between the points.

Definition 1.1.1 *[45] A metric on the set X is a function $d : X \times X \to [0, \infty)$ such that the following conditions are satisfied for all $x, y, z \in X$:*

(M1) Positive property: $d(x, y) = 0$ *if and only if* $x = y$,

(M2) Symmetry property: $d(x,\ y) = d(y,\ x)$ *and,*

(M3) Triangle inequality: $d(x,\ y) \leq d(x,\ z) + d(z,\ y)$.

The pair (X, d) *is called a metric space.*

Example 1.1.1 *The set* $X = \mathbb{R}$ *with* $d(x,\ y) = |x - y|$*, the absolute value of the difference* $x - y$*, for each* $x, y \in$ *Properties* $(M1)$ *and* $(M2)$ *are obvious, and*

$$d(x,\ y) = |x - y| = |(x - z) + (z - y)| \leq |x - z| + |z - y| = d(x, z) + d(z, y).$$

Example 1.1.2 *Let* X *be endowed with the metric* $d : X \longrightarrow \mathbb{R}^+$ *defined by:*

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}$$

for each $x, y \in X$ *is a distance because:*

(1)

$$\begin{aligned} d(x,\ y) = 0 \ &\Rightarrow\ |x - y| = 0 \\ &\Rightarrow\ x = y \end{aligned}$$

and for

$$x = y \ \Rightarrow\ d(x,\ y) = 0.$$

(2) *for the condition of symmetric we have:*

$$\begin{aligned} d(y, x) = &\ \frac{|y - x|}{1 + |y - x|} \\ = &\ \frac{|(-1)(x - y)|}{1 + |(-1)(x - y)|} \\ = &\ \frac{|x - y|}{1 + |x - y|} \\ = &\ d(x,\ y). \end{aligned}$$

(3) *We define* $F : [0,\ \infty[\longrightarrow [0,\ \infty[$ *by* $F(x) = \frac{x}{1+x}$ *then:*
$d(x,\ y) = F(|x-y|)$ *and as* F *is increasing then:*

$$\begin{aligned} d(x,\ y) = F(|x-y|) &\leq F(|x-z|+|z-y|) \\ &= \frac{|x-z|}{1+|x-z|+|z-y|} + \frac{|z-y|}{1+|x-z|+|z-y|} \\ &\leq \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|} \\ &= d(x,z)+d(z,y). \end{aligned}$$

1.1.1 Topology Of Metric Spaces

Definition 1.1.2 *[45] Let* $(X,\ d)$ *be a metric space. Let* $\{x_n\}$ *be a sequence in* X*. We say that the sequence* $\{x_n\}$ *converges to* $x \in X$ *if* $\lim_{n\to\infty} d(x_n,\ x) = 0$*. In this case, we say that* x *is the limit of* $\{x_n\}$*.*
In other words, the sequence $(x_n)_n$ *converges to* $x \in X$ *if and only if:*

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq n_0 \Rightarrow d(x_n, x) < \epsilon.$$

Definition 1.1.3 *[45] Let* (X,d) *be a metric space.* $\{x_n\}_{n=0}^{\infty}$ *a sequence of* X*. The sequence* $\{x_n\}_{n=0}^{\infty}$ *is Cauchy if and only if :*

$$(\ \forall \varepsilon > 0),\ (\exists N \in \mathbb{N}), (\forall n, m \in \mathbb{N}), [n, m > N\ \Rightarrow\ d(x_n, x_m) < \varepsilon].$$

Proposition 1.1.1 *Any convergent sequence is Cauchy sequence.*

Proof 1 *let* $x_n \longrightarrow x$*, then* $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq n_0 \Rightarrow d(x_n, x) < \frac{\epsilon}{2}$*,*
For $n, m > n_0$*, we have:*
$d(x_n, x) < \frac{\epsilon}{2}$ *and* $d(x_m, x) < \frac{\epsilon}{2}$*, so*

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Definition 1.1.4 *[45] The metric space* (X,d) *is said to be complete if any sequence de Cauchy in* X *is convergent.*

Definition 1.1.5 *[45] A subset* E *of a metric space* (X,d) *is complete if for any Cauchy sequence* $\{x_n\}$ *in* E *there exists* $x \in E$ *such that* $\lim_{n\to\infty} d(x_n,\ x) = 0$*.*

Definition 1.1.6 *[45] Let $(X,\ d)$ be a metric space. A non-empty subset A of X is said to be closed if for any sequence $\{x_n\}$ of A, we have*

$$\lim_{n\to\infty} d(x_n, x) = 0,\ x \in X \ \Rightarrow \ x \in A.$$

Proposition 1.1.2 *Let (X, d) be a metric space, and let E be a subset of X.*

(i) *If E is complete then E is closed.*

(ii) *If X is complete and E closed, then E is complete.*

Definition 1.1.7 *[45] Let (X_1, d_1) and (X_2, d_2) be two metric spaces and $T : X_1 \longrightarrow X_2$ is an application, we say that T is continue in $a \in X_1$ if and only if,*

$$\forall\varepsilon > 0, \exists\alpha > 0, \forall x \in X_1, d_1(x, a) < \alpha \Longrightarrow d_2(T(x), T(a)) < \varepsilon.$$

Remark 1.1.1 *The mapping T is continuous on X_1 if and only if it is continues at any point a of X_1.*

Definition 1.1.8 *[45] Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A mapping $T : X_1 \to X_2$ is said to be continuous at point $x \in X_1$ if and only if for every sequence $\{x_n\} \subset X_1$ which is convergent to x in X_1, we have the sequence $\{Tx_n\}$ is convergent to Tx in X_2 .*

1.2 b-Metric Spaces

It is wellknown that in some functional spaces, the natural norm cannot satisfy the triangle inequality for example, on $L^p(\mathbb{R}^n)$ when $p \in]0, 1[$. In view of this observation, Bakhtin [11] proposed a generalization of metric spaces called b-metric spaces in a way that allows the extension of fixed point theory to cover these spaces of functions. As a result, Czerwik [25] used the notion of b-metric to generalize Banach's principle of contraction.

Definition 1.2.1 *[11] For a nonempty set X and a given real number with $s \geq 1$. A function $\boldsymbol{b}_d : X \times X \to [0, \infty)$ is a b-metric on X if for all $x, y, z \in X$, the following conditions hold:*

- $\boldsymbol{b}_d(x,\ y) = 0$ *if and only if $x = y$,*

- $\boldsymbol{b}_d(x,\ y) = \boldsymbol{b}_d(y,\ x)$,
- $\boldsymbol{b}_d(x,\ z) \leq s[\boldsymbol{b}_d(x,\ y) + \boldsymbol{b}_d(y,\ z)]$.

A triplet $(X, \boldsymbol{b}_d, s)$ is called a b-metric space.

Remark 1.2.1 *Also, every metric space is a b-metric space but the converse is not necessarily true.*

Example 1.2.1 *Let $X = [0; 1]$ and $\boldsymbol{b}_d :\ X \times X \to \mathbb{R}^+$ such that $\boldsymbol{b}_d(x,\ y) = |x-y|^2$ for all $x,\ y \in X$. Clearly, $(X,\ \boldsymbol{b}_d,\ 2)$ is a b-metric space, but it is not metric.*

Example 1.2.2 *[11] Space $l^p(\mathbb{R}) = \left\{ \{x_n\} \subset \mathbb{R} :\ \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$ (where $p \in]0, 1[$), such as the function $\boldsymbol{b}_d :\ l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to \mathbb{R}$ is given by:*

$$\boldsymbol{b}_d(\{x_n\}, \{y_n\}) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}},$$

for all $\{x_n\},\ \{y_n\} \in l^p(\mathbb{R})$ is a b-metric space.
Indeed, by an elementary calculation we get:

$$\boldsymbol{b}_d(x,\ z) \leq 2^{\frac{1}{p}}[\boldsymbol{b}_d(x,\ y) + \boldsymbol{b}_d(y,\ z)].$$

Therefore, $s = 2^{\frac{1}{p}} > 1$ in this case.
But $(l^p(\mathbb{R}), \boldsymbol{b}_d)$ is not metric.

Definition 1.2.2 *[22] Let $(X, \boldsymbol{b}_d, s)$ be a b-metric space. The following concepts are natural deductions from the corresponding metric versions:*

1. *A sequence $\{x_n\}$ in X converges to a point $x \in X$ if*
$$\lim_{n\to\infty} \boldsymbol{b}_d(x_n,\ x) = 0.$$
2. *A sequence $\{x_n\}$ in X is Cauchy if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that*
$$\forall n,\ m \geq N\ \boldsymbol{b}_d(x_n, x_m) < \epsilon.$$
3. *We say that $(X,\ \boldsymbol{b}_d,\ s)$ is a complete b-metric space if every Cauchy sequence is convergent in X.*

Proposition 1.2.1 *[19] In a b-metric space $(X, \boldsymbol{b}_d)$, the following assertions hold:*

1. *A convergent sequence has a unique limit.*
2. *Every convergent sequence is a Cauchy sequence.*

Proof 2 *1. By contradiction.*

We hope to prove "For all convergent sequences the limit is unique. The negation of this is "There exists at least one convergent sequence which does not have a unique limit". This is what we assume. On the basis of this assumption let $\{x_n\}$ denote a sequence with more than one limit, two of which are labelled as z_1 and z_2 with $z_1 \neq z_2$.

Choose $\epsilon = \frac{1}{3s}\boldsymbol{b}_d(z_1, z_2)$ which is greater than zero since $z_1 \neq z_2$. Since z_1 is a limit of $\{x_n\}$ we can apply the definition of limit with our choice of ϵ to find $N_1 \in \mathbb{N}$ such that

$$\boldsymbol{b}_d(x_n, z_1) < \epsilon \qquad \forall n \geq N_1.$$

Similarly, as z_2 is a limit of $\{x_n\}$ we can apply the definition of limit with our choice of ϵ to find $N_2 \in \mathbb{N}$ such that

$$\boldsymbol{b}_d(x_n, z_2) < \epsilon \qquad \forall n \geq N_2.$$

There is no reason to assume that in the two uses of the definition of limit we should find the same $N \in \mathbb{N}$ for the different z_1 and z_2. Choose any $m_0 > \max(N1, N2)$, then $\boldsymbol{b}_d(x_{m_0}, z_1) < \epsilon$ and $\boldsymbol{b}_d(x_{m_0}, z_2) < \epsilon$. Using the b-triangle inequality, we have

$$\begin{aligned}
\boldsymbol{b}_d(z_1, z_2) &\leq s[\boldsymbol{b}_d(z_1, x_{m_0}) + \boldsymbol{b}_d(x_{m_0}, z_2)] \\
&< s[\epsilon + \epsilon] \\
&= 2s\epsilon \\
&= \frac{2s}{3s}\boldsymbol{b}_d(z_1, z_2) \\
&= \frac{2}{3}\boldsymbol{b}_d(z_1, z_2).
\end{aligned}$$

So we find that $\boldsymbol{b}_d(z_1, z_2)$, which is not zero, satisfies $\boldsymbol{b}_d(z_1, z_2) \leq \frac{2}{3}\boldsymbol{b}_d(z_1, z_2)$, which is a contradiction.

Hence our assumption must be false, that is, there does not exists a sequence with more than one limit. Hence for all convergent sequences the limit is unique.

2. *Suppose $\{x_n\}$ is a convergent sequence with limit z. For $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $\boldsymbol{b}_d(x_n,\ z) < \frac{\epsilon}{2}$.*
   We introduce x_m by $\boldsymbol{b}_d(x_n, x_m)$ and use the b-triangle inequality:
$$\begin{aligned}\boldsymbol{b}_d(x_n, x_m) &\leq s[\boldsymbol{b}_d(x_n, z) + \boldsymbol{b}_d(z, x_m)] \\ &< s[\frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= s\epsilon \\ &= \acute{\epsilon},\end{aligned}$$
   whenever $n, m \geq \mathbb{N}$.
   Thus the convergent $\{x_n\}$ is Cauchy.

Definition 1.2.3 *[59] Let $(X_1, \boldsymbol{b}_{d_1})$ and $(X_2, \boldsymbol{b}_{d_2})$ be two b-metric spaces with coefficient s and $\acute{s}$, respectively. A mapping $T : X_1 \to X_2$ is called continuous if each sequence $\{x_n\}$ in X_1, which converges to $x \in X_1$ with respect to $\boldsymbol{b}_{d_1}$, then $\{Tx_n\}$ converges to $Tx \in X_2$ with respect to $\boldsymbol{b}_{d_2}$.*

Remark 1.2.2 *In the general case, a b-metric is not continuous.*

1.3 Partial Metric Spaces

Partial metric spaces were introduced by Matthews in [48] as a part of the study of denotational semantics of dataflow networks. These spaces are a generalization of usual metric spaces where the self distance for any point need not be equal to zero.

Definition 1.3.1 *[51] Let X be a nonempty set. A mapping $p :\ X \times X \to [0, +\infty)$ is said to be a p-metric if the following conditions hold for all $x,\ y,\ z \in X$:*

(1) $x = y$ *if and only if* $p(x, x) = p(y, x) = p(y, y)$;

(2) $p(x, x) \leq p(x, y)$;

(3) $p(x, y) = p(y, x)$;

(4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

Then, the pair (X, p) is called a partial metric space.

Observe that from (1) and (2) it follows that if $p(x, y) = 0$ then $x = y$. But if $x = y$, $p(x, y)$ may not be 0.

Example 1.3.1 *Let $X = \mathbb{R}^+$ and p on X defined by $p(x,\ y) = \max\{x,\ y\}$ for all $x,\ y \in X$. Then $(X,\ p)$ is a partial metric space.*

Example 1.3.2 *Let $X = \mathbb{R}$ and p on X defined by $p(x,\ y) = \exp^{\max\{x,\ y\}}$ for all $x,\ y \in X$. Then $(X,\ p)$ is a partial metric space.*

Each dualistic partial metric p on X generates a topology τ_p on X which has as a base the family of open p-balls $B_p(x, \epsilon) : \{x \in X, \epsilon > 0\}$, where

$$B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\},$$

for all $x \in X$ and $\epsilon > 0$

Definition 1.3.2 *[7] Let $(X,\ p)$ be a partial metric space.*

1. *a sequence $\{x_n\}$ in a partial metric space $(X,\ p)$ converges to a point $x \in X$ if and only if $p(x,\ x) = \lim_{n\to\infty}\ p(x,\ x_n)$.*

2. *A sequence $\{x_n\}$ in a partial metric space $(X,\ p)$ is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$. exists and is finite.*

3. *$(X,\ p)$ is called complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,\ x) = \lim_{n,m\to\infty}\ p(x_n,\ x_m)$.*

If p is a partial metric on X, then the functions $p^s, p^w :\ X \times X \to \mathbb{R}^+$; given by

$$p^s(x,\ y) = 2p(x,\ y) - p(x,\ x) - p(y,\ y),$$

and

$$\begin{aligned} p^w(x,\ y) &= \max\{p(x,\ y) - p(x,\ x), p(x,\ y) - p(y,\ y)\} \\ &= p(x,\ y) - \min\{p(x,\ x), p(y,\ y)\}, \end{aligned}$$

are ordinary metrics on X. It is easy to see that p^s and p^w are equivalent metrics on X.

Lemma 1.3.1 *[7] Let (X, p) be a partial metric space.*

1. *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).*

2. *(X, p) is complete if and only if $(X,\ p^s)$ is complete.*

Lemma 1.3.2 *[7] Let (X, p) be a partial metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is a Cauchy sequence in (X, p) if and only if it satisfies the following condition:*
for each $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$p(x_n,\ x_m) - p(x_n,\ x_n) < \epsilon\ ,$$

whenever $n_0 \leq n \leq m$.

Definition 1.3.3 *Let $(X,\ p)$ be a partial metric space, A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a partial metric space $(X,\ p)$ is called 0- Cauchy if*

$$\lim_{n,m\to\infty} p(x_n,\ x_m) = 0.$$

and $(X,\ p)$ is called 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p, to a point $z \in X$ such that $p(z,\ z) = 0$.

Remark 1.3.1 *It is clear that every complete partial metric space is 0-complete, but as it was shown in [57] the converse is not true.*

1.4 Dislocated Metric Spaces

In 2012, Amini Harrandi [8] introduced a generalization to the partial metric spaces, which is called metric- like spaces (dislocated metric spaces). The motivation of defining this new notion is to get better results the fixed point theory in such spaces.

Definition 1.4.1 *[8] Let X be a nonempty set, a function: $\sigma :\ X \times X \to \mathbb{R}_+$ is said to be a dislocated metric on X if the following conditions satisfied:*

- $\sigma(x, y) = 0$ *implies that* $x = y$.
- $\sigma(x, y) = \sigma(y, x)$.
- $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$.

The space (X, σ) *is said to be a dislocated metric space.*

Remark 1.4.1 *Every partial metric is a dislocated metric.*

Example 1.4.1 *Let* $X = \mathbb{R}$ *and define* σ *for all* $x,\ y \in X$ *by:*

$$\sigma(x,\ y) = \frac{|x-y| + |x| + |y|}{2},$$

Then $(X,\ \sigma)$ *is a dislocated metric space.*
if $X = (0, \infty)$, *so clearly that* $\sigma(x, y) = \max\{x, y\}$ *and also* σ *is a partial metric.*

Example 1.4.2 *[8] Let* $X = \{0,\ 1\}$, *define* $\sigma:\ X \times X \to \mathbb{R}_+$ *as follows:*

$$\sigma(x,\ y) = \begin{cases} 2, & \text{if } x = y = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Then $(X,\ \sigma)$ *is a dislocated metric space, but since* $\sigma(0,0) > \sigma(0,1)$ *then* (X, σ) *is not partial metric space.*

Each dislocated metric σ on X generates a topology τ_σ on X which has as a base the family of open σ-balls $\{B_\sigma(x, \varepsilon); x \in X, \varepsilon > 0\}$, where $B_\sigma(x, \varepsilon) = \{y \in X, |\sigma(x, y) - \sigma(x, x)| < \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

Definition 1.4.2 *[14] Let* (X, σ) *be a dislocated metric space.*

1. *A sequence* $\{x_n\}$ *in dislocated metric space* $(X,\ \sigma)$ *is said to be convergent to a point* $x \in X$, *if and only if*
$$\lim_{n\to\infty} \sigma(x,\ x_n) = \sigma(x,\ x).$$
2. *A sequence* $\{x_n\}$ *in* X *is said to be a* σ*-Cauchy sequence if* $\lim\limits_{n\to\infty} \sigma(x_n,\ x_m)$ *exists and is finite.*

3. (X, σ) *is said to be* σ*-complete if every* σ*- Cauchy sequence* $\{x_n\}$ *in* X *is convergent to a point* $x \in X$ *such that* $\lim\limits_{n\to\infty} \sigma(x, x_n) = \sigma(x, x)$.

Definition 1.4.3 *[14] Let* (X, σ) *and* (Y, λ) *be a dislocated metric spaces, and let* $T : X \to Y$ *be a continuous mapping at point* $x \in X$ *if and only if for every sequence* $\{x_n\} \subset X$ *which is convergent to* x *in* X, *we have the sequence* $\{Tx_n\}$ *is convergent to* Tx *in* Y *.*

$$\lim_{n\to\infty} x_n = x \Rightarrow \lim_{n\to\infty} T(x_n) = T(x).$$

1.5 Dislocated b-Metric Spaces

[6] introduced a new generalization of dislocated metric space and partial metric space which is called a dislocated b-metric space. Then, given some fixed point results in such spaces. fixed point theorems, generalize and improve some well-known results in the literature.

Definition 1.5.1 *[8] Let $s \geq 1$ be a constant and X be a nonempty set. A function $\sigma_b : X \times X \to \mathbb{R}_+$ is referred to as dislocated b-metric if the subsequent requirements are true for every $x,\ y,\ z \in X$:*

- $\sigma_b(x,\ y) = 0$ *implies* $x = y$;
- $\sigma_b(x,\ y) = \sigma_b(y,\ x)$;
- $\sigma_b(x,\ z) \leq s[\sigma_b(x,\ y) + \sigma_b(y,\ ,z)]$.

With parameter $s \geq 1$, the pair (X, σ_b, s) is referred to as a dislocated b-metric space.

In the dislocated b-metric space $(X,\ \sigma_b, s)$, $x =\ y$ if $x,\ y \in X$ and $\sigma_b(x,\ y) = 0$; on the other hand, the opposite may not hold as $\sigma_b(x,\ x)$ may be positive for some $x \in X$.

Example 1.5.1 *[8] Let $p \geq 1$ be a constant and $X = \mathbb{R}_+$. Create a function $\sigma_b :\ X \times X \to \mathbb{R}_+$ by defining it as either $\sigma_b(x,\ y) = (max\{x,\ y\})^p$ or $\sigma_b(x,\ y) = (x+\ y)^p$. then a dislocated b- metric space with parameter 2^{p-1} is (X, σ_b, s).*

Definition 1.5.2 *[6] Given a dislocated b-metric space (X, σ_b, s), $\{x_n\}$, a sequence in X, and $x \in X$, let $s \geq 1$. We say:*

1. *A sequence $\{x_n\}$ in X is said to be σ_b-convergent sequence if*
$$\lim_{n\to\infty} \sigma_b(x_n,\ x) = \sigma_b(x,\ x);$$
2. *A sequence $\{x_n\}$ in X is said to be a σ_b-Cauchy sequence if $\lim_{n,m\to\infty} \sigma_b(x_n,\ x_m)$ exists and is finite;*
3. *(X, σ_b, s) is said to be σ_b-complete if every σ_b-Cauchy sequence $\{x_n\}$ in X, there exists $x \in X$ such that*
$$\sigma_b(x,\ x) = \lim_{n\to\infty} \sigma_b(x_n,\ x) = \lim_{n\to\infty} \sigma_b(x_n,\ x_m).$$

Definition 1.5.3 *[6] Given a dislocated b-metric space (X, σ_b, s) and a mapping $\mathcal{T}: X \to X$, let $s \geq 1$. If for each sequence $\{\xi_n\}$ in X such $\sigma_b(\xi_n, xi) \to \sigma_b(\xi, \xi)$ as $n \to \infty$, we have $\sigma_b(\mathcal{T}\xi_n, \mathcal{T}\xi) \to \sigma_b(Tx, Tx)$ as $n \to \infty$. Then $\mathcal{T}$ is σ_b-continuous.*

Remark 1.5.1 *If $\lim\limits_{n\to\infty} \sigma_b(x_n,\ x_m) = 0$ and the limit of $\{x_n\}$ exist in a dislocated b-metric space, then the limit is unique.*

Let $(X,\ \sigma_b,\ s)$ be a dislocated b-metric space. Define $\sigma_b^s : X^2 \to [0, \infty)$ by

$$\sigma_b^s(\xi, \zeta) = |2\sigma_b(x,\ y) - \sigma_b(x,\ x) - \sigma_b(y,\ y)|. \tag{1.1}$$

Clearly, $\sigma_b^s(x,\ x) = 0$ for all $x \in X$.

1.6 Quasi Metric Spaces

[63], Wilson initiated the concept of quasi- metric space (also known as a symmetric metric space) as an extension of the metric space. This is defined as a metric space $(X,\ q)$, but q is not required to be symmetric. The intuitive example of a quasi-metric space is a circle, where the distance q between two points A and B on the circle is defined as the length of the shortest arc measured in the clockwise direction. Clearly, $q(A, B) \neq q(B, A)$ unless A and B are diametrically opposite on the circle. Recent developments in applied mathematics have seen a wide range of applications for quasi- metric spaces.

Definition 1.6.1 *[63] Let X be a non-empty and let $q : X \times X \to [0, \infty)$ be a function which satisfies:*

1. *$q(x,\ y) = 0$ if and only if $x = y$;*

2. *$q(x,\ y) \leqslant q(x,\ z) + q(z,\ y)$.*

Then q is called a quasi- metric and the pair $(X,\ q)$ is called a quasi -metric space.

Remark 1.6.1 *Any metric space is a quasi- metric space, but the converse is not true in general.*

Definition 1.6.2 *[35] Let $(X,\ q)$ be a quasi-metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. The sequence $\{x_n\}$ converges to x if and only if*

$$\lim_{n\to\infty} q\left(x_n,\ x\right) = \lim_{n\to\infty} q\left(x,\ x_n\right) = 0.$$

Example 1.6.1 *Let X be a subset of $\mathbb{R}$ containing $[0,\ 1]$ and define, for all $x,\ y \in X$,*

$$q(x,\ y) = \begin{cases} x - y & \text{if } x \geqslant y, \\ 1 & \text{otherwise.} \end{cases}$$

Then (X, q) is a quasi- metric space. Notice that $\{q(1/n, 0)\} \to 0$ but $\{q(0, 1/n)\} \to 1$. Therefore, $\{1/n\}$ right-converges to 0 but it does not converge from the left.

We also point out that this quasi-metric verifies the following property: if a sequence $\{x_n\}$ has a right-limit x, then it is unique.

Definition 1.6.3 *[35] Let (X, q) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is left-Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $q\left(x_n, x_m\right) < \varepsilon$ for all $n \geqslant m > N$.*

Definition 1.6.4 *[35] Let (X, q) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is right-Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $q\left(x_n,\ x_m\right) < \varepsilon$ for all $m \geqslant n > N$.*

Definition 1.6.5 *[35] Let (X, q) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $q\left(x_n, x_m\right) < \varepsilon$ for all $m, n > N$.*

Remark 1.6.2 *A sequence $\{x_n\}$ in a quasi -metric space is Cauchy if and only if it is left-Cauchy and right-Cauchy.*

Definition 1.6.6 *[35] Let (X, q) be a quasi-metric space. We say that:*

1. *(X, q) is left- complete if and only if each left-Cauchy sequence in X is convergent.*
2. *(X, q) is right-complete if and only if each right- Cauchy sequence in X is convergent.*
3. *(X, q) is complete if and only if each Cauchy sequence in X is convergent.*

Definition 1.6.7 *[35] Let (X, q) be a quasi-metric space. The map $T:\ X \to X$ is continuous if for each sequence $\{x_n\}$ in X converging to $x \in X$, the sequence $\{Tx_n\}$ converges to Tx, that is,*

$$\lim_{n\to\infty} q\left(Tx_n, Tx\right) = \lim_{n\to\infty} q\left(Tx, Tx_n\right) = 0.$$

1.7 Quasi-Partial Spaces

In 2006, Kunzi et al. [46] provided the concept of quasi-partial metric spaces and showed that the partial metric space due to Matthews [48] can be connected to complexity analysis. In 2016, Mohammadia and Valero [50] presented results more generalized than the fixed point results due to Kunzi et al. They proved some results related to the fixed point for continuous and monotone self-mappings in 0-complete quasi-partial metric spaces.

Definition 1.7.1 *[39] A quasi-partial metric is a function* $q: \ X \times X \to \mathbb{R}^{+}$ *satisfying*

1. $q(x,x) \leq q(y,x)$ *(small self-distances),*
2. $q(x,\ x) \leq q(x,\ y)$ *(small self-distances),*
3. $x = y$ *iff* $q(x,\ x) = q(x,\ y)$ *and* $q(y,\ y) = q(y,\ x)$ *(implies equality and vice versa),*
4. $q(x,z) + q(y,y) \leq q(x,y) + q(y,z)$ *(triangularity),*

for all $x, y, z \in X$. The pair (X, q) is called a quasi-partial metric space.
Karapinar et al have taken:
($\grave{3}$) if $0 \leq q(x,\ x) = q(x,\ y) = q(y,\ y)$, then $x = y$ (equality), in place of (3).

If q satisfies all these conditions except possibly (1), then q is called a lopsided partial quasi-metric. It is interesting to see here that for $q(x,y) = q(y,x)$, (X,q) becomes a partial metric space. Also for a quasi-partial metric q on X, the function $d_q: \ X \times X \to \mathbb{R}^{+}$ defined by

$$d_q(x,\ y) = q(x,\ y) + q(y,\ x) - q(x,\ x) - q(y,\ y),$$

is a (usual) metric on X.

Example 1.7.1 *[39] The pair* $(\mathbb{R}^{+}, q)$ *with*

1. $q(x,\ y) = |x - y| + |x|$;
2. $q(x,\ y) = \max\{y - x, 0\} + x$;

is a quasi-partial metric space.

Definition 1.7.2 *[39] Let (X, q) be a quasi-partial metric space.*

1. *A sequence $\{x_n\} \subset X$ in a quasi-partial metric space converges to a point $x \in X$ iff $q(x, x) = \lim_{n\to\infty} q(x, x_n) = \lim_{n\to\infty} q(x_n, x)$.*

2. *A sequence $\{x_n\} \subset X$ is called a Cauchy sequence if and only if $\lim_{n,m\to\infty} q(x_n,\ x_m)$ and $\lim_{n,m\to\infty} q(x_m, x_n)$ exist (and are finite);*

3. *The quasi-partial metric space (X, q) is said to be complete if every Cauchy sequence $\{x_n\} \subset X$ converges, with respect to τ_q, to a point $x \in X$ such that $q(x, x) = \lim_{n,m\to\infty} q(x_m,\ x_n) = \lim_{n,m\to\infty} q(x_n,\ x_m)$.*

4. *A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \epsilon)$.*

Lemma 1.7.1 *[38] Let (X, q) be a quasi-partial metric space. Then following hold*

1. *If $q(x,\ y) = 0$, then $x = y$;*

2. *If $x \neq y$, then $q(x,\ y) > 0$ and $q(y, x) > 0$.*

1.8 Dislocated Quasi Metric Spaces

Zeyada et al. [64] initiated the concept of dislocated quasi-metric space and generalized the result of Hitzler and Seda in dislocated quasi-metric spaces. Recently, the study in such spaces followed by Isufati and Aage and Salunke [1]. In [33], the author proved some fixed point results in dislocated quasi-metric spaces involving continuous contraction mappings exist in the literature of usual complete metric space.

Definition 1.8.1 *[64] Let X be a nonempty set and let $d_q : X \times X \to [0, \infty)$ be a function satisfying following conditions:*

1. *$d_q(x,\ y) = d(y,\ x) = 0$, implies $x = y$,*

2. *$d_q(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.*

Then d_q is called a dislocated quasi-metric on X. If d_q satisfies $d_q(x,\ y) = d_q(y,\ x)$, then it is called dislocated metric.

Definition 1.8.2 *[64] A sequence $\{x_n\}$ in d_q-metric space (dislocated quasi-metric space) (X, d_q) is called Cauchy sequence if for given $\varepsilon > 0, \exists n_0 \in N$, such that $\forall m, n \geq n_0$,implies* $d_q(x_m, x_n) < \varepsilon$ *or* $d_q(x_n, x_m) < \varepsilon$ *i.e.* $\min\{d_q(x_m, x_n), d_q(x_n, x_m\} < \varepsilon$.

Definition 1.8.3 *[64] A sequence $\{x_n\}$ dislocated quasi-converges to x if*

$$\lim_{n\to\infty} d_q(x_n, x) = \lim_{n\to\infty} d_q(x, x_n) = 0.$$

In this case x is called a d_q-limit of $\{x_n\}$ and we write $x_n \to x$.

Lemma 1.8.1 *[64] d_q-$d_q$limits in a d_q-metric space are unique.*

Definition 1.8.4 *[64] A d_q-metric space (X, d_q) is called complete if every Cauchy sequence in it is a d_q-convergent.*

Definition 1.8.5 *[64] Let (X, d_q1) and (Y, d_q2) be d_q-metric spaces and let $T : X \to Y$ be a function. Then T is continuous to $x_0 \in X$, if for each sequence $\{x_n\}$ which is d_q1-q convergent to x_0, the sequence $\{T(x_n)\}$ is d_q2-q convergent to $T(x_0)$ in Y.*

1.9 Dislocated Quasi b-Metric Spaces

In 2013, Shukla [60] generalized the notion of b-metric spaces and introduced the concept of partial b-metric spaces. Recently, Rahman and Sarwar [55] further generalized the concept of b-metric space and initiated the notion of dislocated quasi-b-metric space

Definition 1.9.1 *[25] Let X be a nonempty set, and $s \geq 1$ a constant. A function: $\sigma_q : X \times X \to \mathbb{R}_+$ is a dislocated quasi b-metric if for all $x, y, z \in X$, the following conditions are satisfied:*

- $\sigma_q(x, y) = 0$ *implies* $x = y$,
- $\sigma q \leq s[\sigma_q(x, y) + \sigma_q(y, z)]$.

A triplet (X, σ_q) is called a dislocated quasi b-metric space.
The pair (X, σ_q, s) is a dislocated quasi b-metric space with parameter $s \geq 1$.

In a dislocated quasi b-metric space (X, σ_q, s), if $x, y \in X$ and $\sigma_q(x, y)$. Then $x = y$, however, the converse need not be true, since $\sigma_q(x, x)$ may be positive for some $x \in X$.

Example 1.9.1 *Let $X = \mathbb{R}$, and $p \geq 1$ be a constant. Define a function $\sigma_q : X \times X \to \mathbb{R}_+$ by $\sigma_q(x, y) = (x - y)^2 + 2|y|$ or $\sigma_q(x, y) = (\max\{|x|, |y|\}) + |x|$. Then (X, σ_q, s) is a dislocated quasi b-metric space with parameter $s = 2$.*

Example 1.9.2 *For $X = \mathbb{R}_+$, the function $\sigma_q : X \times X \to \mathbb{R}_+$ defined by $\sigma_q(x, y) = (x-y)^2 + (2x + y)^2$ is dislocated quasi b-metric with parameter $s = 2$.*

Example 1.9.3 *[56] For $X = \mathbb{R}$, the function $\sigma_q : X \times X \to \mathbb{R}_+$ defined by $\sigma_q(x, y) = (2x - y)^2 + (2x + y)^2$ is dislocated quasi b-metric with parameter $s = 2$.*

Definition 1.9.2 *[25] Let (X, σ_q, s) be a dislocated quasi b-metric space with parameter $s \geq 1$, $\{x_n\}$ a sequence in X and $x \in X$. We say:*

1. *A sequence $\{x_n\}$ in X is said to be convergent sequence if*

$$\lim_{n\to\infty} \sigma_q(x_n,\ x) = \lim_{n\to\infty} \sigma_q(x,\ x_n) = \sigma_q(x,\ x).$$

2. *A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim\limits_{n,m\to\infty} \sigma_q(x_n,\ x_m)$ and $\lim\limits_{n,m\to\infty} \sigma_q(x_m, x_n)$ exist and finite.*

3. *(X, σ_q, s) is said to be complete if every σ_q-Cauchy sequence $\{x_n\}$ in X, there exists $x \in X$ such that*

$$\sigma_q(x, x) = \lim_{n\to\infty} \sigma_q(x_n, x) = \lim_{n\to\infty} \sigma_q(x, x_n) = \lim_{n,m\to\infty} \sigma_q(x_n, x_m) = \lim_{n,m\to\infty} \sigma_q(x_m, x_n).$$

4. *A sequence $\{x_n\}$ in (X, b) is called a 0-Cauchy sequence if*

$$\lim_{n,m\to+\infty} \sigma_q\left(x_n, x_m\right) = \lim_{n,m\to+\infty} \sigma_q\left(x_m, x_n\right) = 0.$$

5. *A dislocated quasi-b-metric space (X, b) is called to be 0-complete if for every 0-Cauchy sequence $\{x_n\}$ in X, there exists some $x \in X$ such that*

$$\lim_{n\to+\infty} \sigma_q\left(x_n,\ x\right) = \lim_{n\to+\infty} \sigma_q\left(x,\ x_n\right) = \sigma_q(x,\ x) = 0 = \lim_{n,m\to+\infty} \sigma_q\left(x_n, x_m\right)$$

$$= \lim_{n,m\to+\infty} \sigma_q\left(x_m,\ x_n\right).$$

Remark 1.9.1 *It is obvious that every* 0*-Cauchy sequence is a Cauchy sequence in the dislocated quasi* b*-metric space* (X, σ_q)*, and every complete dislocated quasi-* b*-metric space is a* 0 *-complete dislocated quasi-* b *-metric space, but the converse assertions of these facts may not be true.*

Definition 1.9.3 *[56] Let* (X, σ_q, s) *be a dislocated quasi* b*-metric space with parameter* $s \geq 1$ *and* $\mathcal{T}: X \to X$ *be a mapping. We say that* $\mathcal{T}$ *is continuous if for each sequence* $\{x_n\}$ *in* X *with*

$$\lim_{n\to\infty} \sigma_q(x_n,\ x) = \sigma_q(x,\ x) = \lim_{n,m\to\infty} \sigma_q(x_n,\ x_m),$$

we have

$$\lim_{n\to\infty} \sigma_q(Tx_n,\ Tx) = \sigma_q(Tx,\ Tx) = \lim_{n,m\to\infty} \sigma_q(Tx_n,\ Tx_m).$$

CHAPTER

2

SOME FIXED POINT THEOREMS IN VARIOUS METRICAL STRUCTURE

The famous Banach contraction principle is one of the powerful tools in fixed point theory, which is an important tool in the study of nonlinear analysis, as it serves as the main link between pure and applied mathematics. The Banach contraction principle has been extended and generalized in various directions by different researchers. Fixed point is a useful tool in mathematics which can be used to prove the existence of solution of a differential equation, integral equation and eigen value equation. Present section is providing the de definition of fixed point.

2.1 Banach Principle

In 1922, Banach[12] proved a fixed point theorem for metric spaces, which later on came to be known as the famous "Banach contraction principle". Since then generalizations of the contraction principle in different directions as well as many new fixed point results with applications.

Definition 2.1.1 *A fixed point of a mapping $T : X \to X$ of a set X into itself is an $x_0 \in X$ which is mapped onto itself (is kept fixed by T),*
that is,

$$Tx_0 = x_0,$$

the image $T(x_0)$ coincides with x_0.

Example 2.1.1 *Consider $T : \mathbb{R} \to \mathbb{R}$ be a mapping which is defined as*

$$T(p) = x^2 - 3x + 3;$$

then $x = 1; 3$ are the fixed point of the given mapping.

Definition 2.1.2 ***(Contraction)***
Let $(X,\ d)$ be a metric space. A mapping $T : X \to X$ is called a contraction on X if there exists a positive constant $k < 1$ such that

$$d(Tx,\ Ty) \leq kd(x,\ y)$$

for all $x,\ y \in X$,

Theorem 2.1.1 ***(Banach's Fixed Point Theorem)*** *[44]*
Consider a metric space $(X,\ d)$. Suppose that X is complete and let $T : X \to X$ be a contraction on X. Then T has a unique fixed point $x \in X$.

Proof 3 *Choose $x_0 \in X$ and define $\{x_n\}$ inductively by iterating process*

$$x_{n+1} = Tx_n. \tag{2.1}$$

From (2.1), starting with x_0, we have

$$x_1 = Tx_0$$

$$x_2 = Tx_1 = T^2x_0$$

$$x_3 = Tx_2 = T^2x_1 = T^3x_0$$

$$.$$

$$.$$

$$x_n = T^nx_0.$$

Since, T is a contraction, it follows that

$$d(x_n,\ x_{n+1}) = d(Tx_{n-1},\ Tx_n) \leq kd(x_{n-1},\ x_n)$$

$$= kd(Tx_{n-2},\ Tx_{n-1})$$

$$\leq k^2d(x_{n-2},\ x_{n-1})$$

$$\leq k^3\ d(x_{n-3},\ x_{n-2})$$

$$.$$

$$.$$

$$\leq k^nd(x_0, x_1)$$

$$\Rightarrow d(x_n,\ x_{n+1}) \leq k^nd(x_0, x_1).$$

Now suppose that $m > n$ and then by triangular inequality, we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + ... + d(x_{m-1}, x_m)$$

$$\leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + ... + k^{m-1} d(x_0, x_1)$$

$$\leq (k^n + k^{n+1} + ..) d(x_0, x_1)$$

$$= \frac{k^n}{1-k} d(x_0, x_1).$$

Since, $k < 1 \Rightarrow \frac{k^n}{1-k} \to 0$ *as* $n \to \infty$.

It follows that the sequence $\{x_n\}$ *is Cauchy. But* X *is also complete. Then, an* $x \in X$ *exists there such that* $x_n \to x$.

Then we claim the point x *is a fixed point of the mapping* T.

Since

$$\lim_{n\to\infty} x_n = x,$$

and

$$\lim_{n\to\infty} x_{n+1} = x,$$

Now,

$$Tx = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n$$
$$= \lim_{n\to\infty} x_{n+1} = x.$$

Hence, x *is a fixed point of* T.

Example 2.1.2 *Consider the complete metric space* $(\mathbb{R}, d)$ *where* d *is the Euclidean distance metric, i.e.*

$$d(x, y) = |x - y|.$$

The function $T : \mathbb{R} \to \mathbb{R}$ *where*

$$T(x) = \frac{x}{a} + b$$

T *is a contraction if* $a > 1$.

In this specific case we can find a fixed point, it is given us $\frac{ab}{a-1}$.

2.2 Czerwik Fixed Point Theorem

Definition 2.2.1 *Let (X, d) be a b-metric space and $T : X \to X$ be a self mapping on X. We say that T is an ψ-contractive mapping if satisfies*

$$d(Tx, Ty) \leq \psi(d(x, y)),$$

for all $x, y \in X$,
where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a increasing function such that

$$\lim_{n\to\infty} \psi^n(t) = 0,$$

for all $t \geq 0$.

Theorem 2.2.1 *[25] Let (X, d) be a complete b-metric space with constant $s > 1$, and suppose that $T : X \to X$ be a ψ-contractive mapping.*
Then, T has a unique fixed point $\widetilde{x}$ and

$$\lim_{n\to\infty} T^n(x) = \widetilde{x}.$$

Example 2.2.1 *Let $X = [0, +\infty)$ and $d(x, y) = |x - y|^2$, for all $x, y \in X$.*
It is easy to see that d is a b-metric with parameter $s = 2$. Also, it is easy to see that d is not a metric.
Define a mapping $T : X \to X$ as follows

$$Tx = \frac{1}{3}x + 1, \qquad \forall x \in X,$$

and let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ where

$$\psi(t) = \frac{1}{9}t.$$

Clearly, we have

$$d(Tx, Ty) \leq \frac{1}{9}d(x, y) = \psi(d(x, y)),$$

for all $x, y \in X$,
Then, T has a unique fixed point $x = \frac{3}{2}$.

2.3 Matthews Fixed Point Theorem

Definition 2.3.1 *Let (X, p) be a partial metric space and $T: X \to X$ be a self mapping on X. We say that T is a contraction mapping if there exists $k \in [0,\ 1)$ such that*

$$p(Tx,\ Ty) \leq \max\{kp(x,\ y), p(x,\ x), p(y,\ y)\},$$

for all $x, y \in X$.

Theorem 2.3.1 *[48] Let $(X,\ p)$ be a complete partial metric space and $T: X \to X$ be a contraction mapping. Then T has a unique fixed point.*

Example 2.3.1 *Let $X = [0, 2]$ and define $p: X \times X \to \mathbb{R}$ by*

$$p(x, y) = \max\{x, y\}, \qquad \textit{for all} \quad x, y \in X.$$

Then $(X,\ p)$ is a complete partial metric space.
Define $T: X \to X$ by

$$Tx = \frac{x}{5}, \qquad \textit{for all} \quad x \in X.$$

Now

$$p(Tx,\ Ty) \leq \max\{kp(x,\ y), p(x,\ x), p(y,\ y)\}, \qquad \forall x, y \in X,$$

where $\frac{1}{5} < k < 1$.
By theorem (2.3.1) the mapping T has a unique fixed point $x = 0$.

2.4 Amini Herandi Fixed Point Theorem

Let Ψ be the set of all functions $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ such that:

1. ψ is non decreasing.

2. $\psi(t) < t$ for all $t \geq 0$.

3. $\lim_{n\to\infty} (t - \psi(t)) = \infty$.

Example 2.4.1

1. $\psi = kt$, $k \in [0, \infty)$.

2. $\psi(t) = t - \ln t$, *for all* $t > 0$.

Definition 2.4.1 *Let* (X, σ) *be a dislocated metric space, and let* $T : X \to X$ *be a self mapping on* X. *We say that* T *is a* σ*-contraction mapping if satisfy*

$$\sigma(Tx, Ty) \leq \psi(M(x, y)),$$

for all $x, y \in X$,
where $\psi \in \Psi$ *and*

$$M(x, y) = \max\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty), \sigma(x, Ty), \sigma(y, Tx)\}.$$

Theorem 2.4.1 *[8] Let* (X, σ) *be a complete dislocated metric space, and* $T : X \to X$ *be a* σ*-contraction mapping. Then* T *has a fixed point.*

Example 2.4.2 *Let* $X = \{0, 1, 2\}$ *and define* σ *as follows*

$$\sigma(0,\ 0) = 0,\ \ \sigma(1,\ 1) = 3,\ \ \sigma(2,\ 2) = 1,$$

$$\sigma(0,\ 1) = \sigma(1,\ 0) = 7,\ \ \sigma(0,\ 2) = \sigma(2,\ 0) = 3,\ \ \sigma(1,\ 2) = \sigma(2,\ 1) = 4.$$

Define $T :\ X \to X$ *as*

$$T(0) = 0,\ \ T(1) = 2,\ \ T(2) = 0.$$

We have

$$\sigma(T0,\ T0) = \sigma(1,\ 1) = 0 = \sigma(0,\ 0)$$

$$\sigma(T1,\ T1) = \sigma(2,\ 2) = 1 \leq \sigma(1,\ 1)$$

$$\sigma(T2,\ T2) = \sigma(0,\ 0) = 0 = \sigma(2,\ 2)$$

$$\sigma(T0,\ T1) = \sigma(T1,\ T0) = \sigma(0,\ 2) = 3 \leq \frac{3}{4}\sigma(0,\ 1)$$

$$\sigma(T1,\ T2) = \sigma(T2,\ T1) = \sigma(0,\ 2) = 3 \leq \frac{3}{4}\sigma(1,\ 2)$$

$$\sigma(T0,\ T2) = \sigma(T0,\ T1) = \sigma(0,\ 2) = 3 \leq \frac{3}{4}\sigma(0,\ 1).$$

Then for all $x,\ y \in X$, $x \neq y$ we have

$$\sigma(Tx,\ Ty) = 3 \leq \frac{3}{4}\sigma(x,\ y),$$

and for all $x \in X$ we have

$$\sigma(Tx,\ Tx) = \sigma(x,\ x).$$

Then for all $x,\ y \in X$ we have

$$\sigma(Tx,\ Ty) \leq \frac{3}{4}\sigma(x,\ y) \leq \frac{3}{4}M(x,\ y).$$

Hence T has a fixed point which is 0.

2.5 Fixed Points For Quasi Contraction Type Mappings

2.5.1 Contractive Conditions And Fixed Point Results

It is well known that a self-map T on a metric space $(X,\ d)$ is said to be a Banach contraction mapping, if there exists a number $k \in [0,\ 1)$ such that

$$d(Tx,\ Ty) \leq kd(x,\ y),$$

for all $x,\ y \in X$.

A mapping $T:\ X \to X$ is called a quasi contraction if for some constant $\alpha \in [0,\ 1)$ and for every $x,\ y \in X$

$$d(Tx,\ Ty) \leq \alpha\ \max\{d(x,\ y), d(x,\ Tx), d(y,\ Ty), d(x,\ Ty), d(y,\ Tx)\}.$$

This concept was introduced and studied by Ciric in 1974 [24]. A result of Ciric shows that every quasi contraction in a complete metric space T has a unique fixed point.

Theorem 2.5.1 *[31] Let $(X, \preceq, \sigma_b)$ be a complete partially ordered dislocated b-metric space. If $T : X \to X$ is a nondecreasing and continuous self-map on X such that, for all elements $x,\ y \in X$ with $x \preceq y$,*

$$\sigma_b(Tx, Ty) \leq \alpha M(x, y), \tag{2.2}$$

where $\alpha \in [0, \frac{1}{2s^2})$, and

$$M(x, y) = \max\{\sigma_b(x, y), \sigma_b(x, Tx), \sigma_b(y, Ty), \sigma_b(x, Ty), \sigma_b(y, Tx)\}.$$

Then T has a fixed point.

Example 2.5.1 *[31] Let $X = [0,\ 1]$ be endowed with the usual order. Define $\sigma_b :\ X \times X \to \mathbb{R}^+$ by*

$$\sigma_b(x,\ y) = x^2 + y^2 + |x - y|^2\ ,$$

for all $x,\ y \in X$. Then $(X,\ \sigma_b)$ is a complete dislocated b-metric space with coefficient $s = 2$ Let $T :\ X \times X$ be defined by $Tx = \ln(1 + \frac{x}{4})$. It is easy to see that T is a nondecreasing and continuous self-map on X, and $0 \in Fix(T)$. Using the Mean Value Theorem for any $x,\ y \in X$ with $x \leq y$ and that $Tx \leq x/4$, we have

$$\sigma_b(Tx,\ Ty) = T^2x + T^2y + |Tx - Ty|^2$$

$$= \left(\ln(1 + \frac{x}{4})\right)^2 + \left(\ln(1 + \frac{y}{4})\right)^2 + |\ln(1 + \frac{x}{4}) - \ln(1 + \frac{y}{4})|^2$$

$$\leq \left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2 + |\frac{x}{4} - \frac{y}{4}|^2$$

$$= \frac{1}{16}\left(x^2 + y^2 + |x - y|^2\right)$$

$$= \frac{1}{16}\sigma_b(x, y) \leq \frac{1}{16}M(x, y).$$

Thus (2.2) is satisfied with $\alpha = 1/16 \in [0, 1/8)$.

Thus all conditions of Theorem (2.5.1) are satisfied. Moreover, 0 is the unique fixed point of T.

2.6 Banach Contraction Principle On Quasi Metric Space

Remark 2.6.1 *Every quasi-metric induces a metric, that is, if $(X,\ \sigma)$ is a quasi-metric space, then the function $d:\ X \times X \to [0, \infty)$ defined by*

$$d(x,\ y) = \max\{\sigma(x,\ y), \sigma(y, x)\ \},$$

for all $x, y \in X$,

is a metric on X; see Jleli-Samet [43].

Theorem 2.6.1 *[52] A self map $T : X \to X$ of a quasi-metric space (X, σ) has a fixed point $z \in X$ if and only if z is a fixed point of the self map T of the induced metric space (X, d).*

Proof 4 *If $z = Tz$ in (X, σ), then*

$$d(z,\ Tz) = \max\{\sigma(z,\ Tz), \sigma(Tz,\ z)\} = 0,$$

and hence $d(z,\ Tz) = 0$. The converse is true for $d = \sigma$.

Definition 2.6.1 *[43] Let (X, σ) be a quasi-metric space and $T : X \to X$ be a mapping. We say that T is a contraction if,*

$$\sigma(Tx, Ty) \leq \alpha\sigma(x, y)$$

for all $x, y \in X$, with $0 < \alpha < 1$.

Theorem 2.6.2 *Let (X, σ) be a complete quasi-metric space and let $T : X \to X$ be a contraction. Then T has a unique fixed point $x \in X$.*

Example 2.6.1 *Let $X = [0, 1]$ endowed with the quasi-metric*

$$\sigma(x, y) = \begin{cases} x - y & \text{if } x \geq y, \\ x & \text{if } x < y. \end{cases}$$

It is clear that (X, σ) *is a complete quasi-metric space. Consider the mapping* $T : X \to X$ *defined by*

$$Tx = \frac{x}{2} \qquad \textit{for all} \quad x \in X,$$

1. *if* $x \geq y$ *then* $\sigma(x, y) = x - y$ *and* $Tx \geq Ty$ *where*

$$\sigma(Tx, Ty) = \frac{1}{2}(x - y) \leq \frac{3}{4}\sigma(x, y),$$

2. *if* $x < y$ *then* $\sigma(x, y) = x$ *and* $Tx < Ty$ *where*

$$\sigma(Tx, Ty) = \frac{1}{2}x \leq \frac{3}{4}\sigma(x, y).$$

So, for all $x \in X$

$$\sigma(Tx, Ty) \leq \frac{3}{4}\sigma(x, y)$$

therefore T *has a unique fixed point is* $x = 0$.

2.7 The Fixed Point On Partial Quasi-Metric

Definition 2.7.1 *[46] Let* X *be a set equipped with some partial quasi-metric* $q : X \times X \to [0, \infty)$. *We say that a map* $T : X \to X$ *is a contraction if there is a positive real constant* $k < 1$ *such that*

$$q(Tx,\ Ty) \leq kq(x,\ y),$$

whenever $x, y \in X$.

Theorem 2.7.1 *[46] Let* $T : (X, q) \to (X, q)$ *be a contraction defined on a complete partial quasi-metric space* (X, q). *Then* T *has a unique fixed point.*

Example 2.7.1 *Let* $X = [0, 1]$. *Define* $q : X \times X \to [0, \infty)$ *by*

$$q(x, y) = |x - y| + |x| \qquad \forall x, y \in X.$$

Clear that (X, q) *is a complete partial quasi-metric space. Consider the mapping* $T : X \to X$ *defined by*

$$Tx = \frac{x}{3} \qquad \textit{for all} \quad x \in X,$$

then

$$q(Tx,\ Ty) = \frac{1}{3}(|x-y|+|x|) \leq \frac{1}{2}q(x,\ y).$$

Then T has a unique fixed point is $x=0$.

2.8 Banach Contraction Principle On Dislocated Quasi Metric Space

Definition 2.8.1 *[64] Let (X, d_q) be a dislocated quasi metric space. A map $T:\ X \to X$ is called d_q-contraction if there exists $0 \leq \lambda < 1$ such that*

$$d_q(Tx, Ty) \leq \lambda d_q(x, y),$$

for all $x,\ y \in X$.

Theorem 2.8.1 *[64] Let (X, d_q) be a complete dislocated quasi metric space and let $T:\ X \to X$ be a d_q-contraction mapping. Then T has unique fixed point.*

Example 2.8.1 *Let $X = \mathbb{R}$ and $d_q : X \times X \to [0, \infty)$ defined by*

$$d_q(x, y) = |x| \qquad \forall x, y \in X.$$

Then (X, q) is complete dislocated quasi-metric space.
Consider the mapping $T : X \to X$ defined by

$$Tx = \frac{x-1}{2} \qquad \text{for all} \quad x \in X,$$

then

$$d_q(Tx,\ Ty) = \frac{1}{2}|x-1| \leq \frac{1}{2}|x| = \frac{1}{2}d_q(x,\ y).$$

Then T has a unique fixed point is $x=-1$.

2.9 Fixed Point On Dislocated Quasi- b-Metric Space

Definition 2.9.1 *Let (X, σ_q) be a dislocated quasi-b-metric space with the coefficient $s \geq 1$, and let $T : X \to X$ be a mapping, T is called σ_q-contraction if satisfy*

$$\sigma_q(Tx, Ty) \leq \varphi(\sigma_q(x, y)),$$

for all $x,\ y \in X$,

where $\varphi\ : [0, +\infty) \to [0, +\infty)$ *satisfy:*

1. φ *is a continuous mapping,*
2. $\varphi(t) = 0$ *if and only if* $t = 0$,
3. $\varphi(t) < t$ *for all* $t > 0$.
4. $\sum_{n=1}^{\infty} s^n \varphi^n(t)$ *converges for all* $t > 0$, *where* φ^n *is the nth iterate of* φ.

Theorem 2.9.1 *[65] Let* (X, σ_q) *be a* 0*-complete dislocated quasi-b-metric space with the coefficient* $s \geq 1$, *and let* $T : X \to X$ *be a* σ_q*-contraction mapping. Then* T *has a unique fixed point. Moreover, for any* $x_0 \in X$, *the iterative sequence* $\{T^n(x_0)\}$ *converges to the fixed point.*

Example 2.9.1 *Let* $X = [-1, 1]]$. *Define* $\sigma_q : X \times X \to \mathbb{R}_+$ *by*

$$\sigma_q(x, y) = |x - y|^2 + 3|x| + 2|y|,$$

for all $x, y \in X$. *Then* (X, σ_q) *is a complete dislocated quasi b-metric space with the coefficient* $s = 2$.

Define the mapping $T : X \to X$ *defined by*

$$Tx = \frac{x}{6} \qquad \text{for all} \quad x \in X,$$

and $\varphi\ : [0, +\infty) \to [0, +\infty)$ *where*

$$\varphi(t) = \frac{1}{6}t \qquad \forall t \in [0, +\infty),$$

then

$$\sigma_q(Tx,\ Ty) = \frac{1}{36}|x - y|^2 + \frac{3}{6}|x| + \frac{2}{6}|y|,$$

$$\leq \frac{1}{6}\left(|x - y|^2 + 3|x| + 2|y|\right),$$

$$= \frac{1}{6}\sigma_q(x,\ y) = \varphi(\sigma_q(x, y)).$$

Then T *has a unique fixed point is* $x = 0$.

CHAPTER

3

ALMOST JS-CONTRACTIONS AND FIXED POINT RESULTS IN DISLOCATED b-METRIC SPACES

In this chapter, we combine the notion of α-admissible mappings with the concepts of ϑ-contraction and Berinde type contraction to develop a novel class of contractions and related fixed point results in complete dislocated b-metric spaces. Additionally, we demonstrate the presence of fixed points in partially ordered metric spaces using our primary findings. In order to illustrate the importance of the findings, we provide an example.

3.1 Almost Contractions

Definition 3.1.1 *[16]Consider a metric space (X, d). If $\gamma \in [0, 1)$ and $L \geq 0$ exist such that*

$$d(\mathcal{T}x,\ \mathcal{T}y) \leq \gamma d(x,\ y) + Ld(y,\ \mathcal{T}x),$$

for all $x, y\ \in X$, then $\mathcal{T} : X \to X$ is considered almost ((δ-L) weak).

Remark 3.1.1 *[16] due to the symmetry of the distance, the weak contraction previous condition implicitly includes the following dual inequality*

$$d(Tx,\ Ty) \leq \delta d(x,\ y) + Ld(x, Ty),$$

for all $x, y\ \in X$:

Theorem 3.1.1 *[16] Let $(X,\ d)$ be a complete metric space and $T : X \to X$ be a weak contraction such that there exists $\delta \in [0,\ 1[$ and $L \geq 0$ such that, for all $x, y\ \in X$,*

$$d(Tx,\ Ty) \leq \delta d(x,\ y) + Ld(y,\ Tx);$$

Then T has a unique fixed point in X.

Definition 3.1.2 *[10] If there exist $\gamma\ \in\ [0,\ 1)$ and $L\ \geq\ 0$ such that for any $x, y\ \in\ X$, we have*

$$d(\mathcal{T}x,\ \mathcal{T}y) \leq \gamma d(x,\ y) + L\min(d(x,\ \mathcal{T}x),\ d(y,\ \mathcal{T}y),\ d(x,\ \mathcal{T}y),\ d(y,\ \mathcal{T}x),$$

then a self mapping $\mathcal{T}$ on a metric space $(X,\ d)$ is said to satisfy the condition (B).

Theorem 3.1.2 *[10] Let $(X,\ d)$ be a complete metric space and $T :\ X \to X$ be a satisfy the condition(B) such that there exists $\delta \in [0,\ 1[$ and $L \geq 0$ such that, for all $x,\ y \in\ X$,*

$$d(\mathcal{T}x,\ \mathcal{T}y) \leq \gamma d(x,\ y) + L\min(d(x,\ \mathcal{T}x), d(y, \mathcal{T}y),\ d(x, \mathcal{T}y), d(y,\ \mathcal{T}x),$$

Then T has a unique fixed point in X.

Definition 3.1.3 *[17] For each $x,\ y\ \in\ X$, a self mapping $\mathcal{T}$ on metric space $(X,\ d)$ is said to have a Ćiric type strong contraction if there exist $\delta \geq 0$ and $L \geq 0$ such that*

$$d(\mathcal{T}x,\ \mathcal{T}y) \leq \delta M(x, y) + Ld(y,\ \mathcal{T}x),$$

for any $x, y \in X$ where

$$M(x,\ y) = \max\{d(x,\ y), d(x,\ \mathcal{T}x), d(y,\ \mathcal{T}y), \frac{1}{2}(d(x,\ \mathcal{T}y) + d(y,\ \mathcal{T}x))\}.$$

Theorem 3.1.3 *[17] Let $(X,\ d)$ be a complete metric space and $T : X \to X$ be a strong Ćiric almost contraction, that is, a mapping for which there exist two constants $\delta \geq 0$ and $L \geq 0$ such that*

$$d(\mathcal{T}x,\ \mathcal{T}y) \leq \delta M(x,\ y) + Ld(y,\ \mathcal{T}x),$$

for any $x, y \in X$ where

$$M(x,\ y) = \max\{d(x,\ y), d(x,\ \mathcal{T}x), d(y,\ \mathcal{T}y), \frac{1}{2}(d(x,\ \mathcal{T}y) + d(y,\ \mathcal{T}x))\}.$$

$Fix(T) = x \in X : Tx = x \neq \emptyset$.

3.2 Hardy And Rogers Contraction

Definition 3.2.1 *[28] Let $(X,\ d)$ be a metric space. A mapping $T :\ X \to X$ is said to be a Hardy and Rogers contraction if there exist nonnegative numbers $\alpha,\ \beta,\ \gamma,\ \delta,\ \eta \geq 0$ with $\alpha + \beta + \gamma + \delta + \eta < 1$, and the following inequality holds:*

$$d(Tx,\ Ty) \leq\ \alpha\ d(x,\ y) + \beta d(x,\ Tx) + \gamma d(y,\ Ty) + \delta d(x,\ Ty) + \eta d(y,\ Tx),$$

for each $x, y \in X$.

Theorem 3.2.1 *[28] Let $(X,\ d)$ be a complete metric space and $T :\ X \to X$ be a Hardy and Rogers contraction.*
Then, T has a unique fixed point in X.

3.3 Admissible Mappings

3.3.1 α-Admissible Mappings

Denote with Ψ the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ.

Lemma 3.3.1 *[58] For every function $\psi : [0, +\infty) \to [0, +\infty)$ the following holds:*
if ψ is nondecreasing, then for each $t > 0$, $\lim_{n\to+\infty} \psi^n(t) = 0$ implies $\psi(t) < t$.

Definition 3.3.1 *[58] Let $(X,\ d)$ be a metric space and $T : X \to X$ be a given mapping. We say that T is an α - ψ-contractive mapping if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x,\ y)d(Tx,\ Ty) \leq \psi(d(x,\ y)), \qquad (1)$$

for all $x, y \in X$.

Remark 3.3.1 *If $T : X \to X$ satisfies the Banach contraction principle, then T is an $\alpha -\psi$-contractive mapping, where $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$ for all $t \geq 0$ and some $k \in [0, 1)$.*

Definition 3.3.2 *[58] Let $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that T is α-admissible if*

$$x,\ y \in X, \quad \alpha(x,\ y) \geq 1 \implies \alpha(Tx,\ Ty) \geq 1.$$

Example 3.3.1 *Let $X = (0, +\infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ by $Tx = \ln x$ for all $x \in X$ and*

$$\alpha(x,\ y) = \begin{cases} 2 & \text{if } x \geq y, \\ 0 & \text{if } x < y. \end{cases}$$

Then, T is α-admissible.

Example 3.3.2 *Let $X = [0, +\infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ by $Tx = \sqrt{x}$ for all $x \in X$ and*

$$\alpha(x,\ y) = \begin{cases} e^{x-y} & \text{if } x \geq y, \\ 0 & \text{if } x < y. \end{cases}$$

Then, T is α-admissible.

Remark 3.3.2 *In the setting of Examples (2.1) and (2.2) every nondecreasing self-mapping T is α-admissible.*

Theorem 3.3.1 *[58] Let $(X,\ d)$ be a complete metric space and $T : X \to X$ be an $\alpha - \psi$-contractive mapping satisfying the following conditions:*

1. *T is α-admissible;*

2. *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$;

3. T *is continuous.*

Then, T *has a fixed point, that is, there exists* $x^* \in X$ *such that* $Tx^* = x^*$.

Proof 5 *Let* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$.
Define the sequence $\{x_n\}$ *in* X *by*

$$x_{n+1} = Tx_n, \qquad \text{for all } n \in \mathbb{N}.$$

If $x_n = x_{n+1}$ *for some* $n \in \mathbb{N}$, *then* $x^* = x_n$ *is a fixed point for* T. *Assume that* $x_n \neq x_{n+1}$ *for all* $n \in \mathbb{N}$. *Since* T *is* α*-admissible, we have*

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1, \qquad \text{for all } n \in \mathbb{N}. \qquad (2)$$

Applying the inequality (1) with $x = x_{n-1}$ *and* $y = x_n$, *and using (2), we obtain*

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n)\, d(Tx_{n-1}, Tx_n) \leq \psi(d(x_{n-1}, x_n)).$$

By induction, we get $d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)), \quad$ *for all* $n \in \mathbb{N}$.
Fix $\varepsilon > 0$ *and let* $n(\varepsilon) \in \mathbb{N}$ *such that* $\sum_{n \geq n(\varepsilon)} \psi^n(d(x_0, x_1)) < \varepsilon$. *Let* $n, m \in \mathbb{N}$ *with* $m > n > n(\varepsilon)$, *using the triangular inequality, we obtain*

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \psi^k(d(x_0, x_1)) \\ &\leq \sum_{n \geq n(\varepsilon)} \psi^n(d(x_0, x_1)) < \varepsilon. \end{aligned}$$

Thus we proved that $\{x_n\}$ *is a Cauchy sequence in the metric space* (X, d). *Since* (X, d) *is complete, there exists* $x^* \in X$ *such that* $x_n \to x^*$ *as* $n \to +\infty$. *From the continuity of* T, *it follows that* $x_{n+1} = Tx_n \to Tx^*$ *as* $n \to +\infty$. *By the uniqueness of the limit, we get* $x^* = Tx^*$, *that is,* x^* *is a fixed point of* T.

In the next theorem, we omit the continuity hypothesis of T.

Theorem 3.3.2 *[58] Let (X, d) be a complete metric space and $T: X \to X$ be an $\alpha - \psi$-contractive mapping satisfying the following conditions:*

1. *T is α-admissible;*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
3. *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then, T has a fixed point.

Proof 6 *Following the proof of Theorem (3.3.1), we know that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). Then, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to +\infty$. On the other hand, from (2) and the hypothesis (iii), we have*

$$\alpha(x_n, x^*) \geq 1, \quad \text{for all } n \in \mathbb{N}. \qquad (3)$$

Now, using the triangular inequality, (1) and (3), we get

$$\begin{aligned} d(Tx^*, x^*) &\leq d(Tx^*, Tx_n) + d(x_{n+1}, x^*) \\ &\leq \alpha(x_n, x^*)\, d(Tx_n, Tx^*) + d(x_{n+1}, x^*) \\ &\leq \psi(d(x_n, x^*)) + d(x_{n+1}, x^*.) \end{aligned}$$

Letting $n \to +\infty$, since ψ is continuous at $t = 0$, we obtain $d(Tx^*, x^*) = 0$, that is, $Tx^* = x^*$.

Example 3.3.3 *Let $X = \mathbb{R}$ endowed with the standard metric $d(x, y) = \|x - y\|$ for all $x, y \in \mathbb{R}$. Define the mapping $T: X \to X$ by*

$$Tx = \begin{cases} 2x - \frac{3}{2} & \text{if } x > 1, \\ \frac{x}{2} & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x < 0. \end{cases}$$

At first, we observe that the Banach contraction principle cannot be applied in this case since we have

$$d(T1, T2) = 2 > 1 = d(2, 1).$$

Now, we define the mapping $\alpha: \ X \times X \to [0, +\infty)$ *by*

$$\alpha(x,\ y) = \begin{cases} 1 & \text{if } x, y \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly T *is an* $\alpha - \psi$*-contractive mapping with* $\psi(t) = t/2$ *for all* $t \geq 0$*. In fact, for all* $x, y \in X$*, we have*

$$\alpha(x,y)d(Tx,Ty) \leq \frac{1}{2}d(x,y).$$

Moreover, there exists $x_0 \in X$ *such that* $\alpha(x_0,\ Tx_0) \geq 1$*. In fact, for* $x_0 = 1$*, we have* $\alpha(1, T1) = \alpha(1, 1/2) = 1$

Obviously T *is continuous and so it remains to show that* T *is* α*-admissible. In so doing, let* $x, y \in X$ *such that* $\alpha(x,y) \geq 1$*. This implies that* $x, y \in [0,1]$ *and by the definitions of* T *and* α*, we have*

$$Tx = x/2 \in [0,1], \quad Ty = y/2 \in [0,1], \quad \textit{and} \quad \alpha(Tx,Ty) = 1.$$

Then, T *is* α*-admissible. Now, all the hypotheses of Theorem (3.3.1) are satisfied. Consequently,* T *has a fixed point. Note that Theorem (3.3.1) (also Theorem (3.3.2) guarantees only the existence of a fixed point but not the uniqueness. In this example, 0 and* $3/2$ *are two fixed points of* T*.*

Now, we give an example involving a function T that is not continuous.

Example 3.3.4 *Let* $X = \mathbb{R}$ *endowed with the standard metric* $d(x,\ y) = |x - y|$ *for all* $x, y \in \mathbb{R}$*. Define the mapping* $T: X \to X$ *by*

$$Tx = \begin{cases} 2x - \frac{3}{2} & \text{if } x > 1, \\ \frac{x}{4} & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x < 0. \end{cases}$$

It is clear that T *is not continuous at 1. Then, the Banach contraction principle and also Theorem (2.1) are not applicable in this case. Define the mapping* $\alpha: \ X \times X \to [0, \ +\infty)$ *by*

$$\alpha(x,\ y) = \begin{cases} 1 & \text{if } , x, y \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly T is an $\alpha-\psi$-contractive mapping with $\psi(t) = t/4$ for all $t \geq 0$. In fact, for all $x,\ y \in X$, we have

$$\alpha(x,\ y)d(Tx,\ Ty) \leq \frac{1}{4}d(x,\ y).$$

Moreover, there exists $x_0 \in X$ such that $\alpha(x_0,\ Tx_0) \geq 1$, and so, for $x_0 = 1$ we have

$$\alpha(1,\ T1) = \alpha(1, 1/4) = 1.$$

Now, let $x,\ y \in X$ such that $\alpha(x,\ y) \geq 1$. This implies that $x, y \in [0,1]$ and by the definitions of T and α, we have

$$Tx = x/4 \in [0,1], \quad Ty = y/4 \in [0,1], \quad \textit{and} \quad \alpha(Tx,\ Ty) = 1,$$

that is, T is α-admissible. Finally, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n,\ x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to +\infty$. Since $\alpha(x_n,\ x_{n+1}) \geq 1$ for all n, by the definition of α, we have $x_n \in [0,\ 1]$ for all n and $x \in [0,1]$. Then, $\alpha(x_n, x) = 1$. Therefore, all the required hypotheses of Theorem (3.3.2) are satisfied, and so T has a fixed point. Here, 0 and $3/2$ are two fixed points of T.

Now, we will show that Banach contraction principle can be deduced easily from our theorems.

Theorem 3.3.3 *[58] Let $(X,\ d)$ be a complete metric space and $T:\ X \to X$ be a given mapping satisfying*

$$d(Tx, Ty) \leq kd(x, y) \qquad \textit{for all} \qquad x, y \in X,$$

where $k \in [0,1)$. Then T has a fixed point.

Proof 7 *Let $\alpha:\ X \times X \to [0, +\infty)$ be the mapping defined by*

$$\alpha(x,\ y) = 1,$$

for all $x, y\ \in X$ and $\psi : [0,\ +\infty) \to [0,\ +\infty)$ defined by

$$\psi(t) = kt.$$

Then T is an $\alpha-\psi$-contractive mapping. It is easy to show that all the hypotheses of Theorems (3.3.1) are satisfied. Consequently, T has a fixed point.

Remark 3.3.3 *In Example (3.3.3), as we have said before, so we repeat, the Banach contraction principle cannot be applied since $d(T1, T2) > d(2,1)$. However, using our Theorem (3.3.1), we obtain the existence of a fixed point of T.*

3.4 Fixed Point Results In Ordered Metric Spaces

Theorem 3.4.1 *[58] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that the metric space (X, d) is complete. Let $T : X \to X$ be a continuous and nondecreasing mapping w.r.t. $\preceq$. Suppose that the following two assertions hold:*

1. *There exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq k(d(x, y))$ for all $x, y \in X$ with $x \preceq y$;*
2. *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
3. *T is continuous.*

Then T has a fixed point.

Proof 8 *Define the mapping $\alpha : X \times X \to [0, \ +\infty)$ by*

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

From (1), we have

$$\alpha(x, y)d(Tx, Ty) \leq kd(x, y), \quad \text{for all } x, y \in X.$$

Then T is an $\alpha - \psi$-contractive mapping with $\psi(t) = kt$ for all $t \geq 0$. Now, let $x, y \in X$ such that $\alpha(x, y) \geq 1$. By the definition of α, this implies that $x \preceq y$. Since T is a nondecreasing mapping w.r.t. $\preceq$, we have $Tx \preceq Ty$, which gives us that $\alpha(Tx, Ty) = 1$. Then T is α-admissible. From (2), there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. This implies that $\alpha(x_0, Tx_0) = 1$. Therefore, all the hypotheses of Theorem (3.3.1) are satisfied, and so T has a fixed point.

Remark 3.4.1 *A generalization of Theorem (3.4.1) can be given, replacing condition (1) with the following: (1′) There exists $\psi \in \Psi$ such that $d(Tx, Ty) \leq \psi(d(x, \ y))$ for all $x, y \in X$ with $x \preceq y$.*

Example 3.4.1 *Let $X = [0, \ +\infty)$ endowed with the standard metric $d(x, \ y) = |x - y|$ for all $x, \ y \in X$, and $T : \ X \to X$ be the mapping defined by*

$$Tx = 2x, \quad \text{for all } x \in X.$$

Clearly T is a continuous mapping. We endow X with the usual order $\leq$ in $\mathbb{R}$. Now, condition(1) of Theorem (3.4.1) is not satisfied for $x = 1 \leq 3 = y$. Then, we cannot apply Theorem (3.4.1) to prove the existence of a fixed point of T.

Define $\alpha: X \times X \to [0, +\infty)$ by

$$\alpha(x,\ y) = \begin{cases} 1/4 & \text{if} \quad (x, y) \neq (0, 0), \\ 1 & \text{if} \quad (x, y) = (0, 0). \end{cases}$$

It is clear that

$$\alpha(x,\ y)d(Tx,\ Ty) \leq \frac{1}{2}d(x,\ y), \quad \text{for all } x, y \in X.$$

Then T is an $\alpha - \psi$-contractive mapping with $\psi(t) = t/2$ for all $t \geq 0$. Now, let $x, y \in X$ such that $\alpha(x, y) \geq 1$. By the definition of T, this implies that $x = y = 0$. Then we have $\alpha(Tx,\ Ty) = \alpha(0,\ 0) = 1$, and so T is α-admissible. Also, for $x_0 = 0$, we have $\alpha\left(x_0,\ Tx_0\right) = 1$. Consequently, all the hypotheses of Theorem 3.3.1 are satisfied, then we deduce the existence of a fixed point of T. Here 0 is a fixed point of T.

Theorem 3.4.2 *[58] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric din X such that the metric space (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping w.r.t. $\preceq$. Suppose that the following assertions hold:*

1. *There exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$ with $x \preceq y$;*
2. *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
3. *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$ as $n \to +\infty$, then $x_n \preceq x$ for all n.*

Then T has a fixed point.

Proof 9 *Define the mapping $\alpha: X \times X \to [0, +\infty)$ by*

$$\alpha(x,\ y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

The reader can show easily that T is $\alpha - \psi$-contractive and α-admissible. Now, let $\{x_n\}$ be a sequence in X such that $\alpha\left(x_n, x_{n+1}\right) \geq 1$ for all n and $x_n \to x \in X$ as $n \to +\infty$. By the definition of α, we have $x_n \preceq x_{n+1}$ for all n. From (3), this implies that $x_n \preceq x$ for all

n, which gives us that $\alpha(x_n,\ x) = 1$ for all n. Thus all the hypotheses of Theorem (3.3.2) are satisfied, and T has a fixed point.

Remark 3.4.2 *Again, a generalization of Theorem (3.4.2) can be also given, replacing condition (1) with the following: (1′) There exists $\psi \in \Psi$ such that $d(Tx,\ Ty) \leq \psi(d(x,\ y))$ for all $x,\ y \in X$ with $x \preceq y$.*

Remark 3.4.3 *In the previous Example (3.4.1), also Theorem (3.4.2) cannot be applied since condition (1) is not satisfied.*

3.5 JS-Contractions Mappings

In 2014, Jleli and Samet [34] introduced a new type of single-valued contraction called θ-contraction and established new fixed point theorems for such a contraction in the context of generalized metric spaces.

Definition 3.5.1 *[34] Assume that Δ the set of all functions $\vartheta : (0, +\infty) \to (1, +\infty)$ satisfing:*

(ϑ_1) : *ϑ is non decreasing,*

(ϑ_2) : *for each sequence $\{\varepsilon_n\}$ in $(0, +\infty)$, $\lim\limits_{n\to\infty} \varepsilon_n = 1$ if and only if $\lim\limits_{n\to\infty} \varepsilon_n = 0$,*

(ϑ_3) : *there exist $\rho \in (0,\ 1)$ and $\varrho \in [0,\ \infty)$ such that $\lim\limits_{t\to 0^+} \dfrac{\vartheta(t) - 1}{t^\rho} = \varrho$.*

Example 3.5.1 *The functions listed below are components of Δ.*

1) $\vartheta_1(t) = e^t$.

2) $\vartheta_2(t) = e^{te^t}$.

3) $\vartheta_3(t) = e^{\sqrt{t}}$.

4) $\vartheta_4(t) = e^{\sqrt{t}e^t}$.

Definition 3.5.2 [34] *Let $(X,\ d)$ be a metric space. A map $T :\ X \to X$ is called θ-contraction, if there exists $k \in\ ]0,\ 1[$ such that for all $x, y \in X$,*

$$d(Tx,\ Ty) \neq 0 \Longrightarrow \theta(d(Tx,\ Ty)) \leqslant [\theta(d(x,\ y))]^k, \tag{3.1}$$

for each $x,\ y \in X$ with $d(Tx,\ Ty) > 0$.

Theorem 3.5.1 [34] *Let (X, d) be a complete metric space and $T : X \to X$ is θ-contraction, Then T has a fixed point in X.*

Definition 3.5.3 [34] *Let (X, d) be a metric space. a map $T : X \to X$ is called θ-contraction if there exists $\theta \in \Theta$ and $k \in (0, 1)$ such that*

$$\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k . \tag{3.2}$$

for all $x, y \in X, d(Tx, Ty) > 0$.

Theorem 3.5.2 [34] *Let (X, d) be a complete metric space, and $T : X \to X$ is a θ-contraction. Then T has a single fixed point in X.*

Remark 3.5.1 *The theorem 3.5.2 is a modified version of Banach's principle of contraction. Indeed, if $T : X \to X$ is a contractive Banach map with a contractive constant $\rho \in (0, 1)$, i.e.,*

$$d(Tx, Ty) \leq \lambda d(x, y), \forall x, y \in X.$$

We have $\theta(t) = e^{\sqrt{t}} \in \Delta, t > 0$, passing to the above inequality, we find

$$e^{\sqrt{d(Tx,\ Ty)}} \leq e^{\sqrt{\rho d(x,\ y)}} = \left[e^{\sqrt{d(x,\ y)}}\right]^{\sqrt{\rho}} = \left[e^{\sqrt{d(x,\ y)}}\right]^{k}, \forall x, y \in X,$$

where $k = \sqrt{\rho}$. We follow from theorem 3.5.2 that T has a fixed point in X.

3.6 Existence Of Fixed Point For Almost JS-Contractions In Dislocated b-Metric Space

Definition 3.6.1 *If there exist $\psi \in \Psi$, $\vartheta \in \Delta$, then a self mapping $\mathcal{T}$ on a dislocated b-metric space (X, σ_b, s) is called an almost $(\alpha, \psi, \vartheta)$-contraction of Hardy-Rogers type, where $\varepsilon \geq 1$ and $\kappa \in [0, 1)$, $L \geq 0$, $\alpha : X \times X \to [0, +\infty)$ and non-negative real numbers $a_i, i \in \{1, .., 5\}$ with $a_1 + a_2 + a_3 + 2sa_4 + 2sa_5 = 1$, $a_3 \neq 1$, such that $\sigma_b(\mathcal{T}x, \mathcal{T}y) > 0$ implies*

$$\alpha(x, y)\psi(\vartheta(s^{\varepsilon}\sigma_b(\mathcal{T}x, \mathcal{T}y))) \leq \psi\big([\vartheta(M(x, y))]^{\kappa}\big) + LN(x, y), \tag{3.3}$$

for all $x, y \in X$, with $\alpha(x, y) \geq 1$ where

$$M(x, y) = a_1\sigma_b(x, y) + a_2\sigma_b(x, \mathcal{T}x) + a_3\sigma_b(y, \mathcal{T}y) + a_4\sigma_b(x, \mathcal{T}y) + a_5\sigma_b(y, \mathcal{T}x),$$

and $N(x, y) = \min\{\sigma_b^s(x, \mathcal{T}y)), \sigma_b^s(y, \mathcal{T}x)\}$.

Theorem 3.6.1 *For a complete dislocated b-metric space (X, σ_b, s) and an almost $(\alpha, \psi, \vartheta)$-contraction of Hardy-Rogers type, if the following assertions satisfied:*

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, \mathcal{T}x_0) \geq 1$;*

(ii) *$\mathcal{T}$ is α-admissible;*

(iii) *X is α-regular, that is, for every sequence $\{x_n\}$ in X such that $x_n \to x$ and $\alpha(x_n,\ x_{n+1}) \geq 1$, then $\alpha(x_n,\ x) \geq 1$.*

Then, $\mathcal{T}$ has fixed point in X.

Proof 10 *From (i) there is x_0 such $\alpha(x_0, x_1) \geq 1$, where $x_1 = \mathcal{T}x_0$. If $x_0 = x_1$ then x_1 is a fixed point, suppose the contrary so $\sigma_b(\mathcal{T}x_0,\ \mathcal{T}x_1) = \sigma_b(x_1,\ \mathcal{T}x_1) > 0$, then* (4.1) *gives*

$$\psi(\vartheta(s^{\varepsilon}\sigma_b(\mathcal{T}x_0,\ \mathcal{T}x_1))) \leq \psi\big([\vartheta(M(x_0,\ x_1))]^{\kappa}\big) + L(N(x_0, x_1)).$$

$M(x_0,\ x_1) = a_1\sigma_b(x_0,\ x_1) + a_2\sigma_b(x_0,\ \mathcal{T}x_0) + a_3\sigma_b(x_1,\ \mathcal{T}x_1) + a_4\sigma_b(x_0,\ \mathcal{T}x_1) + a_5\sigma_b(x_1,\ \mathcal{T}x_0)$,

with $x_2 = \mathcal{T}x_1$, we get

$$\begin{aligned} M(x_0, x_1) &\leq (a_1 + a_2 + sa_4)\sigma_b(x_0, x_1) + (a_3 + sa_4)\sigma_b(x_1, x_2) + a_5\sigma_b(x_1, x_1) \\ &\leq (a_1 + a_2 + sa_4)\sigma_b(x_0, x_1) + (a_3 + sa_4 + 2sa_5)\sigma_b(x_1, x_2), \end{aligned}$$

and

$$N(x_0, x_1) = \min\{\sigma_b^s(x_0, x_2), \sigma_b^s(x_1, x_1)\} = 0.$$

Then we have

$$\psi(\vartheta(s^{\varepsilon}\sigma_b((x_1, x_2)x_1, x_2))) \leq \psi\big([\vartheta(M(x_0, x_1))]^{\kappa}\big) < \psi\big(\vartheta(M(x_0, x_1))\big),$$

since ψ and ϑ are non decreasing functions, we get

$$s^{\varepsilon}\sigma_b(x_1, x_2) < M(x_0, x_1) \leq (a_1 + a_2 + sa_4)\sigma_b(x_0, x_1) + (a_3 + sa_4 + 2sa_5)\sigma_b(x_1, x_2),$$

which implies that

$$\sigma_b(x_1, x_2) < \frac{(a_1 + a_2 + sa_4)}{s^{\varepsilon} - a_3 - sa_4 - 2sa_5}\sigma_b(x_0, x_1) \leq \frac{(a_1 + a_2 + sa_4)}{1 - a_3 - sa_4 - 2sa_5}\sigma_b(x_0, x_1) = \sigma_b(x_0, x_1).$$

Hence

$$\vartheta(\sigma_b(x_1, x_2) = \vartheta(\sigma_b(\mathcal{T}x_0,\ \mathcal{T}x_1)) < \big[\vartheta(\sigma_b(x_0,\ x_1))\big]^{\kappa},$$

which implies that

$$\psi\big(\vartheta(\sigma_b(x_1,\ x_2)\big) = \psi\big(\vartheta(\sigma_b(\mathcal{T}x_0,\ \mathcal{T}x_1))\big) < \psi(\Big[\vartheta(\sigma_b(x_0,\ x_1))\Big]^{\kappa}\big).$$

Since $\mathcal{T}$ is α-admissible we have $\alpha(x_1,\ x_2) \geq 1$, where $x_2 = \mathcal{T}x_1$. If $x_1 \neq x_2$ we get $\sigma_b(\mathcal{T}x_1,\ \mathcal{T}x_2) > 0$, then by using (4.1) *we get*

$$\psi(\vartheta(s^{\varepsilon}\sigma_b(\mathcal{T}x_1,\ \mathcal{T}x_2))) \leq \psi\big([\vartheta(M(x_1,x_2))]^{\kappa}\big) + L(N(x_1,x_2)),$$

as in the first step with $x_3 = \mathcal{T}x_2$, we obtain

$$\vartheta(\sigma_b(x_2,x_3) = \vartheta(\sigma_b(\mathcal{T}x_1,\mathcal{T}x_2) < \Big[\vartheta(\sigma_b(x_1,x_2))\Big]^{\kappa} < \Big[\vartheta(\sigma_b(x_0,x_1))\Big]^{\kappa^2}.$$

Which implies that

$$\psi(\vartheta(\sigma_b(x_2,x_3)) = \psi(\vartheta(\sigma_b(\mathcal{T}x_1,\mathcal{T}x_2)) < \psi(\Big[\vartheta(\sigma_b(x_0,x_1))\Big]^{\kappa^2}).$$

Maintaining this approach, we create a sequence (x_n) defined as $x_{n+1} = \mathcal{T}x_n$ for all n in $\mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, x_{n_0} is a fixed point. If $x_n \neq x_{n+1}$ for all n in $\mathbb{N}$ verifies $\alpha(x_n,\ x_{n+1}) \geq 1$ and $\sigma_b(x_n,\ x_{n+1}) > 0$, so we have

$$1 < \psi(\vartheta(\sigma_b(x_n,x_{n+1})) = \psi(\vartheta(\sigma_b(\mathcal{T}x_{n-1},\mathcal{T}x_n)) < \psi\Big(\big[\vartheta(\sigma_b(x_0,x_1))\big]^{\kappa^n}\Big).$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} \psi(\vartheta(\sigma_b(x_n,x_{n+1}))) = 1,$$

which implies that

$$\lim_{n\to\infty} \sigma_b(x_n,\ x_{n+1}) = 0.$$

We establish that $\{x_n\}$ is a Cauchy sequence by showing that, given (ϑ_3), there exist $\rho \in (0,1)$ and $\varrho \in [0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\vartheta(\sigma_b(x_n,x_{n+1})) - 1}{(\sigma_b(x_n,x_{n+1}))^{\rho}} = \varrho.$$

If $\varrho < \infty$, let $2\varepsilon = \varrho$, the definition of limit implied the existence of $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$\varepsilon = \varrho - \varepsilon \leq \frac{\vartheta(\sigma_b(x_n,x_{n+1}) - 1)}{(\sigma_b(x_n,x_{n+1})^{\rho})} = \varrho,$$

hence

$$\sigma_b(x_n,x_{n+1})^{\rho} \leq \frac{\vartheta(\sigma_b(x_n,x_{n+1})) - 1}{\varepsilon},$$

which implies

$$ns^{n\rho}(\sigma_b(x_n, x_{n+1}))^{\rho} \leq \frac{ns^{n\rho}(\vartheta(\sigma_b(x_0, x_1))^{\kappa^n} - 1)}{\varepsilon}. \tag{3.4}$$

If $\varrho = \infty$, if A is any positive real number, then according to the definition of the limit there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have

$$\frac{\vartheta(\sigma_b(x_n, x_{n+1})) - 1}{(\sigma_b(x_n, x_{n+1}))^{\rho}} > A,$$

which implies that

$$ns^{n\rho}(\sigma_b(x_n, x_{n+1}))^{\rho} \leq \frac{ns^{n\rho}(\vartheta(\sigma_b(x_0, x_1))^{\kappa^n} - 1)}{A}. \tag{3.5}$$

Letting $n \to \infty$ in (3.4)*(resp in* (3.5) *), we obtain*

$$\lim_{n\to\infty} ns^{n\rho}(\sigma_b(x_n, x_{n+1}))^{\rho} = 0.$$

According to the definition of the limit, for any $n \geq n_2$, there exists $n_2 \geq \max\{n_0, n_1\}$ such that

$$s^n\sigma_b(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{\rho}}}.$$

Then the series $\sum_{n=1}^{\infty} s^n\sigma_b(x_n, x_{n+1})$ is convergent, so its rest tends to 0, which implies that

for all $n \geq m \geq n_0$, we have

$$\sigma_b(x_n, x_m) \leq \sum_{i=n}^{m-1} s^{i+1-n}\sigma_b(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} s^{i+1-n}\sigma_b(x_i, x_{i+1}) \to 0.$$

This yields $\{x_n\}$ is a Cauchy sequence.
Since (X, σ_b) is complete, so $\{x_n\}$ converges to some $x \in X$ and we have

$$\lim_{n,m\to\infty} \sigma_b(x_n, x_m) = \lim_{n\to\infty} \sigma_b(x_n, x) = \sigma_b(x, x) = 0.$$

By (iii), we have $\alpha(x_n, x) \geq 1$, then using (4.1) we get

$$1 < \psi(\vartheta(\sigma_b(x_{n+1},\ \mathcal{T}x))) \leq \psi(\vartheta(s^{\epsilon}\sigma_b(x_{n+1}, \mathcal{T}x))) \leq \psi(\Big[\vartheta(M(x_n, x))\Big]^{\kappa}) < \psi(\vartheta(\sigma_b(x_n, x))).$$

Letting $n \to \infty$ we obtain

$$\lim_{n\to\infty} \psi(\vartheta(\sigma_b(x_{n+1}, \mathcal{T}x))) = 1,$$

so

$$\lim_{n\to\infty} \vartheta(\sigma_b(x_{n+1}, \mathcal{T}x)) = 1,$$

applying (ϑ_2) we obtain

$$\lim_{n\to\infty} \sigma_b(x_{n+1}, \mathcal{T}x)) = \sigma_b(x, \mathcal{T}x) = 0.$$

Then $x = \mathcal{T}x$.
So, $\mathcal{T}$ has a unique fixed point $x^* \in X$

If $\alpha(x,\ y) = 1$, for all $x,\ y \in X$ we get the following corollary.

Corollary 3.6.1 *Let $(X,\ \sigma_b, s)$ be a dislocated b-metric space and let $\mathcal{T} : X \to X$ be a self mapping such that for all $x,\ y \in X$, we have*

$$\vartheta(s^{\epsilon}\sigma_b(\mathcal{T}x,\ \mathcal{T}y)) \leq \big[\vartheta(M(x,\ y)\big]^{\kappa} + LN(x,y),$$

where $\epsilon \geq 1$, $\vartheta \in \Delta$ and $\kappa \in (0,1)$. Then $\mathcal{T}$ has a fixed point in X.

Corollary 3.6.2 *Let $(X, \sigma_b, \preceq)$ be a partially ordered dislocated b-metric space and let $\mathcal{T} : X \to X$ be an increasing self mapping such that for all $x,\ y \in X$ with $x \preceq y$, we have*

$$\vartheta(s^{\epsilon}\sigma_b(\mathcal{T}x,\ \mathcal{T}y) \leq \big[\vartheta(M(x,\ y)\big]^{\kappa} + L\ \min\{\sigma_b^s(x,\ \mathcal{T}y),\ \sigma_b^s(y, \mathcal{T}x)\},$$

where $\epsilon \geq 1$, $\vartheta \in \Delta$ and $\kappa \in (0,1)$.
If the following assertions hold:

1. *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.*

2. *For every nondecreasing sequence $\{x_n\}$ in X with $x_n \to x$, we have $x_n \preceq x$.*

Then $\mathcal{T}$ has a fixed point in X.

Proof 11 *Define a function $\alpha: \ X \times X \to \mathbb{R}_+$ by*

$$\alpha(x,\ y) = \begin{cases} 1, \text{if } \ x \preceq y, \\ 0, \text{otherwise.} \end{cases}$$

Now, we check the conditions of Theorem (4.3.1), in fact, the existence of $x_0 \in X$ with $x_0 \preceq \mathcal{T}x_0$ implies that $\alpha(x_0, \mathcal{T}x_0) = 1$, also the monotony of $\mathcal{T}$ implies that it is admissible. Consequently, all the conditions of Theorem (4.3.1) are satisfied, then $\mathcal{T}$ has a fixed point.

Example 3.6.1 *Let $X = \mathbb{R}$, $\epsilon \geq 1$ and $\sigma_b(x,\ y) = (x+y)^2$, then (X, σ_b, s) is a complete dislocated b-metric space with $s = 2$. Define $\mathcal{T}\colon X \to X$ and $\alpha : X \times X \to [0, +\infty)$ by*

$$\mathcal{T}x = \frac{2x}{3 \times 2^{(\frac{\epsilon}{2})}},$$

and

$$\alpha(x, y) = 1, \qquad \forall (x, y) \in X.$$

Taking $\psi(t) = \sqrt{t}$, $\vartheta(t) = e^t$, $a_1 = \frac{5}{6}$, $a_2 = \frac{1}{6}$, $a_3 = a_4 = a_5 = 0$, $L = 0$ and $k = \frac{4}{5}$.
Let x, y in X such that $\alpha(x, y) \geq 1$, so we have: $\mathcal{T}x, \mathcal{T}y \in X$ which implies that

$$\alpha(\mathcal{T}x, \mathcal{T}y) = 1 \geq 1.$$

Then $\mathcal{T}$ is α-admissible.
For $x, y \in X$ we have

$$\sqrt{e^{2^{\epsilon}\sigma_b\left(\mathcal{T}x,\mathcal{T}y\right)}} = \sqrt{e^{2^{\epsilon}\left(\frac{2x}{3\times 2^{(\frac{\epsilon}{2})}}+\frac{2y}{3\times 2^{(\frac{\epsilon}{2})}}\right)^2}} = \sqrt{e^{\frac{4}{9}\left(x+y\right)^2}} \leq \sqrt{e^{\frac{2}{3}\left(x+y\right)^2}} \leq \sqrt{\left(e^{\frac{5}{6}\left(x+y\right)^2}\right)^{\frac{4}{5}}}$$

$$\leq \sqrt{\left(e^{M(x,y)}\right)^{\frac{4}{5}}}.$$

Then all hypotheses of Theorem 4.3.1 hold, so $\mathcal{T}$ admits a fixed point, which is 0.

Example 3.6.2 *Let $X = [\frac{-1}{3}, 5]$, $\epsilon \geq 1$ and $\sigma_b(x, y) = (x+y)^2$, then (X, σ_b, s) is a complete dislocated b-metric space with $s = 2$. Define $\mathcal{T} : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ by*

$$\mathcal{T}x = \frac{x-1}{2^{(\frac{\epsilon}{2}+1)}} \qquad \forall x \in X,$$

and

$$\alpha(x, y) = 1, \qquad \forall (x, y) \in X.$$

Taking $\psi(t) = t$, $\vartheta(t) = e^t$, $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{2}$, $a_3 = a_4 = a_5 = 0$, $L = 0$ and $k = \frac{1}{2}$.
Let x, y in X such that $\alpha(x, y) \geq 1$, so we have: $\mathcal{T}x, \mathcal{T}y \in X$ which implies that

$$\alpha(\mathcal{T}x, \mathcal{T}y) = 1 \geq 1.$$

Then $\mathcal{T}$ is α-admissible.
For $x, y \in X$ we have

$$e^{2^{\epsilon}\sigma_b\left(\mathcal{T}x, \mathcal{T}y\right)} = e^{2^{\epsilon}\left(\frac{x-1}{2^{(\frac{\epsilon}{2}+1)}} + \frac{y-1}{2^{(\frac{\epsilon}{2}+1)}}\right)^2} \leq e^{2^{\epsilon}\left(\frac{x}{2^{(\frac{\epsilon}{2}+1)}} + \frac{y}{2^{(\frac{\epsilon}{2}+1)}}\right)^2} \leq e^{\frac{1}{4}\left(x+y\right)^2}$$

$$\leq e^{\left(\frac{1}{2}(x+y)^2\right)^{\frac{1}{2}}} \leq \left(e^{M(x,y)}\right)^{\frac{1}{2}}.$$

Then all hypotheses of Theorem (4.3.1) hold, so $\mathcal{T}$ has a fixed point, which is $\frac{-1}{2^{(\frac{\epsilon}{2}+1)}-1}$.

CHAPTER

4

FIXED POINT RESULTS IN DISLOCATED QUASI B-METRIC SPACES

In this chapter, using a class of functions in dislocated quasi b-metric spaces, we show a fixed point theorem for a self mapping satisfying a generalized contractive condition which the utilization of a class of simulation functions.

4.1 Some Basic Definitions And Concepts

4.1.1 Simulation Functions

Khojasteh et al. [41] defined a new class of simulation functions as follows.

Definition 4.1.1 *let $\Gamma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a function which satisfies the following assertions:*

(Γ_1) : $\Gamma(0,0) = 0$,

$(\Gamma_2):$ $\Gamma(\mathrm{s},\mathrm{t}) < \mathbf{t} - \mathbf{s}$,

$(\Gamma_3):$ *For two sequences* $(s_n),(t_n) \subset \mathbb{R}_+$ *such that* $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n > 0$, *then* $\lim_{n\to\infty} \sup \Gamma(\mathrm{s_n},\mathrm{t_n}) < 0$.

We denote the set of all simulation functions by $\widetilde{\Delta}$.

Example 4.1.1 *[58] The following functions* $\Gamma_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *satisfying* $(\Gamma_1)-(\Gamma_3)$, *for all* $\mathrm{s},\mathrm{t} \in [0,\infty)$:

1. $\Gamma_1(\mathrm{s},\mathrm{t}) = \underline{\vartheta}_1(t) - \underline{\vartheta}_2(s)$, *where* $\underline{\vartheta}_1; \underline{\vartheta}_2$ *are two continuous functions from* $\mathbb{R}_+$ *into itself such that* $\underline{\vartheta}_1(s) = \underline{\vartheta}_2(s) = 0$ *if and only if* $s = 0$ *and* $\underline{\vartheta}_1(s) < s < \underline{\vartheta}_2(s)$, *for all* $s > 0$.

2. $\Gamma_2(\mathrm{s},\mathrm{t}) = \eta(t) - s$, *where* η *is an upper semi continuous function from* $\mathbb{R}_+$ *into itself such that* $\eta(s) < s$ *and* $\eta(0) = 0$.

3. $\Gamma_3(\mathrm{s},\mathrm{t}) = \lambda t - s$, *where* $0 \leq \lambda < 1$ *and* $\mathrm{t},\mathrm{s} \in \mathbb{R}_+$.

4. $\Gamma_4(\mathrm{s},\mathrm{t}) = t - \frac{\varpi_1(\mathrm{s},\mathrm{t})}{\varpi_2(\mathrm{s},\mathrm{t}}s$, $\varpi_1, \varpi_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *are two continuous functions such that* $\varpi_1(\mathrm{s},\mathrm{t}) > \varpi_2(\mathrm{s},\mathrm{t})$, *for all* $s,t > 0$.

5. $\Gamma_5(\mathrm{s},\mathrm{t}) = t - \int_0^s \omega(\sigma)d\sigma$, *where* $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ *is a continuous function such that* $\int_0^\mu \omega(\sigma)d\sigma > \mu$ *for each* $\mu > 0$.

Definition 4.1.2 *[41] Let* (X,d) *be a metric space,* T *is a self-mapping and* $\Gamma \in \widetilde{\Delta}$. *Then* T *is called a* $\widetilde{\Delta}$*-contraction by respect to* Γ *if the following condition holds:*

$$\Gamma(d(Tx,Ty),\ d(x,y)) \geq 0,$$

where $x,\ y \in X$, *with* $x \neq y$.

Remark 4.1.1 *[41] If we take* $\Gamma(s,t) = \lambda t - s$ *for all* $s,t \in \mathbb{R}_+$ *with* $\lambda \in [0,1)$ *in above definition, then the* $\widetilde{\Delta}$*-contraction become to the Banach contraction.*

Remark 4.1.2 *It is clear from the definition simulation function that* $\Gamma(s,t) < 0$ *for all* $s \geq t > 0$. *Therefore, if* T *is a* $\widetilde{\Delta}$*-contraction with respect to* $\Gamma \in \widetilde{\Delta}$ *then*

$$d(Tx,\ Ty) < d(x,\ y) \qquad \text{for all distinct} \qquad x,y\ \in X.$$

This shows that every $\widetilde{\Delta}$ *mapping is contractive, therefore it is continuous.*

Theorem 4.1.1 *[41] Let (X, d) be a complete metric space and $T : X \to X$ be a $\widetilde{\Delta}$-contraction with respect to Γ. Then T has a unique fixed point u in X and for every $x_0 \in X$ the Picard sequence $\{x_n\}$; where $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ converges to the fixed point of T.*

Following example shows that the above theorem is a proper generalization of Banach contraction principle.

Example 4.1.2 *Let $X = [0,\ 1]$ and $d :\ X \times X \to \mathbb{R}$ be defined by $d(x,\ y) = |x - y|$. Then $(X,\ d)$ is a complete metric space. Define a mapping $T :\ X \to X$ as $Tx = \frac{x}{x+1}$ for all $x \in X$. T is a continuous function but it is not a Banach contraction. But it is a $\widetilde{\Delta}$-contraction with respect to $\Gamma \in \widetilde{\Delta}$, where*

$$\Gamma(s, t) = \frac{t}{t+1} - s \text{ for all } t, s \in [0, \infty).$$

Indeed, if $x, y \in X$, then

$$\Gamma(d(Tx, Ty), d(x, y)) = \frac{d(x, y)}{1 + d(x, y)} - d(Tx, Ty)$$

$$= \frac{|x - y|}{1 + |x - y|} - \left| \frac{x}{x+1} - \frac{y}{y+1} \right|$$

$$= \frac{|x - y|}{1 + |x - y|} - \left| \frac{|x - y|}{(x+1)(y+1)} \right| \geq 0.$$

Note that, all the conditions of Theorem (4.1.1) are satisfied and T has a unique fixed point $u = 0 \in X$.

Corollary 4.1.1 *(Banach Contraction principle)[41]*
Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying the following condition:

$$d(Tx, Ty) \leq \lambda d(x, y) \text{ for all } x, y \in X,$$

where $\lambda \in [0, 1)$. Then T has a unique fixed point in X.

Proof 12 *Define $\Gamma_B : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by*

$$\Gamma_B(s, t) = \lambda t - s \text{ for all } s, t \in [0, \infty).$$

Note that, the mapping T is a $\widetilde{\Delta}$-contraction with respect to $\Gamma_B \in \widetilde{\Delta}$. Therefore the result follows by taking $\Gamma = \Gamma_B$ in Theorem (4.1.1).

4.1.2 Admissible Mappings

Definition 4.1.3 *[58] Let $\alpha : X \times X \to [0, \infty)$ be a function and let T be a self-mapping on X. If*

$$\alpha(\xi, \zeta) \geq 1 \Rightarrow \alpha(\mathcal{T}\xi, \mathcal{T}\zeta) \geq 1,$$

for all $\xi, \zeta \in X$ then $\mathcal{T}$ is a α-admissible mapping.

Example 4.1.3 *Let $X = \mathbb{R}$. Define $T : X \to X$ and $\alpha :\ X \times X \to \mathbb{R}_+$ by*

$$Tx = \begin{cases} \ln|x|, & \text{if } x \neq= 0, \\ 3, & \text{if } x = 0, \end{cases}$$

and

$$\alpha(x,\ y) = \begin{cases} 3, & \text{if } x \geq y, \\ 0, & \text{if } x < y. \end{cases}$$

Then, T is α-admissible.

Example 4.1.4 *Let $X = [0,\ +\infty)$. Define $T :\ X \to X$ et $\alpha : X \times X \to [0,\ +\infty)$ by*

$$Tx = \sqrt{x} \qquad \text{for all} \qquad x \in X,$$

and

$$\alpha(x,\ y) = \begin{cases} e^{x-y}, & \text{if } x \geq y, \\ 0, & \text{if } x < y. \end{cases}$$

Then, T is α-admissible.

4.1.2.1 (α, β)-Admissible Mappings

Definition 4.1.4 *[23] Let X be a nonempty set, $\mathcal{T} :\ X \to X$ and $\alpha, \beta :\ X \times X \to [0, \infty)$. $\mathcal{T}$ is said to be an (α, β)-admissible if $\alpha(x,\ y) \geq 1$ and $\beta(x,\ y) \geq 1$ implies $\alpha(\mathcal{T}x,\ \mathcal{T}y) \geq 1$ and $\beta(\mathcal{T}x,\ \mathcal{T}y) \geq 1$, for all $x,\ y \in X$.*

Example 4.1.5 *Let $X = [0, \infty)$ endowed with the usual metric $d(x,\ y) = |x - y|$. Define*

$$Tx = \begin{cases} \frac{1-x^2}{8}, \text{if } & x \in [0, 1], \\ 9x, \text{if } & x \in (1, \infty), \end{cases}$$

$$\alpha(x,\ y) = \begin{cases} 1, \textit{if} \quad x \in [0,1], \\ 0, \textit{if} \quad x \in (1,\infty), \end{cases}$$

and

$$\beta(x,\ y) = \begin{cases} 1, \textit{if} \quad x \in [0,1], \\ 0, \textit{if} \quad x \in (1,\infty). \end{cases}$$

For $x \in [0,\ 1]$ *we have* $\alpha(x,y) \geq 1$ *and* $\beta(x,\ y) \geq 1$*.*
If $x \in [0,\ 1]$ *we have* $Tx < 1$*, so* $\alpha(x,y) \geq 1$ *and* $\beta(x,y) \geq 1$*.*

Lemma 4.1.1 *[18] Let* $\{x_n\}$ *be a sequence in a metric space* (X,d)*. If* $\{x_n\}$ *is not a Cauchy sequence, then there exist an* $\epsilon > 0$ *and two subsequences* $\{x_{n(k)}\}$ *and* $\{x_{m(k)}\}$ *of* $\{x_n\}$ *such that* $k \leq m(k) < n(k)$*,* $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$ *and* $d(x_{m(k)}, x_{n(k)-1})) < \epsilon$*,* $\forall k \in \mathbb{N}$*.*
Furthermore, $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon$*, provided* $\lim_{k\to\infty} d(x_n, x_{n+1}) = 0$*.*

Theorem 4.1.2 *[61] Let* (X,σ) *be a complete dislocated metric space and a continuous mapping* $T : X \to X$ *be a* $(\alpha,\ \beta)$*-admissible* $\widetilde{\Delta}$*-contraction with respect to* Γ *simulation function satisfying as*

$$\Gamma\left(\alpha(Tx,\ Ty)\beta(Tx,\ Ty)\sigma(Tx,\ Ty), \sigma(x,y)\right) \geq 0, \tag{1}$$

and there exist $x_0 \in X$ *such that* $\alpha\left(x_0,\ Tx_0\right) \geq 1, \beta\left(x_0,\ Tx_0\right) \geq 1$*. Then* T *has a unique fixed point* $u \in X$*.*

Proof 13 *Let* $\{x_n\}$ *be a sequence in* X *such that* $x_{n+1} = Tx_n$ *for all* $n = 0,1,2,\ldots$*. If* $x_n = x_{n+1}$ *then* $fx_n = x_{n+1} = x_n$ *i.e.* x_n *is a fixed point of* T*. So proof is trivial. Now, we consider* $x_n \neq x_{n+1}$ *for all* $n \in \mathbb{N}$*. Since* $\alpha\left(x_0,\ Tx_0\right) \geq 1 \Rightarrow \alpha\left(x_0, x_1\right) \geq 1$ *and* f *is an* (α,β)*-admissible, so*

$$\alpha\left(Tx_0, Tx_1\right) \geq 1 \textit{ implies } \alpha\left(x_1, x_2\right) \geq 1.$$

Continuing, we have for all $n \geq 0$

$$\alpha\left(x_n,\ x_{n+1}\right) \geq 1. \tag{2}$$

Similarly for all $n \geq 0$*, we obtain*

$$\beta\left(x_n,\ x_{n+1}\right) \geq 1. \tag{3}$$

From (1), we have

$$0 \leq \Gamma\Big(\alpha\left(Tx_{n-1}, Tx_n\right)\beta\left(Tx_{n-1}, Tx_n\right)\sigma\left(Tx_{n-1}, Tx_n\right), \sigma\left(x_{n-1}, x_n\right)\Big)$$

$$= \Gamma\Big(\alpha\left(x_n, x_{n+1}\right)\beta\left(x_n, x_{n+1}\right)\sigma\left(x_n, x_{n+1}\right), \sigma\left(x_{n-1}, x_n\right)\Big)$$

$$< \sigma\left(x_{n-1}, x_n\right) - \alpha\left(x_n, x_{n+1}\right)\beta\left(x_n, x_{n+1}\right)\sigma\left(x_n, x_{n+1}\right)$$

$$\alpha\left(x_n, x_{n+1}\right)\beta\left(x_n, x_{n+1}\right)\sigma\left(x_n, x_{n+1}\right) < \sigma\left(x_{n-1}, x_n\right). \tag{4}$$

We know

$$\sigma\left(x_n, x_{n+1}\right) \leq \alpha\left(x_n, x_{n+1}\right)\beta\left(x_n, x_{n+1}\right)\sigma\left(x_n, x_{n+1}\right), \tag{5}$$

since $\alpha\left(x_n, x_{n+1}\right) \geq 1$ *and* $\beta\left(x_n, x_{n+1}\right) \geq 1$ *From (4) and (5) for all* $n \geq 0$*, we have*

$$\sigma\left(x_n, x_{n+1}\right) \leq \alpha\left(x_n, x_{n+1}\right)\beta\left(x_n, x_{n+1}\right)\sigma\left(x_n, x_{n+1}\right) < \sigma\left(x_{n-1}, x_n\right), \tag{6}$$

then

$$\sigma\left(x_n, x_{n+1}\right) < \sigma\left(x_{n-1}, x_n\right). \tag{7}$$

The sequence $\{\sigma\left(x_n, x_{n+1}\right)\}$ *is non increasing i.e. decreasing. So, there exist* $r \geq 0$ *such that*

$$\lim_{n\to\infty} \sigma\left(x_{n-1}, x_n\right) = r.$$

We prove that

$$\lim_{n\to\infty} \sigma\left(x_{n-1}, x_n\right) = 0. \tag{8}$$

Now, we assume on the contrary such that $r > 0$*. By (6), we have*

$$\lim_{n\to\infty} \{\alpha\left(x_n, x_{n+1}\right)\beta\left(x_n, x_{n+1}\right)\sigma\left(x_n, x_{n+1}\right)\} = r.$$

Since $r > 0$ *and letting* $s_n = \alpha\left(x_n, x_{n+1}\right)\beta\left(x_n, x_{n+1}\right)\sigma\left(x_n, x_{n+1}\right)$ *and* $t_n = \sigma\left(x_n, x_{n+1}\right)$ *such that* $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = r$*. then by* (Γ_3)

$$\lim_{n\to\infty} \sup \Gamma\left(s_n, t_n\right) < 0.$$

Since $\Gamma\left(s_n, t_n\right) \geq 0$*, so*

$$0 \leq \lim_{n\to\infty} \sup \Gamma\left(s_n, t_n\right) < 0.$$

Which is contradiction. So our assumption is false. Hence $r = 0$. Again we show that $\{x_n\}$ is a cauchy sequence in (X, σ) i.e.

$$\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0. \tag{9}$$

Suppose on the contrary i.e. $\{x_n\}$ is not a cauchy sequence. Then there exist $\varepsilon > 0$ for which we can assume subsequences $x_{n(k)}$ and $x_{m(k)}$ of x_n with $n(k) > m(k) > k$ such that for every $k \in \mathbb{N}$

$$\sigma\left(x_{n(k)}, x_{m(k)}\right) \geq \varepsilon, \tag{10}$$

and $n(k)$ is the smallest number such that (10) holds. From (10), we get

$$\sigma\left(x_{n(k)-1},\ x_{m(k)}\right) < \varepsilon. \tag{11}$$

Then by triangular inequality and (10), we have

$$\varepsilon \leq \sigma\left(x_{n(k)},\ x_{m(k)}\right) \leq \sigma\left(x_{n(k)},\ x_{n(k)-1}\right) + \sigma\left(x_{n(k)-1},\ x_{m(k)}\right)$$

$$< \sigma\left(x_{n(k)},\ x_{n(k)-1}\right) + \varepsilon.$$

Taking $n \to \infty$ in above equation and applying (8), we get

$$\varepsilon \leq \lim_{n\to\infty} \sigma\left(x_{n(k)}, x_{m(k)}\right) < \varepsilon.$$

Hence

$$\lim_{n\to\infty} \sigma\left(x_{n(k)}, x_{m(k)}\right) = \varepsilon. \tag{12}$$

From the triangular inequality, we have

$$\sigma\left(x_{n(k)+1}, x_{m(k)}\right) \leq \sigma\left(x_{n(k)+1}, x_{n(k)}\right) + \sigma\left(x_{n(k)}, x_{m(k)}\right)$$

taking limit $n \to \infty$ and using (8), (10) and (12) we have

$$\lim_{n\to\infty} \sigma\left(x_{n(k)+1}, x_{m(k)}\right) = \varepsilon. \tag{13}$$

Similarly, it is easy to show that

$$\lim_{n\to\infty} \sigma\left(x_{n(k)+1}, x_{m(k)+1}\right) = \varepsilon. \tag{14}$$

Since T is an (α, β)-admissible $\widetilde{\Delta}$-contraction with respect to Γ and using (Γ_3) we have

$$0 \leq \lim_{n\to\infty} \sup \Gamma\Big(\alpha\left(Tx_{n(k)}, Tx_{m(k)}\right)\beta\left(Tx_{n(k)}, Tx_{m(k)}\right)\sigma\left(Tx_{n(k)}, Tx_{m(k)}\right), \sigma\left(x_{n(k)}, x_{m(k)}\right)\Big) < 0.$$

Hence

$$0 \leq \lim_{n\to\infty} \sup \Gamma\Big(\alpha\left(x_{n(k)+1}, x_{m(k)+1}\right)\beta\left(x_{n(k)+1}, x_{m(k)+1}\right)\sigma\left(x_{n(k)+1}, x_{m(k)+1}\right), \sigma\left(x_{n(k)}, x_{m(k)}\right)\Big) < 0.$$

Which is contradict due to our assumption. So $\{x_n\}$ is a cauchy sequence. Since (X, σ) be a complete metric-like space, then there exist $x \in X$ and using (9) such that

$$\lim_{n\to\infty} \sigma\left(x_n, x\right) = \sigma(x, x) = \lim_{n,m\to\infty} \sigma\left(x_n, x_m\right) = 0. \tag{15}$$

We show that x is fixed point of T. Since T is continuous and $x_n \to x$ as $n \to \infty$. So, from (15)

$$\lim_{n\to\infty} \sigma\left(x_{n+1},\ Tx\right) = \lim_{n\to\infty} \sigma\left(Tx_n,\ Tx\right) = \sigma(Tx,\ Tx) = 0. \tag{16}$$

Using (16), we have

$$\lim_{n\to\infty} \sigma\left(x_{n+1}, Tx\right) = \sigma(x, Tx). \tag{17}$$

From (16) and (17)

$$\sigma(x, Tx) = \sigma(Tx, Tx) = 0. \tag{18}$$

Hence $Tx = x$, that is x is a fixed point of T. Now, we shall show that x is unique. We argue by contrary. Assume that there exist $y \in X$ such that $Ty = y$ and $x \neq y$. Now,

$$0 \leq \Gamma\Big(\alpha(Tx, Ty)\beta(Tx, Ty)\sigma(Tx, Ty), \sigma(x, y)\Big)$$

$$\leq \sigma(x, y) - \alpha(Tx, Ty)\beta(Tx, Ty)\sigma(Tx, Ty)$$

$$= \sigma(x, y) - \alpha(x, y)\beta(x, y)\sigma(x, y)$$

$$= \sigma(x, y)\left[1 - \alpha(x, y)\beta(x, y)\right] < 0.$$

Since $\alpha(x,\ y) \geq 1, \beta(x,\ y) \geq 1$. Which is contradiction. So $x = y$, hence T has a unique fixed point.

Corollary 4.1.2 *Let (X, σ) be a complete metric-like space and let T be a self-mapping on X satisfying the following conditions:*

1. *T is (α, β)-admissible $\widetilde{\Delta}$-contraction;*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;*
3. *$\alpha(Tx,\ Ty)\beta(Tx,\ Ty)\sigma(Tx,\ Ty) \leq \lambda\sigma(x, y)$, for all $x, y \in X$ and $\lambda \in [0, 1)$;*
4. *T is continuous.*

Then T has a unique fixed point $u \in X$ with $\sigma(u, u) = 0$.

Example 4.1.6 *let $X = [0, \infty)$ endowed with the metric like $\sigma(x, y) = x + y$. Consider the mapping $T : X \to X$ given by*

$$Tx = \begin{cases} \frac{x}{3}, & \text{if } x \in [0, 1], \\ 3x, & \text{otherwise}. \end{cases}$$

Note that (X, σ) is complete metric-like space. Define mappings $\alpha, \beta\ : X \times X \to \mathbb{R}_+$ by

$$\alpha(x,\ y) = \beta(x,\ y) = \begin{cases} 1, & \text{if } x \in [0, 1], \\ 0, & \text{otherwise}. \end{cases}$$

Note that T is an $(\alpha,\ \beta)$-admissible if $\alpha(x,\ y) \geq 1$ and $\beta(x,\ y) \geq 1 \Rightarrow \alpha(Tx,\ Ty) \geq 1$ and $\beta(Tx,\ Ty) \geq 1$, for all $x,\ y \in X$. By definition of α, β and $x,\ y \in [0,\ 1]$, we have $\alpha(Tx,\ Ty) = \beta(Tx,\ Ty) = 1$ From above, it is clear that T is an (α, β)-admissible mapping. Let $\Gamma(t, s) = \lambda s - t,\ \lambda \in [0, 1]$, for all $s, t \geq 0$. Also, for $x, y \in X$ such that $\alpha(x, y) \geq 1$ and $\beta(x, y) \geq 1$. So, $x, y \in [0, 1]$. In this case we have

$$\Gamma\Big(\alpha(Tx, Ty)\beta(Tx, Ty)\sigma(Tx, Ty), \sigma(x, y)\Big) = \lambda(x + y) - (\frac{x}{3} + \frac{y}{3}).$$

If we take $\lambda = \frac{1}{2}$, we get

$$\frac{x + y}{2} - (\frac{x}{3} + \frac{y}{3}) \geq 0,$$

ie

$$\Gamma\Big(\alpha(Tx, Ty)\beta(Tx, Ty)\sigma(Tx, Ty), \sigma(x, y)\Big) \geq 0.$$

Thus, all the conditions of corollary are verified. Here $x = 0$ is the unique fixed point of T.

Let Ψ be the set of all continuous functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

- $(\psi 1) : \psi(0) = 0$;
- $(\psi 2) : \psi(\text{t}) < \text{t}$, for all $t > 0$;
- $(\psi 3) : \psi$ is non decreasing.

Example 4.1.7 *Let $\psi_j : [0, +\infty) \to [0, +\infty)$, for $j = 1, 2$ be continuous functions:*

1. *The function $\psi_1(t) = at$ where $0 < a < 1$.*
2. *$\psi_2(t) = \frac{t}{t+1}$ for all $t > 0$.*

Then $\psi_1, \psi_2 \in \Psi$.

Let Φ denote the set of functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

- ϕ is continuous.
- Satisfying $\phi(0) = 0$.

Example 4.1.8 *Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be functions such that*

- *$\phi_1(t) = \alpha t$ with $0 < \alpha < 1$.*
- *$\phi_2(t) = \sqrt{t}$.*

So ϕ_1, $\phi_2 \in \Phi$.

4.2 Fixed Point Theorems For (ψ, φ)-Contractions

In 1969, Boyd and Wong [20] proved the following generalization of the Banach contraction principle.

Theorem 4.2.1 *(Boyd–Wong) Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping satisfying the following contractive condition:*

$$d(Tx, Ty) \leq \varphi(d(x, y)) \qquad \textit{for all} \quad x, y \in X,$$

where $\varphi : [0, \infty) \to [0, \infty)$ satisfies the following conditions:

1. $\varphi(t) < t$ *for all* $t > 0$;

2. $\lim_{s\to t^+} \sup \varphi(s) < t$ *for all* $t > 0$.

Then T *has a unique fixed point.*

Theorem 4.2.2 *[54] Let* (X, d) *be a metric space and* $T : X \to X$ *be a mapping satisfying a contractive-type condition*

$$\psi(d(Tx, Ty)) \leq \varphi(d(x, y)),$$

for all $x, y \in X$ *with* $d(Tx, Ty) > 0$,
where the functions $\psi, \varphi : (0, \infty) \to \mathbb{R}$ *satisfy the following conditions:*

1. ψ *is nondecreasing;*

2. $\varphi(t) < \psi(t)$ *for any* $t > 0$;

3. $\lim_{t\to\epsilon^+} \sup \varphi(t) < \psi(\epsilon^+)$ *for any* $\epsilon > 0$.

Then T *has a unique fixed point.*

4.3 Existence Of Fixed Point For Generalized Contractions In Dislocated Quasi b-Metric Space

Definition 4.3.1 *Let* $(X,\ \sigma_q, s)$ *be a complete dislocated b-metric space and* $\epsilon > 1$. *And* $\alpha, \beta : X \times X \to \mathbb{R}_+$ *be two functions, and* $\mathcal{T} : X \to X$ *be a given self map. We say that* T *is called a generalized* $(\alpha, \beta, \epsilon, \psi, \phi, \Gamma)$ *contraction mapping, if there exist a function* $\psi \in \Psi$, $\phi \in \Phi$ *and* $\Gamma \in \widetilde{\Delta}$ *such that:*

$$\alpha(x, y)\beta(x, y)\psi(s^{\epsilon}\sigma_q(\mathcal{T}x, \mathcal{T}y)) \leq \Gamma\Big(\phi(\sigma_q(x, y)), \psi(\sigma_q(x, y))\Big), \tag{4.1}$$

.

Theorem 4.3.1 *Let* (X, σ_q, s) *be a complete dislocated b-metric space and* $\mathcal{T} : X \to X$ *generalized* $(\alpha, \beta, \epsilon, \psi, \phi, \Gamma)$ *contraction mapping, with* $\psi \in \Psi$ *and* $\Gamma \in \widetilde{\Delta}$.
Suppose that the following conditions are satisfied.

(H_1) : $\mathcal{T}$ *is* (α, β)*-admissible.*

(H_2) : *There exists* $x_0 \in X$ *such that* $\alpha(x_0, \mathcal{T}x_0) \geq 1$ *and* $\beta(x_0, \mathcal{T}x_0) \geq 1$.

(H_3) : $\mathcal{T}$ *is continuous.*

Then $\mathcal{T}$ *has a fixed point.*

Proof 14 *From hypothesis* (H_2) *there is* $x_0 \in X$ *such that* $\alpha\left(x_0,\ x_1\right) \geq 1$ *and* $\beta\left(x_0,\ \mathcal{T}x_0\right) \geq 1$*, where* $x_1 = \mathcal{T}x_0$*. If* $x_1 = x_0$ *so* x_0 *is a fixed point. Assume* $x_0 \neq x_1$*, then by putting* $x_2 = \mathcal{T}x_1$*, the equation (4.1) gives*

$$\begin{aligned}
\psi(s^{\epsilon}\sigma_q(x_1,\ x_2)) = \psi(s^{\epsilon}\sigma_q(\mathcal{T}x_0,\ \mathcal{T}x_1)) &\leq \alpha(x_0,\ x_1)\beta(x_0,\ x_1)\psi(s^{\epsilon}\sigma_q(\mathcal{T}x_0,\ \mathcal{T}x_1)) \\
&\leq \Gamma\Big(\phi(\sigma_q(x_0,\ x_1)), \psi(\sigma_q(x_0, x_1))\Big) \\
&< \psi(\sigma_q(x_0,\ x_1)) - \phi(\sigma_q(x_0, x_1)) \\
&\leq \psi(\sigma_q(x_0,\ x_1)).
\end{aligned}$$

Since ψ *is non decreasing, we get*

$$\sigma_q(x_1, x_2) \leq \frac{1}{s^{\epsilon}}\sigma_q(x_0, x_1).$$

Since $\mathcal{T}$ *is* (α, β)*-admissible, for* $\alpha(x_0, x_1) \geq 1$ *and* $\beta(x_0, x_1) \geq 1$ *implies that* $\alpha(x_1, x_2) \geq 1$ *and* $\beta(x_1, x_2) \geq 1$*; then by putting* $x_3 = \mathcal{T}x_2$*, the equation (4.1) gives*

$$\begin{aligned}
\psi(s^{\epsilon}\sigma_q(x_2, x_3)) = \psi(s^{\epsilon}\sigma_q(\mathcal{T}x_1, \mathcal{T}x_2)) &\leq \alpha(x_1, x_2)\beta(x_1, x_2)\psi(s^{\epsilon}\sigma_q(\mathcal{T}x_1, Tx_2)) \\
&\leq \Gamma\Big(\phi(\sigma_q(x_1, x_2)), \psi(\sigma_q(x_1, x_2))\Big) \\
&< \psi(\sigma_q(x_1, x_2)) - \phi(\sigma_q(x_1, x_2)) \\
&\leq \psi(\sigma_q(x_1, x_2)),
\end{aligned}$$

which implies that

$$\sigma_q(x_2, x_3) < \frac{1}{s^{\epsilon}}\sigma_q(x_1, x_2) < \frac{1}{s^{2\epsilon}}\sigma_q(x_0, x_1).$$

Continuing in this manner, we construct a sequence (x_n) *with* $x_{n+1} = \mathcal{T}x_n$ *for all* n *in* $\mathbb{N}$. *If there exists* $n_0 \in \mathbb{N}$ *such that* $x_{n_0} = x_{n_0+1}$, *then* x_{n_0} *is a fixed point. If* $x_n \neq x_{n+1}$ *for all* n *in* $\mathbb{N}$, *we have* $\alpha(x_n, x_{n+1}) \geq 1 \quad \beta(x_n, x_{n+1}) \geq 1$ *and* $\sigma_q(x_n,\ x_{n+1}) > 0$, *then from (4.1), we get*

$$\begin{aligned}\psi(s^{\epsilon}\sigma_q(x_n, x_{n+1})) &\leq \alpha(x_{n-1}, x_n)\beta(x_{n-1}, x_n)\psi(s^{\epsilon}\sigma_q(\mathcal{T}x_{n-1}, \mathcal{T}x_n))\\ &\leq \Gamma\Big(\phi(\sigma_q(x_{n-1}, x_n)), \psi(\sigma_q(x_{n-1}, x_n))\Big)\\ &< \psi(\sigma_q(x_{n-1}, x_n)) - \phi(\sigma_q(x_{n-1}, x_n))\\ &\leq \psi(\sigma_q(x_{n-1}, x_n)).\end{aligned}$$

Since ψ *is non decreasing, we get:*

$$\sigma_q(x_n, x_{n+1}) < \frac{1}{s^{\epsilon}}\sigma_q(x_{n-1}, x_n),$$

which implies that

$$\sigma_q(x_n, x_{n+1}) < \frac{1}{s^{n\epsilon}}\sigma_q(x_0, x_1).$$

Passing to the limit we get

$$\lim_{n\to\infty} \sigma_q(x_n,\ x_{n+1}) = 0.$$

Using the same argument, we get

$$\sigma_q(x_{n+1}, x_n) < \frac{1}{s^{n\epsilon}}\sigma_q(x_1, x_0),$$

which gives

$$\lim_{n\to\infty} \sigma_q(x_{n+1}, x_n) = 0.$$

Now we show (x_n) is a Cauchy sequence, in fact, for each $n, m \in \mathbb{N}$ with $m > n$ we get

$$\sigma_q(x_n, x_m) \leq s\sigma_q(x_n, x_{n+1}) + s^2\sigma_q(x_{n+1}, x_{n+2}) + s^3\sigma_q(x_{n+2}, x_{n+3}) + ... + s^{m-n}\sigma_q(x_{m-1}, x_m)$$

$$\leq \sigma_q(x_0, x_1)s^{-n\epsilon+1}\big(1 + (\frac{s}{s^\epsilon}) + (\frac{s}{s^\epsilon})^2 + ... + (\frac{s}{s^\epsilon})^{m-n-1}\big)$$

$$= \sigma_q(x_0, x_1)s^{-n\epsilon+1}\sum_{i=0}^{m-n-1}(\frac{s}{s^\epsilon})^i \leq \sigma_q(x_0, x_1)s^{-n\epsilon+1}\sum_{i=0}^{\infty}(\frac{1}{s^{\epsilon-1}})^i.$$

Since $\varepsilon > 1$ then the geometric series $\sum_{i=0}^{\infty}(\frac{s}{s^\epsilon})^i$ converges and so $\sigma_q(x_n, x_m)) \to 0$. Consequently, (x_n) is a left 0-Cauchy sequence.
By the same argument, we get

$$\lim_{n,m\to\infty}\sigma_q(x_m, x_n) = 0,$$

which implies that $\{x_n\}$ is a right 0-Cauchy sequence. Hence (x_n) is a Cauchy sequence.

The completeness of (X, σ_q) implies the convergence of (x_n) to some point $x \in X$.
So, we have

$$\sigma_q(x, x) = \lim_{n,m\to\infty}\sigma_q(x_n, x_m) = \lim_{n\to\infty}\sigma_q(x_n, x) = \lim_{n\to\infty}\sigma_q(x, x_n) = 0.$$

Since T is continuous, we get

$$\mathcal{T}x = \lim_{n\to\infty}\mathcal{T}x_n = \lim_{n\to\infty}x_{n+1} = x,$$

Then $x = \mathcal{T}x$.

Remark 4.3.1 *Under the hypotheses of Theorem4.3.1 the fixed point is unique provided, one of the following conditions is satisfied:*

1. *For all $x, y \in X$ we have $\alpha(x, y) \geq 1$ and $\beta(x, y) \geq 1$;*

2. *for each $x, y \in Fix(\mathcal{T})$, we have $\alpha(x, z) \geq 1$ and $\beta(x, y) \geq 1$.*

In fact, suppose there exists two fixed point, x and $\overline{x}$, then from additionally condition and (4.1), we get

$$\begin{aligned}\psi(s^{\epsilon}\sigma_q(\mathcal{T}x,\mathcal{T}\overline{x})) \leq \alpha(x,\overline{x})\beta(x,\overline{x})\psi(s^{\epsilon}\sigma_q(\mathcal{T}x,\mathcal{T}\overline{x})) &\leq \Gamma\Big(\phi(\sigma_q(x,\overline{x})),\psi(\sigma_q(x,\overline{x}))\Big)\\ &< \psi(\sigma_q(x,\overline{x}))-\phi(\sigma_q(x,\overline{x}))\\ &\leq \psi(\sigma_q(x,\overline{x})),\end{aligned}$$

which implies that

$$\sigma_q(x,\overline{x}) \leq \frac{1}{s^{\epsilon}}\sigma_q(x,\overline{x}),$$

so

$$\sigma_q\left(x,\overline{x}\right)=0.$$

Then $x=\overline{x}$.

Corollary 4.3.1 *Let (X,σ_q,s) be a complete dislocated quasi b-metric space and $\epsilon>1$. Let $\mathcal{T}: X\to X$ be a mapping such that for $x,y\in X$:*

$$\psi(s^{\epsilon}\sigma_q(\mathcal{T}x,\mathcal{T}y)) \leq \psi(\sigma_q(x,y))-\phi(\sigma_q(x,y)),$$

suppose that the following conditions are satisfied.

1. *$\mathcal{T}$ is (α,β)-admissible.*
2. *There exists $x_0\in X$ such that $\alpha(x_0,\mathcal{T}x_0)\geq 1$ and $\beta(x_0,\mathcal{T}x_0)\geq 1$.*
3. *$\mathcal{T}$ is continuous.*

Then $\mathcal{T}$ has a fixed point.

Corollary 4.3.2 *Let (X,σ_q,s) be a complete dislocated quasi b-metric space, and $\epsilon>1$. Let $\mathcal{T}: X\to X$ be a mapping such that for $x,y\in X$:*

$$s^{\epsilon}\sigma_q(\mathcal{T}x,\mathcal{T}y) \leq \psi(\sigma_q(x,y)),$$

suppose that the following conditions are satisfied.

1. *$\mathcal{T}$ is (α,β)-admissible;*

2. *There exists* $x_0 \in X$ *such that* $\alpha(x_0, \mathcal{T}x_0) \geq 1$. *and* $\beta(x_0, \mathcal{T}x_0) \geq 1$;

3. $\mathcal{T}$ *is continuous.*

Then $\mathcal{T}$ *has a fixed point.*

Example 4.3.1 *Let* $X = [0, \infty)$ *and* $\sigma_q(x,\ y) = (x-y)^2 + (2x+y)^2$, *then* (X, σ_q, s) *is a complete dislocated b-metric space with* $s = 2$. *Define* $\mathcal{T}\colon X \to X$ *and*
$\alpha, \beta : X \times X \to [0, +\infty)$ *by*

$$\mathcal{T}x = \frac{2x+1}{2^{3\epsilon+1}},$$

with $\epsilon \geq 1$ *and*

$$\alpha(x,\ y) = \beta(x,\ y) = \begin{cases} 1, & x + 2y \geq 2, \\ 0, & \textit{otherwise.} \end{cases}$$

Taking $\psi(t) = \frac{1}{4}t$, $\phi(t) = \frac{1}{2^4}t$, *and* $\Gamma(r,t) = \frac{1}{2}t - r$.
Let x, y *in* X *such that* $\alpha(x,y) = \beta(x,y) \geq 1$, *so* $2x + y \geq 2$ *and*

$$\begin{aligned} \alpha(x,y)\beta(x,y)\psi\left(s^{\epsilon}\sigma_q(\mathcal{T}x, \mathcal{T}y)\right) &= \psi\left(2^{\epsilon}\sigma_q(\mathcal{T}x, \mathcal{T}y)\right) \\ &= \psi\left(2^{\epsilon}\left(\frac{(x-y)^2 + (4x+2y+3)^2}{2^{6\epsilon+2}}\right)\right) \\ &= \psi\left(\frac{1}{2^{5\epsilon+2}}((x-y)^2 + (\frac{4x+2y+3}{2})^2\right) \\ &= \frac{1}{2^{5\epsilon+4}}\left((x-y)^2 + (\frac{4x+2y+3}{2})^2\right), \end{aligned}$$

since

$$\left(\frac{4x+2y+3}{2}\right)^2 \leq 4(2x+y)^2,$$

for $2x + y \geq 2$,

we have

$$\begin{aligned}
\alpha(x,y)\beta(x,y)\psi\left(s^{\epsilon}\sigma_q(\mathcal{T}x,\mathcal{T}y)\right) &= \frac{1}{2^{5\epsilon+2}}\left((x-y)^2+4(2x+y)^2\right) \\
&\leq \frac{1}{2^{5\epsilon+2}}\left(4(x-y)^2+4(2x+y)^2\right) = \frac{1}{2^{5\epsilon}}\sigma_q(x,y) \\
&\leq \frac{1}{2^{5}}\sigma_q(x,y) \\
&\leq \Gamma\left(\phi(\sigma_q(x,y)),\psi(\sigma_q(x,y))\right) = \frac{1}{2^{3}}\sigma_q(x,y) - \frac{1}{2^{4}}\sigma_q(x,y) \\
&= \frac{1}{2^{4}}\sigma_q(x,y).
\end{aligned}$$

Then all hypotheses of Corollary (4.3.1) hold, so $\mathcal{T}$ has a fixed point, which is $\frac{1}{2^{3\epsilon+1}-2}$.

Example 4.3.2 *Let $X=\mathbb{R}$ and $\sigma_q(x,y)=(2x-y)^2+(2x+y)^2$, then (X,σ_q,s) is a complete dislocated quasi b-metric space with $s=2$. Define $\mathcal{T}: X\to X$, and*
α, $\beta: X\times X\to[0,+\infty)$ by

$$\mathcal{T}x=\frac{x}{2^{\epsilon}}.$$

with $\epsilon\geq 1$ and

$$\alpha(x,y)=\beta(x,y)=1 \qquad \forall x,y\in X.$$

Taking $\psi(t)=\frac{1}{2}t$, $\phi(t)=\frac{1}{2^3}t$, and $\Gamma(r,t)=\frac{1}{2}t-r$.

Let x, y in X such that $\alpha(x,y) = \beta(x,y) \geq 1$, so

$$\alpha(x,y)\beta(x,y)\psi(s^{\epsilon}\sigma_q(\mathcal{T}x,\mathcal{T}y)) = \psi\left(2^{\epsilon}\sigma_q(\mathcal{T}x,\mathcal{T}y)\right)$$

$$= \psi\left(2^{\epsilon}\left(\frac{(2x-y)^2}{2^{4\epsilon}} + \frac{(2x+y)^2}{2^{4\epsilon}}\right)\right)$$

$$= \psi\left(\frac{1}{2^{3\epsilon}+1}((2x-y)^2 + (2x+y)^2)\right)$$

$$= \frac{1}{2^{3\epsilon+1}}\sigma_q(x,y)$$

$$\leq \frac{1}{2^4}\sigma_q(x,y)$$

$$\leq \Gamma\left(\phi(\sigma_q(x,y)), \psi(\sigma_q(x,y))\right) = \frac{1}{2}\frac{1}{2}\sigma_q(x,y) - \frac{1}{2^3}\sigma_q(x,y)$$

$$= \frac{1}{2^3}\sigma_q(x,y).$$

Consequently all hypotheses of Theorem (4.3.1) hold, so $\mathcal{T}$ has a fixed point. Hence $\mathcal{T}$ has a fixed point 0.

CHAPTER

5

APPLICATIONS TO INTEGRO-DIFFERENTIAL EQUATIONS

Fixed point theorems have numerous applications in mathematics and are applied in diverse fields as biology, chemistry, economics, engineering, game theory, and physics. They constitute an important field of research. It should be noted that most papers dealing with the existence of solutions of nonlinear of several types integral, differential, integro-differential, fractional integrals mainly use the techniques of nonlinear analysis such as fixed point techniques.

In this chapter, we will apply theorem (3.1.1) to study the problem of existence of the solution of integral equation of the Volterra type and boundary value problem fractional differential equation.

5.1 Solvability Of Integral Equations

5.1.1 Integral Equations

An integral equation can be classified as either a linear integral equation or a nonlinear integral equation.

The most commonly used integral equations are the Volterra and Fredhlom integral equations.

Definition 5.1.1 *An integral equation is any equation of the form*

$$\lambda x(t) = f(t) + \int_E K(t, s, x(s))ds, \tag{5.1}$$

with

E : is a measured space, f a measurable function given on E,

λ : a given scalar which can be real or complex (a numeric parameter).

K : a measurable function on E^3 called the kernel of the integral equation.

$x(t)$: is an unknown function.

With all these data, our problem is to find the function x that satisfies equation (5.1), where f, K are known functions and x the unknown function.

Lemma 5.1.1 *Let K be a function of space $L^2(]a, b[\times]a, b[)$, then the integral operator T defined by:*

$$Tx(s) = \int_a^b K(t, s)x(s)ds \qquad t \in [a, b],$$

with kernel K is compact in $L^2(]a, b[)$ in itself.

Remark 5.1.1 *Note that*

1. *To study this equation, we restrict ourselves to the spaces $L^p(E)$ with $(1 \leq p \leq \infty)$. Implicitly, for the functions $f \in L^p(E)$, we seek the functions x in $L^p(E)$ verifying (5.1).*

2. *If we take*

$$K(t, s, x(s)) = K(t, s)x(s).$$

The integral equation (5.1) becomes

$$\lambda x(t) = f(t) + \int_E K(t,s)x(s)ds,$$

which is said to be a linear integral equation.

3. *Note that equation (5.1) can be written in operator form*

$$Tx = \lambda x - f.$$

Where the operator T is written as follows

$$Tx(s) = \int_E K(t,s)x(s)ds.$$

5.1.2 Classification Of Integral Equations

Integral equations are classified by their characteristics according to three dichotomies:

1. **integration limits**

   If we take $E = [a,b]$ an interval of $\mathbb{R}$.

   Any integral equation with constant limits is called a Fredholm equation.

$$\lambda x(t) = f(t) + \int_a^b K(t,s,x(s))ds \qquad a \leq t \leq b.$$

   If one of the two limits of integration is variable, the equation becomes a Volterra equation

$$\lambda x(t) = f(t) + \int_a^t K(t,s,x(s))ds \qquad a \leq t \leq b.$$

   It is clear that the Volterra equation can be considered as a Fredholm equation where the function K satisfies the condition

$$K(t,s) = 0 \qquad s > t.$$

2. **The integral operator T**

   The equation defined by

$$Tx = f,$$

   is said to be an equation of the first species.

   If the equation is defined by

$$x - Tx = f.$$

   This equation is called an equation of the second species.

3. **The second member of the equation**

   If $f = 0$ the equation is a homogeneous equation. Otherwise this equation is called a non-homogeneous equation.

5.1.3 Existence Of Solutions For Integral Equations

Between linearity and non-linearity, the analytical and numerical studies of equations are totally different.

5.1.3.1 Existence Of The Solution Of Linear Integral Equations

To solve an integral equation with classical methods or with numerical methods, it is necessary to know the existence of the solution; Fredholm's alternative theorem gives conditions on the space E and the linear operator K to say that our problem

$$(\lambda - T)x = f \qquad x, f \in E,$$

is solvable.

Theorem 5.1.1 *[49] Let A be a compact operator of a normed space X in itself then the operator $T = I - A$ is injective if and only if it is surjective and consequently it is bijective, moreover the inverse operator $T^{-1} = (I - A)^{-1}$ defined from X to X is bounded.*

Theorem 5.1.2 *(Fredholm Alternative) [3]*
Assume that X is a Banach space
Let $T : X \to X$ be a compact operator, then for the non-homogeneous equation

$$(\lambda - T)x = f \qquad \lambda \neq 0,$$

admits a unique solution $x \in X$, for all $f \in X$, it is necessary and it suffices that the equation homogeneous

$$(\lambda - T)x = 0,$$

admits the trivial solution $x = 0$.
In this case the operator $(\lambda - T)$ is invertible and bounded.

Theorem 5.1.3 *[26] Let X be a Banach space and A be a bounded operator in X with the following property*

$$\|Ax_1 - Ax_2\| \leq r\,\|x_1 - x_2\|\,,$$

then the following equation

$$\lambda x - Ax = f,$$

admits a unique solution for any $f \in X$.

5.1.3.2 Existence Of The Solution Of The Nonlinear Integral Equation Of Volterra:

Let us first consider the nonlinear integral equation of Volterra

$$\varphi(x) = f(x) + \int_{x_0}^{x} K(x,t,\varphi(t))dt \qquad x \in [x_0, x_0 + a] \qquad a > 0. \tag{5.2}$$

Our goal is to prove the existence of solutions for (5.2), at least on a subinterval of $[x_0, x_0 + a]$, say $[x_0, x_0 + \delta]$, with $\delta \leq a$.
we assume that the following conditions are true:

1. f is a continuous function from $[x_0, x_0 + a]$ to $\mathbb{R}$.

2. K is a continuous function of $\Delta \times B_r$, in $\mathbb{R}$ where

   $$\Delta = \{(x,t); x_0 \leq t \leq x_0 + a\},$$

   and

   $$B_r = \{\varphi \in \mathbb{R}; |\varphi| \leq r, \text{ with } r > 0\}.$$

3. K satisfies the Lipschitz condition in $\Delta \times B_r$

   $$|K(x,t,\varphi) - K(x,t,\psi)| \leq L|\varphi - \psi| \qquad L > 0.$$

4. if $b = \sup|f(x)|$, $x \in [x_0, x_0 + a]$ then $b < r$.

Theorem 5.1.4 *[4] If we assume that conditions (1)-(4) hold, then there exists a unique solution of equation (5.2), defined and continuous on the interval $[x_0, x_0 + \delta]$.*

Theorem 5.1.5 *[4] Let the following nonlinear Volterra integral equation be:*

$$\varphi(x) = f(x) + \int_{0}^{x} K(x,t,\varphi(t))dt \qquad 0 \leq x \leq a. \tag{5.3}$$

Suppose the following conditions are true:

1. *$f : [0, a] \to \mathbb{R}$ is continuous.*

2. *$K : [0, a] \times [0, a] \to \mathbb{R}$, is a continuous function satisfies the following Lipschitz condition:*

$$|K(x,t,u) - K(x,t,v)| \leq L|u - v| \qquad \textit{such as} \qquad L > 0,$$

$0 \leq x \leq a \quad 0 \leq t \leq x \quad$ *and* $\quad u, v \in \mathbb{R}$.

Then, equation (5.3) has a unique solution.$\varphi \in C([0, a], \mathbb{R})$.

5.1.3.3 Existence Of The Solution Of The Nonlinear Fredholm Integral Equation

Let the nonlinear integral equation of Fredholm:

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t,\varphi(t))dt \qquad x \in [a, b]. \tag{5.4}$$

Theorem 5.1.6 *[4] assume that*

$$\left\| \int_a^b K(x,t,\varphi(t))dt \right\| \leq M|\varphi(t)| \qquad M > 0,$$

and

$$|K(x,t,u) - K(x,t,v)| \leq N(x,t)|u - v|,$$

where

$$\int_a^b \int_a^b |N(x,t)|^2 \, dxdt = p^2 < \infty.$$

If $f(x) \in L^2[a, b]$ then (5.4) has a unique solution in $L^2[a, b]$, if $|\lambda|p < 1$.

Theorem 5.1.7 *[4] Consider the following conditions on $K(x, t, \varphi)$:*

1. *$\varphi \to K(x, t, \varphi)$ is Lipschitzian with constant k and*

$$|K(x,t,\varphi)| \leq N(|\varphi|, x), \quad \textit{where} \quad \forall e \in \mathbb{R}, N(e, \bullet) \in L^1.$$

2. *$(x, \varphi) \to K(x, t, \varphi)$ is measurable for all x.*

3. *$t \to K(x, t, \varphi)$ is continuous p.p. at x.*

Under assumptions (1), (2) and (3), the equation (5.4) has a unique solution.

Theorem 5.1.8 *[4] If the following conditions are true:*

1. *The function $f(x)$ is bounded, $|f(x)| < R$ in $a \leq x \leq b$;*

2. *The function $K(x,t,\varphi(t))$ is integrable and bounded where*

$$|K(x,t,\varphi(t))| \leq L \qquad a \leq x,t \leq b;$$

3. *The function $K(x,t,\varphi(t))$ satisfies the Lipschitz condition*

$$|K(x,t,\varphi) - K(x,t,\psi)| \leq M|\varphi - \psi|.$$

Then equation (5.4) has a unique solution if $\lambda < \frac{1}{H(b-a)}$, with $H = \max\{L(1 + \frac{R}{|\lambda| L(b-a)}), M\}$.

5.2 Application To Volterra's Nonlinear Integral Equations

We consider the following problem

$$x(t) = g(t) + \int_0^t K(t,s,x(s))ds, \quad s,t \in [0,1], \tag{5.5}$$

where $g \in C([0,1],\mathbb{R})$ and $K : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.
Clearly that $X = C([0,1])$ with the dislocated b-metric
$\sigma_b(x,y) = \sup_{0\leq t\leq 1}(|x(t)| + |y(t)|)^2$ is a complete dislocated b-metric space.
We introduce the following hypotheses with $\epsilon \geq 1$:

H_1 : For all $x_1(t), x_2(t) \in X$ and for each $s,t \in [0,1]$

$$(|K(t,s,x_1(t))| + |K(t,s,x_2(t))|)^2 \leq \frac{1}{2^{(\epsilon+3)}}(|x_1(t)| + |x_2(t)|)^2,$$

H_2 : For all $t \in [0,1]$

$$\sup_{0\leq t\leq 1} |g(t)|^2 \leq \frac{1}{2^{(\epsilon+5)}}(|x_1(t)| + |x_2(t)|)^2.$$

Theorem 5.2.1 *Under (H_1) and (H_2) the problem (5.5) has a solution in $C([0,1])$.*

Proof 15 *Define an operator*

$$\mathcal{T}x(t) = g(t) + \int_0^t K(t,s,x(s))ds \;\; for\ all\ t \in [0,1],$$

It clear that our problem (5.5) has a solution if and only if $\mathcal{T}$ has a fixed point.

For all $x, y \in X$ and $t \in [0,1]$ we have

$$\begin{aligned}
(|\mathcal{T}x(t)| + |\mathcal{T}y(t)|)^2 &= \left(\left|g(t) + \int_0^t K(t,s,x(s))ds\right| + \left|g(t) + \int_0^t K(t,s,y(s))ds\right|\right)^2 \\
&\leq \left(2|g(t)| + |\int_0^t K(t,s,x(s))ds| + |\int_0^t K(t,s,y(s))ds|\right)^2 \\
&\leq \left(2|g(t)| + \int_0^t |K(t,s,x(s))|ds + \int_0^t |K(t,s,y(s))|ds\right)^2 \\
&= \left(2|g(t)| + \int_0^t (|K(t,s,x(s))| + |K(t,s,y(s))|)ds\right)^2 \\
&\leq 2\left(\left(2|g(t)|\right)^2 + \left(\int_0^t (|K(t,s,x(s))| + |K(t,s,y(s))|)ds\right)^2\right) \\
&\leq 2\left(\left(2|g(t)|\right)^2 + \int_0^t \left(|K(t,s,x(s))| + |K(t,s,y(s))|\right)^2 ds\right) \\
&\leq 8|g(t)|^2 + 2\int_0^t \left(|K(t,s,x(s))| + |K(t,s,y(s))|\right)^2 ds \\
&\leq 8 \sup_{0\leq t\leq 1} |g(t)|^2 + 2\frac{1}{2^{(\epsilon+3)}} \sup_{0\leq t\leq 1} (|x(t)| + |y(t)|)^2 \\
&\leq (\frac{8}{2^{(\epsilon+5)}} + \frac{1}{2^{(\epsilon+2)}}) \sup_{0\leq t\leq 1} (|x(t)| + |y(t)|)^2 \\
&\leq \frac{1}{2^{(\epsilon+1)}} \sup_{0\leq t\leq 1} (|x(t)| + |y(t)|)^2.
\end{aligned}$$

Which follows that:

$$\sup_{0\leq t\leq 1} (|\mathcal{T}x(t)| + |\mathcal{T}y(t)|)^2 \leq \frac{1}{2^{(\epsilon+1)}} \sup_{0\leq t\leq 1} (|x(t)| + |y(t)|)^2.$$

Consequently we obtain

$$2^{\epsilon}\sigma_b(\mathcal{T}x, \mathcal{T}y) \leq \frac{1}{2}\sigma_b(x,y) = \frac{2}{3}\frac{3}{4}\sigma_b(x,y) \leq \frac{3}{4}M(x,y).$$

Hance we have

$$e^{2^{\epsilon}\sigma_b(\mathcal{T}x,\mathcal{T}y)} \leq (e^{M(x,y)})^{\frac{3}{4}}.$$

Then all the hypotheses of Corollary (2.1) are satisfied with $\psi(t) = t$, $\vartheta = e^t$, $a_1 = \frac{2}{3}$ and $\kappa = \frac{3}{4}$, $L = 0$. So $\mathcal{T}$ has a fixed point which is a solution of the problem (5.5).

Example 5.2.1 *Let $I = [0,1]$, and $X = C([0,1])$ with the dislocated b-metric $\sigma_b(x,y) = \sup_{0\leq t\leq 1}(|x(t)| + |y(t)|)^2$ is a complete dislocated b-metric space.*

Consider a function $g \in C([0,1], \mathbb{R})$ defined by $g(t) = \frac{1}{6\times 2^{(\frac{\epsilon}{2}}} tx(t)$ with $\epsilon \geq 1$.
Hence, we obtain

$$\sup_{0\leq t\leq 1} |g(t)|^2 \leq \frac{1}{36\times 2^{\epsilon}}(|x(t)| + |y(t)|)^2 \leq \frac{1}{32\times 2^{\epsilon}}(|x(t)| + |y(t)|)^2$$

$$= \frac{1}{2^{(\epsilon+5)}}(|x(t)| + |y(t)|)^2.$$

Now, we define $K : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ by

$$K(t, s, x(s)) = \frac{1}{2^{(\frac{\epsilon}{2}+2)}}(t - s)x(s).$$

Let $\mathcal{T}$ be a self-mapping on X by:

$$\mathcal{T}x(t) = g(t) + \int_0^t K(t, s, x(s))ds, \quad s, t \in I.$$

Observe that,

$$\sigma_b(\mathcal{T}x, \mathcal{T}y) \leq \frac{1}{2^{(\epsilon+1)}}\sigma_b(x, y),$$

so

$$2^{\epsilon}\sigma_b(\mathcal{T}x, \mathcal{T}y) \leq \frac{1}{2}\sigma_b(x, y) \leq \frac{3}{4}M(x, y),$$

for all $x(t), y(t) \in X$.
Hance we have

$$e^{2^{\epsilon}\sigma_b(\mathcal{T}x,\mathcal{T}y)} \leq (e^{M(x,y)})^{\frac{3}{4}}.$$

Thus, all of the hypotheses of Corollary (2.1) are satisfied with $\psi(t) = t$, $\vartheta = e^t$, $a_1 = \frac{2}{3}$, $\kappa = \frac{3}{4}$, $L = 0$.
So $\mathcal{T}$ has a fixed point which is a solution of the problem (5.5).

Example 5.2.2 *Let $I = [0,1]$, and $\mathfrak{X} = C([0,1])$ with the dislocated b-metric $\sigma_b(x, \mathfrak{X}y) = \sup_{0\leq t\leq 1}(|x(t)| + |y(t)|)^2$ is a complete dislocated b-metric space.*
Consider a function $g \in C([0,1], \mathbb{R})$ defined by $g(t) = \frac{1}{\sqrt{2^{(\frac{\epsilon}{2}+5)}}}x(t)$ with $\epsilon \geq 1$.
Hence, we obtain

$$\sup_{0\leq t\leq 1} |g(t)|^2 \leq \frac{1}{2^{(\epsilon+5)}}(|x(t)| + |y(t)|)^2.$$

Now, we define $K : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ by

$$K(t, s, x(s)) = \frac{1}{2^{(\frac{\epsilon}{2}+1)}\sqrt{2}}(t - s)x(s).$$

Let $\mathcal{T}$ be a self-mapping on X by:

$$\mathcal{T}x(t) = g(t) + \int_0^t K(t,s,x(s))ds, \quad s,t \in I.$$

Observe that,

$$\sigma_b(\mathcal{T}x, \mathcal{T}y) \leq \frac{1}{2^{(\epsilon+1)}}\sigma_b(x,y) \leq \frac{3}{4}M(x,y),$$

for all $x(t), y(t) \in X$.

Hance we have

$$e^{2^\epsilon \sigma_b(\mathcal{T}x,\mathcal{T}y)} \leq (e^{M(x,y)})^{\frac{3}{4}}.$$

Thus, all of the hypotheses of Corollary (2.1) are satisfied with $\psi(t) = t$, $\vartheta = e^t$, $a_1 = \frac{2}{3}$, $\kappa = \frac{3}{4}$, $L = 0$.

So $\mathcal{T}$ has a fixed point which is a solution of the problem (5.5).

5.3 Existence Of Solutions Of Boundary Value Problems Of Fractional Differential Equations

5.3.1 Some Elements Of Fractional Calculus

We give an over view around the fractional derivative and the fractional integral with some properties.

5.3.1.1 Gamma Function:

Definition 5.3.1 *[42]*

The gama function $\Gamma(z)$ is defined by the integral :

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt.$$

z is complex number : $Re(z) > 0$.

properties

we have some basic properties

1 The function $\Gamma(z)$ is continuous for $p > 0$.

2 The function $\Gamma(z)$ obeys the property:

$$\Gamma(z+1) = z\Gamma(z).$$

3 The following relations are also valid:

$$\begin{aligned}\Gamma(z+1) =& (z+n-1)\cdots(z+1)z\Gamma(z),\\ \Gamma(1) =& 1,\\ \Gamma(n+1) =& n!,\\ \Gamma(0) =& +\infty.\end{aligned}$$

5.3.1.2 Fractional Integral Of Order α

Definition 5.3.2 *[42] For every $\alpha > 0$ and a local integrable function f, the right of order α is defined:*

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \qquad -\infty \leq a < t < \infty. \tag{5.6}$$

Alternatively, it can be defined also the left as:

$$I_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t-s)^{\alpha-1} f(s) ds, \qquad -\infty < t < b \leqslant \infty. \tag{5.7}$$

properties

we have the following properties:

i) $I^0 f(t) = f(t)$.

ii) $I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$.

iii) integral Operator I^α is:

$$I^\alpha(\lambda f(t) + g(t)) = \lambda I^\alpha f(t) + I^\alpha g(t), \lambda \in \mathbb{R}_+, \lambda \in \mathbb{C}.$$

iiii) $\dfrac{d}{dt}(I^\alpha f)(t) = I^{\alpha-1} f(t)$.

5.3.1.3 Riemann-Liouville Fractional Derivative

For a continuous function $f : (0, \infty) \to \mathbb{R}\ f \in L^1$, the Riemann-Liouville derivative of fractional order $\alpha > 0$, n = [a] + 1 ([a] denotes the integer part of the real number a) is defined as:

$$\begin{aligned} {}^{RL}D^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_{0}^{t} (t-s)^{n-\alpha-1} f(s) ds, \\ &= \left(\frac{d}{dt}\right)^{n} I^{n-\alpha} f(t). \end{aligned}$$

In particular, if $\alpha = 0$, then:

$$\left(D^{0} f\right)(t) = I^{0} f(t) = f(t).$$

If $\alpha = n \in \mathbb{N}$, then:

$$\left(D^{n} f\right)(t) = f^{(n)}(t).$$

If $0 < \alpha < 1$, then $n = 1$,

$${}^{RL}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{a}^{t} (t-s)^{-\alpha} f(s) ds, \qquad t > a.$$

5.3.1.4 Caputo Fractional Derivative

Definition 5.3.3 *[42] The Caputo derivative of order α for a function $f \in AC^{n}(J, E)$ absolutely continuous is defined by:*

$$\begin{aligned} {}^{C}D^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \\ &= I^{n-\alpha} f^{(n)}(t), \qquad t > 0, \qquad n-1 \leq \alpha < n. \end{aligned}$$

If $0 < \alpha < 1$, then :

$${}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} (t-s)^{-\alpha} f'(s).ds$$

Lemma 5.3.1 *[42] If $f \in AC^{n}[0,1]$, then the Caputo derivative ${}^{C}D^{\alpha} f(t)$ exists almost everywhere on [0,1], where $AC^{n}[0,1] = \left\{ f \in C^{n-1}[0,1], f^{(n-1)} \text{is absolutely continuous} \right\}$ and n is the smallest integer greater than or equal to α.*

Lemma 5.3.2 *[42] Let $\alpha \geq 0$ and $n = [\alpha] + 1$. Then:*

$$I^{\alpha}({}^{c}D^{\alpha} f(t)) = f(t) + \sum_{k=0}^{n-1} \frac{f^{k}(0)}{k!} t^{k}.$$

Lemma 5.3.3 *[42] Let $\alpha > 0$. Then the differential equation:*

$${}^{c}D^{\alpha} f(t) = 0,$$

has a solutions :

$$f(t) = c_0 + c_1 t + c_2 t^2 \cdots + c_{n-1} t^{n-1},$$

where

$$c_i \in \mathbb{R}, i = 1, 2, 3 \cdots n-1, n = [\alpha] + 1.$$

Lemma 5.3.4 *[42] Let $\alpha > 0$ and $n = [\alpha] + 1$.*
Then:

$${}^{c}D^{\alpha} x(t) = f(t),$$

equivalent to:

$$I^{\alpha}({}^{c}D^{\alpha} x(t)) = I^{\alpha} f(t) + c_0 + c_1 t + c_2 t^2 \cdots + c_{n-1} t^{n-1}, \tag{5.8}$$

where :

$$c_i \in \mathbb{R}, i = 1, 2, 3 \cdots n-1.$$

5.4 Relationship Between Fractional Derivatives :

1) Let $\alpha \in \mathbb{R}_{+}, n \in \mathbb{N}^{*}$, and $n = [\alpha] + 1$, if ${}^{c}D^{\alpha} f(t))$ and ${}^{RL}D^{\alpha} f(t))$ exist,so:

$$\textbf{i)}\ {}^{C}D^{\alpha} f(x) =^{RL} D^{\alpha} f(x) - \sum_{i=0}^{n-1} \frac{f^{(i)}(a)(x-a)^{i-\alpha}}{\Gamma(i-\alpha+1)},$$

we deduce that if $f^{(i)}(a) = 0$ for all $i = 0, 1, \ldots n-1$, we will have

$${}^{C}D^{\alpha} f(x) =^{RL} D^{\alpha} f(x).$$

$$\textbf{ii)}\ {}^{C}D^{\alpha} f(x) =^{RL} D^{\alpha} \left(f(x) - \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!} (x-a)^{i} \right).$$

2) Let $0 < \alpha < 1$, the fractional derivative of Riemann-Liouville and that of Caputo are defined respectively by:

$$^{RL}D^{\alpha}f(x) = \frac{d}{dx}\left(^{RL}D^{-(1-\alpha)}f(x)\right) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{-\alpha}f(t)dt.$$

$$^{C}D^{\alpha}f(x) =^{RL} D^{-(1-\alpha)}\left(\frac{df(x)}{dx}\right) = \frac{1}{\Gamma(1-\alpha)}\int_a^x (x-t)^{-\alpha}f'(t)dt.$$

5.5 Study Of Problems Of Fraction Differential Equations

Consider the following boundary value problem:

$$\begin{cases} {}^{C}\mathrm{D}^{p}x(\rho) = f(\rho, x(\rho)), 0 < \rho < 1, \\ x(0) = 0, \\ x(1) = \int_0^1 g(t, x(t))dt, \end{cases} \tag{5.9}$$

where ${}^{C}\mathrm{D}^{p}$ with $1 < p \leq 2$ is the Caputo fractional derivative, and $f : \mathcal{J} = [0,1] \times \mathbb{R} \to \mathbb{R}$.

Let $X = \mathcal{C}(\mathcal{J}, \mathbb{R})$ be the space of all continuous functions from $\mathcal{J}$ into $\mathbb{R}$, we consider on X the b-metric-like

$$\sigma_q(x, y) = |x(\rho) - y(\rho)|)^2 + 2|y(\rho)|,$$

for all $x, y \in X$.

Clearly, (X, σ_q, s) is a complete dislocated quasi b-metric space with parameter $s = 2$.

Lemma 5.5.1 *A function x is a solution of the problem (5.9) if and only if, x is a solution of the following integral equation:*

$$x(\rho) = \rho\int_0^1 g(t, x(t))dt + \int_0^1 \boldsymbol{G}(\rho, t)f(t, x(t))dt,$$

for all $t \in [0,1]$, where

$$\boldsymbol{G}(\rho, t) = \frac{1}{\Gamma(p)}\begin{cases} (\rho - t)^{p-1} - \rho(1-t)^{p-1}, & 0 \leq t \leq \rho \leq 1 \\ -\rho(1-t)^{p-1}, & 0 \leq \rho \leq t \leq 1. \end{cases}$$

Proof 16 *We have*

$$I^p({}^{C}\mathrm{D}^{p}x(\rho)) = x(\rho) - c_0 - c_1\rho = \frac{1}{\Gamma(p)}\int_0^{\rho}(\rho - t)^{p-1}f(t, x(t))dt,$$

by the boundary values we get

$$x(0) = c_0 = 0.$$

$$x(1) = c_1 + \frac{1}{\Gamma(p)} \int_0^1 (1-t)^{p-1} f(t, x(t))dt = \int_0^1 g(t, x(t))dt.$$

Then

$$c_1 = -\frac{1}{\Gamma(p)} \int_0^1 (1-t)^{p-1} f(t, x(t))dt + \int_0^1 g(t, x(t))dt.$$

Which gives

$$x(\rho) = -\frac{\rho}{\Gamma(p)} \int_0^1 (1-t)^{p-1} f(t, x(t))dt + \rho \int_0^1 g(t, x(t))dt + \frac{1}{\Gamma(p)} \int_0^\rho (\rho-t)^{p-1} f(t, x(t))dt,$$

which implies that

$$x(\rho) = \int_0^1 \boldsymbol{G}(\rho, t) f(t, x(t))dt + \rho \int_0^1 g(t, x(t))dt,$$

where

$$\boldsymbol{G}(\rho, t) = \frac{1}{\Gamma(p)} \begin{cases} (\rho-t)^{p-1} - \rho(1-t)^{p-1}, & 0 \le t \le \rho \le 1, \\ -\rho(1-t)^{p-1}, & 0 \le \rho \le t \le 1. \end{cases}$$

$$\int_0^1 \boldsymbol{G}(\rho, t)dt = \frac{1}{\Gamma(p)} \left[\int_0^\rho (\rho-t)^{p-1} - \rho(1-t)^{p-1} dt - \int_\rho^1 \rho(1-t)^{p-1} dt \right]$$

$$= \frac{1}{\Gamma(p)} [\rho^p + 1] \le \frac{2}{\Gamma(p)}.$$

Putting $\mathbf{G}_0 = \frac{2}{\Gamma(p)}$.

Assume that the following assumptions hold:

$(\mathbf{A}_1)$: f and g are continuous.

$(\mathbf{A}_2)$: There exist two functions $\kappa_1, \kappa_2 : [0,1] \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, we have

$$|f(\rho, x(\rho))| - f(\rho, y(\rho))| \le \kappa_1^* |x - y|,$$

$$|g(\rho, x(\rho)) - g(\rho, y(\rho))| \le \kappa_2^* |x - y|,$$

where $\kappa_i^* = \sup_{\rho \in [0,1]} |\kappa_i(\rho)|$.

$(\mathbf{A}_3)$: There exists $M_1 > 0$ and $M_2 > 0$ such that

$$|f(\rho, x(\rho)| \leq M_1|x(\rho)|, \text{and } |g(\rho, x(\rho)| \leq M_2|x(\rho)|,$$

for all $\rho \in [0, 1]$ and $x \in \mathbb{R}$ with $M_1\mathbf{G}_0 + M_2 \leq \Lambda$ and $\Lambda = (\kappa_1^*\mathbf{G}_0 + \kappa_2^*)^2 < \frac{1}{2^{3\epsilon}}$, for $\epsilon \geq 1$.

Theorem 5.5.1 *Under the assumptions $(\boldsymbol{A}_1) - (\boldsymbol{A}_3)$, the problem (5.9) has a solution in X.*

Proof 17 *Define a mapping*

$$Tx(\rho) = \int_0^1 \boldsymbol{G}(\rho, t)f(t, x(t))dt + \rho\int_0^1 g(t, x(t))dt,$$

For $x, y \in X$ and for all $\rho \in [0, 1]$ we have

$$\begin{aligned} |\mathcal{T}x(\rho) - \mathcal{T}y(\rho)| &\leq \int_0^1 \boldsymbol{G}(\rho, t)|f(t, x(t)) - f(t, y(t))|dt + \rho\int_0^1 |g(t, x(t)) - g(t, y(t))|dt \\ &\leq \Big(\kappa_1^*\int_0^1 G(\rho, t))dt + \kappa_2^*\Big)|x - y|, \end{aligned}$$

which follows that

$$|\mathcal{T}x(\rho) - \mathcal{T}y(\rho)|^2 \leq (\kappa_1^*\boldsymbol{G}_0 + \kappa_2^*)^2|x - y|^2 = \Lambda|x - y|^2.$$

On other hand, we have

$$\begin{aligned} |\mathcal{T}y(\rho)| &= |\int_0^1 \boldsymbol{G}(\rho, t)f(t, y(t))dt + \int_0^1 g(t, y(t)dt| \\ &\leq M_1\boldsymbol{G}_0|y| + M_2|y| = (M_1\boldsymbol{G}_0 + M_2)|y|. \end{aligned}$$

Then we have

$$\sigma_q(Tx, Ty) = |Tx - Ty|^2 + 2|Ty| \leq \Lambda|x - y|^2 + 2(M_1\boldsymbol{G}_0 + M_2)|y|,$$

since

$$M_1\boldsymbol{G}_0 + M_2 \leq \Lambda,$$

we get

$$\sigma_q(Tx, Ty) \leq \Lambda(|x - y|^2 + 2|y|) = \Lambda\sigma_q(x, y) \leq \frac{1}{2^{3\epsilon}}\sigma_q(x, y).$$

So we have

$$2^{\epsilon-1}\sigma_q(\mathcal{T}x,\mathcal{T}y) \leq \frac{1}{2^{2\epsilon+1}}\sigma_q(x,y).$$

Then all the hypotheses of Theorem (4.3.1) are satisfied with

$$\psi(\rho) = \frac{1}{2}\rho,\ \phi(\rho) = \frac{1}{2^{2\epsilon+1}},\ \Gamma(\mathrm{r},\rho) = \frac{1}{2^{2\epsilon-1}}\rho - \mathrm{r}\ \textit{and}\ \alpha = \beta = 1.$$

So $\mathcal{T}$ has a fixed point which is a solution of problem (5.9).

CONCLUSION AND PERSPECTIVES

The general conclusion of this study is the interest in finding fixed points of some theorems in a generalized metric space, which is the dislocated quasi b-metric space. By utilizing a some class of simulation functions, we have provided a new generalized contractions type and related fixed point results which enhance and generalize some findings from previous research. To demonstrate our findings, two examples were offered. Finally, we have used our findings to demonstrate the existence of solutions for a boundary value problem of fractional differential equations with integral boundary conditions.

In the future, we will look at the following issues as examples:

1. We will try to produce some results in the field of fixed point theory, by making changes in the used spaces or the conditions of contraction.

2. We will try to apply the obtained results to study the existence problem of the solution of integral equations.

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