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Initial-value problems for nonlinear hybrid implicit Caputo fractional differential equations

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Abstract

We study the existence and uniqueness of solutions of a nonlinear hybrid implicit Caputo fractional differential equation

$$ {}^{c}D^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right)=h\left(t,x(t),{}^{c}D^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right)\right) $$

$$ \left(\frac{x(0)-f(0,x(0))}{g(0,x(0))}\right)=\theta,\theta\in\mathbb{R}, $$

where ${}^{c}D^{\alpha}$ is the Caputo fractional derivative of order $0<\alpha<1$. In the process we convert the given fractional differential equation into an equivalent integral equation. Then we construct an appropriate mapping and employe the Banach's fixed point theorem to show the existence and uniqueness of solutions of this equation. We also use the generalization of Gronwall's inequality to show the estimate of the solution.

Keywords: Implicit fractional differential equations, Caputo fractional derivatives, fixed point theorems, existence, uniqueness.

1. Introduction

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see the references.

2. Our approach

In this section we study the existence and uniqueness of solution for the following nonlinear hybrid implicit Caputo fractional differential equation

$$ {}^{c}D^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right)=h\left(t,x(t),{}^{c}D^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right)\right) \tag{1} $$

$$ \left(\frac{x(0)-f(0,x(0))}{g(0,x(0))}\right)=\theta,\theta\in\mathbb{R}, \tag{2} $$

Where $f:[0,T]\times\mathbb{R}\to\mathbb{R}$, $g:[0,T]\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ and $h:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are nonlinear continuous function and ${}^{c}D^{\alpha}$ is the Caputo fractional derivative of order $0<\alpha<1$.

We start by present some necessary definitions, lemmas and theorems which will be used throughout this paper.

Let $C([0,T])$ be the Banach space of all real-valued continuous functions defined on the compact interval $[0,T]$, endowed with the maximum norm.

Definition 2.1:

The fractional integral of order $\alpha>0$ for a continuous function $x:\mathbb{R}^+\to\mathbb{R}^+$ is defined as

$$I^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds,$$

Definition 2.2:

The Caputo fractional derivative of order $\alpha>0$ for a continuous function $x:\mathbb{R}^+\to\mathbb{R}^+$ is defined as

$$^{c}D^{\alpha}x(t)=I^{n-a}D^{(n)}x(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x(s)ds,$$

where $n=[\alpha]+1$, provided the right side is pointwise defined on $\mathbb{R}^+$.

Lemma 2.3:

Let $\Re(\alpha)>0$. Suppose $x\in C^{n-1}(0,+\infty]$ and $x^{(n)}$ exists almost every-where on any bounded interval of $(0,+\infty]$ then $I^{\alpha}\left[{}^{c}D^{\alpha}x\right](t)=x(t)-\sum_{k=0}^{n-1}\frac{x^{(k)}(0)}{k!}t^k$.

In particular, when $0<\Re(\alpha)<1, I^{\alpha}\left[{}^{c}D^{\alpha}x\right](t)==x(t)-x(0).$

Theorem 2.4: (Banach's fixed point theorem)

Let Ω be a non-empty closed subset of a Banach space $(S,\|.\|)$ then any contraction mapping Φ of Ω into itself has a unique fixed point.

The following generalization of Gronwall's lemma for singular kernels plays an important role in obtaining our main results.

Lemma 2.5: Let $x:[0,T]\to[0,\infty)$ be real function and w is a nonnegative locally integrable function on $[0,T]$.

Assume that there is a constant $a>0$ such that $0<\alpha<1$

$$x(t)\le w(t)+a\int_0^t (t-s)^{-\alpha}x(s)ds,$$

Then, there exist a constant $k=k(\alpha)$ such that

$$x(t)\le w(t)+ka\int_0^t (t-s)^{-\alpha}w(s)ds.$$

3. Results and Discussion

In this section, we give the equivalence of the initiale value problem (1)-(2) and prove the existence, uniqueness and estimate of solution of (1)-(2)

The proof of the following lemma is close to the proof Lemma 6.2 given in [2].

Lemma 3.1: If the functions $f:[0,T]\times\mathbb{R}\to\mathbb{R}$, $g:[0,T]\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ and $h:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous, then the initial value problem (1)-(2) is equivalent to nonlinear fractional Volterra integro-differential equation

$$x(t)=f(t,x(t))+\theta g(t,x(t))+\frac{g(t,x(t))}{\Gamma(\alpha)}\int_0^t (t-s)^{a-1}h\left(s,x(s),{}^{c}D^{\alpha}\left(\frac{x(s)-f(s,x(s))}{g(s,x(s))}\right)\right)ds \qquad (3)$$

For $t\in[0,T]$.

Theorem 3.2: Let $T>0$. Assume that the continuous function $f:[0,T]\times\mathbb{R}\to\mathbb{R}$, $g:[0,T]\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ and $h:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfy the following conditions

(H1) There exists a constant $\mathrm{M}_g\in\mathbb{R}^+$ such that

$$|g(t,u)|\leq M_g,$$

for all $u\in\mathbb{R}$ and $t\in[0,T]$.

(H2) There exists a constant $\mathrm{M}_h\in\mathbb{R}^+$ such that

$$|\mathrm{h(t,u,v)}|\leq \mathrm{M}_h,$$

For all $u,v\in\mathbb{R}$ and $t\in[0,T]$.

(H3) There exist $k_1,k_2,k_3\in\mathbb{R}^+$, $k_4\in(0,1)$ with $k_1+|\theta|k_2\in(0,1)$ such that

$$|f(t,u)-f(t,u^*)|\leq k_1|u-u^*|,$$

$$|g(t,u)-g(t,u^*)|\leq k_2|u-u^*|,$$

and

$$|h(t,u,v)-h(t,u^*,v^*)|\leq k_3|u-u^*|+k_4|v-v^*|,$$

for all $u,v,u^*,v^*\in\mathbb{R}$ and $t\in[0,T]$.

If

$$\beta=(K_1+K_2|\theta|+(M_hK_2+\frac{M_gK_3}{(1-K_4)})\frac{T^{\alpha}}{\Gamma(\alpha+1)})<1. \qquad (4)$$

The probelm (1)-(2) has a unique solution in $C([0,T],\mathbb{R})$.

Proof: Let

$${}^{c}D^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right)=z_x(t),x(0)=\theta g(0,x(0))+f(0,x(0)),$$

then by Lemma (3.1)

$$x(t)=f(t,x(t))+\theta g(t,x(t))+\frac{g(t,x(t))}{\Gamma(\alpha)}\int_0^t (t-s)^{a-1}z_x(s)ds,$$

where $z_x(t)=h(t,\ f(t,x(t))+\theta g(t,x(t))+g(t,x(t))I^{a}z_x(t),z_x(t))$.

That is $x(t)=f(t,x(t))+\theta g(t,x(t))+g(t,x(t))I^{a}z_x(t)$ .Define the mapping $P:C([0,T],\mathbb{R})\to C([0,T],\mathbb{R})$ as follows

$$(Px)(t)=f(t,x(t))+\theta g(t,x(t))+\frac{g(t,x(t))}{\Gamma(\alpha)}\int_0^t(t-s)^{a-1}z_x(s)ds.$$

It is clear that the fixed points of P are solution of (1). Let $\mathrm{x},\ \mathrm{y}\in\mathrm{C}([0,\mathrm{T}],\mathbb{R})$, then we have

$$|(Px)(t)-(Py)(t)|\le K_1|x(t)-y(t)|+K_2|\theta||x(t)-y(t)|+K_2|x(t)-y(t)|\frac{M_h}{\Gamma(\alpha)}\int_0^t(t-s)^{a-1}ds$$

$$+\frac{M_g}{\Gamma(\alpha)}\int_1^t(t-s)^{a-1}|z_x(s)-z_y(s)|ds, \qquad (5)$$

and

$$|z_x(t)-z_y(t)|\le K_3|x(t)-y(t)|+K_4|z_x(t)-z_y(t)|$$

$$\le\frac{K_3}{1-K_4}|x(t)-y(t)|. \qquad (6)$$

By replacing (6) in the inequality (5) we get

$$|(Px)(t)-(Py)(t)|\le(K_1+K_2|\theta|+(M_hK_2+\frac{M_gK_3}{(1-K_4)})\frac{T^\alpha}{\Gamma(\alpha+1)})\|x-y\|.$$

Then

$$\|(Px)(t)-(Py)(t)\|\le\beta\|x-y\|.$$

By (4), the mapping P is a contraction in $C([0,T],\mathbb{R})$ .Hence P has a unique fixed point $x\in C([0,T],\mathbb{R})$ which is a unique solution of the problem (1)-(2).

Theorem 3.3: Assume that $f:[0,T]\times\mathbb{R}\to\mathbb{R}$, $g:[0,T]\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ and $h:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfy (H1), (H2) and (H3). If x is a solution of (1)-(2), then

$$|x(t)|\le\left(\frac{(1-K_4)(1-(K_1+K_2|\theta|))\Gamma(a+1)+M_gK_3KT^a}{(1-K_4)(1-(K_1+K_2|\theta|))^2\Gamma(a+1)}\right)$$

$$\times\left((Q_1+|\theta|Q_2+\frac{M_gQ_3T^a}{(1-K_4)\Gamma(a+1)})\right),$$

where $Q_1=\sup_{t\in[0,T]}|f(t,0)|,\ Q_2=\sup_{t\in[0,T]}|g(t,0)|, Q_3=\sup_{t\in[0,T]}|h(t,0,0)|$ and $K\in\mathbb{R}^+$ is a constant.

Proof: Let

$${}^cD^\alpha\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right)=z_x(t), x(0)=\theta g(0,x(0))+f(0,x(0)),$$

then by Lemma (3.1)

$$x(t)=f(t,x(t))+\theta g(t,x(t))+\frac{g(t,x(t))}{\Gamma(\alpha)}\int_0^t(t-s)^{a-1}z_x(s)ds,$$

Then by (H1)-(H3) and for any $t\in[0,T]$ we have

$$|x(t)|\le|f(t,x(t))-f(t,0)|+|f(t,0)|+|\theta|(|g(t,x(t))-g(t,0)|+|g(t,0)|)+M_g|I^az_x(t)|$$

$$\le K_1|x(t)|+Q_1+|\theta|(K_2|x(t)|+Q_2)+M_g|I^az_x(t)|,$$

On the other hand, for any $\mathrm{t}\in[0,\mathrm{T}]$ we get

$$|I^a z_x(t)| \le \frac{K_3}{1\text{-}K_4}|x(t)| + \frac{Q_3}{1\text{-}K_4},$$

therefore

$$|x(t)| \le K_1|x(t)| + Q_1 + |\theta|(K_2|x(t)| + Q_2) + M_g I^a\left(\frac{K_3}{1\text{-}K_4}|x(t)| + \frac{Q_3}{1\text{-}K_4}\right),$$

thus

$$(1\text{-}(K_1+K_2|\theta|))|x(t)| \le Q_1 + |\theta|Q_2 + \frac{M_g Q_3 T^a}{(1-K4)\Gamma(a+1)} + \frac{M_g K_3}{(1\text{-}K_4)(1\text{-}(K_1+K_2|\theta|))}(I^a\{(1-(K_1+K_2|\theta|))|x(t)|\})$$

By Lemma (2.5), there is a constant $K = K(\alpha)$ such that

$$(1\text{-}(K_1+K_2|\theta|))|x(t)| \le \left(\frac{(1-K_4)(1-(K_1+K_2|\theta|))\Gamma(a+1) + M_g K_3 K T^a}{(1-K_4)(1-(K_1+K_2|\theta|))\Gamma(a+1)}\right)$$

$$\times\left((Q_1 + |\theta|Q_2 + \frac{M_g Q_3 T^a}{(1-K_4)\Gamma(a+1)})\right).$$

Hence

$$|x(t)| \le \left(\frac{(1-K_4)(1-(K_1+K_2|\theta|))\Gamma(a+1) + M_g K_3 K T^a}{(1-K_4)(1-(K_1+K_2|\theta|))^2\Gamma(a+1)}\right)$$

$$\times\left((Q_1 + |\theta|Q_2 + \frac{M_g Q_3 T^a}{(1-K_4)\Gamma(a+1)})\right).$$

This completes the proof.

4. Conclusion

We can conclude that the main results of this paper have been successfully achieved, that is, through the Banach fixed point theorem, we got an important results within the mathematical analysis, we scrutinized the existence and uniqueness of solution of the given fractional differential equation and we also used the generalization of Gronwall's inequality to show the estimate of the solution.

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