THE SOLUTION OF THE SPACE FRACTIONAL SCHRÖDINGER EQUATION BASED ON NON-SINGULAR CAPUTO-FABRIZIO DERIVATIVE

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Abstract. In this work, we solve the space fractional Schrödinger equation based on non singular Caputo-Fabrizio derivative definition for 1D linear potential. To reach this goal, we first work out the fractional differential equation defined in terms of Caputo-Fabrizio derivative. Hence, the wave functions of fractional Schrödinger equations are derived.

1. Introduction

The theory of the fractional derivation and integration has long been considered as a branch of mathematics without any real or practical explanation. Over the last three decades, considerable attention has been paid to the fractional calculus by applying these concepts in different fields of physics and engineering [1]. Recently, the fractional notion enters the world of quantum mechanics for the purpose of generalization without any contradiction with the postulates of standard quantum mechanics. The possibilities of this generalization was proven by Laskin [2], who developed a new fractional quantum mechanics, and was realized using the Feynman's path-integral approach [3]. In the standard quantum mechanics, Feynman's approach is based on the path integral using the measure generated by Brownian motion. The natural generalization of Brownian motion is Lévy's motion. As long as the integral of path on Brownian trajectories lead to the standard Schrödinger equation, the path integral on Lévy trajectories leads to the fractional Schrödinger equation. This equation includes the fractional order derivative α instead of the second derivative $\alpha = 2$ in the standard Schrödinger equation. The results of fractional quantum mechanics have been discussed by several authors[4, 5, 6, 7, 8, 9, 10]. The objective of this work is solving the fractional Schrödinger equation for a linear potential based on Caputo-Fabrizio derivative [11].

2. Fractional Schrödinger equation for a linear potential

The fractional Schrödinger equation with Caputo-Fabrizio derivative for a linear potential is given by

$$\left(\frac{\hbar^{2\gamma} D_{2\gamma}}{a_0^{2\gamma}}\right) {}^{CF}D_x^{2\gamma}\psi(x) + kx\psi(x) = -E\psi(x), \quad \frac{1}{2} < \gamma \leq 1, \tag{2.1}$$

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where $x = X/a_0$ is the reduced space coordinate, k is a constant coefficient and ${}^{CF}D_x^{2\gamma}$ is the Caputo-Fabrizio derivative defined as [11]

$$ {}^{CF}D_x^{\alpha} f(x) = \frac{1}{(1-\alpha)} \int_0^x \exp\left[\frac{-\alpha}{1-\alpha}(x-s)\right] f'(s)\mathrm{d}s, \quad 0 < \alpha \le 1 \quad \& \quad x \le 0. \tag{2.2} $$

We can rewrite the fractional Schrödinger equation (2.1) as follows

$$ \begin{aligned} {}^{CF}D_x^{2\gamma}\psi(x) &= \frac{-a_0^{2\gamma}}{\hbar^{2\gamma} D_{2\gamma}} (kx + E)\,\psi(x) \\ &= \frac{-a_0^{2\gamma} k}{2\hbar^{2\gamma} D_{2\gamma}} \left(x + \frac{E}{k}\right) \psi(x) \\ &= A_{2\gamma}\,(x + \varepsilon)\,\psi(x), \end{aligned} \tag{2.3} $$

where

$$ A_{2\gamma} = \frac{-a_0^{2\gamma}}{\hbar^{2\gamma} D_{2\gamma}} \quad \& \quad \varepsilon = \frac{E}{k}. \tag{2.4} $$

Assuming $\psi(x) = f(x)$ and $A_{2\gamma}\,(x+\varepsilon)\,\psi(x) = g(x)$, the formula (2.3) has the form

$$ {}^{CF}D_x^{2\gamma} f(x) = g(x). \tag{2.5} $$

First, let us take: ${}^{CF}D_x^{\gamma} f(x) = u(x)$ and ${}^{CF}D_x^{2\gamma} f(x) = {}^{CF}D_x^{\gamma} u(x) \equiv g(x)$, then, according to [12], we have

$$ u(x) = (1-\gamma)\,(g(x) - g(0)) + \gamma \int_0^x g(s)\mathrm{d}s + u(0), \tag{2.6} $$

and

$$ f(x) = (1-\gamma)\,(u(x) - u(0)) + \gamma \int_0^x u(s)\mathrm{d}s + f(0). \tag{2.7} $$

When we substitute (2.6) into (2.7), we obtain

$$ \begin{aligned} f(x) = (1-\gamma)^2\,(g(x) - g(0)) + 2\gamma(1-\gamma) \int_0^x g(\tau)\mathrm{d}s + \gamma^2 \int_0^x \mathrm{d}s \int_0^s g(s)\mathrm{d}s \\ + \left[u(0) - (1-\gamma)g(0)\right]\gamma x + f(0). \end{aligned} \tag{2.8} $$

Taking the derivative twice, the last equation yields

$$ f''(x) = (1-\gamma)^2 g''(x) + 2\gamma(1-\gamma) g'(x) + \gamma^2 g(x). \tag{2.9} $$

By putting $f(x) = \psi(x)$ and $g(x) = A_{2\gamma}\,(x+\varepsilon)\,\psi(x)$, we find

$$ \begin{aligned} &\left[(1-\gamma)^2 x + (1-\gamma)^2 \varepsilon - \frac{1}{A_{2\gamma}}\right] \psi''(x) + \left[2\gamma(1-\gamma)x + 2\gamma(1-\gamma)\varepsilon + 2(1-\gamma)^2\right] \psi'(x) \\ &+ \left[\gamma^2 x + \gamma^2 \varepsilon + 2\gamma(1-\gamma)\right] \psi(x) = 0, \end{aligned} \tag{2.10} $$

or

$$ (\frac{c^2}{4} x + \frac{c^2}{4}\varepsilon + k)\psi''(x) + (cx + c\varepsilon + \frac{c^2}{2})\psi'(x) + (x + c + \varepsilon)\psi(x) = 0, \tag{2.11} $$

or equivalently

$$ f_2(x)\psi''(x) + f_1(x)\psi'(x) + f_0(x)\psi(x) = 0, \tag{2.12} $$

where

$$k = -\frac{1}{\gamma^2 A_{2\gamma}}, \quad \text{and} \quad c = 2\left(\frac{1-\gamma}{\gamma}\right). \tag{2.13}$$

The solution of the equation (2.11) can be find as the following: put

$$\psi(x) = y(x)\exp(-\frac{1}{2}\int^x \frac{f_1(s)}{f_2(s)}ds). \tag{2.14}$$

The integral in the exponent is easy to perform and we get a novel equation governing $y(x)$

$$y''(x) + F(x)y(x) = 0, \tag{2.15}$$

where

$$F(x) = \frac{f_0(s)}{f_2(s)} - \frac{1}{4}\left(\frac{f_1(x)}{f_2(x)}\right)^2 - \frac{1}{2}\frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right) = \frac{Ax+B}{(x+E)^2}, \tag{2.16}$$

such that

$$A = \frac{16k}{c^4}, \quad B = \frac{16k}{c^4}\left(\frac{1}{2}c + \varepsilon\right), \quad E = \varepsilon + \frac{4k}{c^2}. \tag{2.17}$$

The solution of the last differential equation is given by:

$$y(x) = \sqrt{x+E}\left[C_1 J_{\sqrt{1+4AE-4B}}(2\sqrt{A}\sqrt{x+E}) + C_2 Y_{\sqrt{1+4AE-4B}}(2\sqrt{A}\sqrt{x+E})\right]. \tag{2.18}$$

Then the final solution giving the wave function of the problem is given by

$$\psi(x) = \sqrt{x+E}\exp\left[-\frac{1}{2}\int^x \frac{f_1(s)}{f_2(s)}ds\right] \times \left[C_1 J_{\sqrt{1+4AE-4B}}(2\sqrt{A}\sqrt{x+E}) + C_2 Y_{\sqrt{1+4AE-4B}}(2\sqrt{A}\sqrt{x+E})\right], \tag{2.19}$$

where $J_\nu(X), Y_\nu(X)$ are the well known Bessel functions and C_1, C_2 are constants.

Conclusion

In conclusion, we have solved the fractional Schrödinger equation with Caputo-Fabrizio derivative for a linear potential. We have transformed this fractional differential equation to a second ordinary differential equation. The wave function then is easily calculated.

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