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THÈSE
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Spécialité: Mathématiques Appliquées

Comportement asymptotique de quelques problèmes thermoélastiques des matériaux poreux et non-simples

Présentée Par : AHMIMA Afaf

Devant le jury composé de :

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THESIS
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Asymptotic behavior of some problems of porous thermoelastic and non-simple materials

Presented by: AHMIMA Afaf

In front of the jury composed of:

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Dedication

I dedicate this work to:

My dear parents

My dear husband

My lovely parents-in-law

My dear daughter

My dear family

All the teachers in the Department of Mathematics.

All my friends

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All praise is due to Allah, who gave me the strength, patience, and ability to complete this research.

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ملخص

في هذه الرسالة قمنا بدراسة السلوك التقاربي عند الزمن اللانهائي لبعض المسائل الخاصة بالأوساط المسامية والمواد غير البسيطة.
بالنسبة للاوساط المسامية درسنا ثلاث مسائل: الاولى تتعلق بجملة مسامية حرارية لزجة مع معامل تأخير حيث اثبتنا الاستقرار الاسي للحل تحت شرط بين معاملي الاضمحلال والتأخير.

في المسألة الثانية تعطى معادلة انتشار الحرارة وفق قانون Gurtin-Pipkin اما المسالة الثالثة فتحوي حرارة دقيقة وفي كلا المسألتين اثبتنا التقارب الاسي.

المسائل الثلاث الأخيرة تتعلق بمواد غير بسيطة بوجود او غياب المسامية ووجود تاثير حراري ذو موجة ثانية حيث توصلنا الى نتائج مختلفة بالنسبة الى الاستقرار.
لاثبات وجود ووحدانية الحلول استخدمنا نظرية انصاف الزمر ونظرية Hille-Yosida و Lax-Millgram .

الكلمات المفتاحية: المسامية، المرونة، مواد غير بسيطة، وجود ووحدانية، استقرار غير اسي، اضمحلال بطيء.

Abstract

In this thesis, we are concerned with the long time behavior of some systems in porous and non-simple thermoelasticity. For porous systems, we studied three problems. The first problem involves a delay term, for which we proved exponential decay under appropriate assumptions on the coefficients of the dissipation and the delay. The second problem is concerned with a porous system with the Gurtin-Pipkin law. The third system is a porous system with microtemperature. We proved exponential stability for both systems. The fourth, fifth, and sixth problems are concerned with non-simple systems with and without porosity, as well as the presence of thermal effects of second sound type. We proved various stability results.

For the well-posedness, we used a semigroup approach and the Hille-Yosida and Lax-Milgram theorems. Additionally, for stability, we employed the multiplier method, Lyapunov functionals, and the Gearhart-Prüs theorem.

Keywords: Porosity, elasticity, non-simple materials, well-posedness, exponential decay, lack of exponential decay, slow decay.

Résumé

Dans cette thèse, nous nous intéressons au comportement à long terme de certains systèmes de thermoélasticité poreuse et non simple. Pour les systèmes poreux, nous avons étudié trois problèmes. Le premier problème concerne un terme de retard, pour lequel nous avons montré un résultat de décroissance exponentielle sous des hypothèses sur les coefficients de la dissipation et du retard. Le deuxième problème concerne un système thermoélastique poreux dont la température est modilarisé par la loi de Gurtin-Pipkin. Le troisième problème est un système poreux avec microtempérature. Pour les deux systèmes, nous avons prouvé la stabilité exponentielle, sous certains conditions sur les coefficients. Le quatrième, le cinquième et le sixième problèmes concernaient des systèmes de matériaux non simples avec et sans porosité, ainsi que en présence d'effets thermiques de deuxième son. Pour ces problèmes, nous avons prouvé quelques résultats de stabilité.

Pour la bien-posé, nous avons utilisé une approche de semi-groupe et les théorèmes de Hille-Yosida et Lax-Milgram. De plus, pour la stabilité, nous avons utilisé la méthode des multiplicateurs, des fonctionnelles de Lyapunov et le théorème de Gearhart-Prüs.

Mots clés: Porosité, élastique, matériaux non simples, existence unicité, décroissance exponentielle, absence de décroissance exponentielle, décroissance lente.

Notation

$H^1, H_0^1, H^2, H^4, W^P, W_0^P$	Sobolev Spaces,
$\langle \cdot, \cdot \rangle$	The inner Product,
∂	The operator of partial differentiation,
$D(\mathcal{A})$	The domain of the operator $\mathcal{A}$,
$\rho(\mathcal{A})$	The resolvent set of the operator $\mathcal{A}$,
$\sigma(\mathcal{A})$	The spectrum of operator $\mathcal{A}$,
$\lvert \cdot \rvert$	The euclidean norm on $\mathbb{R}^d$,
$N(\mathcal{A})$	The kernel of the operator $\mathcal{A}$,
$R(\mathcal{A})$	The range of the operator $\mathcal{A}$,
$\lvert \cdot \rvert$	The euclidean norm on $\mathbb{R}^d$,
$\lVert \cdot \rVert_X$	The norm on a normed space X,
$\mathcal{L}(X)$	The space of bounded linear operators from $X \text{into} X$,
X'	The dual space of X,
$Re\langle ., . \rangle$	The real part of the inner product,
$C_0^\infty(\Omega)$	The test functions space,
$(T(t))_{t \geqslant 0}$	A semigroup,
$C(X, Y)$	The space of all continuous functions from X into Y,
$\Omega \subset \mathbb{R}^N$	A domain of $\mathbb{R}^N$,
$\partial\Omega = \Gamma$	The boundary of Ω.

Contents

Introduction

The theory of porous materials is a significant generalization of the classical theory of elasticity. It is used to analyze and process elastic solids, where the skeleton of the material is thermoelastic and the interstices are empty of material. This theory is concerned with materials that contain small pores or vacuums.

The fundamental concept of this theory is that the apparent density is determined by the product of two factors: the density of the material and the volume fraction. This representation of the apparent density defines an supplimentary level of kinematic freedom in the theory. It was early used by Goodman and Cowin [36] to address the failing of the classical theory of elasticity in describing the deformation caused by the micro-structure contribution.

The theory of materials with voids, formulated by Goodeman and Cowin [36], is also valid for porous materials and it was motivated by physical reasons. In this theory, a high order stress and body force were introduced to calculate for the energy flux and energy supply combined with the time rate of the volume fraction. Similar terms are also present in the higher level elasticity theories formulated by Mindlin [61], Toupin [94], Green and Rivlin [38].

Nunziato and Cowin [63] utilized the constitutive equations established by Goodman and Cowin [36] to propose a nonlinear theory for the comportment of porous solids. This theory allows for both finite deformations and non-linear constitutive equations. Jarić and Golubović [44] and Jarić and Ranković [45] discussed the non-linear theory of thermoelastic materials with vacuums. Cowin and Nunziato [16] formulated a linear theory of elastic materials with vacuums to mathematically study the mechanical comportment of porous solids. An expansion of this theory to linear thermoelastic materials was suggested by Ieşan [41]. Furthermore, Ieşan [42], [43] included the micro-temperature elements to this theory.

Based on the micro-morphic continua theory, Grot [39] formulated a theory of the thermodynamics of elastic materials with internal structure. In this theory, the micro-elements not only undergo micro-deformations but also have micro-temperatures.

The importance of materials with micro-structure has been showen by the great number of papers that have been published in different fields of application, such as biology, petroleum industry, material science, and many others.

As these type of materials have both microscopic and macroscopic structures, scientists have examined the coupling and its strength. In addition, mathematicians have shown an increasing interest in analyzing the long-time behavior of solutions to thermoelastic and porous systems.

One of the earlier contribution in this field was the thermoelastic coupling suggested by Slemrod [86]. In the one-dimensional case, it was observed that the solutions decay in an exponential rate. Since then, many problems have been examined by considering various dissipation mechanisms at microscopic and/or the macroscopic states. Many papers have appeared in which the authors attempted to determine both the type and rate of decay of solutions in porous elasticity theory.

In continuum mechanics, when describing the motion of a material body, the constitutive equations characterize the properties of the material. Typically, a constitutive equation describes the response of the material to various stimuli applied to it.

In the classical theory of elasticity, the microscopic structure is ignored, only the macroscopic properties are analyzed; consequently, the field equations obtained are in the form of strain-stress relations. However, when dealing with non-classical elasticity, such as in the case of non-simple or porous materials, the microstructure takes part in the constitutive equations.

Thermal effects

In this section, we will give some explanations about the different laws which govern the transfer of the heat within a material body.

Classical Fourier law

The transfer of the heat within a material body is generally formulated by Fourier's law of thermal conductivity. It relates the heat flux vector q to the temperature gradient $\nabla\theta$

through the equation

$$q = -\kappa \nabla \theta.$$

Along with the conservation of energy law

$$c\theta_t + divq = 0.$$

These lead to the well-known parabolic heat equation

$$c\theta_t = \kappa \triangle \theta$$

where c is the specific heat capacity and κ is the thermal conductivity of the material.

The parabolic heat equation predicts the paradox of infinite speed: that is, the thermal signals propagate with infinite speed, which is physically inexact.

To address this paradox, many theories have been introduced, such as: second sound thermoelasticty, Green and Naghdi three theories, Gurtin-Pipkin theory and Coleman-Gurtin theory.

Second sound

The first theory suggested to correct the defect of infinite speed is to replace Fourier's law by the following Cattaneo law:

$$q + \tau q_t = -\kappa \nabla \theta$$

where τ is a non-negative constant that represents the time delay in the reaction of the heat flux to the temperature gradient. The corresponding heat transport system reads as:

$$\begin{cases} c_0 \rho \theta_t + \nabla \cdot q = 0 \\ q + \tau q_t + \kappa \nabla \theta = 0, \end{cases}$$

which is hyperbolic type system that predicts a finite speed equal to $\nu = \sqrt{\frac{\kappa}{c_0 \rho \tau}}$.

Green and Naghedi theories

The basic feature of Green and Naghdi is the introduction of a new independent variable called the thermal displacement given by

$$\alpha(x,t) = \alpha_0 + \int_0^t \theta(x,s)ds,$$

where α_0 is the initial value of the thermal displacement α at the reference time t_0 and $\frac{d\alpha}{dt} = \theta$. The theory is subdivided into three types, labelled types I, II, and III.

Thermoelasticity of type I

The standard assumption is $q = T_0 h$, which relates the heat flux vector q to the entropy flux vector $h = -c\nabla\theta$. Here, T_0 represents the absolute temperature.
This equation leads to the same law deduced from the classical theory

$$q = -\kappa\nabla\theta.$$

As a result, the heat equations of the classical and of the type I theory look the same from the mathematical point of view. However, the physical meaning is not the same.

This, with $\rho T_0 \theta_t = -divq$, gives the parabolic type equation

$$\rho T_0 \theta_t = \kappa\Delta\theta.$$

Thermoelasticity of type II

The constitutive relation for the entropy flux h is given by

$$h = -\kappa_2\nabla\alpha.$$

The resulting hyperbolic heat equation

$$\rho c\theta_t = \rho c\alpha_{tt} = \kappa_2\Delta\alpha.$$

This shows that the heat propagates without energy dissipation at a wave speed of $\sqrt{\frac{\kappa_2}{c\rho}}$.

Thermoelasticity of type III

The heat flux is expressed as follows:

$$q = -\kappa_3\nabla\alpha - \kappa_4\nabla\theta.$$

The heat equation reads:

$$\rho c\theta_t = \rho c\alpha_{tt} = \kappa_3\Delta\alpha - \kappa_4\Delta\theta.$$

This implies that the heat propagates with energy dissipation at a wave speed of $\nu = \sqrt{\frac{\kappa_3}{c\rho}}$.

Gurtin and Pipkin

Gurtin and Pipkin have formulated a general non-linear theory of heat conduction that admits the second sound thermoelasticity. They assumed that the heat flux depends on the accumulated history of the temperature gradient. In the linear approximation, the law of heat conduction of this theory is given by

$$q(t) = -\theta_0 \int_{-\infty}^{t} \kappa(t-s)\nabla\theta(s)ds,$$

where κ is relaxation function that depends on the material.

Some literature results

The study of the long-time behavior for porous systems was investigated previously by many scientists for example, Quintanilla [72] studied the system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + \beta\phi_x, & \text{in } (0,L)\times(0,+\infty), \\ \rho\kappa\phi_{tt} = \alpha\phi_{xx} - \beta u_x - \xi\phi - \tau\phi_t, & \text{in } (0,L)\times(0,+\infty), \end{cases} \tag{1}$$

and showed that the solution decays slowly. Precisely, he proved that the frictional damping $\tau\phi_t$ is not sufficient to create an exponential stability. The same result was obtained by Magaña and Quintanilla [54], when the porous damping was replaced by the visco-elastic damping γu_{txx} acting on the first equation of (1), and they obtained an exponential decay when the two equations were linearly controlled.

Quintanilla and co-authors [14, 15, 78] coupled of the system (1) with thermal effects. More specifically, they considered the following system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x, & \text{in } (0,\pi)\times(0,+\infty), \\ J\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi + m\theta - \tau\varphi_t, & \text{in } (0,\pi)\times(0,+\infty), \\ c\theta_t = k^*\theta_{xx} - \beta u_{xt} - m\varphi_t, & \text{in } (0,\pi)\times(0,+\infty), \end{cases} \tag{2}$$

with the boundary conditions

$$u(0,t) = u(\pi,t) = \varphi_x(0,t) = \varphi_x(\pi,t) = \theta_x(0,t) = \theta_x(\pi,t) = 0, \tag{3}$$

where θ represents the temperature variation from the equilibrium reference value. They obtained an exponential rate of decay for $\tau \neq 0$ in [14], and polynomial rate for $\tau = 0$ in [78].

A porous thermoelasticity coupling with a weaker dissipation of memory type was considered by Messaoudi and Fareh [56],[57]. They studied the system

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x + \theta_x = 0 & \text{in } (0,1)\times\mathbb{R}_+, \\ \rho_2\psi_{tt} - \alpha\psi_{xx} + k\left(\varphi_x + \psi\right) - \theta + \int_0^t g(t-s)\psi_{xx}(s)ds = 0 & \text{in } (0,1)\times\mathbb{R}_+, \\ \rho_3\theta_t - \kappa\theta_{xx} + \varphi_{xt} + \psi_t = 0 & \text{in } (0,1)\times\mathbb{R}_+, \end{cases}$$

where g is a relaxation function satisfying $g'(t) \leq -\xi(t)g(t)$, for a non-increasing positive function ξ and established general decay results depending on the rate of decay of g, for equal and non-equal speeds.

Later, in 2017, Apalara [9] and Santos *et al.* [82] improved, separately, the result of [72]. They acheived an exponential stability of the solution of (1) provided that

$$\frac{\mu}{\rho} = \frac{\alpha}{\rho\kappa}.$$

The same result was obtained by Santos *et al.* [82] if the damping $-\tau\phi_t$ is replaced by the elastic damping γu_t in the first equation.

Apalara [10] investigated , in a bounded domain, the following porous system with memory

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \displaystyle\int_0^t g(t-s)\phi_{xx}(s)ds = 0, \end{cases} \tag{4}$$

where g is a C^1- decreasing relaxation function satisfying $g'(s) \leq -\eta(s)g(s)$ for a nonincreasing function η. In the case where the speeds are equal $\frac{\mu}{\rho} = \frac{\delta}{J}$, he established a general decay result depending on the relaxation function g. Feng and Apalara [26] improved the result of [10] by assuming a more general relaxation function $g \in L^1(0,\infty)$ that satisfies $g'(s) \leq -\eta(s)H(g(s))$, for an increasing and convex function H. They established optimal explicit and general energy decay results. In fact, the authors of [9, 10, 26, 82] improved significantly the results of Quintanilla and co-authors [14, 54]. Precisely, they proved that a single dissipative mechanism can exponentially stabilize a porous system provided the wave speeds are equal.

In (2) the heat conduction is given through Fourier's law

$$q + \kappa\theta_x = 0, \tag{5}$$

which leads to a coupled system of hyperbolic-parabolic type. Several results on existence, uniqueness and asymptotic behavior of thermoelastic systems in the framework of aforementioned law can be found in [14, 15, 25, 28, 29, 56, 57, 58, 66, 88, 89, 90].

We would like to point out that the model based on Fourier's law (5) predicts the paradox of infinite-speed heat propagation. To address this physical paradox, numerous alternative models of heat conduction have been proposed. Lord and Shulman [52] suggested replacing (5) with Cattaneo's law

$$\tau q_t + q + \kappa\nabla\theta = 0, \tag{6}$$

which is an improvement of the classical Fourier law.

In the context of this theory, Messaoudi and Fareh [24, 55] considered the following porous thermoelastic system of second sound

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x - \gamma u_t, \\ J\phi_{tt} = \alpha\phi_{xx} - bu_x - \xi\phi - \beta\theta_x, \\ c\theta_t = -q_x - \beta\phi_{tx} - \delta\theta, \\ \tau_0 q_t + q + \kappa\theta_x = 0, \end{cases} \tag{7}$$

where $\rho, \mu, J, \alpha, \xi, \beta, c, \kappa$ and τ_0 are positive constants. for $\gamma \neq 0$, they obtained an exponential decay result without any restrictions on the coefficients. However, for $\gamma = 0$ and $\mu\xi = b^2$, the exponential decay was obtained provided that

$$\chi = \beta^2 - \left(\frac{c\alpha\mu}{\rho} - \frac{\alpha\kappa}{\tau_0}\right)\left(\frac{J}{\alpha} - \frac{\rho}{\mu}\right) = 0.$$

Regarding the nonsimple elasticity theory, Liu *et al.* [50] combined the non-simple elasticity with porosity. They considered the system

$$\begin{cases} \rho u_{tt} = au_{xx} + b\phi_x - cu_{xxxx} - d\phi_{xxx}, & \text{in } (0,\pi)\times(0,+\infty), \\ J\phi_{tt} = du_{xxx} + \beta\phi_{xx} - \xi\phi - bu_x, & \text{in } (0,\pi)\times(0,+\infty), \end{cases} \tag{8}$$

subjected to the boundary conditions

$$u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = \phi_x(0,t) = \phi_x(\pi,t) = 0, \quad \text{in} \quad (0,+\infty),$$

with various dissipation mechanisms. They obtained exponential decay results when they employed the dissipation δu_{txx}, $-\alpha u_{txxxx}$ and $m\varphi_{txx}$ provided that $b + dn^2 \neq 0$ for all $n \in \mathbb{N}$, and slow decay for the dissipation $-\tau\varphi_t$. In fact, the exponential stability is a consequence of the strong coupling between the equations of system (8) reached by the presence of ϕ_{xxx}. Whereas, if $d = 0$ the coupling is weak and the solution decays slowly.

Fernández Sare *et al.* [34] studied the system

$$\begin{cases} \rho u_{tt} - \mu u_{xx} + \alpha u_{xxxx} - \beta\theta_x = 0, \\ c\theta_t - \kappa\theta_{xx} - \beta u_{tx} = 0, \end{cases} \tag{9}$$

with several boundary conditions. They proved that the thermal dissipation alone suffices to bring the whole system to an exponential decay.

Recently, Fernàndez *et al.* [31] examined the system

$$\begin{cases} \rho u_{tt} = -cu_{xxxx} - d\phi_{xxx} + au_{xx} + b\phi_x - \beta^*\theta_x, \\ J\phi_{tt} = \beta\phi_{xx} + du_{xxx} - \xi\varphi - bu_x + m\theta - \tau\phi_t, \\ c^*\theta_t = -\beta^* u_{tx} - m\phi_t + k^*\theta_{xx}, \end{cases} \tag{10}$$

and obtained an exponential decay result.

In the framework of second-sound thermoelasticity, Fernàndez Sare and Muñoz Rivera [33] studied the time behavior of the following plate system in $\mathbb{R}^2$

$$\begin{cases} \rho u_{tt} - \mu\Delta u_{tt} + \gamma\Delta^2 u + \alpha\theta = 0, & \text{in } \Omega, \\ c\theta_t + \kappa divq - \alpha\Delta u_t = 0, & \text{in } \Omega, \\ \tau q + \kappa_0 q + \kappa\nabla\theta = 0, & \text{in } \Omega. \end{cases} \tag{11}$$

They established an exponential rate of decay for $\mu > 0$. The same result for $\gamma > 0$ was obtained in [80] for the system

$$\begin{cases} \rho u_{tt} - \gamma u_{ttxx} - \mu u_{xx} + \alpha u_{xxxx} - \beta\theta_x = 0, & \text{in } (0,\ell) \times \mathbb{R}_+, \\ c\theta_t + q_x - \beta u_{tx} = 0, & \text{in } (0,\ell) \times \mathbb{R}_+, \\ \tau q_t + q + \kappa\theta_x = 0, & \text{in } (0,\ell) \times \mathbb{R}_+. \end{cases} \tag{12}$$

This thesis is divided into two parts and is organized as follows: In the first chapter, we give some functional preliminaries and tools that we used in the subsequent chapters. The first chapter of the part one is devoted to the study of a porous thermoelastic system with delay. In the second chapter of this part, we investigate a porous system with thermal effects given by Gurtin-Pipkin law. The third chapter is devoted to the study of a porous thermoelastic system with microtemperature. In the first chapter of the second part, we consider a non-simple system with thermoelasticity of second sound. The second chapter is assigned to the study of some non-simple porous system. The third chapter is devoted to the study of non-simple porous systems with second-sound thermoelasticity.

chapter 1 Preliminear

In this chapter, we recall some mathematical concepts that will be used throughout this thesis.

Hereafter in this chapter, we denote by Ω a bounded domain of $\mathbb{R}^n$ and by X a Hilbert space.

1.1 Sobolev Spaces

Let's begin by the L^p spaces

Definition 1.1. *[13] Let $p \in \mathbb{R}$ with $1 \leq p < +\infty$, we define the space*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}; f \quad \text{is measurable and} \quad \int_\Omega |f(x)|^p \, dx < +\infty \right\},$$

which is a Banach space with respect to the norm

$$\|f\|_{L^p} = \|f\|_p = \left[\int_\Omega |f(x)|^p \, dx \right]^{1/p},$$

for $p = \infty$,

$$L^\infty(\Omega) = \left\{ \begin{array}{ll} f : \Omega \to \mathbb{R}; f & \text{is measurable and} \quad \exists c > 0 \text{ suth that} \\ & |f(x)| \leq c \text{ a.e. on } \Omega \end{array} \right\},$$

which is a Banach space with respect to the norm

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf \{ C; \; |f(x)| \leq C \quad \text{a.e. on } \Omega \}.$$

Remark 1.1. *For $p = 2$, $L^2(\Omega)$ is a Hilbert space with respect to the inner product*

$$\langle f, g \rangle_{L^2} = \int_\Omega f(x) \, g(x) dx.$$

Lemma 1.1. *The subspace $L^2_*(\Omega)$ of $L^2(\Omega)$, defined by*

$$L^2_*(\Omega) = \left\{ f \in L^2(\Omega); \ \int_\Omega f(x)dx = 0 \right\},$$

is a Hilbert space.

Proof. Let $(\varphi_n) \subset L^2_*(\Omega)$ be a convergent sequence to φ in $L^2(\Omega)$, then

$$\begin{aligned} \left| \int_\Omega \varphi(x)\, dx \right| &= \left| \int_\Omega \varphi(x)\, dx - \int_\Omega \varphi_n(x)\, dx \right| \\ &= \left| \int_\Omega [\varphi(x) - \varphi_n(x)]\, dx \right|. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\left| \int_\Omega \varphi(x)\, dx \right| \leq mes(\Omega) \left[\int_\Omega |\varphi(x) - \varphi_n(x)|^2\, dx \right]^{1/2}.$$

Since $\lim\limits_{n\to\infty} \int_\Omega |\varphi(x) - \varphi_n(x)|^2\, dx = 0$, we get

$$\int_\Omega \varphi(x)\, dx = 0,$$

which implies that $\varphi \in L^2_*(\Omega)$. Consequently $L^2_*(\Omega)$ is closed in $L^2(\Omega)$, which completes the proof. □

Definition 1.2. *[95] For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, we define the Sobolev space $W^{k,p}(\Omega)$ by*

$$W^{k,p}(\Omega) = \left\{ u \in L^p(\Omega); \ D^\alpha u \in L^p(\Omega), \ \forall \alpha \in \mathbb{N}^n \ \textit{with} \ |\alpha| \leq k \right\},$$

where $|\alpha| = \alpha_1 + ... + \alpha_n$ and $D^\alpha u$ is the $\alpha-th$ weak derivative of u which is defined as

$$\int_\Omega u(x)\, D^\alpha v(x) dx = |-1|^\alpha \int_\Omega \varphi(x) v(x)\, dx, \qquad \forall v \in C_c^\infty(\Omega)$$

and

$$\varphi = D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}.$$

The space $W^{k,p}(\Omega)$, equipped with the norm

$$\|u\|_{k,p} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty$$

and

$$\|u\|_{k,\infty} = \max_{|\alpha|\leq k} \|D^\alpha u\|_\infty$$

are Banach spaces. The space $W^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$ and it is a Hilbert space with respect to the inner product

$$\langle u, v\rangle_{H^k} = \sum_{|\alpha|\leq k} \int_\Omega D^\alpha u\,(x)\, D^\alpha v(x)dx, \qquad \forall u, v \in H^k\,(\Omega)\,.$$

Remark 1.2. *We denote by $H^1_*(\Omega)$ and $H^2_*(\Omega)$, the Hilbert spaces*

$$H^1_*(\Omega) = H^1\,(\Omega) \cap L^2_*\,(\Omega) = \left\{u \in H^1(\Omega);\ \int_\Omega u(x)dx = 0\right\},$$

$$H^2_*(\Omega) = \left\{u \in H^2(\Omega);\ \ \frac{\partial u}{\partial \eta} = 0 \ in\ \partial\Omega\right\} = \left\{\nabla u \in \left(H^1(\Omega)\right)^n;\ \ \frac{\partial u}{\partial \eta} = 0 \ in\ \partial\Omega\right\},$$

where η is the unit outward vector on $\partial\Omega$.

Definition 1.3. *[13] Given $1 \leq p < \infty$, we denote by $W^{k,p}_0(\Omega)$ the closure of $C^\infty_c(\Omega)$ in $W^{k,p}(\Omega)$.*

For $p = 2$, we note

$$H^k_0(\Omega) = W^{k,2}_0(\Omega).$$

Notation 1.1. *For $1 \leq p < \infty$ and $1/p + 1/q = 1$. The dual space of $W^{k,p}_0(\Omega)$ is denoted by $W^{-k,q}(\Omega)$ and the dual space of $H^k_0(\Omega)$ is denoted by $H^{-k}(\Omega)$.*

Theorem 1.2. *[13]* ***(Lax Milgram)***

Assume that $b(x, y)$ is a continuous and coercive bilinear form on X. Then, given any $\phi \in X'$, there exists a unique element $x \in X$ such that

$$b\,(x, y) = \langle\phi, y\rangle, \quad \forall y \in X.$$

Moreover, if a is symmetric, then x is characterized by the property

$$\frac{1}{2}b(x, x) - \langle\phi, x\rangle = \min_{y\in X}\left\{\frac{1}{2}b(y, y) - \langle\phi, y\rangle\right\}.$$

Theorem 1.3. ***(Rellich-Kondrachov)****[1] Suppose that Ω is bounded and of class C^1. Then, we have the following compact embedding:*

$$H^m(\Omega) \subset H^j(\Omega),\ \forall j < m.$$

1.2 Some inequalities

In this section, we recall some inequalities which are of major importance and to be frequently used in the rest of the thesis.

Theorem 1.4. ***(Young's inequality)****[13] Let $0 < p, q < \infty$ be real conjugates, then*

$$\forall a, b \in \mathbb{R}, \quad |ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}.$$

As a consequence, if $p = q = 2$, we have

$$|ab| \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \quad \forall \varepsilon > 0.$$

Theorem 1.5. *[13]****(Hölder's inequality)*** *Suppose that $u \in L^p$ and $v \in L^q$ where $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, $uv \in L^1$ and*

$$\int |uv| \leq \|u\|_p \|v\|_q .$$

Theorem 1.6. ***(Cauchy-Schwarz inequality)****[13] Let X be a Hilbert space and $\langle \cdot, \cdot \rangle$ be its inner product, then*

$$|\langle f, g\rangle| \leq \langle f, f\rangle^{\frac{1}{2}} \langle g, g\rangle^{\frac{1}{2}}, \quad \forall f, g \in X.$$

Theorem 1.7. ***(Poincaré's inequality)*** *Suppose that Ω is bounded in $\mathbb{R}^n$ and let $u \in H_0^1(\Omega)$, then there exists a constant $C > 0$, depending only on $mes(\Omega)$ such that*

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

Remark 1.3. *Poincaré's inequality also holds for all $u \in H^1(\Omega)$ that satisfy*

$$\int_\Omega u(x) dx = 0.$$

1.3 Some Semigroup arguments

Definition 1.4. *Let $\mathcal{B} \in \mathcal{L}(X)$ be a bounded operator. The resolvent set $\rho(\mathcal{B})$ of $\mathcal{B}$ is the set of all λ in $\mathbb{C}$ for which $(\mathcal{B} - \lambda I)^{-1}$ exists and bounded*

$$\rho(\mathcal{B}) = \left\{ \lambda \in \mathbb{C}; \quad (\mathcal{B} - \lambda I)^{-1} \in \mathcal{L}(X) \right\}.$$

The spectrum of $\mathcal{B}$, denoted by $\sigma(\mathcal{B})$, is the complement of the resolvent set, i.e.,

$$\sigma(\mathcal{B}) = \mathbb{C}\backslash\rho(\mathcal{B}).$$

A complex number λ is an eigenvalue of $\mathcal{B}$ if

$$N(\mathcal{B} - \lambda I) \neq \{0\}.$$

Theorem 1.8. *Let X be a Banach space and $\mathcal{B} \in \mathscr{L}(X)$. If the operator $\mathcal{B}$ satisfies $\|\mathcal{B}\| < 1$, then $I - \mathcal{B}$ is invertible and its inverse is given by*

$$(I - \mathcal{B})^{-1} = \sum_{n=0}^{\infty} \mathcal{B}^n.$$

Definition 1.5. *[67] Let X be a Banach space. A semigroup of bounded linear operators is a family of bounded linear operators $T(t) \in \mathcal{L}(X)$, which depend on a parameter $0 \leq t < \infty$ and that satisfies the following properties*

$T(0) = I$, (I is the identity operator on X).

$T(t + s) = T(t)T(s)$ for every $t, s \geq 0$ (the semigroup property).

A semigroup of bounded linear operators $T(t)$ is uniformly continuous if

$$\lim_{t\to 0} \|T(t) - I\| = 0.$$

The infinitesimal generator of a semigroup $T(t)$ is the operator $\mathcal{B}$ defined on

$$D(\mathcal{B}) = \left\{ x \in X : \lim_{t\to 0} \frac{T(t)x - x}{t} \qquad \text{exists} \right\}$$

by

$$\mathcal{B}x = \lim_{t\to 0} \frac{T(t)x - x}{t}, \quad \text{for} \quad x \in D(\mathcal{B}).$$

Definition 1.6. *[13](**Maximal Monotone Operators**): Let X be a Hilbert space, an unbounded linear operator $\mathcal{B} : D(\mathcal{B}) \subset X \to X$ is said to be monotone (accretive) if it satisfies*

$$Re\, \langle \mathcal{B}v, v\rangle \geq 0, \quad \forall v \in D(\mathcal{B}).$$

In addition, the operator is called maximal monotone, if $R(I + \mathcal{B}) = X$ i.e.,

$$\forall f \in X,\ \exists u \in D(\mathcal{B}), \text{such that } u + \mathcal{B}u = f.$$

Remark 1.4. *If $-\mathcal{B}$ is monotone, we say that $\mathcal{B}$ is dissipative.*

Proposition 1.1. *[13] Let $\mathcal{B}$ be a maximal monotone operator on a Hilbert space. Then*

1) $D(\mathcal{B})$ is dense in X.

2) $\mathcal{B}$ is a closed operator.

3) For every $\lambda > 0$, $(I + \lambda\mathcal{B})$ is bijective from $D(\mathcal{B})$ into X, $(I + \lambda\mathcal{B})^{-1}$ is a bounded operator, and $\|(I + \lambda\mathcal{B})^{-1}\|_{\mathcal{L}(X)} \leq 1$.

Theorem 1.9. ***(Hille-Yosida)****[13] Let $\mathcal{B}$ be a maximal monotone operator. Then, given any $u_0 \in D(\mathcal{B})$ there exists a unique function*

$$u \in C^1([0, +\infty); X) \cap C([0, +\infty); D(\mathcal{B}))$$

satisfying

$$\begin{cases} \dfrac{du}{dt} + \mathcal{B}u = 0 \quad \text{on } [0, +\infty), \\ u(0) = u_0. \end{cases} \tag{1.1}$$

Theorem 1.10. ***(Lumer-Phillips)*** *[67, 95] Let $\mathcal{B} : D(\mathcal{B}) \subset X \longrightarrow X$ be a densely defined operator on a Hilbert space X. Then $\mathcal{B}$ generates a C_0-semigroup of contractions on X if and only if*

(i) $\mathcal{B}$ is dissipative;

(ii) there exists $\lambda > 0$ such that $\lambda I - \mathcal{B}$ is surjective.

Remark 1.5. *Suppose that assumptions (i) and (ii) of Theorem 1.10 are satisfied, then $D(\mathcal{B})$ is dense in X.*

For the proof see [13]

Theorem 1.11. *[51] Let $\mathcal{B} : D(\mathcal{B}) \subset X \longrightarrow X$, is a linear operator and X Hilbert space.*

If $D(\mathcal{B})$ is dense in X, $\mathcal{B}$ is dissipative and $0 \in \rho(\mathcal{B})$. Then, $\mathcal{B}$ is the infinitesimal generator of a C_0-semi-group of contractions on X.

For the proof see [51]

Theorem 1.12. *[95] Let X be a Banach space and $\mathcal{B} : D(\mathcal{B}) \subset X \longrightarrow X$ be the infinitesimal generator of a C_0-semigroup $\{S(t); t \geq 0\}$ on X. Then, for each $\xi \in D(\mathcal{B})$ and each $t \geq 0$, we have $S(t)\xi \in D(\mathcal{B})$, and the mapping*

$$t \longrightarrow S(t)\xi$$

is class C^1 on $[0, +\infty)$ and satisfies

$$\frac{d}{dt}(S(t)\xi) = \mathcal{B}S(t)\xi = S(t)\mathcal{B}\xi. \tag{1.2}$$

1.4 Exponential stability notions

Definition 1.7. *The solution $U(t) = e^{\mathcal{B}}U_0$ of (1.1) is said to be exponentially stable if there exist two positive constants α and $M > 1$ such that*

$$\|U(t)\| \leq Me^{-\alpha t}, \quad \forall t \geq 0.$$

Theorem 1.13. *[35, 51] A C_0-semigroup of contractions $S(t) = e^{\mathcal{B}t}$ generated by an operator $\mathcal{B}$ in a Hilbert space X is exponentially stable if and only if*

i) $i\mathbb{R} = \{i\lambda, \lambda \in \mathbb{R}\} \subset \rho(\mathcal{B})$,

ii) $\overline{\lim\limits_{|\lambda|\to\infty}} \|(i\lambda I - \mathcal{B})^{-1}\| < \infty$.

Theorem 1.14. *[7] Let $\mathcal{B}$ (unbounded operator) be the infinitesimal generator of a semigroup of contractions $S(t) = e^{\mathcal{B}t}$. Then, $S(t)$ is exponentially stable if and only if there exists a positive constant m such that*

$$\inf_{\lambda\in\mathbb{R}} \|(i\lambda I - \mathcal{B})U\| \geq m\|U\|, \ \forall U \in D(\mathcal{B}). \tag{1.3}$$

Theorem 1.15. *[35, 51] A C_0-semigroup of contractions $S(t) = e^{\mathcal{B}t}$, generated by an operator $\mathcal{B}$ in a Hilbert space $\mathcal{H}$, is analytic if and only if*

i) $i\mathbb{R} \subset \rho(\mathcal{B})$,

ii) $\overline{\lim\limits_{|\lambda|\to\infty}} \|\lambda(i\lambda I - \mathcal{B})^{-1}\| < \infty$.

Theorem 1.16. (Hurwitz) *[22] Let*

$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n, \ (a_0 > 0).$$

All the zeros of the polynomial $f(x)$ have negative real parts if and only if all the determinants

$$\Lambda_i = \det \begin{pmatrix} a_1 & a_0 & 0 & 0 & .. & 0 \\ a_3 & a_2 & a_1 & a_0 & .. & 0 \\ a_5 & a_4 & a_3 & a_2 & .. & .. \\ .. & .. & .. & .. & .. & .. \\ a_{2i-1} & a_{2i-2} & a_{2i-3} & a_{2i-4} & .. & a_i \end{pmatrix}, \forall i > 1$$

are positive.

Part I

Stability of some porous thermoelastic systems

chapter 2

A porous thermoelastic system with delay

2.1 Introduction

The results of this chapter were published in [4].

In this chapter, we investigate the long-time behavior of the following system

$$\begin{cases} \rho u_{tt}(x,t) = a u_{xx}(x,t) + b\phi_x(x,t), & \text{in } (0,\pi)\times(0,\infty), \\ J\phi_{tt}(x,t) = \alpha\phi_{xx}(x,t) - b u_x(x,t) - \xi\phi(x,t) - \beta\theta_x(x,t), & \text{in } (0,\pi)\times(0,\infty), \\ c\theta_t(x,t) = \kappa\theta_{xx}(x,t) + k\theta_{xx}(x,t-\tau) - \beta\phi_{xt}(x,t), & \text{in } (0,\pi)\times(0,\infty), \end{cases} \tag{2.1}$$

where u,ϕ and θ are, respectively, the transversal displacement, the volume fraction and the difference of the temperature. The coefficient ρ is the mass density, J is the product of the mass density by the equilibrated inertia, a is a constant that depends on the geometry of the material, α is the volume distributed density, c is the thermal conductivity and b and β are non zero coupling constants such that

$$\rho>0, a>0,\ J>0,\ \alpha>0,\ \xi>0, c>0,\ \kappa>0, \text{ and } a\xi > b^2. \tag{2.2}$$

The system (2.1) is supplemented with the following initial, history and boundary conditions

$$\begin{cases} u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ \phi(x,0)=\phi_0(x),\ \phi_t(x,0)=\phi_1(x), & x\in(0,\pi), \\ \theta(x,0)=\theta_0(x),\ \theta(x,t-\tau)=f_0(x,t-\tau), & t\in(0,\tau), \\ u(0,t)=u(\pi,t)=\phi_x(0,t)=\phi_x(\pi,t)=\theta(0,t)=\theta(\pi,t)=0, & t\geq 0. \end{cases} \tag{2.3}$$

In the real world, the current state of a material depends on some past occurrences, because in most instances, the response of the material to an applied force is often delayed. Therefore, time-delay appears in many control systems in practical engineering, chemistry, biology and economy. To the best of our knowledge, the first contribution that studied the effect of the time delay on the stability of an equation was made by Datko [20]. He considered the equation

$$u_{tt}(x,t) = u_{xx}(x,t) - 2u_t(x,t-\tau), \text{ in } (0,1)\times(0,+\infty),$$

where the delay is acting on the velocity. He proved that the time delay in the damping term can destabilize the system. He also considered the equation

$$u_{tt}(x,t) = u_{xx}(x,t) - 3u_t(x,t) - u_t(x,t-\tau), \text{ in } (0,1)\times(0,+\infty),$$

and showed that the solution is exponentially stable. Also, Datko *et al.* [19] considered the problem

$$\begin{cases} u_{tt} - u_{xx} + 2au_t + a^2u_t = 0, & \text{in } \quad (0,1)\times(0,+\infty), \\ u\left(0,t\right) = 0,\ u_x\left(1,t\right) = -ku_t(1,t-\tau), & \text{in } \quad (0,+\infty), \end{cases} \tag{2.4}$$

where the delay is acting on the boundary. They proved that for $a > 0$ and $k \geq \dfrac{1-e^{-2a}}{1+e^{-2a}}$, there exists a sequence of delay terms for which the solutions of (2.4) can not be exponentially stable. Nicaise and Pignotti [62] considered the wave equation with dissipation and delay:

$$u_{tt}(x,t) - \Delta u(x,t) + a(x)\left[\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)\right] = 0, \text{ in } \Omega\times(0,+\infty),$$

where $a \in L^\infty(\Omega)$ and $a(x) \geq 0$, $a.e$ in Ω, and showed that the energy decays exponentially to zero, provided that $0 \leq \mu_2 < \mu_1$. In the case where $\mu_2 \geq \mu_1 > 0$, they constructed a sequence of small delays that destabilize the system. They, also, obtained a similar result when the delay is acting on the part of the boundary of Ω.

For coupled thermoelastic systems, Racke [75] considered the system

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t-\tau_1) + b\theta_x(x,t) = 0, & \text{in } (0,L)\times(0,\infty), \\ \theta_t - d\theta_{xx}(x,t-\tau_2) + bu_{tx}(x,t) = 0, & \text{in } (0,L)\times(0,\infty), \end{cases} \tag{2.5}$$

for $\tau_1 > 0$ or $\tau_2 > 0$, and showed that system (2.5) is not well-posed and instable. However, for $\tau_1 = \tau_2 = 0$, the system is exponentially stable. Liu and Chen [49] considered the following porous thermoelastic system with second sound and time-varying delay

$$\begin{cases} \rho_0 u_{tt} - au_{xx} - b\phi_x + \gamma_1 u_t + \gamma_2 u_t\left(\cdot, t-\tau\left(t\right)\right) = 0, & \text{in } (0,1)\times(0,\infty), \\ J\phi_{tt} - \alpha\phi_{xx} + bu_x + \xi\phi - \beta\theta_x = 0, & \text{in } (0,1)\times(0,\infty), \\ c\theta_t + q_x - \beta\phi_{tx} + \delta\theta = 0, & \text{in } (0,1)\times(0,\infty), \\ \tau_0 q_t + q + k\theta_x = 0, & \text{in } (0,1)\times(0,\infty), \end{cases}$$

and established an exponential decay result under specifie assumptions on the weights of the damping and delay. Borges and Santos [12] investigated the porous elastic system with time-varying delay term

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, & \text{in } (0,1) \times (0,\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \mu_1\phi_t + \mu_2\phi_t\left(t - \tau\left(t\right)\right) = 0, & \text{in } (0,1) \times (0,\infty), \end{cases}$$

with Dirichlet boundary conditions, and established an exponential rate of decay. Ouchenane *et al.* [64] considered the following system

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b\phi_x, \\ \rho_2\phi_{tt} = \delta\phi_{xx} - bu_x - \xi\phi - \beta\theta_x - \mu_1\phi_t - \displaystyle\int_{\tau_1}^{\tau_2} \left|\mu_2\left(s\right)\right| \phi_t\left(x, t-s\right) ds, \\ \rho_3\theta_{tt} = l\theta_{xx} - \gamma\phi_{ttx} + k\theta_{txx}, \end{cases}$$

under the assumption

$$\int_{\tau_1}^{\tau_2} \left|\mu_2\left(s\right)\right| ds < \mu_1, \tag{2.6}$$

and proved an exponential stability result in the case when $\dfrac{\mu}{\rho_1} = \dfrac{\delta}{\rho_2}$ and a polynomial decay result otherwise.

In this chapter, we are concerned with a porous thermoelastic system that includes a single dissipation induced by thermal effects, as described by Fourier's law. In addition, the system includes a delay term that impacts the thermal velocity. Under the assumption $|k| < \kappa$, we establish an exponential decay result if $\dfrac{a}{\rho} = \dfrac{\alpha}{J}$.

To the best of our knowledge, porous thermoelastic problems with only macrothermal dissipation were never considered before. Our result improves those of [14, 54], in which they considered two dissipative mechanisms, and extends the results of [9, 82] to the case of thermal dissipation and in the presence of delay.

The rest of this chapter is organized as follows: In Section 2, we use a semigroup approach to prove the existence of unique solution to problem (2.1), (2.3). Section 3 is devoted to the proof of the exponential decay. In Section 4, we prove the lack of the exponential decay in the case of nonequal speed of propagation.

2.2 Well posedness

In this section, we adopt a semigroup approach to prove the existence of a unique solution to the problem (2.1),(2.3). Note, first, that the Neumann boundary conditions, assumed for ϕ, prevent the use of Poincaré's inequality. To overcome this defect, we proceed as follows. From the second equation of (2.1) and the boundary conditions (2.3), we have

$$J\frac{d^2}{dt^2}\int_0^\pi \phi\left(x,t\right)dx+\xi\int_0^\pi \phi\left(x,t\right)dx=0,$$

which gives

$$\int_0^\pi \phi\left(x,t\right)dx=\left(\int_0^\pi \phi_0\left(x\right)dx\right)\cos\left(\sqrt{\frac{\xi}{J}}t\right)+\sqrt{\frac{J}{\xi}}\left(\int_0^\pi \phi_1\left(x\right)dx\right)\sin\left(\sqrt{\frac{\xi}{J}}t\right).$$

Consequently, if we set

$$\overline{\phi}\left(x,t\right)=\phi\left(x,t\right)-\frac{1}{\pi}\left(\int_0^\pi \phi_0\left(x\right)dx\right)\cos\left(\sqrt{\frac{\xi}{J}}t\right)-\frac{1}{\pi}\sqrt{\frac{J}{\xi}}\left(\int_0^\pi \phi_1\left(x\right)dx\right)\sin\left(\sqrt{\frac{\xi}{J}}t\right),$$

then, $\left(u,\overline{\phi},\theta\right)$ satisfies (2.1), the boundary and initial conditions (2.3) with

$$\overline{\phi}(x,0)=\overline{\phi}_0(x)=\phi_0(x)-\frac{1}{\pi}\int_0^\pi \phi_0(x)dx,$$

$$\overline{\phi}_t(x,0)=\overline{\phi}_1(x)=\phi_1(x)-\frac{1}{\pi}\int_0^\pi \phi_1(x)dx$$

and, more importantly,

$$\int_0^\pi \overline{\phi}\left(x,t\right)dx=0,$$

which allows the use of Poincaré's inequality for $\overline{\phi}$.

In the sequel, we work with $(u,\overline{\phi},\theta)$ but, for convenience, we write (u,ϕ,θ).

To deal with the delay term, we follow [62] and introduce the new variable

$$z\left(x,\omega,t\right)=\theta(x,t-\tau\omega),\qquad \omega\in(0,1),\ x\in(0,\pi),\ t>0.$$

Clearly, we have

$$\tau z_t\left(x,\omega,t\right)+z_\omega\left(x,\omega,t\right)=0,\qquad \omega\in(0,1),\ x\in(0,\pi),\ t>0. \tag{2.7}$$

Then, (2.7), together with (2.1), can be written as

$$\begin{cases} \rho u_{tt}(x,t) = a u_{xx}(x,t) + b\phi_x(x,t), & \text{in } (0,\pi)\times(0,+\infty), \\ J\phi_{tt}(x,t) = \alpha\phi_{xx}(x,t) - b u_x(x,t) - \xi\phi(x,t) - \beta\theta_x(x,t), & \text{in } (0,\pi)\times(0,+\infty), \\ c\theta_t(x,t) = \kappa\theta_{xx}(x,t) + k z_{xx}(x,1,t) - \beta\phi_{xt}(x,t), & \text{in } (0,\pi)\times(0,+\infty), \\ \tau z_t(x,\omega,t) + z_\omega(x,\omega,t) = 0, & \text{in } (0,\pi)\times(0,1)\times(0,+\infty), \end{cases} \tag{2.8}$$

and

$$\begin{cases} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x), \\ \theta(x,0) = \theta_0(x),\ z(x,\omega,0) = z_0(x,\omega) = \theta(x,-\tau\omega) = f_0(x,-\tau\omega), \\ z(x,0,t) = \theta(x,t),\ z(x,1,t) = \theta(x,t-\tau),\ \ t\in(0,\tau), \\ u(0,t) = u(\pi,t) = \phi_x(0,t) = \phi_x(\pi,t) = 0, \\ \theta(0,t) = \theta(\pi,t) = z(0,\omega,t) = z(\pi,\omega,t) = 0, \ \ t-\tau\omega > 0. \end{cases} \tag{2.9}$$

Next, in order to apply the Hille-Yosida Theorem 1.9, we introduce the new variables $v = u_t$ and $\psi = \phi_t$ and define the Hilbert space

$$\mathcal{H} := H_0^1(0,\pi)\times L^2(0,\pi)\times H_*^1(0,\pi)\times L_*^2(0,\pi)\times L^2(0,\pi)\times L^2\left((0,1);H_0^1(0,\pi)\right).$$

The space $\mathcal{H}$ is equipped with the inner product

$$\begin{aligned} \langle U,U^*\rangle_{\mathcal{H}} = \ & \rho\int_0^\pi vv^*dx + a\int_0^\pi u_x u_x^* dx + \xi\int_0^\pi \phi\phi^* dx + \alpha\int_0^\pi \phi_x\phi_x^* dx + J\int_0^\pi \psi\psi^* dx \\ & + b\int_0^\pi \phi u_x^* dx + b\int_0^\pi u_x\phi^* dx + c\int_0^\pi \theta\theta^* dx + \tau|k|\int_0^\pi\int_0^1 z_x(x,\omega,t) z_x^*(x,\omega,t)d\omega dx, \end{aligned}$$

where $U = (u,v,\phi,\psi,\theta,z)^T$, $U^* = (u^*,v^*,\phi^*,\psi^*,\theta^*,z^*)^T \in \mathcal{H}$.
Thus, the problem (2.8)-(2.9) can be written as follows:

$$\begin{cases} U_t + \mathcal{A}U = 0, \ \ t>0, \\ U(0) = U_0 = (u_0,u_1,\phi_0,\phi_1,\theta_0,z_0)^T, \end{cases} \tag{2.10}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the operator given by

$$\mathcal{A}U = \begin{pmatrix} -v \\ -\frac{a}{\rho}u_{xx} - \frac{b}{\rho}\phi_x \\ -\psi \\ -\frac{\alpha}{J}\phi_{xx} + \frac{b}{J}u_x + \frac{\xi}{J}\phi + \frac{\beta}{J}\theta_x \\ -\frac{\kappa}{c}\theta_{xx} - \frac{k}{c}z_{xx}(.,1) + \frac{\beta}{c}\psi_x \\ \frac{1}{\tau}z_w \end{pmatrix}, \tag{2.11}$$

with domain

$$D\left(\mathcal{A}\right) = \left\{ \begin{array}{c} U \in \mathcal{H} : u \in H^2(0,\pi) \cap H_0^1(0,\pi),\ v \in H_0^1(0,\pi),\ \phi \in H_*^2(0,\pi) \cap H_*^1(0,\pi), \\ \psi \in H_*^1(0,\pi),\ z \in H^1\Big((0,1); H_0^1(0,\pi)\Big) \end{array} \right\}.$$

Our well-posedness result reads as follows:

Theorem 2.1. *Assume that $|k| \leq \kappa$. Then, for any $U_0 \in D(\mathcal{A})$, there exists a unique solution $U = (u, \phi, \theta, z)$ to the problem* (2.10) *such that*

$$U \in C\left([0, +\infty[; D(\mathcal{A})\right) \cap C^1\left([0, +\infty[; \mathcal{H}\right).$$

The proof of Theorem 2.1 is based on Hille-Yosida theorem.

Proof of Theorem 2.1. In the light of Theorem 1.9, it suffices to show that the operator $\mathcal{A}$ is monotone and maximal.

First, a direct calculation, using integration by parts and the boundary conditions, yields, $\forall U \in D(\mathcal{A})$,

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= a\int_0^\pi u_x v_x dx - b\int_0^\pi \phi_x v dx - a\int_0^\pi v_x u_x dx - \xi\int_0^\pi \psi\phi dx - \alpha\int_0^\pi \psi_x\phi_x dx \\ &+ \alpha\int_0^\pi \phi_x\psi_x dx + \xi\int_0^\pi \phi\psi dx + b\int_0^\pi u_x\psi dx + \beta\int_0^\pi \theta_x\psi dx - b\int_0^\pi \psi u_x dx \\ &+ b\int_0^\pi v\phi_x dx + \kappa\int_0^\pi \theta_x\theta_x dx + \beta\int_0^\pi \psi_x\theta dx + k\int_0^\pi z_x\left(x,1,t\right)\theta_x dx \\ &+ |k|\int_0^\pi\int_0^1 z_{x\omega}\left(x,\omega,t\right) z_x\left(x,\omega,t\right) d\omega dx \\ &= \kappa\int_0^\pi \theta_x^2 dx + k\int_0^\pi z_x\left(x,1,t\right)\theta_x dx + \frac{|k|}{2}\int_0^\pi z_x^2\left(x,1,t\right) dx - \frac{|k|}{2}\int_0^\pi \theta_x^2 dx. \end{aligned}$$

Applying Young's inequality to the second term in the right-hand side, we get

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \geq (\kappa - |k|) \int_0^\pi \theta_x^2 dx \geq 0, \tag{2.12}$$

which shows the monotonicity of the operator $\mathcal{A}$.

Secondly, to establish the maximality of $\mathcal{A}$, we take $F = (f^1, f^2, f^3, f^4, f^5, f^6)^T \in \mathcal{H}$ and seek $U = (u, v, \phi, \psi, \theta, z)^T \in D(\mathcal{A})$ such that $(I + \mathcal{A})U = F$; that is,

$$\begin{cases} u - v &= f^1, \\ \rho v - a u_{xx} - b\phi_x &= \rho f^2, \\ \phi - \psi &= f^3, \\ J\psi - \alpha\phi_{xx} + \xi\phi + bu_x + \beta\theta_x &= Jf^4, \\ c\theta - \kappa\theta_{xx} - kz_{xx}(x,1,t) + \beta\psi_x &= cf^5, \\ \tau z + z_\omega &= \tau f^6. \end{cases} \tag{2.13}$$

Form the first and third equations of (2.13), we get

$$v = u - f^1, \ \ \psi = \phi - f^3. \tag{2.14}$$

Solving the sixth equation of (2.13) with the initial condition $z(0) = \theta$, we infer that

$$z(x, \omega, t) = e^{-\tau\omega}\theta(x,t) + \tau \int_0^\omega e^{\tau(y-\omega)} f^6(x,y,t)dy. \tag{2.15}$$

Next, plugging (2.14) and (2.15) into the second, fourth and fifth equations of (2.13), respectively, we obtain

$$\begin{cases} \rho u - a u_{xx} - b\phi_x &= g_1, \\ J\phi - \alpha\phi_{xx} + \xi\phi + bu_x + \beta\theta_x &= g_2, \\ c\theta - (\kappa + ke^{-\tau})\theta_{xx} + \beta\phi_x &= g_3, \end{cases} \tag{2.16}$$

where

$$g_1 = \rho(f^1 + f^2) \in L^2(0,\pi), \qquad g_2 = J(f^3 + f^4) \in L^2_*(0,\pi) \tag{2.17}$$

and

$$g_3 = cf^5 + \beta f_x^3 + k\tau e^{-\tau} \int_0^1 e^{\tau y} f_{xx}^6(x,y,t)dy \in H^{-1}(0,\pi). \tag{2.18}$$

Introducing the space

$$\mathcal{W} = H_0^1(0,\pi) \times H_*^1(0,\pi) \times H_0^1(0,\pi)$$

and considering the following variational problem

$$B(U,U^*) = L(U^*), \qquad \forall U^* \in \mathcal{W}, \tag{2.19}$$

where $U = (u,\phi,\theta) \in \mathcal{W}$, B is the bilinear form defined on $\mathcal{W}$ by

$$\begin{aligned} B(U,U^*) = \; & a\int_0^\pi u_x u_x^* dx + \rho\int_0^\pi u u^* dx + b\int_0^\pi (\phi u_x^* + u_x\phi^*)dx + (J+\xi)\int_0^\pi \phi\phi^* dx \\ & +\beta\int_0^\pi (\theta_x\phi^* - \phi\theta_x^*)dx + \alpha\int_0^\pi \phi_x\phi_x^* dx + (\kappa + ke^{-\tau})\int_0^\pi \theta_x\theta_x^* dx + c\int_0^\pi \theta\theta^* dx, \end{aligned}$$

and L is the linear form

$$L(U^*) = \int_0^\pi g_1 u^* dx + \int_0^\pi g_2\phi^* dx + \langle g_3, \theta^*\rangle_{H^{-1}\times H_0^1},$$

it is easy to check that B and L are continuous. Furthermore, straightforward calculations show that

$$\begin{aligned} B(U,U) = \; & a\int_0^\pi u_x^2 dx + \rho\int_0^\pi u^2 dx + 2b\int_0^\pi \phi u_x dx + \alpha\int_0^\pi \phi_x^2 dx \\ & +(J+\xi)\int_0^\pi \phi^2 dx + (\kappa + ke^{-\tau})\int_0^\pi \theta_x^2 dx + c\int_0^\pi \theta^2 dx. \end{aligned}$$

Young's inequality and (2.2) yield

$$\begin{aligned} B(U,U) \; & \geq \rho\int_0^\pi u^2 dx + \frac{1}{2}\Big(a - \frac{b^2}{\xi}\Big)\int_0^\pi u_x^2 dx + J\int_0^\pi \phi^2 dx + \alpha\int_0^\pi \phi_x^2 dx \\ & +c\int_0^\pi \theta^2 dx + (\kappa + ke^{-\tau})\int_0^\pi \theta_x^2 dx \\ & \geq C\|U\|_{\mathcal{W}}^2. \end{aligned}$$

Thus, B is coercive. Consequently, Lax-Milgram theorem guarantees the existence of a unique $(u,\phi,\theta) \in \mathcal{W}$ satisfying (2.19), for all $U^* \in \mathcal{W}$.

By taking $(u^*,\phi^*,\theta^*) = (u^*,0,0)$ in (2.19), we arrive at

$$a\int_0^\pi u_x u_x^* dx = \int_0^\pi [g_1 - \rho u + b\phi_x]u^* dx, \forall u^* \in H_0^1(0,\pi), \tag{2.20}$$

By the elliptic regularity theory, we infer

$$u \in H^2(0,\pi) \cap H_0^1(0,\pi).$$

Moreover, integration by parts of the left-hand side of (2.20) yields

$$\rho u - a u_{xx} - b\phi_x = g_1.$$

Using (2.14) and (2.17), we get

$$\rho v - au_{xx} - b\phi_x = \rho f^2.$$

Similarly, by taking $(u^*, \phi^*, \theta^*) = (0, 0, \theta^*)$ in (2.19), we obtain

$$(\kappa + ke^{-\tau}) \int_0^\pi \theta_x \theta_x^* dx = \int_0^\pi [g_3 - c\theta - \beta\phi_x]\theta^* dx, \ \forall \theta^* \in H_0^1(0,\pi), \tag{2.21}$$

therefore,

$$\theta \in H^2(0,\pi) \cap H_0^1(0,\pi).$$

Again, the integration by parts of the left hand side of (2.21) leads to

$$-(\kappa + ke^{-\tau})\theta_{xx} = g_3 - c\theta - \beta\phi_x, \quad \text{in } L^2(0,\pi), \tag{2.22}$$

which solves the third equation of (2.16).
Inserting g_3 from (2.18) and recalling (2.14),(2.15), we get

$$c\theta - \kappa\theta_{xx} - kz_{xx}(x,1,t) + \beta\psi_x = cf^5.$$

Finally, inserting $(u^*, \phi^*, \theta^*) = (0, \phi^*, 0)$ into (2.19), we get

$$\alpha \int_0^\pi \phi_x \phi_x^* dx = \int_0^\pi [g_2 - bu_x - (J+\xi)\phi - \beta\theta_x]\phi^* dx, \ \forall \phi^* \in H_*^1(0,\pi). \tag{2.23}$$

Here, we are not able to apply the elliptic regularity directly. To overcome this difficulty, let $\Phi \in H_0^1(0,\pi)$ and set $\phi^* = \Phi - \dfrac{1}{\pi}\displaystyle\int_0^\pi \Phi(x)dx$.

Clearly $\phi^* \in H_*^1(0,\pi)$. Substituting ϕ^* in (2.23), bearing in mind that $g_2 \in L_*^2(0,\pi)$, we infer that

$$\alpha \int_0^\pi \phi_x \Phi_x dx = \int_0^\pi [g_2 - bu_x - (J+\xi)\phi - \beta\theta_x]\Phi dx, \ \forall \Phi \in H_0^1(0,\pi),$$

which shows that

$$\phi \in H^2(0,\pi).$$

An integration by parts leads to

$$-\alpha\phi_{xx} = g_2 - bu_x - (J+\xi)\phi - \beta\theta_x = Q \in L^2(0,\pi),$$

hence, ϕ solves the third equation of (2.19). We use (2.14) and (2.17) to arrive at

$$J\psi - \alpha\phi_{xx} + bu_x + \xi\phi + \beta\theta_x = Jf^4.$$

On the other hand, since $-\alpha\phi_{xx} = Q \in L^2(0,\pi)$, then

$$-\alpha \int_0^\pi \phi_{xx}\Psi dx = \int_0^\pi Q\Psi dx, \qquad \forall \Psi \in H^1(0,\pi).$$

Integration by parts gives

$$-\alpha\phi_x\Psi\Big|_0^\pi + \alpha \int_0^\pi \phi_x\Psi_x dx = \int_0^\pi Q\Psi dx, \qquad \forall \Psi \in H^1(0,\pi),$$

hence,

$$-\alpha\phi_x\phi^*\Big|_0^\pi + \alpha \int_0^\pi \phi_x\phi_x^* dx = \int_0^\pi Q\phi^* dx, \qquad \forall \phi^* \in H_*^1(0,\pi),$$

and therefore, by (2.23), we obtain

$$-\alpha\phi_x(\pi)\phi^*(\pi) + \alpha\phi_x(0)\phi^*(0) = 0, \qquad \forall \phi^* \in H_*^1(0,\pi).$$

As ϕ^* is arbitrary, we get

$$\phi_x(\pi) = \phi_x(0) = 0,$$

which means that

$$\phi \in H_*^2(0,\pi).$$

Substituting u, ϕ, θ in (2.14) and (2.15), we obtain

$$v \in H_0^1(0,\pi),\ \ \psi \in H_*^1(0,\pi) \text{ and } z \in H^1\Big((0,1); H_0^1(0,\pi)\Big). \tag{2.24}$$

Therefore $\mathcal{A}$ is maximal, which completes the proof of Theorem 2.1. □

2.3 Exponential Stability

In this section, we state and prove our main stability result. First, we define the energy associated to the solution of (2.1) by

$$E(t) := \frac{1}{2}\int_0^\pi \left[\rho u_t^2 + J\phi_t^2 + au_x^2 + \xi\phi^2 + 2bu_x\phi + \alpha\phi_x^2 + c\theta^2 + \tau\,|k| \int_0^1 z_x^2(x,\omega,t)\,d\omega\right] dx, \tag{2.25}$$

and introduce the stability number

$$\chi = \frac{\rho}{a} - \frac{J}{\alpha}. \tag{2.26}$$

Our stability result reads as follows.

Theorem 2.2. *Assume that $\chi = 0$ and suppose that $|k| < \kappa$. Then, the energy functional $E(t)$ satisfies, along the solution of (2.8),(2.9), the estimate*

$$E(t) \leq \sigma e^{-\varpi t}, \ \forall t \geq 0,$$

where σ and ϖ are two positive constants.

The proof of Theorem 2.2 will be established through several lemmas.

Lemma 2.1. *Assume that $|k| < \kappa$, then the energy $E(t)$ satisfies, along the solution of (2.8),(2.9), the estimate*

$$E'(t) \leq -(\kappa - |k|) \int_0^\pi |\theta_x(x,t)|^2 \, dx \leq 0. \tag{2.27}$$

Proof. Taking the L^2-inner product of the three first equations of (2.8) by u_t, ϕ_t, θ respectively, we get

$$\frac{\rho}{2}\frac{d}{dt}\int_0^\pi u_t^2 dx + \frac{a}{2}\frac{d}{dt}\int_0^\pi u_x^2 dx - b\int_0^\pi \phi_x u_t dx = 0, \tag{2.28}$$

$$\frac{J}{2}\frac{d}{dt}\int_0^\pi \phi_t^2 dx + \frac{\alpha}{2}\frac{d}{dt}\int_0^\pi \phi_x^2 dx + b\int_0^\pi u_x\phi_t dx + \frac{\xi}{2}\frac{d}{dt}\int_0^\pi \phi^2 dx + \beta\int_0^\pi \theta_x\phi_t dx = 0 \tag{2.29}$$

and

$$\frac{c}{2}\frac{d}{dt}\int_0^\pi \theta^2 dx + \kappa\int_0^\pi \theta_x^2 dx + k\int_0^\pi z_x(x,1,t)\,\theta_x dx + \beta\int_0^\pi \phi_{xt}\theta dx = 0. \tag{2.30}$$

Next, we perform the inner product of the fourth equation of (2.8) with $|k|\, z(x,\omega,t)$ in $L^2((0,1); H_0^1(0,\pi))$, we arrive at

$$\frac{\tau|k|}{2}\frac{d}{dt}\int_0^\pi\int_0^1 z_x^2(x,\omega,t)\, d\omega dx + |k|\int_0^\pi\int_0^1 z_{x\omega}(x,\omega,t)\, z_x(x,\omega,t)\, d\omega dx = 0,$$

which gives

$$\frac{\tau|k|}{2}\frac{d}{dt}\int_0^\pi\int_0^1 z_x^2(x,\omega,t)\, d\omega dx + \frac{|k|}{2}\int_0^\pi z_x^2(x,1,t)\, dx - \frac{|k|}{2}\int_0^\pi \theta_x^2(x,t)\, dx = 0. \tag{2.31}$$

Adding (2.28)-(2.31), we entail that

$$E'(t) = -\left(\kappa - \frac{|k|}{2}\right)\int_0^\pi \theta_x^2(x,t)\,dx - \frac{|k|}{2}\int_0^\pi z_x^2(x,1,t)\,dx - k\int_0^\pi z_x(x,1,t)\,\theta_x(x,t)\,dx.$$

Thanks to Young's inequality, we infer that

$$E'(t) \le -(\kappa - |k|)\int_0^\pi \theta_x^2(x,t)\,dx.$$

□

Remark 2.1. *We note that the presence of the delay weakens the dissipation caused by thermal damping. Thanks to the assumption $|k| < \kappa$, the energy $E(t)$ is still dissipative at a lower rate, similarly to the case of frictional damping. If $|k| > \kappa$, the system (2.1) becomes non–dissipative.*

To the best of our knowledge, there is no result on the stability or instability related to this situation. However, in the light of [62], we expect that the energy will lose its exponential stability.

Lemma 2.2. *Assume that $\chi = 0$, then, the functional*

$$F_1(t) = \frac{\alpha\rho}{b}\int_0^\pi \phi_x u_t dx + \frac{Ja}{b}\int_0^\pi \phi_t u_x dx + 2J\int_0^\pi \phi\phi_t dx - \frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi(y)dy\right)dx \tag{2.32}$$

satisfies, along the solution of (2.8),(2.9) and for any $\varepsilon_2 > 0$, the estimate

$$\begin{aligned} F_1'(t) \le & -\gamma_1\left\|\sqrt{a}u_x + \frac{b}{\sqrt{a}}\phi\right\|^2 - \gamma_2\|\phi\|^2 - \frac{\alpha}{2}\|\phi_x\|^2 + \varepsilon_2\|u_t\|^2 \\ & + M\|\theta_x\|^2 + M\left(1 + \frac{1}{\varepsilon_2}\right)\|\phi_t\|^2, \end{aligned} \tag{2.33}$$

where γ_1, γ_2 and M are positive constants.

Proof. Direct differentiation, using $(2.8)_1$ and integration by parts, yields

$$\begin{aligned} \frac{d}{dt}\frac{\alpha\rho}{b}\int_0^\pi \phi_x u_t dx &= \frac{\alpha\rho}{b}\int_0^\pi \phi_{tx}u_t dx + \frac{\alpha\rho}{b}\int_0^\pi \phi_x u_{tt}dx \\ &= \frac{\alpha\rho}{b}\int_0^\pi \phi_{tx}u_t dx + \frac{\alpha}{b}\int_0^\pi \phi_x(au_{xx} + b\phi_x)\,dx \\ &= \frac{\alpha\rho}{b}\int_0^\pi \phi_{tx}u_t dx + \frac{a\alpha}{b}\int_0^\pi \phi_x u_{xx}dx + \alpha\|\phi_x\|^2. \end{aligned} \tag{2.34}$$

Similarly, using $(2.8)_2$, we have

$$\begin{aligned}\frac{d}{dt}\frac{Ja}{b}\int_0^\pi \phi_t u_x dx &= \frac{Ja}{b}\int_0^\pi \phi_{tt}u_x dx + \frac{Ja}{b}\int_0^\pi \phi_t u_{xt}dx\\ &= \frac{a}{b}\int_0^\pi \left[\alpha\phi_{xx} - bu_x - \xi\phi - \beta\theta_x\right]u_x dx + \frac{Ja}{b}\int_0^\pi \phi_t u_{xt}dx,\\ &= -\frac{a\alpha}{b}\int_0^\pi \phi_x u_{xx}dx - a\left\|u_x\right\|^2 - \frac{a\xi}{b}\int_0^\pi \phi u_x dx\\ &\quad - \frac{a\beta}{b}\int_0^\pi \theta_x u_x dx + \frac{Ja}{b}\int_0^\pi \phi_t u_{xt}dx \end{aligned}\tag{2.35}$$

and

$$\begin{aligned}2\frac{d}{dt}J\int_0^\pi \phi\phi_t dx &= 2J\left\|\phi_t\right\|^2 + 2J\int_0^\pi \phi\phi_{tt}dx\\ &= 2J\left\|\phi_t\right\|^2 + 2\int_0^\pi \phi\left[\alpha\phi_{xx} - bu_x - \xi\phi - \beta\theta_x\right]dx\\ &= 2J\left\|\phi_t\right\|^2 - 2\alpha\left\|\phi_x\right\|^2 - 2b\int_0^\pi \phi u_x dx - 2\xi\left\|\phi\right\|^2 - 2\beta\int_0^\pi \phi\theta_x dx. \end{aligned}\tag{2.36}$$

The exploitation of $(2.8)_1$ leads to

$$\begin{aligned}-\frac{d}{dt}\frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi(y)dy\right)dx &= -\frac{\rho\xi}{b}\int_0^\pi u_{tt}\left(\int_0^x \phi(y)dy\right)dx - \frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx\\ &= -\frac{\xi}{b}\int_0^\pi \left[au_{xx} + b\phi_x\right]\left(\int_0^x \phi(y)dy\right)dx\\ &\quad - \frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx\\ &= \frac{a\xi}{b}\int_0^\pi u_x\phi dx + \xi\left\|\phi\right\|^2 - \frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx. \end{aligned}\tag{2.37}$$

The addition of (2.34)-(2.37) yields

$$\begin{aligned}F_1'(t) &= -a\left\|u_x\right\|^2 - \xi\left\|\phi\right\|^2 - 2b\int_0^\pi \phi u_x dx - \alpha\left\|\phi_x\right\|^2 + 2J\left\|\phi_t\right\|^2\\ &\quad - 2\beta\int_0^\pi \phi\theta_x dx - \frac{a\beta}{b}\int_0^\pi \theta_x u_x dx - \frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx\\ &\quad + \left(\frac{Ja - \alpha\rho}{b}\right)\int_0^\pi u_{xt}\phi_t dx. \end{aligned}$$

Therefore, since $\chi = 0$, we obtain

$$\begin{aligned}F_1'(t) &= -\left\|\sqrt{a}u_x + \frac{b}{\sqrt{a}}\phi\right\|^2 - \left(\xi - \frac{b^2}{a}\right)\left\|\phi\right\|^2 - \alpha\left\|\phi_x\right\|^2 + 2J\left\|\phi_t\right\|^2\\ &\quad + 2\beta\int_0^\pi \phi_x\theta dx - \frac{a\beta}{b}\int_0^\pi \theta_x u_x dx - \frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx. \end{aligned}\tag{2.38}$$

Cauchy-Schwarz and Young's inequalities yield

$$\beta \int_0^\pi \theta \phi_x dx \leq \frac{\alpha}{2} \|\phi_x\|^2 + \frac{\beta^2 \pi^2}{2\alpha} \|\theta_x\|^2. \tag{2.39}$$

Similarly, for any $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$,

$$\frac{\rho \xi}{b} \int_0^\pi u_t \left(\int_0^x \phi_t(y) dy \right) dx \leq \varepsilon_2 \|u_t\|^2 + \frac{M}{\varepsilon_2} \|\phi_t\|^2 \tag{2.40}$$

and

$$\begin{aligned} \frac{\sqrt{a}\beta}{b} \int_0^\pi \sqrt{a} u_x \theta_x dx &\leq \varepsilon_3 \left\|\sqrt{a} u_x\right\|^2 + \frac{M}{\varepsilon_3} \|\theta_x\|^2 \\ &\leq 2\varepsilon_3 \left\| \sqrt{a} u_x + \frac{b}{\sqrt{a}} \phi \right\|^2 + 2\frac{b^2}{a} \varepsilon_3 \|\phi\|^2 + \frac{M}{\varepsilon_3} \|\theta_x\|^2. \end{aligned} \tag{2.41}$$

The substitution of (2.39)-(2.41) into (2.38) gives

$$\begin{aligned} F_1'(t) \leq & -(1 - 2\varepsilon_3) \left\| \sqrt{a} u_x + \frac{b}{\sqrt{a}} \phi \right\|^2 - \left(\xi - \frac{b^2}{a} - 2\frac{b^2}{a} \varepsilon_3 \right) \|\phi\|^2 - \frac{\alpha}{2} \|\phi_x\|^2 \\ & + M \left(1 + \frac{1}{\varepsilon_2} \right) \|\phi_t\|^2 + \varepsilon_2 \|u_t\|^2 + M \left(1 + \frac{1}{\varepsilon_3} \right) \|\theta_x\|^2. \end{aligned} \tag{2.42}$$

Since $\xi - \frac{b^2}{a} > 0$, we can choose ε_3, small enough, such that $\xi - \frac{b^2}{a} - 2\frac{b^2}{a}\varepsilon_3 > 0$ and $1 - 2\varepsilon_3 > 0$, and, hence, the estimate (2.33) follows. □

Lemma 2.3. *Let (u, ϕ, θ, z) be the solution of (2.8),(2.9). Then, the functional*

$$F_2(t) := -\rho \int_0^\pi u_t u dx - \frac{b\rho}{a} \int_0^\pi u_t \left(\int_0^x \phi(y) dy \right) dx$$

satisfies the estimate

$$F_2'(t) \leq -\frac{\rho}{2} \|u_t\|^2 + \left\| \sqrt{a} u_x + \frac{b}{\sqrt{a}} \phi \right\|^2 + M \|\phi_t\|^2, \tag{2.43}$$

where M is a positive constant.

Proof. By differentiating $F_2(t)$ and using $(2.8)_1$, we arrive at

$$\begin{aligned} F_2'(t) = & -\rho \|u_t\|^2 - \int_0^\pi [a u_{xx} + b\phi_x] u dx - \frac{b}{a} \int_0^\pi [a u_{xx} + b\phi_x] \left(\int_0^x \phi(y) dy \right) dx \\ & - \frac{b\rho}{a} \int_0^\pi u_t \left(\int_0^x \phi_t(y) dy \right) dx \end{aligned}$$

$$= -\rho \|u_t\|^2 + a\|u_x\|^2 + 2b\int_0^\pi u_x\phi dx + \frac{b^2}{a}\|\phi\|^2 - \frac{b\rho}{a}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx$$
$$= -\rho \|u_t\|^2 + \left\|\sqrt{a}u_x + \frac{b}{\sqrt{a}}\phi\right\|^2 - \frac{b\rho}{a}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx.$$

Young's and Cauchy Schwarz inequalities give

$$-\frac{b\rho}{a}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx \le \frac{\rho}{2}\|u_t\|^2 + M\|\phi_t\|^2$$

and (2.43) follows immediately. □

Lemma 2.4. *Let (u,ϕ,θ,z) be the solution of (2.8), (2.9). Then, the functional*

$$F_3(t) = -\frac{cJ}{\beta}\int_0^\pi \theta\left(\int_0^x \phi_t(y)dy\right)dx$$

satisfies, for any $\varepsilon_1 > 0$, the estimate

$$F_3'(t) \le -\frac{J}{2}\|\phi_t\|^2 + \varepsilon_1\left\|\sqrt{a}u_x + \frac{b}{\sqrt{a}}\phi\right\|^2 + \varepsilon_1\|\phi\|^2$$
$$+ \varepsilon_1\|\phi_x\|^2 + \frac{Jk^2}{\beta^2}\|z_x(1,t)\|^2 + M\left(1+\frac{1}{\varepsilon_1}\right)\|\theta_x\|^2, \tag{2.44}$$

where M is a positive constant independent of ε_1.

Proof. Differentiating $F_3(t)$, then using $(2.8)_2$, $(2.8)_3$ and integration by parts, we get

$$F_3'(t) = -\frac{cJ}{\beta}\int_0^\pi \theta_t\left(\int_0^x \phi_t(y)dy\right)dx - \frac{cJ}{\beta}\int_0^\pi \theta\left(\int_0^x \phi_{tt}(y)dy\right)dx$$
$$= -\frac{J}{\beta}\int_0^\pi [\kappa\theta_{xx} + kz_{xx}(1,t) - \beta\phi_{xt}]\left(\int_0^x \phi_t(y)dy\right)dx$$
$$-\frac{c}{\beta}\int_0^\pi \theta\left(\int_0^x (\alpha\phi_{xx} - bu_x - \xi\phi - \beta\theta_x)\,dy\right)dx$$
$$= \frac{J\kappa}{\beta}\int_0^\pi \theta_x\phi_t dx + \frac{Jk}{\beta}\int_0^\pi z_x(1,t)\phi_t dx - J\|\phi_t\|^2 - \frac{c\alpha}{\beta}\int_0^\pi \theta\phi_x dx + c\|\theta\|^2$$
$$+\frac{c}{\beta}\int_0^\pi \theta\left(\int_0^x (bu_x + \xi\phi)\,dy\right)dx.$$

Thus,

$$F_3'(t) = -J\|\phi_t\|^2 + \frac{J\kappa}{\beta}\int_0^\pi \theta_x\phi_t dx + \frac{Jk}{\beta}\int_0^\pi z_x(1,t)\phi_t dx$$
$$+\frac{cb}{\beta\sqrt{a}}\int_0^\pi \theta\left(\int_0^x \left(\sqrt{a}u_x + \frac{b}{\sqrt{a}}\phi\right)(y)\,dy\right)dx$$
$$+\frac{c}{\beta}\left(\xi - \frac{b^2}{a}\right)\int_0^\pi \theta\left(\int_0^x \phi(y)dy\right)dx - \frac{c\alpha}{\beta}\int_0^\pi \theta\phi_x dx + c\|\theta\|^2. \tag{2.45}$$

Young's inequality yields

$$\frac{J\kappa}{\beta}\int_0^\pi \theta_x\phi_t dx \le \frac{J}{4}\left\|\phi_t\right\|^2 + M\left\|\theta_x\right\|^2 \tag{2.46}$$

and

$$\frac{Jk}{\beta}\int_0^\pi z_x(1,t)\phi_t dx \le \frac{J}{4}\left\|\phi_t\right\|^2 + \frac{Jk^2}{\beta^2}\|z_x(1,t)\|^2.$$

Similarly, for any $\varepsilon_1 > 0$, we have

$$\frac{cb}{\beta\sqrt{a}}\int_0^\pi \theta\left(\int_0^x\left(\sqrt{a}u_x+\frac{b}{\sqrt{a}}\phi\right)(y)\,dy\right)dx \le \frac{M}{\varepsilon_1}\left\|\theta_x\right\|^2+\varepsilon_1\left\|\sqrt{a}u_x+\frac{b}{\sqrt{a}}\phi\right\|^2, \tag{2.47}$$

$$\frac{c}{\beta}\left(\xi-\frac{b^2}{a}\right)\int_0^\pi \theta\left(\int_0^x \phi(y)dy\right)dx \le \frac{M}{\varepsilon_1}\left\|\theta_x\right\|^2+\varepsilon_1\left\|\phi\right\|^2 \tag{2.48}$$

and

$$-\frac{c\alpha}{\beta}\int_0^\pi \theta\phi_x dx \le \frac{M}{\varepsilon_1}\left\|\theta_x\right\|^2+\varepsilon_1\|\phi_x\|^2. \tag{2.49}$$

The substitution of (2.46)-(2.49) into (2.45) yields (2.44). □

Lemma 2.5. *Let (u,ϕ,θ,z) be the solution of (2.8),(2.9). Then, the functional*

$$F_4(t)=\tau\int_0^1\int_0^\pi e^{-\omega\tau}z_x^2\left(x,\omega,t\right)dxd\omega$$

satisfies the estimate

$$F_4'(t)\le\left\|\theta_x\right\|^2-e^{-\tau}\int_0^\pi z_x^2\left(x,1,t\right)dx-\tau e^{-\tau}\int_0^1\int_0^\pi z_x^2\left(x,\omega,t\right)dxd\omega. \tag{2.50}$$

Proof. Differentiating $F_4(t)$, then using $(2.8)_3$ and integration by parts, we infer that

$$F_4'(t)=2\tau\int_0^1\int_0^\pi e^{-\omega\tau}z_x\left(x,\omega,t\right)z_{xt}\left(x,\omega,t\right)dxd\omega.$$

Consequently, we get, from (2.7),

$$\begin{aligned}
F_4'\left(t\right) &= -2\int_0^1\int_0^\pi e^{-\omega\tau}z_x\left(x,\omega,t\right)z_{x\omega}\left(x,\omega,t\right)dxd\omega\\
&= -\int_0^1\int_0^\pi e^{-\omega\tau}\frac{d}{d\omega}z_x^2\left(x,\omega,t\right)dxd\omega\\
&= -\int_0^1\int_0^\pi \frac{d}{d\omega}\left(e^{-\omega\tau}z_x^2\left(x,\omega,t\right)\right)dxd\omega-\tau\int_0^1\int_0^\pi e^{-\omega\tau}z_x^2\left(x,\omega,t\right)dxd\omega\\
&= -e^{-\tau}\int_0^\pi z_x^2\left(x,1,t\right)dx+\int_0^\pi \theta_x^2\left(x,t\right)dx-\tau\int_0^1\int_0^\pi e^{-\omega\tau}z_x^2\left(x,\omega,t\right)dxd\omega
\end{aligned}$$

and, finally, we obtain

$$F_4'(t) \leq -e^{-\tau} \int_0^\pi z_x^2(x,1,t)\,dx + \int_0^\pi \theta_x^2(x,t)\,dx - \tau e^{-\tau} \int_0^1 \int_0^\pi z_x^2(x,\omega,t)\,dxd\omega \tag{2.51}$$

and (2.50) follows immediately. □

At this stage, we introduce the Lyapunov functional

$$\mathcal{L}(t) := NE(t) + \frac{\rho}{4\varepsilon_2} F_1(t) + F_2(t) + \frac{1}{\varepsilon_1} F_3(t) + N_4 F_4(t),$$

where N, N_4, ε_1 and ε_2 to be determined later.

Lemma 2.6. *The functional E and $\mathcal{L}$ satisfy, for a positive constant $C_{\varepsilon_1,\varepsilon_2}$, the estimate*

$$(N - C_{\varepsilon_1,\varepsilon_2})E(t) \leq \mathcal{L}(t) \leq (N + C_{\varepsilon_1,\varepsilon_2})E(t), \quad \forall t \geq 0.$$

Proof. Let $\mathcal{K}(t) := \mathcal{L}(t) - NE(t)$, then

$$\mathcal{K}(t) = \frac{\rho}{4\varepsilon_2} F_1(t) + F_2(t) + \frac{1}{\varepsilon_1} F_3(t) + N_4 F_4(t)$$

and

$$\begin{aligned} |\mathcal{K}(t)| \leq{} & C_{\varepsilon_1,\varepsilon_2} \left[\left| \int_0^\pi \phi_x u_t dx \right| + \left| \int_0^\pi \phi_t u_x dx \right| + \left| \int_0^\pi \phi \phi_t dx \right| + \left| \int_0^\pi u_t u dx \right| \right] \\ & + C_{\varepsilon_1,\varepsilon_2} \left[\left| \int_0^\pi u_t \left(\int_0^x \phi(y) dy \right) dx \right| + \left| \int_0^\pi \theta \left(\int_0^x \phi_t(y) dy \right) dx \right| \right] \\ & + C_{\varepsilon_1,\varepsilon_2} \left| \int_0^1 \int_0^\pi e^{-\omega\tau} z_x^2(x,\omega,t)\,dxd\omega \right|. \end{aligned}$$

Therefore, Cauchy-Schwarz and Young's inequalities yield

$$|\mathcal{L}(t) - NE(t)| \leq C_{\varepsilon_1,\varepsilon_2} E(t), \tag{2.52}$$

where $C_{\varepsilon_1,\varepsilon_2} > 0$. Thus,

$$(N - C_{\varepsilon_1,\varepsilon_2}) E(t) \leq \mathcal{L}(t) \leq (N + C_{\varepsilon_1,\varepsilon_2}) E(t).$$

□

Proof of Theorem 2.2

By differentiating $\mathcal{L}(t)$ and using (2.27), (2.33), (2.43), (2.44), (2.50), we get

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left(\frac{\rho\gamma_1}{4\varepsilon_2} - 2\right)\left\|\sqrt{a}u_x + \frac{b}{\sqrt{a}}\phi\right\|^2 \\
& - \left(\frac{\rho\gamma_2}{4\varepsilon_2} - 1\right)\|\phi\|^2 - \frac{\rho}{4}\|u_t\|^2 - \left(\frac{\alpha\rho}{8\varepsilon_2} - 1\right)\|\phi_x\|^2 \\
& - \left(\frac{J}{2\varepsilon_1} - \frac{M\rho}{4\varepsilon_2}\left(1 + \frac{1}{\varepsilon_2}\right) - M\right)\|\phi_t\|^2 \\
& - \left(N\left(\kappa - |k|\right) - \frac{M}{\varepsilon_1}\left(1 + \frac{1}{\varepsilon_1}\right) - M\frac{\rho}{4\varepsilon_2} - N_4\right)\|\theta_x\|^2 \\
& - \left(e^{-\tau}N_4 - \frac{Jk^2}{\varepsilon_1\beta^2}\right)\int_0^\pi z_x^2(x,1,t)dx \\
& - \tau e^{-\tau}N_4\int_0^1\int_0^\pi z_x^2(x,\omega,t)dxd\omega.
\end{aligned}$$

First, we choose $\varepsilon_2 > 0$, small enough, such that

$$\varepsilon_2 \leq \min\left\{\frac{\rho\gamma_1}{8}, \frac{\rho\gamma_2}{4}, \frac{\rho\alpha}{8}\right\}.$$

Then, we pick $\varepsilon_1 > 0$, small enough, such that

$$\varepsilon_1 \leq \frac{J}{2\left(M\left(\frac{\rho}{4\varepsilon_2^2} + \frac{\rho}{4\varepsilon_2}\right) + M\right)}.$$

Next, we take N_4 so large

$$N_4 > \frac{Jk^2}{\varepsilon_1\beta^2 e^{-\tau}}.$$

Finally, we select N such that

$$N\left(\kappa - |k|\right) > \frac{M}{\varepsilon_1}\left(1 + \frac{1}{\varepsilon_1}\right) + M\frac{\rho}{4\varepsilon_2} + N_4$$

and

$$N - C_{\varepsilon_1,\varepsilon_2} > 0. \tag{2.53}$$

Thus, there exist $\lambda, m_1, m_2 > 0$, such that

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\lambda\left(\left\|\sqrt{a}u_x + \frac{b}{\sqrt{a}}\phi\right\|^2 + \|\phi\|^2 + \|u_t\|^2 + \|\phi_x\|^2 + \|\phi_t\|^2 + \|\theta_x\|^2\right) \\
& - \lambda\int_0^\pi\int_0^1 z_x^2\left(x,\omega,t\right)d\omega dx - \lambda\int_0^\pi z_x^2(x,1,t)dx
\end{aligned} \tag{2.54}$$

and

$$m_1 E\left(t\right) \leq \mathcal{L}\left(t\right) \leq m_2 E(t), \quad \forall t \geq 0. \tag{2.55}$$

Therefore, there exists $\delta > 0$, such that

$$\mathcal{L}'(t) \leq -\delta E(t), \ t \geq 0.$$

Using (2.55), we have, for a constant $\varpi > 0$,

$$\mathcal{L}'(t) \leq -\varpi \mathcal{L}(t), \ t \geq 0.$$

An integration over $(0, t)$ and the use of (2.55) again lead to

$$E\left(t\right) \leq \sigma e^{-\varpi t}, \ t \geq 0,$$

for some $\sigma > 0$, which completes the proof of Theorem 2.2.

2.4 Lack of Exponential Stability

In this section, we suppose that $\chi \neq 0$ and prove the lack of the exponential decay. The proof is based on the Theorem 1.14.

Our result is the following:

Theorem 2.3. *Suppose that $\chi \neq 0$. Then, the solution (u, ϕ, θ, z) of problem (2.8)-(2.9) can not be exponentially stable.*

Proof. Choose $F_n = (0, \dfrac{sin(nx)}{\rho}, 0, 0, 0, 0)^T \in \mathcal{H}$ and let $U_n = (u_n, v_n, \phi_n, \psi_n, \theta_n, z_n)^T \in D(\mathcal{A})$ be the solution of $(i\lambda_n I + \mathcal{A})U_n = F_n$, then

$$\|F_n\|_{\mathcal{H}} = \sqrt{\frac{\pi}{2\rho}},$$

and

$$\left\{ \begin{aligned} i\lambda_n u_n - v_n &= 0, \\ i\rho\lambda_n v_n - a u_{nxx} - b\phi_{nx} &= sin(nx), \\ i\lambda_n \phi_n - \psi_n &= 0, \\ iJ\lambda_n \psi_n - \alpha\phi_{nxx} + \xi\phi_n + b u_{nx} + \beta\theta_{nx} &= 0, \\ ic\lambda_n \theta_n - \kappa\theta_{nxx} - k z_{nxx}(x, 1, t) + \beta\psi_{nx} &= 0, \\ i\tau\lambda_n z_n + z_{n\omega} &= 0. \end{aligned} \right. \tag{2.56}$$

Substituting the first and third equation of (2.56) into the second, the fourth and the fifth equation of (2.56), we obtain

$$\begin{cases} \rho\lambda_n^2 u_n + au_{nxx} + b\phi_{nx} &= -sin(nx), \\ (J\lambda_n^2 - \xi)\phi_n + \alpha\phi_{nxx} - bu_{nx} - \beta\theta_{nx} &= 0, \\ ic\lambda_n\theta_n + i\beta\lambda_n\phi_{nx} - \kappa\theta_{nxx} - kz_{nxx}(x,1,t) &= 0, \\ i\tau\lambda_n z_n + z_{n\omega} &= 0. \end{cases} \tag{2.57}$$

Taking into account the boundary conditions, we set

$$\begin{aligned} u_n &= A_n \sin(nx), & \phi_n &= B_n \cos(nx), \\ \theta_n &= C_n \sin(nx), & z_n &= D_n(\omega)\sin(nx), \end{aligned} \tag{2.58}$$

where A_n, B_n, C_n, and $D_n\ : (0,1) \longrightarrow \mathbb{C}$, such that $D_n(0) = C_n$. Substituting (2.58) into (2.57), we get

$$\begin{cases} (an^2 - \rho\lambda_n^2)\, A_n + bnB_n &= 1, \\ bnA_n + (\alpha n^2 + \xi - J\lambda_n^2)\, B_n + \beta nC_n &= 0, \\ (ic\lambda_n + \kappa n^2)\, C_n - i\beta\lambda_n nB_n + kn^2 D_n(1) &= 0, \\ i\tau\lambda_n D_n(\omega) + D_{n\omega}(\omega) &= 0. \end{cases} \tag{2.59}$$

Solving the fourth equation of (2.59), we have

$$D_n(\omega) = C_n e^{-i\lambda_n\tau\omega}.$$

The substitution of $D_n(\omega)$ into the third equation of (2.59) yields

$$\begin{cases} (an^2 - \rho\lambda_n^2)\, A_n + bnB_n &= 1, \\ bnA_n + (\alpha n^2 + \xi - J\lambda_n^2)\, B_n + \beta nC_n &= 0, \\ -i\beta\lambda_n nB_n + \left(ic\lambda_n + \kappa n^2 + kn^2 e^{-i\lambda_n\tau}\right) C_n &= 0, \end{cases} \tag{2.60}$$

which can be written as

$$\begin{pmatrix} P_1(\lambda) & bn & 0 \\ bn & P_2(\lambda) & \beta n \\ 0 & -i\beta\lambda_n n & P_3(\lambda) \end{pmatrix} \begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{2.61}$$

where

$$P_1(\lambda_n) = an^2 - \rho\lambda_n^2, \qquad P_2(\lambda_n) = \alpha n^2 + \xi - J\lambda_n^2,$$

$$P_3(\lambda_n) = ic\lambda_n + \kappa n^2 + kn^2 e^{-i\lambda_n \tau}.$$

From (2.61), we entail that

$$A_n = \frac{P_2(\lambda_n) P_3(\lambda_n) + i\lambda_n n^2 \beta^2}{P_1(\lambda_n)\left[P_2(\lambda_n) P_3(\lambda_n) + i\lambda_n n^2 \beta^2\right] - n^2 b^2 P_3(\lambda_n)}.$$

Taking $\lambda_n = \sqrt{\dfrac{an^2}{\rho}}$, then $P_1(\lambda_n) = 0$ and

$$A_n = \frac{P_2(\lambda_n) P_3(\lambda_n) + i\lambda_n n^2 \beta^2}{-n^2 b^2 P_3(\lambda_n)} = \frac{K}{R},$$

where

$$K = \frac{\alpha\rho}{a}\chi\left(\kappa - ke^{-i\tau\lambda_n}\right)\lambda_n^4 + i\left(\frac{1}{a}\beta^2\rho + \frac{1}{a}c\alpha\rho - Jc\right)\lambda_n^3 + \frac{\xi\rho}{a}\left(\kappa + ke^{-i\tau\lambda_n}\right)\lambda_n^2 + ic\xi\lambda_n$$

and

$$R = \frac{\rho^2 b^2}{a^2}\left(\kappa - ke^{-i\tau\lambda_n}\right)\lambda_n^4 - i\frac{b^2 c\rho}{a}\lambda_n^3.$$

Therefore,

$$A_n = \frac{\frac{\alpha\rho}{a}\chi\left(\kappa - ke^{-i\tau\lambda_n}\right) + i\left(\frac{1}{a}\beta^2\rho + \frac{1}{a}c\alpha\rho - Jc\right)\frac{1}{\lambda_n} + \frac{\xi\rho}{a}\left(\kappa + ke^{-i\tau\lambda_n}\right)\frac{1}{\lambda_n^2} + ic\xi\frac{1}{\lambda_n^3}}{\frac{\rho^2 b^2}{a^2}\left(\kappa - ke^{-i\tau\lambda_n}\right) - i\frac{b^2 c\rho}{a}\frac{1}{\lambda_n}}.$$

As $n \longrightarrow +\infty$, $\lambda_n \longrightarrow +\infty$, we get $e^{-i\tau\lambda_n} \longrightarrow 0$, then

$$A_n \longrightarrow \frac{\alpha a}{\rho b^2}\chi = A \neq 0, \text{ as } n \longrightarrow \infty.$$

Consequently

$$\begin{aligned}
\|U_n\|^2 &\geq \rho\int_0^\pi |v_n|^2\,dx = \rho\int_0^\pi |\lambda_n u_n|^2\,dx = \rho\int_0^\pi |\lambda_n A_n \sin nx|^2\,dx \\
&\geq \rho\,|A_n|^2\,|\lambda_n|^2\int_0^\pi |\sin nx|^2\,dx = \frac{\rho\pi\,|A_n|^2\,|\lambda_n|^2}{2} \longrightarrow \infty.
\end{aligned}$$

Thus,

$$\frac{\|((i\lambda_n I + A)U_n)\|}{\|U_n\|} = \frac{\|F_n\|}{\|U_n\|} \geq \frac{1}{\rho|A_n||\lambda_n|} \longrightarrow 0,$$

which contradicts the statements of Theorem 1.14. Therefore, Theorem 2.3 is established.

□

chapter

3 A porous thermoelastic system with Gurtin-Pipkin law

3.1 Introduction

In this chapter, we are concerned with the following porous thermoelastic system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x, & \text{in } (0,\pi)\times(0,+\infty), \\ J\phi_{tt} = \alpha\phi_{xx} - bu_x - \xi\phi - \beta\theta_x, & \text{in } (0,\pi)\times(0,+\infty), \\ c\theta_t = -q_x - \beta\phi_{xt}, & \text{in } (0,\pi)\times(0,+\infty), \\ q = -\displaystyle\int_{-\infty}^{t} g(t-s)\theta_x(x,s)ds, & \text{in } (0,\pi)\times(0,+\infty). \end{cases} \tag{3.1}$$

together with the initial and boundary conditions:

$$u(x,0) = u_0(x),\ \phi(x,0) = \phi_0(x),\ \theta(x,0) = \theta_0(x),$$
$$u_t(x,0) = u_1(x),\ \phi_t(x,0) = \phi_1(x),$$
$$u(0,t) = u(\pi,t) = \phi_x(0,t) = \phi_x(\pi,t) = \theta(0,t) = \theta(\pi,t) = 0. \tag{3.2}$$

This system models a one-dimensional porous elastic material of length π, subjected to thermal effects modelled by the Gurtin-Pipkin law of thermal conductivity. Here, u is the transversal displacement, ϕ is the volume fraction, θ is the difference of temperature from an equilibrium reference value and q is the heat flux.

The coefficients $\rho, J, c, \mu, b, \alpha$ and ξ are positive constitutive constants. Moreover, to ensure that the energy associated with the solution of (3.1) is a positive definite form, we assume that $\mu\xi > b^2$. The coefficient β is a non-zero coupling constant, but its sign does not matter in the analysis.

The basic evolution equations in the theory of thermoelastic materials with voids were developed by Iesan [41, 43]. In the one dimensional case these equations are written as follows

$$\rho u_{tt} = T_x, \quad J\phi_{tt} = H_x - G, \quad \rho\eta_t = q_x, \tag{3.3}$$

where T is the stress tensor, H is the equilibrated stress vector, G is the equilibrated body force and η is the entropy.

In the linear theory, and assuming that the body is free from stresses and has zero intrinsic equilibrated body force, the constitutive equations are ([41, 43])

$$\begin{aligned} T &= \mu u_x + b\phi, \quad & H &= \alpha\phi_x - \beta\theta, \\ G &= b u_x + \xi\phi, \quad & \rho\eta &= -c\theta - \beta\phi_x, \end{aligned} \tag{3.4}$$

The substitution of the constitutive equation (3.4) into the evolution equations (3.3) leads to the first three equations of the system (3.1). The fourth equation of (3.1) represents the heat conduction modeled by the Gurtin-Pipkin thermal law (3.7), which will be introduced later.

Note that the heat conduction described by Fourier's law

$$q = -\kappa\theta_x, \tag{3.5}$$

leads to a parabolic equation. Consequently, the heat propagates at an infinite speed, which is unrealistic. To overcome this physical paradox, many alternative theories have been developed. Lord and Shulman [52] replaced Fourier's law (3.5) with Cattaneo's law

$$\tau_0 q_t + q + \kappa\theta_x = 0, \tag{3.6}$$

where the positive constant τ_0 represents the time lag in the response of the heat flux to the temperature gradient. According to this theory, called second sound thermoelasticity, the system becomes fully hyperbolic, and heat propagates as a wave with finite speed. Consequently, the speed of the heat equation is involved in the exponential stability condition.

Moreover, the theory of thermoelasticity with second sound is incapable of depicting the memory effect that prevails in some materials, particularly at low temperature. This fact leads to the search for a more general constitutive hypothesis that links the heat flux to the thermal memory. Gurtin and Pipkin [40] proposed that the heat flux depends on the cumulative history of the temperature gradient weighted by a relaxation function called the heat flux kernel. They formulated a general nonlinear theory in which thermal perturbations disseminate with finite speed. According to this theory, the linearized constitutive equation for q is written as follows

$$q = -\int_{-\infty}^{t} g\left(t-s\right)\theta_x\left(x,s\right)ds, \tag{3.7}$$

where $g(s)$ is the heat conductivity relaxation kernel. The presence of the convolution term (3.7) in the heat equation renders the system into a fully hyperbolic system. This allows heat to propagate at a finite speed and enables the description of the memory effect of heat conduction. We note that several constitutive models arise from different choices of $g\left(s\right)$. In particular, if we take $g\left(s\right) = \kappa\delta\left(s\right)$, where δ is the Dirac mass weighted at 0, then (3.7) becomes Fourier's law (3.5), and if we let

$$g\left(s\right) = \frac{\kappa}{\tau_0}e^{-\frac{s}{\tau_0}},\ \tau > 0,$$

we recover Cattaneo's law (3.6). Therefore, (3.7) is a generalized form of Fourier's and Cattaneo's laws.

In the context of Gurtin-Pipkin theory, Pata and Vuk [66] considered the linear thermoelastic system

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) - \theta_x(x,t), \\ \theta_t\left(x,t\right) = -u_{tx}\left(x,t\right) - q_x(x,t), \end{cases}$$

where the heat flux q is modeled by (3.7). They used the semigroup theory and established an exponential stability result under certain assumptions on $\mu\left(s\right) = -g'(s)$. Fatori and Muñoz Rivera [25] studied the system

$$\begin{cases} u_{tt} - au_{xx} + \alpha\theta_x = 0, & \text{in } \ (0;L)\times\mathbb{R}_+, \\ \theta_t - k*\theta_{xx} + \alpha u_{xt} = 0, & \text{in } \ (0;L)\times\mathbb{R}_+, \end{cases}$$

where

$$\left(k*\theta_{xx}\right)(t) = \int_0^t k\left(t-\tau\right)\theta_{xx}\left(\tau\right)d\tau.$$

They obtained an exponential decay result, provided that the kernel k is positive definite and decays exponentially.

Concerning Timoshenko type systems with heat flux given by Gurtin-Pipkin law, Dell'Oro and Pata [21] considered the following system

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) + \delta\theta_x = 0, \\ \rho_3\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^\infty g\left(s\right)\theta_{xx}\left(t-s\right)ds + \delta\psi_{tx} = 0 \end{cases} \tag{3.8}$$

and proved that the solution of system (3.8) is exponentially stable if and only if

$$\chi = \left[\frac{\rho_1}{\rho_3\kappa} - \frac{\beta}{g\left(0\right)}\right]\left[\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right] - \frac{\beta}{g\left(0\right)}\frac{\rho_1\delta^2}{\rho_3\kappa b} = 0.$$

Recently, Fareh [23] considered the porous thermoelastic system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x, & \text{in } (0,\pi) \times (0,+\infty), \\ J\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi + m\theta - \tau\varphi_t, & \text{in } (0,\pi) \times (0,+\infty), \\ c\theta_t = -\displaystyle\int_{-\infty}^0 g(t-s)\theta_{xx}(s)ds - \beta u_{tx} - m\varphi_t, & \text{in } (0,\pi) \times (0,+\infty), \end{cases} \tag{3.9}$$

and proved the exponential stability of the solution of (3.9) without any restrictions on the coefficients of the system.

The system (3.1) considered in the present paper is a (weak) version of (3.9), where the coupling porosity-temperature is direct, however, the coupling elasticity-temperature is through the porosity equation. Moreover, system (3.1) contains only one dissipative mechanism, which is produced by the heat equation of hyperbolic type. Our result improves that of [21] obtained for Timoshenko system and generalizes the results of [24] and [78] since (3.7) subsumes (3.5) and (3.6).

The rest of this chapter is organized as follow: in Section 2, we introduce some preliminaries. Section 3 is devoted to the proof of the existence of a unique solution to (3.1),(3.2). In Section 4, we state and prove our stability result provided that $\chi_g = 0$. Finally, in Section 5, we prove the lack of the exponential decay whenever $\gamma_g = 0$ or $\chi_g \neq 0$.

3.2 Preliminaries

Before we establish our well-posedness result, we introduce the new variables

$$\theta^t(x,s) := \theta(x,t-s),\ s \geq 0,$$

and

$$\eta(x,s)=\eta^t(x,s):=\int_0^s \theta^t(x,\tau)d\tau,\ s\geq 0,$$

which denote the past history and the integrated past history of θ up to t, respectively.

Clearly, $\eta^t(x,s)$ satisfies the boundary conditions

$$\eta(0,s)=\eta(\pi,s)=0.$$

Moreover, we assume that $g(\infty)=0$ and $\eta(x,0)=\lim_{s\longrightarrow 0}\eta^t(x,s)=0$, then

$$q=-\int_{-\infty}^t g(t-s)\,\theta_x(x,s)\,ds=\int_0^{+\infty} g'(s)\,\eta_x^t(x,s)\,ds.$$

Further, we have

$$\eta_t(x,s)=\theta-\eta_s(x,s). \tag{3.10}$$

Setting $\kappa(s)=-g'(s)$, system (3.1) becomes

$$\begin{cases} \rho u_{tt}=\mu u_{xx}+b\phi_x, & \text{in } (0,\pi)\times(0,+\infty),\\ J\phi_{tt}=\alpha\phi_{xx}-bu_x-\xi\phi-\beta\theta_x, & \text{in } (0,\pi)\times(0,+\infty),\\ c\theta_t=-\beta\phi_{xt}+\displaystyle\int_0^{+\infty}\kappa(s)\eta_{xx}^t(x,s)ds, & \text{in } (0,\pi)\times(0,+\infty),\\ \eta_t^t=\theta-\eta_s^t, & \text{in } (0,\pi)\times(0,+\infty). \end{cases} \tag{3.11}$$

From $(3.11)_2$ and the boundary conditions (3.2), we have

$$J\frac{d^2}{dt^2}\int_0^\pi \phi(x,t)\,dx+\xi\int_0^\pi \phi(x,t)\,dx=0,$$

which gives

$$\int_0^\pi \phi(x,t)\,dx=\left(\int_0^\pi \phi_0(x)\,dx\right)\cos\left(\sqrt{\frac{\xi}{J}}t\right)+\sqrt{\frac{J}{\xi}}\left(\int_0^\pi \phi_1(x)\,dx\right)\sin\left(\sqrt{\frac{\xi}{J}}t\right).$$

Consequently, if we set

$$\overline{\phi}(x,t)=\phi(x,t)-\frac{1}{\pi}\left(\int_0^\pi \phi_0(x)\,dx\right)\cos\left(\sqrt{\frac{\xi}{J}}t\right)-\frac{1}{\pi}\sqrt{\frac{J}{\xi}}\left(\int_0^\pi \phi_1(x)\,dx\right)\sin\left(\sqrt{\frac{\xi}{J}}t\right),$$

we find

$$\int_0^\pi \overline{\phi}(x,t)\,dx=0.$$

Moreover, we easily check that $\left(u,\overline{\phi},\theta,\eta\right)$ solves (3.11) subjected to the following boundary and initial conditions

$$\begin{cases} u(x,0)=u_0(x),\ \overline{\phi}(x,0)=\phi_0(x)-\dfrac{1}{\pi}\displaystyle\int_0^{\pi}\phi_0(x)dx,\ \theta(x,0)=\theta_0(x), \\ u_t(x,0)=u_1(x),\ \overline{\phi}_t(x,0)=\phi_1(x)-\dfrac{1}{\pi}\displaystyle\int_0^{\pi}\phi_1\left(x\right)dx,\ \eta^0(x,s)=\eta_0(x,s), \\ u(0,t)=u(\pi,t)=\phi_x(0,t)=\phi_x(\pi,t)=\theta(0,t)=\theta(\pi,t)=\eta(0,s)=\eta(\pi,s)=0, \end{cases} \tag{3.12}$$

and, more importantly, Poincaré's inequality can be applied for $\overline{\phi}$. In the sequel, we work with $(u,\overline{\phi},\theta,\eta)$, but we write (u,ϕ,θ,η) for convenience.

Regarding the memory kernel, we assume the following set of hypotheses:

(h1) $\kappa\in C\left(\mathbb{R}^+\right)\cap L^1(\mathbb{R}^+)$.

(h2) $\kappa\left(s\right)>0,\ \kappa^{'}\left(s\right)\le 0,\ \forall s\ge 0$.

(h3) $\int_0^{\infty}\kappa(s)ds=g(0)$.

(h4) κ is differentiable almost everywhere and there exists $\delta>0$ such that $\kappa'\left(s\right)\le -\delta\kappa(s),\ \forall s\ge 0$.

At this point, let $A:D(A)\subset L^2\left(0,\pi\right)\to L^2\left(0,\pi\right)$ be the positive self-adjoint operator defined by $Au=-\partial_{xx}u$, with domain $D(A)=H^2(0,\pi)\cap H_0^1(0,\pi)$. Therefore, it is possible to define the powers A^r of A for $r\in\mathbb{R}$ and the Hilbert space $V_r=D\left(A^{r/2}\right)$ equipped with the inner product

$$\left\langle u,v\right\rangle_r=\langle A^{r/2}u,A^{r/2}v\rangle,$$

and let $\|u\|_r$ be the associated norm. In particular, $V_0=L^2(0,\pi)$, $V_{-1}=H^{-1}(0,\pi)$, $V_1=H_0^1(0,\pi)$ and

$$\left\langle A^{1/2}u,A^{1/2}v\right\rangle=\left\langle Du,Dv\right\rangle,\ \forall u,v\in H_0^1(0,\pi).$$

For $r_1>r_2$ the injection $V_{r_1}\hookrightarrow V_{r_2}$ is continuous.

Next, we introduce the weighted Hilbert space

$$\mathcal{V}=L_\kappa^2((0,+\infty);H_0^1(0,\pi)),$$

with inner product

$$\langle\eta,\zeta\rangle_{\mathcal{V}}=\int_0^{\infty}\kappa(s)\langle\eta_x(s),\zeta_x(s)\rangle ds,$$

and the norm

$$\|\eta\|_{\mathcal{V}}^2 = \int_0^\infty \kappa(s)\|\eta_x(s)\|^2 ds.$$

We introduce the space

$$\mathcal{H} := H_0^1(0,\pi) \times L^2(0,\pi) \times H_*^1(0,\pi) \times L_*^2(0,\pi) \times L^2(0,\pi) \times \mathcal{V}.$$

It is a Hilbert space with respect to the inner product

$$\begin{aligned}\langle U, U^* \rangle = \quad & \rho \int_0^\pi v v^* dx + \mu \int_0^\pi u_x u_x^* dx + \xi \int_0^\pi \phi\phi^* dx + \alpha \int_0^\pi \phi_x \phi_x^* dx + J \int_0^\pi \psi\psi^* dx \\ & + b \int_0^\pi \phi u_x^* dx + b \int_0^\pi u_x \phi^* dx + c \int_0^\pi \theta\theta^* dx + \int_0^\infty \kappa(s) \int_0^\pi \eta_x(s)\eta_x^*(s) dx ds,\end{aligned}$$

where $U = (u, v, \phi, \psi, \eta)^T$ and $U^* = (u^*, v^*, \phi^*, \psi^*, \eta^*)^T$.

Remark 3.1. *Under the hypothesis $\mu\xi > b^2$, we have*

$$\begin{aligned}\int_0^\pi \left(\mu u_x^2 + \xi\phi^2 + 2b u_x \phi\right) dx \quad &= \left\| \sqrt{\mu} u_x + \frac{b}{\sqrt{\mu}}\phi \right\|^2 + \left(\xi - \frac{b^2}{\mu}\right) \|\phi\|^2 \\ &= \left\| \sqrt{\xi}\phi + \frac{b}{\sqrt{\xi}} u_x \right\|^2 + \left(\mu - \frac{b^2}{\xi}\right) \|u_x\|^2.\end{aligned} \tag{3.13}$$

3.3 Well-posedness

In this section, we prove that the problem given by (3.11), (3.12) has a unique solution. The proof is based on the theory of semigroups and Hille-Yoside Theorem.

To rewrite the problem in the semigroup setting, we introduce two new independent variables $u_t = v$ and $\phi_t = \psi$. Then, the system (3.11), (3.12) can be written as follows

$$\begin{cases} U_t + \mathcal{A}U = 0, \\ U(0) = U_0 = (u_0, u_1, \phi_0, \phi_1, \theta_0, \eta_0)^T, \end{cases} \tag{3.14}$$

where $\mathcal{A}$ is the operator defined on $\mathcal{H}$ by

$$\mathcal{A}U = \begin{pmatrix} -v \\ -\frac{\mu}{\rho}u_{xx} - \frac{b}{\rho}\phi_x \\ -\psi \\ -\frac{\alpha}{J}\phi_{xx} + \frac{\xi}{J}\phi + \frac{b}{J}u_x + \frac{\beta}{J}\theta_x \\ \frac{\beta}{c}\psi_x - \frac{1}{c}\int_0^{+\infty} \kappa(s)\eta_{xx}(s)\, ds \\ -\theta + \eta_s(s) \end{pmatrix}, \tag{3.15}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{c} U \in \mathcal{H}: \ u, \phi \in H^2(0,\pi), \ v, \theta \in H_0^1(0,\pi), \ \psi \in H_*^1(0,\pi), \\ \eta, \eta_s \in L^2_\kappa((0,+\infty); H_0^1), \ \displaystyle\int_0^{+\infty} \kappa(s)\eta_{xx}(s)ds \in L^2(0,\pi), \ \eta(0) = 0 \end{array} \right\}.$$

The well-posedness result reads as follows:

Theorem 3.1. *Suppose that κ satisfies the hypothesis (h1)-(h4). Then, for any $U_0 \in D(\mathcal{A})$, the problem (3.14) has a unique solution $U \in C([0,+\infty[; D(\mathcal{A})) \cap C^1([0,+\infty[; \mathcal{H})$.*

The proof of Theorem 3.1 is based on the Hille-Yosida Theorem and will be established through several lemmas.

Lemma 3.1. *Let T be the operator defined on $\mathcal{V}$ by*

$$T\eta = -\eta_s,$$

with domain

$$D(T) = \left\{ \eta \in \mathcal{V}: \ \eta_s \in \mathcal{V}, \ \lim_{s\to 0} \|\eta_x\| = 0 \right\}.$$

Then, for every $\eta \in D(T)$, we have

$$\langle T\eta, \eta\rangle_{\mathcal{V}} = \frac{1}{2}\int_0^{+\infty} \kappa'(s)\, \|\eta_x(s)\|^2\, ds \leq -\frac{\delta}{2}\, \|\eta\|^2_{\mathcal{V}} \leq 0. \tag{3.16}$$

Proof. By the use of integration by parts, we have

$$\begin{aligned} 2\langle T\eta, \eta\rangle_{\mathcal{V}} = & -2\int_0^{+\infty} \kappa(s) \int_0^{\pi} \eta_{xs}(x,s)\eta_x(x,s)dxds \\ = & -\int_0^{+\infty} \kappa(s) \frac{d}{ds}\|\eta_x(s)\|^2 ds \\ = & -\kappa(s)\|\eta_x(s)\|^2\Big|_0^{+\infty} + \int_0^{+\infty} \kappa'(s)\|\eta_x(s)\|^2 ds, \end{aligned}$$

then,

$$2\langle T\eta, \eta\rangle_{\mathcal{V}} = \lim_{s\to 0} \kappa(s)\|\eta_x(s)\|^2 - \lim_{s\to\infty} \kappa(s)\|\eta_x(s)\|^2 + \int_0^{+\infty} \kappa'(s)\|\eta_x(s)\|^2 ds. \tag{3.17}$$

Since $\eta \in H^1_\kappa(0,+\infty; H_0^1(0,\pi))$ and $\eta_x(0) = 0$, then $\kappa(\cdot)\, \|\eta_{xs}(\cdot)\|^2 \in L^1(\mathbb{R}^+)$ and consequently, the first term in the right hand side of (3.17) worths

$$\begin{aligned} \lim_{s\to 0} \kappa(s)\|\eta_x(s)\|^2 &= \lim_{s\to 0} \kappa(s) \left\| \int_0^s \eta_{xs}(\tau)d\tau \right\|^2 \\ &\leq \limsup_{s\to 0} \left(\int_0^s \kappa(s)^{1/2}\, \|\eta_{xs}(\tau)\|\, d\tau \right)^2. \end{aligned}$$

Applying Cauchy-Schwarz inequality and the fact that $\kappa(s) \leq \kappa(\tau)$ for $\tau \in [0, s]$, we arrive at

$$\lim_{s\to 0} \kappa(s)\|\eta_x(s)\|^2 \leq \limsup_{s\to 0} s \int_0^s \kappa(\tau) \|\eta_{xs}(\tau)\|^2 d\tau = 0.$$

Therefore,

$$2\langle T\eta, \eta\rangle_{\mathcal{V}} = -\lim_{s\to\infty} \kappa(s)\|\eta_x(s)\|^2 + \int_0^{+\infty} \kappa'(s)\|\eta_x(s)\|^2 ds.$$

The left-hand side of the last equation is bounded, and according to (h2), both terms on the right-hand side are non-positive. Therefore, we infer that the limit mentioned above exists and is finite. Consequently, it equals zero. Thus

$$\langle T\eta, \eta\rangle_{\mathcal{V}} = \frac{1}{2}\int_0^\infty \kappa'(s)\|\eta_x(s)\|^2 ds.$$

Moreover, the use of the assumption (h4) on κ leads to (3.16), which completes the proof. □

Lemma 3.2. *The operator $\mathcal{A}$ defined by (3.15) is monotone.*

Proof. A direct calculation, using integration by parts and the boundary conditions, yields, $\forall U \in D(\mathcal{A})$,

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} =& -\mu \int_0^\pi u_{xx} v dx - \int_0^\pi b\phi_x v dx - \mu \int_0^\pi v_x u_x dx - \xi \int_0^\pi \psi\phi dx - \alpha \int_0^\pi \psi_x \phi_x dx \\
& - \alpha \int_0^\pi \phi_{xx}\psi dx + \xi \int_0^\pi \phi\psi dx + b \int_0^\pi u_x \psi dx + \beta \int_0^\pi \theta_x \psi dx - b \int_0^\pi \psi u_x dx \\
& - b \int_0^\pi v_x \phi dx + \beta \int_0^\pi \psi_x \theta dx + \int_0^\pi \int_0^\infty \kappa(s)\eta_x(s)\,\theta_x ds dx \\
& - \int_0^{+\infty} \kappa(s) \int_0^\pi \theta_x \eta_x(s)\, ds dx + \int_0^{+\infty} \kappa(s)\eta_{sx}(s)\eta_x(s)\, dx ds, \\
=& \int_0^{+\infty} \kappa(s) \int_0^\pi \eta_{xs}(s)\eta_x(s) ds dx, \\
=& -\langle T\eta, \eta\rangle_{\mathcal{V}}.
\end{aligned}$$

From Lemma 3.1 and the hypothesis (h4), we infer that

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\frac{1}{2}\int_0^{+\infty} \kappa'(s)\|\eta_x(s)\|^2 ds \geq \frac{\delta}{2}\|\eta\|^2_{\mathcal{V}}.$$

□

Lemma 3.3. *The operator $\mathcal{A}$ defined by (3.15) is maximal.*

Proof. Let $F = (f^1, f^2, f^3, f^4, f^5, f^6)^T \in \mathcal{H}$, we seek $U \in D(\mathcal{A})$ such that $(I + \mathcal{A})U = F$; that is,

$$u - v = f^1, \tag{3.18}$$

$$\rho v - \mu u_{xx} - b\phi_x = \rho f^2, \tag{3.19}$$

$$\phi - \psi = f^3, \tag{3.20}$$

$$J\psi - \alpha\phi_{xx} + \xi\phi + bu_x + \beta\theta_x = Jf^4, \tag{3.21}$$

$$c\theta + \beta\psi_x - \int_0^\infty \kappa(s)\eta_{xx}(s)ds = cf^5, \tag{3.22}$$

$$\eta(s) - \theta + \eta_s(s) = f^6. \tag{3.23}$$

By solving (3.23) subject to the initial condition $\eta(0) = 0$, we get

$$\eta(s) = (1 - e^{-s})\theta + \int_0^s e^{y-s} f^6(y)dy. \tag{3.24}$$

Next, inserting v and ψ from (3.18), (3.20) into (3.19), (3.21) and (3.23), we get

$$\begin{cases} \rho u - \mu u_{xx} - b\phi_x = \rho(f^1 + f^2), \\ (J + \xi)\phi - \alpha\phi_{xx} + bu_x + \beta\theta_x = J(f^3 + f^4), \\ c\theta + \beta\phi_x = \beta f_x^3 + cf^5 + \displaystyle\int_0^{+\infty} \kappa(s)\eta_{xx}(s)ds, \end{cases} \tag{3.25}$$

At this point, we define the space

$$\mathcal{W} = H_0^1(0, \pi) \times H_*^1(0, \pi) \times H_0^1(0, \pi),$$

and consider the following variational problem associated with the system (3.25),

$$B\Big((u, \phi, \theta), (u^*, \phi^*, \theta^*)\Big) = L(u^*, \phi^*, \theta^*), \ \forall (u^*, \phi^*, \theta^*) \in \mathcal{W}, \tag{3.26}$$

where B and L are the bilinear and the linear form defined over $\mathcal{W}$ by

$$\begin{aligned} B((u, \phi, \theta), (u^*, \phi^*, \theta^*)) = \ & \mu \int_0^\pi u_x u_x^* dx + b \int_0^\pi \phi u_x^* dx + \rho \int_0^\pi uu^* dx \\ & + \alpha \int_0^\pi \phi_x \phi_x^* dx + b \int_0^\pi u_x \phi^* dx + \beta \int_0^\pi \theta_x \phi^* dx \\ & + (J + \xi) \int_0^\pi \phi\phi^* dx + c_\kappa \int_0^\pi \theta_x \theta_x^* dx \\ & + c \int_0^\pi \theta\theta^* dx + \beta \int_0^\pi \phi_x \theta^* dx \end{aligned}$$

with

$$c_\kappa = \int_0^{+\infty} \kappa(s)(1-e^{-s})ds < \int_0^{+\infty} \kappa(s)ds < \infty,$$

and

$$\begin{aligned} L(u^*,\phi^*,\theta^*) = \ & \rho\int_0^\pi (f^1+f^2)u^*dx + J\int_0^\pi \left(f^3+f^4\right)\phi^*dx + \int_0^\pi (\beta f_x^3 + cf^5)\theta^* dx \\ & - \int_0^\pi \theta_x^* \int_0^{+\infty} \kappa(s)\Big(\int_0^s e^{y-s} f_x^6(y)dy\Big)dsdx. \end{aligned}$$

Clearly, B and L are bounded. Moreover,

$$\begin{aligned} B\Big((u,\phi,\theta),(u,\phi,\theta)\Big) \ & = \mu\int_0^\pi u_x^2dx + 2b\int_0^\pi \phi u_x dx + \rho\int_0^\pi u^2dx + \alpha\int_0^\pi \phi_x^2 dx \\ & + (J+\xi)\int_0^\pi \phi^2dx + c_\kappa\int_0^\pi \theta_x^2dx + c\int_0^\pi \theta^2dx, \end{aligned}$$

which, by (3.13), leads to

$$\begin{aligned} B\Big((u,\phi,\theta),(u,\phi,\theta)\Big) \ & \geq \left(\mu - \frac{b^2}{\xi}\right)\int_0^\pi u_x^2dx + c_\kappa\int_0^\pi \theta_x^2dx + \alpha\int_0^\pi \phi_x^2dx \\ & \geq C\|(u,\phi,\theta)\|_{\mathcal{W}}^2, \end{aligned}$$

for some $C>0$; hence, B is coercive. Consequently, Lax-Milgram theorem guarantees the existence of a unique solution $(u,\phi,\theta)\in\mathcal{W}$ satisfying (3.26).

Next, we take $(u^*,\phi^*,\theta^*) = (u^*,0,0)$ in (3.26) to get

$$\mu\int_0^\pi u_x u_x^* dx = \int_0^\pi (b\phi_x - \rho u + \rho\left(f^1+f^2\right))u^*dx, \ \forall u^*\in H_0^1(0,\pi).$$

Standard arguments of elliptic equation infer that $u\in H^2(0,\pi)\cap H_0^1(0,\pi)$ and

$$\mu u_{xx} = -(b\phi_x - \rho u + \rho\left(f^1+f^2\right)).$$

Therefore, by using (3.18), we arrive at

$$\rho v - \mu u_{xx} - b\phi_x = \rho f^2.$$

Similarly, by taking $(u^*,\phi^*,\theta^*) = (0,\phi^*,0)$ in (3.26), we get

$$\alpha\int_0^\pi \phi_x\phi_x^*dx = \int_0^\pi \left(-bu_x + \beta\theta_x - (J+\xi)\phi + J\left(f^3+f^4\right)\right)\phi^*dx, \ \forall\phi^*\in H_*^1(0,\pi). \quad (3.27)$$

Here, we cannot apply the elliptic regularity directly. To do so, we proceed as follows: Let $\varphi \in H_0^1(0,\pi)$ and set $\phi^* = \varphi - \frac{1}{\pi}\int_0^\pi \varphi(x)dx$. We easily check that $\phi^* \in H_*^1(0,\pi)$. Plugging ϕ^* in (3.27), taking into account that

$$-bu_x + \beta\theta_x - (J+\xi)\phi + J\left(f^3 + f^4\right) \in L_*^2(0,\pi),$$

we obtain

$$\alpha\int_0^\pi \phi_x\varphi_x dx = \int_0^\pi \left(-bu_x + \beta\theta_x - (J+\xi)\phi + J\left(f^3+f^4\right)\right)\varphi dx, \quad \forall \varphi \in H_0^1(0,\pi);$$

which means that

$$\phi \in H^2\left(0,\pi\right) \cap H_*^1(0,\pi).$$

An integration by parts leads to

$$\alpha\phi_{xx} = bu_x - \beta\theta_x + (J+\xi)\phi - J\left(f^3+f^4\right) = R \in L^2(0,\pi). \tag{3.28}$$

The use of (3.20) leads to

$$J\psi - \alpha\phi_{xx} + \xi\phi + bu_x + \beta\theta_x = Jf^4.$$

Thus, ϕ satisfies (3.21).

On the other hand, since $\alpha\phi_{xx} = R \in L^2(0,\pi)$, then

$$\alpha\int_0^\pi \phi_{xx}\Phi dx = \int_0^\pi R\Phi dx, \quad \forall \Phi \in H^1(0,\pi).$$

Integration by parts gives

$$\alpha\phi_x\Phi\Big|_0^\pi - \alpha\int_0^\pi \phi_x\Phi_x dx = \int_0^\pi R\Phi dx, \quad \forall \Phi \in H^1(0,\pi).$$

Since $H_*^1(0,\pi) \subset H^1(0,\pi)$, we have

$$\alpha\phi_x\Phi\Big|_0^\pi - \alpha\int_0^\pi \phi_x\Phi_x dx = \int_0^\pi R\Phi dx, \quad \forall \Phi \in H_*^1(0,\pi).$$

The use of (3.27) leads to

$$\alpha\phi_x(\pi)\Phi(\pi) - \alpha\phi_x(0)\Phi(0) = 0, \quad \forall \Phi \in H_*^1(0,\pi).$$

As Φ is arbitrary, then

$$\alpha\phi_x(\pi) = \alpha\phi_x(0) = 0,$$

and, hence,

$$\phi \in H_*^2(0,\pi).$$

Back to (3.18),(3.20) and (3.24) taking into account the last results, we get

$$v \in H_0^1(0,\pi), \quad \psi \in H_*^1(0,\pi) \quad \text{and} \quad \eta \in H_\kappa^1((0,+\infty); H_0^1).$$

Furthermore, (3.22) yields

$$\int_0^{+\infty} \kappa(s)\,\eta_{xx}^t(s)\,ds \in L^2(0,\pi).$$

Hence, the solution U belongs to $D(\mathcal{A})$, then $\mathcal{A}$ is maximal, which completes the proof of the Lemma. □

Proof of Theorem 3.1 From the Lemmas 3.2, 3.3 and by Hille-Yoside Theorem 1.9, the problem (3.14) has a unique solution.

3.4 Exponential stability

In this section, we state and prove the stability result of our problem.
First, let

$$\gamma_g = c\mu - \rho g(0) \tag{3.29}$$

and for $\gamma_g \neq 0$, we introduce the stability number

$$\chi_g = \frac{\rho}{\mu} - \frac{J}{\alpha} + \frac{\rho\beta^2}{\alpha\gamma_g}. \tag{3.30}$$

Next, we define the energy associated with the solution (u,ϕ,θ,η) of problem (3.11), (3.12), by

$$E(t) := \frac{1}{2}\int_0^\pi \left[\rho u_t^2 + J\phi_t^2 + \mu u_x^2 + \xi\phi^2 + 2bu_x\phi + \alpha\phi_x^2 + c\theta^2 + \int_0^\infty \kappa(s)\eta_x^2(s)ds\right]dx. \tag{3.31}$$

Lemma 3.4. *The energy $E(t)$, defined by (3.31), satisfies, along the solution (u,ϕ,θ,η), the estimate*

$$\frac{d}{dt}E(t) = \langle T\eta, \eta\rangle. \tag{3.32}$$

Proof. Taking the L^2-inner product of the first three equations of (3.11) by u_t, ϕ_t, θ, respectively, and the $\mathcal{V}-$inner product of the forth equation by η_t, then adding the obtained equations and using integration by parts, (3.32) follows immediately. □

Remark 3.2. *According to (3.16) and (3.32), the energy $E(t)$ satisfies*

$$\frac{d}{dt}E(t) = \frac{1}{2}\int_0^{+\infty} \kappa'(s)\|\eta_x(s)\|^2 ds \leq 0. \tag{3.33}$$

The main result reads as follow:

Theorem 3.2. *Let (u, ϕ, θ, η) be the solution of problem (3.11), (3.12) and assume that (h1)-(h4) and that $\gamma_g \neq 0$ and $\chi_g = 0$. Then, the energy functional (3.31) satisfies*

$$E(t) \leq \sigma E(0)e^{-\omega t}, \ \forall t > 0,$$

where σ and ω are two positive constants.

The proof of Theorem 3.2 will be established by the multipliers method through several lemmas.

Lemma 3.5. *Let (u, ϕ, θ, η) be the solution of (3.11), (3.12). Then, the functional*

$$F_1(t) = -\frac{2c}{g(0)}\int_0^{+\infty} \kappa(s)\int_0^{\pi} \theta(x,t)\eta^t(x,s)dxds$$

satisfies, for any $\varepsilon_1 > 0$, the estimate

$$\frac{d}{dt}F_1(t) \leq -c\|\theta\|^2 - \|\eta\|_\nu^2 + \varepsilon_1\|\phi_t\|^2 - M\left(1+\frac{1}{\varepsilon_1}\right)\int_0^{+\infty} \kappa'(s)\|\eta_x(s)\|^2 ds, \tag{3.34}$$

where M is a positive constant independent of ε_1.

Proof. Differentiating $F_1(t)$, using the third and fourth equation of (3.11) and integration by parts, we obtain

$$\begin{aligned}\frac{d}{dt}F_1(t) &= -\frac{2c}{g(0)}\int_0^{+\infty}\kappa(s)\int_0^{\pi}\theta_t(x,t)\eta(x,s)dxds - \frac{2c}{g(0)}\int_0^{+\infty}\kappa(s)\int_0^{\pi}\theta(x,t)\eta_t(x,s)dxds\\
&= -\frac{2}{g(0)}\int_0^{+\infty}\kappa(s)\int_0^{\pi}\left(-\beta\phi_{xt}(x,t)+\int_0^{+\infty}\kappa(s)\eta_{xx}(x,s)ds\right)\eta(x,s)dxds\\
&- \frac{2c}{g(0)}\int_0^{+\infty}\kappa(s)\int_0^{\pi}(\theta(x,t)-\eta_s(x,s))\,\theta(x,t)dxds\\
&= -\frac{2\beta}{g(0)}\int_0^{+\infty}\kappa(s)\int_0^{\pi}\phi_t(x,t)\eta_x(x,s)dxds + \frac{2}{g(0)}\left\|\int_0^{+\infty}\kappa(s)\eta_x(s)ds\right\|^2\\
&- 2c\|\theta\|^2 + \frac{2c}{g(0)}\int_0^{+\infty}\kappa(s)\int_0^{\pi}\theta(x,t)\eta_s(x,s)dxds.\end{aligned} \tag{3.35}$$

Now, integrating by parts in the last term of (3.35) with respect to s, using Cauchy Schwarz, Young's and Poincaré's inequalities, we get

$$\begin{aligned} \frac{2c}{g(0)}\int_0^{+\infty}\kappa(s)\int_0^{\pi}\theta(x,t)\eta_s(x,s)dxds &= -\frac{2c}{g(0)}\int_0^{\pi}\theta(x,t)\int_0^{+\infty}\kappa'(s)\eta(x,s)dsdx \\ &\leq c\,\|\theta\|^2 - M\int_0^{+\infty}\kappa'(s)\|\eta_x(s)\|^2 ds. \end{aligned} \tag{3.36}$$

Next, Young's inequality and (3.16) yield

$$\begin{aligned} -\frac{2\beta}{g(0)}\int_0^{+\infty}\kappa(s)\int_0^{\pi}\phi_t(x,t)\eta_x(x,s)dxds &= -\frac{2\beta}{g(0)}\int_0^{\pi}\phi_t(x,t)\int_0^{+\infty}\kappa(s)\eta_x(x,s)dsdx \\ &\leq \varepsilon_1\,\|\phi_t\|^2 - \frac{M}{\varepsilon_1}\int_0^{+\infty}\kappa'(s)\,\|\eta_x(s)\|^2\,ds. \end{aligned} \tag{3.37}$$

Also, Cauchy-Schwarz' inequality and (3.16) give

$$\begin{aligned} \frac{2}{g(0)}\left\|\int_0^{+\infty}\kappa(s)\eta_x(s)ds\right\|^2 &\leq 2\,\|\eta\|_{\mathcal{V}}^2 \leq -\,\|\eta\|_{\mathcal{V}}^2 - \frac{3}{\delta}\int_0^{+\infty}\kappa'(s)\,\|\eta_x(s)\|^2\,ds \\ &\leq -\,\|\eta\|_{\mathcal{V}}^2 - M\int_0^{+\infty}\kappa'(s)\,\|\eta_x(s)\|^2\,ds. \end{aligned} \tag{3.38}$$

Substituting (3.36)-(3.38) into (3.35), estimate (3.34) follows immediately. $\square$

Lemma 3.6. *Let (u,ϕ,θ,η) be the solution of (3.11), (3.12). Then, the functional*

$$F_2(t) = -\frac{cJ}{\beta}\int_0^{\pi}\theta(x,t)\int_0^{x}\phi_t(y,t)dydx$$

satisfies, for any $\varepsilon_2>0$, the estimate

$$\begin{aligned} \frac{d}{dt}F_2(t) \leq &-\frac{J}{2}\,\|\phi_t\|^2 - M\int_0^{+\infty}\kappa'(s)\,\|\eta_x(s)\|^2\,ds + M\left(1+\frac{1}{\varepsilon_2}\right)\|\theta\|^2 \\ &+\varepsilon_2\left\|\sqrt{a}u_x + \frac{b}{\sqrt{a}}\phi\right\|^2 + \varepsilon_2\,\|\phi\|^2 + \varepsilon_2\|\phi_x\|^2, \end{aligned} \tag{3.39}$$

where M is a positive constant independent of ε_2.

Proof. Differentiating $F_2(t)$, using $(3.11)_2$, $(3.11)_3$ and integration by parts, we get

$$\begin{aligned} \frac{d}{dt}F_2(t) &= -\frac{cJ}{\beta}\int_0^{\pi}\theta_t(x,t)\int_0^{x}\phi_t(y,t)dydx - \frac{cJ}{\beta}\int_0^{\pi}\theta(x,t)\int_0^{x}\phi_{tt}(y,t)dydx \\ &= -\frac{J}{\beta}\int_0^{\pi}\left(-\beta\phi_{xt}(x,t) + \int_0^{+\infty}\kappa(s)\eta_{xx}^t(x,s)ds\right)\int_0^{x}\phi_t(y,t)dydx \\ &\quad - \frac{c}{\beta}\int_0^{\pi}\theta(x,t)\int_0^{x}\left(\alpha\phi_{xx} - bu_x - \xi\phi - \beta\theta_x\right)(y,t)dydx \end{aligned}$$

$$= -J\left\|\phi_t\right\|^2 + \frac{J}{\beta}\int_0^{+\infty}\kappa(s)\int_0^{\pi}\eta_x^t(x,s)\phi_t(x,t)dds - \frac{c\alpha}{\beta}\int_0^{\pi}\theta(x,t)\phi_x(x,t)dx$$
$$+ c\left\|\theta\right\|^2 + \frac{c}{\beta}\int_0^{\pi}\theta(x,t)\int_0^{x}\left(bu_x+\xi\phi\right)(y,t)dydx.$$

Notice that

$$\frac{c}{\beta}\int_0^{\pi}\theta(x,t)\int_0^{x}\left(bu_x+\xi\phi\right)(y,t)\,dydx = \frac{cb}{\beta\sqrt{\mu}}\int_0^{\pi}\theta(x,t)\int_0^{x}\left(\sqrt{\mu}u_x+\frac{b}{\sqrt{\mu}}\phi\right)(y,t)\,dydx$$
$$+ \frac{c}{\beta}\left(\xi-\frac{b^2}{\mu}\right)\int_0^{\pi}\theta(x,t)\int_0^{x}\phi(y,t)dydx.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}F_2(t) &= -J\left\|\phi_t\right\|^2 + \frac{J}{\beta}\int_0^{+\infty}\kappa(s)\int_0^{\pi}\eta_x^t(x,s)\phi_t(x,t)dxds\\ &+ \frac{cb}{\beta\sqrt{\mu}}\int_0^{\pi}\theta(x,t)\int_0^{x}\left(\sqrt{\mu}u_x+\frac{b}{\sqrt{\mu}}\phi\right)(y,t)\,dydx\\ &+ \frac{c}{\beta}\left(\xi-\frac{b^2}{\mu}\right)\int_0^{\pi}\theta(x,t)\int_0^{x}\phi(y,t)dydx - \frac{c\alpha}{\beta}\int_0^{\pi}\theta(x,t)\phi_x(x,t)dx + c\|\theta\|^2.\end{aligned} \tag{3.40}$$

Young's and Cauchy Schwarz inequalities and (3.16) give

$$\begin{aligned}\frac{J}{\beta}\int_0^{+\infty}\kappa(s)\int_0^{\pi}\eta_x(x,s)\phi_t(x,t)dxds &\le \frac{J}{\beta}\left\|\phi_t\right\|\int_0^{+\infty}\kappa(s)\left\|\eta_x\right\|ds\\ &\le \frac{J}{2}\left\|\phi_t\right\|^2 + M\left\|\eta\right\|_{\mathcal{V}}^2\\ &\le \frac{J}{2}\left\|\phi_t\right\|^2 - M\int_0^{+\infty}\kappa'(s)\left\|\eta_x(s)\right\|^2ds.\end{aligned} \tag{3.41}$$

Similarly, for any $\varepsilon_2 > 0$, we have

$$\frac{cb}{\beta\sqrt{\mu}}\int_0^{\pi}\theta(x,t)\int_0^{x}\left(\sqrt{\mu}u_x+\frac{b}{\sqrt{\mu}}\phi\right)(y,t)\,dydx \le \frac{M}{\varepsilon_2}\left\|\theta\right\|^2 + \varepsilon_2\left\|\sqrt{\mu}u_x+\frac{b}{\sqrt{\mu}}\phi\right\|^2, \tag{3.42}$$

$$\frac{c}{\beta}\left(\xi-\frac{b^2}{a}\right)\int_0^{\pi}\theta(x,t)\int_0^{x}\phi(y,t)dydx \le \frac{M}{\varepsilon_2}\left\|\theta\right\|^2 + \varepsilon_2\left\|\phi\right\|^2 \tag{3.43}$$

and

$$-\frac{c\alpha}{\beta}\int_0^{\pi}\theta\phi_x dx \le \frac{M}{\varepsilon_2}\left\|\theta\right\|^2 + \varepsilon_2\|\phi_x\|^2. \tag{3.44}$$

The substitution of (3.41)-(3.44) into (3.40) yields (3.39). □

Lemma 3.7. *Let*

$$I_1(t) = \frac{\alpha\rho}{b}\int_0^{\pi}\phi_x u_t dx + \frac{J\mu}{b}\int_0^{\pi}\phi_t u_x dx + 2J\int_0^{\pi}\phi\phi_t dx - \frac{\rho\xi}{b}\int_0^{\pi}u_t\int_0^{x}\phi(y)dydx,$$

$$I_2(t) = -\frac{1}{\sqrt{\mu}} \int_0^{+\infty} \kappa(s) \int_0^{\pi} \eta_x(x,s) \left(\sqrt{\mu} u_x(x,t) + \frac{b}{\sqrt{\mu}} \phi(x,t) \right) dxds,$$

$$I_3(t) = -\frac{\beta\rho}{b} \int_0^{\pi} \theta u_t dx,$$

and

$$F_3(t) = I_1(t) + \frac{\rho\beta\mu}{b\gamma_g} I_2(t) + \frac{\mu c}{\gamma_g} I_3(t). \tag{3.45}$$

Suppose that $\chi_g = 0$, then, for all $\varepsilon > 0$, the functional F_3 satisfies, along the solution of (3.11), (3.12)

$$\begin{aligned} \frac{d}{dt} F_3(t) \leq & -\frac{1}{2} \left\| \sqrt{\mu} u_x + \frac{b}{\sqrt{\mu}} \phi \right\|^2 - \left(\xi - \frac{b^2}{a} \right) \|\phi\|^2 - \frac{\alpha}{2} \|\phi_x\|^2 + M \left(1 + \frac{1}{\varepsilon_3} \right) \|\phi_t\|^2 \\ & + \varepsilon_3 \|u_t\|^2 + M \|\theta\|^2 - M(1+\varepsilon_3) \int_0^{+\infty} \kappa'(s) \|\eta_x\|^2 ds, \end{aligned} \tag{3.46}$$

where χ_g and γ_g are given in (3.29) *and (3.30).*

Proof. Direct differentiation, using $(3.11)_1$ and integration by parts, yields

$$\begin{aligned} \frac{d}{dt} \frac{\alpha\rho}{b} \int_0^{\pi} \phi_x u_t dx &= \frac{\alpha\rho}{b} \int_0^{\pi} \phi_{tx} u_t dx + \frac{\alpha\rho}{b} \int_0^{\pi} \phi_x u_{tt} dx \\ &= \frac{\alpha\rho}{b} \int_0^{\pi} \phi_{tx} u_t dx + \frac{\alpha}{b} \int_0^{\pi} \phi_x (\mu u_{xx} + b\phi_x) dx \\ &= \frac{\alpha\rho}{b} \int_0^{\pi} \phi_{tx} u_t dx + \frac{\mu\alpha}{b} \int_0^{\pi} \phi_x u_{xx} dx + \alpha \|\phi_x\|^2. \end{aligned} \tag{3.47}$$

Similarly, using $(3.11)_2$, we have

$$\begin{aligned} \frac{d}{dt} \frac{J\mu}{b} \int_0^{\pi} u_x \phi_t dx &= \frac{J\mu}{b} \int_0^{\pi} u_{tx} \phi_t dx + \frac{\mu}{b} \int_0^{\pi} J\phi_{tt} u_x dx \\ &= \frac{\mu}{b} \int_0^{\pi} u_x (\alpha\phi_{xx} - bu_x - \xi\phi - \beta\theta_x) dx + \frac{J\mu}{b} \int_0^{\pi} u_{tx} \phi_t dx \\ &= \frac{\mu\alpha}{b} \int_0^{\pi} u_x \phi_{xx} dx - \mu \|u_x\|^2 - \frac{\mu\xi}{b} \int_0^{\pi} u_x \phi dx - \frac{\mu\beta}{b} \int_0^{\pi} u_x \theta_x dx \\ &+ \frac{J\mu}{b} \int_0^{\pi} u_{tx} \phi_t dx \end{aligned} \tag{3.48}$$

and

$$\begin{aligned} 2\frac{d}{dt} J \int_0^{\pi} \phi\phi_t dx &= 2J \|\phi_t\|^2 + 2J \int_0^{\pi} \phi\phi_{tt} dx \\ &= 2J \|\phi_t\|^2 + 2 \int_0^{\pi} \phi (\alpha\phi_{xx} - bu_x - \xi\phi - \beta\theta_x) dx \\ &= 2J \|\phi_t\|^2 - 2\alpha \|\phi_x\|^2 - 2\xi \|\phi\|^2 - 2b \int_0^{\pi} \phi u_x dx - 2\beta \int_0^{\pi} \phi\theta_x dx. \end{aligned} \tag{3.49}$$

By exploiting $(3.11)_1$, we infer that

$$\begin{aligned}
-\frac{d}{dt}\frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi(y)dy\right)dx &= -\frac{\rho\xi}{b}\int_0^\pi u_{tt}\left(\int_0^x \phi(y)dy\right)dx - \frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx \\
&= -\frac{\xi}{b}\int_0^\pi (\mu u_{xx} + b\phi_x)\left(\int_0^x \phi(y)dy\right)dx \\
&- \frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx \\
&= \frac{\mu\xi}{b}\int_0^\pi u_x\phi dx + \xi\left\|\phi\right\|^2 - \frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx.
\end{aligned} \tag{3.50}$$

The addition of (3.47)-(3.50) yields

$$\begin{aligned}
\frac{d}{dt}I_1(t) &= -\mu\left\|u_x\right\|^2 - \xi\left\|\phi\right\|^2 - 2b\int_0^\pi \phi u_x dx - \alpha\left\|\phi_x\right\|^2 + 2J\left\|\phi_t\right\|^2 - 2\beta\int_0^\pi \phi\theta_x dx \\
&- \frac{\mu\beta}{b}\int_0^\pi \theta_x u_x dx - \frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx + \left(\frac{J\mu}{b} - \frac{\alpha\rho}{b}\right)\int_0^\pi u_{tx}\phi_t dx.
\end{aligned}$$

Therefore, (3.13) leads to

$$\begin{aligned}
\frac{d}{dt}I_1(t) &= -\left\|\sqrt{\mu}u_x + \frac{b}{\sqrt{\mu}}\phi\right\|^2 - \left(\xi - \frac{b^2}{\mu}\right)\left\|\phi\right\|^2 - \alpha\left\|\phi_x\right\|^2 + 2J\left\|\phi_t\right\|^2 - 2\beta\int_0^\pi \phi\theta_x dx \\
&- \frac{\mu\beta}{b}\int_0^\pi \theta_x u_x dx - \frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx + \left(\frac{J\mu}{b} - \frac{\alpha\rho}{b}\right)\int_0^\pi u_{tx}\phi_t dx.
\end{aligned} \tag{3.51}$$

The differentiation of $I_2(t)$ and integration by parts yield

$$\begin{aligned}
\frac{d}{dt}I_2(t) &= -\frac{1}{\sqrt{\mu}}\int_0^{+\infty}\kappa(s)\int_0^\pi \eta_{tx}(x,s)\left(\sqrt{\mu}u_x(x,t) + \frac{b}{\sqrt{\mu}}\phi(x,t)\right)dxds \\
&- \frac{1}{\sqrt{\mu}}\int_0^{+\infty}\kappa(s)\int_0^\pi \eta_x(x,s)\left(\sqrt{\mu}u_{tx}(x,t) + \frac{b}{\sqrt{\mu}}\phi_t(x,t)\right)dxds \\
&= \frac{1}{\sqrt{\mu}}\int_0^{+\infty}\kappa(s)\int_0^\pi \eta_t(x,s)\left(\sqrt{\mu}u_{xx}(x,t) + \frac{b}{\sqrt{\mu}}\phi_x(x,t)\right)dxds \\
&+ \frac{1}{\sqrt{\mu}}\int_0^{+\infty}\kappa(s)\int_0^\pi \sqrt{\mu}u_t(x,t)\eta_{xx}(x,s)dxds \\
&- \frac{1}{\sqrt{\mu}}\int_0^{+\infty}\kappa(s)\int_0^\pi \frac{b}{\sqrt{\mu}}\eta_x(x,s)\phi_t(x,t)dxds.
\end{aligned}$$

By using $(3.11)_3$, $(3.11)_4$ and integration by parts, we get

$$\begin{aligned}
\frac{d}{dt}I_2(t) =& \frac{g(0)}{\sqrt{\mu}}\int_0^\pi \theta(x,t)\left(\sqrt{\mu}u_{xx}(x,t)+\frac{b}{\sqrt{\mu}}\phi_x(x,t)\right)dx \\
&-\frac{1}{\sqrt{\mu}}\int_0^\infty \kappa(s)\int_0^\pi \eta_s(x,s)\left(\sqrt{\mu}u_{xx}+\frac{b}{\sqrt{\mu}}\phi_x\right)dxds \\
&+\int_0^\pi (c\theta_t+\beta\phi_{xt})\,u_t dx-\frac{b}{\mu}\int_0^{+\infty}\kappa(s)\int_0^\pi \eta_x(x,s)\phi_t(x,t)dxds \\
&= -g(0)\int_0^\pi \theta_x u_x dx+\frac{b}{\mu}g(0)\int_0^\pi \theta\phi_x dx \\
&-\frac{1}{\sqrt{\mu}}\int_0^{+\infty}\kappa(s)\int_0^\pi \eta_s(x,s)\left(\sqrt{\mu}u_{xx}+\frac{b}{\sqrt{\mu}}\phi_x\right)dxds \\
&+c\int_0^\pi \theta_t u_t dx-\beta\int_0^\pi \phi_t u_{xt}dx-\frac{b}{\mu}\int_0^{+\infty}\kappa(s)\int_0^\pi \eta_x(x,s)\phi_t(x,t)dxds.
\end{aligned}$$

The integration by parts with respect to s then with respect to x gives

$$\begin{aligned}
&-\frac{1}{\sqrt{\mu}}\int_0^{+\infty}\kappa(s)\int_0^\pi \eta_s(x,s)\left(\sqrt{\mu}u_{xx}(x,t)+\frac{b}{\sqrt{\mu}}\phi_x(x,t)\right)dxds \\
&=\frac{1}{\sqrt{\mu}}\int_0^{+\infty}\kappa'(s)\int_0^\pi \eta(x,s)\left(\sqrt{\mu}u_{xx}(x,t)+\frac{b}{\sqrt{\mu}}\phi_x(x,t)\right)dxds \\
&=-\frac{1}{\sqrt{\mu}}\int_0^{+\infty}\kappa'(s)\int_0^\pi \eta_x(x,s)\left(\sqrt{\mu}u_x(x,t)+\frac{b}{\sqrt{\mu}}\phi(x,t)\right)dxds.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\frac{d}{dt}I_2(t) =& -g(0)\int_0^\pi \theta_x u_x dx+\frac{b}{\mu}g(0)\int_0^\pi \theta\phi_x dx \\
&-\frac{1}{\sqrt{\mu}}\int_0^{+\infty}\kappa'(s)\int_0^\pi \eta_x(x,s)\left(\sqrt{\mu}u_x(x,t)+\frac{b}{\sqrt{\mu}}\phi(x,t)\right)dxds \\
&+c\int_0^\pi \theta_t u_t dx+\beta\int_0^\pi \phi_{xt}u_t dx-\frac{b}{\mu}\int_0^{+\infty}\kappa(s)\int_0^\pi \eta_x(x,s)\phi_t(x,t)dxds. \qquad (3.52)
\end{aligned}$$

Finally, using $(3.11)_1$, we get

$$\begin{aligned}
\frac{d}{dt}I_3(t) &= -\frac{\beta\rho}{b}\int_0^\pi \theta_t u_t dx-\frac{\beta\rho}{b}\int_0^\pi \theta u_{tt}dx \\
&= -\frac{\beta\rho}{b}\int_0^\pi \theta_t u_t dx-\frac{\beta}{b}\int_0^\pi \theta\left(\mu u_{xx}+b\phi_x\right)dx \\
&= -\frac{\beta\rho}{b}\int_0^\pi \theta_t u_t dx+\frac{\beta\mu}{b}\int_0^\pi \theta_x u_x dx-\beta\int_0^\pi \theta\phi_x dx. \qquad (3.53)
\end{aligned}$$

Combining (3.51)-(3.53) with (3.45), we infer

$$\begin{aligned}
\frac{d}{dt}F_3(t) = &-\left\|\sqrt{\mu}u_x+\frac{b}{\sqrt{\mu}}\phi\right\|^2-\left(\xi-\frac{b^2}{\mu}\right)\|\phi\|^2-\alpha\|\phi_x\|^2+2J\|\phi_t\|^2\\
&-2\beta\int_0^\pi \phi\theta_x dx-\frac{\mu\beta}{b}\int_0^\pi \theta_x u_x dx-\frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx\\
&+\left(\frac{J\mu}{b}-\frac{\alpha\rho}{b}\right)\int_0^\pi u_{tx}\phi_t dx-\frac{\rho\beta\mu g(0)}{b\gamma_g}\int_0^\pi \theta_x u_x dx+\frac{\rho\beta g(0)}{\gamma_g}\int_0^\pi \theta\phi_x dx\\
&-\frac{\rho\beta\sqrt{\mu}}{b\gamma_g}\int_0^{+\infty}\kappa'(s)\int_0^\pi \eta_x(x,s)\left(\sqrt{\mu}u_x(x,t)+\frac{b}{\sqrt{\mu}}\phi(x,t)\right)dxds\\
&+\frac{\rho c\beta\mu}{b\gamma_g}\int_0^\pi \theta_t u_t dx+\frac{\rho\beta^2\mu}{b\gamma_g}\int_0^\pi \phi_{xt}u_t dx-\frac{\rho\beta}{\gamma_g}\int_0^{+\infty}\kappa(s)\int_0^\pi \eta_x(x,s)\phi_t(x,t)\,dxds\\
&-\frac{\mu c\beta\rho}{b\gamma_g}\int_0^\pi \theta_t u_t dx+\frac{\beta\mu^2 c}{b\gamma_g}\int_0^\pi \theta_x u_x dx-\frac{\mu c\beta}{\gamma_g}\int_0^\pi \theta\phi_x dx,
\end{aligned}$$

then,

$$\begin{aligned}
\frac{d}{dt}F_3(t) = &-\left\|\sqrt{\mu}u_x+\frac{b}{\sqrt{\mu}}\phi\right\|^2-\left(\xi-\frac{b^2}{\mu}\right)\|\phi\|^2-\alpha\|\phi_x\|^2+2J\|\phi_t\|^2\\
&+\frac{\mu}{b}\left(\frac{\mu c\beta}{\gamma_g}-\frac{\rho\beta g(0)}{\gamma_g}-\beta\right)\int_0^\pi \theta_x u_x dx-\frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx\\
&+\left(\frac{\rho\beta g(0)}{\gamma_g}-\frac{\mu c\beta}{\gamma_g}+2\beta\right)\int_0^\pi \theta\phi_x dx-\frac{\mu\alpha}{b}\chi_g\int_0^\pi \phi_t u_{tx}dx\\
&-\frac{\rho\beta\sqrt{\mu}}{b\gamma_g}\int_0^{+\infty}\kappa'(s)\int_0^\pi \eta_x(x,s)\left(\sqrt{\mu}u_x(x,t)+\frac{b}{\sqrt{\mu}}\phi(x,t)\right)dxds\\
&-\frac{\rho\beta}{\gamma_g}\int_0^{+\infty}\kappa(s)\int_0^\pi \eta_x(x,s)\phi_t(x,t)dxds.
\end{aligned}$$

Taking into account that $\chi_g = 0$, we obtain

$$\begin{aligned}
\frac{d}{dt}F_3(t) = &-\left\|\sqrt{\mu}u_x+\frac{b}{\sqrt{\mu}}\phi\right\|^2-\left(\xi-\frac{b^2}{\mu}\right)\|\phi\|^2-\alpha\|\phi_x\|^2+2J\|\phi_t\|^2\\
&-\frac{\rho\beta\sqrt{\mu}}{b\gamma_g}\int_0^{+\infty}\kappa'(s)\int_0^\pi \eta_x(x,s)\left(\sqrt{\mu}u_x(x,t)+\frac{b}{\sqrt{\mu}}\phi(x,t)\right)dxds\\
&-\frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx+\beta\int_0^\pi \theta\phi_x dx\\
&-\frac{\rho\beta}{\gamma_g}\int_0^{+\infty}\kappa(s)\int_0^\pi \eta_x(x,s)\phi_t(x,t)dxds. \qquad (3.54)
\end{aligned}$$

Cauchy-Schwarz and Young's inequalities yield

$$-\frac{\rho\beta\sqrt{\mu}}{b\gamma_g}\int_0^{+\infty}\kappa'(s)\int_0^{\pi}\eta_x(x,s)\left(\sqrt{\mu}u_x(x,t)+\frac{b}{\sqrt{\mu}}\phi(x,t)\right)dxds \leq$$
$$\frac{1}{2}\left\|\sqrt{\mu}u_x+\frac{b}{\sqrt{\mu}}\phi\right\|^2 - M\int_0^{+\infty}\kappa^{'}(s)\left\|\eta_x\right\|^2 ds. \tag{3.55}$$

Similarly, for any $\varepsilon_3 > 0$,

$$\frac{\rho\xi}{b}\int_0^{\pi}u_t\left(\int_0^x \phi_t(y)dy\right)dx \leq \varepsilon_3\left\|u_t\right\|^2+\frac{M}{\varepsilon_3}\|\phi_t\|^2, \tag{3.56}$$

$$\beta\int_0^{\pi}\theta\phi_x dx \leq \frac{\alpha}{2}\left\|\phi_x\right\|^2 + M\|\theta\|^2. \tag{3.57}$$

Using (3.16), we have, for any $\varepsilon_3 > 0$,

$$-\frac{\rho\beta}{\gamma_g}\int_0^{+\infty}\kappa(s)\int_0^{\pi}\eta_x(x,s)\phi_t(x,t)ds \leq \frac{1}{\varepsilon_3}\left\|\phi_t\right\|^2+\varepsilon_3 M\left\|\eta\right\|^2_{\mathcal{V}}$$
$$\leq \frac{1}{\varepsilon_3}\left\|\phi_t\right\|^2 - \varepsilon_3 M\int_0^{+\infty}\kappa^{'}(s)\left\|\eta_x\right\|^2 ds. \tag{3.58}$$

Therefore, (3.46) follows from (3.54)-(3.58). □

Lemma 3.8. *Let (u,ϕ,θ,η) be the solution of (3.11), (3.12). Then, the functional*

$$F_4(t) := -\rho\int_0^{\pi}u_t u dx - \frac{b\rho}{\mu}\int_0^{\pi}u_t\left(\int_0^x\phi(y)dy\right)dx$$

satisfies,

$$\frac{d}{dt}F_4(t) \leq -\frac{\rho}{2}\left\|u_t\right\|^2+\left\|\sqrt{\mu}u_x+\frac{b}{\sqrt{\mu}}\phi\right\|^2+M\|\phi_t\|^2, \tag{3.59}$$

where M is a positive constant.

Proof. By differentiating $F_4(t)$ and using (3.11), we arrive at

$$\begin{aligned}\frac{d}{dt}F_4(t) = &-\rho\left\|u_t\right\|^2 - \int_0^{\pi}\left(\mu u_{xx}+b\phi_x\right)udx - \frac{b}{\mu}\int_0^{\pi}\left(\mu u_{xx}+b\phi_x\right)\int_0^x\phi(y)dydx\\ &-\frac{b\rho}{\mu}\int_0^{\pi}u_t\left(\int_0^x\phi_t(y)dy\right)dx\\ =&-\rho\left\|u_t\right\|^2+\mu\left\|u_x\right\|^2+2b\int_0^{\pi}u_x\phi dx+\frac{b^2}{\mu}\left\|\phi\right\|^2-\frac{b\rho}{\mu}\int_0^{\pi}u_t\left(\int_0^x\phi_t(y)dy\right)dx\\ =&-\rho\left\|u_t\right\|^2+\left\|\sqrt{\mu}u_x+\frac{b}{\sqrt{\mu}}\phi\right\|^2-\frac{b\rho}{\mu}\int_0^{\pi}u_t\left(\int_0^x\phi_t(y)dy\right)dx.\end{aligned}$$

Young's and Cauchy Schwarz inequalities yield

$$-\frac{b\rho}{a}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx \leq \frac{\rho}{2}\left\|u_t\right\|^2 + M\left\|\phi_t\right\|^2$$

and (3.59) follows immediately. □

At this stage, we introduce the Lyapunov functional

$$\mathscr{L}(t) = NE\left(t\right) + N_1F_1\left(t\right) + \frac{1}{\varepsilon_2}F_2\left(t\right) + \frac{\rho}{4\varepsilon_3}F_3\left(t\right) + F_4(t),$$

where N, N_1, ε_2 and ε_3 are positive constants that will be determined later.

Lemma 3.9. *The functional E and $\mathscr{L}$ satisfy*

$$\left(N - C_{\varepsilon_2,\varepsilon_3}\right)E\left(t\right) \leq \mathscr{L}\left(t\right) \leq \left(N + C_{\varepsilon_2,\varepsilon_3}\right)E(t),\ \forall t > 0,$$

for a positive constant $C_{\varepsilon_2,\varepsilon_3}$, depending on ε_2, ε_3 only.

Proof. Let $\mathscr{K}\left(t\right) := \mathscr{L}\left(t\right) - NE(t)$, then

$$\mathscr{K}\left(t\right) := N_1F_1\left(t\right) + \frac{1}{\varepsilon_2}F_2\left(t\right) + \frac{\rho}{4\varepsilon_3}F_3\left(t\right) + F_4(t),$$

and

$$\begin{aligned}
\left|\mathscr{K}\left(t\right)\right| \leq C_{\varepsilon_2,\varepsilon_3}&\left[\left|\int_0^{+\infty}\kappa(s)\int_0^\pi \theta(x,t)\eta^t(x,s)dxds\right| + \left|\int_0^\pi \theta\left(\int_0^x \phi_t(y)dy\right)dx\right|\right]\\
+C_{\varepsilon_2,\varepsilon_3}&\left[\left|\int_0^\pi \phi_x u_t dx\right| + \left|\int_0^\pi \phi_t u_x dx\right| + \left|\int_0^\pi \phi\phi_t dx\right| + \left|\int_0^\pi u_t u dx\right|\right]\\
+C_{\varepsilon_2,\varepsilon_3}&\left[\left|\int_0^\pi u_t\left(\int_0^x \phi(y)dy\right)dx\right| + \left|\int_0^\pi \theta u_t dx\right|\right]\\
+C_{\varepsilon_2,\varepsilon_3}&\left[\left|\int_0^\pi\int_0^\infty \kappa(s)\eta_x(x,s)ds\left(\sqrt{\mu}u_x(x,t) + \frac{b}{\sqrt{\mu}}\phi(x,t)\right)dx\right|\right].
\end{aligned}$$

Therefore, Cauchy-Schwarz and Young's inequalities yield

$$\left|\mathscr{L}\left(t\right) - NE\left(t\right)\right| \leq C_{\varepsilon_2,\varepsilon_3}E(t),$$

where $C_{\varepsilon_2,\varepsilon_3}$ is a positive constant. Consequently,

$$\left(N - C_{\varepsilon_2,\varepsilon_3}\right)E\left(t\right) \leq \mathscr{L}\left(t\right) \leq \left(N + C_{\varepsilon_2,\varepsilon_3}\right)E(t).$$

□

Proof of Theorem 3.2 By differentiating $\mathscr{L}(t)$ and using (3.33), (3.34), (3.39), (3.46),(3.59), we get

$$\begin{aligned}\frac{d}{dt}\mathscr{L}(t) \leq &-\left(\frac{\rho}{8\varepsilon_3}-2\right)\left\|\sqrt{\mu}u_x+\frac{b}{\sqrt{\mu}}\phi\right\|^2-\left[\left(\xi-\frac{b^2}{\mu}\right)\frac{\rho}{4\varepsilon_3}-1\right]\|\phi\|^2-\frac{\rho}{4}\|u_t\|^2\\ &-\left(\frac{\rho\alpha}{8\varepsilon_3}-1\right)\|\phi_x\|^2-\left(\frac{J}{2\varepsilon_2}-\varepsilon_1N_1-M\left(\frac{\rho}{4\varepsilon_3^2}+\frac{\rho}{4\varepsilon_3}+1\right)\right)\|\phi_t\|^2\\ &-\left(cN_1-M\left(1+\frac{1}{\varepsilon_2}\right)\frac{1}{\varepsilon_2}-M\frac{\rho}{4\varepsilon_3}\right)\|\theta\|^2-N_1\|\eta\|_{\mathcal{V}}^2\\ &+\left(\frac{N}{2}-MN_1\left(1+\frac{1}{\varepsilon_1}\right)-\frac{M}{\varepsilon_2}-\frac{M\rho}{4\varepsilon_3}(1+\varepsilon_3)\right)\int_0^{+\infty}\kappa'(s)\|\eta_x\|^2\,ds.\end{aligned}$$

First, we choose $\varepsilon_3>0$, small enough, such that

$$\varepsilon_3<\min\left\{\frac{\rho}{16},\frac{\rho\alpha}{8},\left(\xi-\frac{b^2}{\mu}\right)\frac{\rho}{4}\right\}.$$

Then, we select ε_2,

$$\varepsilon_2=\frac{J\varepsilon_3^2}{M\left[\rho\varepsilon_3+\rho+4\varepsilon_3^2\right]},$$

to get

$$\frac{J}{2\varepsilon_2}-M\left(\frac{\rho}{4\varepsilon_3^2}+\frac{\rho}{4\varepsilon_3}+1\right)=M\left(\frac{\rho}{4\varepsilon_3^2}+\frac{\rho}{4\varepsilon_3}+1\right).$$

Next, we pick N_1 so large

$$cN_1>M\left[\left(1+\frac{1}{\varepsilon_2}\right)\frac{1}{\varepsilon_2}+\frac{\rho}{4\varepsilon_3}\right].$$

After that, we choose ε_1 small enough, such that

$$M\left(\frac{\rho}{4\varepsilon_3^2}+\frac{\rho}{4\varepsilon_3}+1\right)>\varepsilon_1N_1.$$

Finally, we choose N large enough, such that

$$N>MN_1\left(1+\frac{1}{\varepsilon_1}\right)+\frac{M}{\varepsilon_2}+\frac{M\rho}{4\varepsilon_3}(1+\varepsilon_3)$$

and

$$N>C_{\varepsilon_2,\varepsilon_3}.$$

Thus, there exists $\varpi,\lambda,m_1,m_2>0$, such that

$$\begin{aligned}\frac{d}{dt}\mathscr{L}(t)\ \leq\ &-\varpi\left(\left\|\sqrt{\mu}u_x+\frac{b}{\sqrt{\mu}}\phi\right\|^2+\|\phi\|^2+\|u_t\|^2+\|\phi_x\|^2+\|\phi_t\|^2+\|\theta\|^2+\|\eta\|_\nu^2\right)\\ &+\lambda\int_0^\infty\kappa^{'}(s)\|\eta_x\|^2\,ds.\end{aligned}\tag{3.60}$$

and

$$m_1 E(t) \leq \mathscr{L}(t) \leq m_2 E(t), \qquad \forall t > 0. \tag{3.61}$$

Therefore, there exists $\gamma > 0$, such that

$$\frac{d}{dt}\mathscr{L}(t) \leq -\gamma E(t),\ t \geq 0.$$

Using (3.61), we infer that there exists $\omega > 0$, such that

$$\frac{d}{dt}\mathscr{L}(t) \leq -\omega \mathscr{L}(t),\ t \geq 0.$$

An integration over $(0,t)$ leads to

$$\mathscr{L}(t) \leq \mathscr{L}(0)\, e^{-\omega t},\ t \geq 0.$$

Using (3.61) again, we get

$$E(t) \leq \sigma E(0) e^{-\omega t},\ t \geq 0,$$

for $\sigma > 0$, which completes the proof of Theorem 3.2.

3.5 Lack of exponential stability

In this section, we examine the cases when $\gamma_g = 0$ or $\chi_g \neq 0$ and prove that the solution (u, ϕ, θ, η) looses its exponential stability.

Theorem 3.3. *Suppose that $\chi_g \neq 0$. Then, the solution (u, ϕ, θ, η) can not be exponentially stable.*

Proof. Taking $F_n = (0, \frac{\sin(n\pi)}{\rho}, 0, 0, 0, 0)^T$, then

$$\|F_n\|_{\mathcal{H}} = \sqrt{\frac{\pi}{2\rho}}.$$

Let $U_n = (u_n, v_n, \phi_n, \psi_n, \theta_n, \eta_n) \in D(\mathcal{A})$ be the solution of $(i\lambda_n I + \mathcal{A})U_n = F_n$, then

$$\begin{cases} i\lambda_n u_n - v_n = & 0, \\ i\rho\lambda_n v_n - \mu u_{nxx} - b\phi_{nx} = & \sin(n\pi), \\ i\lambda_n \phi_n - \psi_n = & 0, \\ iJ\lambda_n \psi_n - \alpha\phi_{nxx} + \xi\phi_n + bu_{nx} + \beta\theta_{nx} = & 0, \\ ic\lambda_n \theta_n + \beta\psi_{nx} - \displaystyle\int_0^{+\infty} \kappa(s)\eta_{nxx}(s)ds = & 0, \\ i\lambda_n \eta_n - \theta_n + \eta_{ns} = & 0. \end{cases} \tag{3.62}$$

Removing v_n, ψ_n from (3.62), we obtain

$$\begin{cases} \rho\lambda_n^2 u_n + \mu u_{nxx} + b\phi_{nx} = & -\sin(n\pi), \\ (J\lambda_n^2 - \xi)\phi_n + \alpha\phi_{nxx} - bu_{nx} - \beta\theta_{nx} = & 0, \\ ic\lambda_n\theta_n + i\lambda_n\beta\phi_{nx} - \int_0^{+\infty} \kappa(s)\eta_{nxx}(s)ds = & 0, \\ i\lambda_n\eta_n - \theta_n + \eta_{ns} = & 0. \end{cases} \tag{3.63}$$

Taking into account the boundary conditions (3.2), we set

$$\begin{cases} u_n = A_n \sin(nx), \\ \phi_n = B_n \cos(nx), \\ \theta_n = C_n \sin(nx), \\ \eta_n = D_n(s)\sin(nx), \end{cases} \tag{3.64}$$

where $A_n, B_n, C_n \in \mathbb{C}$ and $D_n : \mathbb{R}_+ \to \mathbb{C}$ such that $\displaystyle\int_0^\infty \kappa(s)n^2|D_n(s)|^2 ds < \infty$, and $D_n(0) = 0$. Plugging (3.64) into (3.63), we infer

$$\begin{cases} (\rho\lambda_n^2 - \mu n^2)\, A_n - bnB_n = & -1, \\ (J\lambda_n^2 - \xi - \alpha n^2)B_n - bnA_n - \beta nC_n = & 0, \\ ic\lambda_n C_n - i\beta n\lambda_n B_n + n^2 \int_0^{+\infty} \kappa(s)D_n(s)ds = & 0, \\ i\lambda_n D_n\,(s) - C_n + D_n'\,(s) = & 0. \end{cases} \tag{3.65}$$

Solving $(3.65)_4$, we get

$$D_n\,(s) = \frac{C_n}{i\lambda_n}(1 - e^{-i\lambda_n s}).$$

The substitution of $D_n\,(s)$ into $(3.65)_3$ yields

$$\begin{aligned} (\rho\lambda_n^2 - \mu n^2)\, A_n - bnB_n \quad &= -1, \\ (J\lambda_n^2 - \xi - \alpha n^2)B_n - bnA_n - \beta nC_n \quad &= 0, \\ \beta n\lambda_n^2 B_n - \left[c\lambda_n^2 - n^2\,(g\,(0) - \widehat{\kappa}\,(\lambda_n))\right] C_n \quad &= 0, \end{aligned}$$

where $\widehat{\kappa}(\lambda_n) = \int_0^{+\infty} \kappa(s)e^{-i\lambda_n s}ds$ and $|\widehat{\kappa}(\lambda_n)| \le \int_0^{+\infty} \kappa(s)ds < \infty$, which can be written as

$$\begin{pmatrix} p_1\,(\lambda_n) & -bn & 0 \\ -bn & p_2\,(\lambda_n) & -\beta n \\ 0 & \beta n\lambda_n^2 & p_3\,(\lambda_n) \end{pmatrix} \begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \tag{3.66}$$

where

$$p_1(\lambda_n) = (\rho\lambda_n^2 - \mu n^2), \quad p_2(\lambda_n) = (J\lambda_n^2 - \xi - \alpha n^2), \quad p_3(\lambda_n) = -c\lambda_n^2 + n^2(g(0) - \widehat{\kappa}(\lambda_n)).$$

It follows that

$$A_n = \frac{-K_1}{p_1(\lambda_n) K_1 - K_2},$$

where

$$K_1 = p_2(\lambda_n) p_3(\lambda_n) + (\beta n \lambda_n)^2, \; K_2 = (bn)^2 p_3(\lambda_n).$$

Taking λ_n such that $p_1(\lambda_n) = 0$, that is $n^2 = \frac{\rho}{\mu}\lambda_n^2$, then

$$A_n = \frac{K_1}{K_2}. \tag{3.67}$$

where

$$K_1 = \frac{\alpha}{\mu}\left(\frac{\rho}{\mu} - \frac{J}{\alpha} + \frac{\xi}{\alpha\lambda_n^2}\right)[c\mu - \rho g(0) + \rho\widehat{\kappa}(\lambda_n)]\lambda_n^4 + \frac{\rho\beta^2}{\mu}\lambda_n^4,$$

An $\lambda_n \longrightarrow +\infty$, then $\frac{\xi}{\alpha\lambda_n^2} \longrightarrow 0$, consequently,

$$K_1 \simeq \frac{\alpha}{\mu}\left[\left(\frac{\rho}{\mu} - \frac{J}{\alpha}\right)\gamma_g + \frac{\rho\beta^2}{\alpha} + \left(\frac{\rho}{\mu} - \frac{J}{\alpha}\right)\rho\widehat{\kappa}(\lambda_n)\right]\lambda_n^4 \tag{3.68}$$

and

$$\begin{aligned} K_2 &= \frac{b^2\rho}{\mu^2}\lambda_n^4(-c\mu + \rho g(0) - \rho\widehat{\kappa}(\lambda_n)) \\ &= -\frac{b^2\rho}{\mu^2}(\gamma_g + \rho\widehat{\kappa}(\lambda_n))\lambda_n^4. \end{aligned} \tag{3.69}$$

Substituting (3.68) and (3.69) into (3.67), we get

$$A_n \simeq -\frac{\alpha\mu\left[\chi_g\gamma_g + \left(\frac{\rho}{\mu} - \frac{J}{\alpha}\right)\rho\widehat{\kappa}(\lambda_n)\right]}{b^2\rho(\gamma_g + \rho\widehat{\kappa}(\lambda_n))}.$$

- If $\frac{\rho}{\mu} - \frac{J}{\alpha} = 0$ and $\gamma_g \neq 0$, since $\lim_{n\longrightarrow\infty}\widehat{\kappa}(\lambda_n) = 0$, we have

$$A_n \simeq \frac{\beta^2\mu}{-b^2(\gamma_g + \rho\widehat{\kappa}(\lambda_n))} \longrightarrow \frac{\beta^2\mu}{-b^2\gamma_g} = A \neq 0,$$

where A is a constant.

- If $\frac{\rho}{\mu} - \frac{J}{\alpha} \neq 0$ and $\gamma_g = 0$, then

$$A_n \simeq -\frac{\alpha\mu\left(\frac{\rho}{\mu} - \frac{J}{\alpha}\right)}{b^2\rho} = A \neq 0.$$

- If $\gamma_g = 0$ and $\frac{\rho}{\mu} - \frac{J}{\alpha} = 0$, then

$$K_1 = \frac{\rho\xi}{\mu}\widehat{\kappa}(\lambda_n)\lambda_n^2 + \frac{\rho\beta^2}{\mu}\lambda_n^4, \; K_2 = -\frac{b^2\rho^2}{\mu^2}\widehat{\kappa}(\lambda_n)\lambda_n^4.$$

  Since $\lim\limits_{n\longrightarrow\infty} \widehat{\kappa}(\lambda_n) = 0$, we obtain

$$A_n \simeq \frac{\mu\beta^2}{-b^2\rho\widehat{\kappa}(\lambda_n)} \longrightarrow \infty.$$

- If $\gamma_g \neq 0$ and $\chi_g \neq 0$, then

$$A_n \simeq -\frac{\alpha\mu\left[\chi_g + \left(\frac{\rho}{\mu} - \frac{J}{\alpha}\right)\rho\frac{\widehat{\kappa}(\lambda_n)}{\gamma_g}\right]}{b^2\rho\left(1 + \rho\frac{\widehat{\kappa}(\lambda_n)}{\gamma_g}\right)}$$

  and

$$A_n \longrightarrow -\frac{\alpha\mu\chi_g}{b^2\rho} = A \neq 0.$$

Therefore, if $A_n \longrightarrow A$, we have

$$\begin{aligned}\|U_n\|^2 \geq &\rho\int_0^\pi |v_n|^2\,dx = \rho\int_0^\pi |\lambda_n u_n|^2\,dx = \rho\int_0^\pi |\lambda_n A_n \sin nx|^2\,dx \\ \geq &\rho\,|\lambda_n A|^2\int_0^\pi |\sin nx|^2\,dx = \frac{\rho\pi\,|\lambda_n A|^2}{2} \longrightarrow \infty,\end{aligned}$$

and if $A_n \longrightarrow \infty$, we have

$$\|U_n\|^2 \geq \mu\int_0^\pi |u_n|^2\,dx = \mu\int_0^\pi |A_n \sin nx|^2\,dx = \mu|A_n|^2\int_0^\pi |\sin nx|^2\,dx = \frac{\mu\pi\,|A_n|^2}{2} \longrightarrow \infty,$$

which completes the proof of Theorem 3.3. □

chapter

4 A porous thermoelastic system with micro-temperature

4.1 Introduction

The results of this chapter were published in [3].

In this chapter, we consider the following porous thermoelastic system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x & \text{in } (0,\pi)\times(0,\infty), \\ J\phi_{tt} = \delta\phi_{xx} - bu_x - \xi\phi - \beta w_x & \text{in } (0,\pi)\times(0,\infty), \\ \alpha w_t = \kappa w_{xx} - \beta\phi_{xt} & \text{in } (0,\pi)\times(0,\infty), \end{cases} \tag{4.1}$$

where u is the transversal displacement, ϕ is the volume fraction and w denotes the micro-temperature at the point of abscissa x and time t of a porous material of length π. The coefficient ρ is the mass density, J is the product of the mass density by the equilibrated inertia, μ is a constant that depends on the geometry of the material, δ is the volume distributed density, α the micro-thermal conductivity and b and β are the coupling constants.

In order for the stored energy of the system (4.1) to be positive definite, we require that the constitutive coefficients satisfy the following conditions

$$\rho > 0,\ \mu > 0,\ \alpha > 0,\ J > 0,\ \xi > 0,\ \delta > 0,\ \kappa > 0,\ b \neq 0,\ \beta \neq 0,\ \text{and}\ \ \mu\xi > b^2.$$

We also assume that the unknown functions u, ϕ and w satisfy the following initial and boundary conditions

$$\begin{gathered} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \phi(x,0) = \phi_0(x), \\ \phi_t(x,0) = \phi_1(x), \quad w(x,0) = w_0(x), \end{gathered} \tag{4.2}$$

$$u(0,t) = u(\pi,t) = \phi_x(0,t) = \phi_x(\pi,t) = w(0,t) = w(\pi,t) = 0. \tag{4.3}$$

Since the boundary conditions on ϕ are of Neumann type, then they prevent the application of Poincaré's inequality. To overcome this inconvenience, we make some changes on ϕ as follows:

From the second equation of (4.1) and the boundary conditions (4.3), we obtain

$$J\frac{d^2}{dt^2}\int_0^\pi \phi(x,t)\,dx + \xi\int_0^\pi \phi(x,t)\,dx = 0,$$

which entails

$$\int_0^\pi \phi(x,t)\,dx = \left(\int_0^\pi \phi_0(x)\,dx\right)\cos\left(\sqrt{\frac{\xi}{J}}t\right) + \sqrt{\frac{J}{\xi}}\left(\int_0^\pi \phi_1(x)\,dx\right)\sin\left(\sqrt{\frac{\xi}{J}}t\right).$$

Consequently, if we set

$$\begin{aligned}\overline{\phi}(x,t) =&\phi(x,t) - \frac{1}{\pi}\left(\int_0^\pi \phi_0(x)\,dx\right)\cos\left(\sqrt{\frac{\xi}{J}}t\right)\\ &- \frac{1}{\pi}\sqrt{\frac{J}{\xi}}\left(\int_0^\pi \phi_1(x)\,dx\right)\sin\left(\sqrt{\frac{\xi}{J}}t\right),\end{aligned}$$

then, $\left(u,\overline{\phi},w\right)$ solves (4.1)-(4.3), with the following initial condition for $\overline{\phi}$:

$$\overline{\phi}(x,0) = \overline{\phi}_0(x) = \phi_0(x) - \frac{1}{\pi}\left(\int_0^\pi \phi_0(x)\,dx\right),$$

$$\overline{\phi}_t(x,0) = \overline{\phi}_1(x) = \phi_1(x) - \frac{1}{\pi}\left(\int_0^\pi \phi_1(x)\,dx\right).$$

and more importantly,

$$\int_0^\pi \overline{\phi}(x,t)\,dx = 0,$$

which allows the application of Poincaré's inequality for $\overline{\phi}$. In the sequel, we work with $(u,\overline{\phi},w)$, but for convenience, we write (u,ϕ,w).

The coupling with micro-temperature was first considered by Ieşan [42] for microstretch elastic materials and later extended by Quintanilla and co-authors [54, 15] to porous-elastic materials. Magaña and Quintalilla [54] studied the system

$$\begin{cases}\rho u_{tt} = \mu u_{xx} + b\varphi_x + \gamma u_{txx},\\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi + dw_x - \tau\varphi_t,\\ \alpha w_t = \kappa_4 w_{xx} - d\varphi_{xt} - \kappa_2 w.\end{cases} \tag{4.4}$$

Under the assumption $\dfrac{\mu}{\rho} = \dfrac{\delta}{J}$, they established an exponential rate of decay in the case where $\tau = 0$ and $\gamma > 0$, and a slow decay in the case where $\gamma = 0$ and $\tau > 0$.
Santos *et al.* [81] considered (4.4) with $\gamma = 0$. They established an exponential rate of decay when $\dfrac{\mu}{\rho} = \dfrac{\delta}{J}$ and an optimal rate of decay $t^{-1/2}$ in the opposite case.

Apalara [11] improved the results of [54, 81] and established the exponential stability of the solution of the system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x, \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi + dw_x, \\ \alpha w_t = \kappa_4 w_{xx} - d\varphi_{xt} - \kappa_2 w, \end{cases} \tag{4.5}$$

provided that

$$\chi = \frac{\mu}{\rho} - \frac{\delta}{J} = 0. \tag{4.6}$$

He also obtained an optimal polynomial rate of decay of $t^{-1/2}$ in the case when (4.6) does not hold.

Recently, Feng *et al.* [29] considered an inhomogeneous porous-thermoelastic system with micro-temperatures. In their system, the dissipation was produced from the temperature and the micro-temperature dampings. They proved an exponential stability result regardless of any relation between the wave speeds of the system.

From a thermomechanical perspective, the coefficient κ_2 in the third equation of (4.5) represents the difference $\kappa_2 = k_1 - k_2$ between the coefficients of the heat flux q and the mean heat flux Q.

In this work, we assume $\kappa_2 = 0$ and aim to enhance the findings of [11, 81]. Specifically, we demonstrate that the dissipation produced by the gradient of the micro-temperature alone can exponentially stabilize the system (4.1), given that condition (4.6) is satisfied.

In this chapter, we study the well-posedness and the exponential stability of system (4.1)-(4.3). By using the Hille-Yosida theorem, we prove the existence and uniqueness of the result. On the other hand, we utilize the multiplier method to demonstrate that the single dissipation produced by the gradient of the micro-temperature can exponentially stabilize the system. Our result improves the results of [11, 81] and extends the results of [9, 84] to the case of micro-thermal dissipation.

This chapter is organized as follows: In Section 2, we prove the well-posedness of the system. In Section 3, we show that the system is exponential decay.

4.2 Well-posedness

In this section, we use a semigroup approach to prove that the problem (4.1)-(4.3) has a unique solution.

First, we define the Hilbert space

$$\mathcal{H} := H_0^1(0,\pi) \times L^2(0,\pi) \times H_*^1(0,\pi) \times L_*^2(0,\pi) \times L^2(0,\pi),$$

equipped with the following inner product

$$\begin{aligned}\langle U, U^* \rangle_{\mathcal{H}} = \ & \rho \int_0^\pi v v^* dx + \mu \int_0^\pi u_x u_x^* dx + \xi \int_0^\pi \phi \phi^* dx + \delta \int_0^\pi \phi_x \phi_x^* dx \\ & + J \int_0^\pi \psi \psi^* dx + b \int_0^\pi \phi u_x^* dx + b \int_0^\pi u_x \phi^* dx + \alpha \int_0^\pi w w^* dx,\end{aligned}$$

where $U = (u, v, \phi, \psi, w)^T$, $U^* = (u^*, v^*, \phi^*, \psi^*, w^*)^T \in \mathcal{H}$.

Next, introducing the new variables $v = u_t$ and $\psi = \phi_t$. The system (4.1)-(4.3) becomes

$$\begin{cases} U_t + \mathcal{A}U = 0, \quad t > 0, \\ U(0) = U_0 = (u_0, u_1, \phi_0, \phi_1, w_0)^T, \end{cases} \tag{4.7}$$

where $\mathcal{A}$ is the operator defined on $\mathcal{H}$ by

$$\mathcal{A} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -\frac{\mu}{\rho}\partial_{xx} & 0 & -\frac{b}{\rho}\partial_x & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \frac{b}{J}\partial_x & 0 & -\frac{\delta}{J}\partial_{xx} + \frac{\xi}{J} & 0 & \frac{\beta}{J}\partial_x \\ 0 & 0 & 0 & \frac{\beta}{\alpha}\partial_x & -\frac{k}{\alpha}\partial_{xx} \end{pmatrix}, \tag{4.8}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{c} U \in \mathcal{H} : \ u, w \in H^2(0,\pi) \cap H_0^1(0,\pi), \ v \in H_0^1(0,\pi), \\ \phi \in H_*^2(0,\pi), \ \psi \in H_*^1(0,\pi) \end{array} \right\},$$

where H_*^1 and H_*^2 are given in Remark 1.2.

The existence and uniqueness result reads as follows:

Theorem 4.1. *Let $U_0 \in D(\mathcal{A})$. Then, the problem* (4.1)-(4.3) *has a unique solution* $U \in C\left((0,+\infty); D(\mathcal{A})\right) \cap C^1\left((0,+\infty); \mathcal{H}\right)$.

The proof of Theorem 4.1 is based on the Hille-Yosida theorem.

Proof. By virtue of Theorem 1.9, it suffices to prove that $\mathcal{A}$ is maximal and monotone.

First, suppose that $U \in D(\mathcal{A})$, then a direct calculation implies that

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= \mu \int_0^\pi u_x v_x dx - b\int_0^\pi \phi_x v dx - \mu \int_0^\pi v_x u_x dx - \xi \int_0^\pi \psi\phi dx \\
&- \delta \int_0^\pi \psi_x \phi_x dx + \delta \int_0^\pi \phi_x \psi_x dx + \xi \int_0^\pi \phi\psi dx + b\int_0^\pi u_x \psi dx \\
&+ \beta \int_0^\pi w_x \psi dx - b \int_0^\pi \psi u_x dx - b\int_0^\pi v_x \phi dx + \kappa \int_0^\pi w_x^2 dx \\
&+ \beta \int_0^\pi \psi_x w dx \\
&= \mu \int_0^\pi u_x v_x dx - b\int_0^\pi \phi_x v dx - \mu \int_0^\pi v_x u_x dx - \delta \int_0^\pi \psi_x \phi_x dx \\
&+ \delta \int_0^\pi \phi_x \psi_x dx + b\int_0^\pi u_x \psi dx + \beta \int_0^\pi w_x \psi dx - b\int_0^\pi \psi u_x dx \\
&- b\int_0^\pi v_x \phi dx + \kappa \int_0^\pi w_x^2 dx + \beta \int_0^\pi \psi_x w dx,
\end{aligned}$$

using the integration by parts and the boundary conditions, we get

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \kappa \int_0^\pi w_x^2 dx \geq 0. \tag{4.9}$$

Therefore, the operator $\mathcal{A}$ is monotone.

Next, to prove that $\mathcal{A}$ is maximal, we have to show that for any $F = \left(f^1, f^2, f^3, f^4, f^5\right)^T \in \mathcal{H}$, there exists $U = (u, v, \phi, \psi, w)^T \in D(\mathcal{A})$ such that $(I + \mathcal{A})U = F$; that is,

$$\left\{\begin{aligned}
u - v &= f^1, \\
\rho v - \mu u_{xx} - b\phi_x &= \rho f^2, \\
\phi - \psi &= f^3, \\
J\psi - \delta\phi_{xx} + \xi\phi + bu_x + \beta w_x &= Jf^4, \\
\alpha w - \kappa w_{xx} + \beta\psi_x &= \alpha f^5.
\end{aligned}\right. \tag{4.10}$$

From the first and third equation of (4.10), we get

$$v = u - f^1, \;\; \psi = \phi - f^3, \tag{4.11}$$

Substituting (4.11) into the second, fourth and fifth equations of (4.10), we obtain

$$\begin{cases} \rho u - \mu u_{xx} - b\phi_x = h_1, \\ J\phi - \delta\phi_{xx} + \xi\phi + bu_x + \beta w_x = h_2, \\ \alpha w - \kappa w_{xx} + \beta\phi_x = h_3, \end{cases} \tag{4.12}$$

where

$$h_1 = \rho(f^1 + f^2) \in L^2(0,\pi), \quad h_2 = J(f^4 + f^3) \in L^2_*(0,\pi), \quad h_3 = \alpha f^5 + \beta f^3_x \in L^2(0,\pi). \tag{4.13}$$

To solve (4.12), we consider the following variational formulation

$$B((u,\phi,w),(\widetilde{u},\widetilde{\phi},\widetilde{w})) = L(\widetilde{u},\widetilde{\phi},\widetilde{w}) \qquad \forall(\widetilde{u},\widetilde{\phi},\widetilde{w}) \in \mathcal{W}, \tag{4.14}$$

where $\mathcal{W} = H^1_0(0,\pi) \times H^1_*(0,\pi) \times H^1_0(0,\pi)$, B is bilinear form and L is a linear form on $\mathcal{W}$ given by

$$\begin{aligned} B((u,\phi,w),(\widetilde{u},\widetilde{\phi},\widetilde{w})) &= \rho\int_0^\pi u\widetilde{u}dx + \mu\int_0^\pi u_x\widetilde{u}_x dx + b\int_0^\pi \left(\phi\widetilde{u}_x + \widetilde{\phi}u_x\right)dx + \delta\int_0^\pi \phi_x\widetilde{\phi}_x dx \\ &+ (J+\xi)\int_0^\pi \phi\widetilde{\phi}dx + \kappa\int_0^\pi w_x\widetilde{w}_x dx + \alpha\int_0^\pi w\widetilde{w}dx + \beta\int_0^\pi \left(w_x\widetilde{\phi} + \phi_x\widetilde{w}\right)dx \end{aligned}$$

and

$$L((\widetilde{u},\widetilde{\phi},\widetilde{w})) = \int_0^\pi h_1\widetilde{u}dx + \int_0^\pi h_2\widetilde{\phi}dx + \int_0^\pi h_3\widetilde{\theta}dx.$$

We can easily show, by using Cauchy-Schwarz inequality, that B and L are continuous forms over $\mathcal{W}$.

Moreover,

$$\begin{aligned} B((u,\phi,w),(u,\phi,w)) &= \rho\int_0^\pi u^2 dx + \mu\int_0^\pi u_x^2 dx + 2b\int_0^\pi \phi u_x dx + \delta\int_0^\pi \phi_x^2 dx \\ &\quad + (J+\xi)\int_0^\pi \phi^2 dx + \kappa\int_0^\pi w_x^2 dx + \alpha\int_0^\pi w^2 dx \end{aligned}$$

Noting that

$$\mu\int_0^\pi |u_x|^2 dx + \xi\int_0^\pi |\phi|^2 dx + 2b\int_0^\pi \phi u_x dx \geq \frac{1}{2}\left(\mu - \frac{b^2}{\xi}\right)\int_0^\pi |u_x|^2 dx + \frac{1}{2}\left(\xi - \frac{b^2}{\mu}\right)\int_0^\pi |\phi|^2 dx,$$

and using the fact $\mu\xi > b^2$, we have

$$\begin{aligned} B\left((u,\phi,w),(u,\phi,w)\right) &\geq \rho\int_0^\pi |u|^2 dx + \frac{1}{2}\left(\mu - \frac{b^2}{\xi}\right)\int_0^\pi |u_x|^2 dx + \frac{1}{2}\left(2J + \xi - \frac{b^2}{\mu}\right)\int_0^\pi |\phi|^2 dx \\ &\quad + \delta\int_0^\pi |\phi_x|^2 dx + \alpha\int_0^\pi |w|^2 dx + \kappa\int_0^\pi |w_x|^2 dx, \end{aligned}$$

Thus, there exists $C > 0$, such that

$$\begin{aligned} B\left((u,\phi,w),(u,\phi,w)\right) &\geq C\left(\int_0^\pi |u_x|^2 dx + \int_0^\pi |\phi|^2\, dx + \int_0^\pi |\phi_x|^2\, dx + \int_0^\pi |w_x|^2\, dx\right) \\ &= C\|(u,\phi,w)\|^2_{\mathcal{W}}, \end{aligned}$$

which shows that B is coercive. Therefore, Lax-Milgram theorem guarantees the existence of a unique $(u,\phi,w) \in \mathcal{W}$ satisfies (4.14), for every $\left(\widetilde{u},\widetilde{\phi},\widetilde{w}\right) \in \mathcal{W}$.

By taking $(\widetilde{u},\widetilde{\phi},\widetilde{w}) = (\tilde{u},0,0)$ in (4.14), we obtain

$$\mu\int_0^\pi u_x\widetilde{u}_x dx = \int_0^\pi (h_1 - \rho u + b\phi_x)\,\widetilde{u} dx, \quad \forall \widetilde{u} \in H_0^1(0,\pi). \tag{4.15}$$

By the elliptic regularity theory, we arrive at

$$u \in H^2\,(0,\pi) \cap H_0^1(0,\pi).$$

Moreover, integration by parts of the left-hand side of (4.15) yields

$$\rho u - \mu u_{xx} - b\phi_x = h_1.$$

Using (4.11) and (4.13), we get

$$\rho v - a u_{xx} - b\phi_x = \rho f^2.$$

Similarly, by taking $(\widetilde{u},\widetilde{\phi},\widetilde{w}) = (0,0,\widetilde{w})$ in (4.14), we obtain

$$\kappa\int_0^\pi w_x\widetilde{w}_x dx = \int_0^\pi (h_3 - \alpha w - \beta\phi_x)\,\widetilde{w} dx, \;\; \forall \widetilde{w} \in H_0^1(0,\pi), \tag{4.16}$$

therefore,

$$w \in H^2(0,\pi) \cap H_0^1(0,\pi).$$

Again, the integration by parts of the left-hand side of (4.16) leads to

$$-\kappa w_{xx} = h_3 - \alpha w - \beta\phi_x.$$

Using (4.11) and (4.13), we obtain

$$\alpha w - \kappa w_{xx} + \beta\psi_x = \alpha f^5.$$

Finally, by taking $\left(\widetilde{u},\widetilde{\phi},\widetilde{w}\right)=(0,\widetilde{\phi},0)$ in (4.14), we get

$$\delta\int_0^\pi \phi_x\widetilde{\phi}_x dx=\int_0^\pi \left(h_2-(J+\xi)\,\phi-bu_x-\beta w_x\right)\widetilde{\phi}dx,\quad \forall\widetilde{\phi}\in H_*^1(0,\pi). \tag{4.17}$$

Here, we are not able to apply the elliptic regularity directly. To do so, we proceed as follows:

Let $\varphi\in H_0^1(0,\pi)$, and set $\widetilde{\phi}=\varphi-\dfrac{1}{\pi}\displaystyle\int_0^\pi\varphi(x)\,dx$. Then $\widetilde{\phi}\in H_*^1(0,\pi)$. Plugging $\widetilde{\phi}$ in (4.17), taking into account that

$$h_2-(J+\xi)\,\phi-bu_x-\beta w_x\in L_*^2(0,\pi),$$

we obtain

$$\delta\int_0^\pi \phi_x\varphi_x dx=\int_0^\pi \left(h_2-(J+\xi)\,\phi-bu_x-\beta w_x\right)\varphi dx,\quad \forall\varphi\in H_0^1(0,\pi), \tag{4.18}$$

which means that

$$\phi\in H^2(0,\pi)\cap H_*^1(0,\pi).$$

An integration by parts leads to

$$\delta\phi_{xx}=\beta w_x+(J+\xi)\,\phi+bu_x-h_2=R\in L^2(0,\pi),$$

hence, ϕ solves the third equation of (4.12). We use (4.11) and (4.13) to arrive at

$$J\psi-\delta\phi_{xx}+bu_x+\xi\phi+\beta w_x=Jf^4.$$

On the other hand, since $\delta\phi_{xx}=R\in L^2(0,\pi)$, then

$$\delta\int_0^\pi \phi_{xx}\Psi dx=\int_0^\pi R\Psi dx,\quad \forall\Psi\in H^1(0,\pi).$$

Integration by parts gives

$$\delta\phi_x\Psi\Big|_0^\pi-\delta\int_0^\pi\phi_x\Psi_x dx=\int_0^\pi R\Psi dx,\quad \forall\Psi\in H^1(0,\pi).$$

Since $H_*^1(0,\pi)\subset H^1(0,\pi)$, we have

$$\delta\phi_x\Psi\Big|_0^\pi-\delta\int_0^\pi\phi_x\Psi_x dx=\int_0^\pi R\Psi dx,\quad \forall\Psi\in H_*^1(0,\pi).$$

The use of (4.18) leads to

$$\alpha\phi_x(\pi)\,\Psi(\pi) - \alpha\phi_x(0)\,\Psi(0) = 0.$$

As Ψ is arbitrary, we obtain

$$\phi_x(\pi) = \phi_x(0) = 0,$$

and, hence,

$$\phi \in H_*^2(0,\pi).$$

Substituting u, ϕ, w in (4.11), we get

$$v \in H_0^1(0,\pi) \text{ and } \psi \in H_*^1(0,\pi). \tag{4.19}$$

Finally, we conclude that $\mathcal{A}$ is maximal and monotone. Consequently, the well-posedness result follows from Theorem 1.9. □

4.3 Exponential stability

In this section, we prove the exponential stability of the solution of system (4.1). We use the multiplier method to achieve our goal.

First, we define the energy associated to the solution of (4.1)–(4.3) by

$$E(t) := \frac{1}{2}\int_0^\pi \left[\rho u_t^2 + J\phi_t^2 + \mu u_x^2 + \xi\phi^2 + 2bu_x\phi + \delta\phi_x^2 + \alpha w^2\right] dx. \tag{4.20}$$

Recalling that $\mu\xi > b^2$, we have

$$E(t) := \frac{1}{2}\int_0^\pi \left[\rho u_t^2 + J\phi_t^2 + \left(\sqrt{\mu}u_x + \frac{b}{\sqrt{\mu}}\phi\right)^2 + \left(\xi - \frac{b^2}{\mu}\right)\phi^2 + \delta\phi_x^2 + \alpha w^2\right] dx. \tag{4.21}$$

Thus, $E(t)$ is positive definite form. Moreover, we have the following:

Lemma 4.1. *The energy $E(t)$, defined by (4.20), satisfies, along the solution (u, ϕ, w) of (4.1), the estimate*

$$\frac{d}{dt}E(t) = -\kappa\int_0^\pi |w_x|^2\, dx \le 0. \tag{4.22}$$

Proof. Taking the L^2-inner product of the equations of (4.1) by u_t, ϕ_t and θ, respectively, then, using integration by parts and boundary conditions (4.3), we get

$$\frac{\rho}{2}\frac{d}{dt}\int_0^\pi u_t^2 dx + \frac{\mu}{2}\frac{d}{dt}\int_0^\pi u_x^2 dx - b\int_0^\pi \phi_x u_t dx = 0, \tag{4.23}$$

$$\frac{J}{2}\frac{d}{dt}\int_0^\pi \phi_t^2 dx + \frac{\delta}{2}\frac{d}{dt}\int_0^\pi \phi_x^2 dx + b\int_0^\pi u_x\phi_t dx + \frac{\xi}{2}\frac{d}{dt}\int_0^\pi \phi^2 dx + \beta\int_0^\pi w_x\phi_t dx = 0 \tag{4.24}$$

and

$$\frac{\alpha}{2}\frac{d}{dt}\int_0^\pi w^2 dx + \kappa\int_0^\pi w_x^2 dx + \beta\int_0^\pi \phi_{xt} w dx = 0. \tag{4.25}$$

The addition of (4.23)-(4.25) gives (4.22). □

Our stability result reads as follows:

Theorem 4.2. *Assume that (4.6) holds and let (u, ϕ, w) be the solution of the problem (4.1)-(4.3), then, the energy functional (4.20) satisfies*

$$E(t) \leq \sigma e^{-\varpi t}, \ \forall t \geq 0.$$

for two positive constants σ and ϖ.

The proof of Theorem 4.2 will be established through several lemmas.

Lemma 4.2. *Let (u, ϕ, w) be the solution of (4.1). Then, the functional*

$$F_1(t) = -\frac{\alpha J}{\beta}\int_0^\pi w\left(\int_0^x \phi_t(y)dy\right)dx$$

satisfies, for any $\varepsilon_1 > 0$, the estimate

$$\begin{aligned} F_1'(t) \leq & -\frac{J}{2}\|\phi_t\|^2 + \varepsilon_1\left\|\sqrt{\mu}u_x + \frac{b}{\sqrt{\mu}}\phi\right\|^2 + \varepsilon_1\|\phi\|^2 \\ & + \varepsilon_1\|\phi_x\|^2 + M\left(1 + \frac{1}{\varepsilon_1}\right)\|w_x\|^2, \end{aligned} \tag{4.26}$$

where M is a positive constant independent of ε_1.

Proof. Differentiating $F_1(t)$, then, using the second and third equations of (4.1) and integration by parts, we get

$$\begin{aligned}
F_1'(t) =& -\frac{\alpha J}{\beta}\int_0^\pi w_t\left(\int_0^x \phi_t(y)dy\right)dx - \frac{\alpha J}{\beta}\int_0^\pi w\left(\int_0^x \phi_{tt}(y)dy\right)dx \\
=& -\frac{J}{\beta}\int_0^\pi (\kappa w_{xx} - \beta\phi_{xt})\int_0^x \phi_t(y)dydx \\
& -\frac{\alpha}{\beta}\int_0^\pi w\left(\int_0^x (\delta\phi_{xx} - bu_x - \xi\phi - \beta w_x)\,dy\right)dx \\
=& \frac{J\kappa}{\beta}\int_0^\pi w_x\phi_t dx - J\|\phi_t\|^2 - \frac{\alpha\delta}{\beta}\int_0^\pi w\phi_x dx + \alpha\|w\|^2 + \frac{\alpha}{\beta}\int_0^\pi w\left(\int_0^x (bu_x + \xi\phi)\,dy\right)dx.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{\alpha}{\beta}\int_0^\pi w\left(\int_0^x (bu_x+\xi\phi)(y)\,dy\right)dx =& \frac{\alpha b}{\beta\sqrt{\mu}}\int_0^\pi w\left(\int_0^x\left(\sqrt{\mu}u_x + \frac{b}{\sqrt{\mu}}\phi\right)(y)\,dy\right)dx \\
& + \frac{\alpha}{\beta}\left(\xi - \frac{b^2}{\mu}\right)\int_0^\pi w\left(\int_0^x \phi(y)dy\right)dx,
\end{aligned}$$

consequently, we infer that

$$\begin{aligned}
F_1'(t) =& -J\|\phi_t\|^2 + \frac{J\kappa}{\beta}\int_0^\pi w_x\phi_t dx + \frac{\alpha b}{\beta\sqrt{\mu}}\int_0^\pi w\left(\int_0^x\left(\sqrt{\mu}u_x + \frac{b}{\sqrt{\mu}}\phi\right)(y)dy\right)dx \\
& + \frac{\alpha}{\beta}\left(\xi - \frac{b^2}{\mu}\right)\int_0^\pi w\left(\int_0^x \phi(y)dy\right)dx - \frac{\alpha\delta}{\beta}\int_0^\pi w\phi_x dx + \alpha\|w\|^2. \qquad (4.27)
\end{aligned}$$

Young's and Cauchy Schwarz inequalities lead to

$$\begin{aligned}
\frac{J\kappa}{\beta}\int_0^\pi w_x\phi_t dx \leq& \frac{J\kappa}{\beta}\|\phi_t\|\,\|w_x\| \\
\leq& \frac{J}{2}\|\phi_t\|^2 + M\|w_x\|^2. \qquad (4.28)
\end{aligned}$$

Similarly, for any $\varepsilon_1 > 0$, we have

$$\frac{\alpha b}{\beta\sqrt{\mu}}\int_0^\pi w\left[\int_0^x\left(\sqrt{\mu}u_x + \frac{b}{\sqrt{\mu}}\phi\right)(y)\,dy\right]dx \leq \frac{M}{\varepsilon_1}\|w_x\|^2 + \varepsilon_1\left\|\sqrt{\mu}u_x + \frac{b}{\sqrt{\mu}}\phi\right\|^2, \qquad (4.29)$$

$$\frac{\alpha}{\beta}\left(\xi - \frac{b^2}{\mu}\right)\int_0^\pi w\int_0^x (\phi(y)dy)\,dx \leq \frac{M}{\varepsilon_1}\|w_x\|^2 + \varepsilon_1\|\phi\|^2 \qquad (4.30)$$

and

$$-\frac{\alpha\delta}{\beta}\int_0^\pi w\phi_x dx \leq \frac{M}{\varepsilon_1}\|w_x\|^2 + \varepsilon_1\|\phi_x\|^2. \qquad (4.31)$$

The substitution of (4.28)-(4.31) into (4.27) yields (4.26). □

Lemma 4.3. *Assume that $\chi = 0$, then, the functional*

$$F_2(t) = \frac{\delta\rho}{b}\int_0^\pi \phi_x u_t dx + \frac{J\mu}{b}\int_0^\pi \phi_t u_x dx + 2J\int_0^\pi \phi\phi_t dx - \frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi(y)dy\right)dx \quad (4.32)$$

satisfies, for any $\varepsilon_2 > 0$, the estimate

$$\begin{aligned} F_2'(t) \leq & -\gamma_1\left\|\sqrt{\mu}u_x + \frac{b}{\sqrt{\mu}}\phi\right\|^2 - \gamma_2\|\phi\|^2 - \frac{\delta}{2}\|\phi_x\|^2 + \varepsilon_2\|u_t\|^2 \\ & + M\|w_x\|^2 + M\left(1+\frac{1}{\varepsilon_2}\right)\|\phi_t\|^2, \end{aligned} \quad (4.33)$$

where γ_1, γ_2 are positive constants.

Proof. Direct differentiation, using $(4.1)_1$ and integration by parts yields

$$\begin{aligned} \frac{d}{dt}\frac{\delta\rho}{b}\int_0^\pi \phi_x u_t dx &= \frac{\delta\rho}{b}\int_0^\pi \phi_{tx} u_t dx + \frac{\delta\rho}{b}\int_0^\pi \phi_x u_{tt} dx \\ &= \frac{\delta\rho}{b}\int_0^\pi \phi_{tx} u_t dx + \frac{\delta}{b}\int_0^\pi \phi_x\left(\mu u_{xx} + b\phi_x\right)dx \\ &= \frac{\delta\rho}{b}\int_0^\pi \phi_{tx} u_t dx + \frac{\mu\delta}{b}\int_0^\pi \phi_x u_{xx} dx + \delta\|\phi_x\|^2. \end{aligned} \quad (4.34)$$

Similarly, using $(4.1)_2$, we get

$$\begin{aligned} \frac{d}{dt}\frac{J\mu}{b}\int_0^\pi \phi_t u_x dx &= \frac{J\mu}{b}\int_0^\pi \phi_{tt} u_x dx + \frac{J\mu}{b}\int_0^\pi \phi_t u_{xt} dx \\ &= \frac{\mu}{b}\int_0^\pi \left(\delta\phi_{xx} - bu_x - \xi\phi - \beta w_x\right)u_x dx + \frac{J\mu}{b}\int_0^\pi \phi_t u_{xt} dx \\ &= -\frac{\mu\delta}{b}\int_0^\pi \phi_x u_{xx} dx - \mu\|u_x\|^2 - \frac{\mu\xi}{b}\int_0^\pi \phi u_x dx - \frac{\mu\beta}{b}\int_0^\pi w_x u_x dx \\ &+ \frac{J\mu}{b}\int_0^\pi \phi_t u_{xt} dx, \end{aligned} \quad (4.35)$$

and

$$\begin{aligned} 2\frac{d}{dt}J\int_0^\pi \phi\phi_t dx &= 2J\|\phi_t\|^2 + 2J\int_0^\pi \phi\phi_{tt} dx \\ &= 2J\|\phi_t\|^2 + 2\int_0^\pi \phi\left(\delta\phi_{xx} - bu_x - \xi\phi - \beta w_x\right)dx \\ &= 2J\|\phi_t\|^2 - 2\delta\|\phi_x\|^2 - 2b\int_0^\pi \phi u_x dx - 2\xi\|\phi\|^2 - 2\beta\int_0^\pi \phi w_x dx. \end{aligned} \quad (4.36)$$

The exploitation of $(4.1)_1$ and integration by parts lead to

$$\begin{aligned}
-\frac{d}{dt}\frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi(y)dy\right)dx &= -\frac{\rho\xi}{b}\int_0^\pi u_{tt}\left(\int_0^x \phi(y)dy\right)dx - \frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx \\
&= -\frac{\xi}{b}\int_0^\pi [\mu u_{xx}+b\phi_x]\left(\int_0^x \phi(y)dy\right)dx \\
&\quad -\frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right) \\
&= \frac{\mu\xi}{b}\int_0^\pi u_x\phi dx + \xi\,\|\phi\|^2 - \frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right).
\end{aligned} \tag{4.37}$$

The addition of (4.34)-(4.37) yields

$$\begin{aligned}
F_2'(t) = &-\mu\,\|u_x\|^2 - \xi\,\|\phi\|^2 - 2b\int_0^\pi \phi u_x dx - \delta\,\|\phi_x\|^2 + 2J\,\|\phi_t\|^2 - 2\beta\int_0^\pi \phi w_x dx \\
&- \frac{\mu\beta}{b}\int_0^\pi w_x u_x dx - \frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right) + \left(\frac{J\mu-\delta\rho}{b}\right)\int_0^\pi u_{xt}\phi_t dx.
\end{aligned}$$

Therefore, since $\chi = 0$, we end up with

$$\begin{aligned}
F_2'(t) = &-\left\|\sqrt{\mu}u_x + \frac{b}{\sqrt{\mu}}\phi\right\|^2 - \left(\xi - \frac{b^2}{\mu}\right)\|\phi\|^2 - \alpha\,\|\phi_x\|^2 + 2J\,\|\phi_t\|^2 \\
&+ 2\beta\int_0^\pi \phi_x w dx - \frac{\mu\beta}{b}\int_0^\pi w_x u_x dx - \frac{\rho\xi}{b}\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right).
\end{aligned} \tag{4.38}$$

Cauchy-Schwarz, Young's and Poincaré's inequalities give

$$\beta\int_0^\pi w\phi_x dx \le \frac{\delta}{4}\|\phi_x\|^2 + \frac{\beta^2 C_P}{\delta}\|w_x\|^2, \tag{4.39}$$

where C_P is the Poincaré constant.

Similarly, for any $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$, we infer

$$\frac{\rho\xi}{b}\left|\int_0^\pi u_t\left(\int_0^x \phi_t(y)dy\right)dx\right| \le \varepsilon_2\,\|u_t\|^2 + \frac{M}{\varepsilon_2}\,\|\phi_t\|^2 \tag{4.40}$$

and, using the inequality $a^2 \le 2(a+b)^2 + 2b^2$, we get

$$\begin{aligned}
\left|\frac{\mu\beta}{b}\int_0^\pi w_x u_x dx\right| =& \frac{\sqrt{\mu}\beta}{b}\left|\int_0^\pi \sqrt{\mu}w_x u_x dx\right| \le \varepsilon_3\,\|\sqrt{\mu}u_x\|^2 + \frac{M}{\varepsilon_3}\,\|w_x\|^2 \\
\le& 2\varepsilon_3\left\|\sqrt{\mu}u_x + \frac{b}{\sqrt{\mu}}\phi\right\|^2 + 2\frac{b^2}{\mu}\varepsilon_3\,\|\phi\|^2 + \frac{M}{\varepsilon_3}\|w_x\|^2.
\end{aligned} \tag{4.41}$$

The insertion of (4.39)-(4.41) into (4.38) yields

$$\begin{aligned} F_2'(t) = &-(1-2\varepsilon_3)\left\|\sqrt{\mu}u_x+\frac{b}{\sqrt{\mu}}\phi\right\|^2-\left(\xi-\frac{b^2}{\mu}-2\frac{b^2}{\mu}\varepsilon_3\right)\|\phi\|^2-\frac{\delta}{2}\|\phi_x\|^2 \\ &+M\left(1+\frac{1}{\varepsilon_2}\right)\|\phi_t\|^2+\varepsilon_2\|u_t\|^2+M\left(1+\frac{1}{\varepsilon_3}\right)\|w_x\|^2. \end{aligned} \tag{4.42}$$

Since $\xi-\frac{b^2}{\mu}>0$, we can choose ε_3, small enough such that $\xi-\frac{b^2}{\mu}-2\frac{b^2}{\mu}\varepsilon_3>0$ and $1-2\varepsilon_3>0$. Therefore the estimate (4.33) follows immediately. □

Lemma 4.4. *For (u,ϕ,w), the solution of (4.1), the functional*

$$F_3(t):=-\rho\int_0^\pi u_t u dx-\frac{b\rho}{\mu}\int_0^\pi u_t\left(\int_0^x\phi(y)dy\right)dx$$

satisfies,

$$F_3'(t)\le-\frac{\rho}{2}\|u_t\|^2+\left\|\sqrt{\mu}u_x+\frac{b}{\sqrt{\mu}}\phi\right\|^2+M\|\phi_t\|^2, \tag{4.43}$$

where M is a positive constant.

Proof. By differentiating $F_3(t)$ and using the first equation of (4.1), we arrive at

$$\begin{aligned} F_3'(t)&=-\rho\|u_t\|^2-\int_0^\pi(\mu u_{xx}+b\phi_x)udx-\frac{b}{\mu}\int_0^\pi[\mu u_{xx}+b\phi_x]\left(\int_0^x\phi(y)dy\right)dx \\ &\quad-\frac{b\rho}{\mu}\int_0^\pi u_t\left(\int_0^x\phi_t(y)dy\right)dx \\ &=-\rho\|u_t\|^2+\mu\|u_x\|^2+2b\int_0^\pi u_x\phi dx+\frac{b^2}{\mu}\|\phi\|^2-\frac{b\rho}{\mu}\int_0^\pi u_t\left(\int_0^x\phi_t(y)dy\right)dx \\ &=-\rho\|u_t\|^2+\left\|\sqrt{\mu}u_x+\frac{b}{\sqrt{\mu}}\phi\right\|^2-\frac{b\rho}{\mu}\int_0^\pi u_t\left(\int_0^x\phi_t(y)dy\right)dx. \end{aligned}$$

Young's and Cauchy Schwarz' inequalities give

$$\frac{b\rho}{\mu}\left|\int_0^\pi u_t\left(\int_0^x\phi_t(y)dy\right)dx\right|\le\frac{\rho}{2}\|u_t\|^2+M\|\phi_t\|^2$$

and (4.43) follows immediately. □

At this stage, we introduce the Lyapunov functional

$$\mathcal{L}(t)=NE(t)+\frac{1}{\varepsilon_1}F_1(t)+\frac{\rho}{4\varepsilon_2}F_2(t)+F_3(t),$$

where $N,\varepsilon_1,\varepsilon_2$ are positive constants to be specified later.

Lemma 4.5. *For N sufficiently large, there exist two positive constants m_1 and m_2 such that the functional $\mathcal{L}$ satisfies*

$$m_1 E(t) \leq \mathcal{L}(t) \leq m_2 E(t), \quad \forall t \geq 0. \tag{4.44}$$

Proof. Let $\mathcal{K}(t) := \mathcal{L}(t) - NE(t)$, then

$$\mathcal{K}(t) = \frac{1}{\varepsilon_1} F_1(t) + \frac{\rho}{4\varepsilon_2} F_2(t) + F_3(t)$$

and

$$\begin{aligned} |\mathcal{K}(t)| \leq C & \left[\left| \int_0^\pi w \left(\int_0^x \phi_t(y) dy \right) dx \right| + \left| \int_0^\pi \phi_x u_t dx \right| + \left| \int_0^\pi \phi_t u_x dx \right| + \left| \int_0^\pi \phi \phi_t dx \right| \right] \\ & + C \left[\left| \int_0^\pi u_t u dx \right| + \left| \int_0^\pi u_t \left(\int_0^x \phi(y) dy \right) dx \right| \right]. \end{aligned}$$

Therefore, Cauchy-Schwarz and Young's inequalities yield

$$|\mathcal{L}(t) - NE(t)| \leq CE(t), \tag{4.45}$$

for a positive constant $C_{\varepsilon_1,\varepsilon_2}$. Thus,

$$(N - C_{\varepsilon_1,\varepsilon_2}) E(t) \leq \mathcal{L}(t) \leq (N + C_{\varepsilon_1,\varepsilon_2}) E(t).$$

It suffices to pick N large such that

$$N > C_{\varepsilon_1,\varepsilon_2}$$

and take $m_1 = N - C_{\varepsilon_1,\varepsilon_2}$, $m_2 = N + C_{\varepsilon_1,\varepsilon_2}$. □

4.3.1 Proof of Theorem 4.2

By differentiating $\mathcal{L}$ and using (4.22),(4.26),(4.33) and (4.43), we get

$$\begin{aligned} \mathcal{L}'(t) \leq & - \left(\frac{\rho\gamma_1}{4\varepsilon_2} - 2 \right) \left\| \sqrt{\mu} u_x + \frac{b}{\sqrt{\mu}} \phi \right\|^2 - \left(\frac{\rho\gamma_2}{4\varepsilon_2} - 1 \right) \|\phi\|^2 - \frac{\rho}{4} \|u_t\|^2 \\ & - \left(\frac{\rho\delta}{8\varepsilon_2} - 1 \right) \|\phi_x\|^2 - \left(\frac{J}{2\varepsilon_1} - \frac{\rho M}{4} \left(\frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_2^2} \right) - M \right) \|\phi_t\|^2 \\ & - \left(\kappa N - \frac{M}{\varepsilon_1} \left(1 + \frac{1}{\varepsilon_1} \right) - \frac{\rho M}{4\varepsilon_2} \right) \|w_x\|^2. \end{aligned}$$

Now, we choose the constants $\varepsilon_1, \varepsilon_2$ and N carefully.

First, we choose ε_2, small enough, such that

$$\varepsilon_2 < \min\left\{\frac{\rho\gamma_1}{8}, \frac{\rho\gamma_2}{4}, \frac{\rho\delta}{8}\right\}. \tag{4.46}$$

Next, we choose

$$\varepsilon_1 < \frac{J}{2\left(\rho M\left(\frac{1}{4\varepsilon_2^2} + \frac{1}{4\varepsilon_2}\right) + M\right)}. \tag{4.47}$$

Finally, we pick N large such that

$$N > \max\left\{\frac{M}{\kappa\varepsilon_1}\left(1 + \frac{1}{\varepsilon_1}\right) + \frac{\rho M}{4\kappa\varepsilon_2}, C_{\varepsilon_1,\varepsilon_2}\right\}.$$

Therefore, there exist $\lambda > 0$, such that

$$\mathcal{L}'(t) \leq -\lambda\left(\left\|\sqrt{\mu}u_x + \frac{b}{\sqrt{\mu}}\phi\right\|^2 + \|\phi\|^2 + \|u_t\|^2 + \|\phi_x\|^2 + \|\phi_t\|^2 + \|w_x\|^2\right).$$

Using Poincaré's inequality, we infer that there exists $\omega > 0$, such that

$$\mathcal{L}'(t) \leq -\omega E(t),\ t \geq 0.$$

Using (4.44), we arrive at

$$\mathcal{L}'(t) \leq -\varpi\mathcal{L}(t),\ t \geq 0,$$

for some $\varpi > 0$.

An integration over $(0, t)$ gives

$$\mathcal{L}(t) \leq \mathcal{L}(0)\, e^{-\varpi t},\ t \geq 0.$$

Again, the fact that $\mathcal{L}(t) \sim E(t)$ yields

$$E(t) \leq \sigma e^{-\varpi t},\ t \geq 0,$$

for some constant $\sigma > 0$, which completes the proof of Theorem 4.2.

Part II

Stability of Some Non-simple Systems

Introduction

In this second part of the thesis, we investigate the coupling between non-simple elasticity, porosity and thermal effects of hyperbolic type. The non-simple elasticity is characterized by the presence of higher-order derivatives of the displacement u in basic postulates, which provides more details on the configuration of the material, the thermal effect is characterized by wave-like heat conduction.

The general non-linear form of the theory of non-simple materials (also called strain-gradient theory) was introduced by Toupin [94]. Mindlin [61] and Mindlin and Eshel [60] used the basis of the conservation principle and obtained the linear theory in which the potential energy density depends on the stain and also on the gradient of the strain.

The study of the long-time behavior of the solutions of thermoelastic systems has been of interest to scientists since Dafermos published his pioneer paper [18]. As a result, a lot of papers have appeared with the aim to discuss the asymptotic behavior of the solutions to elastic equations coupled with porosity and/or thermoelasticity.

chapter

5 A non-simple system with thermoelasticity of second-sound

5.1 Introduction

In this chapter, we consider the following one-dimensional thermoelastic system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} - \alpha u_{xxxx} + \beta \theta_x & \text{in } (0, L) \times (0, \infty), \\ c\theta_t = -\gamma q_x + \beta u_{xt} & \text{in } (0, L) \times (0, \infty), \\ \tau q_t + q + \kappa \theta_x = 0 & \text{in } (0, L) \times (0, \infty), \end{cases} \tag{5.1}$$

where u is the transversal displacement of a nonsimple material of length L, q is the heat flux, θ is the difference of the temperature, κ is the coefficient of thermal conductivity and τ represents the time lag in the response of the heat flux to the temperature gradient.

The system (5.1) is supplemented with the following initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \theta(x,0) = \theta_0(x), & \text{for } x \in (0, L), \\ q\,(x,0) = q_0(x), & \text{for } x \in (0, L), \\ u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = q(0,t) = q(L,t) = 0 & \text{in } (0, \infty). \end{cases} \tag{5.2}$$

Integrating $(5.1)_2$ with respect to x and bearing in mind the boundary conditions on u and q, we obtain

$$\frac{d}{dt}\int_0^L \theta(x,t)dx = 0, \tag{5.3}$$

which entails

$$\int_0^L \theta\,(x,t)\,dx = \int_0^L \theta_0\,(x)\,dx.$$

Thus, if we set

$$\overline{\theta}(x,t) = \theta(x,t) - \frac{1}{L}\int_0^L \theta_0\,(x)\,dx, \tag{5.4}$$

then, $(u, \overline{\theta}, q)$ solves (5.1)-(5.2), with the initial condition

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \overline{\theta}(x,0) = \theta_0(x) - \frac{1}{L}\int_0^L \theta_0(x)\,dx,\ q(x,0) = q_0(x) \quad (5.5)$$

and, more importantly,

$$\int_0^L \overline{\theta}(x,t)\,dx = 0,$$

which enables the application of Poincaré's inequality on $\overline{\theta}$. In the sequel, we work with $(u, \overline{\theta}, q)$, but for convenience, we write (u, θ, q).

The rest of this chapter is organized as follows. In Section 2, we state and prove a well-posedness result by the use of a semigroup approach. In Section 3, we prove the lack of analyticity of the solution of problem (5.1),(5.2).

5.2 Well-posedness

In this section, we state and prove the existence and the uniqueness of a solution to the problem (5.1), (5.2), (5.5).

First, we introduce the space

$$\mathcal{H} := (H^2(0,L) \cap H_0^1(0,L)) \times L^2(0,L) \times L_*^2(0,L) \times L^2(0,L),$$

The space $\mathcal{H}$ equipped with the inner product

$$\langle U, U^* \rangle_{\mathcal{H}} = \int_0^L \left(\rho v v^* + \mu u_x u_x^* + \alpha u_{xx} u_{xx}^* + c\theta\theta^* + \frac{\gamma\tau}{\kappa} q q^* \right) dx,$$

for $U = (u, v, \theta, q)^T$ and $U^* = (u^*, v^*, \theta^*, q^*)^T$, is a Hilbert space. The associated norm is

$$\|U\|_{\mathcal{H}}^2 = \rho \|v\|^2 + \mu \|u_x\|^2 + \alpha \|u_{xx}\|^2 + c \|\theta\|^2 + \frac{\gamma\tau}{\kappa} \|q\|^2 .$$

To rewrite the problem (5.1) in the semigroup setting, we introduce the new variable $v = u_t$, then the system (5.1) becomes

$$\begin{cases} u_t - v = 0, \\ v_t - \dfrac{1}{\rho}(\mu u_{xx} - \alpha u_{xxxx} + \beta\theta_x) = 0, \\ \theta_t + \dfrac{1}{c}(\gamma q_x - \beta v_x) = 0, \\ q_t + \dfrac{1}{\tau}(q + \kappa\theta_x) = 0. \end{cases} \quad (5.6)$$

The system (5.6) with the initial conditions (5.5), can be written in the form

$$\begin{cases} \Phi_t + \mathcal{A}\Phi_t = 0, \\ \Phi(0) = \Phi_0, \end{cases} \tag{5.7}$$

where $\Phi = (u, v, \theta, q)^T$, $\Phi_0 = (u_0, u_1, \theta_0, q_0)^T$ and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is the operator given by

$$\mathcal{A} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -\frac{\mu}{\rho}\partial_{xx} + \frac{\alpha}{\rho}\partial_{xxxx} & 0 & -\frac{\beta}{\rho}\partial_x & 0 \\ 0 & -\frac{\beta}{c}\partial_x & 0 & \frac{\gamma}{c}\partial_x \\ 0 & 0 & \frac{\kappa}{\tau}\partial_x & \frac{1}{\tau} \end{pmatrix}, \tag{5.8}$$

with domain

$$D(\mathcal{A}) = \left\{ \Phi \in \mathcal{H} \middle| \begin{array}{c} u \in H^4(0,L) \cap H_0^1(0,L),\ v \in H^2(0,L) \cap H_0^1(0,L), \\ \theta \in H_*^1(0,L),\ q \in H_0^1(0,L),\ u_{xx}(0) = u_{xx}(L) = 0 \end{array} \right\},$$

Our well-posedness result is the following theorem.

Theorem 5.1. *For any $\Phi_0 \in D(\mathcal{A})$, the problem (5.7) has a unique solution $\Phi \in C([0, +\infty[\,; D(\mathcal{A})) \cap C^1([0, +\infty[\,; \mathcal{H})$.*

Remark 5.1. *If $\Phi_0 \in \mathcal{H}$. Then, the solution of the problem (5.7) satisfies $\Phi \in C([0, +\infty[\,; \mathcal{H})$.*

Our main tool for the proof of Theorem 5.1 is Hille-Yosida theorem 1.9.

Proof of Theorem 5.1. According to Theorem 1.9, it suffices to prove that $\mathcal{A}$ is monotone and maximal.

A straightforward calculation shows that, $\forall \Phi \in D(\mathcal{A})$,

$$\begin{aligned} \langle \mathcal{A}\Phi, \Phi \rangle_{\mathcal{H}} = & \mu \int_0^L u_x v_x dx + \alpha \int_0^L u_{xx} v_{xx} dx - \beta \int_0^L \theta_x v dx - \mu \int_0^L v_x u_x dx \\ & - \alpha \int_0^L v_{xx} u_{xx} dx + \gamma \int_0^L q_x \theta dx - \beta \int_0^L v_x \theta dx + \gamma \int_0^L \theta_x q dx \\ & + \frac{\gamma}{\kappa} \int_0^L q^2 dx = \frac{\gamma}{\kappa} \int_0^L |q|^2 dx \geq 0, \end{aligned}$$

which shows the monotonicity of the operator $\mathcal{A}$.

On the other hand, to prove that $\mathcal{A}$ is maximal, we have to show that for any $F = (f^1, f^2, f^3, f^4)^T \in \mathcal{H}$, there exists $\Phi = (u, v, \theta, q)^T \in D(\mathcal{A})$ such that $(I + \mathcal{A})\,\Phi = F$; that is,

$$\begin{cases} u - v = f^1, \\ \rho v - \mu u_{xx} + \alpha u_{xxxx} - \beta\theta_x = \rho f^2, \\ c\theta + \gamma q_x - \beta v_x = c f^3, \\ \tau q + q + \kappa\theta_x = \tau f^4. \end{cases} \tag{5.9}$$

From the fourth equation of (5.9), we get

$$q\,(x) = \frac{\tau}{1+\tau} f^4\,(x) - \frac{\kappa}{1+\tau}\theta_x\,(x)\,. \tag{5.10}$$

Substituting $v = u - f^1$ and (5.10) into $(5.9)_2$ and $(5.9)_3$, respectively, we get

$$\begin{cases} \rho u - \mu u_{xx} + \alpha u_{xxxx} - \beta\theta_x = g_1, \\ c\theta - \dfrac{\kappa\gamma}{1+\tau}\theta_{xx} - \beta u_x = g_2, \end{cases} \tag{5.11}$$

where

$$g_1 = \rho(f^1 + f^2) \in L^2(0, L) \tag{5.12}$$

and

$$g_2 = c f^3 - \frac{\tau\gamma}{1+\tau} f^4_x - \beta f^1_x \in H^{-1}\,(0, L)\,. \tag{5.13}$$

To solve (5.11), we consider the variational formulation

$$a(V, \widetilde{V}) = L(\widetilde{V}),\ \forall \widetilde{V} \in \mathcal{W}, \tag{5.14}$$

where, for $V = (u, \theta)$ and $\widetilde{V} = (\widetilde{u}, \widetilde{\theta})$,

$$\begin{aligned} a\left(V, \widetilde{V}\right) = {} & \rho \int_0^L u\widetilde{u}dx + \mu \int_0^L u_x\widetilde{u}_x dx + \alpha \int_0^L u_{xx}\widetilde{u}_{xx} dx + \beta \int_0^L \theta\widetilde{u}_x dx \\ & + c \int_0^L \theta\widetilde{\theta} dx + \frac{\kappa\gamma}{1+\tau} \int_0^L \theta_x\widetilde{\theta}_x dx - \beta \int_0^L u_x\widetilde{\theta} dx \end{aligned}$$

and

$$L\left(\widetilde{V}\right) = \int_0^L g_1\widetilde{u}dx + c \int_0^L f^3\widetilde{\theta} dx + \frac{\tau\gamma}{1+\tau} \int_0^L f^4\widetilde{\theta}_x dx - \beta \int_0^L f^1_x\widetilde{\theta} dx$$

are, respectively, bilinear and linear forms defined on the Hilbert space

$$\mathcal{W} = (H^2(0, L) \cap H^1_0(0, L)) \times H^1_*(0, L),$$

equipped with the norm

$$\|V\|_{\mathcal{W}}^2 = \rho \|u\|^2 + \mu \|u_x\|^2 + \alpha \|u_{xx}\|^2 + c \|\theta\|^2 + \frac{\kappa\gamma}{1+\tau} \|\theta_x\|^2 .$$

A simple calculation, using the Cauchy-Schwarz inequality, shows that

$$\left| a\left(V, \widetilde{V}\right) \right| \leq C \|V\|_{\mathcal{W}} \left\| \widetilde{V} \right\|_{\mathcal{W}}$$

and

$$\left| L\left(\widetilde{V}\right) \right| \leq C^* \left\| \widetilde{V} \right\|_{\mathcal{W}},$$

for positive constants C and C^*. Moreover,

$$a(V,V) = \rho \|u\|^2 + \mu \|u_x\|^2 + \alpha \|u_{xx}\|^2 + \frac{\kappa\gamma}{\tau+1} \|\theta_x\|^2 + c\|\theta\|^2 = \|V\|_{\mathcal{W}}^2 .$$

Therefore, $a(\cdot,\cdot)$ and L fulfil the assumptions of Lax-Milgram theorem and, consequently, there exists a unique $(u,\theta) \in (H^2(0,L) \cap H_0^1(0,L)) \times H_*^1(0,L)$, such that

$$a\left((u,\theta), \left(\widetilde{u}, \widetilde{\theta}\right)\right) = L\left(\widetilde{u}, \widetilde{\theta}\right), \ \forall \left(\widetilde{u}, \widetilde{\theta}\right) \in \mathcal{W}.$$

From $(5.9)_1$, we get

$$v = u - f^1 \in H^2(0,L) \cap H_0^1(0,L).$$

Furthermore, if we take $\left(\widetilde{u}, \widetilde{\theta}\right) = (\widetilde{u}, 0)$ in (5.14), we arrive at

$$\int_0^L u_{xx}\widetilde{u}_{xx}dx = \frac{1}{\alpha}\int_0^L \left(g^1 - \rho u + \mu u_{xx} + \beta\theta_x\right)\widetilde{u}dx, \ \forall \widetilde{u} \in H^2(0,L) \cap H_0^1(0,L). \tag{5.15}$$

Then, the regularity theory implies that

$$u \in H^4(0,L) \cap H_0^1(0,L).$$

Moreover, since $C_0^\infty(0,L) \subset H^2(0,L) \cap H_0^1(0,L)$ then, simple integrations yield

$$\int_0^L u_{xxxx}\widetilde{u}dx = \frac{1}{\alpha}\int_0^L (g_1 - \rho u + \mu u_{xx} + \beta\theta_x)\widetilde{u}dx, \ \forall \widetilde{u} \in C_0^\infty(0,L).$$

Therefore,

$$u_{xxxx} = \frac{1}{\alpha}(g_1 - \rho u + \mu u_{xx} + \beta\theta_x) \in L^2(0,L). \tag{5.16}$$

Substituting g_1 from (5.12), we obtain

$$\rho v - \mu u_{xx} + \alpha u_{xxxx} - \beta\theta_x = \rho f^2.$$

Thus, u satisfies $(5.9)_2$.

Next, an integration by parts in (5.15) yields, for all $\widetilde{u} \in H^2(0,L) \cap H_0^1(0,L)$,

$$u_{xx}(L)\widetilde{u}_x(L) - u_{xx}(0)\widetilde{u}_x(0) + \int_0^L u_{xxxx}\widetilde{u}dx = \frac{1}{\alpha}\int_0^L (g_1 - \rho u + \mu u_{xx} + \beta\theta_x)\,\widetilde{u}dx.$$

The use of (5.16) leads to

$$u_{xx}(L)\widetilde{u}_x(L) - u_{xx}(0)\widetilde{u}_x(0) = 0.$$

As $\widetilde{u} \in H^2(0,L) \cap H_0^1(0,L)$ is arbitrary, then

$$u_{xx}(L) = u_{xx}(0) = 0.$$

From the third equation of (5.9), we get

$$q_x = \frac{1}{\gamma}\left(cf^3 + \beta v_x - c\theta\right) \in L^2(0,L). \tag{5.17}$$

An integration over $(0,x)$ gives

$$q\,(x) = \frac{c}{\gamma}\int_0^x f^3\,(y)\,dy + \frac{\beta}{\gamma}v(x) - \frac{c}{\gamma}\int_0^x \theta\,(y)\,dy,$$

then,

$$q\,(0) = q\,(L) = 0, \tag{5.18}$$

which implies that

$$q \in H_0^1(0,L).$$

Thus, (v,q,θ) solves $(5.9)_3$, and, consequently, $\Phi = (u,v,\theta,q)^T \in D(\mathcal{A})$. Therefore, $\mathcal{A}$ is maximal and monotone and Theorem 1.9 ensures the existence of a unique solution to (5.7), which, in turns, proves that the problem (5.1), (5.2) and (5.5) has a unique solution (u,θ,q) with the desired regularity. □

5.3 Lack of analyticity

In this section, we prove that the semigroup associated to the system (5.1) is not analytic. The proof is based on Theorem 1.15.

Our result is reads as follows:

Theorem 5.2. *The semigroup* $S(t) = e^{-\mathcal{A}t}$ *associated to the problem* (5.7) *is not analytic.*

Proof. It suffices to show that, there exists a bounded sequence $(F_n) \subset \mathcal{H}$ and a sequence $(\lambda_n) \subset \mathbb{R}$, such that

$$\overline{\lim_{|\lambda_n|\to\infty}} \left\| \lambda_n (i\lambda_n I + \mathcal{A})^{-1} F_n \right\| = \infty.$$

Let $F_n = (0, \sin(\frac{n\pi}{L}x), 0, 0)^T \in \mathcal{H}$ and for $\lambda_n \in \mathbb{R}$, let $U_n \in D(\mathcal{A})$ be the solution of $(i\lambda_n I + \mathcal{A})U_n = F_n$, then

$$\begin{cases} i\lambda_n u_n - v_n = 0, & H^2(0,L) \cap H_0^1(0,L)\ , \\ i\rho\lambda_n v_n - \mu u_{nxx} + \alpha u_{nxxxx} - \beta\theta_{nx} = \sin(\frac{n\pi}{L}x), & L^2(0,L), \\ ic\lambda_n\theta_n + \gamma q_{nx} - \beta v_{nx} = 0, & L^2_*(0,L), \\ i\lambda\tau q_n + q_n + \kappa\theta_{nx} = 0, & L^2(0,L). \end{cases} \tag{5.19}$$

Bearing in mind the boundary conditions (5.2), we take

$$\begin{cases} u_n = A_n \sin(\frac{n\pi}{L}x), \\ \theta_n = B_n \cos(\frac{n\pi}{L}x), \\ q_n = C_n \sin(\frac{n\pi}{L}x). \end{cases}$$

The substitution into (5.19) gives

$$\begin{cases} -\rho\lambda_n^2 A_n + \mu\left(\frac{n\pi}{L}\right)^2 A_n + \alpha\left(\frac{n\pi}{L}\right)^4 A_n + \beta\frac{n\pi}{L}B_n = 1, \\ ic\lambda_n B_n + \gamma\frac{n\pi}{L}C_n - \beta i\lambda_n\frac{n\pi}{L}A_n = 0, \\ (i\lambda\tau + 1)\,C_n - \kappa\frac{n\pi}{L}B_n = 0. \end{cases} \tag{5.20}$$

Using the third equation of (5.20) and eliminating C_n, we obtain

$$\begin{aligned} p_1(\lambda_n)A_n - \beta\frac{n\pi}{L}B_n &= -1, \\ (-i\beta\lambda_n\frac{n\pi}{L})A_n + p_2(\lambda_n)B_n &= 0, \end{aligned} \tag{5.21}$$

where

$$p_1(\lambda_n) = \rho\lambda_n^2 - \mu\frac{n^2\pi^2}{L^2} - \alpha\frac{n^4\pi^4}{L^4}, \quad p_2(\lambda_n) = i\lambda_n c + \gamma\frac{n^2\pi^2\kappa}{(i\lambda_n\tau+1)L^2}.$$

Solving (5.21), we have

$$A_n = \frac{-p_2(\lambda)}{\left(p_1(\lambda)p_2(\lambda) - i\left(\frac{\pi n}{L}\right)^2\beta^2\lambda_n\right)}$$

and

$$B_n = \frac{-\beta i\lambda_n\frac{n\pi}{L}}{\left(p_1(\lambda)p_2(\lambda) - i\left(\frac{\pi n}{L}\right)^2\beta^2\lambda_n\right)}.$$

Let λ_n be a solution to $p_1(\lambda)=0$, for instance $\lambda_n = \frac{n\pi}{L}\sqrt{\frac{\mu}{\rho}+\frac{\alpha}{\rho}\frac{n^2\pi^2}{L^2}}$, then, we get

$$A_n = \frac{L^2(c+K_1+iK_2)}{\beta^2\pi^2n^2} \quad \text{and} \quad B_n = \frac{L}{\pi n\beta}, \tag{5.22}$$

where

$$K_1 = -\frac{\tau\kappa\rho L^2\gamma}{\tau^2(\mu L^2+\alpha n^2\pi^2)+\frac{\rho L^4}{n^2\pi^2}}$$

and

$$K_2 = -\frac{\rho\sqrt{\rho}L^4\kappa\gamma}{n\pi\sqrt{(L^2\mu+\alpha n^2\pi^2)}\left(\tau^2(\mu L^2+\alpha n^2\pi^2)+\frac{\rho L^4}{n^2\pi^2}\right)}.$$

Recalling that

$$\|U_n\|^2 = \rho\|v_n\|^2+\mu\|u_{nx}\|^2+\alpha\|u_{nxx}\|^2+c\|\theta_n\|^2+\frac{\gamma\tau}{\kappa}\|q_n\|^2 \geq \alpha\int_0^L|u_{nxx}|^2\,dx,$$

we get

$$\begin{aligned}\|U_n\|^2 &\geq \alpha\left(\frac{n\pi}{L}\right)^4|A_n|^2\int_0^L\left|\sin\frac{n\pi x}{L}\right|^2dx \\ &\geq \frac{\alpha L}{2\beta^2}\left((c+K_1)^2+K_2^2\right)\frac{n\pi}{L}^2 \simeq \frac{\alpha L}{2\beta^2}c^2\frac{n\pi}{L}^2 \to \infty, \text{ as } n\longrightarrow\infty,\end{aligned}$$

since $K_1\to 0$ and $K_2\to 0$ as $n\to\infty$. By noting that $\lambda_n\to\infty$, we deduce that

$$\lim_{n\to\infty}\|\lambda_n U_n\|^2 = \infty.$$

This completes the proof of Theorem 5.2. □

chapter

6 Non-simple porous thermoelastic systems with weak coupling

6.1 Introduction

In this chapter, we study two problems where the coupling elasticity-porosity is weak and establish an exponential stability result. The first problem is a nonsimple porous elastic system with viscoelastic and porous dissipative mechanisms, whereas in the second problem the viscoelastic dissipation is replaced by a thermal dissipation. The coupling between nonsimple elasticity and porosity was considered by Liu *et al.* [50], where they studied the longtime behavior of the system

$$\begin{cases} \rho u_{tt} = a u_{xx} + b\phi_x - c u_{xxxx} - d\phi_{xxx}, & \text{in } (0,\pi)\times(0,\infty), \\ J\phi_{tt} = d u_{xxx} + \beta\phi_{xx} - \xi\phi - b u_x - \tau\phi_t, & \text{in } (0,\pi)\times(0,\infty), \end{cases} \tag{6.1}$$

subjected to the boundary conditions

$$u(t,0) = u(t,\pi) = u_{xx}(t,0) = u_{xx}(t,\pi) = \phi_x(t,0) = \phi_x(t,\pi) = 0, \quad \text{in } (0,\infty),$$

and proved that the solution decays slowly. They also examined the case of a viscoelastic dissipation acting on the first equation and obtained an exponential stability result provided that $b+dn^2 \neq 0$ for all $n \in \mathbb{N}$. In fact, the exponential stability is a consequence of the strong coupling between the equations of system (6.1) reached by the presence of ϕ_{xxx}. Whereas, if $d = 0$, the coupling is weak and the solution decays slowly.

The rest of the chapter is organized as follows: in the second section, we prove the well-posedness of the first problem. Section 3 is devoted to the exponential decay of the first problem. Finally in the fourth section, we prove the exponential decay of the second problem.

6.2 A Non-Simple Porous Elastic Problem

In this section, we study the well-posedness and exponential stability for the following problem

$$\begin{cases} \rho u_{tt} = au_{xx} + b\phi_x - cu_{xxxx} + \delta u_{txx}, & \text{in } (0,\pi)\times(0,\infty), \\ J\phi_{tt} = \beta\phi_{xx} - \xi\phi - bu_x - \tau\phi_t, & \text{in } (0,\pi)\times(0,\infty), \end{cases} \tag{6.2}$$

where $a, b, c, \rho, \beta, \delta, \xi, J$ and τ are constitutive coefficients satisfy

$$\rho > 0,\ a > 0,\ c > 0,\ J > 0,\ \beta > 0,\ \xi > 0,\ \delta > 0\ \tau > 0,\ \text{and } a\xi > b^2 \neq 0. \tag{6.3}$$

The unknown functions in system (6.2) are assumed to be subjected to the following boundary and initial conditions

$$\begin{cases} u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = \phi_x(0,t) = \phi_x(\pi,t) = 0, & \text{in } (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x), & \text{in } x \in (0,\pi). \end{cases} \tag{6.4}$$

We notice that system (6.2) is the merging of two slowly decaying systems.

It is worth noting that whenever boundary conditions (6.4) are imposed, there exists a solution that do not decay. To avoid this possibility we solve the differential equation

$$J\frac{d^2}{dt^2}\int_0^\pi \phi(x,t)dx + \tau\frac{d}{dt}\int_0^\pi \phi(x,t)dx + \xi\int_0^\pi \phi(x,t)dx = 0,$$

and introduce the new variable

$$\overline{\phi}(x,t) = \phi(x,t) - \frac{1}{\pi}\int_0^\pi \phi(x,t)dx.$$

6.2.1 Well-Posedness

In this subsection, we will prove that the problem (6.2), (6.4) has a unique solution. To do this, we first introduce the new variables $v = u_t$ and $\psi = \phi_t$ and define the Hilbert space

$$\mathcal{H} := \left(H^2(0,\pi)\cap H_0^1(0,\pi)\right)\times L^2(0,\pi)\times H_*^1(0,\pi)\times L_*^2(0,\pi),$$

endowed by the inner product

$$\langle U, U^*\rangle_{\mathcal{H}} = \int_0^\pi \left[\rho vv^* + au_xu_x^* + cu_{xx}u_{xx}^* + J\psi\psi^* + \beta\phi_x\phi_x^* + \xi\phi\phi^* + b\left(u_x\phi^* + \phi u_x^*\right)\right]dx,$$

for $U=(u,v,\phi,\psi)^T$ and $U^*=(u^*,v^*,\phi^*,\psi^*)^T$, and the associated norm

$$\|U\|_{\mathcal{H}}^2 := \rho\int_0^\pi |v|^2\,dx + a\int_0^\pi |u_x|^2\,dx + c\int_0^\pi |u_{xx}|^2\,dx + J\int_0^\pi |\psi|^2\,dx$$
$$+\xi\int_0^\pi |\phi|^2\,dx + \beta\int_0^\pi |\phi_x|^2\,dx + 2b\int_0^\pi u_x\phi dx.$$

According to the above notation, the problem (6.2)-(6.4) can be written

$$\begin{cases} U_t+\mathcal{A}U=0, \\ U\left(0\right)=\left(u_0,u_1,\phi_0,\phi_1\right)^T, \end{cases} \tag{6.5}$$

where $\mathcal{A}$ is the operator defined on $\mathcal{H}$ by

$$\mathcal{A}=\begin{pmatrix} 0 & -1 & 0 & 0 \\ -\frac{a}{\rho}\partial_{xx}+\frac{c}{\rho}\partial_{xxxx} & -\frac{\delta}{\rho}\partial_{xx} & -\frac{b}{\rho}\partial_x & 0 \\ 0 & 0 & 0 & -1 \\ \frac{b}{J}\partial_x & 0 & \frac{\xi}{J}-\frac{\beta}{J}\partial_{xx} & \frac{\tau}{J} \end{pmatrix}, \tag{6.6}$$

with domain

$$D\left(\mathcal{A}\right)=\left\{\begin{array}{c} U\in\mathcal{H}: u\in H^4(0,\pi)\cap H_0^1(0,\pi), v\in H^2(0,\pi)\cap H_0^1(0,\pi), \\ \phi\in H_*^2(0,\pi)\cap H_*^1(0,\pi), \psi\in H_*^1(0,\pi), u_{xx}(0)=u_{xx}(\pi)=0 \end{array}\right\}.$$

Our well-posedness result reads as follows:

Theorem 6.1. *For any $U_0\in D(\mathcal{A})$, the problem (6.2),(6.4) has a unique weak solution $U\in C\left([0,+\infty[\,;D(\mathcal{A})\right)\cap C^1\left([0,+\infty[\,;\mathcal{H}\right).$*

Proof. We use the semigroup approach (see [67], [51]). So, we prove that $\mathcal{A}$ is a maximal monotone operator.

1) First, we prove that $\mathcal{A}$ is monotone, thus for any $U \in D(\mathcal{A})$, we have

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= -a\int_0^\pi u_{xx}vdx - b\int_0^\pi \phi_x vdx + c\int_0^\pi u_{xxxx}vdx - \delta\int_0^\pi v_{xx}vdx \\
&- a\int_0^\pi v_x u_x dx - c\int_0^\pi v_{xx}u_{xx}dx - \beta\int_0^\pi \phi_{xx}\psi dx + \xi\int_0^\pi \phi\psi dx + b\int_0^\pi u_x\psi dx \\
&+ \tau\int_0^\pi \psi\psi dx - \xi\int_0^\pi \psi\phi dx - b\int_0^\pi v_x\phi dx - b\int_0^\pi \psi u_x dx - \beta\int_0^\pi \psi_x\phi_x dx. \\
&= -a\int_0^\pi u_{xx}vdx - b\int_0^\pi \phi_x vdx + c\int_0^\pi u_{xxxx}vdx - \delta\int_0^\pi v_{xx}vdx \\
&- a\int_0^\pi v_x u_x dx - c\int_0^\pi v_{xx}u_{xx}dx - \beta\int_0^\pi \phi_{xx}\psi dx + \tau\int_0^\pi \psi\psi dx \\
&- b\int_0^\pi v_x\phi dx - \beta\int_0^\pi \psi_x\phi_x dx.
\end{aligned}$$

Using integration by parts and the boundary conditions, we obtain

$$Re\,\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \delta\int_0^\pi |v_x|^2 dx + \tau\int_0^\pi |\psi|^2\,dx \geq 0.$$

Therefore, $\mathcal{A}$ is monotone.

2) Secondly, we prove that the operator $(I + \mathcal{A})$ is surjective.

Let $F = (f^1, f^2, f^3, f^4)^T \in \mathcal{H}$ and find $U = (u, v, \phi, \psi)^T \in D(\mathcal{A})$, such that $(I + \mathcal{A})U = F$; that is,

$$\begin{cases} u - v &= f^1, \\ \rho v - au_{xx} + cu_{xxxx} - b\phi_x - \delta v_{xx} &= \rho f^2, \\ \phi - \psi &= f^3, \\ bu_x + (J+\tau)\,\psi - \beta\phi_{xx} + \xi\phi &= Jf^4. \end{cases} \tag{6.7}$$

Substituting v and ψ from $(6.7)_1$, $(6.7)_3$ into $(6.7)_2$ and $(6.7)_4$, we get

$$\begin{cases} \rho u - (a+\delta)\,u_{xx} + cu_{xxxx} - b\phi_x &= \rho\,(f^2 + f^1) - \delta f^1_{xx}, \\ bu_x + (J+\tau+\xi)\,\phi - \beta\phi_{xx} &= J\,(f^4 + f^3) + \tau f^3. \end{cases} \tag{6.8}$$

At this stage, we consider the following variational formulation

$$B\left((u,\phi), (u^*,\phi^*)\right) = L\left(u^*,\phi^*\right), \quad \forall\,(u^*,\phi^*) \in \mathcal{W}, \tag{6.9}$$

where B and L are the bilinear and linear forms defined over the Hilbert space $\mathcal{W} = \left(H^2(0,\pi) \cap H^1_0(0,\pi)\right) \times H^1_*(0,\pi)$ by

$$B\left((u,\phi), (u^*,\phi^*)\right) = \rho\int_0^\pi uu^* dx + (a+\delta)\int_0^\pi u_x u^*_x dx + c\int_0^\pi u_{xx}u^*_{xx}dx$$

$$-b\int_0^\pi \phi_x u^* dx + b\int_0^\pi u_x\phi^* dx + (J+\tau+\xi)\int_0^\pi \phi\phi^* dx + \beta\int_0^\pi \phi_x\phi_x^* dx$$

and

$$L\left((u^*,\phi^*)\right) = \int_0^\pi g_1 u^* dx + \int_0^\pi g_2\phi^* dx,$$

where

$$g_1 = \rho\left(f^2+f^1\right) - \delta f_{xx}^1 \in L^2(0,\pi) \text{ and } g_2 = Jf^4 + (J+\tau)f^3 \in L^2(0,\pi). \tag{6.10}$$

We can easily show, by using Cauchy-Schwarz inequality, that B and L are continuous. Moreover,

$$B\left((u,\phi),(u,\phi)\right) = \rho\int_0^\pi u^2 dx + (a+\delta)\int_0^\pi u_x^2 dx + c\int_0^\pi u_{xx}^2 dx$$
$$+2b\int_0^\pi u_x\phi dx + (J+\tau+\xi)\int_0^\pi \phi^2 dx + \beta\int_0^\pi \phi_x^2 dx.$$

Recalling that

$$a\int_0^\pi u_x^2 dx + \xi\int_0^\pi \phi^2 dx + 2b\int_0^\pi u_x\phi dx =$$
$$\frac{1}{2}\left(a-\frac{b^2}{\xi}\right)\int_0^\pi u_x^2 dx + \frac{1}{2}\left(\xi-\frac{b^2}{a}\right)\int_0^\pi \phi^2 dx$$
$$+\frac{a}{2}\int_0^\pi\left(u_x+\frac{b}{a}\phi\right)^2 dx + \frac{\xi}{2}\int_0^\pi\left(\phi+\frac{b}{\xi}u_x\right)^2 dx \geq 0,$$

we infer that

$$B\left((u,\phi),(u,\phi)\right) \geq \rho\int_0^\pi u^2 dx + \delta\int_0^\pi u_x^2 dx + c\int_0^\pi u_{xx}^2 dx$$
$$+(J+\tau)\int_0^\pi |\phi|^2 dx + \beta\int_0^\pi |\phi_x|^2 dx.$$

Thus, there exists $\alpha > 0$ such that

$$B\left((u,\phi),(u,\phi)\right) \geq \alpha\left(\|u_x\|^2 + \|u_{xx}\|^2 + \|\phi\|^2 + \|\phi_x\|^2\right) = \alpha\,\|(u,\phi)\|_{\mathcal{W}}^2,$$

which shows that B is coercive. Therefore, Lax-Milgram theorem guarantees the existence of a unique $(u,\phi)\in\mathcal{W}$ satisfies (6.9), for every $(u^*,\phi^*)\in\mathcal{W}$.

Back to the system (6.7), we get

$$v = u - f^1 \in H^2(0,\pi)\cap H_0^1(0,\pi),$$
$$\psi = \phi - f^3 \in H_*^1(0,\pi),$$

If we take $(u^*, \phi^*) = (0, \phi^*)$ in (6.9), we get

$$\beta \int_0^\pi \phi_x \phi_x^* dx = \int_0^\pi \left[Jf^4 + (J+\tau)f^3 - bu_x - (J+\tau+\xi)\phi \right] \phi^* dx, \quad \forall \phi^* \in H_*^1(0,\pi). \tag{6.11}$$

We cannot use the regularity theory directly here, because $\phi^* \in H_*^1(0,\pi)$. So, let $\Phi \in H_0^1(0,\pi)$ and set

$$\phi^* = \Phi - \frac{1}{\pi} \int_0^\pi \Phi(x) dx.$$

It's clear that $\phi^* \in H_*^1(0,\pi)$. Substituting ϕ^* in (6.11) leads to

$$\beta \int_0^\pi \phi_x \Phi_x dx = \int_0^\pi r \Phi dx, \quad \forall \Phi \in H_0^1(0,\pi), \tag{6.12}$$

where

$$r = Jf^4 + (J+\tau)f^3 - bu_x - (J+\tau+\xi)\phi \in L_*^2(0,\pi).$$

The regularity theory, then, implies that

$$\phi \in H^2(0,\pi)$$

and

$$-\beta\phi_{xx} = Jf^4 + (J+\tau)f^3 - bu_x - (J+\tau+\xi)\phi = r.$$

We use the first and third equation of (6.7) to get

$$bu_x + (\tau+J)\psi - \beta\phi_{xx} + \xi\phi = Jf^4.$$

On the other hand, since $-\beta\phi_{xx} = r$, then

$$-\beta \int_0^\pi \phi_{xx} \Psi dx = \int_0^\pi r \Psi dx, \quad \forall \Psi \in H^1(0,\pi).$$

Integration by parts yields

$$\beta \int_0^\pi \phi_x \Psi_x dx - \beta \phi_x \Psi \Big|_0^\pi = \int_0^\pi r \Psi dx, \quad \forall \Psi \in H^1(0,\pi).$$

As $H_*^1(0,\pi) \subset H^1(0,\pi)$, we have

$$\beta \int_0^\pi \phi_x \Psi_x dx - \beta \phi_x \Psi \Big|_0^\pi = \int_0^\pi r \Psi dx, \quad \forall \Psi \in H_*^1(0,\pi),$$

therefore, by (6.11), we obtain

$$-\beta\phi_x(\pi)\Psi(\pi)+\beta\phi_x(0)\Psi(0)=0, \qquad \forall\Psi\in H^1_*(0,\pi).$$

Since Ψ is arbitrary, then

$$\phi_x(0)=\phi_x(\pi)=0.$$

Therefore,

$$\phi\in H^2_*(0,\pi).$$

Next, taking $\phi^*=0$ in (6.9), we arrive at

$$c\int_0^\pi u_{xx}u^*_{xx}dx=\int_0^\pi\left(g_1-\rho u+(a+\delta)\,u_{xx}+b\phi_x\right)u^*dx, \qquad \forall u^*\in H^2(0,\pi)\cap H^1_0(0,\pi). \tag{6.13}$$

Consequently, the regularity theory implies that

$$u\in H^4(0,\pi)\cap H^1_0(0,\pi).$$

Furthermore, since $C^\infty_0(0,\pi)\subset H^2(0,\pi)\cap H^1_0(0,\pi)$ then, simple integration lead to

$$c\int_0^\pi u_{xxxx}u^*dx=\int_0^\pi\left(g_1-\rho u+(a+\delta)\,u_{xx}+b\phi_x\right)u^*dx, \qquad \forall u^*\in C^\infty_0(0,\pi).$$

Therefore,

$$cu_{xxxx}=g_1-\rho u+(a+\delta)\,u_{xx}+b\phi_x. \tag{6.14}$$

We use $(6.7)_1$ and (6.10) to get

$$\rho v-au_{xx}+cu_{xxxx}-b\phi_x-\delta v_{xx}=\rho f^2.$$

On the other hand, an integration by parts in (6.13), leads to

$$cu_{xx}u^*_x\Big|_0^\pi+c\int_0^\pi u_{xxxx}u^*dx=\int_0^\pi\left(g_1-\rho u+(a+\delta)\,u_{xx}+b\phi_x\right)u^*dx, \qquad \forall u^*\in H^2(0,\pi)\cap H^1_0(0,\pi),$$

we use (6.14), we obtain

$$cu_{xx}(\pi)u^*_x(\pi)-cu_{xx}(0)u^*_x(0)=0.$$

Since $u^*\in H^2(0,\pi)\cap H^1_0(0,\pi)$ is arbitrary, so

$$u_{xx}(\pi)=u_{xx}(0)=0.$$

Therefore, we deduce that $U=(u,v,\phi,\psi)^T\in D(\mathcal{A})$ and consequently, the well-posedness result follows from Theorem 6.1. $\square$

6.2.2 Exponential stability

In this subsection, we prove the exponential decay of the solution of Problem (6.2)-(6.4). Our stability result reads as follows

Theorem 6.2. *Let (u, ϕ) be the solution of (6.2)-(6.4), then, there exists two positive constants η and ω, independent of t and the initial data, such that*

$$E(t) \leq \eta e^{-\omega t},\ t \geq 0.$$

The proof of Theorem 6.2 is based on Gearhart Prüss [51], and will be established through the following two lemmas.

Lemma 6.1. *The operator $\mathcal{A}$ defined by (6.6) satisfies the following property,*

$$i\mathbb{R} = \{i\lambda, \lambda \in \mathbb{R}\} \subset \rho(\mathcal{A}).$$

Proof. First, notice that by a similar method as in the proof of Theorem 6.1, we easily check that $0 \in \rho(\mathcal{A})$. Thus, $\mathcal{A}^{-1} \in \mathcal{L}(\mathcal{H})$.

Secondly, we prove that $\mathcal{A}^{-1}$ is compact. Indeed, let (F_n) be a bounded sequence in $\mathcal{H}$ such that $\|F_n\| \leq C_2$ and (U_n) the sequence in $D(\mathcal{A})$ that satisfies $F_n = \mathcal{A}U_n$, where $U_n = (u_n, v_n, \phi_n, \psi_n)^T$. Since $\mathcal{A}^{-1} \in \mathcal{L}(\mathcal{H})$, we have $\|\mathcal{A}^{-1}\| \leq C_1$ and

$$\|U_n\|_{D(\mathcal{A})} = \|\mathcal{A}^{-1}F_n\|_{\mathcal{H}} + \|F_n\|_{\mathcal{H}} \leq (C_1 + 1)C_2.$$

Therefore $U_n = (u_n, v_n, \phi_n, \psi_n)^T$ is bounded in $D(\mathcal{A})$.

Recalling that, the injection of $H^m(0;\pi)$ into $H^j(0;\pi)$ is compact for $m > j$ then, there exists a subsequence $U_r = (u_r, v_r, \phi_r, \psi_r)^T$ and a function $U = (u, v, \phi, \psi)^T$ such that:

$$(u_r, v_r, \phi_r, \psi_r)^T \xrightarrow{\mathcal{H}} (u, v, \phi, \psi)^T,$$

which shows that the subsequence $U_r = \mathcal{A}^{-1}F_r$ converges in $\mathcal{H}$. Thus, $\mathcal{A}^{-1}$ is compact.

Suppose that there exists $\lambda \in \mathbb{R}$ $(\lambda \neq 0)$ such that $i\lambda$ is in the spectrum of $\mathcal{A}$. As $\mathcal{A}^{-1}$ is compact, $i\lambda$ must be an eigenvalue of $\mathcal{A}$. Then, there exists a vector $U = (u; v, \phi, \psi)^T \neq 0$

such that $i\lambda U + \mathcal{A}U = 0$; that is,

$$\begin{gathered} i\lambda u - v = 0, \\ i\lambda \rho v - a u_{xx} - b\phi_x + c u_{xxxx} - \delta v_{xx} = 0, \\ i\lambda \phi - \psi = 0, \\ i\lambda J\psi - \beta\phi_{xx} + \xi\phi + b u_x + \tau\psi = 0. \end{gathered} \tag{6.15}$$

First, we have

$$Re\langle (i\lambda I + \mathcal{A})U, U\rangle_{\mathcal{H}} = \delta \int_0^\pi |v_x|^2 dx + \tau \int_0^\pi |\psi|^2 \, dx = 0,$$

therefore,

$$\psi = 0, \;\; v_x = 0,$$

Poincaré's inequality leads to

$$v = 0.$$

Substituting $v = 0$ in $(6.15)_1$ and $\psi = 0$ in $(6.15)_3$, we get

$$u = 0 \;\;\text{ and }\;\; \phi = 0,$$

this is a contradiction with $U \neq 0$, and the proof is complete. $\square$

Lemma 6.2. *Let $\mathcal{A}$ be the operator defined in (6.6). Then,*

$$\overline{\lim_{|\lambda|\to\infty}} \left\| (i\lambda I + \mathcal{A})^{-1} \right\|_{\mathscr{L}(\mathcal{H})} < \infty. \tag{6.16}$$

Proof. Let $\lambda \in \mathbb{R}$ be given and let $F = (f, g, p, h)^T \in \mathcal{H}$, then, there exists a unique $U = (u, v, \phi, \psi)^T \in D(\mathcal{A})$, such that $(i\lambda I + \mathcal{A})U = F$, that is,

$$\begin{cases} i\lambda u - v = f, \\ i\lambda \rho v - a u_{xx} - b\phi_x + c u_{xxxx} - \delta v_{xx} = \rho g, \\ i\lambda \phi - \psi = p, \\ i\lambda J\psi - \beta\phi_{xx} + \xi\phi + b u_x + \tau\psi = Jh. \end{cases} \tag{6.17}$$

First, recall that

$$Re\langle (i\lambda I + \mathcal{A})U, U\rangle_{\mathcal{H}} = \delta \int_0^\pi |v_x|^2 dx + \tau \int_0^\pi |\psi|^2 \, dx = Re\langle F, U\rangle_{\mathcal{H}}.$$

Therefore,

$$\delta\int_0^\pi |v_x|^2dx+\tau\int_0^\pi |\psi|^2\,dx\leq \|F\|\,\|U\|. \tag{6.18}$$

Further, Poincaré's inequality leads to

$$\int_0^\pi |v|^2dx\leq C_p\int_0^\pi |v_x|^2dx\leq C\,\|F\|\,\|U\|. \tag{6.19}$$

Taking the L^2-product of $(6.17)_2$ by u, and $(6.17)_4$ by ϕ and, using $(6.17)_1$ and $(6.17)_3$, we obtain

$$\begin{aligned} &a\left\|u_x\right\|^2+c\left\|u_{xx}\right\|^2+\beta\left\|\phi_x\right\|^2+\xi\left\|\phi\right\|^2+b(\langle\phi,u_x\rangle+\langle u_x,\phi\rangle)\\ =\;&\rho\left\langle g,u\right\rangle+\rho\left\langle v,f\right\rangle+J\left\langle h,\phi\right\rangle+J\left\langle\psi,p\right\rangle+\rho\left\|v\right\|^2+J\left\|\psi\right\|^2-\delta\left\langle v_x,u_x\right\rangle-\tau\langle\psi,\phi\rangle. \end{aligned}$$

Cauchy-Schwarz and Young's inequalities and (6.18) entails that

$$\frac{a}{2}\|u_x\|^2+c\|u_{xx}\|^2+\beta\|\phi_x\|^2+\frac{\xi}{2}\|\phi\|^2+b\Big(\langle\phi,u_x\rangle+\langle u_x,\phi\rangle\Big)\leq C\|F\|\|U\|. \tag{6.20}$$

Adding (6.18),(6.19) and (6.20), we obtain

$$\rho\left\|v\right\|^2+\frac{a}{2}\left\|u_x\right\|^2+c\left\|u_{xx}\right\|^2+\beta\left\|\phi_x\right\|^2+\frac{\xi}{2}\left\|\phi\right\|^2+b\Big(\langle\phi,u_x\rangle+\langle u_x,\phi\rangle\Big)+\|\psi\|^2\leq C\|F\|\left\|U\right\|, \tag{6.21}$$

which gives

$$\|U\|\leq C\|F\|.$$

Thus, (6.16) is satisfied and the proof of Lemma 6.2 is completed. □

The assertion of Theorem 6.2 follows from the two above Lemmas.

6.3 Exponential Stability of a Porous thermoelastic system

The second problem, we consider in this chapter, is the following

$$\begin{cases} \rho u_{tt}=au_{xx}+b\phi_x-cu_{xxxx}-\gamma\theta_x, & \text{in } (0,\pi)\times(0,\infty),\\ J\phi_{tt}=\beta\phi_{xx}-\xi\phi-bu_x+m\theta-\tau\phi_t, & \text{in } (0,\pi)\times(0,\infty),\\ c^*\theta_t=\kappa\theta_{xx}-\gamma u_{xt}-m\phi_t, & \text{in } (0,\pi)\times(0,\infty). \end{cases} \tag{6.22}$$

Here, θ represents the temperature variation from an equilibrium reference value. The unknown functions are subjected to the following boundary and the initial conditions

$$u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = \phi_x(0,t) = \phi_x(\pi,t) = \theta_x(0,t) = \theta_x(\pi,t) = 0, \tag{6.23}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x),\ \theta(x,0) = \theta_0(x). \tag{6.24}$$

The coefficients of the system are assumed to satisfy $\gamma, m, b \neq 0$ and

$$\rho > 0,\ a > 0,\ c > 0,\ c^* > 0,\ J > 0,\ \beta > 0,\ \xi > 0,\ \kappa > 0,\ \tau > 0,\ a\xi > b^2.$$

Here, we consider the Hilbert space

$$\mathcal{H} := \Big(H^2(0,\pi) \cap H_0^1(0,\pi)\Big) \times L^2(0,\pi) \times H_*^1(0,\pi) \times L_*^2(0,\pi) \times L_*^2(0,\pi),$$

with the inner product

$$\langle U, U^* \rangle_{\mathcal{H}} = \int_0^\pi \left[\rho v v^* + a u_x u_x^* + c u_{xx} u_{xx}^* + J\psi\psi^* + \beta\phi_x\phi_x^* + \xi\phi\phi^* + c^*\theta\theta^* + b\left(u_x\phi^* + \phi u_x^*\right)\right] dx,$$

where $U = (u, v, \phi, \psi, \theta)^T$, $U^* = (u^*, v^*, \phi^*, \psi^*, \theta^*)^T$. The energy associated to the solution of (6.22)-(6.24) is given by

$$E(t) := \frac{1}{2}\int_0^\pi \Big[\rho u_t^2 + a u_x^2 + c u_{xx}^2 + J\phi_t^2 + \xi\phi^2 + \beta\phi_x^2 + c^*\theta^2 + 2b u_x \phi\Big] dx.$$

We have the following estimate

$$\frac{dE}{dt}(t) = -\tau \int_0^\pi \phi_t^2 dx - \kappa \int_0^\pi \theta_x^2 dx.$$

Before stating the stability result let us rewrite problem (6.22)-(6.24) in the setting of an abstract Cauchy problem. First, we introduce the new variables $v = u_t$ and $\psi = \phi_t$, then problem (6.22)-(6.24) can be written as the abstract Cauchy problem,

$$\begin{cases} U_t = \mathcal{B}U,\ t > 0, \\ U(0) = (u_0, u_1, \phi_0, \phi_1, \theta_0)^T, \end{cases}$$

where $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is the operator defined by

$$\mathcal{B} = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ \frac{a}{\rho}\partial_{xx} - \frac{c}{\rho}\partial_{xxxx} & 0 & \frac{b}{\rho}\partial_x & 0 & -\frac{\gamma}{\rho}\partial_x \\ 0 & 0 & 0 & I & 0 \\ -\frac{b}{J}\partial_x & 0 & \frac{\beta}{J}\partial_{xx} - \frac{\xi}{J} & -\frac{\tau}{J} & \frac{m}{J} \\ 0 & -\frac{\gamma}{c^*}\partial_x & 0 & -\frac{m}{c^*} & \frac{\kappa}{c^*}\partial_{xx} \end{pmatrix}, \tag{6.25}$$

with domain

$$D(\mathcal{B}) := \Big\{ U \in \mathcal{H} : u \in H^4(0,\pi), v \in H^2(0,\pi) \cap H_0^1(0,\pi), \phi, \theta \in H_*^2(0,\pi), \psi \in H_*^1(0,\pi) \Big\}.$$

Our well-posedness result is the following theorem.

Theorem 6.3. *For any $U_0 \in D(\mathcal{B})$, the problem (6.22)-(6.24) has a unique solution $U \in C([0,+\infty[\,; D(\mathcal{B})) \cap C^1([0,+\infty[\,; \mathcal{H})$.*

The proof of the Theorem 6.3 is familiar, we will omit it for simplicity.

Our stability result reads as follows:

Theorem 6.4. *Let (u, ϕ, θ) be the solution of (6.22)-(6.24), then, there exist two positive constants η and ω, independent of t and the initial data, such that*

$$E(t) \leq \eta e^{-\omega t},\, t > 0.$$

The proof of Theorem 6.4 will be done again, by the use of Theorem 1.14.
First, we have

Lemma 6.3. *The operator $\mathcal{B}$ defined by (6.25) satisfies the following property*

$$i\mathbb{R} = \{i\lambda, \lambda \in \mathbb{R}\} \subset \rho(\mathcal{B}).$$

Proof. We can show easily that $0 \in \rho(\mathcal{B})$ the resolvent set of $\mathcal{B}$, then $\mathcal{B}^{-1} \in \mathcal{L}(\mathcal{H})$ and $\|\mathcal{B}^{-1}\| \leq C_1$.

Next, we prove that $\mathcal{B}^{-1}$ is compact. Indeed, let (F_n) be a bounded sequence in $\mathcal{H}$ and (U_n) the sequence in $D(\mathcal{B})$ such that, $F_n = \mathcal{B}U_n$, where $U_n = (u_n, v_n, \phi_n, \psi_n, \theta_n)^T$. Thus,

$$\|U_n\|_{D(\mathcal{B})} = \|U_n\|_{\mathcal{H}} + \|\mathcal{B}U_n\|_{\mathcal{H}} = \|\mathcal{B}^{-1}F_n\| + \|F_n\| \leq C,$$

for a positive constant C. Therefore $U_n = (u_n, v_n, \phi_n, \psi_n, \theta_n)^T$ is bounded in $D(\mathcal{B})$.

Since the injection $H^m(0,\pi) \hookrightarrow H^j(0,\pi)$ is compact for $m > j$ then, there exists a subsequence $(U_r) = (u_r, v_r, \phi_r, \psi_r, \theta_r)^T = (\mathcal{B}^{-1}F_r)$ and a vector $U = (u, v, \phi, \psi, \theta)^T$ such that:

$$(u_r, v_r, \phi_r, \psi_r, \theta_r)^T \xrightarrow{\mathcal{H}} (u, v, \phi, \psi, \theta)^T,$$

which shows that $\mathcal{B}^{-1}$ is compact.

Suppose that there exists $\lambda \in \mathbb{R}$ ($\lambda \neq 0$) such that $i\lambda$ is in the spectrum of $\mathcal{B}$. Since $\mathcal{B}^{-1}$ is compact, $i\lambda$ must be an eigenvalue of $\mathcal{B}$.
Therefore, there exists a vector $U = (u; v, \phi, \psi, \theta)^T \neq 0$ such that $i\lambda U = \mathcal{B}U$; that is,

$$\begin{aligned}
i\lambda u - v &= 0,\\
i\lambda\rho v - a u_{xx} - b\phi_x + c u_{xxxx} + \gamma\theta_x &= 0,\\
i\lambda\phi - \psi &= 0,\\
i\lambda J\psi - \beta\phi_{xx} + \xi\phi + b u_x - m\theta + \tau\psi &= 0,\\
i\lambda c^*\theta - \kappa\theta_{xx} + \delta v_x + m\psi &= 0.
\end{aligned} \tag{6.26}$$

We have

$$Re\langle (i\lambda I - \mathcal{B})U, U\rangle_{\mathcal{H}} = \tau\int_0^\pi \psi^2 dx + \gamma\int_0^\pi \theta_x^2 dx = 0.$$

therefore

$$\psi = 0 \quad \text{and} \quad \theta_x = 0, \tag{6.27}$$

Poincaré's inequality leads to

$$\theta = 0.$$

Substituting $\theta = 0$ and $\psi = 0$ in $(6.26)_5$, we infer that

$$-\kappa\theta_{xx} + \delta v_x = 0.$$

Integration over $(0, x)$ and (6.27) yield

$$v = 0.$$

Plugging $v = 0$ in $(6.26)_1$ and $\psi = 0$ in $(6.26)_3$, we get

$$u = 0 \quad \text{and} \quad \phi = 0.$$

Thus, $U = 0$, which contradicts the fact that U is an eigenvector, and then the proof of Lemma 6.3 is complete. □

Secondly, we have

Lemma 6.4. *Let $\mathcal{B}$ be the operator defined in (6.25). Then*

$$\overline{\lim_{|\lambda|\to\infty}} \left\|(i\lambda I - \mathcal{B})^{-1}\right\|_{\mathscr{L}(\mathcal{H})} < \infty. \tag{6.28}$$

Proof. Let $\lambda \in \mathbb{R}$ be given and let $F = (f, g, p, h, q)^T \in \mathcal{H}$, then there exists a unique $U = (u, v, \phi, \psi, \theta)^T \in D(\mathcal{B})$, such that $(i\lambda I - \mathcal{B})U = F$; that is,

$$\begin{cases} i\lambda u - v = f, \\ i\lambda\rho v - au_{xx} - b\phi_x + cu_{xxxx} + \gamma\theta_x = \rho g, \\ i\lambda\phi - \psi = p, \\ i\lambda J\psi - \beta\phi_{xx} + \xi\phi + bu_x - m\theta + \tau\psi = Jh, \\ i\lambda c^*\theta - \kappa\theta_{xx} + \delta v_x + m\psi = c^* q. \end{cases} \tag{6.29}$$

Recall that

$$Re\langle (i\lambda I - \mathcal{B})U, U\rangle_{\mathcal{H}} = \tau \int_0^\pi \psi^2 dx + \kappa \int_0^\pi \theta_x^2 dx = Re\langle F, U\rangle_{\mathcal{H}}.$$

Therefore,

$$\tau \int_0^\pi \psi^2 dx + \kappa \int_0^\pi \theta_x^2 dx \leq \|F\| \|U\|. \tag{6.30}$$

Poincaré's inequality leads to

$$\tau \int_0^\pi \psi^2 dx + \kappa \int_0^\pi \theta^2 dx \leq C \|F\| \|U\|. \tag{6.31}$$

Taking the L^2-product of $(6.29)_2$ by u, $(6.29)_4$ by ϕ, then, adding the obtained equations and using $(6.29)_1$,$(6.29)_3$, we arrive at

$$\begin{aligned} &a\|u_x\|^2 + c\|u_{xx}\|^2 + \beta\|\phi_x\|^2 + \xi\|\phi\|^2 + b(\langle\phi, u_x\rangle + \langle u_x, \phi\rangle) \\ &= \rho\langle g, u\rangle + \rho\langle v, f\rangle + J\langle h, \phi\rangle + J\langle\psi, p\rangle + \rho\|v\|^2 + J\|\psi\|^2 + \gamma\langle\theta, u_x\rangle + m\langle\theta, \phi\rangle - \tau\langle\psi, \phi\rangle. \end{aligned}$$

Cauchy-Schwarz and Young's inequalities and (6.30) lead to

$$\frac{a}{2}\|u_x\|^2 + c\|u_{xx}\|^2 + \beta\|\phi_x\|^2 + \frac{\xi}{2}\|\phi\|^2 + b\Big(\langle\phi, u_x\rangle + \langle u_x, \phi\rangle\Big) \leq \rho\|v\|^2 + C\|F\|\|U\|. \tag{6.32}$$

At this point, we estimate $\|v\|_{L^2}$. For this, we define the functions φ, w, z and y as solutions of the following problems:

$$-\varphi_{xx} = v, \qquad \varphi_x(0) = \varphi_x(\pi) = 0, \tag{6.33}$$

$$-w_{xx} = \theta, \qquad w_x(0) = w_x(\pi) = 0, \tag{6.34}$$

$$-z_{xx} = q, \qquad z(0) = z(\pi) = 0, \tag{6.35}$$

$$-y_{xx} = g, \qquad y(0) = y(\pi) = 0. \tag{6.36}$$

Taking the L^2-inner product of (6.35) by z and (6.36) by y, using integration by parts, the Poincaré and Cauchy-Schwarz inequalities, we get

$$\|z_x\| \le C_p \|q\| \quad \text{and} \quad \|y_x\| \le C_p \|g\|. \tag{6.37}$$

Multiplying $(6.29)_5$ by φ_x in $L^2(0,\pi)$, we obtain

$$\langle i\lambda c^*\theta, \varphi_x\rangle - \kappa \langle \theta_{xx}, \varphi_x\rangle + \delta \langle v_x, \varphi_x\rangle + m\langle \psi, \varphi_x\rangle = c^*\langle q, \varphi_x\rangle. \tag{6.38}$$

Now we estimate (6.38) term by term.

First, integrating by parts, using (6.33) and (6.34), we infer that

$$I_1 = \langle i\lambda c^*\theta, \varphi_x\rangle = c^*\langle i\lambda w_x, \varphi_{xx}\rangle = c^*\langle w_x, i\lambda v\rangle.$$

Replacing $i\lambda v$ from $(6.29)_2$ and using (6.36), we obtain

$$I_1 = \frac{ac^*}{\rho}\langle w_x, u_{xx}\rangle + \frac{bc^*}{\rho}\langle w_x, \phi_x\rangle - \frac{cc^*}{\rho}\langle w_x, u_{xxxx}\rangle - \frac{\gamma c^*}{\rho}\langle w_x, \theta_x\rangle - c^*\langle w_x, y_{xx}\rangle.$$

Integration by parts and (6.34) lead to

$$I_1 = \frac{ac^*}{\rho}\langle \theta, u_x\rangle + \frac{bc^*}{\rho}\langle \theta, \phi\rangle + \frac{cc^*}{\rho}\langle \theta_x, u_{xx}\rangle - \frac{\gamma c^*}{\rho}\|\theta\|^2 - c^*\langle \theta, y_x\rangle. \tag{6.39}$$

Similarly, we get

$$I_2 = -\kappa\langle \theta_{xx}, \varphi_x\rangle = \kappa\langle \theta_x, \varphi_{xx}\rangle = -\kappa\langle \theta_x, v\rangle, \tag{6.40}$$

$$I_3 = \delta\langle v_x, \varphi_x\rangle = -\delta\langle v, \varphi_{xx}\rangle = \delta\|v\|^2 \tag{6.41}$$

and

$$I_4 = c^*\langle q, \varphi_x\rangle = -c^*\langle z_{xx}, \varphi_x\rangle = c^*\langle z_x, \varphi_{xx}\rangle = -c^*\langle z_x, v\rangle. \tag{6.42}$$

Substituting (6.39)-(6.42) into (6.38), we get

$$\delta \|v\|^2 = -\frac{ac^*}{\rho} \langle \theta, u_x \rangle - \frac{bc^*}{\rho} \langle \theta, \phi \rangle - \frac{cc^*}{\rho} \langle \theta_x, u_{xx} \rangle + \frac{\gamma c^*}{\rho} \|\theta\|^2$$
$$+ c^* \langle \theta, y_x \rangle + \kappa \langle \theta_x, v \rangle - m\langle \psi, \varphi_x \rangle - c^* \langle z_x, v \rangle . \tag{6.43}$$

Cauchy-Schwarz and Young inequalities, (6.30) and (6.31) yield

$$\frac{ac^*}{\rho} |\langle \theta, u_x \rangle| \leq \frac{a\delta}{16\rho} \|u_x\|^2 + C\|F\|\|U\|, \tag{6.44}$$
$$\frac{bc^*}{\rho} |\langle \theta, \phi \rangle| \leq \frac{\xi\delta}{16\rho} \|\phi\|^2 + C\|F\|\|U\| \tag{6.45}$$

and

$$\frac{cc^*}{\rho} |\langle \theta_x, u_{xx} \rangle| \leq \frac{c\delta}{8\rho} \|u_{xx}\|^2 + C\|F\|\|U\|. \tag{6.46}$$

Similarly, we have

$$c^* \Big(|\langle \theta, y_x \rangle| + |\langle z_x, v \rangle| \Big) \leq c^* C_p \Big(\|\theta\|\|g\| + \|q\|\|v\| \Big) \leq C\|F\|\|U\| \tag{6.47}$$

and

$$|\kappa \langle \theta_x, v \rangle| \leq \frac{\delta}{4} \|v\|^2 + C \|F\| \|U\| . \tag{6.48}$$

Finally, we arrive at

$$m |\langle \psi, \varphi_x \rangle| \leq \varepsilon \|\varphi_x\|^2 + \frac{m^2}{4\varepsilon} \|\psi\|^2 \leq \varepsilon C_p \|\varphi_{xx}\|^2 + \frac{m^2}{4\varepsilon} \|\psi\|^2$$
$$\leq \frac{\delta}{4} \|v\|^2 + \frac{m^2 C_p}{\delta} \|\psi\|^2 \leq \frac{\delta}{4} \|v\|^2 + C\|F\|\|U\| \tag{6.49}$$

Substituting (6.44)-(6.49) into (6.43), we get

$$\frac{\delta}{2} \|v\|^2 \leq \frac{a\delta}{16\rho} \|u_x\|^2 + \frac{c\delta}{8\rho} \|u_{xx}\|^2 + \frac{\xi\delta}{16\rho} \|\phi\|^2 + C \|F\| \|U\| .$$

Thus,

$$2\rho \|v\|^2 \leq \frac{a}{4} \|u_x\|^2 + \frac{c}{2} \|u_{xx}\|^2 + \frac{\xi}{4} \|\phi\|^2 + C \|F\| \|U\| . \tag{6.50}$$

The addition of (6.31), (6.32) and (6.50) yield

$$\frac{1}{4} \|U\|^2 \leq \frac{a}{4} \|u_x\|^2 + \frac{c}{2} \|u_{xx}\|^2 + \beta \|\phi_x\|^2 + \frac{\xi}{4} \|\phi\|^2 + b\Big(\langle \phi, u_x \rangle + \langle u_x, \phi \rangle \Big) + \rho \|v\|^2 + \tau \|\psi\|^2 + \kappa \|\theta\|^2$$
$$\leq C\|F\|\|U\|$$

which yields

$$\|U\| \leq C \|F\| .$$

Thus, (6.28) is satisfied and the proof of Lemma 6.4 is completed. □

The assertion of Theorem 6.4 is a consequence of Lemmas 6.3, 6.4.

chapter 7

Non-Simple porous thermoelasticity systems with second sound

7.1 Introduction

The results of this chapter were published in [2]

In this chapter, we consider a non-simple porous elastic bar with thermoelasticity of second sound. The basic equations in the one-dimensional theory are given by

$$\begin{cases} \rho u_{tt} = a u_{xx} + b\phi_x - c u_{xxxx} - d\phi_{xxx} - \delta\theta_x, & \text{in } (0,\pi)\times(0,\infty), \\ J\phi_{tt} = d u_{xxx} + \beta\phi_{xx} - \xi\phi - b u_x + m\theta - \mu\phi_t, & \text{in } (0,\pi)\times(0,\infty), \\ c^*\theta_t = -q_x - \delta u_{xt} - m\phi_t, & \text{in } (0,\pi)\times(0,\infty), \\ \tau q_t + q + \kappa\theta_x = 0, & \text{in } (0,\pi)\times(0,\infty), \end{cases} \tag{7.1}$$

where u, ϕ, θ and q are, respectively, the transversal displacement, the volume fraction, the difference of temperature from an equilibrium reference value and the heat flux of an elastic bar of length π. The coefficients $\rho, J, a, c, c^*, \beta, \xi, \tau, \mu$ and κ are positive constitutive constants. The coefficients b, d, m and δ are the coupling constants that are different from zero but their signs do not matter in the analysis. In addition, to ensure that the energy functional associated to the system (7.1) is a positive definite form, we assume that the coefficients a, b, c, d, β and ξ satisfy the inequalities

$$a\xi > b^2 \text{ and } c\beta > d^2.$$

In addition, to make the problem given by the system (7.1) well determined, we impose the following boundary and initial conditions

$$\begin{gathered} u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0,\ t \geq 0, \\ \phi_x(0,t) = \phi_x(\pi,t) = q(0,t) = q(\pi,t) = 0,\ t \geq 0, \end{gathered} \tag{7.2}$$

and

$$\begin{cases} u(x,0)=u_0(x), u_t(x,0)=v_0(x), \phi(x,0)=\phi_0(x), & \text{in } \ x\in(0,\pi), \\ \phi_t(x,0)=\varphi_0(x), \theta(x,0)=\theta_0(x), q(x,0)=q_0(x), & \text{in } x\in(0,\pi). \end{cases} \tag{7.3}$$

Here, we will study the problem (7.1) in two case, $\mu \neq 0$ and $\mu = 0$. In both cases, we use the semigroup approach and prove the existence of a unique solution to (7.1). In addition, we use Gearhart Prüss theorem to prove the exponential stability of the solution in the first case and the lack exponential stability in the second case.

First, let us deal with the case where the porous dissipation is present, that is $\mu \neq 0$. Note that since the Neumann boundary conditions are assumed for ϕ and θ, we are not able to apply Poincaré's inequality for them. To allow the application of the aforementioned inequality, we proceed as follows.

Integrating $(7.1)_2$ and $(7.1)_3$ with respect to x and taking into account the boundary conditions, we obtain

$$\frac{d^2}{dt^2}\int_0^\pi \phi(x,t)\,dx = -\frac{\xi}{J}\int_0^\pi \phi(x,t)\,dx + \frac{m}{J}\int_0^\pi \theta(x,t)\,dx - \frac{\mu}{J}\frac{d}{dt}\int_0^\pi \phi(x,t)\,dx \tag{7.4}$$

and

$$c^*\frac{d}{dt}\int_0^\pi \theta(x,t)dx = -m\frac{d}{dt}\int_0^\pi \phi(x,t)dx. \tag{7.5}$$

Solving (7.5), we get

$$\int_0^\pi \theta(x,t)\,dx = -\frac{m}{c^*}\int_0^\pi \phi(x,t)\,dx + C. \tag{7.6}$$

The use of initial conditions leads to

$$C = \int_0^\pi \theta_0(x)\,dx + \frac{m}{c^*}\int_0^\pi \phi_0(x)\,dx.$$

Plugging (7.6) into (7.4), we infer

$$\frac{d^2}{dt^2}\int_0^\pi \phi(x,t)\,dx + \frac{\mu}{J}\frac{d}{dt}\int_0^\pi \phi(x,t)\,dx + \left(\frac{\xi}{J} + \frac{m^2}{Jc^*}\right)\int_0^\pi \phi(x,t)\,dx = \frac{m}{J}C. \tag{7.7}$$

The solution of (7.7) is then

$$\int_0^\pi \phi(x,t)\,dx = y_H(t) + y_P(t),$$

where y_P and y_H are, respectively, a particular solution of (7.7) and the general solution of the corresponding homogeneous equation of (7.7). Therefore,

$$\int_0^\pi \theta(t,x)\,dx = -\frac{m}{c^*}\left(y_H(t) + y_P(t)\right) + C.$$

Thus, if we set

$$\widetilde{\phi}(x,t) = \phi(x,t) - \frac{1}{\pi}\Big(y_H(t) + y_P(t)\Big)$$

and

$$\widetilde{\theta}(x,t) = \theta(x,t) + \frac{m}{c^*\pi}\Big(y_H(t) + y_P(t)\Big) - \frac{C}{\pi},$$

we reach

$$\int_0^\pi \widetilde{\phi}(x,t)\,dx = 0 \quad \text{and} \quad \int_0^\pi \widetilde{\theta}(x,t)\,dx = 0,$$

which allows the application of Poincaré's inequality on $\widetilde{\phi}$ and $\widetilde{\theta}$. Furthermore, $(u, \widetilde{\phi}, \widetilde{\theta}, q)$ satisfies the system (7.1) with the boundary conditions (7.3). In the sequel, we work with $(u, \widetilde{\phi}, \widetilde{\theta}, q)$, but for simplicity, we write (u, ϕ, θ, q).

7.2 Well-posedness

In this section, we use the semigroup approach to prove the existence and the uniqueness of a solution to the problem (7.1)-(7.3). First, we consider the Hilbert space

$$\mathcal{H} := \Big(H^2(0,\pi) \cap H_0^1(0,\pi)\Big) \times L^2(0,\pi) \times H_*^1(0,\pi) \times L_*^2(0,\pi) \times L_*^2(0,\pi) \times L^2(0,\pi).$$

The Hilbert space $\mathcal{H}$ is equipped with an inner product

$$\begin{aligned}\langle U, U^* \rangle_{\mathcal{H}} &= \int_0^\pi \Big[\rho v v^* + a u_x u_x^* + c u_{xx} u_{xx}^* + J\psi\psi^* + \beta\phi_x\phi_x^* + \xi\phi\phi^* + c^*\theta\theta^*\Big] dx \\ &\quad + \int_0^\pi \Big[\frac{\tau}{\kappa} q q^* + b\,(u_x\phi^* + \phi u_x^*) + d\,(u_{xx}\phi_x^* + \phi_x u_{xx}^*)\Big]\, dx, \end{aligned} \tag{7.8}$$

where $U = (u, v, \phi, \psi, \theta, q)^T$ and $U^* = (u^*, v^*, \phi^*, \psi^*, \theta^*, q^*)^T$.

Remark 7.1. *Under the hypothesis $a\xi > b^2$ and $c\beta > d^2$, it's easy to check that (7.8) defines the inner product.*

In fact; since $a\xi > b^2$ and $c\beta > d^2$, we obtain

$$\int_0^\pi \Big[a u_x^2 + \xi\phi^2 + 2b u_x\phi\Big] dx = \int_0^\pi \Big(\sqrt{a}\,u_x + \frac{b}{\sqrt{a}}\phi\Big)^2 dx + \Big(\xi - \frac{b^2}{a}\Big)\int_0^\pi \phi^2 dx$$

and

$$\int_0^\pi \Big[c u_{xx}^2 + \beta\phi_x^2 + 2d u_{xx}\phi_x\Big] dx = \int_0^\pi \Big(\sqrt{c}\,u_{xx} + \frac{d}{\sqrt{c}}\phi_x\Big)^2 dx + \Big(\beta - \frac{d^2}{c}\Big)\int_0^\pi \phi_x^2 dx.$$

Therefore,

$$\begin{aligned}\|U\|_{\mathcal{H}}^2 = \langle U, U\rangle_{\mathcal{H}} =&\rho \int_0^\pi v^2 dx + \left(\xi - \frac{b^2}{a}\right) \int_0^\pi \phi^2 dx + \left(\beta - \frac{d^2}{c}\right) \int_0^\pi \phi_x^2 dx + J \int_0^\pi \psi^2 dx \\ &+ c^* \int_0^\pi \theta^2 dx + \frac{\tau}{\kappa} \int_0^\pi q^2 dx + \int_0^\pi \left(\sqrt{a} u_x + \frac{b}{\sqrt{a}}\phi\right)^2 dx \\ &+ \int_0^\pi \left(\sqrt{c} u_{xx} + \frac{d}{\sqrt{c}}\phi_x\right)^2 dx.\end{aligned}$$

Thus, we deduce that $\langle U, U^*\rangle_{\mathcal{H}}$ defines an "equivalent" inner product over $\mathcal{H}$.

Now, we introduce the new variables $v = u_t$ and $\psi = \phi_t$. Then, the system (7.1) becomes

$$\begin{cases} u_t = v, \\ v_t = \dfrac{1}{\rho}(a u_{xx} + b\phi_x - c u_{xxxx} - d\phi_{xxx} - \delta\theta_x), \\ \phi_t = \psi, \\ \psi_t = \dfrac{1}{J}(d u_{xxx} + \beta\phi_{xx} - \xi\phi - b u_x + m\theta - \mu\psi), \\ \theta_t = -\dfrac{1}{c^*}(q_x + \delta v_x + m\psi), \\ q_t = -\dfrac{1}{\tau}(q + \kappa\theta_x), \end{cases}$$

and, with the initial conditions (7.3), it can be written in the form

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0 = (u_0, u_1, \phi_0, \phi_1, \theta_0, q_0)^T, \end{cases} \tag{7.9}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is the operator given by

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{a}{\rho}\partial_{xx} - \frac{c}{\rho}\partial_{xxxx} & 0 & \frac{b}{\rho}\partial_x - \frac{d}{\rho}\partial_{xxx} & 0 & -\frac{\delta}{\rho}\partial_x & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{d}{J}\partial_{xxx} - \frac{b}{J}\partial_x & 0 & \frac{\beta}{J}\partial_{xx} - \frac{\xi}{J} & -\frac{\mu}{J} & \frac{m}{J} & 0 \\ 0 & -\frac{\delta}{c^*}\partial_x & 0 & -\frac{m}{c^*} & 0 & -\frac{1}{c^*}\partial_x \\ 0 & 0 & 0 & 0 & -\frac{\kappa}{\tau}\partial_x & -\frac{1}{\tau} \end{pmatrix}, \tag{7.10}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U \in \mathcal{H} : v \in H^2(0,\pi) \cap H_0^1(0,\pi),\ \phi \in H^2(0,\pi) \cap L_*^2(0,\pi),\ \psi, \theta \in H_*^1(0,\pi), \\ q \in H_0^1(0,\pi),\ (c u_x + d\phi) \in H^3(0,\pi) \cap L_*^2(0,\pi),\ (d u_x + \beta\phi)_x \in H_0^1(0,\pi) \end{array} \right\},$$

The well-posedness of the problem (7.1)-(7.3) is given by the following theorem.

Theorem 7.1. *For any $U_0 = (u_0, u_1, \phi_0, \phi_1, \theta_0, q_0) \in D(\mathcal{A})$, there exists a unique solution $U \in C([0, +\infty[\,; D(\mathcal{A})) \cap C^1([0, +\infty[\,; \mathcal{H})$ of the problem* (7.1)-(7.3).

The proof of Theorem 7.1 will be established through the following lemma.

Lemma 7.1. *The operator $\mathcal{A}$ defined by* (7.10) *is the infinitesimal generator of a C_0-semi-group of contractions on $\mathcal{H}$.*

Proof. By virtue of Theorem 1.11, we just need to prove that $\mathcal{A}$ is dissipative and maximal.

First, a simple computation implies that, for any $U \in D(\mathcal{A})$,

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = {} & a\int_0^\pi u_{xx} v dx + b\int_0^\pi \phi_x v dx - c\int_0^\pi u_{xxxx} v dx - d\int_0^\pi \phi_{xxx} v dx - \delta\int_0^\pi \theta_x v dx \\
& + a\int_0^\pi v_x u_x dx + c\int_0^\pi v_{xx} u_{xx} dx + d\int_0^\pi u_{xxx}\psi dx + \beta\int_0^\pi \phi_{xx}\psi dx - \xi\int_0^\pi \phi\psi dx \\
& - b\int_0^\pi u_x\psi dx + m\int_0^\pi \theta\psi dx - \mu\int_0^L \psi^2 dx + \beta\int_0^\pi \psi_x\phi_x dx + \xi\int_0^\pi \psi\phi dx - \int_0^\pi q_x\theta dx \\
& - \delta\int_0^\pi v_x\theta dx - m\int_0^\pi \psi\theta dx - \frac{1}{\kappa}\int_0^\pi q^2 dx - \int_0^\pi \theta_x q dx + b\int_0^\pi v_x\phi dx + b\int_0^\pi \psi u_x dx \\
& + d\int_0^\pi v_{xx}\phi_x dx + d\int_0^\pi \psi_x u_{xx} dx
\end{aligned}$$

$$\begin{aligned}
= {} & a\int_0^\pi u_{xx} v dx + b\int_0^\pi \phi_x v dx - c\int_0^\pi u_{xxxx} v dx - d\int_0^\pi \phi_{xxx} v dx - \delta\int_0^\pi \theta_x v dx \\
& + a\int_0^\pi v_x u_x dx + c\int_0^\pi v_{xx} u_{xx} dx + d\int_0^\pi u_{xxx}\psi dx + \beta\int_0^\pi \phi_{xx}\psi dx - \mu\int_0^L \psi^2 dx \\
& + \beta\int_0^\pi \psi_x\phi_x dx - \int_0^\pi q_x\theta dx - \delta\int_0^\pi v_x\theta dx - \frac{1}{\kappa}\int_0^\pi q^2 dx - \int_0^\pi \theta_x q dx \\
& + b\int_0^\pi v_x\phi dx + d\int_0^\pi v_{xx}\phi_x dx + d\int_0^\pi \psi_x u_{xx} dx.
\end{aligned}$$

Using integration by parts and the boundary conditions, we get

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\frac{1}{\kappa}\int_0^\pi q^2 dx - \mu\int_0^\pi \psi^2 dx \le 0,$$

Therefore, $\mathcal{A}$ is dissipative.

Next, we prove that $0 \in \rho(\mathcal{A})$. Let $F = (f^1, f^2, f^3, f^4, f^5, f^6)^T \in \mathcal{H}$ be given, and find

$U = (u, v, \phi, \psi, \theta, q)^T \in D(\mathcal{A})$ such that $\mathcal{A}U = F$; that is,

$$\begin{cases} v = f^1, \\ au_{xx} + b\phi_x - cu_{xxxx} - d\phi_{xxx} - \delta\theta_x = \rho f^2, \\ \psi = f^3, \\ du_{xxx} + \beta\phi_{xx} - \xi\phi - bu_x + m\theta - \mu\psi = Jf^4, \\ q_x + \delta v_x + m\psi = -c^* f^5, \\ q + \kappa\theta_x = -\tau f^6. \end{cases} \tag{7.11}$$

From $(7.11)_1$ and $(7.11)_3$, we have

$$v = f^1 \in H^2(0,\pi) \cap H_0^1(0,\pi), \tag{7.12}$$

and

$$\psi = f^3 \in H_*^1(0,\pi). \tag{7.13}$$

The fifth equation of (7.11) leads to

$$q_x = -c^* f^5 - \delta f_x^1 - mf^3 \in L^2(0,\pi). \tag{7.14}$$

Simple integration yields

$$q(x) = -c^\star \int_0^x f^5(s)ds - \delta f^1(x) - m \int_0^x f^3(s)ds + c_0 \in H^1(0,\pi).$$

It suffices to take $c_0 = 0$ and recall (7.12) to get $q(0) = 0$ and to recall (7.12), (7.13) and that $f^5 \in L_*^2(0,\pi)$, tu see that

$$q(\pi) = -c^\star \int_0^\pi f^5(s)ds - \delta f^1(\pi) - m \int_0^\pi f^3(s)ds = 0.$$

Hence,

$$q \in H_0^1(0,\pi). \tag{7.15}$$

On the other hand, plugging q in the sixth equation of (7.11), we get

$$\theta_x = -\frac{1}{\kappa}(\tau f^6 + q) \in L^2(0,\pi).$$

Consequently, we conclude that

$$\theta \in H^1(0,\pi).$$

By substituting the values of ψ and θ in $(7.11)_2$ and $(7.11)_4$, we get

$$\begin{cases} au_{xx} + b\phi_x - cu_{xxxx} - d\phi_{xxx} = g_1, \\ du_{xxx} + \beta\phi_{xx} - \xi\phi - bu_x = g_2, \end{cases} \tag{7.16}$$

where

$$g_1 = \rho f^2 + \delta\theta_x \in L^2(0,\pi) \quad \text{and} \quad g_2 = Jf^4 - m\theta + \mu f^3 \in L^2(0,\pi).$$

To prove the solvability of (7.16), we consider the variational problem:

Find (u,ϕ) in $\mathcal{W}$, such that

$$B\left((u,\phi),(u^*,\phi^*)\right) = L\left((u^*,\phi^*)^T\right), \qquad \forall (u^*,\phi^*) \in \mathcal{W}, \tag{7.17}$$

where $\mathcal{W} = ((H^2(0,\pi) \cap H_0^1(0,\pi)) \times H_*^1(0,\pi)$, B is a bilinear form and L is a linear form on $\mathcal{W}$ given by

$$\begin{aligned} B\left((u,\phi),(u^*,\phi^*)\right) &= c\int_0^\pi u_{xx}u_{xx}^* dx + a\int_0^\pi u_x u_x^* dx + \beta\int_0^\pi \phi_x\phi_x^* dx \\ &+ b\int_0^\pi \left(\phi u_x^* + u_x\phi^*\right) dx + d\int_0^\pi \left(\phi_x u_{xx}^* + u_{xx}\phi_x^*\right) dx \\ &+ \xi\int_0^\pi \phi\phi^* dx \end{aligned}$$

and

$$L\left((u^*,\phi^*)^T\right) = \int_0^\pi \left(\rho f^2 + \delta\theta_x\right) u^* dx + \int_0^\pi \left(Jf^4 - m\theta + \mu f^3\right)\phi^* dx.$$

Clearly, $B(\cdot,\cdot)$ and $L(\cdot)$ are continuous forms on $\mathcal{W}$.

In fact:

1. Continuity of $B(\cdot,\cdot)$:

$$\begin{aligned} \Big| B\left((u,\phi),(u^*,\phi^*)\right) \Big| &\le |c|\int_0^\pi |u_{xx}u_{xx}^*| dx + |a|\int_0^\pi |u_x u_x^*| dx + |\beta|\int_0^\pi |\phi_x\phi_x^*| dx \\ &+ |b|\int_0^\pi \left(|\phi u_x^*| + |u_x\phi^*|\right) dx + |d|\int_0^\pi \left(|\phi_x u_{xx}^*| + |u_{xx}\phi_x^*|\right) dx \\ &+ |\xi|\int_0^\pi |\phi\phi^*| dx. \end{aligned}$$

Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \Big| B\left((u,\phi),(u^*,\phi^*)\right) \Big| &\le |c|\|u_{xx}\|\|u_{xx}^*\| + |a|\|u_x\|\|u_x^*\| + |\beta|\|\phi_x\|\|\phi_x^*\| + |b|\|\phi\|\|u_x^*\| \\ &+ |b|\|u_x\|\|\phi^*\| + |d|\|\phi_x\|\|u_{xx}^*\| + |d|\|u_{xx}\|\|\phi_x^*\| + |\xi|\|\phi\|\|\phi^*\| \\ &\le max\Big(|c|,|a|,|\beta|,|b|,|d|,|\xi|\Big)\Big(\|u_{xx}\| + \|u_x\| + \|\phi_x\| + \|\phi\|\Big) \\ &\times\Big(\|u_{xx}^*\| + \|u_x^*\| + \|\phi_x^*\| + \|\phi^*\|\Big) \\ &\le C\|(u,\phi)\|_{\mathcal{W}}\|(u^*,\phi^*)\|_{\mathcal{W}}. \end{aligned}$$

So, $B(\cdot,\cdot)$ is continuous.

2. Continuity of $L(\cdot)$: we use Cauchy-Schwarz inequality, we get

$$\begin{aligned}\Big|L\left((u^*,\phi^*)\right)\Big| &\le |\rho|\|f^2\|\|u^*\| + |\delta|\|\theta_x\|\|u^*\| + |J|\|f^4\|\|\phi^*\| + |\mu|\|f^3\|\|\phi^*\| \\ &+|m|\|\theta\|\|\phi^*\| \\ &\le max\Big(|\rho|\|f^2\|, |\delta|\|\theta_x\|, |J|\|f^4\|, |\mu|\|f^3\|, |m|\|\theta\|\Big)\Big(\|u^*\| + \|\phi^*\|\Big) \\ &\le C\|(u^*,\phi^*)\|_{\mathcal{W}}.\end{aligned}$$

Moreover,

$$\begin{aligned}B\left((u,\phi),(u,\phi)\right) =& c\int_0^\pi u_{xx}^2dx + a\int_0^\pi u_x^2dx + \beta\int_0^\pi \phi_x^2dx \\ &+ \xi\int_0^\pi \phi^2dx + 2b\int_0^\pi \phi u_xdx \\ &+ 2d\int_0^\pi \phi_x u_{xx}dx.\end{aligned}$$

Noting that

$$\begin{aligned}&a\int_0^\pi u_x^2dx + \xi\int_0^\pi \phi^2dx + 2b\int_0^\pi \phi u_xdx \\ \ge& \frac{1}{2}\left(a - \frac{b^2}{\xi}\right)\int_0^\pi u_x^2dx + \frac{1}{2}\left(\xi - \frac{b^2}{a}\right)\int_0^\pi \phi^2dx\end{aligned}$$

and

$$\begin{aligned}&c\int_0^\pi u_{xx}^2dx + \beta\int_0^\pi \phi_x^2dx + 2d\int_0^\pi \phi_x u_{xx}dx \\ \ge& \frac{1}{2}\left(c - \frac{d^2}{\beta}\right)\int_0^\pi u_{xx}^2dx + \frac{1}{2}\left(\beta - \frac{d^2}{c}\right)\int_0^\pi \phi_x^2dx\end{aligned}$$

and using the fact $a\xi > b^2$ and $c\beta > d^2$, we arrive at

$$\begin{aligned}B\left((u,\phi)^T,(u,\phi)^T\right) \ge& \frac{1}{2}\left(a - \frac{b^2}{\xi}\right)\int_0^\pi u_x^2dx + \frac{1}{2}\left(\xi - \frac{b^2}{a}\right)\int_0^\pi \phi^2dx \\ &+ \frac{1}{2}\left(c - \frac{d^2}{\beta}\right)\int_0^\pi u_{xx}^2dx + \frac{1}{2}\left(\beta - \frac{d^2}{c}\right)\int_0^\pi \phi_x^2dx,\end{aligned}$$

This demonstrates that $B(\cdot,\cdot)$ is coercive. Then, using the Lax-Milgram theorem, we infer that the problem (7.16) has a unique solution $(u,\phi) \in \mathcal{W}$.

Moreover, by setting $(u^*,\phi^*) = (u^*,0)$ in equation (7.17), we obtain

$$\int_0^\pi (cu_x + d\phi)_x u_{xx}^*dx = \int_0^\pi \left(au_{xx} + b\phi_x + \rho f^2 + \delta\theta_x\right)u^*dx,\ \forall u^* \in H^2(0,\pi) \cap H_0^1(0,\pi).$$

This implies that

$$cu_x + d\phi \in H^3(0,\pi).$$

Moreover, as $C_0^\infty(0,\pi) \subset H^2(0,\pi) \cap H_0^1(0,\pi)$, then, simple integration yield

$$\int_0^\pi (cu_{xxxx} + d\phi_{xxx})u^* dx = \int_0^\pi \left(au_{xx} + b\phi_x + \rho f^2 + \delta\theta_x\right) u^* dx, \ \forall u^* \in C_0^\infty(0,\pi).$$

Therefore

$$cu_{xxxx} + d\phi_{xxx} = au_{xx} + b\phi_x + \rho f^2 + \delta\theta_x.$$

Thus, u satisfies $(7.11)_2$.

On the other hand, clearly $\int_0^\pi (cu_x + d\phi)dx = 0$. Therefore,

$$cu_x + d\phi \in H^3(0,\pi) \cap L_*^2(0,\pi). \tag{7.18}$$

Similarly, if we take $(u^*, \phi^*) = (0, \phi^*)$, we get

$$\int_0^\pi (du_x + \beta\phi)_x \phi_x^* dx = -\int_0^\pi \left(bu_x + \xi\phi - Jf^4 + m\theta - \mu f^3\right) \phi^* dx, \ \forall \phi^* \in H_*^1(0,\pi). \tag{7.19}$$

Here, we are unable to apply directly the theory of elliptic regularity. To address this difficulty, we proceed as follows:

Let $\Psi \in H_0^1(0,\pi)$ and set $\phi^* = \Psi - \frac{1}{\pi}\int_0^\pi \Psi(x,t)\,dx$. Then, $\phi^* \in H_*^1(0,\pi)$. Inserting ϕ^* in (7.19), bearing in mind that $\int_0^\pi (bu_x + \xi\phi - Jf^4 + m\theta - \mu f^3)\,dx = 0$, we infer that

$$\int_0^\pi (du_x + \beta\phi)_x \Psi_x dx = -\int_0^\pi \left(bu_x + \xi\phi - Jf^4 + m\theta - \mu f^3\right) \Psi dx, \ \forall \Psi \in H_0^1(0,\pi). \tag{7.20}$$

This shows that

$$du_x + \beta\phi \in H^2(0,\pi).$$

Moreover, integration by parts of equation (7.20) yields

$$du_{xxx} + \beta\phi_{xx} = bu_x + \xi\phi - Jf^4 + m\theta - \mu f^3 = r.$$

By virtue of (7.13), ϕ satisfies $(7.11)_4$.

Since $\int_0^\pi (du_x + \beta\phi)dx = 0$, we get

$$du_x + \beta\phi \in H^2(0,\pi) \cap L_*^2(0,\pi). \tag{7.21}$$

From (7.18) and (7.21), we deduce that

$$\phi \in H_*^2(0,\pi). \tag{7.22}$$

On other hand, since $(du_x+\beta\phi)_{xx} = r \in L^2(0,\pi)$, then

$$\int_0^\pi (du_x+\beta\phi)_{xx}\,\varphi dx = \int_0^\pi r\varphi dx,\ \forall\varphi\in H^1(0,\pi).$$

Integration by parts and the fact that $H_*^1(0,\pi) \subset H^1(0,\pi)$ yield

$$(du_x+\beta\phi)_x\,\varphi\Big|_0^\pi - \int_0^\pi (du_x+\beta\phi)_x\,\varphi_x dx = \int_0^\pi r\varphi dx, \quad \forall\varphi\in H_*^1(0,\pi). \tag{7.23}$$

Thus, (7.19) leads to

$$(du_x+\beta\phi)_x(\pi)\,\varphi(\pi) - (du_x+\beta\phi)_x(0)\,\varphi(0) = 0,\ \forall\varphi\in H_*^1(0,\pi).$$

As φ is arbitrary, we obtain

$$(du_x+\beta\phi)_x(\pi) = (du_x+\beta\phi)_x(0) = 0. \tag{7.24}$$

Using $(7.11)_4$, (7.24), the fact that $(u,\phi)\in\mathcal{W}$ and $(\psi, f^4)\in[L_*^2(0,\pi)]^2$, we infer that

$$\int_0^\pi \theta(x,t)\,dx = \frac{1}{m}\int_0^\pi \left(Jf^4 - du_{xxx} - \beta\phi_{xx} + \xi\phi + bu_x + \mu\psi\right)dx = 0.$$

Thus,

$$\theta\in H_*^1(0,\pi). \tag{7.25}$$

Therefore, there exists a unique $U\in D(\mathcal{A})$ that satisfies (7.11). which proves that $0\in\rho(\mathcal{A})$. Thus $\mathcal{A}$ is invertible. Consequently, by virtue of Theorem 1.8, the operator $\lambda I-\mathcal{A} = \mathcal{A}(\lambda\mathcal{A}^{-1}-I)$ is invertible for $0<\lambda<\|\mathcal{A}^{-1}\|^{-1}$.

By using Lumer Phillips Theorem, the equation $U_t = \mathcal{A}U$ has a unique solution, which completes the proof. □

7.3 Exponential stability

In this section, we prove the exponential stability of the solution of the problem (7.1)-(7.3). To achieve this, we utilize a tool based on the theorems of Prüss and Gearhart [35]. We start with the following lemma.

Lemma 7.2. *For any (u, ϕ, θ, q) solution of (7.1)-(7.3), the energy functional*

$$E(t) := \frac{1}{2}\int_0^\pi \left[\rho u_t^2 + au_x^2 + cu_{xx}^2 + J\phi_t^2 + \xi\phi^2 + \beta\phi_x^2 + c^*\theta^2 + \frac{\tau}{\kappa}q^2 + 2bu_x\phi + 2du_{xx}\phi_x\right]dx$$

satisfies

$$\frac{dE}{dt}(t) = -\frac{1}{\kappa}\int_0^\pi |q|^2 dx - \mu\int_0^\pi |\phi_t|^2\, dx \leq 0.$$

Proof. Multiplying the equations of (7.1) by u_t, ϕ_t, θ and $\dfrac{1}{\kappa}q$, respectively, and integrating with respect of x over $(0, \pi)$, we obtain

$$\frac{\rho}{2}\frac{d}{dt}\int_0^\pi u_t^2 dx - a\int_0^\pi u_{xx}u_t dx - b\int_0^\pi \phi_x u_t dx + c\int_0^\pi u_{xxxx}u_t dx + d\int_0^\pi \phi_{xxx}u_t dx + \delta\int_0^\pi \theta_x u_t dx = 0, \tag{7.26}$$

$$\begin{gathered}\frac{J}{2}\frac{d}{dt}\int_0^\pi \phi_t^2 dx - d\int_0^\pi u_{xxx}\phi_t dx - \beta\int_0^\pi \phi_{xx}\phi_t dx + \frac{\xi}{2}\frac{d}{dt}\int_0^\pi \phi^2 dx + b\int_0^\pi u_x\phi_t dx \\ -m\int_0^\pi \theta\phi_t dx + \mu\int_0^\pi \phi_t^2 dx = 0,\end{gathered} \tag{7.27}$$

$$\frac{c^*}{2}\frac{d}{dt}\int_0^\pi \theta^2 dx + \int_0^\pi q_x\theta dx + \delta\int_0^\pi u_{xt}\theta dx + m\int_0^\pi \phi_t\theta dx = 0, \tag{7.28}$$

$$\frac{\tau}{2\kappa}\frac{d}{dt}\int_0^\pi q^2 dx + \frac{1}{\kappa}\int_0^\pi q^2 dx + \int_0^\pi \theta_x q dx = 0. \tag{7.29}$$

Adding (7.26)-(7.29), using integration by parts and the boundary conditions, we obtain

$$\frac{d}{dt}E(t) = -\mu\int_0^\pi \phi_t^2 dx - \frac{1}{\kappa}\int_0^\pi q^2 dx \leq 0.$$

□

Now, we state our main result.

Theorem 7.2. *Let (u, ϕ, θ, q) be the solution of* (7.1)-(7.3). *Then, there exist two positive constants η and ω, independent of t, such that*

$$E\left(t\right) \leq \eta E(0)e^{-\omega t},\ t > 0.$$

The proof of Theorem 7.2 will be established through the following two lemmas.

Lemma 7.3. *The set $i\mathbb{R} = \{i\lambda; \lambda \in \mathbb{R}\}$ is contained in $\rho(\mathcal{A})$.*

Proof. First, note that since $0 \in \rho(\mathcal{A})$, then $\mathcal{A}^{-1} \in \mathcal{L}(\mathcal{A})$. Let's prove that $\mathcal{A}^{-1}$ is compact.

Let (F_n) be a bounded sequence in $\mathcal{H}$ and let (U_n) be the sequence in $D(\mathcal{A})$ that satisfies $F_n = \mathcal{A}U_n$, where $U_n = (u_n, v_n, \phi_n, \psi_n, \theta_n, q_n)$.

Since $\mathcal{A}^{-1} \in \mathcal{L}(\mathcal{H})$ and F_n is bounded, we deduce that the sequence $(U_n) = (\mathcal{A}^{-1}F_n)$ is bounded in $\mathcal{H}$. Consequently, there exists a constant $C > 0$ such that

$$\|U_n\|_{D(\mathcal{A})} = \|U_n\|_{\mathcal{H}} + \|\mathcal{A}U_n\|_{\mathcal{H}} \leq C.$$

Therefore, $(U_n) = (u_n, v_n, \phi_n, \psi_n, \theta_n, q_n)$ is bounded in $D(\mathcal{A})$. Using the fact that the embedding of $H^m(0;\pi)$ into $H^i(0;\pi)$ is compact for $m > i$, we can extract a subsequence $U_r = (u_r, v_r, \phi_r, \psi_r, \theta_r, q_r)$ and find $U = (u, v, \phi, \psi, \theta, q) \in \mathcal{H}$ such that

$$(u_r, v_r, \phi_r, \psi_r, \theta_r, q_r) \longrightarrow (u, v, \phi, \psi, \theta, q) \quad \text{in} \quad \mathcal{H}. \tag{7.30}$$

Thus, the subsequence $(\mathcal{A}^{-1}F_r)$ converges in $\mathcal{H}$. This shows that $\mathcal{A}^{-1}$ is compact.

Suppose that there exists $\lambda \in \mathbb{R}$ $(\lambda \neq 0)$ such that $i\lambda \in \sigma(\mathcal{A})$. As $\mathcal{A}^{-1}$ is compact, $i\lambda$ must be an eigenvalue of $\mathcal{A}$. Then, there exists a vector $U = (u, v, \phi, \psi, \theta, q) \neq 0$ such that $i\lambda U - \mathcal{A}U = 0$; that is,

$$\begin{aligned} i\lambda u - v &= 0, \\ i\lambda\rho v - au_{xx} - b\phi_x + cu_{xxxx} + d\phi_{xxx} + \delta\theta_x &= 0, \\ i\lambda\phi - \psi &= 0, \\ i\lambda J\psi - du_{xxx} - \beta\phi_{xx} + \xi\phi + bu_x - m\theta + \mu\psi &= 0, \\ i\lambda c^*\theta + q_x + \delta v_x + m\psi &= 0, \\ i\lambda\tau q + q + \kappa\theta_x &= 0. \end{aligned} \tag{7.31}$$

Next, we perform the inner product of $(i\lambda I - \mathcal{A})U$ and U in $\mathcal{H}$, we obtain

$$\langle i\lambda U - \mathcal{A}U, U\rangle_{\mathcal{H}} = i\lambda\|U\|^2_{\mathcal{H}} - \langle \mathcal{A}U, U\rangle_{\mathcal{H}} = 0,$$

then

$$\operatorname{Re}\langle i\lambda U - \mathcal{A}U, U\rangle_{\mathcal{H}} = \frac{1}{\kappa}\int_0^\pi |q|^2 dx + \mu\int_0^\pi |\psi|^2\, dx = 0. \tag{7.32}$$

Consequently,

$$q = 0 \quad \text{and} \quad \psi = 0. \tag{7.33}$$

Moreover, $(7.31)_3$ and $(7.31)_6$ give

$$\phi = 0, \quad \text{and} \quad \theta = 0. \tag{7.34}$$

In addition, $(7.31)_5$ yields $v_x = 0$ and the integration over $(0, x)$, using the boundary conditions, gives

$$v = 0. \tag{7.35}$$

Finally, the substitution in $(7.31)_1$ leads to

$$u = 0. \tag{7.36}$$

Thus, $U = 0$, which contradicts the fact that λ is an eigenvalue. This completes the proof of Lemma 7.3. $\square$

Lemma 7.4. *The operator $\mathcal{A}$ defined by* (7.10) *satisfies*

$$\overline{\lim_{|\lambda|\to\infty}} \left\|(i\lambda I - \mathcal{A})^{-1}\right\| < \infty.$$

Proof. It suffices to prove that there exists a positive constant C such that

$$\|\,(i\lambda I - \mathcal{A})^{-1}\,F\| \leq C\|F\|, \quad \forall\lambda \in \mathbb{R}, \quad \forall F \in \mathcal{H}.$$

Let $\lambda \in \mathbb{R}$ and let $F = \left(f^1, f^2, f^3, f^4, f^5, f^6\right)^T \in \mathcal{H}$ be given. Note that since $\mathcal{A}$ is a contraction, so $i\lambda I - \mathcal{A}$ is invertible, then there exists a unique $U = (u, v, \phi, \psi, \theta, q)^T \in D\left(\mathcal{A}\right)$, such that $(i\lambda I - \mathcal{A})U = F$; that is,

$$\begin{cases} i\lambda u - v = f^1, \\ i\lambda\rho v - au_{xx} - b\phi_x + cu_{xxxx} + d\phi_{xxx} + \delta\theta_x = \rho f^2, \\ i\lambda\phi - \psi = f^3, \\ i\lambda J\psi - du_{xxx} - \beta\phi_{xx} + \xi\phi + bu_x - m\theta + \mu\psi = Jf^4, \\ i\lambda c^*\theta + q_x + \delta v_x + m\psi = c^* f^5, \\ i\lambda\tau q + q + \kappa\theta_x = \tau f^6. \end{cases} \tag{7.37}$$

First, recall that

$$Re\langle(i\lambda I - \mathcal{A})U, U\rangle_{\mathcal{H}} = \frac{1}{\kappa}\int_0^\pi |q|^2 dx + \mu\int_0^\pi |\psi|^2\, dx = Re\langle F, U\rangle_{\mathcal{H}}.$$

Therefore,

$$\frac{1}{\kappa}\int_0^\pi |q|^2 dx + \mu \int_0^\pi |\psi|^2 \, dx \le \|F\| \, \|U\| \, . \tag{7.38}$$

where $C > 0$ is a constant. Performing the L^2-inner product of $(7.37)_2$ by u and $(7.37)_4$ by ϕ, using equations $(7.37)_1$, $(7.37)_3$, and integration by parts, we obtain

$$a \|u_x\|^2 + c \|u_{xx}\|^2 + \beta \|\phi_x\|^2 + \xi \|\phi\|^2 + 2b\langle u_x, \phi\rangle + 2d\langle u_{xx}, \phi_x\rangle$$

$$= \rho \left\langle f^2, u\right\rangle + \rho \left\langle v, f^1\right\rangle + J \left\langle f^4, \phi\right\rangle + J \left\langle \psi, f^3\right\rangle + \rho \|v\|^2 + J \|\psi\|^2$$

$$+\delta \langle \theta, u_x\rangle + m \langle \theta, \phi\rangle - \mu \langle \psi, \phi\rangle \, .$$

Equation (7.38), Cauchy-Schwarz and Young's inequalities lead to

$$\begin{aligned}&\frac{a}{2}\|u_x\|^2 + c\|u_{xx}\|^2 + \beta\|\phi_x\|^2 + \frac{\xi}{2}\|\phi\|^2 + 2b\langle u_x, \phi\rangle + 2d\langle u_{xx}, \phi_x\rangle \\ &\qquad \le C\|F\|\|U\| + \rho \|v\|^2 + (\frac{\delta^2}{2a} + \frac{m^2}{\xi})\|\theta\|^2.\end{aligned} \tag{7.39}$$

At this point, we need to estimate $\|v\|_{L^2}$. To this end, we introduce the functions φ, w, z and y as solutions to the following problems:

$$-\varphi_{xx} = v, \qquad \varphi_x(0) = \varphi_x(\pi) = 0, \tag{7.40}$$

$$-w_{xx} = \theta, \qquad w_x(0) = w_x(\pi) = 0, \tag{7.41}$$

$$-z_{xx} = f^5, \qquad z(0) = z(\pi) = 0, \tag{7.42}$$

$$-y_{xx} = f^2, \qquad y(0) = y(\pi) = 0. \tag{7.43}$$

Multiplying (7.42) by z in $L^2(0,\pi)$ and applying Cauchy-Schwarz and Poincaré's inequalities, we obtain

$$\|z_x\| \le C_p \left\|f^5\right\| , \tag{7.44}$$

where C_p is the Poincaré constant. Similarly, we have

$$\|y_x\| \le C_p \left\|f^2\right\| . \tag{7.45}$$

Next, we multiply $(7.37)_5$ by φ_x in $L^2(0,\pi)$ to obtain

$$\langle i\lambda c^*\theta, \varphi_x\rangle + \langle q_x, \varphi_x\rangle + \delta \langle v_x, \varphi_x\rangle + m\langle \psi, \varphi_x\rangle = c^*\langle f^5, \varphi_x\rangle \, . \tag{7.46}$$

Now, we estimate (7.46) term by term. Integration by parts, and equations (7.40), (7.41), lead to

$$I_1 = \langle i\lambda c^*\theta, \varphi_x\rangle = -c^*\langle i\lambda w_{xx}, \varphi_x\rangle = c^*\langle i\lambda w_x, \varphi_{xx}\rangle = c^*\langle w_x, i\lambda v\rangle. \tag{7.47}$$

From $(7.37)_2$, we get

$$i\lambda v = \frac{1}{\rho}\left(au_{xx} + b\phi_x - cu_{xxxx} - d\phi_{xxx} - \delta\theta_x + \rho f^2\right).$$

Plugging $i\lambda v$ in (7.47), and using (7.43), we obtain

$$\begin{aligned} I_1 = \;& \frac{ac^*}{\rho}\langle w_x, u_{xx}\rangle + \frac{bc^*}{\rho}\langle w_x, \phi_x\rangle - \frac{cc^*}{\rho}\langle w_x, u_{xxxx}\rangle \\ & -\frac{dc^*}{\rho}\langle w_x, \phi_{xxx}\rangle - \frac{\delta c^*}{\rho}\langle w_x, \theta_x\rangle - c^*\langle w_x, y_{xx}\rangle . \end{aligned}$$

Integration by parts and equation (7.41) give

$$I_1 = \frac{ac^*}{\rho}\langle \theta, u_x\rangle + \frac{bc^*}{\rho}\langle \theta, \phi\rangle + \frac{cc^*}{\rho}\langle \theta_x, u_{xx}\rangle + \frac{dc^*}{\rho}\langle \theta_x, \phi_x\rangle - \frac{\delta c^*}{\rho}\|\theta\|^2 - c^*\langle \theta, y_x\rangle . \tag{7.48}$$

Similarly, we get

$$I_2 = \langle q_x, \varphi_x\rangle = -\langle q, \varphi_{xx}\rangle = \langle q, v\rangle , \tag{7.49}$$

$$I_3 = \delta\langle v_x, \varphi_x\rangle = -\delta\langle v, \varphi_{xx}\rangle = \delta\|v\|^2 \tag{7.50}$$

and

$$I_4 = c^*\left\langle f^5, \varphi_x\right\rangle = -c^*\langle z_{xx}, \varphi_x\rangle = c^*\langle z_x, \varphi_{xx}\rangle = -c^*\langle z_x, v\rangle . \tag{7.51}$$

Substituting equations (7.48)-(7.51) into equation (7.46), we obtain

$$\begin{aligned} \delta\|v\|^2 = \;& -\frac{ac^*}{\rho}\langle \theta, u_x\rangle - \frac{bc^*}{\rho}\langle \theta, \phi\rangle - \frac{cc^*}{\rho}\langle \theta_x, u_{xx}\rangle - \frac{dc^*}{\rho}\langle \theta_x, \phi_x\rangle \\ & +\frac{\delta c^*}{\rho}\|\theta\|^2 + c^*\langle \theta, y_x\rangle - \langle q, v\rangle - m\langle \psi, \varphi_x\rangle - c^*\langle z_x, v\rangle . \end{aligned} \tag{7.52}$$

Young's inequality yields

$$\frac{ac^*}{\rho}|\langle \theta, u_x\rangle| \le \frac{a\delta}{24\rho}\|u_x\|^2 + \frac{6ac^{*2}}{\delta\rho}\|\theta\|^2$$

and

$$\frac{bc^*}{\rho}|\langle \theta, \phi\rangle| \le \frac{\xi\delta}{24\rho}\|\phi\|^2 + \frac{6b^2c^{*2}}{\xi\delta\rho}\|\theta\|^2.$$

Replacing θ_x with the value obtained from equation $(7.37)_6$, we get

$$\begin{aligned}\frac{cc^*}{\rho}\left|\langle\theta_x,u_{xx}\rangle\right| = & \frac{cc^*}{\rho\kappa}\left|\langle\tau f^6-i\lambda\tau q-q,u_{xx}\rangle\right|\\ & \leq \frac{cc^*\tau}{\rho\kappa}\left|\langle f^6,u_{xx}\rangle\right|+\left(\frac{cc^*\lambda\tau}{\rho\kappa}+\frac{cc^*}{\rho\kappa}\right)\left|\langle q,u_{xx}\rangle\right|\end{aligned}$$

and

$$\begin{aligned}\frac{dc^*}{\rho}\left|\langle\theta_x,\phi_x\rangle\right| = & \frac{dc^*}{\rho\kappa}\left|\langle\tau f^6-i\lambda\tau q-q,\phi_x\rangle\right|\\ & \leq \frac{dc^*\tau}{\rho\kappa}\left|\langle f^6,\phi_x\rangle\right|+\left(\frac{dc^*\lambda\tau}{\rho\kappa}+\frac{dc^*}{\rho\kappa}\right)\left|\langle q,\phi_x\rangle\right|.\end{aligned}$$

Thus, Cauchy-Schwarz and Young's inequalities lead to

$$\frac{cc^*}{\rho}\left|\langle\theta_x,u_{xx}\rangle\right| \leq \frac{cc^*\tau}{\rho\kappa}\left\|f^6\right\|\left\|u_{xx}\right\|+\frac{4cc^{*2}\left(1+\lambda\tau\right)^2}{\delta\rho\kappa^2}\left\|q\right\|^2+\frac{c\delta}{16\rho}\left\|u_{xx}\right\|^2$$

and

$$\frac{dc^*}{\rho}\left|\langle\theta_x,\phi_x\rangle\right| \leq \frac{dc^*\tau}{\rho\kappa}\left\|f^6\right\|\left\|\phi_x\right\|+\frac{4d^2c^{*2}\left(1+\lambda\tau\right)^2}{\beta\delta\rho\kappa^2}\left\|q\right\|^2+\frac{\beta\delta}{16\rho}\left\|\phi_x\right\|^2.$$

Using equation (7.38), we infer that

$$\frac{cc^*}{\rho}\left|\langle\theta_x,u_{xx}\rangle\right| \leq \frac{c\delta}{16\rho}\left\|u_{xx}\right\|^2+C\|F\|\|U\|$$

and

$$\frac{dc^*}{\rho}\left|\langle\theta_x,\phi_x\rangle\right| \leq \frac{\beta\delta}{16\rho}\left\|\phi_x\right\|^2+C\|F\|\|U\|.$$

Similarly, the Cauchy-Schwarz inequality leads to

$$c^*\Big(|\langle\theta,y_x\rangle|+|\langle z_x,v\rangle|\Big) \leq c^*C_p\Big(\|\theta\|\|y_x\|+\|z_x\|\|v\|\Big).$$

Therefore, (7.44) and (7.45) yield

$$c^*\Big(|\langle\theta,y_x\rangle|+|\langle z_x,v\rangle|\Big) \leq c^*C_p\Big(\|\theta\|\|f^2\|+\|f^5\|\|v\|\Big) \leq C\|F\|\|U\|.$$

Next, Young's and Poincaré's inequalities, (7.38) and (7.40) imply

$$|\langle q,v\rangle| \leq \frac{\delta}{4}\left\|v\right\|^2+C\left\|F\right\|\left\|U\right\|$$

and

$$\begin{aligned}m|\langle\psi,\varphi_x\rangle| & \leq \varepsilon\|\varphi_x\|^2+\frac{m^2}{4\varepsilon}\|\psi\|^2 \leq \varepsilon C_p\|\varphi_{xx}\|^2+\frac{m^2}{4\varepsilon}\|\psi\|^2\\ & \leq \frac{\delta}{4}\|v\|^2+C\|F\|\|U\|.\end{aligned}$$

Finally, we substitute the above estimates into equation (7.52), we obtain

$$\begin{aligned} 2\rho \|v\|^2 \leq\ & \frac{a}{6}\|u_x\|^2 + \frac{\xi}{6}\|\phi\|^2 + \frac{c}{4}\|u_{xx}\|^2 + \frac{\beta}{4}\|\phi_x\|^2 \\ & + \left(\frac{24ac^{*2}}{\delta^2} + \frac{24b^2c^{*2}}{\xi\delta^2} + 4c^*\right)\|\theta\|^2 + C\|F\|\|U\|. \end{aligned} \tag{7.53}$$

The addition of (7.39) and (7.53) gives

$$\begin{aligned} \rho\|v\|^2 + \frac{a}{3}\|u_x\|^2 + \frac{3c}{4}\|u_{xx}\|^2 + \frac{3\beta}{4}\|\phi_x\|^2 + \frac{\xi}{3}\|\phi\|^2 + c^*\|\theta\|^2 \\ + J\|\psi\|^2 + \frac{\tau}{\kappa}\|q\|^2 + 2b\langle u_x, \phi\rangle + 2d\langle u_{xx}, \phi_x\rangle \\ \leq \left(\frac{24ac^{*2}}{\delta^2} + \frac{24b^2c^{*2}}{\xi\delta^2} + \frac{\delta^2}{2a} + \frac{m^2}{\xi} + 5c^*\right)\|\theta\|^2 + C\|F\|\|U\|. \end{aligned} \tag{7.54}$$

The next step is to estimate $\|\theta\|_{L^2}$. To this end, we define the functions w and z as in (7.41) and (7.42), and α, η to be the solutions to the following problems:

$$-\alpha_{xx} = q, \qquad \alpha(0) = \alpha_x(\pi) = 0, \tag{7.55}$$

$$-\eta_{xx} = f^6, \qquad \eta(0) = \eta(\pi) = 0. \tag{7.56}$$

Multiplying the equations (7.55) by α and (7.56) by η in $L^2(0,\pi)$, using integration by parts, Cauchy-Schwarz and Poincaré's inequalities, we obtain

$$\|\alpha_x\| \leq C_p\|q\|, \tag{7.57}$$

$$\|\eta_x\| \leq C_p\left\|f^6\right\|. \tag{7.58}$$

Performing the L^2-inner product of equation $(7.37)_6$ by w_x, using (7.55), (7.56) and (7.41), integration by parts, we get

$$\tau\langle\alpha_x, i\lambda\theta\rangle - \langle\alpha_x, \theta\rangle + \kappa\|\theta\|^2 = -\tau\langle\eta_x, \theta\rangle.$$

Inserting $i\lambda\theta$ obtained from $(7.37)_5$, we arrive at

$$\begin{aligned} \tau\langle\alpha_x, i\lambda\theta\rangle &= \tau\left\langle\alpha_x, f^5\right\rangle - \frac{\tau}{c^*}\langle\alpha_x, q_x\rangle - \frac{\delta\tau}{c^*}\langle\alpha_x, v_x\rangle - \frac{m\tau}{c^*}\langle\alpha_x, \psi\rangle \\ &= -\tau\langle q, z_x\rangle - \frac{\tau}{c^*}\|q\|^2 - \frac{\delta\tau}{c^*}\langle q, v\rangle - \frac{m\tau}{c^*}\langle\alpha_x, \psi\rangle. \end{aligned}$$

Thus,

$$\kappa\|\theta\|^2 = -\tau\langle\eta_x, \theta\rangle + \tau\langle q, z_x\rangle + \frac{\tau}{c^*}\|q\|^2 + \frac{\delta\tau}{c^*}\langle q, v\rangle + \frac{m\tau}{c^*}\langle\alpha_x, \psi\rangle + \langle\alpha_x, \theta\rangle.$$

Let us estimate each term in the obtained relation. First, we have

$$\tau\Big(|\langle \eta_x, \theta\rangle| + |\langle q, z_x\rangle|\Big) \le C\,\|F\|\,\|U\|\,.$$

Young's inequality, (7.57) and (7.38) give

$$\left|\frac{\tau\delta}{c^*}\langle q, v\rangle\right| \le C_\varepsilon\|F\|\|U\| + \varepsilon\|v\|^2,$$

$$|\langle \alpha_x, \theta\rangle| \le C\,\|F\|\,\|U\| + \frac{\kappa}{2}\,\|\theta\|^2$$

and

$$\frac{m\tau}{c^*}|\langle \alpha_x, \psi\rangle| \le C\,\|F\|\,\|U\|\,.$$

Therefore,

$$\frac{\kappa}{2}\,\|\theta\|^2 \le C_\varepsilon\,\|F\|\,\|U\| + \varepsilon\|v\|^2. \tag{7.59}$$

Multiplying (7.59) by $\left(\dfrac{48ac^{*2}}{\kappa\delta^2} + \dfrac{48b^2c^{*2}}{\kappa\xi\delta^2} + \dfrac{\delta^2}{a\kappa} + \dfrac{2m^2}{\kappa\xi} + \dfrac{10c^*}{\kappa}\right)$ and choosing ε such that

$$\Big(\frac{48ac^{*2}}{\kappa\delta^2} + \frac{48b^2c^{*2}}{\kappa\xi\delta^2} + \frac{\delta}{a\kappa} + \frac{2m^2}{\kappa\xi} + \frac{10c^*}{\kappa}\Big)\varepsilon = \frac{\rho}{2},$$

we obtain

$$\Big(\frac{24ac^{*2}}{\delta^2} + \frac{24b^2c^{*2}}{\xi\delta^2} + \frac{\delta}{2a} + \frac{m^2}{\xi} + 5c^*\Big)\,\|\theta\|^2 \le C\,\|F\|\,\|U\| + \frac{\rho}{2}\,\|v\|^2\,. \tag{7.60}$$

Substituting (7.60) into (7.54), we get

$$\frac{\rho}{2}\,\|v\|^2 + \frac{a}{3}\,\|u_x\|^2 + \frac{3c}{4}\,\|u_{xx}\|^2 + \frac{3\beta}{4}\,\|\phi_x\|^2 + \frac{\xi}{3}\,\|\phi\|^2 + c^*\,\|\theta\|^2 + J\|\psi\|^2 + \frac{\tau}{\kappa}\|q\|^2$$
$$+2b\,\langle u_x, \phi\rangle + 2d\,\langle u_{xx}, \phi_x\rangle \le C\,\|F\|\,\|U\|\,,$$

hence,

$$\frac{1}{3}\,\|U\|^2 \le C\,\|F\|\,\|U\|\,,$$

which gives

$$\|U\| \le C\,\|F\|\,,$$

and the proof of Lemma 7.4 is completed. □

Proof of theorem 7.2 In view of lemmas 7.3-7.4, the result of Theorem 7.2 is established

.

7.4 Slow decay

In this section, we consider the problem (7.1) without porous dissipation ($\mu = 0$), and prove that the solution of

$$\begin{cases} \rho u_{tt} = au_{xx} + b\phi_x - cu_{xxxx} - d\phi_{xxx} - \delta\theta_x & \text{in } (0,\pi)\times(0,\infty), \\ J\phi_{tt} = du_{xxx} + \beta\phi_{xx} - \xi\phi - bu_x + m\theta & \text{in } (0,\pi)\times(0,\infty), \\ c^*\theta_t = -q_x - \delta u_{xt} - m\phi_t & \text{in } (0,\pi)\times(0,\infty), \\ \tau q_t + q + \kappa\theta_x = 0 & \text{in } (0,\pi)\times(0,\infty), \end{cases} \tag{7.61}$$

decays slowly.

Our main result is given by the following theorem.

Theorem 7.3. *The solution of the problem determined by the system (7.61) and the boundary and initial conditions (7.2),(7.3) decays slowly.*

Proof. The purpose is to prove that for all sufficiently small ε, there exists a solution of (7.61) in the form

$$\begin{aligned} u = K_1 \exp(wt)\sin(nx), \quad \varphi = K_2 \exp(wt)\cos(nx), \\ \theta = K_3 \exp(wt)\cos(nx), \quad q = K_4 \exp(wt)\sin(nx). \end{aligned} \tag{7.62}$$

such that $Re(w) > -\varepsilon$. Thus, we can always find a solution w as near as we want of the imaginary axis.

By substituting the values of u, φ, θ and q from (7.62) into (7.61), we get to the following homogeneous system with the unknowns K_1, K_2, K_3 and K_4

$$\begin{cases} \rho w^2 K_1 = -an^2K_1 - nbK_2 - cn^4K_1 - n^3dK_2 + n\delta K_3, \\ Jw^2K_2 = -n^3dK_1 - \beta n^2K_2 - \xi K_2 - nbK_1 + mK_3, \\ c^*\ wK_3 = -nK_4 - n\delta wK_1 - mwK_2, \\ \tau wK_4 + K_4 - n\kappa K_3 = 0. \end{cases}$$

Therefore,

$$\begin{cases} (\rho w^2 + an^2 + cn^4)\,K_1 + (bn + dn^3)\,K_2 - \delta nK_3 = 0, \\ (dn^3 + nb)\,K_1 + (Jw^2 + \beta n^2 + \xi)\,K_2 - mK_3 = 0, \\ n\delta wK_1 + mwK_2 + c^*\ wK_3 + nK_4 = 0, \\ -n\kappa K_3 + (\tau w + 1)\,K_4 = 0, \end{cases}$$

which can be written as

$$\begin{pmatrix} (\rho w^2+an^2+cn^4) & (bn+dn^3) & -\delta n & 0 \\ (dn^3+nb) & (Jw^2+\beta n^2+\xi) & -m & 0 \\ n\delta w & mw & c^*\ w & n \\ 0 & 0 & -n\kappa & (\tau w+1) \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In order to obtain a non-trivial solution. We assume that the determinant is equal to zero, that is

$$\Delta = P_0 + P_1 w + P_2 w^2 + P_3 w^3 + P_4 w^4 + P_5 w^5 + P_6 w^6 = 0,$$

where

$$\begin{aligned} P_0 &= \left(c\kappa\beta - d^2\kappa\right) n^8 + \left(a\kappa\beta + c\kappa\xi - 2bd\kappa\right) n^6 + \left(a\kappa\xi - b^2\kappa\right) n^4, \\ P_1 &= \left(c\beta c^* - d^2 c^*\right) n^6 + \left(cm^2 - 2dm\delta - 2bdc^* + a\beta c^* + c\xi c^* + \beta\delta^2\right) n^4 \\ &\quad + \left(\delta^2\xi - 2bm\delta + am^2 + a\xi c^* - b^2 c^*\right) n^2, \\ P_2 &= \left(c\beta\tau c^* + Jc\kappa - d^2\tau c^*\right) n^6 + \left(\beta\tau\delta^2 - 2dm\tau\delta - 2bd\tau c^* + a\beta\tau c^* + Ja\kappa\right. \\ &\quad \left.+\kappa\beta\rho + cm^2\tau + c\tau\xi c^*\right) n^4 + \left(a\tau\xi c^* + \tau\delta^2\xi - b^2\tau c^* - 2bm\tau\delta + \kappa\xi\rho + am^2\tau\right) n^2, \\ P_3 &= m^2\rho + \xi\rho c^* + \left(J\delta^2 + \beta\rho c^* + Jac^*\right) n^2 + Jcc^* n^4, \\ P_4 &= m^2\tau\rho + \tau\xi\rho c^* + \left(J\kappa\rho + Ja\tau c^* + \beta\tau\rho c^* + J\tau\delta^2\right) n^2 + Jc\tau c^* n^4, \\ P_5 &= J\rho c^*, \\ P_6 &= J\tau\rho c^*. \end{aligned}$$

We can write the determinant equation in the form

$$\Delta = P_0 + P_1 X + P_2 X^2 + P_3 X^3 + P_4 X^4 + P_5 X^5 + P_6 X^6 = 0.$$

We want to prove that the equation (7.61) has solutions as near as we want to the imaginary axis, that is for all $\varepsilon > 0$, the equation (7.61) has solutions at the right of the line $\mathrm{Re}(w) = -\varepsilon$. This is equivalent to prove that the equation

$$\Delta = P_0 + P_1(X-\varepsilon) + P_2(X-\varepsilon)^2 + P_3(X-\varepsilon)^3 + P_4(X-\varepsilon)^4 + P_5(X-\varepsilon)^5 + P_6(X-\varepsilon)^6 = 0$$

has all the solutions with positive real parts. We utilize the Hurwitz theorem 1.16, which states that the necessary and sufficient condition to ensure that the equation

$$q_0 X^6 + q_1 X^5 + q_2 X^4 + q_3 X^3 + q_4 X^2 + q_5 X + q_6 = 0$$

has solutions with negative real part is

$$\Lambda_0 = q_1 > 0, \quad \Lambda_1 = \det \begin{pmatrix} q_1 & q_0 \\ q_3 & q_2 \end{pmatrix} > 0, \quad \Lambda_2 = \det \begin{pmatrix} q_1 & q_0 & 0 \\ q_3 & q_2 & q_1 \\ q_5 & q_4 & q_3 \end{pmatrix} > 0$$

and

$$\Lambda_3 = \det \begin{pmatrix} q_1 & q_0 & 0 & 0 \\ q_3 & q_2 & q_1 & q_0 \\ q_5 & q_4 & q_3 & q_2 \\ 0 & q_6 & q_5 & q_4 \end{pmatrix} > 0, \quad \Lambda_4 = \det \begin{pmatrix} q_1 & q_0 & 0 & 0 & 0 \\ q_3 & q_2 & q_1 & q_0 & 0 \\ q_5 & q_4 & q_3 & q_2 & q_1 \\ 0 & q_6 & q_5 & q_4 & q_3 \\ 0 & 0 & 0 & q_6 & q_5 \end{pmatrix} > 0,$$

$$\Lambda_5 = \det \begin{pmatrix} q_1 & q_0 & 0 & 0 & 0 & 0 \\ q_3 & q_2 & q_1 & q_0 & 0 & 0 \\ q_5 & q_4 & q_3 & q_2 & q_1 & q_0 \\ 0 & q_6 & q_5 & q_4 & q_3 & q_2 \\ 0 & 0 & 0 & q_6 & q_5 & q_4 \\ 0 & 0 & 0 & 0 & 0 & q_6 \end{pmatrix} > 0,$$

where

$$
\begin{aligned}
q_0 = \;& J\tau\rho c^*,\\
q_1 = \;& J\rho c^*\left(1-6\tau\varepsilon\right),\\
q_2 = \;& \rho\tau\left(m^2+\xi c^*\right)+5J\rho c^*\left(3\tau\varepsilon-1\right)\varepsilon+\left(J\kappa\rho+Ja\tau c^*+\beta\tau\rho c^*+J\tau\delta^2\right)n^2+Jc\tau c^* n^4,\\
q_3 = \;& \left(J\delta^2+\beta\rho c^*+Jac^*-4\left(J\kappa\rho+Ja\tau c^*+\beta\tau\rho c^*+J\tau\delta^2\right)\varepsilon\right)n^2+Jcc^*\left(1-4\tau\varepsilon\right)n^4\\
& +\rho\left(m^2+\xi c^*\right)-4\tau\rho\left(m^2+\xi c^*\right)\varepsilon+10J\rho c^*\left(1-2\tau\varepsilon\right)\varepsilon^2,\\
q_4 = \;& \left(c\beta\tau c^*+Jc\kappa-d^2\tau c^*\right)n^6+\left(a\tau\xi c^*+\tau\delta^2\xi-b^2\tau c^*-2bm\tau\delta+\kappa\xi\rho+am^2\tau\right)n^2\\
& +\left(\beta\tau\delta^2-2dm\tau\delta-2bd\tau c^*+a\beta\tau c^*+Ja\kappa+\kappa\beta\rho+cm^2\tau+c\tau\xi c^*\right)n^4+J\rho c^*\left(15\tau\varepsilon-10\right)\varepsilon^3\\
& +6\left(m^2\tau\rho+\tau\xi\rho c^*+\left(J\kappa\rho+Ja\tau c^*+\beta\tau\rho c^*+J\tau\delta^2\right)n^2+Jc\tau c^* n^4\right)\varepsilon^2\\
& -3\left(m^2\rho+\xi\rho c^*+\left(J\delta^2+\beta\rho c^*+Jac^*\right)n^2+Jcc^* n^4\right)\varepsilon,\\
q_5 = \;& \left(-2bm\delta+am^2+a\xi c^*+\delta^2\xi-b^2c^*\right)n^2+\left(cm^2-2dm\delta-2bdc^*+a\beta c^*+c\xi c^*+\beta\delta^2\right)n^4\\
& +\left(c\beta c^*-d^2c^*\right)n^6+3\left(m^2\rho+\xi\rho c^*+\left(J\delta^2+\beta\rho c^*+Jac^*\right)n^2+Jcc^* n^4\right)\varepsilon\\
& -4\left(m^2\tau\rho+\tau\xi\rho c^*+\left(J\kappa\rho+Ja\tau c^*+\beta\tau\rho c^*+J\tau\delta^2\right)n^2+Jc\tau c^* n^4\right)\varepsilon^3+J\rho c^*\varepsilon^4\left(5-6\tau\varepsilon\right)\\
& -2\left(\beta\tau\delta^2-2dm\tau\delta-2bd\tau c^*+a\beta\tau c^*+Ja\kappa+\kappa\beta\rho+cm^2\tau+c\tau\xi c^*\right)n^4\varepsilon\\
& -2\left(\left(a\tau\xi c^*+\tau\delta^2\xi-b^2\tau c^*-2bm\tau\delta+\kappa\xi\rho+am^2\tau\right)n^2+\left(c\beta\tau c^*+Jc\kappa-d^2\tau c^*\right)n^6\right)\varepsilon,
\end{aligned}
$$

$$
\begin{aligned}
q_6 = \;& \left(c\kappa\beta-d^2\kappa\right)n^8+\left(a\kappa\beta+c\kappa\xi-2bd\kappa\right)n^6+\left(a\kappa\xi-b^2\kappa\right)n^4+\left(c\beta c^*-d^2c^*\right)n^6\varepsilon\\
& -\left[\left(am^2-2bm\delta+a\xi c^*+\delta^2\xi-b^2c^*\right)n^2+\left(cm^2-2dm\delta-2bdc^*+a\beta c^*+c\xi c^*+\beta\delta^2\right)n^4\right]\varepsilon\\
& +\left[\left(a\tau\xi c^*+\tau\delta^2\xi-b^2\tau c^*-2bm\tau\delta+\kappa\xi\rho+am^2\tau\right)n^2+\left(c\beta\tau c^*+Jc\kappa-d^2\tau c^*\right)n^6\right]\varepsilon^2\\
& +\left(\beta\tau\delta^2-2dm\tau\delta-2bd\tau c^*+a\beta\tau c^*+Ja\kappa+\kappa\beta\rho+cm^2\tau+c\tau\xi c^*\right)n^4\varepsilon^2\\
& -\left[m^2\rho+\xi\rho c^*+\left(J\delta^2+\beta\rho c^*+Jac^*\right)n^2+Jcc^* n^4\right]\varepsilon^3\\
& +\left[m^2\tau\rho+\tau\xi\rho c^*+\left(J\kappa\rho+Ja\tau c^*+\beta\tau\rho c^*+J\tau\delta^2\right)n^2+Jc\tau c^* n^4\right]\varepsilon^4\\
& -J\rho c^*\varepsilon^5+J\tau\rho c^*\varepsilon^6.
\end{aligned}
$$

□

We have the following lemma

Lemma 7.5. *For any $\varepsilon > 0$ sufficiently small, there exists n sufficiently large such that $\Lambda_2 < 0$.*

Proof. A direct calculation shows that

$$\Lambda_2 = \alpha_\varepsilon n^4 + P\left(\varepsilon, n^2\right),$$

where

$$\begin{aligned} P\left(\varepsilon, n^2\right) = & -J\rho c^* \left(2J\tau^2\delta^2\varepsilon + 2aJ\tau^2\varepsilon c^* + 2\kappa J\tau\varepsilon\rho - \kappa J\rho + 2\beta\tau^2\varepsilon\rho c^*\right) n^2 \\ & - 70J^2\tau^2\varepsilon^3\rho^2(c^*)^2 + 35J^2\tau\varepsilon^2\rho^2(c^*)^2 - 5J^2\varepsilon\rho^2(c^*)^2 - 2Jm^2\tau^2\varepsilon\rho^2 c^* \\ & - 2\xi J\tau^2\varepsilon\rho^2(c^*)^2 \end{aligned}$$

and

$$\alpha_\varepsilon = -2cJ^2\tau^2\rho(c^*)^2\varepsilon.$$

Clearly, for any $\varepsilon > 0$ sufficiently small, the term α_ε satisfies

$$\alpha_\varepsilon = -2cJ^2\tau^2\rho(c^*)^2\varepsilon < 0.$$

By taking n sufficiently large, we note that

$$\Lambda_2 = \alpha_\varepsilon n^4 + P\left(\varepsilon, n^2\right) < 0.$$

□

Proof of Theorem 7.1 From the lemma 7.5 the conditions of Hurwitz theorem are not satisfied. Therefore, there exists a solution to the problem such that w is to the right of the line $\mathrm{Re}\{z\} = -\varepsilon$. Consequently, we cannot obtain a uniform rate of exponential decay for all the solutions. As a result, the decay of the solutions is slow.

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