Common fixed point theorems for a multivalued closed mappings on ordered Banach space

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Abstract

The purpose of this paper is to present and study some common fixed point theorems for a finite family of multivalued closed mappings of Caristi and Pata type on ordered Banach space.

Keywords: Cones, common fixed point, multivalued mappings, closed mappings, Caristi type mapping, Pata type mapping, absolute derivative.

1 Introduction

Many authors studied the existence of a common fixed point for pairs of single and multivalued functions in various types of spaces such as metric spaces, ordered spaces and in recent years in complex valued metric space with weak or strong topology [3], [2] and [4]. Concerning multivalued mappings, the theory also developed in a parallel way starting from the extension of the Lery-Schauder topological degree to some classes of multifunctions by Petryshyn, Fitzpatrick, and Nussbaum in the 1970 s [5].

In [1] Dehage proved some common fixed point theorems for pairs of weakly isotone condensing mappings in an ordered Banach space and in [2] authors extended and generalized these results for a finite family of multivalued functions on an ordered Banach space using the measure of non compactness. Authors in [4] have obtained a common fixed point theorem of Caristi type mapping for two mappings using the absolute derivative in a complete metric space X. Motivated by the above results, in this paper we introduce a common fixed point theorems for a finite family of multivalued mappings of Caristi type using an explicit estimate of the convergence rate and also of Pata type using the absolute derivative on ordered Banach spaces.

2 Abstract background

Let X be a Banach space. We denote by 2^X the space of all nonempty subset of X. A nonempty closed convex subset C of X is said to be a cone if $(tC) \subset C$ for all $t \geq 0$ and $C \cap (-C) = \{0_X\}$. It is well known that a cone C induces a partial order in the Banach space X. We write for all $x, y \in X : x \preceq y$ if $y - x \in C$, $x \prec y$ if $y - x \in C$.

A cone C is said to be normal with a constant $n_C > 0$ if for all $u, v \in C$, $u \preceq v$ implies $\|u\| \leq n_C \|v\|$. We can also define a partial order on 2^X by : $A \leq B \iff x \leq y, \forall (x, y) \in A \times B$.

A multivalued function on X is a mapping from X into 2^X and a point $x^* \in X$ is called a fixed point of T, if $x^* \in T(x^*)$.

Let M be a nonempty subset of an ordered Banach space X, and let $T_1, T_2, ..., T_p : M \to 2^M$, be p-operators on M. We say that:

1. T_1 is $(T_k)_{2 \leq k \leq p}$ weakly-isotone increasing if for all $x \in M$, such that $x_1 \in T_1(x), x_2 \in T_2(x_1), ..., x_p \in T_p(x_{p-1})$, imply

$$T_1(x) \leq T_2(x_1) \leq ... \leq T_p(x_{p-1}) \leq T_1(x_p)$$

2. T_1 is $(T_k)_{2 \leq k \leq p}$ weakly-isotone decreasing if for all $x \in M$, such that $x_1 \in T_1(x), x_2 \in T_2(x_1), ..., x_p \in T_p(x_{p-1})$, imply

$$T_1(x_p) \leq T_p(x_{p-1}) \leq T_{p-1}(x_{p-2}) \leq ... \leq T_1(x).$$

3. T_1 is $(T_k)_{2 \leq k \leq p}$ weakly-isotone if it is either weakly-isotone increasing or weakly-isotone decreasing.

3 Main results

In this section, some common fixed point theorems are proved.

3.1 Common fixed point theorem of Caristi-type mappings

Let X be an ordered Banach space and $K \subset X$. Some authors have mentioned that a mapping $f : K \to K$ is called Caristi-type mappings if there exists a lower semicontinuous function $\varphi : K \to [0, +\infty)$ such that the inequalities

$$\|fx - x\| \leq \varphi(x) - \varphi(fx)$$

is satisfied for each $x \in K$.

Theorem 3.1. *Let X be an ordered Banach space, $p \geq 2$ be an integer. Let M be a nonempty closed subsets of X and $T_1, T_2, ..., T_p : M \to 2^M$ be p-closed mappings satisfying:*
For all $i \in \{1, ..., p\}$ there exists selections $f_i \in T_i$ such that every f_i is normally differentiable on M with the absolute derivative $f_i' : M \to \mathbb{R}^+$, satisfying:

$$\forall x, y \in M, \exists u \in T_i(x), \exists v \in T_j(y), \quad \|u - v\| \leq f_i'(x) - f_{i+1}'(u) + f_j'(y) - f_{j+1}'(v)$$

for all $i, j \in \{1, ..., p-1\}$.
Then $T_1, T_2, ..., T_p$ have a common fixed point.

Let M be a nonempty subset of an ordered Banach space X, and let $T_1, T_2, ..., T_p : M \to M$ be p mappings. Note that in this case the set $T(x)$ becomes single $\{T(x)\}$ and $y \in \{T(x)\}$ becomes $y = T(x)$. According to the previous theorem, we obtain the following result

Corollary 1. *Let X be an ordered Banach space, $p \geq 2$ be an integer. Let M be a nonempty closed subsets of X and $T_1, T_2, ..., T_p : M \to M$ be p-continuous mappings satisfying:*
For all $i \in \{1, ...p\}$ the function T_i is metrically differentiable on X with the absolute derivative $f_i' : X \to \mathbb{R}^+$, satisfying:

$$\forall x, y \in X, \quad \|T_i x - T_j y\| \leq f_i'(x) - f_{i+1}'(x) + f_j'(y) - f_{j+1}'(y)$$

for all $i, j \in \{1, ..., p-1\}$.
Then $T_1, T_2, ..., T_p$ have a common fixed point.

3.2 Common fixed point theorem of Pata-type mappings

Pata extended the Banach contraction principle with weaker hypotheses than those of the Banach contraction principle.

Theorem 3.2. *Let X be an ordered Banach space, $p \geq 2$ be an integer. Let M be a nonempty closed subsets of X and $T_1, T_2, ..., T_p : M \to 2^M$ be p-closed mappings satisfying:*

1. *T_1 is $(T_k)_{2\leq k\leq p}$ weakly isotone.*
2. *$\forall x, y \in M, \exists u \in T_i(x), \exists v \in T_j(y)$ we have*

$$\|u - v\| \leq (1 - \varepsilon)\|x - y\| + c\ \varepsilon^{\alpha}[1 + \|x\| + \|y\|]^{\beta}$$

for all $i, j \in \{1, ..., p\}$, where $c > 0$, $\alpha > 1$ et $\beta \in [0, \alpha]$.
Then $T_1, T_2, ..., T_p$ have a common fixed point.

By theorem 2, we obtain the following result.

Corollary 2. *Let X be an ordered Banach space, $p \geq 2$ be an integer. Let M be a nonempty closed subsets of X and $T_1, T_2, ..., T_p : M \to M$ be p-closed mappings satisfying:*

1. *T_1 is $(T_k)_{2\leq k\leq p}$ weakly isotone.*
2. *$\forall x, y \in M, \exists u \in T_i(x), \exists v \in T_j(y)$ we have*

$$\|T_i(x) - T_j(y)\| \leq (1 - \varepsilon)\|x - y\| + c\ \varepsilon^{\alpha}[1 + \|x\| + \|y\|]^{\beta}$$

for all $i, j \in \{1, ..., p\}$, where $c > 0$, $\alpha > 1$ et $\beta \in [0, \alpha]$.
Then $T_1, T_2, ..., T_p$ have a common fixed point.

4 Application

To validate our main results we consider the system of nonlinear hyperbolic differential inclusions:

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t \partial x} \in F_i(t, x, u(t, x)) & a.e(t, x) \in [0, 1] \times [0, 1] \\ u(t, 0) = \alpha(t) + \alpha(0),\ u(0, x) = \beta(x) \end{cases} \tag{1}$$

where $\alpha, \beta \in C([0,1]\times[0,1], +\infty)$ and $F_i : [0,1]\times[0,1]\times\mathbb{R} \to 2^{\mathbb{R}}$, for all $i \in \{1, 2, ..., p\}$.
By a common solution for the system of integral inclusions (1) , we mean a continuous function $u : [0,1]\times[0,1] \to \mathbb{R}$, such that:

$$u(t,x) = \alpha(t) + \beta(x) + \int_0^t \int_0^x v_i(s,z)dsdz$$

for $v_i \in \mathcal{L}^1([0,1]\times[0,1], \mathbb{R})$ satisfying $v_i(t,x) \in F_i(t,x,u(t,x))$ a.e.$(t,x) \in [0,1]\times[0,1]$, for all $i \in \{1, 2, ..., p\}$.
We introduce notations which are used throughout this section $E = C([0,1]\times[0,1], \mathbb{R})$ is the Banach space consisting of all continuous functions from $[0,1]\times[0,1]$ into $\mathbb{R}$ with the norm

$$\|u\| = \sup_{(t,x)\in[0,1]\times[0,1]} |u(t,x)|, u \in E.$$

By theorem 2.2, we see that $(T_1, T_2, ..., T_p)$ have a common fixed point in M . This implies that system of inclusions (1) has a common solution on E.

References

[1] **B.C. Dhage, D. ORegan and R. P. Agarwal**.*Common fixed point theorems for a pair of countably condensing mappings in ordered Banach spaces*. J. Appl. Math. Stoch. Anal. 16 (3), 243 - 248 (2003).

[2] **M. Farid, K. Chaira, El M. Marhrani and M. Aamri**. *fixed point theorems for a finite family of multivalued mappings in an ordered Banach space*, Fixed point theory and applications, 17, 2018, 1 - 14.

[3] **C. Vetro**. *fixed points in ordered Banach spaces*. Le Mathematiche LXIII, 93- 100 (2008).

[4] **M. Muslikh1, A. Kihman, S. Sapar and N. Bachok**. *Common Fixed-Point Theorems of Caristi-Type Mappings by Using its Absolute Derivative*, Appl. Math. Inf. Sci. 13, No. 1, 17 - 24 (2019).

[5] **R. D. Nussbaum**.*The fixed point index and asymptotic fixed point theorems for k-set contractions*. Bull. Am. Math. Soc. 75, 490 - 495 (1969).