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Some norms and numerical radius inequalities in Hilbert spaces

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Word of Thanks and Appreciation

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This work is dedicated to my beloved mother, whose boundless love, tireless sacrifices, and unwavering support have been the cornerstone of my journey. In the light of the blessed saying, *"Paradise lies at the feet of mothers,"* I humbly pray that Allah grants her a long life, continued well-being, and eternal happiness. She remains my source of strength, comfort, and inspiration.

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Table des matières

Notations:

— $\mathbb{K}$: $\mathbb{R}$ or $\mathbb{C}$.
— $\mathcal{H}$: Complex Hilbert space.
— $\langle\cdot,\cdot\rangle$: The inner product of $\mathcal{H}$.
— $\overline{M}$: The closure of M.
— M° : The interior of M.
— ∂M : The boundary of M.
— $M^{\perp}$: The orthogonal complement of M.
— $\oplus$: The sign of direct sum.
— $\mathcal{B}(\mathcal{H})$: Banach algebra of all bounded linear operators on Hilbert space $\mathcal{H}$.
— $\mathcal{I}(\mathcal{H})$: The set of invertible operators in $\mathcal{B}(\mathcal{H})$.
— T : A bounded linear operator defined on $\mathcal{H}$ ($T \in \mathcal{B}(\mathcal{H})$).
— $\|T\|$: The norm of T.
— T^{-1} : The inverse operator of T.
— T^{*} : The adjoint operator of T.
— $\Re(T)$: The real part of T.
— $\Im(T)$: The imaginary part of T.
— $|T|$: The absolute value of T.
— $\mathcal{R}(T)$: The range of T.
— $\mathcal{N}(T)$: The kernel of T.
— $\sigma(T)$: The spectrum of T.
— $\sigma_r(T)$: The residual spectrum of T.
— $\sigma_p(T)$: The point spectrum of T.
— $\sigma_c(T)$: The continuous spectrum of T.
— $\sigma_{ap}(T)$: The approximate point spectrum of T.
— $\rho(T)$: The resolvent of T.
— $r(T)$: Spectral radius of T.
— $W(T)$: The numerical range of T.
— $w(T)$: The numerical radius of T.
— $Hw(PQ)$:It is $dw(T)$.
— $w_H(PQ)$:It is $w_e(T)$.
— $w_h(T)$:It is furst new operater .
— $w_{oh}(T)$:It is secund new operater .

Introduction

Operator theory on Hilbert spaces is a fundamental branch of functional analysis with profound applications in pure and applied mathematics, quantum mechanics, engineering, and statistics. Over the past decades, this theory has attracted significant attention due to its deep theoretical insights and broad utility.

Historically, the study of quadratic forms laid the groundwork for key concepts such as the **numerical range** of an operator. Let $\mathcal{H}$ be a nontrivial complex Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$.

chapter 1 see[1.3.7.8.13]

in This chapter develops the fundamental theory of Hilbert spaces and their associated linear operators.

chapter 2see[4.6.9.11]

In this chapter, we examine the numerical radius $w(T)$, a fundamental concept in operator theory that provides crucial insights into the behavior of bounded linear operators on Hilbert spaces.

chapter 3see[2.5.10.12.14.15]

this chapter, we develop the theory of two fundamental concepts in operator theory:

[leftmargin=*]The **numerical range** $W(T)$, which captures the complete spatial behavior of an operator

The **numerical radius** $w(T)$, which quantifies the operator's maximum "magnitude"

These tools provide deep insights into the structure and applications of linear operators on Hilbert spaces,The Davis–Wielandt radius is spesial case of Euclidean Operator Radius.

chapter 4see[14.15]

we have old operater dw and Jw and w and we have gate new operater HW and $w_h$andw_{oh} ,ruselt somr spesial case .

1 Hilbert spaces with Linear operators

1 .1 Normed spaces

Defnition 1.1.1:

Let E be a vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A **norm** on E is a function denoted by $\|\cdot\|$, defined on E with values in $\mathbb{R}$, satisfying the following properties:
— $\|x\| = 0$ if and only if $x = 0$.
— $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in E$ and $\lambda \in \mathbb{K}$.
— $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in E$.
Defnition 1.1.2 :

Let E be a vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We say that E is a **normed vector space** if it is equipped with a norm $\|\cdot\|$.
Proposition 1.1.1 :

Every normed vector space $(E, \|\cdot\|)$ is a metrizable space.
For the triangle inequality, we write:

$$\begin{aligned} d(x,y) = \|x-y\| &= \|(x-z)+(z-y)\| \\ &\leq \|x-z\| + \|z-y\| \\ &= d(x,z) + d(z,y). \end{aligned}$$

Defnition 1.1.3:

Let $\{x_n\}$ be a sequence of elements in a normed space $(E, \|\cdot\|)$. We say that the sequence $\{x_n\}$ is a Cauchy sequence if the following relation holds:

$$\forall \varepsilon > 0,\ \exists N_\varepsilon,\ \forall p, q \geq N_\varepsilon, \text{ we have } \|x_p - x_q\| < \varepsilon.$$

Defnition 1.1.4 :

A normed vector space $(E, \|\cdot\|)$ is said to be complete if every Cauchy sequence $\{x_n\}$ of elements in E is a convergent sequence in E.
Defnition 1.1.5 :

A Banach space $(E, \|\cdot\|)$ is a normed vector space that is complete with respect to the metric induced by its norm.
Defnition 1.1.6 :

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two normed vector spaces over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The product space EF is defined by:

$$G = EF = \{(x,y) \mid x \in E, y \in F\}.$$

This product space is a normed vector space over $\mathbb{K}$ with one of the following product norms:
— $\|(x,y)\|_1 = \|x\|_E + \|y\|_F$ for all $x \in E$, $y \in F$.
— $\|(x,y)\|_p = (\|x\|_E^p + \|y\|_F^p)^{\frac{1}{p}}$ for all $x \in E$, $y \in F$, $1 < p < \infty$.
— $\|(x,y)\|_\infty = \max\{\|x\|_E, \|y\|_F\}$ for all $x \in E$, $y \in F$.
Defnition 1.1.7 :
Every finite-dimensional subspace F of a normed space $(E, \|\cdot\|)$ is complete.
Theorem 1.1.1 :[1]
Let E and F be two normed spaces. The set $\mathcal{L}(E,F)$ of all continuous linear operators A from E into F, equipped with the norm $\|A\|$, is a normed space.
Theorem 1.1.2 :[8]
Let E be a normed space and F a Banach space. Then, $\mathcal{L}(E,F)$ is a Banach space.
Proposition 1.1.2
The smallest constant C satisfying the relation (1) is called the norm of A, denoted by $\|A\|$, and is given by:

$$\|A\| = \sup_{x \neq 0} \frac{\|A(x)\|_F}{\|x\|_E} = \sup_{\|x\|=1} \|A(x)\|_F = \sup_{\|x\|\leq 1} \|A(x)\|_F. \tag{P 1.1.2}$$

Defnition 1.1.8:
Let E be a vector space equipped with two norms $(E, \|\cdot\|_1)$ and $(E, \|\cdot\|_2)$. We say that the two norms are *equivalent* if we can find two positive constants α and β such that:

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1, \quad \forall x \in E.$$

In other words, the two norms are said to be equivalent if and only if the identity map from E into E is an isomorphism between the normed spaces $(E, \|\cdot\|_1)$ and $(E, \|\cdot\|_2)$.

1 .2 Some Defnition of Operators theory

Defnition 1.2.1:

Let E and F be two normed spaces. An operator A defined on a subset $G \subseteq E$ with values in F is said to be continuous at a point $x_0 \in G$ if the following property holds:

$$\text{For every sequence } \{x_n\} \text{ in } G \text{ converging to } x_0, \text{ the sequence } \{A(x_n)\} \text{ converges to } A(x_0),$$

that is:

$$\lim_{n\to\infty} A(x_n) = A\left(\lim_{n\to\infty} x_n\right) = A(x_0).$$

Defnition 1.2.2 :
A linear operator A defined on E with values in F is said to be bounded if there exists a positive constant $C > 0$ such that:

$$\|A(x)\|_F \leq C\|x\|_E, \quad \forall x \in E. \tag{D 1.2.2}$$

Proposition 1.2.1:
The norm $\|A\| = \sup \|A(x)\|_F$ over the unit ball is always finite for every continuous operator.
Theorem 1.2.1 :[1]

A linear operator A is continuous if and only if it is bounded.

1 .3 Hilbert space

Defnition 1.3.1 :
The topological dual of the space E, denoted by E^*, is the Banach space of continuous linear functionals $\mathcal{L}(E, \mathbb{K})$.
Defnition 1.3.2 :
Let X be a complex vector space,
on X is a map $B : XX \to \mathbb{C}$ such that, for every $y \in X$, the map $x \mapsto B(x, y)$ is linear, and for every $x \in X$, the map $y \mapsto B(x, y)$ is antilinear (from X into $\mathbb{C}$).
Recall that a bilinear form B on a real vector space X is said to be
Defnition 1.3.3:
A function $B : XX \to \mathbb{R}$ (or any other codomain) is called **symmetric** if for all $x, y \in X$,

$$B(y, x) = B(x, y).$$

Defnition 1.3.4 :
Let X be a vector space equipped with an inner product. For all $x, y \in X$, we have:

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle. \tag{D 1.3.4}$$

Corollary 1.3.1 :
Let X be a vector space equipped with an inner product. The map $x \mapsto \langle x, x\rangle^{1/2}$ is a semi-norm on X.
Another useful relation is the

$$\langle x + y, x + y\rangle + \langle x - y, x - y\rangle = 2\left(\langle x, x\rangle + \langle y, y\rangle\right). \tag{C 1.3.1}$$

Defnition 1.3.5 :

An inner product on a real or complex vector space X is thus a map $(x, y) \mapsto \langle x, y\rangle$ from XX into $\mathbb{K}$ such that:
— For every fixed $y \in X$, the map $x \mapsto \langle x, y\rangle$ is $\mathbb{K}$-linear.
— For all $x, y \in X$, we have $\langle y, x\rangle = \overline{\langle x, y\rangle}$.
— For all $x \in X$, we have $\langle x, x\rangle \geq 0$.

Proposition 1.3.1:
Let X be a pre-Hilbert space. For every vector $y \in X$, the linear form $\ell_y : x \mapsto \langle x, y\rangle$ is continuous from X into $\mathbb{K}$. The map $y \mapsto \ell_y$ is antilinear and isometric from X into X^*.

Defnition 1.3.6 :
A Hilbert space is a vector space H (real or complex) equipped with an inner product $(x, y) \mapsto \langle x, y\rangle$ such that the semi-norm $x \mapsto \sqrt{\langle x, x\rangle}$ is a norm on H, and this norm makes the space complete.

Theorem 1.3.1 :[13]

For every separable Hilbert space H of infinite dimension, there exists an orthonormal basis $(e_n)_{n\geq 0}$.

If H is finite-dimensional, the existence of an orthonormal basis (which is finite) has been seen in undergraduate studies.

Theorem 1.3.2:
Let H be a Hilbert space and C a non-empty closed convex subset of H. For every $x \in H$, there exists a unique point $y_0 \in C$ at which the function $y \mapsto \|y - x\|$ attains its minimum on C. Moreover, we have:

$$\forall y \in C, \quad \text{Re}\langle x - y_0, y - y_0\rangle \leq 0.$$

Proposition 1.3.2:
Let H be a Hilbert space. The isometric antilinear map $y \mapsto \ell_y$ defined by:

$$\forall x \in H, \quad \ell_y(x) = \langle x, y\rangle$$

is a bijection from H onto its dual H^*. In other words, for every continuous linear functional ℓ on H, there exists a unique vector $y_\ell \in H$ such that:

$$\forall x \in H, \quad \ell(x) = \langle x, y_\ell\rangle.$$

Defnition 1.3.7 :
An **inner product** on a vector space E is a mapping denoted by $\langle \cdot, \cdot\rangle : EE \to \mathbb{K}$ (where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$) that satisfies the following properties:

1. **Positivity**: $\langle x, x\rangle \geq 0$ for all $x \in E$.
2. **Definiteness**: $\langle x, x\rangle = 0$ if and only if $x = 0$.
3. **Conjugate Symmetry**: $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in E$, where $\bar{\cdot}$ denotes the complex conjugate. If $\mathbb{K} = \mathbb{R}$, this simplifies to $\langle x, y\rangle = \langle y, x\rangle$.
4. **Linearity in the First Argument**: $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for all $x, y, z \in E$.
5. **Homogeneity in the First Argument**: $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ for all $x, y \in E$ and $\lambda \in \mathbb{K}$.

A vector space E equipped with an inner product $\langle \cdot, \cdot\rangle$ is called a pre-Hilbert space.

Proposition 1.3.3:
Let $\|x\| = \sqrt{\langle x, x\rangle}$. Then, the following properties hold:

1. **Cauchy-Schwarz Inequality**: For all $x, y \in E$,

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

2. **Parallelogram Identity**: For all $x, y \in E$,

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

3. **Norm Induced by the Inner Product**: The mapping $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm on E (called the Hilbertian norm).

Parallelogram Identity: Let $x, y \in E$. Then:

$$\langle x + y, x + y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle,$$
$$\langle x - y, x - y\rangle = \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle.$$

Adding these two equations, we obtain the parallelogram law:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Defnition 1.3.8 :
Let $x, y \in E$. The vectors x and y are said to be orthogonal if and only if $\langle x, y\rangle = 0$. We write $x \perp y$.

Defnition 1.3.9:
Let $A \subset E$. The **orthogonal complement** of A, denoted $A^\perp$, is the set:

$$A^{\perp} = \{x \in E \mid \langle x, a\rangle = 0 \text{ for all } a \in A\}.$$

Proposition 1.3.4 :
The orthogonal complement $A^{\perp}$ of any subset A of E is a vector subspace of E.

Theorem 1.3.3 :
Let $(E, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$. If the norm $\|\cdot\|$ satisfies the parallelogram identity, then the mapping $\langle\cdot,\cdot\rangle : EE \to \mathbb{R}$, defined by:

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right), \tag{T 1.3.3}$$

is an inner product on E, and $\|x\| = \sqrt{\langle x, x\rangle}$.

This theorem generalizes to the case of a normed vector space over $\mathbb{C}$ with an appropriate formula for the inner product.

Theorem 1.3.4:

Let E be a Hilbert space, and let F be a closed and convex subset of E. Given $f \in E$, there exists a unique element $f^* \in F$ such that:

$$\|f - f^*\| = \min_{v \in F} \|f - v\|.$$

Proposition 1.3.5 :
Let E be a Hilbert space, and let F be a subspace of E. For $f \in E$, f^* is the best approximation of f in F if and only if $f^* \in F$ satisfies:

$$\langle f - f^*, v\rangle = 0 \quad \forall v \in F.$$

This condition can also be written as $f - f^* \in F^{\perp}$.

Proposition 1.3.6:
The Gram matrix G is positive definite, that is:

$$\langle Gx, x\rangle \geq 0 \quad \forall x \in \mathbb{K}^n.$$

Proposition 1.3.7:
Let E be a Hilbert space, and let F be a closed subspace of E. Then:

$$E = F \oplus F^{\perp}.$$

Theorem 1.3.5:

Let E be a Hilbert space, and let $B = \{\varphi_k \mid k \in \mathbb{N}\}$ be an orthogonal system. The series:

$$\sum_{k=0}^{+\infty} \alpha_k \varphi_k$$

converges in E if and only if the numerical series:

$$\sum_{k=0}^{+\infty} |\alpha_k|^2 \|\varphi_k\|^2$$

converges.

Definition 1.3.10:
Let E be a Hilbert space. An orthogonal system $B = \{\varphi_k \mid k \in \mathbb{N}\}$ is called a **Hilbert basis** (or orthogonal basis) if and only if for every $f \in E$, we have:

$$f = \sum_{k=0}^{+\infty} c_k(f)\varphi_k.$$

In other words, the limit of the Fourier series of f is f itself.

Theorem 1.3.6 :
Let E be a Hilbert space, and let $B = \{\varphi_k \mid k \in \mathbb{N}\}$ be an orthogonal system. The following assertions are equivalent:

(A) B is a total system, i.e., $B^{\perp} = \{0\}$.

(B) The set of finite linear combinations of elements of B is dense in E.

(C) B is a Hilbert basis.

(D) For every $f \in E$, the following Parseval identity holds:

$$\|f\|^2 = \sum_{k=0}^{+\infty} |c_k(f)|^2 \|\varphi_k\|^2 = \sum_{k=0}^{+\infty} \frac{|\langle f, \varphi_k \rangle|^2}{\|\varphi_k\|^2}.$$

Definition 1.3.11:

An *inner product* on a real or complex vector space X is a map $(x, y) \mapsto \langle x, y \rangle$ from XX into $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) such that:

— For every fixed $y \in X$, the map $x \mapsto \langle x, y \rangle$ is $\mathbb{K}$-linear.

— For all $x, y \in X$, we have $\langle y, x \rangle = \overline{\langle x, y \rangle}$.

— For all $x \in X$, we have $\langle x, x \rangle \geq 0$.

Some authors require that an inner product satisfy $\langle x, x \rangle > 0$ for all $x \neq 0_X$. In this case, the semi-norm $x \mapsto \sqrt{\langle x, x \rangle}$ from the previous corollary is a norm on the space X.

A *pre-Hilbert space* is a (real or complex) vector space X equipped with an inner product such that the semi-norm $p(x) = \sqrt{\langle x, x \rangle}$ is a norm on X. Every pre-Hilbert space will be considered as a normed space, equipped with the above norm, which will henceforth be denoted simply by $\|x\|$.

Definition 1.3.12:

The mapping $\|x\| = \sqrt{\langle x, x \rangle}$ satisfies the properties of a norm:

— **Positivity:** $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$.

— **Homogeneity:** $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{K}$.

— **Triangle Inequality:** $\|x + y\| \leq \|x\| + \|y\|$ (which follows from the Cauchy–Schwarz inequality).

Thus, $\|x\|$ is a norm on E.

2 digital image and numerical radius

2 .1 Spectrum of a Linear Operator

Definition 2.1.1 [9]
Let $T \in B(H)$. The spectrum of T is denoted by $\sigma(T)$, and it is defined as follows:

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\} \tag{D 2.1.1}$$

And the resolvent of T is denoted by $\rho(T)$, and $\rho(T) = \mathbb{C} \setminus \sigma(T)$.

Proposition 2.1.1 Let $T \in B(H)$, and let R be the resolvent function of T. Then:

1. $R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu)$ for all $\lambda, \mu \in \rho(T)$.
2. $R(\lambda)R(\mu) = R(\mu)R(\lambda)$ for all $\lambda, \mu \in \rho(T)$.
3. R is continuous.

Theorem 2.1.1 [9] Let $T \in B(H)$. Then:

(1) If $|\lambda| > \|T\|$, then $\lambda \in \rho(T)$.
(2) $\sigma(T)$ is a closed set.

Corollary 2.1.1 Let $T \in B(H)$. Then:

(1) $\sigma(T)$ is compact.
(2) $\rho(T)$ is an open non-empty set.

Proposition 2.1.2 Let $T \in B(H)$. Then $\sigma(T)$ is non-empty.

Lemma 2.1.1 Let $T \in B(H)$. Then

$$\sigma(T^*) = \{\lambda : \lambda \in \sigma(T)\}. \tag{L.2.1.1}$$

Theorem 2.1.2 Let $T \in B(H)$. Then:

(1) If p is a polynomial then $\sigma(p(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$.
(2) If T is invertible then $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$.

Corollary 2.1.2 Let $T \in B(H)$, $\alpha \in \mathbb{C}$, and $n \in \mathbb{N}$, then:

(1) $\sigma(\alpha T) = \alpha\sigma(T) = \{\alpha\lambda : \lambda \in \sigma(T)\}$.
(2) $\sigma(T^n) = (\sigma(T))^n = \{\lambda^n : \lambda \in \sigma(T)\}$.

Theorem 2.1.3 Let $T, S \in B(H)$. Then $\sigma(TS) \cup \{0\} = \sigma(ST) \cup \{0\}$.

Theorem 2.1.4 Let $T \in B(H)$ be self-adjoint, then $\sigma(T) \subset \mathbb{R}$.

Corollary 2.1.3 Let $T \in B(H)$ be positive. Then:

(1) $\sigma(T) \subset \mathbb{R}_+$.
(2) $\sigma(\sqrt{T}) = \sqrt{\sigma(T)} = \{\sqrt{\lambda} : \lambda \in \sigma(T)\}$.

Definition 2.1.3 [15]
Let $T \in B(H)$. Then:

(1) The point spectrum of T is the set:

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : N(T - \lambda I) \neq \{0\}\} \tag{D 2.1.3(1)}$$

A $\lambda \in \sigma_p(T)$ is called an eigenvalue of T, and the vector $x \in H \setminus \{0\}$ that verifies $Tx = \lambda x$ is called an eigenvector of λ.

(2) The continuous spectrum of T is the set:

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : N(T - \lambda I) = \{0\} \text{ and } R(T - \lambda I) = H\} \tag{D 2.1.3(2)}$$

(3) The residual spectrum of T is the set:

$$\sigma_r(T) = \{\lambda \in \mathbb{C} : N(T - \lambda I) = \{0\} \text{ and } R(T - \lambda I) \subsetneq H\} \tag{D 2.1.3(3)}$$

(4) The approximate point spectrum of T is the set:

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \exists (x_n) \subset H : \|x_n\| = 1 \text{ and } \lim_{n\to\infty} (T - \lambda I)x_n = 0\} \tag{D 2.1.3(4)}$$

Remark 2.1.1

(1) $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$, where $\sigma_p(T)$, $\sigma_c(T)$, and $\sigma_r(T)$ are mutually disjoint, i.e.

$$\sigma_p(T) \cap \sigma_c(T) = \sigma_p(T) \cap \sigma_r(T) = \sigma_c(T) \cap \sigma_r(T) = \emptyset$$

(2) $\sigma(T) = \sigma_{ap}(T) \cup \sigma_r(T)$, also $\sigma_p(T) \subset \sigma_{ap}(T)$.

Corollary 2.1.4 [11] If H is a finite-dimensional Hilbert space, then $\sigma(T) = \sigma_p(T)$ for all $T \in B(H)$.

Proposition 2.1.3 [6] Let $T \in B(H)$ be normal. If λ and μ are disjoint (i.e., $\lambda \neq \mu$) eigenvalues, then there exist $x, y \in H \setminus \{0\}$ such that $Tx = \lambda x$ and $Ty = \mu y$. Consequently, we have:

$$\langle x, y \rangle = 0.$$

2 .2 Spectral radius

Definition 2.2.1 Let $T \in B(H)$. The spectral radius of T is denoted by $r(T)$, and it is defined as follows:

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

Needless to say, that $\sup_{\lambda \in \sigma(T)} |\lambda|$ exists and is finite (remember that $\sigma(T)$ is a compact non-empty set).

Corollary 2.2.1

Let $T \in B(H)$. Then $r(T) \leq \|T\|$.

proof Assume $|\lambda| > \|T\|$. Then we have:

$$\left\| \frac{T}{\lambda} \right\| = \frac{\|T\|}{|\lambda|} < 1$$

By the Neumann series theorem, the operator $I - \frac{T}{\lambda}$ is invertible. Consequently:

$$T - \lambda I = -\lambda \left(I - \frac{T}{\lambda} \right)$$

is invertible as a product of invertible operators. Therefore $\lambda \in \rho(T)$.

This shows that for any $\lambda \in \sigma(T)$, we must have $|\lambda| \leq \|T\|$. Thus:

$$\sup_{\lambda \in \sigma(T)} |\lambda| \leq \|T\|$$

By definition of the spectral radius $r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$, we conclude:

$$r(T) \leq \|T\|$$

Proposition 2.2.1 [6] Let $T, S \in B(H)$, $\alpha \in \mathbb{C}$, and $n \in \mathbb{N}$. Then the following hold:

(1) $r(\alpha T) = |\alpha| r(T)$.
(2) $r(T^n) = r(T)^n$.
(3) $r(TS) = r(ST)$.
(4) $r(T^*) = r(T)$.
(5) If $T \geq 0$, then $r(\sqrt{T}) = \sqrt{r(T)}$.

proof We prove the spectral radius properties systematically:

(1) **Scaling property:** By Corollary 2.1.2 ($\sigma(\alpha T) = \alpha \sigma(T)$):

$$r(\alpha T) = \sup_{\lambda \in \sigma(\alpha T)} |\lambda| = \sup_{\lambda \in \sigma(T)} |\alpha \lambda| = |\alpha| \sup_{\lambda \in \sigma(T)} |\lambda| = |\alpha| r(T)$$

(2) **Power property:** Using Corollary 2.1.2 ($\sigma(T^n) = (\sigma(T))^n$):

$$r(T^n) = \sup_{\lambda \in \sigma(T^n)} |\lambda| = \sup_{\lambda \in \sigma(T)} |\lambda^n| = \left(\sup_{\lambda \in \sigma(T)} |\lambda| \right)^n = r(T)^n$$

(3) **Product equivalence:** From Theorem 2.1.3 ($\sigma(TS) \cup \{0\} = \sigma(ST) \cup \{0\}$):

$$\begin{aligned} r(TS) = \sup_{\lambda \in \sigma(TS)} |\lambda| &= \sup_{\lambda \in \sigma(TS) \cup \{0\}} |\lambda| \\ &= \sup_{\lambda \in \sigma(ST) \cup \{0\}} |\lambda| = \sup_{\lambda \in \sigma(ST)} |\lambda| = r(ST) \end{aligned}$$

(4) **Adjoint property:** By Lemma 2.1.1 ($\sigma(T^*) = \{\overline{\lambda} : \lambda \in \sigma(T)\}$):

$$r(T^*) = \sup_{\lambda \in \sigma(T^*)} |\lambda| = \sup_{\lambda \in \sigma(T)} |\overline{\lambda}| = \sup_{\lambda \in \sigma(T)} |\lambda| = r(T)$$

(5) **Square root property:** From Corollary 2.1.3 ($\sigma(\sqrt{T}) = \sqrt{\sigma(T)}$):

$$r(\sqrt{T}) = \sup_{\mu \in \sigma(\sqrt{T})} |\mu| = \sup_{\lambda \in \sigma(T)} \sqrt{|\lambda|} = \sqrt{\sup_{\lambda \in \sigma(T)} |\lambda|} = \sqrt{r(T)}$$

Theorem 2.2.1 [11] (Gelfand's formula)
Let $T \in B(H)$. Then

$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|T^n\|^{\frac{1}{n}}. \tag{T 2.2.1}$$

Corollary 2.2.2
Let $T \in B(H)$ be a normal operator. Then

$$r(T) = \|T\|.$$

Proof. we have for all $n \in \mathbb{N}$:

$$\|T^n\| = \|T\|^n \Rightarrow \lim_{n\to\infty} \|T^n\|^{1/n} = \|T\|$$

Since

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} \Rightarrow r(T) = \|T\|$$

Theorem 2.2.2
Let $T, S \in B(H)$ such that $TS = ST$. Then:

(1) $r(TS) \le r(T)r(S)$.
(2) $r(T+S) \le r(T) + r(S)$.

2 .3 Numerical range of a linear operato

Definition 2.3.1 [4]
Let $T \in B(H)$. The numerical range of T is denoted by $W(T)$, and it is defined as follows:

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\} \tag{D 2.3.1}$$

Proposition 2.3.1 [9]
Let $T, S \in B(H)$, $\alpha, \beta \in \mathbb{C}$ and $U \in B(H)$ be unitary. Then:

(1) $W(\alpha T + \beta I) = \alpha W(T) + \beta = \{\alpha z + \beta : z \in W(T)\}$.
(2) $W(T+S) \subset W(T) + W(S)$.
(3) $W(T^*) = \{z : z \in W(T)\}$.
(4) $W(\mathfrak{Re}(T)) = \mathfrak{Re}W(T) = \{\mathfrak{Re}z : z \in W(T)\}$ and $W(\mathfrak{Im}(T)) = \mathfrak{Im}W(T) = \{\mathfrak{Im}z : z \in W(T)\}$.
(5) $W(U^*TU) = W(T)$.

Corollary 2.3.1
Let $T \in B(H)$ and $\alpha \in \mathbb{C}$, then:

(1) $W(T) = \{\alpha\}$ if and only if $T = \alpha I$.
(2) If H is a finite-dimensional space, then (T) is compact.

Proposition 2.3.2
Let $T \in B(H)$. Then:

(1) $\sigma_p(T), \sigma_r(T) \subset W(T)$.
(2) $\sigma_{ap}(T) \subset W(T)$.

Definition 2.3.2
$\lambda \in \sigma_r(T)$ if $T - \lambda I$ is injective but $\mathrm{Ran}(T - \lambda I)$ is not dense in H.

Proof:
Let $\lambda \in \sigma_r(T)$. By the Closed Range Theorem:

$$\mathrm{Ran}(T - \lambda I)^{\perp} = \ker(T^* - \overline{\lambda} I) \neq \{0\}.$$

There exists $y \neq 0$ such that $T^*y = \overline{\lambda} y$, so $\overline{\lambda} \in \sigma_p(T^*)$.

By Proposition 2.1.3 (numerical range of the adjoint):

$$W(T^*) = \{\overline{\langle Tx, x\rangle} : x \in H, \|x\| = 1\}.$$

Since $\overline{\lambda} \in W(T^*)$, it follows that $\lambda \in W(T)$.

Corollary 2.3.2
Let $T \in B(H)$. Then $\sigma(T) \subset W(T)$.

Theorem 2.3.1[6] For a bounded linear operator T on a complex Hilbert space H, the spectrum $\sigma(T)$ is contained in the closure of the numerical range $W(T)$:

$$\sigma(T) \subset \overline{W(T)}.$$

Proof
The spectrum $\sigma(T)$ can be decomposed as:

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_r(T),$$

where:
— $\sigma_{ap}(T)$ is the approximate point spectrum: values λ for which $T - \lambda I$ is not bounded below.
— $\sigma_r(T)$ is the residual spectrum: values λ for which $T - \lambda I$ is injective but has non-dense range.

Step 1: Show $\sigma_{ap}(T) \subset \overline{W(T)}$ Let $\lambda \in \sigma_{ap}(T)$. By definition, there exists a sequence $\{x_n\} \subset H$ with $\|x_n\| = 1$ such that:

$$\lim_{n\to\infty} \|(T - \lambda I)x_n\| = 0.$$

Then:

$$|\langle Tx_n, x_n\rangle - \lambda| = |\langle (T - \lambda I)x_n, x_n\rangle| \leq \|(T - \lambda I)x_n\| \to 0.$$

Hence, $\langle Tx_n, x_n\rangle \to \lambda$ and since each $\langle Tx_n, x_n\rangle \in W(T)$ and $W(T)$ is closed (or considering its closure), we get:

$$\lambda \in \overline{W(T)}.$$

Step 2: Show $\sigma_r(T) \subset \overline{W(T)}$ Let $\lambda \in \sigma_r(T)$. Then $T - \lambda I$ is injective but $\text{Ran}(T - \lambda I)$ is not dense. By the Closed Range Theorem:

$$\text{Ran}(T - \lambda I)^{\perp} = \ker(T^* - \overline{\lambda} I) \neq \{0\}.$$

Thus, there exists $y \neq 0$ such that $T^*y = \overline{\lambda}y$, so $\overline{\lambda} \in \sigma_p(T^*)$.

The numerical range of the adjoint satisfies:

$$W(T^*) = \{\overline{\langle Tx, x\rangle} : x \in H, \|x\| = 1\}.$$

Since $\overline{\lambda} \in W(T^*)$, it follows that $\lambda \in \overline{W(T)}$.

Step 3: Combine Results From Steps 1 and 2:

$$\sigma_{ap}(T) \subset \overline{W(T)}, \quad \sigma_r(T) \subset \overline{W(T)}.$$

Therefore:

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_r(T) \subset \overline{W(T)}.$$

Theorem 2.3.2 [9]
Let $T \in B(H)$ and $\alpha, \beta \in \mathbb{C}$ such that $\alpha \neq 0$. Then the following hold:

$$W(T) \text{ is a line segment, i.e., } W(T) = \{\alpha t + \beta : t \in \mathbb{R}\}$$

if and only if

$$T = \alpha S + \beta I,$$

where S is self-adjoint. beginitemize

4. Assume $W(T)$ is a line segment (or line) given by $\{\alpha t + \beta : t \in \mathbb{R}\}$.
5. Define $S = \alpha^{-1}T - \alpha^{-1}\beta I$. For any unit vector x, the inner product $\langle Sx, x\rangle$ is real because it transforms $\langle Tx, x\rangle$ (which lies on the line) to a real number by scaling and shifting.
6. Since $\langle Sx, x\rangle$ is real for all unit vectors x, S is self-adjoint.
7. Therefore, T can be written as $T = \alpha S + \beta I$.

Converse Direction:
— Assume $T = \alpha S + \beta I$ where S is a self-adjoint operator.
— For any unit vector x, the inner product $\langle Tx, x\rangle = \alpha\langle Sx, x\rangle + \beta$. Since $\langle Sx, x\rangle$ is real, $\langle Tx, x\rangle$ lies on the line $\{\alpha t + \beta : t \in \mathbb{R}\}$.
— Hence, the numerical range $W(T)$ is the line segment (or line) $\{\alpha t + \beta : t \in \mathbb{R}\}$.

Corollary 2.3.3
Let $T \in B(H)$. If $W(T)$ is a line segment, then T is normal.

Proof By Theorem 2.3.2, there exist $\alpha, \beta \in \mathbb{C}$ and a self-adjoint operator S such that:

$$T = \alpha S + \beta I.$$

Since S is self-adjoint, we have $S^* = S$, and since scalar multiplication and the identity operator commute with the adjoint, we obtain:

$$T^* = (\alpha S + \beta I)^* = \overline{\alpha} S + \overline{\beta} I.$$

Now compute TT^*:

$$\begin{aligned} TT^* &= (\alpha S + \beta I)(\overline{\alpha} S + \overline{\beta} I) \\ &= \alpha\overline{\alpha} S^2 + \alpha\overline{\beta} S + \beta\overline{\alpha} S + \beta\overline{\beta} I \\ &= |\alpha|^2 S^2 + (\alpha\overline{\beta} + \beta\overline{\alpha}) S + |\beta|^2 I. \end{aligned}$$

Now compute T^*T:

$$\begin{aligned} T^*T &= (\overline{\alpha} S + \overline{\beta} I)(\alpha S + \beta I) \\ &= \overline{\alpha}\alpha S^2 + \overline{\alpha}\beta S + \overline{\beta}\alpha S + \overline{\beta}\beta I \\ &= |\alpha|^2 S^2 + (\overline{\alpha}\beta + \overline{\beta}\alpha) S + |\beta|^2 I. \end{aligned}$$

Note that $\alpha\overline{\beta} + \beta\overline{\alpha} = \overline{\alpha}\beta + \overline{\beta}\alpha$, since these are complex conjugates of each other and thus equal as real numbers. Hence:

$$TT^* = T^*T.$$

2 .4 Numerical radius

Definition 2.4.1 Let $T \in \mathbb{B}(\mathcal{H})$. The *numerical radius* of T, denoted by $w(T)$, is defined as

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \sup_{\|x\|=1} |\langle Tx, x\rangle|, \tag{D 2.4.1}$$

.

Example 2.4.1
Let $T \in \mathbb{B}(\mathbb{C}^2)$ be defined by the matrix

$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We compute $w(T)$ as follows.

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^2$ with $\|x\| = 1$. Then

$$\langle Tx, x\rangle = \left\langle \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle = x_2\overline{x_1}.$$

So,

$$|\langle Tx, x\rangle| = |x_1||x_2| \leq \frac{1}{2}(|x_1|^2 + |x_2|^2) = \frac{1}{2},$$

since $|x_1|^2 + |x_2|^2 = \|x\|^2 = 1$ and the product $|x_1||x_2|$ is maximized when $|x_1| = |x_2| = \frac{1}{\sqrt{2}}$. Hence,

$$w(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle| = \frac{1}{2}.$$

Proposition 2.4.1 [4]
The numerical radius w defines a norm on $B(H)$.

Theorem 2.4.1 [9]
Let T be a bounded linear operator on a complex Hilbert space H. The numerical radius $w(T)$ is defined as:

$$w(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

We claim that $w(T)$ defines a norm on $B(H)$, the space of bounded linear operators on H.
proof

1 If $T = 0$, then for all $x \in H$, we have:

$$\langle Tx, x\rangle = 0.$$

Thus, $w(T) = 0$.

2 If $w(T) = 0$, then for all $x \in H$ with $\|x\| = 1$, we have:

$$\langle Tx, x\rangle = 0.$$

Homogeneity (1)
$w(\alpha T) = |\alpha| w(T)$ for $\alpha \in \mathbb{C}$ For $\alpha \in \mathbb{C}$, the numerical range scales as $W(\alpha T) = \alpha W(T)$. We compute:

$$w(\alpha T) = \sup_{\lambda \in W(\alpha T)} |\lambda| = \sup_{\lambda \in W(T)} |\alpha \lambda| = |\alpha| \sup_{\lambda \in W(T)} |\lambda| = |\alpha| w(T).$$

Subadditivity (2)
$w(T+S) \leq w(T) + w(S)$ The numerical range satisfies $W(T+S) \subset W(T) + W(S)$. Using the triangle inequality for complex numbers, we compute:

$$w(T+S) = \sup_{\lambda \in W(T+S)} |\lambda| \leq \sup_{\lambda_1 \in W(T), \lambda_2 \in W(S)} |\lambda_1 + \lambda_2| \leq \sup_{\lambda_1 \in W(T)} |\lambda_1| + \sup_{\lambda_2 \in W(S)} |\lambda_2| = w(T) + w(S).$$

Non-Negativity (3)
By definition, we have:

$$w(T) \geq 0,$$

since $w(T)$ is the supremum of non-negative values $|\langle Tx, x \rangle|$ for $\|x\| = 1$.

Proposition [9]
Let $T \in B(H)$. Then:

$$w(T) \leq \|T\| \leq 2w(T) \tag{P 2.4.2}$$

Proof

let $x \in H$ with $\|x\| = 1$, we have:

$$|\langle Tx, x \rangle| \leq \|Tx\| \|x\| \leq \|T\| \|x\|^2 = \|T\|.$$

takeing the supremum:

$$w(T) = \sup_{\|x\|=1} |\langle Tx, x \rangle| \leq \|T\|.$$

Thus, we have shown:

$$w(T) \leq \|T\|.$$

For any vectors $x, y \in H$, we apply the polarization identity for complex Hilbert spaces:

$$\langle Tx, y \rangle = \frac{1}{4} \left[\langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle + i \langle T(x+iy), x+iy \rangle - i \langle T(x-iy), x-iy \rangle \right].$$

Taking absolute values and applying the triangle inequality:

$$|\langle Tx, y \rangle| \leq \frac{1}{4} \left(|\langle T(x+y), x+y \rangle| + |\langle T(x-y), x-y \rangle| + |\langle T(x+iy), x+iy \rangle| + |\langle T(x-iy), x-iy \rangle| \right).$$

Each term is bounded by (T) times the squared norm:

$$|\langle T(x+y), x+y \rangle| \leq w(T) \|x+y\|^2, \quad \text{and similarly for the other terms.}$$

Using the identity for complex Hilbert spaces:

$$\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2 = 4(\|x\|^2 + \|y\|^2),$$

we obtain:

$$|\langle Tx, y \rangle| \leq w(T) \cdot \frac{1}{4} \cdot 4(\|x\|^2 + \|y\|^2) = w(T)(\|x\|^2 + \|y\|^2).$$

For unit vectors x, y with $\|x\| = \|y\| = 1$, this simplifies to:

$$|\langle Tx, y \rangle| \leq w(T)(1+1) = 2w(T).$$

Taking the supremum over all unit vectors x, y:

$$\|T\| = \sup_{\|x\|=\|y\|=1} |\langle Tx, y \rangle| \leq 2w(T).$$

Thus, we have shown:

$$\|T\| \leq 2w(T).$$

Corollary 2.4.1
Let $T \in B(H)$. Then:

$$r(T) \leq w(T) \tag{C 2.4.1}$$

proof By Corollary 2.3.3 we have $\sigma(T) \subset W(T)$. Then:

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda| \leq \sup_{\lambda \in W(T)} |\lambda| = w(T).$$

Corollary 2.4.2
Let $T \in B(H)$. If $r(T) = \|T\|$, then $w(T) = \|T\|$.
Defnetion 2.4.2
For any bounded linear operator T, the following inequalities hold:

$$r(T) \leq w(T) \leq \|T\|. \tag{D 2.4.2}$$

Theorem 2.4.2 [4]
Let $T \in B(H)$, and $n \in \mathbb{N}$. Then:

$$w(T^n) \leq (w(T))^n \tag{T 2.4.2}$$

we new For any bounded linear operator T on a complex Hilbert space H, the numerical radius satisfies (T 2.4.2)
Reverse Inequality:
From the general inequality $\frac{1}{2}\|T\| \leq w(T)$, which holds for any bounded operator T, we conclude:

$$w(T) = \frac{1}{2}\|T\|.$$

Theorem 2.4.3 [9]

Let $T \in B(H)$,Then the numerical radius of T satisfies

$$w(T) \leq \frac{1}{2} \| \, |T| + |T^*| \, \| , \tag{T 2.4.3}$$

where $|T| = (T^*T)^{1/2}$ and $|T^*| = (TT^*)^{1/2}$.
proof

Step 1: Numerical Radius Definition.
The numerical radius of T is defined as:

$$w(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

Step 2: Cauchy-Schwarz-Type Inequality.
For any unit vector $x \in H$, the generalized Cauchy-Schwarz inequality for operators implies:

$$|\langle Tx, x\rangle| \leq \sqrt{\langle |T|x, x\rangle \cdot \langle |T^*|x, x\rangle},$$

where $|T| = (T^*T)^{1/2}$ and $|T^*| = (TT^*)^{1/2}$.
Step 3: Apply the AM-GM Inequality.
Let $a = \langle |T|x, x\rangle$ and $b = \langle |T^*|x, x\rangle$. Since $a, b \geq 0$, the arithmetic mean-geometric mean inequality yields:

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Substituting into the earlier inequality:

$$|\langle Tx, x\rangle| \leq \frac{1}{2} \left(\langle |T|x, x\rangle + \langle |T^*|x, x\rangle\right).$$

Step 4: Combine Using Linearity.

$$\frac{1}{2} \left(\langle |T|x, x\rangle + \langle |T^*|x, x\rangle\right) = \frac{1}{2}\langle (|T| + |T^*|)x, x\rangle.$$

Step 5: Take Supremum.
Taking the supremum over all unit vectors x:

$$w(T) \leq \frac{1}{2} \sup_{\|x\|=1} \langle (|T| + |T^*|)x, x\rangle.$$

Step 6: Self-Adjoint Norm Property.
Since $|T| + |T^*|$ is self-adjoint, we know that:

$$\sup_{\|x\|=1} \langle (|T| + |T^*|)x, x\rangle = \| \, |T| + |T^*| \, \|.$$

Remark 2.4.1

$$\frac{1}{2}\||T|+|T^*|\| \leq \frac{1}{2}\left(\||T|\|+\||T^*|\|\right) = \frac{1}{2}\left(\|T\|+\|T\|\right) = \|T\| \tag{R 2.4.1}$$

Lemma 2.4.1
Let $T, S \in B(H)$ be positive operators. Then:

$$r(TS) = \|\sqrt{T}\sqrt{S}\|^2$$

Let T, S be positive bounded operators on a Hilbert space H. Then the spectral radius of the product satisfies:

$$r(TS) = \|TS\|^2.$$

Proposition 2.4.3 [11]
Let $T \in B(H)$. Then:

$$w(T) \leq \frac{1}{2}\left(\|T\| + \sqrt{\|T^2\|}\right) \tag{P 2.4.3}$$

proof
We begin with the previously established inequality:

$$w(T) \leq \frac{1}{2}\|\,|T|+|T^*|\,\|.$$

Step 1: Bounding the Norm of the Sum.
we have:

$$\|\,|T|+|T^*|\,\| \leq \max\{\|\,|T|\,\|, \|\,|T^*|\,\|\} + \|\,|T|^{1/2}|T^*|^{1/2}\,\|.$$

Step 2: Simplify Terms.
We use the fact that:

$$\|\,|T|\,\| = \|\,|T^*|\,\| = \|T\|.$$

Therefore,

$$\max\{\|\,|T|\,\|, \|\,|T^*|\,\|\} = \|T\|.$$

Step 3: Product Norm Estimate.
we have:

$$\|\,|T|^{1/2}|T^*|^{1/2}\,\| \leq \|\,|T||T^*|\,\|^{1/2} = \|T^2\|^{1/2}.$$

Step 4: Combine Results.
Putting everything together:

$$\|\,|T|+|T^*|\,\| \leq \|T\| + \|T^2\|^{1/2}.$$

Thus,

$$w(T) \leq \frac{1}{2}\left(\|T\| + \|T^2\|^{1/2}\right).$$

nots of ruselt:
Let T be a bounded linear operator on a Hilbert space H. Then the numerical radius of T satisfies:

$$w(T) \leq \frac{1}{2}\left(\|T\| + \|T^2\|^{1/2}\right) \leq \frac{1}{2}\left(\|T\| + \|T^2\|\right).$$

Corollary 2.4.3 [11]:
Let $T \in B(H)$. Then:

1. If $T^2 = 0$, then $w(T) = \frac{1}{2}\|T\|$.
2. If $w(T) = \|T\|$, then $\|T^2\| = \|T\|^2$.

Proof

(1) Case: $T^2 = 0$
we know:

$$w(T) \leq \frac{1}{2}\left(\|T\| + \sqrt{\|T^2\|}\right).$$

Since $T^2 = 0$, it follows that $\|T^2\| = 0$, and the bound simplifies to:

$$w(T) \leq \frac{1}{2}\|T\|.$$

It is also known that $w(T) \geq \frac{1}{2}\|T\|$ always holds for any bounded linear operator T. Therefore:

$$w(T) = \frac{1}{2}\|T\|.$$

Hence, we conclude that in this case:

$$\|T^2\| = 0 = \|T\|^2.$$

(2) Case: $w(T) = \|T\|$
Using again the inequality:

$$w(T) \leq \frac{1}{2}\left(\|T\| + \sqrt{\|T^2\|}\right) \leq \|T\|,$$

and assuming $w(T) = \|T\|$, we obtain:

$$\frac{1}{2}\left(\|T\| + \sqrt{\|T^2\|}\right) = \|T\|.$$

Solving for $\sqrt{\|T^2\|}$, we get:

$$\|T\| + \sqrt{\|T^2\|} = 2\|T\| \quad \Rightarrow \quad \sqrt{\|T^2\|} = \|T\|.$$

Therefore:

$$\|T^2\| = \|T\|^2.$$

Theorem 2.4.4 [9]
Let $T \in B(H)$. Then:

$$\frac{1}{4}\|T^*T + TT^*\| \leq w(T)^2 \leq \frac{1}{2}\|T^*T + TT^*\|$$

Proof
Let T be a bounded linear operator on a Hilbert space H. We aim to establish the two-sided bound:

$$\frac{1}{4}\|T^*T + TT^*\| \leq w(T)^2 \leq \frac{1}{2}\|T^*T + TT^*\|.$$

Part 1: Upper Bound $w(T)^2 \leq \frac{1}{2}\|T^*T + TT^*\|$
Let $x \in H$ be a unit vector. By the Cauchy-Schwarz inequality for semi-inner products (Theorem 2.14), we have:

$$|\langle Tx, x\rangle|^2 \leq \langle |T|x, x\rangle \cdot \langle |T^*|x, x\rangle.$$

Applying the AM-GM inequality:

$$\langle |T|x, x\rangle \cdot \langle |T^*|x, x\rangle \leq \frac{1}{2}\left(\langle |T|x, x\rangle^2 + \langle |T^*|x, x\rangle^2\right).$$

Since for self-adjoint operators A, $\langle A^2x, x\rangle = \|Ax\|^2$, we get:

$$\frac{1}{2}\left(\||T|x\|^2 + \||T^*|x\|^2\right) = \frac{1}{2}\langle(|T|^2 + |T^*|^2)x, x\rangle.$$

Recognizing that $|T|^2 = T^*T$ and $|T^*|^2 = TT^*$, we get:

$$|\langle Tx, x\rangle|^2 \leq \frac{1}{2}\langle(T^*T + TT^*)x, x\rangle.$$

Taking the supremum over all $\|x\| = 1$, we obtain:

$$w(T)^2 \leq \frac{1}{2}\|T^*T + TT^*\|.$$

Part 2: Lower Bound $\frac{1}{4}\|T^*T + TT^*\| \leq w(T)^2$
We decompose T into its real and imaginary parts:

$$T = \mathfrak{Re}(T) + i\mathfrak{Im}(T),$$

where:

$$\mathfrak{Re}(T) = \frac{T + T^*}{2}, \quad \mathfrak{Im}(T) = \frac{T - T^*}{2i},$$

are self-adjoint operators.

Then for any unit vector $x \in H$,

$$|\langle Tx, x\rangle|^2 = \langle \mathfrak{Re}(T)x, x\rangle^2 + \langle \mathfrak{Im}(T)x, x\rangle^2.$$

Hence,

$$w(T)^2 = \sup_{\|x\|=1} \left(\langle \mathfrak{Re}(T)x, x\rangle^2 + \langle \mathfrak{Im}(T)x, x\rangle^2 \right).$$

We observe:

$$\langle \mathfrak{Re}(T)x, x\rangle^2 + \langle \mathfrak{Im}(T)x, x\rangle^2 = \frac{1}{2} \left(\langle (\mathfrak{Re}(T) + \mathfrak{Im}(T))x, x\rangle^2 + \langle (\mathfrak{Re}(T) - \mathfrak{Im}(T))x, x\rangle^2 \right),$$

by the parallelogram law.

Taking the supremum and using the operator norm of self-adjoint operators:

$$w(T)^2 \geq \frac{1}{2} \max \left\{ \|(\mathfrak{Re}(T) + \mathfrak{Im}(T))^2\|, \|(\mathfrak{Re}(T) - \mathfrak{Im}(T))^2\| \right\}.$$

Now observe:

$$(\mathfrak{Re}(T) + \mathfrak{Im}(T))^2 + (\mathfrak{Re}(T) - \mathfrak{Im}(T))^2 = 2 \left(\mathfrak{Re}(T)^2 + \mathfrak{Im}(T)^2 \right).$$

So:

$$w(T)^2 \geq \frac{1}{2} \left\| \mathfrak{Re}(T)^2 + \mathfrak{Im}(T)^2 \right\|.$$

Finally, we use the identity:

$$\mathfrak{Re}(T)^2 + \mathfrak{Im}(T)^2 = \frac{1}{4}(T^*T + TT^*),$$

which gives:

$$w(T)^2 \geq \frac{1}{2} \cdot \frac{1}{4} \|T^*T + TT^*\| = \frac{1}{4} \|T^*T + TT^*\|.$$

Remark 2.4.2 we have:

$$\|Tx\|^2 = \langle T^*Tx, x\rangle \leq \langle T^*Tx, x\rangle + \langle TT^*x, x\rangle = \langle (T^*T + TT^*)x, x\rangle$$

we conclude :

$$\|T\|^2 \leq \|T^*T + TT^*\|.$$

Theorem 2.4.5 [11] Let $T \in B(H)$. Then:

$$w(T)^2 \leq \frac{1}{2} \left(w(T^2) + \|T\|^2 \right)$$

Theorem 2.4.6[6]

Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. Then the numerical radius $w(T)$ satisfies:

$$w(T)^2 \leq \frac{1}{2} w(T^2) + \frac{1}{4} \|T^*T + TT^*\|.$$

Remark 2.4.3

$$\frac{1}{2} w(T^2) \leq \frac{1}{2} w(T)^2 \leq \frac{1}{4} \|T^*T + TT^*\|$$

Theorem 2.4.7 [4] Let $T \in B(H)$. Then:

$$w(T) \leq \frac{1}{2} \left(\|T\| + \sqrt{r(|T||T^*|)} \right)$$

Theorem 2.4.8[11]

Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. Then the numerical radius $w(T)$ satisfies:

$$w(T) \leq \frac{1}{2} \left(\|T\| + r(|T||T^*|) \right),$$

where $r(\cdot)$ denotes the spectral radius.

Remark 2.4.4

$$\sqrt{r(|T||T^*|)} \leq \sqrt{\||T||T^*|\|} = \sqrt{\|T^2\|}$$

Corollary 2.4.4 Let $T \in B(H)$. Then:

1. If $r(|T||T^*|) = 0$, then $w(T) = \frac{1}{2}\|T\|$.
2. If $w(T) = \frac{1}{2}(\|T\| + \sqrt{\|T^2\|})$, then $r(|T||T^*|) = \|T^2\|$.

Proposition 2.4.4 Let $T, S \in B(H)$. Then:

1. $w(TS) \leq 4w(T)w(S)$.

2. If $TS = ST$, then $(TS) \leq 2w(T)w(S)$.
3. $w(TS) \leq \frac{1}{2}\|TT^* + S^*S\|$.

Theorem 2.4.9 *[6]* Let $T, S \in B(H)$ such that $TS = ST$ and $T^*S = ST^*$. Then:

$$w(TS) \leq \|T\| w(S) \tag{T 2.4.9}$$

Corollary 2.4.5 Let $T, S \in B(H)$ such that $TS = ST$. Then:

1. If T is normal, then $w(TS) \leq w(T)w(S)$.
2. If T is unitary, then $w(TS) \leq w(S)$.

Proof

(1) Case: T is normal and $TS = ST$

Since T is normal and commutes with S, $\mathrm{T}^*S = ST^*$. Then, by inequality(T 2.4.9), we have:

$$w(TS) \leq \|T\| \cdot w(S).$$

Because T is normal, we know that

$$w(T) = \|T\|,$$

and therefore:

$$w(TS) \leq w(T) \cdot w(S).$$

(2) Case: T is unitary

If T is unitary, then T is also normal. Hence, by part (1), we again have:

$$w(TS) \leq w(T) \cdot w(S).$$

Moreover, since T is unitary, we have:

$$\|T\| = 1 \quad \text{and} \quad w(T) = \|T\| = 1.$$

Therefore,

$$w(TS) \leq w(S).$$

Lemma 2.4.2 [6] Let $T, S \in B(H)$ such that $|T|S = S^*|T|$. Then:

$$|\langle TSx, y\rangle| \leq r(S)\|\sqrt{|T|}x\|\|\sqrt{|T^*|}y\| \quad \text{for all } x, y \in H. \tag{L 2.4.2}$$

Theorem 2.4.10 [9] Let $T, S \in B(H)$ such that $|T|S = S^*|T|$. Then:

$$w(TS) \leq r(S)^2 \left(\|T\| + \sqrt{r(|T||T^*|)}\right).$$

Proof:

Initial Setup: Let $x \in H$ with $\|x\| = 1$. From (L 2.4.2) (which assumes the commutation relations), we have:

$$|\langle TSx, x\rangle| \leq r(S)\|\sqrt{|T|}x\| \, \|\sqrt{|T^*|}x\|.$$

AM-GM Application: Applying the arithmetic mean-geometric mean inequality:

$$\|\sqrt{|T|}x\| \, \|\sqrt{|T^*|}x\| \leq \frac{1}{2}\left(\|\sqrt{|T|}x\|^2 + \|\sqrt{|T^*|}x\|^2\right),$$

so:

$$|\langle TSx, x\rangle| \leq \frac{r(S)}{2}\left(\|\sqrt{|T|}x\|^2 + \|\sqrt{|T^*|}x\|^2\right).$$

Norm Conversion: Convert squared norms to inner products:

$$= \frac{r(S)}{2}\left(\langle |T|x, x\rangle + \langle |T^*|x, x\rangle\right) = \frac{r(S)}{2}\langle (|T| + |T^*|)x, x\rangle.$$

Norm Bound: Taking the supremum over all unit vectors x, we obtain:

$$w(TS) = \sup_{\|x\|=1} |\langle TSx, x\rangle| \leq \frac{r(S)}{2}\||T| + |T^*|\|.$$

Operator Decomposition: Using the triangle inequality and a decomposition argument, we get:

$$\||T| + |T^*|\| \leq \|T\| + \|\sqrt{|T|}\sqrt{|T^*|}\|.$$

Spectral Radius Connection: standard results on positive operators we have:

$$\|\sqrt{|T|}\sqrt{|T^*|}\| = r(|T||T^*|).$$

Theorem 2.4.11 [11] Let $T, S, R \in B(H)$. Then:

$$w(TS \pm RT) \leq 2\sqrt{2}\max\{\|S\|, \|R\|\}w(T).$$

Proof:

Initial Setup: Let $x \in H$ be a unit vector (i.e., $\|x\| = 1$). We analyze the numerical radius using the triangle inequality:

$$|\langle (TS \pm RT)x, x\rangle| \leq |\langle TSx, x\rangle| + |\langle RTx, x\rangle|.$$

Inner Product Decomposition: Express the inner products using adjoints:

$$|\langle TSx, x\rangle| = |\langle Sx, T^*x\rangle|, \quad |\langle RTx, x\rangle| = |\langle Tx, R^*x\rangle|.$$

Cauchy-Schwarz Application: Apply the Cauchy-Schwarz inequality:

$$|\langle Sx, T^*x\rangle| \leq \|Sx\|\,\|T^*x\|, \quad |\langle Tx, R^*x\rangle| \leq \|Tx\|\,\|R^*x\|.$$

Operator Norm Bounds: Since $\|Sx\| \leq \|S\|$, $\|R^*x\| \leq \|R^*\| = \|R\|$, etc., we get:

$$|\langle (TS \pm RT)x, x\rangle| \leq \|S\|\,\|T^*x\| + \|R\|\,\|Tx\| \leq \max\{\|S\|, \|R\|\}(\|Tx\| + \|T^*x\|).$$

Norm Combination: Use the inequality $a + b \leq \sqrt{2(a^2 + b^2)}$ for non-negative a, b:

$$\|Tx\| + \|T^*x\| \leq \sqrt{2(\|Tx\|^2 + \|T^*x\|^2)}.$$

Thus,

$$|\langle (TS \pm RT)x, x\rangle| \leq \sqrt{2}\max\{\|S\|, \|R\|\}\sqrt{\|Tx\|^2 + \|T^*x\|^2}.$$

Norm Conversion: Express norms in terms of inner products:

$$\|Tx\|^2 = \langle T^*Tx, x\rangle, \quad \|T^*x\|^2 = \langle TT^*x, x\rangle,$$

so

$$\|Tx\|^2 + \|T^*x\|^2 = \langle (T^*T + TT^*)x, x\rangle.$$

Numerical Radius Connection:

$$\langle (T^*T + TT^*)x, x\rangle \leq \|T^*T + TT^*\| \leq 2w(T).$$

Remark 2.4.5 Let $T, S \in B(H)$. Then:

$$w(TS \pm ST) \leq 2\sqrt{2}\|T\|w(T) \tag{R 2.4.5}$$

Proposition 2.4.5 *[22]* Let $T, S \in B(H)$. Then:

$$w(TS \pm ST) \leq \frac{1}{2}\|T^*T + TT^* + S^*S + SS^*\|. \tag{P 2.4.5}$$

Proof:

Initial Setup: Let $x \in H$ be a unit vector (i.e., $\|x\| = 1$). We estimate the numerical radius using the triangle inequality:

$$|\langle (TS \pm ST)x, x\rangle| \leq |\langle TSx, x\rangle| + |\langle STx, x\rangle|.$$

Adjoint Formulation: Express each inner product using adjoints:

$$|\langle TSx, x\rangle| = |\langle Sx, T^*x\rangle|, \quad |\langle STx, x\rangle| = |\langle Tx, S^*x\rangle|.$$

Cauchy-Schwarz Application: Apply the Cauchy-Schwarz inequality:

$$|\langle Sx, T^*x\rangle| \leq \|Sx\|\,\|T^*x\|, \quad |\langle Tx, S^*x\rangle| \leq \|Tx\|\,\|S^*x\|.$$

AM-GM Inequality: Apply the inequality $ab \leq \frac{a^2+b^2}{2}$ to each product:

$$\|Sx\|\,\|T^*x\| \leq \frac{1}{2}(\|Sx\|^2 + \|T^*x\|^2), \quad \|Tx\|\,\|S^*x\| \leq \frac{1}{2}(\|Tx\|^2 + \|S^*x\|^2),$$

so combining,

$$|\langle (TS \pm ST)x, x\rangle| \leq \frac{1}{2}\left(\|T^*x\|^2 + \|Sx\|^2 + \|Tx\|^2 + \|S^*x\|^2\right).$$

Norm Conversion: Express norms in terms of inner products:

$$\|T^*x\|^2 = \langle TT^*x, x\rangle, \quad \|Sx\|^2 = \langle S^*Sx, x\rangle, \quad \|Tx\|^2 = \langle T^*Tx, x\rangle, \quad \|S^*x\|^2 = \langle SS^*x, x\rangle,$$

so:

$$|\langle (TS \pm ST)x, x\rangle| \leq \frac{1}{2}\langle (TT^* + S^*S + T^*T + SS^*)x, x\rangle.$$

Operator Norm Bound: Taking the supremum over all unit vectors x, we obtain:

$$w(TS \pm ST) \leq \frac{1}{2}\|TT^* + S^*S + T^*T + SS^*\|.$$

Definition 2.4.3 Let $T \in B(H)$. Then $c(T)$ is called the Crawford number of T, and it is defined as follows:

$$c(T) = \inf_{\|x\|=1} |\langle Tx, x\rangle|$$

Lemma 2.4.3 [6] Let $T \in B(H)$. Then:

$$\|T^*T + TT^*\| \leq 4\left(w(T)^2 - c(\Re(T))^2 + 2c(\Im(T))^2\right) \tag{L 2.4.3}$$

Proof:

Operator Decomposition: We begin by expressing the real and imaginary parts of the operator T:

$$\Re(T) = \frac{1}{2}(T + T^*), \quad \Im(T) = \frac{1}{2i}(T - T^*).$$

Key Identity: Through direct computation, one obtains:

$$\frac{1}{2}\|T^*T + TT^*\| = \|\Re(T)^2 + \Im(T)^2\|.$$

Norm Inequality: By the subadditivity and homogeneity of the operator norm:

$$\frac{1}{2}\|\Re(T)^2 + \Im(T)^2\| \leq \frac{1}{2}(\|\Re(T)\|^2 + \|\Im(T)\|^2).$$

Using the expressions for $\Re(T)$ and $\Im(T)$:

$$\|\Re(T)\| = \frac{1}{2}\|T + T^*\|, \quad \|\Im(T)\| = \frac{1}{2}\|T - T^*\|.$$

Hence,

$$\frac{1}{4}\|T^*T + TT^*\| \leq \frac{1}{8}(\|T + T^*\|^2 + \|T - T^*\|^2).$$

Numerical Radius Relation: For any unit vector $x \in H$, we use the decomposition of T:

$$|\langle Tx, x\rangle|^2 = |\langle \Re(T)x, x\rangle|^2 + |\langle \Im(T)x, x\rangle|^2.$$

Taking the supremum over all unit vectors yields:

$$w(T)^2 = \sup_{\|x\|=1} |\langle Tx, x\rangle|^2 \geq \sup_{\|x\|=1}\left(|\langle \Re(T)x, x\rangle|^2 + |\langle \Im(T)x, x\rangle|^2\right).$$

This implies:

$$c(\Re(T))^2 + \|\Im(T)\|^2 \leq w(T)^2, \quad c(\Im(T))^2 + \|\Re(T)\|^2 \leq w(T)^2,$$

where $c(A) = \inf_{\|x\|=1} |\langle Ax, x\rangle|$ is the **Crawford number** of operator A.

Combining Inequalities: Adding the above two inequalities:

$$c(\Re(T))^2 + c(\Im(T))^2 + \|\Re(T)\|^2 + \|\Im(T)\|^2 \leq 2w(T)^2.$$

Rewriting:

$$\frac{1}{8}(\|T + T^*\|^2 + \|T - T^*\|^2) \leq w(T)^2 - \frac{1}{2}(c(\Re(T))^2 + c(\Im(T))^2).$$

Final Result: Since:

$$\frac{1}{4}\|T^*T + TT^*\| \leq \frac{1}{8}(\|T + T^*\|^2 + \|T - T^*\|^2),$$

we conclude:

$$\frac{1}{4}\|T^*T + TT^*\| \le w(T)^2 - \frac{1}{2}(c(\Re(T))^2 + c(\Im(T))^2).$$

Theorem 2.4.12 [4] Let $T, S \in B(H)$. Then:

$$w(TS \pm ST) \le 2\sqrt{2}\|S\|\sqrt{w(T)^2 - c(\Re(T))^2 + 2c(\Im(T))^2} \qquad \text{(T 2.4.12)}$$

Proof:

Initial Setup: Let $x \in H$ be a unit vector, i.e., $\|x\| = 1$. Then, by the triangle inequality:

$$|\langle (TS \pm ST)x, x\rangle| \le |\langle TSx, x\rangle| + |\langle STx, x\rangle|.$$

Adjoint Formulation: We rewrite the inner products using adjoints:

$$|\langle TSx, x\rangle| = |\langle Sx, T^*x\rangle|, \quad |\langle STx, x\rangle| = |\langle Tx, S^*x\rangle|.$$

So,

$$|\langle (TS \pm ST)x, x\rangle| \le |\langle Sx, T^*x\rangle| + |\langle Tx, S^*x\rangle|.$$

Cauchy-Schwarz Application: Using the Cauchy-Schwarz inequality:

$$|\langle Sx, T^*x\rangle| \le \|Sx\| \cdot \|T^*x\|, \quad |\langle Tx, S^*x\rangle| \le \|Tx\| \cdot \|S^*x\|.$$

Thus,

$$|\langle (TS \pm ST)x, x\rangle| \le \|Sx\|\|T^*x\| + \|Tx\|\|S^*x\|.$$
$$\le \|S\|(\|T^*x\| + \|Tx\|).$$

Norm Combination: Using the inequality $a + b \le \sqrt{2(a^2 + b^2)}$:

$$\le \sqrt{2}\|S\|\sqrt{\|T^*x\|^2 + \|Tx\|^2}.$$

Norm Conversion: We express the norms in terms of inner products:

$$\|T^*x\|^2 = \langle TT^*x, x\rangle, \quad \|Tx\|^2 = \langle T^*Tx, x\rangle.$$

Therefore:

$$\le \sqrt{2}\|S\|\sqrt{\langle (TT^* + T^*T)x, x\rangle}.$$

Numerical Radius Connection: From a previous result (e.g., Theorem 2.2.3), we have:

$$\langle (TT^* + T^*T)x, x\rangle \le 4\left(w(T)^2 - \frac{c(\Re(T))^2 + c(\Im(T))^2}{2}\right).$$

Final Combination: Substituting the bound into our inequality:

$$|\langle (TS \pm ST)x, x\rangle| \le \sqrt{2}\|S\|\sqrt{4\left(w(T)^2 - \frac{c(\Re(T))^2 + c(\Im(T))^2}{2}\right)}.$$

$$= 2\sqrt{2}\|S\|\sqrt{w(T)^2 - \frac{c(\Re(T))^2 + c(\Im(T))^2}{2}}.$$

Taking the supremum over all unit vectors x yields:

$$w(TS \pm ST) \le 2\sqrt{2}\|S\|\sqrt{w(T)^2 - \frac{c(\Re(T))^2 + c(\Im(T))^2}{2}}.$$

Proposition 2.4.6

Let $T, S, R \in B(H)$ such that $TS = ST$ and $TR = RT$. Then the following inequality holds:

$$w(TS \pm RT) \le 2(w(S) + (R))w(T)$$

Proof:

Commutation Assumptions:

Suppose that

$$TS = ST \quad \text{and} \quad T^*S = ST^*,$$

so that Proposition 2.4.4 applies to the operator TS.

Similarly, suppose

$$RT = TR \quad \text{and} \quad R^*T = TR^*,$$

so that Proposition 2.4.4 applies to RT as well.

Proposition 2.4.7

For commuting operators T and S, we have:

$$w(TS) \leq 2w(T)w(S).$$

Likewise, for commuting operators T and R:

$$w(RT) \leq 2w(T)w(R).$$

Triangle Inequality for Numerical Radius:

The numerical radius satisfies the triangle inequality:

$$w(TS \pm RT) \leq w(TS) + w(RT). \tag{P 2.4.7}$$

Combine the Results:

Substituting the bounds obtained from P 2.4.7, and lass ruseit we get:

$$w(TS \pm RT) \leq 2w(T)w(S) + 2w(T)w(R).$$

Factoring out the common term $2w(T)$:

$$w(TS \pm RT) \leq 2w(T)(w(S) + w(R)).$$

Under the commutation conditions

$$TS = ST, \quad T^*S = ST^*, \quad RT = TR, \quad \text{and} \quad R^*T = TR^*,$$

we conclude:

$$w(TS \pm RT) \leq 2(w(S) + w(R))w(T).$$

Theorem 3.4.12 [9] Let $T \in B(H)$. Then:

$$w(T)^2 \leq \frac{1}{2}w(T^2) + \frac{1}{4}\|T^*T + TT^*\|$$

3 Numerical Range and Operator Norms in Hilbert Spaces

Introduction

Let H be a complex Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. The space of all bounded linear operators on H, denoted by $\mathcal{B}(H)$, forms a C^*-algebra. For any $A \in \mathcal{B}(H)$, its adjoint is denoted by A^*, and its modulus is defined as $|A| = (A^*A)^{1/2}$, the unique positive square root of A^*A.

The numerical range of A, denoted by $W(A)$, is the collection of all values $\langle Ax, x \rangle$ where x ranges over the unit vectors in H:

$$W(A) = \{\langle Ax, x \rangle \mid x \in H, \|x\| = 1\}.$$

Two fundamental quantities associated with A are its operator norm $\|A\|$ and its numerical radius $w(A)$, defined respectively as:

$$\|A\| = \sup\left\{|\langle Ax, y \rangle| \mid x, y \in H, \|x\| = \|y\| = 1\right\},$$
$$w(A) = \sup\left\{|\langle Ax, x \rangle| \mid x \in H, \|x\| = 1\right\}.$$

The numerical radius $w(\cdot)$ defines a norm on $\mathcal{B}(H)$ that is equivalent to the operator norm $\|\cdot\|$. Specifically, the following sharp inequalities hold for any $A \in \mathcal{B}(H)$:

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|.$$

The first inequality becomes an equality if $A^2 = 0$.

The second inequality becomes an equality if and only if A is normal.

An improvement to these inequalities was established by Kittaneh [?], who proved:

$$\frac{1}{4}\left\||A|^2 + |A^*|^2\right\| \leq w^2(A) \leq \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|.$$

3 .1 Euclidean Operator Radius and The Davis–Wielandt radius

Euclidean Operator Radius 3.1.1[10],[15]

Given a pair of operators $(C, D) \in \mathcal{B}(H)^2 := \mathcal{B}(H)\mathcal{B}(H)$, the Euclidean operator radius is defined as:

$$w_e(C, D) := \sup_{\substack{x \in H \\ \|x\|=1}} \left(|\langle Cx, x \rangle|^2 + |\langle Dx, x \rangle|^2\right)^{1/2}. \tag{1}$$

$w_e : \mathcal{B}(H)^2 \to [0, \infty)$ is a norm, and the following sharp inequality holds:

$$\frac{\sqrt{2}}{4}\left\||C|^2 + |D|^2\right\|^{1/2} \leq w_e(C, D) \leq \left\||C|^2 + |D|^2\right\|^{1/2}, \tag{2}$$

where the constants $\frac{\sqrt{2}}{4}$ and 1 are optimal.

the second author derived the following lower bound with the best possible constant $\frac{\sqrt{2}}{2}$:

$$\frac{\sqrt{2}}{2}\left\|B^2 + C^2\right\|^{1/2} \leq w_e(B, C). \tag{3}$$

Additionally, the following results were obtained:

Special Cases and Simplifications 3.1.2 [10]:

By selecting $(B, C) = (A, A^*)$ or $(B, C) = (\mathrm{Re}(A), \mathrm{Im}(A))$ for $A \in \mathcal{B}(H)$, where

$$\mathrm{Re}(A) := \frac{A + A^*}{2}, \quad \mathrm{Im}(A) := \frac{A - A^*}{2i},$$

the second author derived several norm and numerical radius inequalities for a single operator A. These results simplify further when B and C are self-adjoint then since $|B|^2 = B^2$, $|C|^2 = C^2$, and $C^*B = CB$. If B and C are orthogonal projections , $B^2 = B$, $C^2 = C$, and $CB = 0$), the bounds become even more tractable.

For recent refinements on Euclidean numerical radius bounds.

DW Radius 3.1.3[15] The Davis–Wielandt radius of an operator $T \in \mathcal{B}(H)$, denoted $dw(T)$, yous the definition 2.4.1:

$$dw(T) = \sup_{\substack{x \in H \\ \|x\|=1}} \left(|\langle Tx, x \rangle|^2 + \|Tx\|^4\right)^{1/2}. \tag{4}$$

Properties:

— $dw(T) \geq 0$, with equality if and only if $T = 0$.
— Homogeneity: For $\lambda \in \mathbb{C}$,

$$dw(\lambda T) = |\lambda| dw(T) \text{ if } |\lambda| = 1,$$

while $dw(\lambda T) > |\lambda| dw(T)$ for $|\lambda| > 1$ and $dw(\lambda T) < |\lambda| dw(T)$ for $|\lambda| < 1$.
— The triangle inequality holds for $dw(\cdot)$.

DW Radius and Associated Inequalities 3.1.4[12]
The DW radius of an operator $T \in \mathcal{B}(\mathcal{H})$, denoted $dw(T)$, is defined as:

$$dw(T) = \sup_{\|x\|=1} \left(|\langle Tx, x \rangle|^2 + \|Tx\|^4 \right)^{1/2}.$$

While the triangle inequality

$$dw(T + S) \leq dw(T) + dw(S) \tag{5}$$

does not hold universally for arbitrary $T, S \in \mathcal{B}(\mathcal{H})$, it is valid under the condition $\operatorname{Re}(T^*S) = 0$

Fundamental Inequalities 3.1.4.(1)

$$\max\{w(T), \|T\|^2\} \leq dw(T) \leq \left(w^2(T) + \|T\|^4 \right)^{1/2}, \tag{6}$$

where both bounds are sharp

Connection to Euclidean Operator Radius 3.1.5
By setting $C = T$ and $D = |T|^2$, we identify:

$$w_e(T, |T|^2) = dw(T). \tag{7}$$

Substituting this into (2) yields the sharp inequalities:

$$\frac{1}{2} \left\| |T|^2 + |T|^4 \right\| \leq dw^2(T) \leq \left\| |T|^2 + |T|^4 \right\|. \tag{8}$$

Refined Upper Bounds Zamani and Shebrawi established:

$$dw(T) \leq \left(w^2(T - |T|^2) + 2\|T\|^2 w(T) \right)^{1/2}. \tag{9}$$

thes:

$$dw^2(T) \leq \|T\|^2 \max\{1, \|T\|^2\} + 2w(|T|^2 T), \tag{10}$$

$$dw^2(T) \leq \frac{1}{2} \left(|T|^4 + |T|^2 + \left| |T|^4 - |T|^2 \right| + 2w(|T|^2 T) \right), \tag{11}$$

$$dw^2(T) \leq \|T\| \max\{w(T), \|T\|^2\} \left(1 + \|T\|^2 + 2w(T) \right)^{1/2}.$$

Recent Contributions 3.1.6
derived the upper bound:

$$dw(T) \leq \left\| |T|^4 + |T|^2 \right\|^{1/2}.$$

Sharp bounds involving $w(B \pm C)$:

$$\frac{\sqrt{2}}{2} \max\{w(B + C), w(B - C)\} \leq w_e(B, C) \leq \frac{\sqrt{2}}{2} \left(w^2(B + C) + w^2(B - C) \right)^{1/2}, \tag{12}$$

where $\frac{\sqrt{2}}{2}$ is optimal in both inequalities.

Sharp upper bounds:

$$w_e^2(B, C) \leq \max\{\|B\|^2, \|C\|^2\} + w(C^*B),$$

$$w_e^2(B, C) \leq \frac{1}{2} \left\| |B|^2 + |C|^2 + \left| |B|^2 - |C|^2 \right| \right\| + w(C^*B), \tag{13}$$

both of which are sharp.

For sums of operators, established:

$$dw(T + S) \leq \left(2(dw^2(T) + dw^2(S)) + 6 \left\| |T|^4 + |S|^4 \right\| \right)^{1/2} \leq 2\sqrt{2} \left(dw^2(T) + dw^2(S) \right)^{1/2}. \tag{14}$$

Motivation and Contribution 3.1.6.(1) Building on these advances, we derive new lower and upper bounds for the Euclidean numerical radius $w_e(B, C)$ of operator pairs $(B, C) \in \mathcal{B}(\mathcal{H})^2$. Our results refine and improve upon recent bounds in the literature, offering sharper estimates under broader conditions.

WD Radius and Sums of Operators 3.1.7
For operators $T, S \in \mathcal{B}(\mathcal{H})$, recent established inequalities for the Davis–Wielandt radius of their sum (14)
Theorem 3.1.1 [10]
Let $B, C \in \mathcal{B}(\mathcal{H})$. Then:

$$\sqrt{2}\,\|B^*B + C^*C\|^{1/2} \leq w_e(B, C) \leq \inf_{\theta\in\mathbb{R}} \left\||B|^2 + |C|^2 + e^{i\theta}(B^*C + C^*B)\right\|^{1/2}. \tag{15}$$

Sharpness: The lower bound becomes equality if B and C are self-adjoint and anticommute ($BC = -CB$). The upper bound is sharp for commuting normal operators.
Comparison with Existing Results 3.1.8
— **Improvement 3.1.8.(1)** For $(B, C) = (T, S)$, our upper bound subsumes the inequality

$$w_e(T, S) \leq \left\||T|^2 + |S|^2\right\|^{1/2},$$

by incorporating an additional cross-term $e^{i\theta}(T^*S + S^*T)$, which tightens the estimate.
— **Refinement 3.1.8.(2)** When $(B, C) = (\mathrm{Re}(A), \mathrm{Im}(A))$, our lower bound strengthens the inequality

$$\sqrt{2}\left\|A^2 + (A^*)^2\right\|^{1/2} \leq w_e(\mathrm{Re}(A), \mathrm{Im}(A)), \tag{16}$$

by replacing $A^2 + (A^*)^2$ with the more general $B^*B + C^*C$.
remark 3.1.1

Self-Adjoint Operators
If B and C are self-adjoint, then $|B|^2 = B^2$ and $|C|^2 = C^2$, and our bounds simplify to:

$$\sqrt{2}\left\|B^2 + C^2\right\|^{1/2} \leq w_e(B, C) \leq \left\|B^2 + C^2 + BC + CB\right\|^{1/2}.$$

remark 3.1.2
Orthogonal Projections
For orthogonal projections B, C with $BC = 0$, we obtain:

$$w_e(B, C) = \sqrt{\|B\|^2 + \|C\|^2}, \tag{17}$$

demonstrating exactness in this case.

3 .2 Lower Bounds for the Euclidean Numerical Radius of Operator Pairs:

theorem 3.2.1[5],[15]
For any $B, C \in \mathcal{B}(H)$, the following lower bounds hold for the Euclidean numerical radius $w_e(B, C)$:

$$w_e^2(B, C) \geq \frac{1}{2}\max\{w(B^2 + C^2),\, w(BC + CB)\} + \frac{1}{2}\max\{w(B), w(C)\}\,|w(B + C) - w(B - C)|, \tag{1}$$

$$w_e^2(B, C) \geq \frac{1}{2}\max\{w(B^2 + C^2),\, w(B^2 - C^2)\} + \frac{1}{2}\max\{w(B + C), w(B - C)\}\,|w(B) - w(C)|. \tag{2}$$

Proof
***Proof of Inequality (1)**

Step 1: From the definition of the Euclidean numerical radius:

$$w_e^2(B, C) = \sup_{\|x\|=1}\left(|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2\right).$$

Using the inequality $a^2 + b^2 \geq \frac{1}{2}(a \pm b)^2$, we obtain:

$$w_e^2(B, C) \geq \frac{1}{2}\sup_{\|x\|=1}|\langle (B \pm C)x, x\rangle|^2 = \frac{1}{2}w^2(B \pm C).$$

Taking the maximum over $\pm$, we get:

$$w_e^2(B, C) \geq \frac{1}{2}\max\{w^2(B + C), w^2(B - C)\}. \tag{3}$$

Step 2: Since $w(T^2) \geq w^2(T)$ for any operator T, applying this to $T = B \pm C$ gives:

$$w((B \pm C)^2) \geq w^2(B \pm C).$$

So by (3), we have:

$$w_e^2(B,C) \geq \frac{1}{2}\max\{w((B+C)^2), w((B-C)^2)\}.$$

Step 3: Expanding the squares:

$$(B+C)^2 = B^2 + C^2 + BC + CB, \quad (B-C)^2 = B^2 + C^2 - BC - CB.$$

Using the triangle inequality:

$$w((B+C)^2) + w((B-C)^2) \geq 2\max\{w(B^2+C^2), w(BC+CB)\}.$$

Also, by the parallelogram law:

$$w(B+C) + w(B-C) \geq 2\max\{w(B), w(C)\}.$$

Step 4: Combining the results:

$$w_e^2(B,C) \geq \frac{1}{4}\left[w((B+C)^2) + w((B-C)^2)\right] + \frac{1}{4}\left[w(B+C) + w(B-C)\right]|w(B+C) - w(B-C)|.$$

Substituting the previous bounds yields inequality (1).

Step 5: Noting that $(B+C)^2 - (B-C)^2 = 4BC$, and using this to estimate $w(B^2 - C^2)$, we similarly derive:

$$w_e^2(B,C) \geq \frac{1}{2}\max\{w(B^2+C^2), w(B^2-C^2)\} + \frac{1}{2}\max\{w(B+C), w(B-C)\}|w(B) - w(C)|.$$

This proves inequality 2).

***Proof of Inequality 2**

Starting Point. From Inequality 1, for any $B, C \in \mathbb{B}(\mathcal{H})$, we have:

$$w_e^2(B,C) \geq \frac{1}{2}\max\{w(B^2+C^2), w(BC+CB)\} + \frac{1}{2}\max\{w(B), w(C)\} \cdot |w(B+C) - w(B-C)|. \tag{4}$$

Step 1: Substitution in last Inequality Apply the transformation $B \mapsto B+C$, $C \mapsto B-C$ in (1). This gives:

$$\begin{aligned} w_e^2(B+C, B-C) \geq{}& \frac{1}{2}\max\left\{w((B+C)^2 + (B-C)^2), w((B+C)(B-C) + (B-C)(B+C))\right\} \\ &+ \frac{1}{2}\max\{w(B+C), w(B-C)\} \cdot |w(B+C) - w(B-C)|. \end{aligned} \tag{5}$$

Step 2: Simplify Operator Expressions. Note that:

$$(B+C)^2 + (B-C)^2 = 2(B^2+C^2), \quad (B+C)(B-C) + (B-C)(B+C) = 2(B^2 - C^2).$$

Substitute into (5):

$$w_e^2(B+C, B-C) \geq \max\{w(B^2+C^2), w(B^2-C^2)\} + \max\{w(B+C), w(B-C)\} \cdot |w(B) - w(C)|. \tag{6}$$

Step 3: Relate $w_e^2(B+C, B-C)$ to $w_e^2(B,C)$. For any $x \in \mathcal{H}$ with $\|x\| = 1$, observe:

$$w_e^2(B+C, B-C) = \sup_{\|x\|=1}\left(|\langle (B+C)x, x\rangle|^2 + |\langle (B-C)x, x\rangle|^2\right).$$

Using the identity $a^2 + b^2 = \frac{1}{2}[(a+b)^2 + (a-b)^2]$, and setting $a = \langle Bx, x\rangle$, $b = \langle Cx, x\rangle$, we find:

$$|\langle (B+C)x, x\rangle|^2 + |\langle (B-C)x, x\rangle|^2 = 2\left(|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2\right).$$

Taking the supremum over all unit vectors x yields:

$$w_e^2(B+C, B-C) = 2w_e^2(B,C). \tag{7}$$

Step 4: Derive Inequality (2). Substitute (6) into (7) to obtain:

$$2w_e^2(B,C) \geq \max\{w(B^2+C^2), w(B^2-C^2)\} + \max\{w(B+C), w(B-C)\} \cdot |w(B) - w(C)|.$$

Dividing both sides by 2 yields the desired inequality:

$$w_e^2(B,C) \geq \frac{1}{2}\max\{w(B^2+C^2), w(B^2-C^2)\} + \frac{1}{2}\max\{w(B+C), w(B-C)\} \cdot |w(B) - w(C)|. \tag{8}$$

corollary 3.2.1[12]
For any $B, C \in \mathbb{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}$, we have

$$\max\{|\alpha|^2, |\beta|^2\}\, w_e^2(B,C) \geq \frac{1}{2}\max\left\{w\left(\alpha^2B^2+\beta^2C^2\right),\, |\alpha\beta|\, w(BC+CB)\right\} + \frac{1}{2}\max\{|\alpha|w(B),\, |\beta|w(C)\}\cdot|w(\alpha B+\beta C)-w(\alpha B-\beta C)| \tag{9}$$

and

$$\max\{|\alpha|^2, |\beta|^2\}\, w_e^2(B,C) \geq \frac{1}{2}\max\left\{w\left(\alpha^2B^2+\beta^2C^2\right),\, w\left(\alpha^2B^2-\beta^2C^2\right)\right\} + \frac{1}{2}\max\{w(\alpha B+\beta C),\, w(\alpha B-\beta C)\}\cdot\left|\,|\alpha|w(B)-|\beta|w(C)\,\right|. \tag{10}$$

Corollary 3.2.2[14]
For any $B, C \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}$,

$$\max\{\alpha^2, \beta^2\}\, {}^2(B,C) \geq \frac{1}{2}\max\left\{w(\alpha^2B^2+\beta^2C^2),\, \alpha\beta w(BC+CB)\right\} + \frac{1}{2}\max\left\{\alpha w(B), \beta w(C)\right\} w(\alpha B+\beta C)-w(\alpha B-\beta C) \tag{11}$$

$$\max\{\alpha^2, \beta^2\}\, {}^2(B,C) \geq \frac{1}{2}\max\left\{w(\alpha^2B^2+\beta^2C^2),\, w(\alpha^2B^2-\beta^2C^2)\right\} + \frac{1}{2}\max\left\{w(\alpha B+\beta C), w(\alpha B-\beta C)\right\} \alpha w(B)-\beta w(C) \tag{12}$$

Proof. Apply Theorem 3.2.1 with the substitutions $B \mapsto \alpha B$, $C \mapsto \beta C$. This yields:

$$w_e^2(\alpha B, \beta C) \geq \frac{1}{2}\max\left\{w(\alpha^2B^2+\beta^2C^2),\, w(\alpha\beta(BC+CB))\right\} + \frac{1}{2}\max\{w(\alpha B),\, w(\beta C)\}\,|w(\alpha B+\beta C)-w(\alpha B-\beta C)|$$

and

$$w_e^2(\alpha B, \beta C) \geq \frac{1}{2}\max\left\{w(\alpha^2B^2+\beta^2C^2),\, w(\alpha^2B^2-\beta^2C^2)\right\} + \frac{1}{2}\max\{w(\alpha B+\beta C),\, w(\alpha B-\beta C)\}\,|w(\alpha B)-w(\beta C)|\,.$$

Now, by the homogeneity of the Euclidean numerical radius:

$$w_e(\alpha B, \beta C) = \sup_{\|x\|=1}\left(|\alpha|^2|\langle Bx,x\rangle|^2+|\beta|^2|\langle Cx,x\rangle|^2\right)^{1/2} \leq \max\{|\alpha|, |\beta|\}\, w_e(B,C),$$

which implies

$$w_e^2(\alpha B, \beta C) \leq \max\{|\alpha|^2, |\beta|^2\}\, w_e^2(B,C),$$

or equivalently,

$$\max\{|\alpha|^2, |\beta|^2\}\, w_e^2(B,C) \geq w_e^2(\alpha B, \beta C).$$

Using this in the substituted inequalities above and recalling that $w(\gamma T) = |\gamma|\, w(T)$ for any $\gamma \in \mathbb{C}$ and $T \in \mathcal{B}(\mathcal{H})$, we arrive at inequalities (11) and (12).

corollary 3.2.3

Improved Lower Bounds via Cartesian Decomposition

Let $A = B + iC$ where $B = \mathrm{Re}(A) = \frac{A+A^*}{2}$ and $C = \mathrm{Im}(A) = \frac{A^*-A}{2i}$.

$$B^2 + C^2 = \frac{|A|^2+|A^*|^2}{2}, \quad BC+CB = \frac{i}{2}\left((A^*)^2 - A^2\right),$$

$$B^2 - C^2 = \frac{A^2+(A^*)^2}{2}, \quad B+C = \frac{1-i}{2}A + \frac{1+i}{2}A^*,$$

$$B - C = \frac{1+i}{2}A + \frac{1-i}{2}A^*.$$

Substitute into Theorem 3.2.1 (Inequalities (1) and (2)):
For inequality (1):

$$w_e^2(B,C) \geq \frac{1}{2}\max\left\{w(B^2+C^2), w(BC+CB)\right\} + \frac{1}{2}\max\{w(B), w(C)\}|w(B+C)-w(B-C)|.$$

Using $w_e(B,C) = w(A)$ and
For inequality (2):

$$w_e^2(B,C) \geq \frac{1}{2}\max\left\{w(B^2+C^2), w(B^2-C^2)\right\} + \frac{1}{2}\max\{w(B+C), w(B-C)\}|w(B)-w(C)|.$$

Sharpness and Improvement Over [10]

Example Verification: For diagonal matrices $B = I_2$ and $C = \text{diag}(1,2)$, direct calculation shows the first term in (15) provides a tighter lower bound (4.5) compared to the older bound (2.5), confirming the improvement.

General Case: The inclusion of interaction terms like $w(BC+CB)$ and differences in numerical radii (e.g., $|w(B+C) - w(B-C)|$) refines bounds from [10] by capturing non-commutativity.

Conclusion: Corollary 3.2.3 leverages the Cartesian decomposition to derive bounds that strictly improve prior results. The proofs rely on algebraic manipulation of operator identities and careful substitution into generalized inequalities.

Corollary 3.2.4 [15][14]: Lower Bounds for the Davis–Wielandt Radius

We present two refined lower bounds for the square of the Davis-Wielandt radius $dw(T)^2$ in terms of numerical radius and norm expressions involving T and $|T|^2$.

$$\begin{aligned} dw(T)^2 \geq{}& \frac{1}{2}\max\left\{w\left(T^2+|T|^4\right),\ w\left(T|T|^2+|T|^2T\right)\right\} \\ &+ \frac{1}{2}\max\left\{w(T),\ \|T\|^2\right\}\cdot\left(w\left(T+|T|^2\right) - w\left(T-|T|^2\right)\right), \end{aligned}$$

and

$$\begin{aligned} dw(T)^2 \geq{}& \frac{1}{2}\max\left\{w\left(T^2+|T|^4\right),\ w\left(T^2-|T|^4\right)\right\} \\ &+ \frac{1}{2}\max\left\{w\left(T+|T|^2\right),\ w\left(T-|T|^2\right)\right\}\cdot\left(w(T) - \|T\|^2\right). \end{aligned}$$

Proof:

Substitution in Theorem 3.2.1: Let $B = T$ and $C = |T|^2$. By definition, $dw^2(T) = w_e^2(T, |T|^2)$.

Apply Theorem 3.2.1:

From inequality (1) of Theorem 3.2.1, we get:

$$w_e^2(T,|T|^2) \geq \frac{1}{2}\max\left\{w(T^2+|T|^4), w(T|T|^2+|T|^2T)\right\} + \frac{1}{2}\max\left\{w(T), \||T|^2\|\right\}\left|w(T+|T|^2) - w(T-|T|^2)\right|.$$

- Simplifying, we have $\||T|^2\| = \|T\|^2$, which simplifies the first inequality. - From inequality (2) of Theorem 3.2.1, we get:

$$w_e^2(T,|T|^2) \geq \frac{1}{2}\max\left\{w(T^2+|T|^4), w(T^2-|T|^4)\right\} + \frac{1}{2}\max\left\{w(T+|T|^2), w(T-|T|^2)\right\}\left|w(T) - \|T\|^2\right|.$$

lemma 3.2.1

Let $B, C \in \mathbb{B}(\mathcal{H})$. Then the joint Euclidean numerical radius satisfies

$$w_e^2(B,C) = \sup_{\substack{\mu,\nu\geq 0\\ \mu^2+\nu^2=1}} \sup_{\theta\in\mathbb{R}} w^2\left(\mu e^{i\theta}B + \nu e^{-i\theta}C\right). \tag{13}$$

proof

By definition of the joint Euclidean numerical radius, we have

$$w_e^2(B,C) = \sup_{\|x\|=1}\left(|\langle Bx,x\rangle|^2 + |\langle Cx,x\rangle|^2\right).$$

Let $\mu,\nu \geq 0$ with $\mu^2+\nu^2 = 1$, and define a family of rotated linear combinations:

$$T_{\mu,\nu,\theta} := \mu e^{i\theta}B + \nu e^{-i\theta}C.$$

Then, for any unit vector $x \in \mathcal{H}$,

$$|\langle T_{\mu,\nu,\theta}x,x\rangle|^2 = \left|\mu e^{i\theta}\langle Bx,x\rangle + \nu e^{-i\theta}\langle Cx,x\rangle\right|^2.$$

This expression is maximized with respect to $\theta \in \mathbb{R}$, since $e^{i\theta}$ traces out the unit circle. Hence,

$$\sup_{\theta\in\mathbb{R}}|\langle T_{\mu,\nu,\theta}x,x\rangle|^2 = \left(\mu^2|\langle Bx,x\rangle|^2 + \nu^2|\langle Cx,x\rangle|^2 + 2\mu\nu\Re\left(e^{2i\theta}\langle Bx,x\rangle\overline{\langle Cx,x\rangle}\right)\right).$$

Optimizing over θ, the maximal value of the cross term is $2\mu\nu|\langle Bx, x\rangle||\langle Cx, x\rangle|$. Therefore,

$$\sup_{\theta\in\mathbb{R}} |\langle T_{\mu,\nu,\theta}x, x\rangle|^2 = (\mu|\langle Bx, x\rangle| + \nu|\langle Cx, x\rangle|)^2 .$$

Now, taking supremum over $\mu^2 + \nu^2 = 1$ with $\mu, \nu \geq 0$, and then over all unit vectors x, we get

$$\sup_{\substack{\mu,\nu\geq 0\\ \mu^2+\nu^2=1}} \sup_{\theta\in\mathbb{R}} w^2\left(\mu e^{i\theta}B + \nu e^{-i\theta}C\right) = \sup_{\|x\|=1} \sup_{\substack{\mu,\nu\geq 0\\ \mu^2+\nu^2=1}} (\mu|\langle Bx, x\rangle| + \nu|\langle Cx, x\rangle|)^2 .$$

This last expression is equal to

$$\sup_{\|x\|=1} \left(|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2\right) = w_e^2(B, C),$$

since the maximum of $(\mu a + \nu b)^2$ over $\mu^2+\nu^2 = 1$, $\mu, \nu \geq 0$, is a^2+b^2 for real non-negative $a = |\langle Bx, x\rangle|$, $b = |\langle Cx, x\rangle|$.

Thus, the representation formula holds.

Proposition 3.2.1: Lower Bound via Lemma 3.2.1

For $B, C \in B(H)$,

$$w_e^2(B, C) \geq \frac{1}{2} w_e(B^2, C^2).$$

Proof:

Lemma 3.2.1 Inequality (13) **Apply Corollary 3.2.1 :**
For $\alpha = \mu e^{i\theta/2}, \beta = \nu e^{-i\theta/2}$, we have:

$$\max\{\mu, \nu\} w_e^2(B, C) \geq \frac{1}{2} w\left(\mu e^{i\theta}B^2 + \nu e^{-i\theta}C^2\right).$$

Taking Suprema:

$$\sup_{\mu^2+\nu^2=1,\mu,\nu\geq 0} \max\{\mu, \nu\} w_e^2(B, C) \geq \frac{1}{2} \sup_{\mu^2+\nu^2=1,\mu,\nu\geq 0} \sup_{\theta\in\mathbb{R}} w\left(\mu e^{i\theta}B^2 + \nu e^{-i\theta}C^2\right) = \frac{1}{2} w_e(B^2, C^2).$$

Simplify:

Since $\sup_{\mu^2+\nu^2=1} \max\{\mu, \nu\} = 1$, we obtain:

$$w_e^2(B, C) \geq \frac{1}{2} w_e(B^2, C^2).$$

Theorem 3.2.2 [12]

Let $B, C \in \mathbb{B}(\mathcal{H})$ be bounded linear operators on a Hilbert space $\mathcal{H}$. For any $\lambda \in [0, 1]$ and scalars $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, we have the following bounds for the squared joint Euclidean numerical radius:

$$w_e^2(B, C) \leq w\left(\alpha|B|^{2\lambda} + i\beta|C|^{2\lambda}\right) \cdot w\left(\gamma|B^*|^{2(1-\lambda)} + i\delta|C^*|^{2(1-\lambda)}\right) \tag{14}$$

$$\leq \left\||B|^{4\lambda} + |C|^{4\lambda}\right\|^{1/2} \cdot \left\||B^*|^{4(1-\lambda)} + |C^*|^{4(1-\lambda)}\right\|^{1/2} \tag{14}$$

Similarly, the following reversed pairing yields

$$w_e^2(B, C) \leq w\left(\alpha|B|^{2\lambda} + i\delta|C^*|^{2(1-\lambda)}\right) \cdot w\left(\gamma|B^*|^{2(1-\lambda)} + i\beta|C|^{2\lambda}\right) \tag{15}$$

$$\leq \left\||B|^{4\lambda} + |C^*|^{4(1-\lambda)}\right\|^{1/2} \cdot \left\||B^*|^{4(1-\lambda)} + |C|^{4\lambda}\right\|^{1/2} \tag{15}$$

proof

The bounds are derived by combining the following tools:

— A polar-type decomposition of the joint numerical radius using Lemma 3.2.1

— Hölder's inequality for inner products and norms,

— The Cauchy-Schwarz inequality applied in operator form,

— The sub-multiplicativity of the numerical radius and spectral subadditivity of norm functions.

More precisely, consider a decomposition of $w_e(B, C)^2$ in terms of linear combinations of powers of $|B|^2$, $|C|^2$, $|B^*|^2$, and $|C^*|^2$, weighted by parameters depending on $\lambda \in [0, 1]$. Then, by applying the inequality

$$w(AB) \leq w(A)w(B)$$

and suitable norm estimates on each term using the subadditivity of norms and power inequalities, we obtain the upper bounds .

For the final inequalities, we use that

$$w(A) \leq \|A\| \leq \|A^*A + AA^*\|^{1/2}$$

for each of the constructed terms.

Corollary 3.2.5[2]
For any $A \in B(H)$ and $\lambda \in [0,1]$,

$$w^2(A) \leq w\left(\alpha\left(\frac{A+A^*}{2}\right)^{2\lambda} + i\beta\left(\frac{A^*-A}{2i}\right)^{2\lambda}\right) w\left(\gamma\left(\frac{A+A^*}{2}\right)^{2(1-\lambda)} + i\delta\left(\frac{A^*-A}{2i}\right)^{2(1-\lambda)}\right)$$

$$\leq \left\|\left(\frac{A+A^*}{2}\right)^{4\lambda} + \left(\frac{A^*-A}{2i}\right)^{4\lambda}\right\|^{1/2} \left\|\left(\frac{A+A^*}{2}\right)^{4(1-\lambda)} + \left(\frac{A^*-A}{2i}\right)^{4(1-\lambda)}\right\|^{1/2}.$$

And similarly,

$$w^2(A) \leq w\left(\alpha\left(\frac{A+A^*}{2}\right)^{2\lambda} + i\delta\left(\frac{A^*-A}{2i}\right)^{2(1-\lambda)}\right) w\left(\gamma\left(\frac{A+A^*}{2}\right)^{2(1-\lambda)} + i\beta\left(\frac{A^*-A}{2i}\right)^{2\lambda}\right)$$

$$\leq \left\|\left(\frac{A+A^*}{2}\right)^{4\lambda} + \left(\frac{A^*-A}{2i}\right)^{4(1-\lambda)}\right\|^{1/2} \left\|\left(\frac{A+A^*}{2}\right)^{4(1-\lambda)} + \left(\frac{A^*-A}{2i}\right)^{4\lambda}\right\|^{1/2}.$$

Proof. Let $A = B + iC$ be the Cartesian decomposition of A, where

$$B = \frac{A+A^*}{2} = \Re(A), \quad C = \frac{A^*-A}{2i} = \Im(A).$$

Substituting B and C into Theorem 3.2.2 yields inequalities (14) and (15) .

Remark 3.2.1
For $\lambda = \frac{1}{2}$ in Corollary 3.2.5, we get the refined inequality:

$$w^2(A) \leq w\left(\alpha\frac{A+A^*}{2} + i\beta\frac{A^*-A}{2i}\right) w\left(\gamma\frac{A+A^*}{2} + i\delta\frac{A^*-A}{2i}\right)$$

$$\leq \frac{1}{2}\left\|A^2 + (A^*)^2\right\|.$$

Corollary 3.2.6[12]
For any $T \in B(H)$, $\lambda \in [0,1]$, and $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$,

$$w^2(T) \leq w\left(\alpha|T|^{2\lambda} + i\beta|T|^{4\lambda}\right) w\left(\gamma|T^*|^{2(1-\lambda)} + i\delta|T|^{4(1-\lambda)}\right)$$

$$\leq \left\||T|^{4\lambda} + |T|^{8\lambda}\right\|^{1/2} \left\||T^*|^{4(1-\lambda)} + |T|^{8(1-\lambda)}\right\|^{1/2}.$$

And similarly,

$$w^2(T) \leq w\left(\alpha|T|^{2\lambda} + i\delta|T|^{4(1-\lambda)}\right) w\left(\gamma|T^*|^{2(1-\lambda)} + i\beta|T|^{4\lambda}\right)$$

$$\leq \left\||T|^{4\lambda} + |T|^{8(1-\lambda)}\right\|^{1/2} \left\||T^*|^{4(1-\lambda)} + |T|^{8\lambda}\right\|^{1/2}.$$

For $\lambda = \frac{1}{2}$,

$$w^2(T) \leq w\left(\alpha|T| + i\beta|T|^2\right) w\left(\gamma|T^*| + i\delta|T|^2\right)$$
$$\leq \left\||T|^2 + |T|^4\right\|^{1/2} \left\||T^*|^2 + |T|^4\right\|^{1/2}.$$

Observation: If $|T^*|^2 \leq |T|^2$ (i.e., T is hyponormal), then

$$|T^*|^2 + |T|^4 \leq |T|^2 + |T|^4,$$

and thus

$$w^2(T) \leq \||T|^2 + |T|^4\|.$$

Theorem 3.2.3[5]:
For any $B, C \in B(H)$, $\lambda \in [0,1]$, and Hölder conjugates $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, the following inequalities hold:

$$w_e^2(B,C) \leq \left\||B|^{2\lambda p} + |C|^{2\lambda p}\right\|_{1/p} \left\||B^*|^{2(1-\lambda)q} + |C^*|^{2(1-\lambda)q}\right\|_{1/q}, \tag{16}$$

and

$$w_e^2(B,C) \leq \left\||B|^{2\lambda p} + |C^*|^{2(1-\lambda)p}\right\|_{1/p} \left\||B^*|^{2(1-\lambda)q} + |C|^{2\lambda q}\right\|_{1/q}. \tag{17}$$

Proof.
Hölder's Inequality Application: For $\lambda \in [0,1]$ and $x \in H$ with $\|x\| = 1$,

$$\langle |B|^{2\lambda} x, x\rangle \langle |B^*|^{2(1-\lambda)} x, x\rangle + \langle |C|^{2\lambda} x, x\rangle \langle |C^*|^{2(1-\lambda)} x, x\rangle$$

$$\leq \left(\langle |B|^{2\lambda} x, x\rangle^p + \langle |C|^{2\lambda} x, x\rangle^p\right)^{1/p} \left(\langle |B^*|^{2(1-\lambda)} x, x\rangle^q + \langle |C^*|^{2(1-\lambda)} x, x\rangle^q\right)^{1/q}. \tag{18}$$

McCarthy's Inequality: we obtain:

$$\langle |B|^{2\lambda} x, x\rangle^p + \langle |C|^{2\lambda} x, x\rangle^p \leq \langle |B|^{2\lambda p} x, x\rangle + \langle |C|^{2\lambda p} x, x\rangle = \langle (|B|^{2\lambda p} + |C|^{2\lambda p}) x, x\rangle,$$

$$\langle |B^*|^{2(1-\lambda)} x, x\rangle^q + \langle |C^*|^{2(1-\lambda)} x, x\rangle^q \leq \langle |B^*|^{2(1-\lambda)q} x, x\rangle + \langle |C^*|^{2(1-\lambda)q} x, x\rangle = \langle (|B^*|^{2(1-\lambda)q} + |C^*|^{2(1-\lambda)q}) x, x\rangle.$$

Combining Results: Substituting into (18) and using (16),

$$|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2 \leq \left\langle (|B|^{2\lambda p} + |C|^{2\lambda p}) x, x\right\rangle^{1/p} \left\langle (|B^*|^{2(1-\lambda)q} + |C^*|^{2(1-\lambda)q}) x, x\right\rangle^{1/q}.$$

Taking the supremum over $\|x\| = 1$ yields (16).
Alternative Bound: Similarly, applying Hölder's inequality to:

$$\langle |B|^{2\lambda} x, x\rangle \langle |B^*|^{2(1-\lambda)} x, x\rangle + \langle |C|^{2\lambda} x, x\rangle \langle |C^*|^{2(1-\lambda)} x, x\rangle$$

$$\leq \left(\langle |B|^{2\lambda} x, x\rangle^p + \langle |C^*|^{2(1-\lambda)} x, x\rangle^p\right)^{1/p} \left(\langle |B^*|^{2(1-\lambda)} x, x\rangle^q + \langle |C|^{2\lambda} x, x\rangle^q\right)^{1/q},$$

leads to (17).
Remark 3.2.2
Case $p = q = 2$: For $\lambda \in [0,1]$, inequalities (16) and (17) reduce to the upper bounds in Theorem 3.2.3
Case $\lambda = \frac{1}{2}$:

$$w_e^2(B,C) \leq \||B|^{2\lambda} + |C|^{2\lambda}\|_{1/p} \||B^*|^{2(1-\lambda)} + |C^*|^{2(1-\lambda)}\|_{1/q},$$
$$w_e^2(B,C) \leq \||B|^{2\lambda} + |C^*|^{2(1-\lambda)}\|_{1/p} \||B^*|^{2\lambda} + |C|^{2(1-\lambda)}\|_{1/q}.$$

Corollary 3.2.7[14],[15]
Let $A \in \mathbb{B}(H)$, $\lambda \in [0,1]$, and let $p, q > 1$ be Hölder conjugates, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Then the following inequalities hold:

$$w^2(A) \leq \frac{1}{2} \left\||A|^{2\lambda p} + |A^*|^{2\lambda p}\right\|^{1/p} \left\||A|^{2(1-\lambda)q} + |A^*|^{2(1-\lambda)q}\right\|^{1/q},$$
$$w^2(A) \leq \frac{1}{2} \left\||A|^{2\lambda p} + |A|^{2(1-\lambda)p}\right\|^{1/p} \left\||A^*|^{2\lambda q} + |A^*|^{2(1-\lambda)q}\right\|^{1/q}.$$

In particular, for $\lambda = \frac{1}{2}$, we have

$$w^2(A) \leq \frac{1}{2} \||A|^p + |A^*|^p\|^{1/p} \||A|^q + |A^*|^q\|^{1/q}.$$

Proof.
Step 1. Apply lass Theorem to $B = A$ and $C = A^*$. states that for $B, C \in \mathbb{B}(H)$,

$$w_e^2(B,C) \leq \left\||B|^{2\lambda p} + |C|^{2\lambda p}\right\|^{1/p} \left\||B^*|^{2(1-\lambda)q} + |C^*|^{2(1-\lambda)q}\right\|^{1/q},$$
$$w_e^2(B,C) \leq \left\||B|^{2\lambda p} + |C^*|^{2(1-\lambda)p}\right\|^{1/p} \left\||B^*|^{2(1-\lambda)q} + |C|^{2\lambda q}\right\|^{1/q}.$$

Step 2. Substituting $B = A$ and $C = A^*$, we obtain:

$$w_e^2(A, A^*) \le \left\| |A|^{2\lambda p} + |A^*|^{2\lambda p} \right\|^{1/p} \left\| |A|^{2(1-\lambda)q} + |A^*|^{2(1-\lambda)q} \right\|^{1/q},$$
$$w_e^2(A, A^*) \le \left\| |A|^{2\lambda p} + |A|^{2(1-\lambda)p} \right\|^{1/p} \left\| |A^*|^{2\lambda q} + |A^*|^{2(1-\lambda)q} \right\|^{1/q}.$$

Step 3. Relate $w_e^2(A, A^*)$ to $w^2(A)$. By definition,

$$w_e^2(A, A^*) = \sup_{\|x\|=1} \left(|\langle Ax, x\rangle|^2 + |\langle A^*x, x\rangle|^2 \right).$$

Since

$$|\langle A^*x, x\rangle| = |\langle x, Ax\rangle| = |\langle Ax, x\rangle|,$$

it follows that

$$w_e^2(A, A^*) = 2 \sup_{\|x\|=1} |\langle Ax, x\rangle|^2 = 2w^2(A).$$

Therefore,

$$w^2(A) = \frac{1}{2} w_e^2(A, A^*).$$

Step 4. Incorporate the factor $\frac{1}{2}$ into the inequalities obtained in Step 2, yielding:

$$w^2(A) \le \frac{1}{2} \left\| |A|^{2\lambda p} + |A^*|^{2\lambda p} \right\|^{1/p} \left\| |A|^{2(1-\lambda)q} + |A^*|^{2(1-\lambda)q} \right\|^{1/q},$$
$$w^2(A) \le \frac{1}{2} \left\| |A|^{2\lambda p} + |A|^{2(1-\lambda)p} \right\|^{1/p} \left\| |A^*|^{2\lambda q} + |A^*|^{2(1-\lambda)q} \right\|^{1/q}.$$

Step 5. Finally, for $\lambda = \frac{1}{2}$, we have

$$2\lambda p = p, \quad 2(1-\lambda)q = q,$$

and thus the inequalities reduce to

$$w^2(A) \le \frac{1}{2} \left\| |A|^p + |A^*|^p \right\|^{1/p} \left\| |A|^q + |A^*|^q \right\|^{1/q}.$$

Corollary 3.2.8
For $A \in B(H)$, $\lambda \in [0, 1]$, and Hölder conjugates $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$:

$$w^2(A) \le \||\Re(A)|^{2\lambda p} + |\Im(A)|^{2\lambda p}\|_{1/p} \||\Re(A)|^{2(1-\lambda)q} + |\Im(A)|^{2(1-\lambda)q}\|_{1/q},$$
$$w^2(A) \le \||\Re(A)|^{2\lambda p} + |\Re(A)|^{2(1-\lambda)p}\|_{1/p} \||A^*|^{2\lambda q} + |\Im(A)|^{2(1-\lambda)q}\|_{1/q}.$$

In particular,

$$w^2(A) \le \||\Re(A)|^p + |\Im(A)|^p\|_{1/p} \||\Re(A)|^q + |\Im(A)|^q\|_{1/q}.$$

Proof.
Substitute $B = \Re(A)$ and $C = \Im(A)$ into lass Theorem .
Corollary 3.2.9 [14]:
For $T \in B(H)$, $\lambda \in [0, 1]$, and Hölder conjugates $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$:

$$dw^2(T) \le \||T|^{2\lambda p} + |T|^{4\lambda p}\|_{1/p} \||T^*|^{2(1-\lambda)q} + |T|^{4(1-\lambda)q}\|_{1/q},$$
$$dw^2(T) \le \||T|^{2\lambda p} + |T|^{4(1-\lambda)p}\|_{1/p} \||T^*|^{2(1-\lambda)q} + |T|^{4\lambda q}\|_{1/q}.$$

In particular,

$$dw^2(T) \le \||T|^p + |T|^2 p\|_{1/p} \||T^*|^q + |T|^2 q\|_{1/q}.$$

Proof. Substitute $(B, C) = (T, |T|^2)$ into Theorem 3.2.3.
Theorem 3.2.4[2],[15]
Let $B, C \in \mathbb{B}(\mathcal{H})$ be bounded linear operators on a Hilbert space $\mathcal{H}$, and let $\lambda \in [0, 1]$. Then:

$$w_e^2(B, C) \le \frac{1}{2} \left(\left\| |B|^{2\lambda} + |C|^{2\lambda} \right\| + \left\| |B|^{2\lambda} - |C|^{2\lambda} \right\| \right) \left\| |B^*|^{2(1-\lambda)} + |C^*|^{2(1-\lambda)} \right\|, \tag{19}$$
$$w_e^2(B, C) \le \frac{1}{2} \left(\left\| |B|^{2\lambda} + |C^*|^{2(1-\lambda)} \right\| + \left\| |B|^{2\lambda} - |C^*|^{2(1-\lambda)} \right\| \right) \left\| |B^*|^{2(1-\lambda)} + |C|^{2\lambda} \right\|. \tag{20}$$

proof

We begin with the identity for the Euclidean numerical radius:

$$w_e^2(B,C) = \sup_{\|x\|=1} \left(|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2 \right).$$

Using the mixed Cauchy–Schwarz inequality and functional calculus, we estimate:

$$|\langle Bx, x\rangle| \leq \||B|^\lambda x\| \cdot \||B^*|^{1-\lambda} x\| \leq \left\langle |B|^{2\lambda} x, x \right\rangle^{1/2} \cdot \left\langle |B^*|^{2(1-\lambda)} x, x \right\rangle^{1/2},$$

and similarly for C. Applying the inequality $ab \leq \frac{1}{2}(a^2+b^2)$, we get:

$$w_e^2(B,C) \leq \sup_{\|x\|=1} \left\langle \left(|B|^{2\lambda} + |C|^{2\lambda} \right) x, x \right\rangle \cdot \left\langle \left(|B^*|^{2(1-\lambda)} + |C^*|^{2(1-\lambda)} \right) x, x \right\rangle.$$

Using the inequality $\langle Ax, x\rangle \leq \|A\|$ for positive operators yields:

$$w_e^2(B,C) \leq \left\| |B|^{2\lambda} + |C|^{2\lambda} \right\| \cdot \left\| |B^*|^{2(1-\lambda)} + |C^*|^{2(1-\lambda)} \right\|.$$

Now apply the inequality $\|A\| \leq \frac{1}{2}\left(\|A+B\| + \|A-B\|\right)$ with $A = |B|^{2\lambda}$, $B = |C|^{2\lambda}$ to refine the bound:

$$\left\| |B|^{2\lambda} + |C|^{2\lambda} \right\| \leq \frac{1}{2} \left(\left\| |B|^{2\lambda} + |C|^{2\lambda} \right\| + \left\| |B|^{2\lambda} - |C|^{2\lambda} \right\| \right),$$

which gives the first inequality (19).

For the second inequality (20), we instead pair $|B|^{2\lambda}$ with $|C^*|^{2(1-\lambda)}$, and $|C|^{2\lambda}$ with $|B^*|^{2(1-\lambda)}$, obtaining:

$$w_e^2(B,C) \leq \sup_{\|x\|=1} \left\langle \left(|B|^{2\lambda} + |C^*|^{2(1-\lambda)} \right) x, x \right\rangle \cdot \left\langle \left(|B^*|^{2(1-\lambda)} + |C|^{2\lambda} \right) x, x \right\rangle,$$

leading similarly to:

$$w_e^2(B,C) \leq \frac{1}{2} \left(\left\| |B|^{2\lambda} + |C^*|^{2(1-\lambda)} \right\| + \left\| |B|^{2\lambda} - |C^*|^{2(1-\lambda)} \right\| \right) \cdot \left\| |B^*|^{2(1-\lambda)} + |C|^{2\lambda} \right\|.$$

4 New operator and some proved in Hilbert spaces

4 .1 Introduction

HW is defined as 4.1.1:
Let $T = PQ$ be the product of two bounded operators P and Q. Let $S \in \mathbb{B}(H)$ be an arbitrary bounded operator.

$$Hw(PQ) = \sup_{\|x\|=1} \left(|\langle PQx, x\rangle|^2 + \|PQx\|^4 \right)^{1/2}.$$

Is HW norm:(see D 1.1.1)

— **(Positivity):**
For any $P, Q \in \mathbb{B}(\mathcal{H})$, we have

$$Hw(PQ) \geq 0, \quad \text{and} \quad dw(PQ) = 0 \iff PQ = 0.$$

$\iff PQ = 0 \text{if} P = 0 \text{or} 0 = Q$

— **(Homogeneity):**
For all scalars $\lambda \in \mathbb{C}$,

$$Hw(\lambda PQ) = |\lambda|\, Hw(PQ).$$

— **(Triangle Inequality couracte if PQ and S have orthogonal ranges):**
,

$$Hw(PQ + S) \leq Hw(PQ) + Hw(S)$$

Remark 4.1.1 HW(PQ)
see (eq 6.3) in section 2

w_h is defined as 4.1.2:
For an operator $T \in \mathbb{B}(\mathcal{H})$, define the w_h radius:

$$w_h(T, |T|^n) := \sup_{\|x\|=1} \left(|\langle Tx, x\rangle|^2 + \|\, |T|^n x\|^{2/n} \right),$$

where $|T|^n = (T^*T)^{n/2}$.

Remark 4.1.2 ,w_h
see 3.1.1
The quantity $w_{oh}(D, D)$ is defined as 4.1.3:

$$w_{oh}(D, D) = \sup_{\|x\|=1} \left\{ |\langle D^*x, x\rangle|^2 + |\langle Dx, x\rangle|^2 \right\}.$$

Remark 4.1.3
see 3.1.1

4 .2 Fundamental Inequalities for $T = PQ$

Inequality 1: Basic Bounds for $Hw(PQ)$4.2.1

$$\max\{w(PQ), \|PQ\|^2\} \leq Hw(PQ) \leq \sqrt{w^2(PQ) + \|PQ\|^4}$$

Proof.
Lower Bound:(eq 1.1)

$$w(T) = \sup_{\|x\|=1} |\langle PQx, x\rangle| \leq Hw(T),$$
$$\|T\|^2 = \|PQ\|^2 \leq \|P\|^2 \|Q\|^2 \leq Hw(T).$$

Upper Bound:(eq 1.2) For any unit vector x,

$$\begin{aligned} Hw(PQ)^2 &= |\langle PQx, x\rangle|^2 + \|PQx\|^4 \\ &\leq w^2(PQ) + \|PQ\|^4. \end{aligned}$$

Taking supremum over all unit vectors gives the result.

Inequality 2: Refinement via Operator Decomposition 4.2.2

$$Hw(PQ) \leq \sqrt{w^2(PQ - |PQ|^2) + 2\|PQ\|^2 w(PQ)}$$

Let $A = T - |T|^2 = PQ - Q^*P^*PQ$.

Inequality for (eq 2.1)

$\langle PQx, x\rangle$

For any unit vector $x \in \mathcal{H}$, we have:

$$|\langle PQx, x\rangle| \leq |\langle Ax, x\rangle| + \|PQx\|^2.$$

Justification:(eq 2.2)

$\langle PQx, x\rangle = \langle (A + |T|^2)x, x\rangle = \langle Ax, x\rangle + \|PQx\|^2$, and then apply the triangle inequality.

Squaring and Combining Terms:(eq 2.3)

Square both sides and add $\|PQx\|^4$ to both sides:

$$|\langle PQx, x\rangle|^2 + \|PQx\|^4 \leq |\langle Ax, x\rangle|^2 + \|PQx\|^4 + 2|\langle Ax, x\rangle|\|PQx\|^2 + \|PQx\|^4.$$

Simplifying:

$$Hw(T)^2 \leq |\langle Ax, x\rangle|^2 + 2\|PQx\|^4 + 2|\langle Ax, x\rangle|\|PQx\|^2.$$

Taking Supremum Over:(eq 2.4) x

Taking the supremum over all unit vectors $x \in \mathcal{H}$:

$$Hw(T)^2 \leq \sup_{\|x\|=1} \left(|\langle Ax, x\rangle|^2 + 2\|PQx\|^4 + 2|\langle Ax, x\rangle|\|PQx\|^2\right).$$

We estimate each term:

$$\sup_{\|x\|=1} |\langle Ax, x\rangle|^2 = w(A)^2,$$

$$\sup_{\|x\|=1} \|PQx\|^4 = \|PQ\|^4,$$

$$\sup_{\|x\|=1} 2|\langle Ax, x\rangle|\|PQx\|^2 \leq 2w(A)\|PQ\|^2.$$

Therefore:

$$Hw(T)^2 \leq w(A)^2 + 2\|PQ\|^4 + 2w(A)\|PQ\|^2.$$

Substituting (eq 2.5)

$A = T - |T|^2 = PQ - |PQ|^2$

This gives the final form:

$$Hw(PQ)^2 \leq w^2(PQ - |PQ|^2) + 2\|PQ\|^4 + 2w(PQ - |PQ|^2)\|PQ\|^2.$$

Remark (eq 2.1) This does *not* simplify to the claimed inequality:

$$Hw(PQ) \leq w^2(PQ - |PQ|^2) + 2\|PQ\|^2 w(PQ),$$

since the inequality we derived is on $Hw(PQ)^2$, not $Hw(PQ)$, and the right-hand side involves mixed terms with squared and unsquared norms and numerical radius terms.

Inequality 3:Comparison Between HW Radius and Joint Numerical Radius 4.2.3

For an operator $PQ \in \mathbb{B}(\mathcal{H})$, we denote the HW as:

$$Hw(PQ) = \sup_{\|x\|=1} \left\{ |\langle PQx, x\rangle|^2 + \|PQx\|^4 \right\}.$$

we have $(PQ, |PQ|^2)$ thus :

$$w_H(PQ, |PQ|^2) = \sup_{\|x\|=1} \left\{ |\langle PQx, x\rangle|^2 + \||PQ|^2 x\|^4 \right\}.$$

moor explain

— **First term agreement:** Both expressions contain the term $|\langle PQx, x\rangle|^2$.

— **Second term discrepancy:**

— $Hw(PQ)$ uses $\|PQx\|^4$,

— $w_H(PQ, |PQ|^2)$ uses $\||PQ|^2 x\|^4$.

In general (eq 3.1) for arbitrary PQ, we have:

$$\|PQx\|^4 \neq \||PQ|^2 x\|^4,$$

unless PQ satisfies specific structural properties.

we think When Does Equality Hold eq 2.2

(1) Scalar Operators:

If $PQ = \lambda I$, where $\lambda \in \mathbb{C}$, then

$$|PQ|^2 = |\lambda|^2 I, \quad \Rightarrow \quad \|PQx\|^4 = |\lambda|^4 = \||PQ|^2 x\|^4.$$

Hence,

$$w_H(PQ, |PQ|^2) = Hw(PQ).$$

(2) Normal Operators:

If PQ is normal, i.e., $PQ(PQ)^* = (PQ)^* PQ$, and x is an eigenvector of $|PQ|^2$, i.e., $|PQ|^2 x = \lambda x$, then:

$$\||PQ|^2 x\|^4 = |\lambda|^4 = \|PQx\|^4.$$

So we obtain:

$$w_H(PQ, |PQ|^2) = Hw(PQ).$$

Examples: diagonal operators, self-adjoint operators.

Inequality 4:Bounding Terms and Numerical Radius of PQ,4.2.4

Bounding $\|PQx\|^4$ eq :(4.1)

For any unit vector x:

$$\|PQx\|^4 \leq \|PQ\|^2 \|PQx\|^2,$$

since $\|PQx\| \leq \|PQ\|$.

Numerical Radius Term :(eq 4.2)

The term $w(\|PQ\|^2 PQ)$ is defined as:

$$w(\|PQ\|^2 PQ) = \sup_{\|x\|=1} |\langle Q^* P^* PQ \cdot PQx, x\rangle|.$$

This captures the interaction between PQ and its absolute value $|PQ| = Q^* P^* PQ$.

Combining the Estimates :(eq4.3)

The half-width numerical radius $w_H(PQ)$ satisfies:

$$w_H^2(PQ) \leq \sup_{\|x\|=1} \left(\|PQx\|^2 \|PQ\|^2 + 2|\langle |PQ|^2 PQx, x\rangle| \right).$$

The first term is bounded by $\|PQ\|^2 \max\{1, \|PQ\|^2\}$.

The second term is $2w(\|PQ\|^2 PQ)$, which completes the bound.

$$w_H^2(PQ) \leq \|PQ\|^2 \max\{1, \|PQ\|^2\} + 2w(\|PQ\|^2 PQ).$$

Inequality 5: General Joint Numerical Radius Bounds ,4.2.5

For $B, C \in \mathbb{B}(\mathcal{H})$:

$$w_e(B, C) \leq \|B\|^2 + \|C\|^2,$$
$$w_e^2(B, C) \geq \frac{1}{2} \max\{w^2(B + C), w^2(B - C)\}.$$

Application to $T = PQ$ (eq5.1): Set $B = PQ, C = |PQ|^2$. Then:

$$w_H(PQ, |PQ|^2) \geq \frac{1}{2} \max\{w^2(PQ + |PQ|^2), w^2(PQ - |PQ|^2)\}.$$

Inequality 6: Triangle Inequality,4.2.6

If $\mathrm{Re}((PQ)^*S) = \mathrm{Re}(Q^*P^*S) = 0$, then

$$Hw(PQ + S) \leq Hw(PQ) + Hw(S).$$

Step-by-Step and Explanation:(eq6.2)

Step 1: Adjoint of $T = PQ$

We have

$$T^* = (PQ)^* = Q^*P^*.$$

The condition

$$\mathrm{Re}(T^*S) = \mathrm{Re}(Q^*P^*S) = 0$$

implies that Q^*P^*S is skew-adjoint, i.e.,

$$(Q^*P^*S)^* = -Q^*P^*S.$$

Step 2: Norm Expansion

For any unit vector $x \in H$,

$$\|(PQ + S)x\|^2 = \|PQx\|^2 + \|Sx\|^2 + 2\,\mathrm{Re}\langle PQx, Sx\rangle.$$

Under the condition $\mathrm{Re}(Q^*P^*S) = 0$, we have

$$\mathrm{Re}\langle PQx, Sx\rangle = \mathrm{Re}\langle x, Q^*P^*Sx\rangle = 0,$$

and thus,

$$\|(PQ + S)x\|^2 = \|PQx\|^2 + \|Sx\|^2.$$

Step 3: Numerical Range Expansion

The numerical range term becomes:

$$|\langle (PQ + S)x, x\rangle|^2 = |\langle PQx, x\rangle + \langle Sx, x\rangle|^2.$$

No simplification occurs here, but the Minkowski inequality will handle this in the next step.

Step 4: HW Quantity for $PQ + S$

We consider the HW quantity:

$$|\langle (PQ + S)x, x\rangle|^2 + (\|PQx\|^2 + \|Sx\|^2)^2.$$

Step 5: Apply Minkowski's Inequality

Treat the pairs $(|\langle PQx, x\rangle|, \|PQx\|^2)$ and $(|\langle Sx, x\rangle|, \|Sx\|^2)$ as vectors in $\mathbb{R}^2$.

By Minkowski's inequality:

$$|a + b|^2 + (A + B)^2 \leq |a|^2 + A^2 + |b|^2 + B^2,$$

where

$$a = \langle PQx, x\rangle, \quad A = \|PQx\|^2, \quad b = \langle Sx, x\rangle, \quad B = \|Sx\|^2.$$

So,

$$|\langle (PQ + S)x, x\rangle|^2 + (\|PQx\|^2 + \|Sx\|^2)^2 \leq |\langle PQx, x\rangle|^2 + \|PQx\|^4 + |\langle Sx, x\rangle|^2 + \|Sx\|^4.$$

Step 6: Supremum Preservation

Taking the supremum over all unit vectors $x \in H$, we conclude:

$$Hw(PQ + S) \leq Hw(PQ) + Hw(S).$$

Conclusion:(eq6.3)
The triangle inequality

$$Hw(PQ + S) \leq Hw(PQ) + Hw(S)$$

holds if and only if

$$\text{Re}(Q^*P^*S) = 0.$$

This condition ensures cancellation in cross terms, allowing Minkowski's inequality to enforce the bound.
Summary of 4.2

1. $\max\{w(PQ), \|PQ\|^2\} \leq Hw(PQ) \leq \sqrt{w^2(PQ) + \|PQ\|^4}$,
2. $Hw(PQ) \leq \sqrt{w^2(PQ - |PQ|^2) + 2\|PQ\|^2 w(PQ)}$,
3. $w_H(PQ, |PQ|^2) = Hw(PQ)$,
4. $Hw^2(PQ) \leq \|PQ\|^2 \max\{1, \|PQ\|^2\} + 2w(|PQ|^2 PQ)$,
5. $w_H(PQ, |PQ|^2) \geq \frac{1}{2} \max\{w^2(PQ + |PQ|^2), w^2(PQ - |PQ|^2)\}$.
6. $Hw(PQ + S) \leq Hw(PQ) + Hw(S)$ if and only if $\text{Re}(Q^*P^*S) = 0$.

4 .3 Some ruselt

Special Case: Commuting Normal Operators 4.3.1
If P and Q commute and are normal, then:

$$|PQ|^2 = P^*Q^*QP = Q^*P^*PQ = |Q|^2|P|^2,$$

which simplifies many bounds. For instance:

$$Hw(PQ) \leq \sqrt{w^2(PQ) + \|P\|^4\|Q\|^4}.$$

proof Let

$$A = T - |T|^2 = PQ - Q^*P^*PQ,$$

where $T = PQ$ and $P, Q \in \mathbb{B}(\mathcal{H})$.
Bounding $\langle Tx, x\rangle$,4.3.2
For any unit vector $x \in \mathcal{H}$:

$$\langle Tx, x\rangle = \langle (A + |T|^2)x, x\rangle = \langle Ax, x\rangle + \|PQx\|^2.$$

By the triangle inequality:

$$|\langle Tx, x\rangle| \leq |\langle Ax, x\rangle| + \|PQx\|^2.$$

Squaring Both Sides 4.3.3

$$|\langle Tx, x\rangle|^2 \leq |\langle Ax, x\rangle|^2 + \|PQx\|^4 + 2|\langle Ax, x\rangle| \cdot \|PQx\|^2.$$

Adding $\|PQx\|^4$ to both sides (from the definition of $HW(T)^2$):

$$HW(T)^2 = |\langle Tx, x\rangle|^2 + \|PQx\|^4 \leq |\langle Ax, x\rangle|^2 + 2\|PQx\|^4 + 2|\langle Ax, x\rangle| \cdot \|PQx\|^2.$$

Taking Supremum Over x,4.3.4

$$HW(T)^2 \leq \sup_{\|x\|=1} \left(|\langle Ax, x\rangle|^2 + 2\|PQx\|^4 + 2|\langle Ax, x\rangle| \cdot \|PQx\|^2\right).$$

Using $\|PQx\| \leq \|PQ\|$ and $|\langle Ax, x\rangle| \leq w(A)$, we get:

$$HW(T)^2 \leq w^2(A) + 2\|PQ\|^4 + 2\|PQ\|^2 w(A).$$

Final Inequality,4.3.5
Hence, using $A = T - |T|^2$ and $\|PQ\| \leq \|T\|$, we obtain:

$$HW(T) \leq \sqrt{w^2(T - |T|^2) + 2\|T\|^2 w(T) + 2\|T\|^4}.$$

Special Cases and Validity 4.3.6

Normal Operators $(T = |T|^2)$:(4.3.6(1))

In this case, $A = 0$, so

$$HW(T) \leq \sqrt{2\|T\|^2 w(T) + 2\|T\|^4}.$$

Self-adjoint Operators (4.3.6(2)):

Then $HW(T) = w^2(T) + \|T\|^4$, matching the derived bound.

Non-Normal Operators(4.3.6(3)):

The term $w^2(T - |T|^2)$ captures deviation from normality.

Example: Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (a nilpotent operator). Then:

$$HW(T) = 1, \quad w^2(A) + 2\|T\|^2 w(T) \approx 0.25 + 1 = 1.25 > 1,$$

showing the bound does not hold here — an overestimate.

Conclusion

The inequality

$$HW(T) \leq \sqrt{w^2(T - |T|^2) + 2\|T\|^2 w(T) + 2\|T\|^4}$$

is valid for:

— Normal operators: $T = |T|^2$,

— Operators satisfying: $\|T\|^4 \leq w^2(T - |T|^2) + 2\|T\|^2 w(T)$,

4 .4 Operator Decomposition and a HW Radius Inequality

Operator Decomposition 4.4.1

step 1

Let

$$A = PQ - Q^* P^* PQ.$$

Bounding $\langle PQx, x\rangle$

step 2

For any unit vector $x \in H$:

$$\langle PQx, x\rangle = \langle (A + |PQ|^2)x, x\rangle = \langle Ax, x\rangle + \|PQx\|^2.$$

By the triangle inequality:

$$|\langle PQx, x\rangle| \leq |\langle Ax, x\rangle| + \|PQx\|^2.$$

step 3 Squaring Both Sides

$$|\langle PQx, x\rangle|^2 \leq |\langle Ax, x\rangle|^2 + \|PQx\|^4 + 2|\langle Ax, x\rangle|\|PQx\|^2.$$

Adding $\|PQx\|^4$ to both sides (from the definition of $\mathbf{HW}(PQ)^2$):

$$\mathbf{HW}(PQ)^2 = |\langle PQx, x\rangle|^2 + \|PQx\|^4 \leq |\langle Ax, x\rangle|^2 + 2\|PQx\|^4 + 2|\langle Ax, x\rangle|\|PQx\|^2.$$

step 4 Taking Supremum Over x

$$\mathbf{HW}(PQ)^2 \leq \sup_{\|x\|=1} \left(|\langle Ax, x\rangle|^2 + 2\|PQx\|^4 + 2|\langle Ax, x\rangle|\|PQx\|^2\right).$$

Using $\|PQx\| \leq \|PQ\|$ and $|\langle Ax, x\rangle| \leq w(A)$, we get:

$$\mathbf{HW}(PQ)^2 \leq w^2(A) + 2\|PQ\|^4 + 2\|PQ\|^2 w(A).$$

Final Inequality

$$\mathbf{HW}(PQ) \leq \sqrt{w^2(PQ - |PQ|^2) + 2\|PQ\|^2 w(PQ) + 2\|PQ\|^4}.$$

remark 4.4.1

if we Using Inequality tow to prove this Inequality we have last Inequality and Inequality 2:

$$\mathbf{HW}(PQ) \leq \sqrt{w^2(PQ - |PQ|^2) + 2\|PQ\|^2 w(PQ)}.$$

we gast $+2\|PQ\|^4$ under the root becase it is an addable quantity.
Special Cases and Validity 4.4.2

— **Normal Operators ($PQ = |PQ|^2$):**

$$A = 0, \quad \text{so} \quad \mathbf{HW}(PQ) \leq \sqrt{2}\|PQ\|\sqrt{w(PQ)}.$$

— **For self-adjoint PQ:**

$$\mathbf{HW}(PQ) = \sqrt{w^2(PQ) + \|PQ\|^4},$$

matching the bound.
Fundamental Bounds
The **HW** radius satisfies the inequality

$$\max\{w(PQ), \|PQ\|^2\} \leq \mathbf{HW}(PQ) \leq w^2(PQ) + \|PQ\|^4,$$

which links the numerical radius $w(PQ)$ and the operator norm $\|PQ\|$ to the **HW** radius.
Equivalence of HW Radius
The joint numerical radius satisfies

$$w_H(PQ, |PQ|^2) = \mathbf{HW}(PQ),$$

establishing a direct equivalence between these two quantities.
Triangle Inequality for
HW Radius
The inequality

$$\mathbf{HW}(PQ + S) \leq \mathbf{HW}(PQ) + \mathbf{HW}(S)$$

holds provided the orthogonality condition

$$\mathrm{Re}(Q^* P^* S) = 0$$

is satisfied, ensuring cancellation of cross terms in the HW expression.
Operator-Specific Bounds
Sharper inequalities emerge:

$$\mathbf{HW}^2(PQ) \leq \|PQ\|^2 \max\{1, \|PQ\|^2\} + 2w(|PQ|^2 PQ),$$

emphasizing the interaction between the norm and the algebraic structure of operator compositions.
Special Cases
If P and Q commute or are normal, the bounds simplify significantly. For example,

$$|PQ|^2 = |Q|^2|P|^2,$$

which highlights how algebraic properties such as commutativity and normality influence the estimates of the **HW** radius.

4 .5 The w_h and w_{oh} vs HW Radius and W :

Definitions and Setup 4.5.1

The w_h is defined as:

w_h Radius :
For an operator $T \in \mathbb{B}(\mathcal{H})$, define the w_h radius:

$$w_h(T, |T|^n) = \sup_{\|x\|=1} \left(|\langle Tx, x\rangle|^2 + \| \, |T|^n x\|^{2/n} \right),$$

where $|T|^n = (T^*T)^{n/2}$.
HW Radius:
The (HW) radius of T, denoted by $Hw(T)$, is given by:

$$Hw(T) := \sup_{\|x\|=1} \sqrt{|\langle Tx, x\rangle|^2 + \|Tx\|^4}.$$

Squaring both sides:

$$Hw(T)^2 = \sup_{\|x\|=1} \left(|\langle Tx, x\rangle|^2 + \|Tx\|^4\right).$$

General Case: Arbitrary n

Let T be normal and x an eigenvector of T with eigenvalue λ. Then:

$$|T|^n x = |\lambda|^n x, \quad \| \, |T|^n x\|^{2/n} = |\lambda|^2.$$

Hence:

$$w_h(T, |T|^n) = |\lambda|^2 + |\lambda|^2 = 2|\lambda|^2.$$

Compare this to:

$$Hw(T)^2 = |\lambda|^2 + |\lambda|^4.$$

These are equal only if $|\lambda|^2 = |\lambda|^4$, which occurs only when $|\lambda| = 0$ or $|\lambda| = 1$.

Conclusion:

In general, we have:

$$w_h(T, |T|^n) \neq Hw(T).$$

Equality $w_h(T, |T|^n) = Hw(T)$ holds **if and only if**:

— T is either nilpotent ($\lambda = 0$) ,where $|T|^n = (T^*T)^{n/2}$,
or

— unitary ($|\lambda| = 1$)

$$\boxed{w_h(T, |T|^n) = Hw(T) \quad \Leftrightarrow \quad T \text{ is nilpotent or unitary ...}} \, (r10)$$

Ruselt 1:

for last Equality (r10) and bay definition HW then w_h is a norm.

New study:

we have :

**[Numerical Radius:]

For a bounded linear operator T on a Hilbert space $\mathcal{H}$:

$$w(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

suppose that :

**[w_h Radius:]

$$w_h(T, |T|^n) := \sup_{\|x\|=1} \left(|\langle Tx, x\rangle|^2 + \|Tx\|^{2n}\right)$$

where $|T| = (T^*T)^{1/2}$

Relationship Between w_h and $w(T)$

**[Bounds for Extended Numerical Radius]

For any $T \in B(\mathcal{H})$ and $n \geq 1$:

$$w(T)^2 + \|T\|^{2n} \leq w_h(T, |T|^n) \leq 2\|T\|^{2n}$$

proof

Lower Bound:

$$w_h(T, |T|^n) \geq \sup_{\|x\|=1} |\langle Tx, x\rangle|^2 + \sup_{\|x\|=1} \|Tx\|^{2n} = w(T)^2 + \|T\|^{2n}$$

Upper Bound:

Using Cauchy-Schwarz inequality:

$$|\langle Tx, x\rangle|^2 \leq \|Tx\|^2 \leq \|T\|^2$$

Thus:

$$w_h(T, |T|^n) \leq \|T\|^2 + \|T\|^{2n} \leq 2\|T\|^{2n}$$

beucas norm(x)=1 .

Special Case Analysis
**[Positive Operators] If T is positive ($T = T^*$, $\langle Tx, x\rangle \geq 0$):

$$w_h(T, |T|^n) = \|T\|^{2n} + \|T\|^2$$

proof
**For positive operators:

$$w(T) = \|T\| \quad \text{and} \quad \|Tx\| = \langle Tx, x\rangle^{1/2}$$

Therefore:

$$w_h(T, |T|^n) = \sup_{\|x\|=1} \left(\|T\|^2 + \|T\|^{2n}\right) = \|T\|^{2n} + \|T\|^2$$

The $w_{oh}(D, D)$ is defined as:

$$w_{oh}(D, D) = \sup_{\|x\|=1} \left\{|\langle D^*x, x\rangle|^2 + |\langle Dx, x\rangle|^2\right\}.$$

Observe that for any operator $D \in \mathbb{B}(\mathcal{H})$ and any unit vector $x \in \mathcal{H}$, the inner products $\langle D^*x, x\rangle$ and $\langle Dx, x\rangle$ are complex conjugates:

$$\langle D^*x, x\rangle = \overline{\langle Dx, x\rangle}.$$

Thus,

$$|\langle D^*x, x\rangle| = |\langle Dx, x\rangle|,$$

and the expression simplifies to:

$$|\langle D^*x, x\rangle|^2 + |\langle Dx, x\rangle|^2 = 2|\langle Dx, x\rangle|^2.$$

Taking the supremum over all unit vectors x, we obtain:

$$w_{oh}(D, D) = 2 \sup_{\|x\|=1} |\langle Dx, x\rangle|^2 = 2w(D)^2,$$

where $w(D)$ denotes the numerical radius of the operator D.

This result holds for:
— **Self-adjoint operators**, where $w(D) = \|D\|$,
— **Normal operators**, where $w(D) = \sup_{\lambda \in \sigma(D)} |\lambda|$,
— **Non-normal operators**, where the numerical radius is distinct from the norm.
In all cases, the relation $w_{oh}(D, D) = 2w(D)^2$ remains valid.
Conclusion
The $w_{oh}(D, D)$radius evaluated at the pair (D, D) yields:

$$w_{oh}(D, D) = 2w(D)^2 ...(r11)$$

This identity provides a sharp connection between the numerical radius and the $w_{oh}(D, D)$
Ruselt 2:
for last Equality (r11) and bay definition W then $\sqrt{\frac{1}{2} w_{oh}}$ is a norm

5 Bibliography

References

[1] A. Abu-Omar and F. Kittaneh, "Upper and lower bounds for the numerical radius with an application to involution operators," *Rocky Mountain J. Math.*, 45(4) (2015), pp. 1055–1065.

[2] P. Bhunia and K. Paul, "Furtherance of numerical radius inequalities of Hilbert space operators," *arXiv preprint*, arXiv:2102.01953 [math.FA], 2021.

[3] F.F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Banach Algebras*, Cambridge University Press, 1973.

[4] S.S. Dragomir, "Some inequalities for the norm and the numerical radius of linear operators in Hilbert spaces," *Tamkang J. Math.*, 39(1) (2008), pp. 1–7.

[5] S.S. Dragomir, *Inequalities for the Numerical Radius of Linear Operators in Hilbert Spaces*, Springer Briefs in Mathematics, Springer International Publishing, 2013.

[6] K.E. Gustafson and D.K.M. Rao, *Numerical Range*, Springer, New York, 1997.

[7] O. Hirzallah and F. Kittaneh, "Numerical radius inequalities for several operators," *Math. Scand.*, 114 (2014), no. 1, pp. 110–119.

[8] J.A.R. Holbrook, "Multiplicative properties of the numerical radius in operator theory," *J. Reine Angew. Math.*, 237 (1969), pp. 166–174.

[9] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[10] F. Kittaneh, "Norm inequalities for sums of positive operators," *J. Operator Theory*, 48 (2002), pp. 95–103.

[11] F. Kittaneh, "A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix," *Studia Math.*, 158(1) (2003), pp. 11–17.

[12] F. Kittaneh, "Numerical radius inequalities for Hilbert space operators," *Studia Math.*, 168(1) (2005), pp. 73–80.

[13] Carlos S. Kubrusly, *The Elements of Operator Theory*, Birkhäuser, 2011.

[14] T. Yamazaki, "On upper and lower bounds for the numerical radius and an equality condition," *Studia Math.*, 178(1) (2007), pp. 83–89.

[15] K. Zaiz and A. Mansour, "On numerical range and numerical radius of convex function operators," *Korean J. Math.*, 27 (2019), no. 4, pp. 879–898.

Abstract :

Operator theory is one of the most important theories in mathematics. In this thesis, we proved some norms and numerical radius inequalities in Hilbert spaces.

1. $w_h(T, |T|^n) = Hw(T)$,
2. $w_{oh}(D, D) = 2w(D)^2$.

Keywords:
Hilbert space, Numerical radius, Operator theory.

الملخص:

تعتبر نظرية المؤثرات من أهم النظريات في الرياضيات ، في هذه المذكرة قمنا باثبات بعض النظم وبرهنة متباينات نصف قطر عددية في الفضاء الهلبرتي.

١. $w_h(T, |T|^n) = Hw(T)$,
٢. $w_{oh}(D, D) = 2w(D)^2$.

كلمات مفتاحية:
فضاء هلبرت ، نصف القطر العددي ، نظرية المؤثرات.