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# THESIS

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# Analytical and numerical solution of fractional boundary value problems

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# Abstract

This thesis is devoted to the study of boundary value problem and some systems for fractional differential equations involving the Caputo-Fabrizio derivative. The existence and uniqueness of the solution leads to the study of a linear integral Volterra-Fredholm equation of the second type.

After that, we used numerical method to obtain an approximate solution. The solution was implemented and the error committed was estimated using Matlab.

Our conclusion is that the presented numerical methods gives an almost perfect approximation through the example that we have provided in the last one.

**Key words:**

Fractional integral, Caputo-Fabrizio fractional derivative, Fractional boundary value problem, Coupled system of fractional differential equations, Volterra-Fredholm integral equation, Banach fixed point Theorem, Adomian decomposition method (ADM).

**Mathematics Subject Classification (2010):**

26A33, 34A08, 34K37, 45B05, 34A12, 47G20, 47Gxx.

# الملـــــــخص

هذه الأطروحة مخصصة لدراسة مسألة حدية و بعض أنظمة المعادلات التفاضلية الكسرية التي تحتوي على مشتق كابوتو- فابريزيو. ان وجود الحل و وحدانيته يؤول إلى دراسة معادلة فولتيرا - فريدهولم التكاملية الخطية من النوع الثاني.
وبعد ذلك، استخدمنا الطرق العددية للحصول على الحل التقريبي. تم تنفيذ الحل و تقدير الخطأ المرتكب باستخدام $MATLAB$.
استنتاجنا هو ان الطرق العددية المقدمة تعطي تقريبا مثاليا وذلك من خلال الأمثلة التي قمنا بتقديمها.

الكلمات المفتاحية:

التكامل الكسري، مشتق كابتو- فابريزيو الكسري، مسألة حدية كسرية، نظام مزدوج للمعادلة التفاضلية الكسرية، معادلة فولتيرا- فريدهولم التكاملية، نظرية بناخ للنقطة الثابتة، طريقة تحليل ادومين (ADM).

# Résumé

Cette thèse est consacrée à l'étude du problème des valeurs limites et de certains systèmes d'équations différentielles fractionnaires impliquant la dérivée de Caputo-Fabrizio. L'existence et l'unicité de la solution conduisent à l'étude d'une équation intégrale linéaire de Volterra-Fredholm du deuxième type.
Nous avons ensuite utilisé la méthode numérique pour obtenir une solution approchée. La solution a été implémentée et l'erreur commise a été estimée sous Matlab.
Notre conclusion est que les méthodes numériques présentées donnent une approximation presque parfaite à travers l'exemple que nous avons fourni dans la dernière.

**Mots clés:**

Intégrale fractionnaire, Dérivé fractionnaire du Caputo-Fabrizio, Problème de valeur limite fractionnaire, Système couplé d'équations différentielles fractionnaires, Équation intégrales de Volterra-Fredholm, Théorème du point fixe de Banach, Méthode de décomposition d'Adomian (ADM).

# Dedication

It is with the help of Almighty **GOD** that this modest work could be carried out, **GOD** who gave me time, reason and lucidity.

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My dear parents ♡ **" Mansouri Abdesselam "** ♡ and ♡ **" Haniche Assia "** ♡ for their love, their patience and their encouragement which never failed to suit me During my student years,

♡ My dear husband **" Said Ghemam Amara "** ♡.

♡ My sisters **" Ichrak, Yousra "** ♡.

♡ My brother **" Mouhamed Akram "** ♡.

♡ My close friends ♡

♡ All my family **" Mansouri, Haniche and Ghemam Amara "** ♡.

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# Publicatuion

1. M. Moumen Bekkouche, **I. Mansouri**, A. Azeb Ahmed. Numerical solution of fractional boundary value problem with caputo-fabrizio and its fractional integral, Journal of Applied Mathematics and Computing (2022).
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2. **Ikram Mansouri**, Mohammed Moumen Bekkouche, Abdelaziz Azeb Ahmed. Numerical Solution of a Fractional Coupled System with the Caputo-Fabrizio Fractional Derivative, Journal of Applied Mathematics and Computational Mechanics (2023).
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3. **I. Mansouri**, M. M. Bekkouche, A. A. Ahmed, F. Yazid and F. S. Djeradi. Analytical and Numerical Study of a Linear Coupled System involving Caputo-Fabrizio Fractional Derivative with Boundary Conditions, Journal of Fractional Differential Calculus (2023).
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# Contents

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# General introduction

Fractional calculus is the calculation of integrals and derivatives of real or complex order and has gained great importance over the past 30 years.The question that arises in concerns the emergence of this concept, and the answer is a simple question about his intellect, in which Marquis Gottfried Wilhelm Leibniz asked whether $\frac{d^n y}{dx^n}$ and $n = \frac{1}{2}$ [26].
The fraction was born on September 30, 1695. Work on this was carried out over many years by Leibniz, L'Hôpital (1695), Bernoulli (1697), Euler (1730), La Grange (1772), Laplace (1812), Fourier (1822), and Abel (1823), Liouville (1832), Riemann (1847), Grunwald (1867), Letnikov (1868), Nekrasov (1888), Hadamard (1892), Heaviside (1892), Hardy (1915), Weyl (1917), Reese (1922), P. Levy (1923), Davis (1924), Kober (1940), Zygmunt (1945), Kuttner (1953), J.L.Lyons (1959) and Liberman (1964). The first book on fractions was published by Oldham and Spanier in his 1974. There is also a monograph by Samko, Kirbas, and Malichev, published in Russian in 1987 and in English in 1993 [42]. Other studies on fractional differential equations are by Miller and Ross (1993) [34], Podlubny (1999) [41], and Kilbas et al. (2006) [26], Diethelm (2010) [19], Oldham et al. [40], etc. Fractional differential equations are of great importance in physics, mathematics, engineering, biology, image processing, electricity, control theory, economics, biophysics, mechanics, etc, ([33, 45, 12, 8, 9, 13]). For many mathematicians, fixed point theory is an extremely significant topic of research and a very useful tool in mathematics. This theory's roots may be found in the continuous approximations produced by Peano and Picard in the 19th century to

demonstrate the uniqueness and existence of differential equation solutions.

Some results have been achieved for additional fractional differential equations ([23, 32, 15, 1, 6, 18, 30, 11] ), using the new Caputo-Fabrizio derivative (see [14, 17, 35]).

We now provide the thesis's outline, which is as follows:

**In the chapter 1:** We use the notation to describe preliminary results, theorems, lemmas, as well as other significant results.

**In the chapter 2:** We investigate the existence and uniqueness of solutions to fractional differential equation problems regarding boundary values by utilizing the Caputo-Fabrizio derivative in the following ways:

$$\begin{cases} {}^{CF}\mathscr{D}_0^{\gamma}\,\phi(t) + r(t)\phi(t) = h(t), \quad 0 \leqslant t \leqslant 1 \\ \phi(0) = a, \quad \phi(1) = b \end{cases}$$

where $1 < \gamma < 2$ is a real number, $r$ is the potential function, and $h : [0,1] \to \mathbb{R}$ is continuous, and ${}^{CF}\mathscr{D}_0^{\gamma}$ is the fractional derivative of Caputo-Fabrizio.

In order to study this problem we used a new definition of fractional integral as an inverse of the conformable fractional derivative of Caputo-Fabrizio, therefore, so we transformed the problem to a equivalent linear Volterra-Fredholm integral equations of the second kind, and taking sufficient conditions existence and uniqueness of this solution is proven based on the results obtained. The analytical study is followed by a complete numerical study. The main results of the problem considered before are in the article [36].

**In the chapter 3:** We will present a currently used method that can be relied upon to introducing linear Caputo-Fabrizio fractional pair systems with the following boundary conditions:

$$\begin{cases} {}^{CF}\mathscr{D}_0^{\alpha}\,\phi(t) = c_1\phi(t) + c_2\psi(t) + h(t), \quad t \in \Lambda := [0,1] \\ {}^{CF}\mathscr{D}_0^{\alpha}\,\psi(t) = c_3\phi(t) + c_4\psi(t) + g(t), \\ \phi(0) = \psi(0) = 0, \end{cases}$$

where $0 < \alpha < 1$ is a real number, ${}^{CF}\mathscr{D}_0^{\alpha}$ is the Caputo-Fabrizio fractional derivative, $h, g : \Lambda \to \mathbb{R}$ are given continuous functions, and $c_i$ real constants and $i = 1, 2, 3, 4$.

We discuss the existence of solutions for a coupled system of linear fractional differential equations involving Caputo-Fabrizio fractional orders. We prove the existence and uniqueness of the solution by using the Picard–Lindelöf method and fixed point theory. Also, to compute an approximate solution of problem, we utilize the Adomian decomposition method (ADM). Numerical examples are presented to illustrate the validity and effectiveness of the proposed method. The main results of the problem considered before are in the article [30].

**In the chapter 4:** We examine the following coupled system of linear fractional differential equations:

$$\begin{cases} {}^{CF}\mathscr{D}_0^{\gamma}\,\phi(t) = c_1\phi(t) + c_2\psi(t) + h(t), \quad t \in \Lambda := [0,1] \\ {}^{CF}\mathscr{D}_0^{\gamma}\,\psi(t) = c_3\phi(t) + c_4\psi(t) + g(t), \\ \phi(0) = \phi(1) = 0,\ \psi(0) = \psi(1) = 0, \end{cases}$$

where $1 < \gamma < 2$ is a real number, ${}^{CF}\mathscr{D}_0^{\gamma}$ is the Caputo-Fabrizio fractional derivative, $h, g : \Lambda \to \mathbb{R}$ are continuous function, and $c_i$ real constants and $i = 1, 2, 3, 4$.

In order to prove the existence and uniqueness of solution, the problem is transformed into an equivalent linear Volterra-Fredholm integral equations of the second kind, and by using the Banach's fixed-point theory the existence and uniqueness of solutions is obtained. Finally, the analytical results are supported by numerical results to illustrate of obtained results. The main results of the problem considered before are in the article [31].

# Generals notations

Certain notations will be used throughout this memoir which we list below:

| | |
|---|---|
| $\mathbb{R}$ | Set of real numbers. |
| $\mathbb{R}^+$ | Set of positive or zero real numbers. |
| $\mathbb{R}^n$ | Real vector space of dimension $n$ built on the field of the reals. |
| $\mathbb{N}$ | Set of natural numbers. |
| $\mathbb{Z}$ | Set of integers numbers. |
| $\mathbb{C}$ | Set of integers complexes numbers. |
| $[a, b)$ | Semi-open interval of $\mathbb{R}$ with endpoints $a$ and $b$. |
| $(a, b)$ | Open interval of $\mathbb{R}$ with endpoints $a$ and $b$. |
| $\mathcal{C} = \mathcal{C}(K, F)$ | Set of continuous functions $K$ to $F$. |
| $\mathcal{C}^n = \mathcal{C}^n([a, b])$ | The space of $n-$times continuously differentiable functions in $[a, b]$. |
| $\mathrm{H}^1(a, b)$ | The Sobolev space of order one |
| $\lvert\,.\,\rvert$ | Absolute value of a real number or modulus of a complex number. |
| $\Gamma(.)$ | Gamma function. |
| $N(.)$ | Normalization function. |
| ${}^{GL}\mathscr{D}^{\gamma}$ | Derivative of order $\gamma$ according to Grünwald–Letnikov's definition. |
| ${}^{RL}\mathscr{D}^{\gamma}$ | Derivative of order $\gamma$ according to Riemann-Liouville's definition. |
| ${}^{C}\mathscr{D}^{\gamma}$ | Derivative of order $\gamma$ according to Caputo's definition. |
| ${}^{CF}\mathscr{D}^{\alpha}$ | Derivative of order $\alpha$ according to Caputo-Fabrizio's definition. |
| $I_a^{\alpha}$ | The fractional integral of order $\alpha$. |

# Chapter 1

# Introduction and preliminaries of fractional calculus

In this chapter, we give the basic elements of fractional differentiation. We mentioned some notions relating to this mathematical tool, some approaches to derivatives fractional are introduced namely, the approach of Riemann-Liouville and that of Caputo as well as their properties.

## 1.1 Some important function in fractional calculus

### 1.1.1 The Gamma function

**Definition 1.1.** *[20] We call Eulerian Gamma function (or Eulerian integral of second species, denoted by $\Gamma$, the capital letter gamma from the Greek alphabet ) is extension of the factorial function to the set of complex numbers.*

*For all complex number $z$ such that $\Re(z) > 0$, we defined the Gamma function by:*

$$\Gamma: \ \mathbb{C} \longrightarrow \mathbb{R}$$
$$z \longrightarrow \Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\,\mathrm{d}x.$$

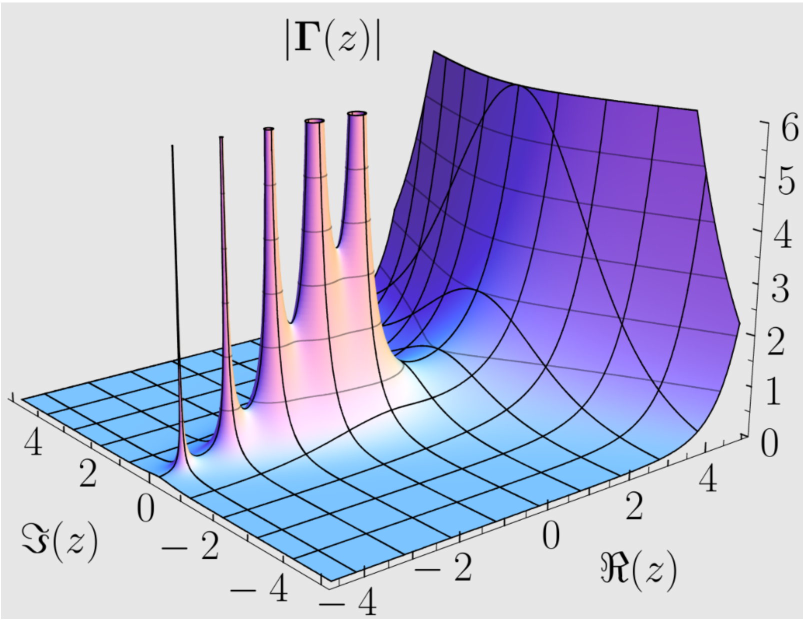


Figure 1.1: The gamma function along part of the real axis.

*We now present one of the basic properties of $\Gamma$ function:*

$$\Gamma(z+1) = z\Gamma(z), \quad \Re(z) > 0.$$

*This relation is used to extend the Euler gamma function to the half-plane $\Re(z) \leqslant 0$ by*

$$\Gamma(z) = \frac{\Gamma(z+n)}{(z)_n} \quad (\Re(z) > -n;\ n \in \mathbb{N}^*;\ z \in \mathbb{Z}^- := \{0, -1, -2, \ldots\}).$$

*The Pochhammer symbol in this case is $(z)_n$, which is defined for the complex $z \in \mathbb{C}$ and the non-negative integer $n \in \mathbb{N}$ by*

$$(z)_0 = 1 \text{ and } (z)_n = z(z+1)\ldots(z+n-1), \ (n \in \mathbb{N}^*).$$

*In particular $\Gamma(1) = 1$ and we deduce that*

$$\Gamma(n+1) = (1)_n = (n)!, \quad \forall n \in \mathbb{N}.$$

## 1.2 Fractional calculus: history, definitions and applications

### 1.2.1 Grünwald–Letnikov fractional derivative

The Grünwald–Letnikov fractional derivative is based on a generalization of the usual differentiation of a function $g(t)$ of the natural derivative to fractional derivative, Anton Karl Grünwald introduced his derivative in 1867, while Aleksey Vasilievich Letnikov presented his derivative in 1868. Hence, the definition was written in [26, 41].

**Definition 1.2.**

$$^{GL}\mathscr{D}^{\gamma}g(t) = \lim_{h\to 0}\frac{1}{h^{\gamma}}\sum_{p=0}^{\infty}(-1)^p\begin{pmatrix}\gamma\\ p\end{pmatrix}g(t-ph)$$

*where $p \in \mathbb{N}$, and the binomial coefficient are calculated by the help of the Gamma function and defined for $\gamma \in \mathbb{C}$ and $p \in \mathbb{N}$ by the formula*

$$\begin{pmatrix}\gamma\\ 0\end{pmatrix} = 1, \quad \begin{pmatrix}\gamma\\ p\end{pmatrix} = \frac{\gamma(\gamma-1)(\gamma-2)\dots(\gamma-p+1)}{p!} = \frac{(-1)^p(-\gamma)_p}{p!}, \quad (p \in \mathbb{N}^*).$$

### 1.2.2 Riemann-Liouville fractional derivative

The Riemann-Liouville fractional derivative acquiring by Riemann in 1847 is considered as defined in [26, 41].

**Definition 1.3.** *Let $g(t)$ be an integrable function on $[a,T]$, $T > a$ and $p \in \mathbb{N}\setminus\{0\}$ satisfies $p-1 < \gamma < p$. Then the Riemann-Liouville derivative of order $\gamma$ is given by*

$$^{RL}\mathscr{D}_a^{\gamma}g(t) = \frac{1}{\Gamma(p-\gamma)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^p\int_a^t\frac{g(x)}{(t-x)^{\gamma-p+1}}\,\mathrm{d}x, \ (t > a)$$

*where $p = [\gamma]+1$, and $[\gamma]$ is the integer part of $\gamma$ real number. Cauchy's integral is extended from the integer number to the real number via this operator. Based on the procedures of* *fractionalization.*

*In particular, when $\gamma = p \in \mathbb{N}$, then*

$$ {}^{RL}\mathscr{D}_a^0 g(t) = g(t), \quad \textit{and} \quad {}^{RL}\mathscr{D}_a^p g(t) = g^{(p)}(t), $$

*where $g^{(p)}(t)$ is the usual derivative of $g(t)$ of order $p$.*

*If $0 < \gamma < 1$, thereafter, the Riemann-Liouville operator became*

$$ {}^{RL}\mathscr{D}_a^\gamma g(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{g(x)}{(t-x)^\alpha} \, \mathrm{d}x. $$

*Here nth derivative is operated outside the integral sign.*

### 1.2.3 Caputo fractional derivative

By substituting the fractional integral operator for the ordinary derivative order, Michel Caputo rewrote the formulation of the Riemann-Liouville fractional derivative in his 1967 study [16].

**Definition 1.4.** *Let $g$ be a function on $[a, T]$, $T > a$, such that $\frac{\mathrm{d}^p}{\mathrm{d}t^p} g$ is integrable, then the fractional derivative of ordre $\gamma$ in the Caputo sens is defined by:*

$$ {}^{C}\mathscr{D}_a^\gamma g(t) := \begin{cases} \dfrac{1}{\Gamma(p-\gamma)} \displaystyle\int_a^t (t-x)^{p-\gamma-1} g^{(p)}(x) \, \mathrm{d}x, & p-1 < \gamma < p \in \mathbb{N}, \\ g^{(p)}(t), & \gamma = p \in \mathbb{N}, \end{cases} \tag{1.1} $$

*where $p = [\gamma] + 1$. If $0 < \gamma < 1$, then the Caputo operator reduced to*

$$ {}^{C}\mathscr{D}_a^\gamma g(t) = \frac{1}{\Gamma(1-\gamma)} \int_a^t \frac{g'(x)}{(t-x)^\gamma} \, \mathrm{d}x. $$

*Here n-th derivative is operated inside the integral sign.*

### 1.2.4 Caputo-Fabrizio fractional derivative

Let $\mathrm{H}^1(a, b)$ denote the order one Sobolev space, which is defined as follows:

$$ \mathrm{H}^1(a, b) = \{ g \in \mathrm{L}^2(a, b) : g' \in \mathrm{L}^2(a, b) \}, $$

where

$$\mathrm{L}^2(a,b) = \{g : \int_a^b |g(t)|^2 \,\mathrm{d}t < \infty\}.$$

**Definition 1.5.** *[17] Let $g \in \mathrm{H}^1(a,b)$, $b > a$ and $\alpha \in ]0,1[$, then the fractional derivative of order $\alpha$ in the Caputo-Fabrizio sens is given as*

$$^{CF}\mathscr{D}_a^{\alpha}\, g(t) = \frac{N(\alpha)}{1-\alpha} \int_a^t g'(x) \exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right] \mathrm{d}x, \tag{1.2}$$

*where $a \in ]-\infty, t)$, and $N$ is the normalization function having the property $N(0) = N(1) = 1$.*

*From the Definition 1.5, we deduce that if $g$ is a constant function, then its fractional time derivative equals zero, as it happens in the case of the ordinary Caputo fractional derivative. It is also possible to use this definition to find the fractional derivatives of functions that are not parts of $\mathrm{H}^1(a,b)$.*

*In fact, we can also write the Definition 1.5 for any $g \in \mathrm{L}^1(-\infty, b)$ and for any $\alpha \in ]0,1[$ in the following way:*

$$^{CF}\mathscr{D}^{\alpha} g(t) = \frac{\alpha N(\alpha)}{1-\alpha} \int_{-\infty}^t (g(t) - g(x)) \exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right] \mathrm{d}x.$$

*Now, it is prudent to note that if we place*

$$\beta = \frac{1-\alpha}{\alpha} \in [0,\infty], \;\; \alpha = \frac{1}{1-\beta} \in [0,1],$$

*the definition of Caputo-Fabrizio fractional derivative assumes the form*

$$^{CF}\tilde{\mathscr{D}}_a^{\beta} g(t) = \frac{M(\beta)}{\beta} \int_a^t g'(x) \exp\left[-\frac{t-x}{\beta}\right] \mathrm{d}x,$$

*where $\beta \in [0,\infty]$, and the term $M(\beta)$ is the function that normalizes $N(\alpha)$ and satisfies $M(0) = M(\infty) = 1$.*

*This is definition on which our study is based.*

**Remark 1.1.** *[17] As per Definition 1.5, we obtain the following properties:*

1. $\lim\limits_{\alpha\to 1} {}^{CF}\mathscr{D}_a^{\alpha}\, g(t) = g'(t)$.
2. $\lim\limits_{\alpha\to 0} {}^{CF}\mathscr{D}_a^{\alpha}\, g(t) = g(t) - g(a)$.

**Definition 1.6.** *If $n \geq 1$, and $\alpha \in [0,1]$ the fractional derivative ${}^{CF}\mathscr{D}_a^{\alpha+n} g(t)$ of order $(n+\alpha)$ is defined by*

$$\begin{aligned} {}^{CF}\mathscr{D}_a^{\alpha+n} g(t) &:= {}^{CF}\mathscr{D}_a^{\alpha}\left(D^{(n)} g(t)\right) \\ &= \frac{N(\alpha)}{1-\alpha}\int_a^t g^{(n+1)}(x)\exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right]\,\mathrm{d}x. \end{aligned}$$

**Theorem 1.1.** *[17] For the fractional time derivative, if the function $g(t)$ is such that*

$$g^{(s)}(a) = 0, \quad s = 1, 2, .., n$$

*then, we have*

$$D^{(n)}\left({}^{CF}\mathscr{D}_a^{\alpha} g(t)\right) = {}^{CF}\mathscr{D}_a^{\alpha}\left(D^{(n)} g(t)\right).$$

*Proof.* [17] Letting $n = 1$, we deduce from Definition 1.6 of $\mathscr{D}^{(\alpha+1)} g(t)$, that

$${}^{CF}\mathscr{D}_a^{\alpha}\left(D^{(1)} g(t)\right) = \frac{N(\alpha)}{1-\alpha}\int_a^t g''(x)\exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right]\,\mathrm{d}x.$$

Hence, after an integration by parts and supposing $g'(a) = 0$, we obtain

$$\begin{aligned} {}^{CF}\mathscr{D}_a^{\alpha}\left(D^{(1)} g(t)\right) &= \frac{N(\alpha)}{(1-\alpha)}\int_a^t \left(\frac{\mathrm{d}}{\mathrm{d}x} g'(x)\right)\exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right]\,\mathrm{d}x \\ &= \frac{N(\alpha)}{(1-\alpha)}\left[\int_a^t \frac{\mathrm{d}}{\mathrm{d}x}\left(g'(x)\exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right]\right)\,\mathrm{d}x \right.\\ &\left. -\frac{\alpha}{1-\alpha}\int_a^t g'(x)\exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right]\,\mathrm{d}x\right] \\ &= \frac{N(\alpha)}{(1-\alpha)}\left[g'(t) - \frac{\alpha}{1-\alpha}\int_a^t g'(x)\exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right]\,\mathrm{d}x\right], \end{aligned}$$

in another hand

$$\begin{aligned} D^{(1)}\left({}^{CF}\mathscr{D}_a^{\alpha} g(t)\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{N(\alpha)}{1-\alpha}\int_a^t g'(x)\exp -\frac{\alpha(t-x)}{1-\alpha}\,\mathrm{d}x\right) \\ &= \frac{N(\alpha)}{1-\alpha}\left[g'(t) - \frac{\alpha}{1-\alpha}\int_a^t g'(x)\exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right]\,\mathrm{d}x\right]. \end{aligned}$$

So $D^{(1)}\left({}^{CF}\mathscr{D}_a^{\alpha} g(t)\right) = {}^{CF}\mathscr{D}_a^{\alpha}\left(D^{(1)} g(t)\right)$.

Similarly

$$D^{(n)}\left({}^{CF}\mathscr{D}_a^{\alpha} g(t)\right) = {}^{CF}\mathscr{D}_a^{\alpha}\left(D^{(n)} g(t)\right), \quad n = 1, 2, \ldots.$$

□

The following MATLAB code calculates the Caputo-Fabrizio derivative for any $\mathcal{C}^{n+1}$-continuous function.

```
% The goal of this program is calculate  the Caputo-Fabrizio derivative
  clc; close; clear all;
  syms t x;
  a=0;  b=7;
  alpha=1.5;
  g(t)=t^1.5;
  K(t)=Da(g,alpha,a)
```

```
function y=Da(g,alpha,a)
   syms t x;    n=floor(alpha);
   y=int(diff(g(x),n+1)*exp(-(alpha-n)*(t-x)/(1-alpha+n))
                ,x,a,t)*Na(alpha-n)/(1-alpha+n);
end
```

## 1.3 Fractional integral

In this part, we present a for the definition of Caputo-Fabrizio fractional integral in this way

**Definition 1.7.** *[37] If $n \geq 1$, $\alpha \in [0,1]$, and $g \in \mathcal{C}[a,b]$, the Caputo-Fabrizio fractional integral $I_a^{n+\alpha} g(t)$ of order $n+\alpha$ is defined by*

$$I_a^{n+\alpha} g(t) = \frac{1}{N(\alpha)\cdot n!}\int_a^t (t-\eta)^{n-1}[\alpha(t-\eta)+n(1-\alpha)]g(\eta)\,\mathrm{d}\eta,$$

*where $N(\alpha)$ denotes a normalization function with $N(0)$ and $N(1)$ are both equal 1.*

If $0<\alpha<1$, then the Caputo-Fabrizio fractional integral reduced to

$$I_a^{\alpha} s(t) = \frac{1}{N(\alpha)}\left[(1-\alpha)(s(t)-s(a))+\alpha\int_a^t s(\eta)\,\mathrm{d}\eta\right], \tag{1.3}$$

where $N(\alpha)$ is a normalization function.

## 1.4 Fundamental lemmas

**Lemma 1.1.** *[37] Let $n \geqslant 1$, and $\gamma \in (n, n+1)$. If we suppose $\phi \in \mathcal{C}^{n+1}[a,b]$, then the fractional equation*

$$^{CF}\mathscr{D}_a^{\gamma}\,\phi(t) = 0,\ \forall t \in [a,b]$$

*has $\phi(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0;\ a_i \in \mathbb{R},\ i = 0,1,\cdots,n,$ as unique solutions.*

*Proof.* In the beginning, considering

$$\phi(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0;\ a_i \in \mathbb{R},\ i = 0,1,\cdots,n,$$

then $\phi^{(n+1)}(t) = 0$ such that $^{CF}\mathscr{D}_a^{\gamma}\,\phi(t) = 0,\ \forall t \in [a,b]$.
Now, let $\gamma \in ]n, n+1[$, it can be expressed in the following form: $\gamma = n + \alpha$ when $\alpha \in ]0,1[$, and $n = [\gamma]$, we suppose $^{CF}\mathscr{D}_a^{\gamma}\,\phi(t) = 0,\ \forall t \in [a,b]$, we have

$$^{CF}\mathscr{D}_a^{\gamma}\,\phi(x) = \frac{N(\alpha)}{1-\alpha}\int_a^t \phi^{(n+1)}(x)\exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right]\mathrm{d}x,$$

and the Leibniz integral rule gives the formula

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(^{CF}\mathscr{D}_a^{\gamma}\,\phi(t)\right) &= \frac{N(\alpha)}{1-\alpha}\phi^{(n+1)}(t)\\ &\quad - \frac{\alpha}{1-\alpha}\frac{N(\alpha)}{1-\alpha}\int_a^t \phi^{(n+1)}(x)\exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right]\mathrm{d}x\\ &= \frac{N(\alpha)}{1-\alpha}\phi^{(n+1)}(t) - \frac{\alpha}{1-\alpha}\ {}^{CF}\mathscr{D}_a^{\gamma}\,\phi(t),\end{aligned}$$

and we have $^{CF}\mathscr{D}_a^{\gamma}\,\phi(t) = 0$, consequently $\phi^{(n+1)}(t) = 0$.
Therefore,

$$\phi(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0;\ a_i \in \mathbb{R},\ i = 0,1,\cdots,n.$$

This completes the proof. □

**Lemma 1.2.** *[2] Let $\gamma$ in the interval $(n, n+1)$ where $n = [\gamma] \geqslant 0$, and assuming that $\phi$ belongs to the space $C^n[a,b]$, then the following assertions remain valid:*

1. $I_a^\gamma\left({}^{CF}\mathscr{D}_a^\gamma\,\phi(t)\right) = \phi(t) + \sum_{i=0}^{n}\sigma_i t^i,\ \sigma_i \in \mathbb{R}\ i = 0, 1, \ldots, n.$

2. *if* $\phi(a) = 0$, *then* ${}^{CF}\mathscr{D}_a^\gamma\left(I_a^\gamma\phi(t)\right) = \phi(t)$.

*Proof.* 1. Let $\gamma \in ]n, n+1[$, it can be expressed in the following form: $\gamma = n + \alpha$ where $\alpha \in ]0, 1[$, and $n = [\gamma] \geqslant 0$, we have

$$
\begin{aligned}
I_a^\gamma\left({}^{CF}\mathscr{D}_a^\gamma\,\phi(t)\right) &= \frac{1}{N(\alpha)\cdot n!}\int_a^t (t-\eta)^n\left[\alpha\ {}^{CF}\mathscr{D}_a^\gamma\,\phi(\eta) + (1-\alpha)\frac{\mathrm{d}}{\mathrm{d}\eta}\left({}^{CF}\mathscr{D}_a^\gamma\,\phi(\eta)\right)\right]\mathrm{d}\eta. \\
&= \frac{\alpha}{N(\alpha)\cdot n!}\int_a^t (t-\eta)^n\ {}^{CF}\mathscr{D}_a^\gamma\,\phi(\eta)\,\mathrm{d}\eta \\
&+ \frac{(1-\alpha)}{N(\alpha)\cdot n!}\int_a^t (t-\eta)^n\frac{\mathrm{d}}{\mathrm{d}\eta}\left({}^{CF}\mathscr{D}_a^\gamma\,\phi(\eta)\right)\mathrm{d}\eta \\
&= \frac{\alpha}{N(\alpha)\cdot n!}\int_a^t (t-\eta)^n\ {}^{CF}\mathscr{D}_a^\gamma\,\phi(\eta)\,\mathrm{d}\eta \\
&+ \frac{(1-\alpha)}{N(\alpha)\cdot n!}\int_a^t (t-\eta)^n\left(\frac{N(\alpha)}{1-\alpha}\phi^{(n+1)}(\eta) - \frac{\alpha}{1-\alpha}\ {}^{CF}\mathscr{D}_a^\gamma\,\phi(\eta)\right)\mathrm{d}\eta \\
&= \frac{1}{n!}\int_a^t (t-\eta)^n\phi^{(n+1)}(\eta)\,\mathrm{d}\eta \\
&= \phi(t) + \sum_{i=0}^{n}\sigma_i t^i,\ \sigma_i \in \mathbb{R},\ i = 0, 1, \ldots, n.
\end{aligned}
$$

2. Let $\gamma$ in the open interval between $n$ and $n+1$, then it can be expressed as $\gamma = n+\alpha$, where $\alpha$ is a real number between 0 and 1, and $n$ is the greatest integer less than or

equal to $\gamma$ (which is non-negative). Thus we get

$$
\begin{aligned}
{}^{CF}\mathscr{D}_a^{\gamma}\left(I_a^{\gamma}\phi(t)\right) &= \frac{\alpha}{(1-\alpha)}\int_a^t \frac{\mathrm{d}^{(n+1)}}{\mathrm{d}\eta^{(n+1)}}\left[\frac{1}{n!}\int_a^x (x-\eta)^n\phi(\eta)\,\mathrm{d}\eta\right]\exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right]\mathrm{d}x \\
&+\int_a^t \frac{\mathrm{d}^{(n+1)}}{\mathrm{d}\eta^{(n+1)}}\left[\frac{1}{n!}\int_a^x (x-\eta)^n\phi'(\eta)\,\mathrm{d}\eta\right]\exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right]\mathrm{d}x \\
&= \frac{\alpha}{(1-\alpha)}\int_a^t \phi(x)\exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right]\mathrm{d}x \\
&+\int_a^t \phi'(x)\exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right]\mathrm{d}x \\
&= \frac{\alpha}{(1-\alpha)}\int_a^t \phi(x)\exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right]\mathrm{d}x \\
&+\phi(x)\exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right]\Bigg|_a^t - \frac{\alpha}{(1-\alpha)}\int_a^t \phi(x)\exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right]\mathrm{d}x \\
&= \phi(t)-\phi(a)\exp\left[-\frac{\alpha(t-a)}{1-\alpha}\right] \\
&= \phi(t).
\end{aligned}
$$

□

## 1.5 Differentiation under the integral sign

Suppose we have a function $\phi$ defined by the formula:

$$\phi(t)=\int_a^b g(t,\eta)\,\mathrm{d}\eta, \quad c\leqslant t\leqslant d,$$

where $c$ and $d$ are constants. If the integral can be explicitly evaluated, then we can find the derivative $\phi'(t)$ through computation. However, even when direct evaluation of the integral is impossible, there are cases where we can still find $\phi'(t)$. This fundamental result is encapsulated in the next theorem, commonly known as **Leibniz' Rule**.

**Theorem 1.2.** *[37] Let $\phi$ be a function that is given by*

$$\phi(t)=\int_a^b g(t,\eta)\,\mathrm{d}\eta, \quad c\leqslant t\leqslant d,$$

*where $a$ and $b$ are constants. If $g$ and $g_t$ are continuous in the rectangle*

$$R = \{(t, \eta) : c \leq t \leq d, \quad a \leqslant \eta \leqslant b\}.$$

*Then*

$$\phi'(t) = \int_a^b g_t(t, \eta)\, \mathrm{d}\eta, \quad c < t < d.$$

*In other words, we can obtain the derivative by applying the differentiation under the integral sign technique.*

We look at a function that is given by:

$$\phi(t) = \int_{a(t)}^{b(t)} g(t, \eta)\, \mathrm{d}\eta, \tag{1.4}$$

given continuously differentiable functions $u(t)$ and $v(t)$ for $c \leqslant t \leqslant d$, where the ranges of $u$ and $v$ lie between $a$ and $b$, we aim to find a formula for the derivative $\phi'(t)$, where $\phi$ is defined by an integral like the one in equation (1.4). To simplify this process, we introduce a new integral that is more general than (1.4). Specifically, we define

$$G(t, y, z) = \int_y^z g(t, \eta)\, \mathrm{d}\eta, \tag{1.5}$$

and derive the following corollary of Leibniz' Rule.

**Theorem 1.3.** *[37] Let us consider a scenario where $g$ satisfies the conditions outlined in Theorem 1.2, and $G$ is defined by equation (1.5) with $a < y,\ z < b$. Under these circumstances, we have the following results:*

$$\frac{\partial G}{\partial t} = \int_y^z g_t(t, \eta)\, \mathrm{d}\eta, \tag{1.6}$$

$$\frac{\partial G}{\partial y} = -g(t, y), \tag{1.7}$$

$$\frac{\partial G}{\partial z} = g(t, z). \tag{1.8}$$

*Proof.* These results follow from the Fundamental Theorem of Calculus and the conditions specified in Theorem 1.2. □

**Theorem 1.4** (General Rule for Differentiation under the Integral Sign). *[37] Assume that both the function $g$ and its partial derivative with respect to $t$ are continuous within the given rectangle*

$$R = \{(t, \eta) : c \leqslant t \leqslant d, \quad a \leqslant \eta \leqslant b\},$$

*and suppose that $u(t)$, $v(t)$ are continuously differentiable for $c \leqslant t \leqslant d$, with the range of $u$ and $v$ in $(a, b)$. If $\phi$ is given by*

$$\phi(t) = \int_{u(t)}^{v(t)} g(t, \eta) \, \mathrm{d}\eta,$$

*then*

$$\phi'(t) = g\left(t, v(t)\right) v'(t) - g\left(t, u(t)\right) u'(t) + \int_{u(t)}^{v(t)} g_t(t, \eta) \, \mathrm{d}\eta. \tag{1.9}$$

*Proof.* We observe that

$$G\left(x, u(t), v(t)\right) = \phi(t).$$

In Theorem 1.6, when we apply the Chain Rule, the outcome is:

$$\phi'(t) = G_t + G_y u'(t) + G_z v'(t).$$

Inserting the values of $G_t, G_y$, and $G_z$ from (1.6) and (1.8), We achieve the expected outcome (1.9). □

**Theorem 1.5** (The Cauchy Formula For Repeated Integration). *[37] The Cauchy formula for repeated integration, attributed to the mathematician Augustin Louis Cauchy, provides a way to consolidate $n$ anti-derivatives of a function into a single integral. Let's break it down: Given a continuous function $g$ defined on the real line, we consider the $n$-th repeated integral of $g$ based at point $c$:*

$$g^{(-n)}(t) = \int_c^t \int_c^{\eta_1} \cdots \int_c^{\eta_{n-1}} g\left(\eta_n\right) \mathrm{d}\eta_n \cdots \mathrm{d}\eta_2 \, \mathrm{d}\eta_1,$$

*this expression represents a nested integration process, where we integrate $g(\eta_n)$ successively with respect to each variable $\eta_1$, $\eta_2, \cdots, \eta_n$ from the base point $c$ up to the final variable $t$. Interestingly, this repeated integral can be expressed more succinctly using a single integration:*

$$g^{(-n)}(t) = \frac{1}{(n-1)!} \int_c^t (t-\eta)^{n-1} g(\eta) \, \mathrm{d}\eta.$$

## 1.6 Fixed point theorem

**Definition 1.8.** *[22] Let $\nu$ be a Banach space and let $U: D(U) \subset \nu \to R(U) \subset \nu$, be an operator, defined on $D(U)$.*
*A point $t^* \in D(U)$ is called a fixed point of $U$ if $U(t^*) = t^*$.*
*Any fixed point of $U$ is therefore solution of the equation: $t = U(t)$.*

### 1.6.1 Banach fixed point theorem

The Banach fixed point theorem, also referred to as the contraction mapping theorem, is a straightforward result that ensures the existence of a single fixed point for any contracting map. In other words, it guarantees that certain types of mappings within a complete metric space have a unique point that remains unchanged under the map's transformation (see [22]).

**Theorem 1.6.** *Consider a Banach space denoted as $(\nu, \| \,.\, \|)$. Suppose we have a contraction mapping $U: \nu \to \nu$. The Banach fixed point theorem ensures that $U$ possesses a single fixed point in $v$, meaning there exists a unique $\exists!\ t^* \in \nu$ such as $Ut^* = t^*$.*
*Additionally, this fixed point can be obtained as the limit of a sequence generated by the iterative process.*

$$t_{n+1} = Ut_n, \quad n \in N,$$

*with $t_0$ an arbitrary element in $\nu$ and*

$$\|t_n - t^*\| \leqslant \frac{k^n}{1-k}\|t_1 - t_0\|.$$

Chapter 2

# Numerical solution of fractional boundary value problem with Caputo-Fabrizio and its fractional integral

## 2.1 Introduction

In our investigation, we delve into the existence and uniqueness of the solution for a fractional boundary value problem. Specifically, we consider the Caputo-Fabrizio type with fractional derivation. To address this problem, we introduce a novel definition of the fractional integral, which serves as the inverse of the conformable fractional derivative of Caputo-Fabrizio. By employing this approach, we transform the original problem into an equivalent set of linear Volterra-Fredholm integral equations of the second kind. The conditions for the existence and uniqueness of the solution are established based on the results obtained. Our study encompasses both analytical and numerical aspects. Notably, recent research has contributed significantly to the development and simulation of C-FFDE (Caputo-Fabrizio Fractional Differential Equations). These equations find applications in various complex biological systems. For instance: The fractional C-F derivative has been employed to model rabies [8], it plays a role in describing cancer treatment by radiotherapy [5], the transmission of Covid-19 has also been studied using this approach [12]. Additionally, a novel model of the human liver has been explored [10], the groundwater flow model [7], ...

Our specific focus lies in investigating the existence and uniqueness of the solution to the fractional differential equation boundary value problem involving the Caputo-Fabrizio operator:

$$\begin{cases} {}^{CF}\mathscr{D}_0^{\gamma}\,\phi(t) + r(t)\phi(t) = h(t), \quad 0 \leqslant t \leqslant 1 \\ \phi(0) = a, \quad \phi(1) = b. \end{cases} \tag{2.1}$$

Here, $1 < \gamma < 2$ is a real number, $r$ is the potential function, and $h : [0,1] \to \mathbb{R}$ is continuous, and ${}^{CF}\mathscr{D}_0^{\gamma}$ is the Caputo-Fabrizio fractional derivative.

## 2.2 Analytic study

Below, we assume the function $N(\alpha) = 1$.

**Lemma 2.1.** *Let $r \in \mathcal{C}[0,1]$. A function $\phi \in \mathcal{C}[0,1]$ is solution of problem*

$$\begin{cases} {}^{CF}\mathscr{D}_0^{\gamma}\,\phi(t) + r(t)\phi(t) = h(t), \quad \forall t \in \Lambda := [0,1],\ 1 < \gamma < 2 \\ \phi(0) = a, \quad \phi(1) = b, \end{cases} \tag{2.2}$$

*if and only if $\phi$ satisfies the following linear integral equation Volterra-Fredholm of the second kind*

$$\phi(t) + \int_0^t L(t,\eta)\phi(\eta)\,\mathrm{d}\eta + \int_0^1 F(t,\eta)\phi(\eta)\,\mathrm{d}\eta = J(t), \tag{2.3}$$

*where* $J(t) = (b-a)t + a + \int_0^t [\alpha(t-\eta) + (1-\alpha)]h(\eta)\,\mathrm{d}\eta + \int_0^1 t(\alpha\eta - 1)h(\eta)\,\mathrm{d}\eta,$

$$L(t,\eta) = [\alpha(t-\eta) + (1-\alpha)]r(\eta) \text{ and } F(t,\eta) = t(\alpha\eta - 1)r(\eta).$$

*Proof.* We can apply Lemma 1.2 to transform the equation (2.2) into equivalent integral equation

$$I_0^{\gamma}\left({}^{CF}\mathscr{D}_0^{\gamma}\,\phi(t)\right) = I_0^{\gamma}\left(h(t) - r(t)\phi(t)\right). \tag{2.4}$$

Following that, the equation (2.4) can be stated as

$$\phi(t)+\sigma_1 t+\sigma_2=\int_0^t[\alpha(t-\eta)+(1-\alpha)]\left(h(\eta)-r(\eta)\phi(\eta)\right)\,\mathrm{d}\eta.$$

Using that $\phi(0)=a,\ \phi(1)=b$, we have

$$\sigma_2=-\phi(0)=-a,\ \textit{and}\ \sigma_1=a-b+\int_0^1(1-\alpha\eta)\left(h(\eta)-r(\eta)\phi(\eta)\right)\,\mathrm{d}\eta.$$

So, we get

$$\begin{aligned}\phi(t)&=(b-a)t+a+\int_0^t[\alpha(t-\eta)+(1-\alpha)]\left(h(\eta)-r(\eta)\phi(\eta)\right)\,\mathrm{d}\eta\\&+\int_0^1 t(\alpha\eta-1)\left(h(\eta)-r(\eta)\phi(\eta)\right)\,\mathrm{d}\eta\\&=(b-a)t+a+\int_0^t[\alpha(t-\eta)+(1-\alpha)]h(\eta)\,\mathrm{d}\eta+\int_0^1 t(\alpha\eta-1)h(\eta)\,\mathrm{d}\eta\\&-\int_0^t[\alpha(t-\eta)+(1-\alpha)]r(\eta)\phi(\eta)\,\mathrm{d}\eta-\int_0^1 t(\alpha\eta-1)r(\eta)\phi(\eta)\,\mathrm{d}\eta\\&=J(t)-\int_0^t L(t,\eta)\phi(\eta)\,\mathrm{d}\eta-\int_0^1 F(t,\eta)\phi(\eta)\,\mathrm{d}\eta.\end{aligned}$$

Conversely, if $\phi$ satisfies ( 2.3), then

$${}^{CF}\mathscr{D}_0^{\gamma}\,\phi(t)+r(t)\phi(t)=h(t),\quad\forall t\in\Lambda:=[0,1],\ 1<\gamma<2\ \textit{and}\ \phi(0)=a,\ \phi(1)=b.$$

This completes the proof. □

## 2.3 The existence and uniqueness theorem

Employing the classical Picard method to demonstrating the existence and uniqueness of the solution of (2.3). This is obtained by successive approximations for $n=1,2,\ldots$

$$\phi_n(t)=J(t)-\int_0^t L(t,\eta)\phi_{n-1}(\eta)\,\mathrm{d}\eta-\int_0^1 F(t,\eta)\phi_{n-1}(\eta)\,\mathrm{d}\eta,\tag{2.5}$$

with $\phi_0(t)=J(t)$. For simplicity of handling, it is preferable to introduce

$$\psi_n(t)=\phi_n(t)-\phi_{n-1}(t),\quad n=1,2,\ldots\tag{2.6}$$

with $\psi_0(t) = J(t)$. On subtracting from (2.5), the same equation with $n$ replaced by $n-1$, and we see that

$$\psi_n(t) = -\int_0^t L(t,\eta)\psi_{n-1}(\eta)\,\mathrm{d}\eta - \int_0^1 F(t,\eta)\psi_{n-1}(\eta)\,\mathrm{d}\eta, \quad n = 1, 2, \dots \tag{2.7}$$

Also, from (2.6)

$$\phi_n(t) = \sum_{i=0}^{n} \psi_i(t). \tag{2.8}$$

Through this iteration, the following theorem substantiates the existence and uniqueness of the solution given very limited conditions, namely that $L(t,\eta)$, $F(t,\eta)$ and $J(t)$ are continuous.

**Theorem 2.1.** *Assume that the following conditions are satisfied:*

*i) The function $J(t)$ is continuous in $[0,1]$, such that $\| J \|_{\mathcal{C}[0,1]} = \max_{0\leqslant t\leqslant 1} |J(t)| \leqslant \delta_1$.*

*ii) The kernels $F(t,\eta)$, $L(t,\eta)$ are continuous with respect to second component in $[0,1]$ and satisfy $|L(t,\eta)| \leqslant \delta_2$ and $|F(t,\eta)| \leqslant \delta_3$, $\forall\ t,\eta \in [0,1]$, and $0 \leqslant \eta \leqslant t \leqslant 1$.*

*iii) $(\delta_2 + \delta_3)\delta_1 < 1$.*

*Then the equation (2.3) has an unique solution $\phi$ in $\mathcal{C}[0,1]$.*

*Proof.* We begin using equation (2.7), we have

$$|\psi_n(t)| \leqslant \left| \int_0^t L(t,\eta)\psi_{n-1}(\eta)\,\mathrm{d}\eta \right| + \left| \int_0^1 F(t,\eta)\psi_{n-1}(\eta)\,\mathrm{d}\eta \right|, \tag{2.9}$$

at $n = 1$, from (2.9) and using the given assumptions of the Theorem 2.1, we get

$$\begin{aligned} |\psi_1(t)| &\leqslant \left| \int_0^t L(t,\eta)\psi_0(\eta)\,\mathrm{d}\eta \right| + \left| \int_0^1 F(t,\eta)\psi_0(\eta)\,\mathrm{d}\eta \right| \\ &\leqslant \int_0^t |L(t,\eta)|\,|\psi_0(\eta)|\,\mathrm{d}\eta + \int_0^1 |F(t,\eta)|\,|\psi_0(\eta)|\,\mathrm{d}\eta. \end{aligned} \tag{2.10}$$

Hence, we obtain

$$|\psi_1(t)| \leqslant (\delta_2 + \delta_3)\delta_1, \tag{2.11}$$

In addition, at $n = 2$, we have

$$\begin{aligned}|\psi_2(t)| &\leqslant \left|\int_0^t L(t,\eta)\psi_1(\eta)\,\mathrm{d}\eta\right| + \left|\int_0^1 F(t,\eta)\psi_1(\eta)\,\mathrm{d}\eta\right| \\ &\leqslant \int_0^t |L(t,\eta)|\,|\psi_1(\eta)|\,\mathrm{d}\eta + \int_0^1 |F(t,\eta)|\,|\psi_1(\eta)|\,\mathrm{d}\eta,\end{aligned} \tag{2.12}$$

which leads to

$$|\psi_2(t)| \leqslant (\delta_2 + \delta_3)^2 \delta_1. \tag{2.13}$$

Subsequently, the mathematical induction is applied to obtain

$$|\psi_n(t)| \leqslant (\delta_2 + \delta_3)^n \delta_1. \tag{2.14}$$

Hence,

$$\| \psi_n \| \leqslant \lambda^n \delta_1, \quad \lambda = \delta_2 + \delta_3.$$

Note that, the series $\sum_{i=0}^{\infty} \psi_i(t)$ is uniformly convergent if and only if the series $\sum_{n=0}^{\infty} \lambda^n \delta_1$ is convergent.

In consideration $\delta_2 + \delta_3 < 1$, we can establish $\lambda < 1$, thus demonstrating the convergence of the series $\sum_{n=0}^{\infty} \lambda^n \delta_1$. It is clear from this bound that the sequence $\phi_n(t)$ in (2.8) converges. Thus, for $n \to +\infty$, we obtain

$$\phi(t) = \sum_{i=0}^{\infty} \psi_i(t). \tag{2.15}$$

We now prove that $\phi(t)$ satisfies equation (2.3).

The equation (2.14) shows that the terms $\psi_i(t)$ are dominated by $(\delta_2 + \delta_3)^i \delta_1$, consequently the series (2.15) is uniformly convergent.

Therefore, we can swap the order of integration and summation in the following expression to get

$$\begin{aligned}\int_0^t L(t,\eta) \sum_{i=0}^{\infty} \psi_i(\eta)\,\mathrm{d}\eta + \int_0^1 F(t,\eta) \sum_{i=0}^{\infty} \psi_i(\eta)\,\mathrm{d}\eta &= \sum_{i=0}^{\infty} \int_0^t L(t,\eta)\psi_i(\eta)\,\mathrm{d}\eta \\ &\quad + \sum_{i=0}^{\infty} \int_0^1 F(t,\eta)\psi_i(\eta)\,\mathrm{d}\eta \\ &= -\sum_{i=0}^{\infty} \psi_{i+1}(\eta) \\ &= -\sum_{i=0}^{\infty} \psi_i(\eta) + J(t).\end{aligned}$$

This proves that $\phi(t)$ defined by (2.14) satisfies equation (2.3).

Each of the $\psi_i(t)$ is clearly continuous. Therefore $\phi(t)$ is continuous, since it is the limit of a uniformly convergent sequence of continuous functions.

Now, to show that $\phi(t)$ is unique continuous solution, we suppose that there exists another continuous solution $\tilde{\phi}(t)$ of equation (2.3). So, we get

$$\phi(t)-\tilde{\phi}(t)=-\int_0^t L(t,\eta)(\phi(\eta)-\tilde{\phi}(\eta))\,\mathrm{d}\eta-\int_0^1 F(t,\eta)(\phi(\eta)-\tilde{\phi}(\eta))\,\mathrm{d}\eta, \tag{2.16}$$

since $\phi(t)$ and $\tilde{\phi}(t)$ are both continuous, there exists a constant $C$ such that

$$|\phi(t)-\tilde{\phi}(t)|\leqslant C,\quad 0\leqslant t\leqslant 1.$$

Substituting this into (2.16)

$$|\phi(t)-\tilde{\phi}(t)|\leqslant C(\delta_2+\delta_3),$$

after repeating the step, this demonstrates that

$$|\phi(t)-\tilde{\phi}(t)|\leqslant C(\delta_2+\delta_3)^n,\quad \text{for any } n.$$

For sufficiently large $n$, the right-hand side can be arbitrarily small.

Therefore, $\parallel \phi-\tilde{\phi} \parallel= 0$ and it is implied that $\phi(t)=\tilde{\phi}(t)$, for $t\in[0,1]$ which means the solution is unique. □

**Theorem 2.2.** *If* $h(t),\ r(t)$ *are continuous in* $[0,1]$, *and* $\max\limits_{0\leqslant t\leqslant 1}|r(t)|<1$, *then the fractional boundary value problem (2.1) possesses a unique continuous solution for* $0\leqslant t\leqslant 1$.

*Proof.* If $h(t),\ r(t)$ are continuous in $[0,1]$, then it is clear that the following functions

$$\begin{aligned}
J(t)&=(b-a)t+a+\int_0^t[\alpha(t-\eta)+(1-\alpha)]h(\eta)\,\mathrm{d}\eta+\int_0^1 t(\alpha\eta-1)h(\eta)\,\mathrm{d}\eta,\\
L(t,\eta)&=[\alpha(t-\eta)+(1-\alpha)]r(\eta),\\
F(t,\eta)&=t(\alpha\eta-1)r(\eta),
\end{aligned}$$

are continuous, and $|F(t,\eta)|=|t(\alpha\eta-1)r(\eta)|\leqslant|r(\eta)|<1,\ \forall\ t,\ \eta\in[0,1]$, which means that integral equation (2.3) possesses a unique continuous solution for $0\leqslant t\leqslant 1$. Therefore, there is a unique continuous solution of the fractional boundary value problem (2.1) for $0\leqslant t\leqslant 1$. □

## 2.4 Convergence of the solution

**Theorem 2.3.** *Assume that $k$ is properly chosen for which there exists $0 < \mu < 1$ so that $\| \psi_{m+1} \| \leqslant \mu \| \psi_m \|$, $\forall m \geqslant m_0$, for some $m_0 \in \mathbb{N}$, then $\phi_n(t,k)$ in (2.8) converges as $n \to +\infty$.*

*Proof.* Define the sequence $\Phi_i$ as

$$\begin{aligned} \Phi_0 &= \psi_0 \\ \Phi_1 &= \psi_0 + \psi_1 \\ &\vdots \\ \Phi_i &= \psi_0 + \psi_1 + \cdots + \psi_i. \end{aligned} \tag{2.17}$$

□

Now, we show that $\Phi_i$ is a Cauchy sequence in the space $\mathcal{C}[0,1]$. Consider

$$\| \Phi_{i+1} - \Phi_i \| = \| \psi_{i+1} \| \leqslant \mu \| \psi_i \| \leqslant \mu^2 \| \psi_{i-1} \| \leqslant \cdots \leqslant \mu^{i-m_0+1} \| \psi_{m_0} \| . \tag{2.18}$$

For every $i \in \mathbb{N}$, $i \geqslant l > m_0$, we have

$$\begin{aligned} \| \Phi_i - \Phi_l \| &= \| (\Phi_i - \Phi_{i-1}) + (\Phi_{i-1} - \Phi_{i-2}) + \cdots + (\Phi_{l+1} - \Phi_l) \| \\ &\leqslant \| \Phi_i - \Phi_{i-1} \| + \| \Phi_{i-1} - \Phi_{i-2} \| + \cdots + \| \Phi_{l+1} - \Phi_l \| \\ &\leqslant \mu^{i-m_0} \| \psi_{m_0} \| + \mu^{i-m_0-1} \| \psi_{m_0} \| + \cdots + \mu^{l-m_0+1} \| \psi_{m_0} \| \\ &= \mu^{l-m_0+1} \left( \frac{1-\mu^{i-l}}{1-\mu} \right) \| \psi_{m_0} \| . \end{aligned} \tag{2.19}$$

Since $0 < \mu < 1$, we get

$$\lim_{i,l \to \infty} \| \Phi_i - \Phi_l \| = 0. \tag{2.20}$$

Therefore, $(\Phi_i)$ is a Cauchy sequence in the space $\mathcal{C}[0,1]$ and $\Phi_n = \phi_n(t,k)$ converges as $n \to \infty$ and the proof is complete.

## 2.5 Stability of the solution

**Definition 2.1.** *The proposed problem (2.1) is Hyers-Ulam stable if at any $\varepsilon > 0$ for the given inequality*

$$|^{CF}\mathscr{D}_0^{\gamma}\,\phi(t) + r(t)\phi(t) - h(t)| < \varepsilon, \quad \forall t \in [0,1], \tag{2.21}$$

*there exists a unique solution $\bar{\phi}(t)$ with a constant $C > 0$ such that*

$$|\phi(t) - \bar{\phi}(t)| \leqslant C\varepsilon, \quad \forall t \in [0,1]. \tag{2.22}$$

*Further, the considered problem (2.1) will generalize Hyers-Ulam stable if there exists non-decreasing function $\kappa : (0,1) \to (0,\infty)$ such that*

$$|\phi(t) - \bar{\phi}(t)| \leqslant C\kappa(\varepsilon), \quad \forall t \in [0,1], \tag{2.23}$$

*with $\kappa(0) = 0$ and $\kappa(1) = 0$.*

In the next theorem, we present sufficient conditions upon which the problem (2.1) is Ulam-Hyers stable.

**Theorem 2.4.** *Suppose that $\delta_1 + \delta_2 < 1$. Let $\phi(t)$ be the solution of the problem (2.1) and $\bar{\phi}(t)$ be such that $\bar{\phi}(0) = a,\ \bar{\phi}(1) = b$ and*

$$|^{CF}\mathscr{D}_0^{\gamma}\,\bar{\phi}(t) + r(t)\bar{\phi}(t) - h(t)| < \varepsilon, \quad \forall t \in [0,1], \tag{2.24}$$

*where $\varepsilon > 0$. Then, there exists a constant $C > 0$ such that*

$$|\phi(t) - \bar{\phi}(t)| \leqslant C\varepsilon, \quad \forall t \in [0,1], \tag{2.25}$$

*which means that the problem (2.1) is Ulam-Hyers stable and also generalized Ulam–Hyers stable.*

*Proof.* By Theorem 2.1, the solution of the problem (2.1) exists and is unique. Let $\phi(t)$ be that unique solution of the problem (2.1) and suppose $\bar{\phi}(t)$ satisfies inequality (2.24). It follows that $\bar{\phi} \in C([0,1])$ is a solution of inequality (2.24) if and only if there exists a function $g \in C([0,1])$, which depends on $\bar{\phi}(t)$ such that

i) $|g(t)| \leqslant \varepsilon, \quad t \in [0,1], \ \varepsilon > 0,$

ii) $g(t) = {}^{CF}\mathscr{D}_0^{\gamma}\,\bar{\phi}(t) + r(t)\bar{\phi}(t) - h(t), \quad t \in [0,1],$

iii) $\bar{\phi}(0) = a$ *and* $\bar{\phi}(1) = b.$

Computing the $\gamma$-order Caputo-Fabrizio fractional integral of each member in (ii), according to Lemma 1.2, we obtain

$$\bar{\phi}(t) + \sigma_1 t + \sigma_2 + I_0^{\gamma}\left(r(t)\bar{\phi}(t)\right) - I_0^{\gamma}\left(h(t) - g(t)\right) = 0. \tag{2.26}$$

Since $\bar{\phi}(0) = a, \ \bar{\phi}(1) = b$, we have

$$\sigma_2 = -\bar{\phi}(0) = -a, \ \ \text{and} \ \sigma_1 = a - b + \int_0^1 (1-\alpha\eta)\left(h(\eta) - g(\eta) - r(\eta)\bar{\phi}(\eta)\right)\,\mathrm{d}\eta,$$

and we conclude that

$$\begin{aligned}
\bar{\phi}(t) &= (b-a)t + a + \int_0^t [\alpha(t-\eta) + (1-\alpha)]\left(h(\eta) - g(\eta) - r(\eta)\bar{\phi}(\eta)\right)\,\mathrm{d}\eta \\
&+ \int_0^1 t(\alpha\eta - 1)\left(h(\eta) - g(\eta) - r(\eta)\bar{\phi}(\eta)\right)\,\mathrm{d}\eta \\
&= (b-a)t + a + \int_0^t [\alpha(t-\eta) + (1-\alpha)]h(\eta)\,\mathrm{d}\eta + \int_0^1 t(\alpha\eta-1)h(\eta)\,\mathrm{d}\eta \\
&- \int_0^t [\alpha(t-\eta) + (1-\alpha)]g(\eta)\,\mathrm{d}\eta - \int_0^1 t(\alpha\eta-1)g(\eta)\,\mathrm{d}\eta \\
&- \int_0^t [\alpha(t-\eta) + (1-\alpha)]r(\eta)\bar{\phi}(\eta)\,\mathrm{d}\eta - \int_0^1 t(\alpha\eta-1)r(\eta)\bar{\phi}(\eta)\,\mathrm{d}\eta \\
&= J(t) - \int_0^t [\alpha(t-\eta) + (1-\alpha)]g(\eta)\,\mathrm{d}\eta - \int_0^1 t(\alpha\eta-1)g(\eta)\,\mathrm{d}\eta - \int_0^t L(t,\eta)\bar{\phi}(\eta)\,\mathrm{d}\eta \\
&- \int_0^1 F(t,\eta)\bar{\phi}(\eta)\,\mathrm{d}\eta.
\end{aligned}$$

Then take

$$\begin{aligned}|\phi(t)-\bar{\phi}(t)| = \Big|J(t) - \int_0^t L(t,\eta)\phi(\eta)\,\mathrm{d}\eta - \int_0^1 F(t,\eta)\phi(\eta)\,\mathrm{d}\eta - [J(t) \\ - \int_0^t [\alpha(t-\eta)+(1-\alpha)]g(\eta)\,\mathrm{d}\eta - \int_0^1 t(\alpha\eta-1)g(\eta)\,\mathrm{d}\eta \\ - \int_0^t L(t,\eta)\bar{\phi}(\eta)\,\mathrm{d}\eta - \int_0^1 F(t,\eta)\bar{\phi}(\eta)\,\mathrm{d}\eta\Big]\Big| \\ \leqslant \int_0^t \mid L(t,\eta)\mid\mid \phi(\eta)-\bar{\phi}(\eta)\mid \,\mathrm{d}\eta + \int_0^1 \mid F(t,\eta)\mid\mid \phi(\eta)-\bar{\phi}(\eta)\mid \,\mathrm{d}\eta \\ + \int_0^t \mid \alpha(t-\eta)+(1-\alpha)\mid\mid g(\eta)\mid \,\mathrm{d}\eta + \int_0^1 t\mid \alpha\eta-1\mid\mid g(\eta)\mid \,\mathrm{d}\eta.\end{aligned} \tag{2.27}$$

Using (ii) in the Theorem 2.1 and (2.27) in the above inequality, then taking maximum on both sides, we have

$$|\phi(t)-\bar{\phi}(t)| \leqslant (\delta_1+\delta_2)\mid \phi(\eta)-\bar{\phi}(\eta)\mid +3\varepsilon.$$

Hence, from the above inequality, we have

$$\parallel \phi-\bar{\phi}\parallel \leqslant \frac{3}{1-\delta_1-\delta_2}\varepsilon. \tag{2.28}$$

Therefore, the solution is Hyers-Ulam stable. Further, let

$$C = \frac{3}{1-\delta_1-\delta_2}, \tag{2.29}$$

and there exist non decreasing function $\kappa \in \mathcal{C}((0,1),(0,\infty))$. Then, from (2.28) we can write as

$$\parallel \phi-\bar{\phi}\parallel \leqslant C\kappa(\varepsilon),\ \ \kappa(0)=a,\ \ \kappa(1)=b. \tag{2.30}$$

Therefore (2.5) implies that the solution is also generalized Ulam–Hyers stable. □

## 2.6 Estimation of error of the approximation solution

**Theorem 2.5.** *If assumptions of Theorem 2.1 are satisfied, $n \in \mathbb{N}$ and $n \geqslant m_0$, then we obtain the estimation of error of the approximate solution defined by*

$$\parallel \phi-\phi_n\parallel \leqslant \frac{\mu^{n+1-m_0}}{1-\mu}\parallel \psi_{m_0}\parallel. \tag{2.31}$$

*Proof.* Let $n \in \mathbb{N}$ and $n \geqslant m_0$, we get

$$\begin{aligned} \| \phi - \phi_n \| &= \sup_{t\in[0,1]} \left| \phi(t) - \sum_{i=0}^{n} \psi_i(x,t) \right| \\ &\leqslant \sup_{t\in[0,1]} \left( \sum_{i=n+1}^{\infty} |\psi_i(x,t)| \right) \\ &\leqslant \sum_{i=n+1}^{\infty} \sup_{t\in[0,1]} (|\psi_i(x,t)|) \\ &\leqslant \sum_{i=n+1}^{\infty} \mu^{i-m_0} \| \psi_{m_0} \| \\ &= \frac{\mu^{n+1-m_0}}{1-\mu} \| \psi_{m_0} \| . \end{aligned} \tag{2.32}$$

□

## 2.7 Numerical study

This section, we present an algorithm based on the method of trapezoidal rule to find a numerical solution of linear Volterra-Fredholm integral equations of the second kind.
For all $N \in \mathbb{N}$, we divide the interval $[0,1]$ into sub intervals with equal length $k = (d-c)/N$.
We denote $t_0 = c,\ t_N = d$ and $t_i = c + i \cdot k,\ 0 \leqslant i \leqslant N$.
Consider, the numerical integration is given in this formula:

$$\int_c^d h(\eta)\, \mathrm{d}\eta \approx \frac{k}{2} \left[ h(c) + 2 \sum_{j=1}^{N} h(t_j) + h(d) \right].$$

We employ this formula in equation (2.3), and we get:

$$\begin{aligned} J(t_i) = \phi(t_i) &+ \frac{k}{2} \left[ L(t_i,t_0)\phi(t_0) + 2 \sum_{j=1}^{i-1} L(t_i,t_j)\phi(t_j) + L(t_i,t_i)\phi(t_i) \right] \\ &+ \frac{k}{2} \left[ F(t_i,t_0)\phi(t_0) + 2 \sum_{j=1}^{N-1} F(t_i,t_j)\phi(t_j) + F(t_i,t_N)\phi(t_N) \right]. \end{aligned}$$

That is equivalent to

$$J_i = \phi_i + \frac{k}{2}\left[L_{i0}\phi_0 + 2\sum_{j=1}^{i-1} L_{ij}\phi_j + L_{ii}\phi_i\right] + \frac{k}{2}\left[F_{i0}\phi_0 + 2\sum_{j=1}^{N-1} F_{ij}\phi_j + F_{iN}\phi_N\right], \ \forall i = 0, \dots, N.$$

This leads to

$$\frac{k}{2}(L_{i0} + F_{i0})\phi_0 + k\sum_{j=1}^{i-1}(L_{ij} + F_{ij})\phi_j + \frac{k}{2}\left(\frac{2}{k} + L_{ii} + 2F_{ii}\right)\phi_i + k\sum_{j=i+1}^{N-1} F_{ij}\phi_j + \frac{k}{2}F_{iN}\phi_N = J_i.$$

So

$$k\sum_{j=1}^{i-1}(L_{ij} + F_{ij})\phi_j + \frac{k}{2}\left(\frac{2}{k} + L_{ii} + 2F_{ii}\right)\phi_i + k\sum_{j=i+1}^{N-1} F_{ij}\phi_j = J_i - \frac{k}{2}(L_{i0} + F_{i0})\phi_0 - \frac{k}{2}F_{iN}\phi_N. \tag{2.33}$$

Finally, we obtain $N-1$ equations in $\phi_i,\ i = 1, \dots, N-1$ that represents the approximate solution to equation (2.3).

Equation (2.33) can be represented as a system of linear equations:

$$A\Phi = B. \tag{2.34}$$

Which is $A = (a_{ij})_{i,j=1,\dots,N-1}$, $\Phi = (\phi_1, \dots, \phi_{N-1})$ and $B = (b_i)_{i=1,\dots,N-1}$.

Where

$$a_{ij} = \begin{cases} k \cdot \left(\dfrac{L_{ii} + 2F_{ii}}{2} + \dfrac{1}{k}\right) & \text{if } j = i, \\ k \cdot (L_{ij} + F_{ij}) & \text{if } j = 1 : i-1, \\ k \cdot F_{ij} & \text{if } j = i+1 : N-1, \end{cases}$$

and

$$b_i = J_i - \frac{k}{2}\left(L_{i0} + F_{i0}\right)a - \frac{k}{2}F_{iN}b, \quad i = 1, \dots, N-1.$$

We have opted to formulate our system as a general matrix

$$A = k\cdot\begin{pmatrix} \frac{L_{11}+2F_{11}}{2}+\frac{1}{k} & F_{12} & F_{13} & \cdots & F_{1N-2} & F_{1N-1} \\ F_{21}+L_{21} & \frac{L_{22}+2F_{22}}{2}+\frac{1}{k} & F_{23} & \cdots & F_{2N-2} & F_{2N-1} \\ F_{31}+L_{31} & F_{32}+L_{32} & \frac{L_{33}+2F_{33}}{2}+\frac{1}{k} & \ddots & \vdots & F_{3N-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \frac{L_{N-2N-2}+2F_{N-2N-2}}{2}+\frac{1}{k} & F_{N-2N-1} \\ F_{N-11}+L_{N-11} & \cdots & \cdots & \cdots & F_{N-1N-2}+L_{N-1N-2} & \frac{L_{N-1N-1}+2F_{N-1N-1}}{2}+\frac{1}{k} \end{pmatrix}$$

and

$$B = \begin{pmatrix} J_1 - \frac{k}{2}(L_{10}+F_{10})a - \frac{k}{2}F_{1N}b \\ \vdots \\ J_{N-1} - \frac{k}{2}(L_{N-10}+F_{N-10})a - \frac{k}{2}F_{N-1N}b \end{pmatrix}.$$

## 2.8 Illustrating examples

In this section, some numerical examples will be given to investigate the efficiency and accuracy of the above methods to solve the Volterra-Fredholm linear integral equations of the second kind. We use MATLAB to solve these examples.

**Example 2.1.** *Consider the following fractional boundary value problem:*

$$\begin{cases} \mathscr{D}_0^{\frac{3}{2}}\phi(t) + \left(1-\frac{t^4}{20}\right)\phi(t) = \frac{t}{4}\left(3e^{-t}-t-1\right), & 0\leqslant t\leqslant 1 \\ \phi(0)=1, \quad \phi(1)=\frac{e^{-1}+2}{2}. \end{cases} \tag{2.35}$$

*The exact solution of problem (2.35) is:*

$$\phi(t) = \frac{e^{-t}+t+1}{2}.$$

**Algorithm 2.1.** *The following algorithm is coded using MATLAB software. This algorithm is solving problem (2.35) numerically, and plot the approximate solution resulting and absolute error between the exact value and approximated value.*

```
% this program which calculates the approximate solution of the problem
```

```
% phi(t)=j(t)+\int(c,t)k1(t,eta,u(eta))deta ...
    +\int(c,d)k2(t,eta,u(eta))deta    t in [c,d]
% print('-dpsc','Example1.eps')
clc  ;  clear;
%%%%%%%%%%%%% variables %%%%%%%%%%%%%%%%%%%
gamma=1.5;
alpha=gamma-floor(gamma);
c=0;
d=1;
a=1;
b=(2+exp(-1))/2;
N=8;
k=(d-c)/N;
tic
%%%%%%%%%%%%%% functions %%%%%%%%%%%%%%%%%%
syms t eta ;
phi(t)=(1+t+exp(-t))/2
r(t)=1-t^{4}/20
h(t)= r(t)*phi(t)+t*exp(-t)
L(t,eta)=(alpha*(t-eta)+(1-alpha))*r(eta);
F(t,eta)=t*(alpha*eta-1)*r(eta);
j(t)=(b-a)*t+a+int(L(t,eta)/r(eta)*h(eta),eta,0,t)+ ...
    int(F(t,eta)/r(eta)*h(eta),eta,0,1);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for n=1:N-1
    X(n)=vpa(c+k*n);
    B(n)=vpa(j(X(n))-k*a*(L(X(n),c)+F(X(n),c))/2-k*F(X(n),d)*b/2);
end
for i=1:N-1
    A(i,i)=vpa(1+k*(L(X(i),X(i))+2*F(X(i),X(i)))/2);
end
for i=1:N-1
   for j=1:i-1
    A(i,j)=vpa(k*(L(X(i),X(j))+F(X(i),X(j))));
   end
end
for i=1:N-1
```

```
    for j=i+1:N-1
     A(i,j)=vpa(k*F(X(i),X(j)));
    end
end
A
Phi=vpa(inv(A))*B';
for i=1:N-1
    x(i+1)=X(i);
    apphi(i+1)=Phi(i);
end
x(1)=c;
apphi(1)=a;
x(N+1)=d;
apphi(N+1)=b;
for i=1:N+1
    er(i)=vpa(abs(apphi(i)-phi(x(i))));
end
toc
subplot(2,1,1);
plot(x,apphi);
title('Approximate Solution of \phi');
xlabel('t','FontSize',10)
ylabel(' \phi(t)','FontSize',10)
subplot(2,1,2);
plot(x,er);
title('The absolute error when N =8');
xlabel('t','FontSize',10)
ylabel('Absolute error','FontSize',10)
```

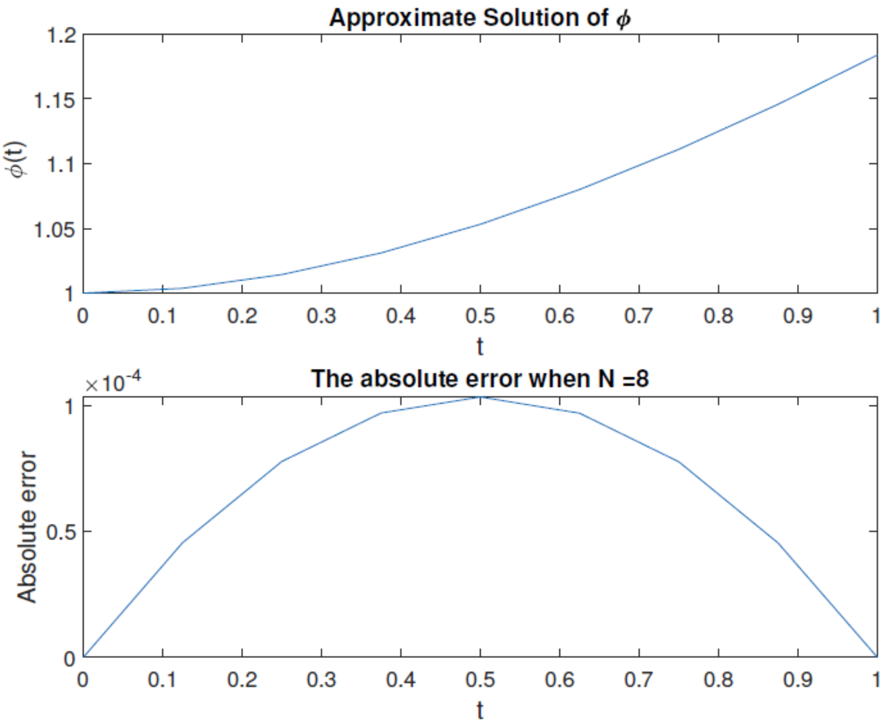


Figure 2.1: The approximate solution and the absolute error in Example 2.1 when $N = 8$.

**Example 2.2.** *Consider the following fractional boundary value problem:*

$$\begin{cases} \mathscr{D}_0^{\frac{3}{2}}\phi(t) + \left(1 - \dfrac{t^{10}}{10000}\right)\phi(t) = te^{-t} - \left(\dfrac{t^{10}}{10000} - 1\right)\left(\dfrac{e^{-t}-1}{2}\right), \quad 0 \leqslant t \leqslant 1 \\ \phi(0) = 1, \quad \phi(1) = \dfrac{e^{-1}+1}{2}. \end{cases} \tag{2.36}$$

*The exact solution of problem (2.36) is:*

$$\phi(t) = \frac{e^{-t}+1}{2}.$$

**Algorithm 2.2.** *The following algorithm is coded using MATLAB software. This algorithm is solving problem (2.36) numerically, and plot the approximate solution resulting and absolute error between the exact value and approximated value.*

```
% this program which calculates the approximate solution of the problem
% phi(t)=j(t)+\int(c,t)k1(t,eta,u(eta))deta ...
    +\int(c,d)k2(t,eta,u(eta))deta     t in [c,d]
```

```
% print('-dpsc','Example21.eps')
clc  ;  clear;
%%%%%%%%%%%%% variables %%%%%%%%%%%%%%%%%%%
gamma=1.5;
alpha=gamma-floor(gamma);
c=0;
d=1;
a=1;
b=(1+exp(-1))/2;
N=16;
k=(d-c)/N;
tic
%%%%%%%%%%%%%% functions %%%%%%%%%%%%%%%%%%
syms t eta ;
phi(t)=(1+exp(-t))/2;
r(t)=1-t^{10}/10000;
h(t)= r(t)*phi(t)+t*exp(-t);
L(t,eta)=(alpha*(t-eta)+(1-alpha))*r(eta);
F(t,eta)=t*(alpha*eta-1)*r(eta);
j(t)=(b-a)*t+a+int(L(t,eta)/r(eta)*h(eta),eta,0,t)+ ...
    int(F(t,eta)/r(eta)*h(eta),eta,0,1);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for n=1:N-1
    X(n)=vpa(c+k*n);
    B(n)=vpa(j(X(n))-k*a*(L(X(n),c)+F(X(n),c))/2-k*F(X(n),d)*b/2);
end
for i=1:N-1
    A(i,i)=vpa(1+k*(L(X(i),X(i))+2*F(X(i),X(i)))/2);
end
for i=1:N-1
   for j=1:i-1
    A(i,j)=vpa(k*(L(X(i),X(j))+F(X(i),X(j))));
   end
end
for i=1:N-1
   for j=i+1:N-1
    A(i,j)=vpa(k*F(X(i),X(j)));
```

```
    end
end
A
Phi=vpa(inv(A))*B';
for i=1:N-1
    x(i+1)=X(i);
    apphi(i+1)=Phi(i);
end
x(1)=c;
apphi(1)=a;
x(N+1)=d;
apphi(N+1)=b;
for i=1:N+1
    er(i)=vpa(abs(apphi(i)-phi(x(i))));
end
toc
subplot(2,1,1);
plot(x,apphi);
title('Approximate Solution of \phi');
xlabel('t','FontSize',10)
ylabel(' \phi(t)','FontSize',10)
subplot(2,1,2);
plot(x,er);
title('The absolute error when N =16');
xlabel('t','FontSize',10)
ylabel('Absolute error','FontSize',10)
```

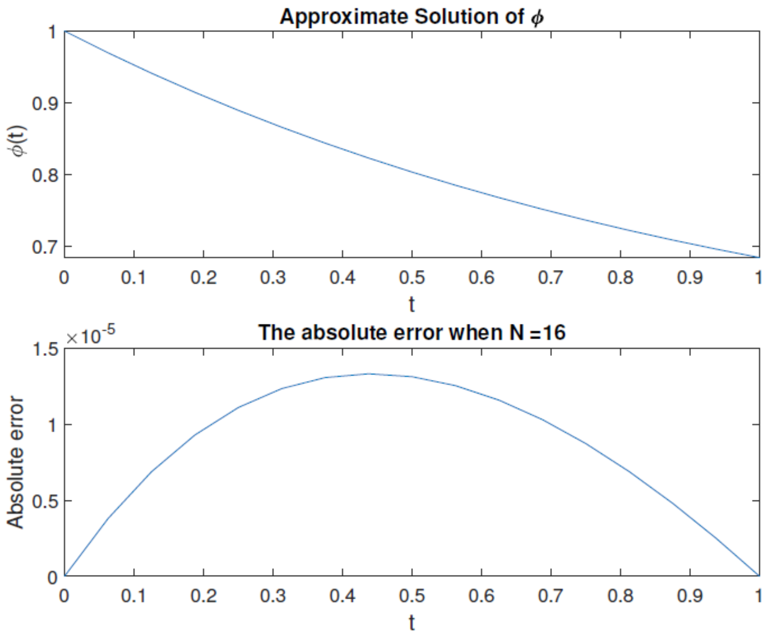


Figure 2.2: The approximate solution and the absolute error in Example 2.2 when $N = 16$.

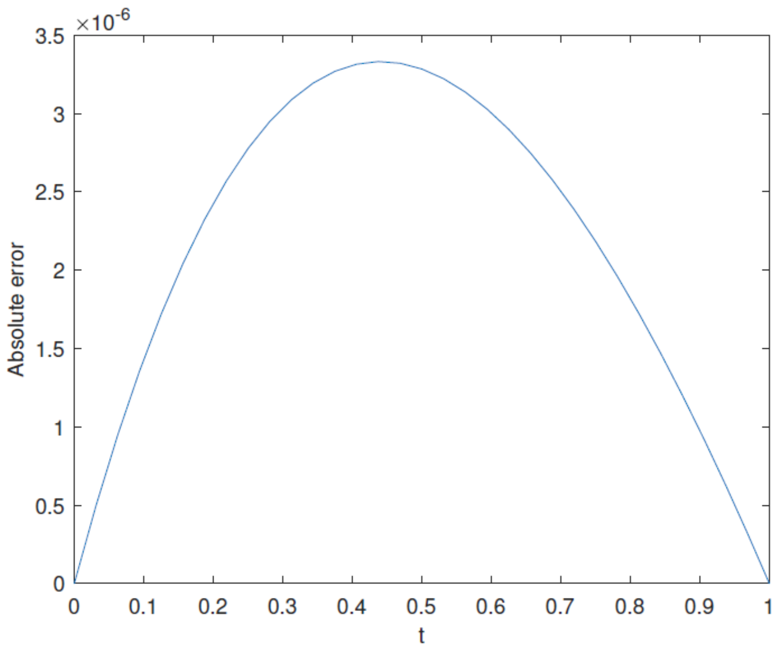


Figure 2.3: The absolute error when $N = 32$ in Example 2.2.

**Example 2.3.** *Consider the following fractional boundary value problem:*

$$\begin{cases} \mathscr{D}_0^{\frac{7}{4}}\phi(t) + \dfrac{1-t}{2}\phi(t) = -10te^{2t}, \quad 0 \leqslant t \leqslant 1 \\ \phi(0) = 1, \quad \phi(1) = 2. \end{cases} \tag{2.37}$$

*In this case, we don't know the exact solution.*

```
% this program which calculates the approximate solution of the problem
% phi(t)=j(t)+\int(c,t)k1(t,eta,u(eta))deta ...
    +\int(c,d)k2(t,eta,u(eta))deta    t in [c,d]
% print('-dpsc','Example3.eps')
clc  ;  clear;
%%%%%%%%%%%%% variables %%%%%%%%%%%%%%%%%%%
gamma=1.75;
alpha=gamma-floor(gamma);
c=0;
d=1;
a=1;
b=2;
N=8;
k=(d-c)/N;
tic
%%%%%%%%%%%%%% functions %%%%%%%%%%%%%%%%%%
syms t eta ;
r(t)=(1-t)/2;
h(t)= -10*t*exp(2*t);
L(t,eta)=(alpha*(t-eta)+(1-alpha))*r(eta);
F(t,eta)=t*(alpha*eta-1)*r(eta);
j(t)=(b-a)*t+a+int(L(t,eta)/r(eta)*h(eta),eta,0,t)+ ...
    int(F(t,eta)/r(eta)*h(eta),eta,0,1);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for n=1:N-1
    X(n)=vpa(c+k*n);
    B(n)=vpa(j(X(n))-k*a*(L(X(n),c)+F(X(n),c))/2-k*F(X(n),d)*b/2);
end
for i=1:N-1
```

```
    A(i,i)=vpa(1+k*(L(X(i),X(i))+2*F(X(i),X(i)))/2);
end
for i=1:N-1
   for j=1:i-1
    A(i,j)=vpa(k*(L(X(i),X(j))+F(X(i),X(j))));
   end
end
for i=1:N-1
   for j=i+1:N-1
    A(i,j)=vpa(k*F(X(i),X(j)));
   end
end
A
Phi=vpa(inv(A))*B';
for i=1:N-1
   x(i+1)=X(i);
   apphi(i+1)=Phi(i);
end
x(1)=c;
apphi(1)=a;
x(N+1)=d;
apphi(N+1)=b;
plot(x,apphi);
xlabel('t','FontSize',10)
ylabel(' \phi(t)','FontSize',10)
```

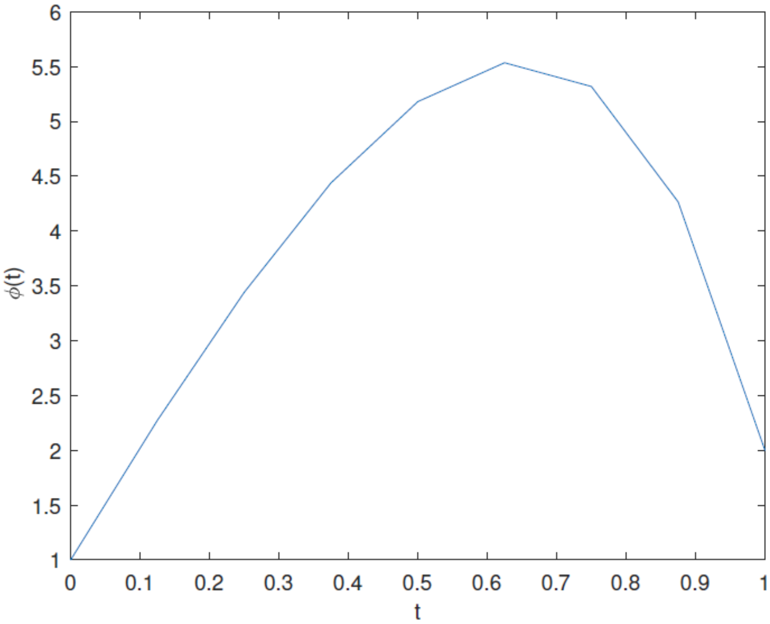


Figure 2.4: The approximate solution of Example 2.3 with $N = 8$.

Chapter 3

# Numerical solution of a fractional coupled system with the Caputo-Fabrizio fractional derivative

## 3.1 Introduction

This chapter delves into the topic of solutions to linear fractional coupled systems of differential equations with Caputo-Fabrizio fractional order. We establish the existence and uniqueness of solutions using the Picard-Lindelöf method and fixed point theory. The Adomian decomposition method (ADM) is employed to compute the approximate solution, as it presents the solution as a series that converges to the exact solution. The method's efficiency and effectiveness are demonstrated through numerical examples.
Fractional differential equations are pivotal in numerous scientific and engineering fields, including physics, economics, chemistry, biology, etc, (refer to [46, 44, 24] for more details). The study of linear and nonlinear problems in the realm of fractional differential equations has been extensively researched and referenced. For a comprehensive study of various linear

and nonlinear problems in fractional differential equations and their applications, refer to [25, 33, 45].

Coupled systems of fractional differential equations have emerged as an intriguing area of recent research. They have proven to be more precise and realistic, with numerous applications in real-world problems (see [29, 47, 4, 38] for examples). However, due to the complexity of nonlinear terms, many fractional differential equations lack an exact analytical solution. This challenge has led researchers to develop numerical schemes for finding approximate solutions. Various methods exist for computing numerical solutions to fractional differential equations and integral equations, such as iterative techniques [39], numerical bifurcation [27], difference methods [43], and others.

We will introduce a technique that serves as a foundation for understanding linear fractional pair systems of the Caputo-Fabrizio type with specific initial conditions

$$\begin{cases} {}^{CF}\mathscr{D}_0^{\alpha}\,\phi(t) = c_1\phi(t) + c_2\psi(t) + h(t), & t \in \Lambda := [0,1] \\ {}^{CF}\mathscr{D}_0^{\alpha}\,\psi(t) = c_3\phi(t) + c_4\psi(t) + g(t), & t \in \Lambda := [0,1] \\ \phi(0) = \psi(0) = 0, \end{cases} \tag{3.1}$$

where $0 < \alpha < 1$ is a real number, ${}^{CF}\mathscr{D}_0^{\alpha}$ is the new fractional derivative of Caputo Fabrizio, $h, g : \Lambda \to \mathbb{R}$ are given continuous functions, and $c_i$ real constants and $i = 1, 2, 3, 4$.

## 3.2 Study of the associated linear system

Below, we assume the function $N(\alpha) = 1$.

**Lemma 3.1.** *Let* $0 < \alpha < 1,\ \phi, \psi \in \mathcal{C}^1(\Lambda),\ h, g : \Lambda \to \mathbb{R}$ *be continuous functions, and* $c_1,\ c_2,\ c_3,\ c_4$ *real constants. Then the solution of couple system (3.1) is given by*

$$\begin{aligned} \phi(t) &= K(t) + (1-\alpha)(c_1\phi(t) + c_2\psi(t)) + \alpha\int_0^t (c_1\phi(\eta) + c_2\psi(\eta))\,\mathrm{d}\eta, \\ \psi(t) &= G(t) + (1-\alpha)(c_3\phi(t) + c_4\psi(t)) + \alpha\int_0^t (c_3\phi(\eta) + c_4\psi(\eta))\,\mathrm{d}\eta, \end{aligned} \tag{3.2}$$

*where*

$$K(t) = (1-\alpha)(h(t)-h(0)) + \alpha \int_0^t h(\eta)\,\mathrm{d}\eta,$$

*and*

$$G(t) = (1-\alpha)(g(t)-g(0)) + \alpha \int_0^t g(\eta)\,\mathrm{d}\eta.$$

**Proof 3.2.1.** *Applying the operator $I_0^\alpha$ on the fractional differential equations in (3.1) and utilizing Lemma 1.2, we acquire*

$$\begin{aligned} \phi(t) + \sigma_1 &= (1-\alpha)(c_1\phi(t) + c_2\psi(t) + h(t) - h(0)) \\ &\quad + \alpha \int_0^t (c_1\phi(\eta) + c_2\psi(\eta) + h(\eta))\,\mathrm{d}\eta, \\ \psi(t) + \sigma_2 &= (1-\alpha)(c_3\phi(t) + c_4\psi(t) + g(t) - g(0)) \\ &\quad + \alpha \int_0^t (c_3\phi(\eta) + c_4\psi(\eta) + g(\eta))\,\mathrm{d}\eta, \end{aligned} \tag{3.3}$$

*where $\sigma_1, \sigma_2 \in \mathbb{R}$.*

*Inserting (3.3) in the initial conditions $\phi(0) = \psi(0) = 0$, we get*

$$\sigma_1 = \sigma_2 = 0.$$

*Thus (3.3) reduces to*

$$\begin{aligned} \phi(t) &= (1-\alpha)(c_1\phi(t) + c_2\psi(t) + h(t) - h(0)) + \alpha \int_0^t (c_1\phi(\eta) + c_2\psi(\eta) + h(\eta))\,\mathrm{d}\eta, \\ \psi(t) &= (1-\alpha)(c_3\phi(t) + c_4\psi(t) + g(t) - g(0)) + \alpha \int_0^t (c_3\phi(\eta) + c_4\psi(\eta) + g(\eta))\,\mathrm{d}\eta. \end{aligned} \tag{3.4}$$

*The equation (3.4) can then be expressed as follows*

$$\begin{aligned} \phi(t) &= (1-\alpha)(h(t)-h(0)) + \alpha \int_0^t h(\eta)\,\mathrm{d}\eta + (1-\alpha)(c_1\phi(t) + c_2\psi(t)) \\ &\quad + \alpha \int_0^t (c_1\phi(\eta) + c_2\psi(\eta))\,\mathrm{d}\eta, \\ \psi(t) &= (1-\alpha)(g(t)-g(0)) + \alpha \int_0^t g(\eta)\,\mathrm{d}\eta + (1-\alpha)(c_3\phi(t) + c_4\psi(t)) \\ &\quad + \alpha \int_0^t (c_3\phi(\eta) + c_4\psi(\eta))\,\mathrm{d}\eta. \end{aligned} \tag{3.5}$$

*Therefore, the solution of couple system (3.1) is obtained as follows*

$$\begin{aligned} \phi(t) &= K(t) + (1-\alpha)(c_1\phi(t) + c_2\psi(t)) + \alpha \int_0^t (c_1\phi(\eta) + c_2\psi(\eta))\,\mathrm{d}\eta, \\ \psi(t) &= G(t) + (1-\alpha)(c_3\phi(t) + c_4\psi(t)) + \alpha \int_0^t (c_3\phi(\eta) + c_4\psi(\eta))\,\mathrm{d}\eta. \end{aligned}$$

*This completes the proof.*

## 3.3 The existence and uniqueness theorem

Here we establish existence of a uniqueness criteria of solution through the Picard–Lindelöf method and the fixed point theory.
Let $(\phi_0(t), \psi_0(t)) = (K(t), G(t))$; then the Picard iteration is defined as follows

$$\begin{aligned} \phi_{i+1}(t) &= (1-\alpha)(c_1\phi_i(t) + c_2\psi_i(t)) + \alpha \int_0^t (c_1\phi_i(\eta) + c_2\psi_i(\eta))\,\mathrm{d}\eta, \\ \psi_{i+1}(t) &= (1-\alpha)(c_3\phi_i(t) + c_4\psi_i(t)) + \alpha \int_0^t (c_3\phi_i(\eta) + c_4\psi_i(\eta))\,\mathrm{d}\eta. \end{aligned} \tag{3.6}$$

To demonstrate the existence of a unique solution, let's establish a definition

$$\begin{aligned} L_1(t,\phi,\psi) &= c_1\phi + c_2\psi, \\ L_2(t,\phi,\psi) &= c_3\phi + c_4\psi, \end{aligned} \tag{3.7}$$

and $\lambda(t) = (\phi(t), \psi(t))$.

**Lemma 3.2.** *Let $L_1, L_2 : \Lambda \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then $L_1(t,\phi,\psi)$ and $L_2(t,\phi,\psi)$ are contraction with respect to $\phi$ and $\psi$ if*

$$\mu_1 < 1 \text{ and } \mu_2 < 1, \tag{3.8}$$

*where $\mu_1 = \max(|c_1|, |c_2|),\ \mu_2 = \max(|c_3|, |c_4|)$.*

**Proof 3.3.1.** *Let $\phi_i,\ \psi_i \in \mathbb{R}$, where $i = 1, 2$ and $\forall t \in \Lambda$, we obtain*

$$\begin{aligned} |L_1(t,\phi_1,\psi_1) - L_1(t,\phi_2,\psi_2)| &\leqslant |c_1| \parallel \phi_1 - \phi_2 \parallel + |c_2| \parallel \psi_1 - \psi_2 \parallel \\ &\leqslant \max(|c_1|, |c_2|)\,(\parallel \phi_1 - \phi_2 \parallel + \parallel \psi_1 - \psi_2 \parallel) \\ &\leqslant \mu_1 \parallel \lambda_1 - \lambda_2 \parallel . \end{aligned}$$

*In a similar manner:*

$$|L_2(t,\phi_1,\psi_1) - L_2(t,\phi_2,\psi_2)| \leqslant \mu_2 \parallel \lambda_1 - \lambda_2 \parallel .$$

*Hence,*

$$\begin{aligned} \| L_1(t,\phi_1,\psi_1) - L_1(t,\phi_2,\psi_2) \| &\leqslant \mu_1 \| \lambda_1 - \lambda_2 \|, \\ \| L_2(t,\phi_1,\psi_1) - L_2(t,\phi_2,\psi_2) \| &\leqslant \mu_2 \| \lambda_1 - \lambda_2 \| . \end{aligned} \tag{3.9}$$

*Which, in view of (3.8), implies that* $L_1(t,\phi,\psi,)$ *and* $L_2(t,\phi,\psi)$ *are contraction with respect to* $\phi$ *and* $\psi$.

**Theorem 3.1.** *Let* $L_1, L_2 : \Lambda \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous function. Then, coupled system (3.1) has a unique solution in* $\Lambda$*, provided by*

$$\mu = \mu_1 + \mu_2 < 1,$$

*where* $\mu_1 = \max(|c_1|, |c_2|),\ \mu_2 = \max(|c_3|, |c_4|)$.

**Proof 3.3.2.** *To prove the existence of a unique solution,* $L_1(t,\phi,\psi)$ *and* $L_2(t,\phi,\psi)$ *must be contractions with respect to* $\phi$ *and* $\psi$*. So, can be defined of the Picard operator :*

$$\Theta(\lambda(t)) = \lambda_0 + (1-\alpha)\xi(t,\lambda(t)) + \alpha \int_0^t \xi(\eta,\lambda(\eta))\,\mathrm{d}\eta, \tag{3.10}$$

*where* $\xi(t,\lambda(t)) = (L_1(t,\phi(t),\psi(t)), L_2(t,\phi(t),\psi(t)))$ *and* $\lambda_0 = (K(t), G(t))$*. One should keep in mind that a fractional problem has a bounded solution. Given that* $L_1$ *and* $L_2$ *are contractions, the following also applies*

$$\| \xi(t,\lambda_1(t)) - \xi(t,\lambda_2(t)) \| \leqslant \mu \| \lambda_1 - \lambda_2 \|, \tag{3.11}$$

*where* $\mu = \mu_1 + \mu_2 < 1$*. Additionally, through the utilization of equation (3.2), we obtain*

$$\begin{aligned} \| \lambda - \lambda_0 \| &= \| (1-\alpha)\xi(t,\lambda(t)) + \alpha \int_0^t \xi(\eta,\lambda(\eta))\,\mathrm{d}\eta \| \\ &\leqslant (1-\alpha) \| \xi(t,\lambda(t)) \| + \alpha \int_0^t \| \xi(\eta,\lambda(\eta)) \| \ \mathrm{d}\eta \\ &\leqslant (1-\alpha+\alpha t)\mu \\ &\leqslant \mu, \end{aligned} \tag{3.12}$$

*where* $\mu < 1$.

*We now illustrate the contraction feature of* $\Theta$ *by using the definition of the Picard operator*

*(3.10), we acquire*

$$\begin{aligned}\| \Theta(\lambda_1(t)) - \Theta(\lambda_2(t)) \| =& \| (1-\alpha)(\xi(t,\lambda_1(t)) - \xi(t,\lambda_2(t))) \\ &+ \alpha \int_0^t (\xi(\eta,\lambda_1(\eta)) - \xi(\eta,\lambda_2(\eta)))\, \mathrm{d}\eta \| \\ &\leqslant (1-\alpha) \| \xi(t,\lambda_1(t)) - \xi(t,\lambda_2(t)) \| \\ &+ \alpha \int_0^t \| \xi(\eta,\lambda_1(\eta)) - \xi(\eta,\lambda_2(\eta)) \| \ \mathrm{d}\eta \\ &\leqslant (1-\alpha)\mu \| \lambda_1(t) - \lambda_2(t) \| \\ &+ \alpha\mu \int_0^t \| \lambda_1(\eta) - \lambda_2(\eta) \| \ \mathrm{d}\eta \\ &\leqslant (1-\alpha+\alpha t)\mu \| \lambda_1(t) - \lambda_2(t) \| \\ &\leqslant \mu \| \lambda_1(t) - \lambda_2(t) \|,\end{aligned} \tag{3.13}$$

*since according to equation (3.12), we have $\mu < 1$. Consequently, the specified operator $\Theta$ is a contraction. Therefore, the operator $\Theta$ has a unique fixed point, which is the unique solution to (3.1), in accordance with Banach's fixed point theorem 1.6.*

## 3.4 Numerical study

In section, we use ADM [3] to fully implement the fractional system (3.1).
In the decomposition method, the solutions $\phi(t)$ and $\psi(t)$ of the integral equation (3.2) are usually expressed in series form and defined as follows:

$$\phi(t) = \sum_{i=0}^{\infty} \phi_i(t) \quad \textit{and } \psi(t) = \sum_{i=0}^{\infty} \psi_i(t). \tag{3.14}$$

Replacing the decomposition (3.14) into both sides of (3.2) results in

$$\begin{aligned}\sum_{i=0}^{\infty} \phi_i(t) = K(t) + (1-\alpha)\left(c_1 \sum_{i=0}^{\infty} \phi_i(t) + c_2 \sum_{i=0}^{\infty} \psi_i(t)\right) \\ + \alpha \int_0^t \left(c_1 \sum_{i=0}^{\infty} \phi_i(\eta) + c_2 \sum_{i=0}^{\infty} \psi_i(\eta)\right) \mathrm{d}\eta,\end{aligned}$$

and

$$\sum_{i=0}^{\infty}\psi_i(t) = G(t) + (1-\alpha)\left(c_3\sum_{i=0}^{\infty}\phi_i(t) + c_4\sum_{i=0}^{\infty}\psi_i(t)\right)$$
$$+\alpha\int_0^t\left(c_3\sum_{i=0}^{\infty}\phi_i(\eta) + c_4\sum_{i=0}^{\infty}\psi_i(\eta)\right)\,\mathrm{d}\eta,$$

or equivalently

$$\begin{aligned}\phi_0(t)+\phi_1(t)+\phi_2(t)+\phi_3(t)+\cdots = {} & K(t) + (1-\alpha)\left(c_1\phi_0(t)+c_2\psi_0(t)\right)\\ & +\alpha\int_0^t\left(c_1\phi_0(\eta)+c_2\psi_0(\eta)\right)\,\mathrm{d}\eta\\ & +(1-\alpha)\left(c_1\phi_1(t)+c_2\psi_1(t)\right)\\ & +\alpha\int_0^t\left(c_1\phi_1(\eta)+c_2\psi_1(\eta)\right)\,\mathrm{d}\eta\\ & (1-\alpha)\left(c_1\phi_2(t)+c_2\psi_2(t)\right)\\ & +\alpha\int_0^t\left(c_1\phi_2(\eta)+c_2\psi_2(\eta)\right)\,\mathrm{d}\eta+\cdots\end{aligned}$$

and

$$\begin{aligned}\psi_0(t)+\psi_1(t)+\psi_2(t)+\psi_3(t)+\cdots = {} & G(t) + (1-\alpha)\left(c_3\phi_0(t)+c_4\psi_0(t)\right)\\ & +\alpha\int_0^t\left(c_3\phi_0(\eta)+c_4\psi_0(\eta)\right)\,\mathrm{d}\eta\\ & +(1-\alpha)\left(c_3\phi_1(t)+c_4\psi_1(t)\right)\\ & +\alpha\int_0^t\left(c_3\phi_1(\eta)+c_4\psi_1(\eta)\right)\,\mathrm{d}\eta\\ & +(1-\alpha)\left(c_3\phi_2(t)+c_4\psi_2(t)\right)\\ & +\alpha\int_0^t\left(c_3\phi_2(\eta)+c_4\psi_2(\eta)\right)\,\mathrm{d}\eta+\cdots\end{aligned}$$

The elements $\phi_0(t),\ \phi_1(t),\ \phi_2(t),\ \phi_3(t),\cdots$ and $\psi_0(t),\ \psi_1(t),\ \psi_2(t),\ \psi_3(t),\cdots$ of the unknown functions $\phi(t)$ and $\psi(t)$ respectively can be fully defined in a recursive sequence, given certain conditions are met:

$$\phi_0(t) = K(t), \quad \textit{and} \;\; \psi_0(t) = G(t),$$

$$\phi_1(t) = (1-\alpha)\left(c_1\phi_0(t)+c_2\psi_0(t)\right) + \alpha\int_0^t\left(c_1\phi_0(\eta)+c_2\psi_0(\eta)\right)\,\mathrm{d}\eta,$$

$$\psi_1(t) = (1-\alpha)\left(c_3\phi_0(t) + c_4\psi_0(t)\right) + \alpha \int_0^t \left(c_3\phi_0(\eta) + c_4\psi_0(\eta)\right)\, \mathrm{d}\eta,$$

$$\phi_2(t) = (1-\alpha)\left(c_1\phi_1(t) + c_2\psi_1(t)\right) + \alpha \int_0^t \left(c_1\phi_1(\eta) + c_2\psi_1(\eta)\right)\, \mathrm{d}\eta,$$

$$\psi_2(t) = (1-\alpha)\left(c_3\phi_1(t) + c_4\psi_1(t)\right) + \alpha \int_0^t \left(c_3\phi_1(\eta) + c_4\psi_1(\eta)\right)\, \mathrm{d}\eta,$$

$$\phi_3(t) = (1-\alpha)\left(c_1\phi_2(t) + c_2\psi_2(t)\right) + \alpha \int_0^t \left(c_1\phi_2(\eta) + c_2\psi_2(\eta)\right)\, \mathrm{d}\eta,$$

$$\psi_3(t) = (1-\alpha)\left(c_3\phi_2(t) + c_4\psi_2(t)\right) + \alpha \int_0^t \left(c_3\phi_2(\eta) + c_4\psi_2(\eta)\right)\, \mathrm{d}\eta,$$

and so on.

In accordance with the previously described scheme for determining the components $\phi_0(t)$, $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t), \cdots$ and $\psi_0(t)$, $\psi_1(t)$, $\psi_2(t)$, $\psi_3(t), \cdots$ of the solution $\phi(t)$ and $\psi(t)$ of equation(3.2) respectively, we can depict them in a recursive manner as follows:

$$\phi_0(t) = K(t),$$

$$\phi_{i+1}(t) = (1-\alpha)(c_1\phi_i(t) + c_2\psi_i(t)) + \alpha \int_0^t \left(c_1\phi_i(\eta) + c_2\psi_i(\eta)\right)\, \mathrm{d}\eta, \quad i \geqslant 0,$$

and

$$\psi_0(t) = G(t),$$

$$\psi_{i+1}(t) = (1-\alpha)(c_3\phi_i(t) + c_4\psi_i(t))\alpha \int_0^t \left(c_3\phi_i(\eta) + c_4\psi_i(\eta)\right)\, \mathrm{d}\eta, \quad i \geqslant 0.$$

## 3.5 Illustrating examples

In this part, we use the procedure given in last section to find the numerical solution of linear integral equations. These results are then compared to exact solutions, by using MATLAB to solve the following examples.

**Example 3.1.** *We then consider the linear fractional differential where $\mu < 1$:*

$$\begin{cases} \mathscr{D}_0^{\frac{3}{4}}\phi(t) = -\frac{1}{5}\phi(t) - \frac{4}{5}\psi(t) + 6\left(t - \frac{8}{5}\right)\sin(t) - 2\left(t + \frac{7}{5}\right)\cos(t) \\ \qquad -\frac{58}{15}e^{-3t} + \frac{20}{3}, \ t \in [0,1] \\ \mathscr{D}_0^{\frac{3}{4}}\psi(t) = \frac{1}{8}\phi(t) - \frac{1}{7}\psi(t) + 2\left(t + \frac{12}{5}\right)\sin(t-1) + 6\left(t - \frac{3}{5}\right)\cos(t-1) \\ \qquad +\frac{6}{5}\left(4\sin(1) + 3\cos(1)\right)e^{-3t}, \quad t \in [0,1] \\ \phi(0) = 0, \ \psi(0) = 0. \end{cases} \tag{3.15}$$

*The exact solution of problem (3.15) is*

$$\phi(t) = 5(t-1)(1-\cos(t)) \quad and \quad \psi(t) = 5t\sin(t).$$

**Algorithm 3.1.** *The following algorithm is coded using MATLAB software. This algorithm is solving problem (3.15) numerically, and plot the absolute error between the exact value and approximated value.*

```
% This program which calculates the approximate solution and absolute ...
    error of the problem
% phi(t)=K(t)+(1-alpha)k1(t,phi(t),psi(t))+alpha\int(c,t) ...
    k1(eta,phi(eta),psi(eta))deta    t in [c,d]
% psi(t)=G(t)+(1-alpha)k2(t,phi(t),psi(t))+alpha\int(c,t) ...
    k2(eta,phi(eta),psi(eta))deta    t in [c,d]
%print('-dpsc','exampl2121.eps')
clc  ;  clear;
%%%%%%%%%%%%% variables %%%%%%%%%%%%%%%%%%%
a=0;
b=1;
alpha=0.75;
c1=-1/5;
c2=-4/5;
c3=1/8;
c4=-1/7;
N=7;
tic
```

```
%%%%%%%%%%%%%% functions %%%%%%%%%%%%%%%%%%
syms t eta;
phi(t)=5*(t-1)*(1-cos(t));
psi(t)=5*t*sin(t-1);
f(t)= 6*t*sin(t) - (14*cos(t))/5 - (48*sin(t))/5 - 2*t*cos(t) - ...
    (58*exp(-3*t))/15 + 20/3-c1*phi(t)-c2*psi(t);
g(t)= (24*sin(t - 1))/5 - (18*cos(t - 1))/5 + (24*exp(-3*t)*sin(1))/5 ...
    + 6*t*cos(t - 1) + 2*t*sin(t - 1) + ...
    (18*cos(1)*exp(-3*t))/5-c3*phi(t)-c4*psi(t);
K(t)=(1-alpha)*(f(t)-f(0))+alpha*int(f(eta),eta,0,t);
G(t)=(1-alpha)*(g(t)-g(0))+alpha*int(g(eta),eta,0,t);
phii(t)=K(t);
apphi(t)=K(t);
psii(t)=G(t);
appsi(t)=G(t);
for i=1:N
    i
    phiii(t)=vpa((1-alpha)*(c2*psii(t)+c1*phii(t)) ...
        +alpha*int(c1*phii(eta)+c2*psii(eta),eta,0,t));
    psiii(t)=vpa((1-alpha)*(c4*psii(t)+c3*phii(t)) ...
        +alpha*int(c3*phii(eta)+c4*psii(eta),eta,a,t));
  phii(t)=phiii(t);
  psii(t)=psiii(t);
 apphi(t)=phii(t)+apphi(t);
 appsi(t)=psii(t)+appsi(t);
end
A(t)=abs(apphi(t)-phi(t));
B(t)=abs(appsi(t)-psi(t));
hold off
subplot(2,2,1), fplot(phi(t),[a b],'Linewidth',2);
hold on
fplot(apphi(t),[a b],'--or');
title(' Exact and approximate solution at N=7 ')
xlabel('t','FontSize',10)
ylabel('\phi(t)','FontSize',10)
legend('Exact solution','approximate solution')
subplot(2,2,2), fplot(psi(t),[a b],'Linewidth',2);
```

```
hold on
fplot(appsi(t),[a b],'--or');
title(' Exact and approximate solution at N=7')
xlabel('t','FontSize',10)
ylabel('\psi(t)','FontSize',10)
legend('Exact solution','approximate solution')
subplot(2,2,[3,4]), fplot(A(t),[a b]);
hold on
subplot(2,2,[3,4]), fplot(B(t),[a b]);
xlabel('t','FontSize',10)
ylabel('Absolute error','FontSize',10)
title(' The Absolute Error with N = 7')
legend('Absolute error of \phi','Absolute error of \psi')
```

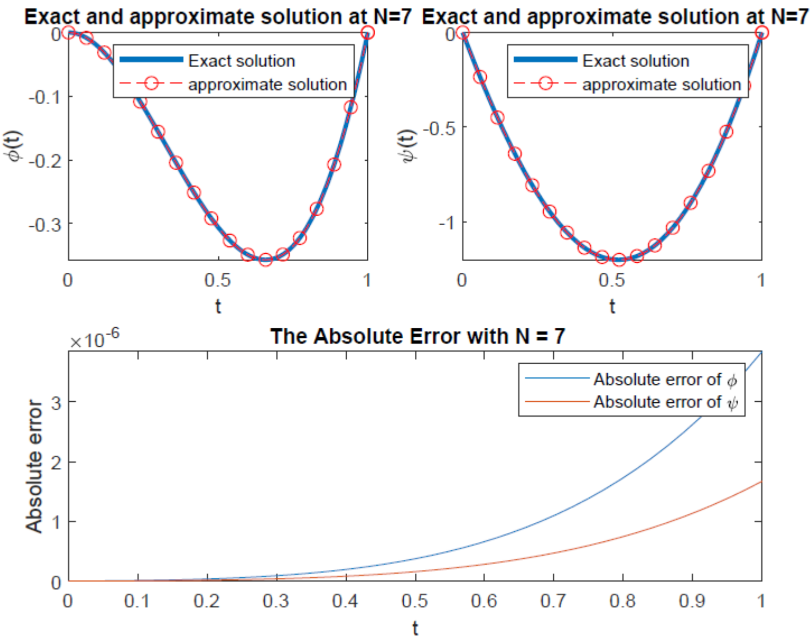


Figure 3.1: A comparison between the exact and approximate solution and the absolute error in Example 3.1.

**Example 3.2.** *Consider the linear fractional differential problem:*

$$\begin{cases} \mathscr{D}_0^{\frac{1}{2}}\phi(t) = -\frac{1}{3}\phi(t) + \frac{2}{3}\psi(t) + ch(t) + \frac{1}{3}\left((2t+4)e^t - t - 7\right), & t \in [0,1] \\ \mathscr{D}_0^{\frac{1}{2}}\psi(t) = -\frac{1}{4}\phi(t) + \frac{1}{5}\psi(t) + sh(t) + \frac{(e^t-1)(t+1)}{4} + \frac{4te^t}{5}, & t \in [0,1] \\ \phi(0) = 0, \ \psi(0) = 0. \end{cases} \tag{3.16}$$

*The exact solution of problem (3.16) is*

$$\phi(t) = (t+1)(e^t - 1) \quad and \quad \psi(t) = te^t$$

**Algorithm 3.2.** *The following algorithm is coded using MATLAB software. This algorithm is solving problem (3.16) numerically, and plot the approximate solution resulting and absolute error between the exact value and approximated value.*

```
% This program which calculates the approximate solution and absolute ...
    error of the problem
% phi(t)=K(t)+(1-alpha)k1(t,phi(t),psi(t)) ...
    +alpha\int(c,t)k1(eta,phi(eta),psi(eta))deta    t in [a,b]
% psi(t)=G(t)+(1-alpha)k2(t,phi(t),psi(t)) ...
    +alpha\int(c,t)k2(eta,phi(eta),psi(eta))deta    t in [a,b]
% print('-dpsc','Exampleart221.eps')
clc  ;  clear;
%%%%%%%%%%%%% variables %%%%%%%%%%%%%%%%%%%
a=0;
b=1;
alpha=0.5;
c1=-1/3;
c2=2/3;
c3=-1/4;
c4=1/5;
N=8;
tic
%%%%%%%%%%%%%% functions %%%%%%%%%%%%%%%%%%
syms t eta;
phi(t)=(t+1)*(exp(t)-1);
psi(t)=t*exp(t);
```

```
f(t)= exp(-t)/2 + (3*exp(t))/2 + t*exp(t) - 2-c1*phi(t)-c2*psi(t);
g(t)= exp(t)/2 - exp(-t)/2 + t*exp(t)-c3*phi(t)-c4*psi(t);
K(t)=(1-alpha)*(f(t)-f(0))+alpha*int(f(eta),eta,0,t);
G(t)=(1-alpha)*(g(t)-g(0))+alpha*int(g(eta),eta,0,t);
phii(t)=K(t);
apphi(t)=K(t);
psii(t)=G(t);
appsi(t)=G(t);
for i=1:N
    i
    phiii(t)=vpa((1-alpha)*(c2*psii(t)+c1*phii(t)) ...
        +alpha*int(c1*phii(eta)+c2*psii(eta),eta,0,t));
    psiii(t)=vpa((1-alpha)*(c4*psii(t)+c3*phii(t)) ...
        +alpha*int(c3*phii(eta)+c4*psii(eta),eta,a,t));
  phii(t)=phiii(t);
  psii(t)=psiii(t);
 apphi(t)=phii(t)+apphi(t);
 appsi(t)=psii(t)+appsi(t);
end
A(t)=abs(apphi(t)-phi(t));
B(t)=abs(appsi(t)-psi(t));
hold off
subplot(2,2,1), fplot(phi(t),[a b],'Linewidth',2);
hold on
fplot(apphi(t),[a b],'--or');
title(' Exact and approximate solution at N=8 ')
xlabel('t','FontSize',10)
ylabel('\phi(t)','FontSize',10)
legend('Exact solution','approximate solution')
subplot(2,2,2), fplot(psi(t),[a b],'Linewidth',2);
hold on
fplot(appsi(t),[a b],'--or');
title(' Exact and approximate solution at N=8')
xlabel('t','FontSize',10)
ylabel('\psi(t)','FontSize',10)
legend('Exact solution','approximate solution')
subplot(2,2,[3,4]), fplot(A(t),[a b]);
```

```
hold on
subplot(2,2,[3,4]), fplot(B(t),[a b]);
xlabel('t','FontSize',10)
ylabel('Absolute error','FontSize',10)
title(' The Absolute Error with N = 8')
legend('Absolute error of \phi','Absolute error of \psi')
```

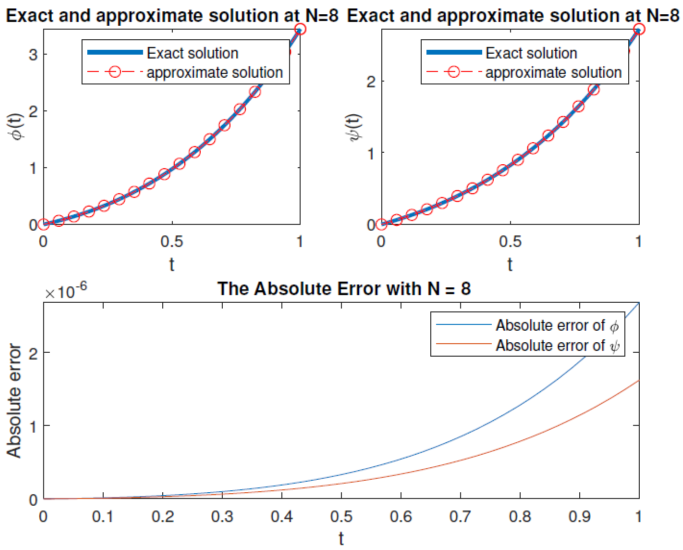


Figure 3.2: A comparison between the exact and approximate solution and the absolute error in Example 3.2.

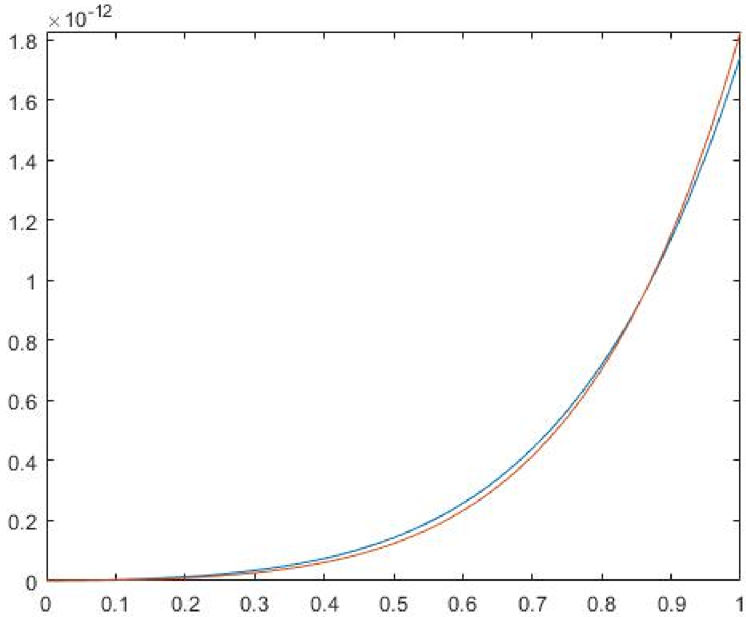


Figure 3.3: The absolute error of Example 3.2 with $N = 16$.

# Chapter 4

# Analytical and numerical study of a linear coupled system involving Caputo-Fabrizio fractional derivative with boundary conditions

## 4.1 Introduction

In this chapter, a coupled system of linear fractional differential equations with boundary conditions of the Caputo-Fabrizio type conformable fractional derivation is discussed. The problem is converted into an equivalent linear Volterra-Fredholm integral equation of the second kind in order to demonstrate the existence and uniqueness of the solution. Then, by applying Banach's fixed point theory, the existence and uniqueness of the solution are established. Lastly, numerical findings are provided to illustrate the acquired results and support the analytical results.
This work aims to present a state-of-the-art that may be used as a foundation to become acquainted with the Caputo-Fabrizio type couple system of linear fractional order. Only the latest developments are covered in this review.

The coupled system of fractional differential equations is an interesting new field of research. It has been demonstrated to be more realistic and accurate, and it has several applications in practical problems as both of these [10, 11].
We examine a system of coupled linear fractional differential equations in the following:

$$\begin{cases} {}^{CF}\mathscr{D}_0^{\gamma}\,\phi(t) = c_1\phi(t) + c_2\psi(t) + h(t), & t \in \Lambda := [0,1] \\ {}^{CF}\mathscr{D}_0^{\gamma}\,\psi(t) = c_3\phi(t) + c_4\psi(t) + g(t), & t \in \Lambda := [0,1] \\ \phi(0) = \phi(1) = 0,\ \psi(0) = \psi(1) = 0, \end{cases} \tag{4.1}$$

where $1 < \gamma < 2$ is a real number, the Caputo-Fabrizio fractional derivative is denoted as ${}^{CF}\mathscr{D}_0^{\gamma}$, $h, g : \Lambda \to \mathbb{R}$ the both functions are continuous, and $c_i$ real constants $(i = 1,2,3,4)$.

## 4.2 Analytic study

Below, we assume the function $N(\alpha) = 1$.

**Lemma 4.1.** *Let $1 < \gamma < 2$, $\phi, \psi \in \mathcal{C}^1[0,1]$, $h, g : [0,1] \to \mathbb{R}$ the both functions are continuous, and $c_i$ real constants $(i = 1,2,3,4)$. Following that, the solution of the system (4.1) satisfies the following linear Volterra integral equations of the second kind*

$$\phi(t) = \int_0^t L(t,\eta)(c_1\phi(\eta) + c_2\psi(\eta))\,\mathrm{d}\eta + \int_0^1 F(t,\eta)(c_1\phi(\eta) + c_2\psi(\eta))\,\mathrm{d}\eta + K(t) \tag{4.2}$$

$$\psi(t) = \int_0^t L(t,\eta)(c_3\phi(\eta) + c_4\psi(\eta))\,\mathrm{d}\eta + \int_0^1 F(t,\eta)(c_3\phi(\eta) + c_4\psi(\eta))\,\mathrm{d}\eta + G(t). \tag{4.3}$$

*where* $L(t,\eta) = \alpha(t-\eta) + 1 - \alpha,\ F(t,\eta) = t(\alpha\eta - 1)$,
$K(t) = \int_0^t (\alpha(t-\eta) + 1 - \alpha))h(\eta)\,\mathrm{d}\eta + \int_0^1 t(\alpha\eta - 1)h(\eta)\,\mathrm{d}\eta$, *and*
$G(t) = \int_0^t (\alpha(t-\eta) + 1 - \alpha))g(\eta)\,\mathrm{d}\eta + \int_0^1 t(\alpha\eta - 1)g(\eta)\,\mathrm{d}\eta$,

*Proof.* Lemma 1.2 may be applied to reduce Eq. (4.1) to a corresponding integral equations

$$I_0^{\gamma}\left({}^{CF}\mathscr{D}_0^{\gamma}\,\phi(t)\right) = I_0^{\gamma}\left(c_1\phi(t) + c_2\psi(t) + h(t)\right),$$

$$I_0^{\gamma}\left({}^{CF}\mathscr{D}_0^{\gamma}\,\psi(t)\right) = I_0^{\gamma}\left(c_3\phi(t) + c_4\psi(t) + g(t)\right),$$

we obtain

$$\phi(t) + a_1 t + b_1 = \int_0^t (\alpha(t-\eta) + 1 - \alpha)(c_1\phi(\eta) + c_2\psi(\eta) + h(\eta))\,\mathrm{d}\eta, \tag{4.4}$$

$$\psi(t) + a_2 t + b_2 = \int_0^t (\alpha(t-\eta) + 1 - \alpha)(c_3\phi(\eta) + c_4\psi(\eta) + g(\eta))\,\mathrm{d}\eta, \tag{4.5}$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

Using conditions $\phi(0) = 0$ and $\psi(0) = 0$ of the problem (4.1) in (4.4) and (4.5) respectively yields, we obtain $b_1 = b_2 = 0$.

Thus (4.4) and (4.5) reduces to

$$\phi(t) + a_1 t = \int_0^t (\alpha(t-\eta) + 1 - \alpha)(c_1\phi(\eta) + c_2\psi(\eta) + h(\eta))\,\mathrm{d}\eta, \tag{4.6}$$

$$\psi(t) + a_2 t = \int_0^t (\alpha(t-\eta) + 1 - \alpha)(c_3\phi(\eta) + c_4\psi(\eta) + g(\eta))\,\mathrm{d}\eta. \tag{4.7}$$

Considering boundary conditions $\phi(1) = 0$ and $\psi(1) = 0$, we get

$$a_1 = \int_0^1 (1 - \alpha\eta)\left(c_1\phi(\eta) + c_2\psi(\eta) + h(\eta)\right)\,\mathrm{d}\eta,$$

and

$$a_2 = \int_0^1 (1 - \alpha\eta)\left(c_3\phi(\eta) + c_4\psi(\eta) + g(\eta)\right)\,\mathrm{d}\eta.$$

Substituting the values of $a_1$ and $a_2$ in (4.6) and (4.7), leads to the next solution

$$\begin{aligned}\phi(t) = &\int_0^t (\alpha(t-\eta) + 1 - \alpha)(c_1\phi(\eta) + c_2\psi(\eta) + h(\eta))\,\mathrm{d}\eta \\ &+ \int_0^1 t(\alpha\eta - 1)(c_1\phi(\eta) + c_2\psi(\eta) + h(\eta))\,\mathrm{d}\eta,\end{aligned}$$

$$\begin{aligned}\psi(t) = &\int_0^t (\alpha(t-\eta) + 1 - \alpha)(c_3\phi(\eta) + c_4\psi(\eta) + g(\eta))\,\mathrm{d}\eta \\ &+ \int_0^1 t(\alpha\eta - 1)(c_3\phi(\eta) + c_4\psi(\eta) + g(\eta))\,\mathrm{d}\eta.\end{aligned}$$

So

$$\begin{aligned}\phi(t) = &\int_0^t (\alpha(t-\eta) + 1 - \alpha)(c_1\phi(\eta) + c_2\psi(\eta))\,\mathrm{d}\eta + \int_0^1 t(\alpha\eta - 1)(c_1\phi(\eta) + c_2\psi(\eta))\,\mathrm{d}\eta \\ &+ \int_0^t (\alpha(t-\eta) + 1 - \alpha)h(\eta)\,\mathrm{d}\eta + \int_0^1 t(\alpha\eta - 1)h(\eta)\,\mathrm{d}\eta,\end{aligned}$$

$$\psi(t) = \int_0^t (\alpha(t-\eta)+1-\alpha)(c_3\phi(\eta)+c_4\psi(\eta))\,\mathrm{d}\eta + \int_0^1 t(\alpha\eta-1)(c_3\phi(\eta)+c_4\psi(\eta))\,\mathrm{d}\eta$$
$$+ \int_0^t (\alpha(t-\eta)+1-\alpha)g(\eta)\,\mathrm{d}\eta + \int_0^1 t(\alpha\eta-1)g(\eta)\,\mathrm{d}\eta.$$

Hence, the unique solution of problem (4.1) is

$$\phi(t) = \int_0^t L(t,\eta)(c_1\phi(\eta)+c_2\psi(\eta))\,\mathrm{d}\eta + \int_0^1 F(t,\eta)(c_1\phi(\eta)+c_2\psi(\eta))\,\mathrm{d}\eta + K(t),$$

$$\psi(t) = \int_0^t L(t,\eta)(c_3\phi(\eta)+c_4\psi(\eta))\,\mathrm{d}\eta + \int_0^1 F(t,\eta)(c_3\phi(\eta)+c_4\psi(\eta))\,\mathrm{d}\eta + G(t).$$

The proof is complete. □

## 4.3 Existence and uniqueness theorem

Let us define the space $\mathcal{C}([0,1],\mathbb{R})$ with the norm $\| \phi \| = \sup_{t\in[0,1]} |\phi(t)|$.
Clearly, $(\mathcal{C}([0,1],\mathbb{R}), \| \, . \, \|)$ is a Banach space.
Indicate with $\nu = \mathcal{C}([0,1],\mathbb{R}) \times \mathcal{C}([0,1],\mathbb{R})$. Then, the product space $(\nu, \| \, . \, \|)$ is actually a Banach space presented with the norm $\| (\phi,\psi) \| = \| \phi \| + \| \psi \| = \sup_{t\in[0,1]} |\phi(t)| + \sup_{t\in[0,1]} |\psi(t)|$, for $(\phi,\psi) \in \nu$.
Based on Lemma 4.1, we define an operator $T : \nu \longrightarrow \nu$ for the problem (4.1) as follows:

$$T(\phi,\psi)(t) := (T_1(\phi,\psi)(t), T_2(\phi,\psi)(t)), \tag{4.8}$$

$$T_1(\phi,\psi)(t) = \int_0^t L(t,\eta)(c_1\phi(\eta)+c_2\psi(\eta))\,\mathrm{d}\eta + \int_0^1 F(t,\eta)(c_1\phi(\eta)+c_2\psi(\eta))\,\mathrm{d}\eta + K(t), \tag{4.9}$$

and

$$T_2(\phi,\psi)(t) = \int_0^t L(t,\eta)(c_3\phi(\eta)+c_4\psi(\eta))\,\mathrm{d}\eta + \int_0^1 F(t,\eta)(c_3\phi(\eta)+c_4\psi(\eta))\,\mathrm{d}\eta + G(t). \tag{4.10}$$

Here we establish the existence of the solutions for the boundary value problem (4.1) by using Banach's contraction mapping principle.

**Theorem 4.1.** *Let $h, g : \Lambda \to \mathbb{R}$ are jointly continuous function. Hence, there exists a unique solution to the problem (4.1) on $\Lambda$ if*

$$\lambda + \varepsilon < \frac{1}{2-\alpha}. \tag{4.11}$$

*where* $\lambda = \max(|c_1|, |c_2|),\ \varepsilon = \max(|c_3|, |c_4|)$,

*Proof.* As $h$ and $g$ are continuous functions in $\Lambda = [0,1]$ is compact then $h$ and $g$ are bounded, so exist real constants $m_1 > 0,\ m_2 > 0$, such that $|h(t)| \leqslant m_1$ and $|g(t)| \leqslant m_2$ for all $t \in \Lambda$.

Considering that the operator $T : \nu \longrightarrow \nu$ as defined by (4.8) and fix

$$k > \frac{(2-\alpha)(m_1+m_2)}{1-(\lambda+\varepsilon)(2-\alpha)}.$$

Then we show that $TB_k \subset B_k$, where $B_k = \{(\phi,\psi) \in \nu : \| (\phi,\psi) \| \leqslant k\}$.

By our assumption, for $(\phi,\psi) \in B_k,\ t \in \Lambda$, we have

$$\begin{aligned}
|T_1(\phi,\psi)(t)| \leqslant & \int_0^t (\alpha(t-\eta)+1-\alpha)|c_1\phi(\eta)+c_2\psi(\eta)|\,\mathrm{d}\eta \\
& + \int_0^1 t|\alpha\eta-1||c_1\phi(\eta)+c_2\psi(\eta)|\,\mathrm{d}\eta + \int_0^t (\alpha(t-\eta)+1-\alpha)|h(\eta)|\,\mathrm{d}\eta \\
& + \int_0^1 t|\alpha\eta-1||h(\eta)|\,\mathrm{d}\eta \\
& \leqslant \max(|c_1|,|c_2|)(|\phi|+|\psi|)\left(\int_0^t (\alpha(t-\eta)+1-\alpha)\,\mathrm{d}\eta + \int_0^1 t(1-\alpha\eta)\,\mathrm{d}\eta\right) \\
& + m_1\left(\int_0^t (\alpha(t-\eta)+1-\alpha)\,\mathrm{d}\eta + \int_0^1 t(1-\alpha\eta)\,\mathrm{d}\eta\right) \\
& \leqslant (\lambda(|\phi|+|\psi|)+m_1)\left(\int_0^t (\alpha(t-\eta)+1-\alpha)\,\mathrm{d}\eta + \int_0^1 t(1-\alpha\eta)\,\mathrm{d}\eta\right) \\
& \leqslant (2-\alpha)(\lambda(|\phi|+|\psi|)+m_1).
\end{aligned}$$

This, when considered the norm for $t \in \Lambda$, results in

$$\| T_1(\phi,\psi)(t) \| \leqslant (2-\alpha)(\lambda(\| \phi \| + \| \psi \|)+m_1).$$

In the same way, for $(\phi,\psi) \in B_k$, one can obtain

$$\| T_2(\phi,\psi)(t) \| \leqslant (2-\alpha)(\varepsilon(\| \phi \| + \| \psi \|)+m_2).$$

Therefore, for any $(\phi,\psi) \in B_k$, we have

$$\begin{aligned}
\| T(\phi,\psi)(t) \| = & \| T_1(\phi,\psi)(t) \| + \| T_2(\phi,\psi)(t) \| \\
& \leqslant (2-\alpha)((\lambda+\varepsilon)(\| \phi \| + \| \psi \|)+m_1+m_2) \\
& \leqslant (2-\alpha)((\lambda+\varepsilon)k+m_1+m_2) \\
& < k,
\end{aligned}$$

which show that $T$ maps $B_k$ into itself.

Let $(\phi_1,\psi_1),\ (\phi_2,\psi_2)\in\nu,\ \forall t\in\Lambda$ to prove that the operator $T$ is a contraction. After that, we get

$$\begin{aligned}
&|T_1(\phi_1,\psi_1)(t)-T_1(\phi_2,\psi_2)(t)|\\
&\quad\leqslant\int_0^t(\alpha(t-\eta)+1-\alpha)|c_1(\phi_1(\eta)-\phi_2(\eta))+c_2(\psi_1(\eta)-\psi_2(\eta))|\,\mathrm{d}\eta\\
&\quad+\int_0^1 t|\alpha\eta-1||c_1(\phi_1(\eta)-\phi_2(\eta))+c_2(\psi_1(\eta)-\psi_2(\eta))|\,\mathrm{d}\eta\\
&\quad\leqslant\int_0^t(\alpha(t-\eta)+1-\alpha)(|c_1(\phi_1(\eta)-\phi_2(\eta))|+|c_2(\psi_1(\eta)-\psi_2(\eta))|)\,\mathrm{d}\eta\\
&\quad+\int_0^1 t|\alpha\eta-1|(|c_1(\phi_1(\eta)-\phi_2(\eta))|+|c_2(\psi_1(\eta)-\psi_2(\eta))|)\,\mathrm{d}\eta\\
&\quad\leqslant\max(|c_1|,|c_2|)(|\phi_1(t)-\phi_2(t)|+|\psi_1(t)-\psi_2(t)|)\left(\int_0^t(\alpha(t-\eta)+1-\alpha)\,\mathrm{d}\eta\right.\\
&\quad\left.+\int_0^1 t(1-\alpha\eta)\,\mathrm{d}\eta\right)\\
&\quad\leqslant\lambda(|\phi_1(t)-\phi_2(t)|+|\psi_1(t)-\psi_2(t)|)\left(\int_0^t(\alpha(t-\eta)+1-\alpha)\,\mathrm{d}\eta+\int_0^1 t(1-\alpha\eta)\,\mathrm{d}\eta\right)\\
&\quad\leqslant(2-\alpha)\lambda(|\phi_1(t)-\phi_2(t)|+|\psi_1(t)-\psi_2(t)|).
\end{aligned}$$

Thus, when considered the norm for $t\in\Lambda$, leads to

$$\parallel T_1(\phi_1,\psi_1)-T_1(\phi_2,\psi_2)\parallel\leqslant(2-\alpha)\lambda(\parallel\phi_1-\phi_2\parallel+\parallel\psi_1-\psi_2\parallel).$$

And likewise, for $(\phi_1,\psi_1)\ (\phi_2,\psi_2)\in\nu,\ t\in\Lambda$, one can obtain

$$\parallel T_2(\phi_1,\psi_1)-T_2(\phi_2,\psi_2)\parallel\leqslant(2-\alpha)\varepsilon(\parallel\phi_1-\phi_2\parallel+\parallel\psi_1-\psi_2\parallel).$$

From the previous inequalities, it is evident that

$$\begin{aligned}
\parallel T(\phi_1,\psi_1)-T(\phi_2,\psi_2)\parallel&=\parallel T_1(\phi_1,\psi_1)-T_1(\phi_2,\psi_2)\parallel+\parallel T_2(\phi_1,\psi_1)-T_2(\phi_2,\psi_2)\parallel\\
&\leqslant(2-\alpha)(\lambda+\varepsilon)(\parallel\phi_1-\phi_2\parallel+\parallel\psi_1-\psi_2\parallel)\\
&=(2-\alpha)(\lambda+\varepsilon)\parallel(\phi_1,\psi_1)-(\phi_2,\psi_2)\parallel\\
&=N\parallel(\phi_1-\phi_2,\psi_1-\psi_2)\parallel.
\end{aligned}$$

Since $N=(2-\alpha)(\lambda+\varepsilon)<1$.

Which means that $T$ is a contraction mapping when considering (4.11). Therefore, the operator $T$ has only a single fixed point according to Banach's fixed-point theorem. Thus, the problem (4.1) has a unique solution on $\Lambda$.

The proof is completed. $\square$

## 4.4 Numerical study

This section presents an algorithm based on the Adomian Decomposition for the numerical solution of linear Volterra integral equations of the second kind.

The solutions $\phi(t)$ and $\psi(t)$ of the integral equations 4.2 and 4.3 are usually expressed in a series form using the decomposition method, which is defined as

$$\phi(t) = \sum_{i=0}^{\infty} \phi_i(t) \quad \text{and } \psi(t) = \sum_{i=0}^{\infty} \psi_i(t).$$

Substituting the decomposition (4.4) into both sides of 2.3 and 4.3 yields

$$\begin{aligned}\sum_{i=0}^{\infty} \phi_i(t) = K(t) &+ \int_0^t L(t,\eta)\left(c_1 \sum_{i=0}^{\infty} \phi_i(\eta) + c_2 \sum_{i=0}^{\infty} \psi_i(\eta)\right) \mathrm{d}\eta \\ &+ \int_0^1 F(t,\eta)\left(c_1 \sum_{i=0}^{\infty} \phi_i(\eta) + c_2 \sum_{i=0}^{\infty} \psi_i(\eta)\right) \mathrm{d}\eta,\end{aligned}$$

and

$$\begin{aligned}\sum_{i=0}^{\infty} \psi_i(t) = G(t) &+ \int_0^t L(t,\eta)\left(c_3 \sum_{i=0}^{\infty} \phi_i(\eta) + c_4 \sum_{i=0}^{\infty} \psi_i(\eta)\right) \mathrm{d}\eta \\ &+ \int_0^1 F(t,\eta)\left(c_3 \sum_{i=0}^{\infty} \phi_i(\eta) + c_4 \sum_{i=0}^{\infty} \psi_i(\eta)\right) \mathrm{d}\eta,\end{aligned}$$

or equivalently

$$\begin{aligned}\phi_0(t)+\phi_1(t)+\phi_2(t)+\cdots = K(t) &+ \int_0^t L(t,\eta)\left(c_1\phi_0(\eta)+c_2\psi_0(\eta)\right)\,\mathrm{d}\eta\\ &+ \int_0^1 F(t,\eta)\left(c_1\phi_0(\eta)+c_2\psi_0(\eta)\right)\,\mathrm{d}\eta\\ &+ \int_0^t L(t,\eta)\left(c_1\phi_1(\eta)+c_2\psi_1(\eta)\right)\,\mathrm{d}\eta\\ &+ \int_0^1 F(t,\eta)\left(c_1\phi_1(\eta)+c_2\psi_1(\eta)\right)\,\mathrm{d}\eta\\ &+ \int_0^t L(t,\eta)\left(c_1\phi_2(\eta)+c_2\psi_2(\eta)\right)\,\mathrm{d}\eta\\ &+ \int_0^1 F(t,\eta)\left(c_1\phi_2(\eta)+c_2\psi_2(\eta)\right)\,\mathrm{d}\eta+\cdots\end{aligned}$$

and

$$\begin{aligned}\psi_0(t)+\psi_1(t)+\psi_2(t)+\cdots = G(t) &+ \int_0^t L(t,\eta)\left(c_3\phi_0(\eta)+c_4\psi_0(\eta)\right)\,\mathrm{d}\eta\\ &+ \int_0^1 F(t,\eta)\left(c_3\phi_0(\eta)+c_4\psi_0(\eta)\right)\,\mathrm{d}\eta\\ &+ \int_0^t L(t,\eta)\left(c_3\phi_1(\eta)+c_4\psi_1(\eta)\right)\,\mathrm{d}\eta\\ &+ \int_0^1 F(t,\eta)\left(c_3\phi_1(\eta)+c_4\psi_1(\eta)\right)\,\mathrm{d}\eta\\ &+ \int_0^t L(t,\eta)\left(c_3\phi_2(\eta)+c_4\psi_2(\eta)\right)\,\mathrm{d}\eta\\ &+ \int_0^1 F(t,\eta)\left(c_3\phi_2(\eta)+c_4\psi_2(\eta)\right)\,\mathrm{d}\eta+\cdots\end{aligned}$$

If we set $\phi_0(t)=K(t), \quad \textit{and} \quad \psi_0(t)=G(t)$, then the components $\phi_0(t),\phi_1(t),\phi_2(t),\cdots$ and $\psi_0(t),\psi_1(t),\psi_2(t),\cdots$ of the unknown functions $\phi(t)$ and $\psi(t)$, respectively, are completely determined in a recurrent manner.

So

$$\begin{aligned}\phi_1(t) &= \int_0^t L(t,\eta)\left(c_1\phi_0(\eta)+c_2\psi_0(\eta)\right)+\int_0^1 F(t,\eta)\left(c_1\phi_0(\eta)+c_2\psi_0(\eta)\right)\,\mathrm{d}\eta,\\ \psi_1(t) &= \int_0^t L(t,\eta)\left(c_3\phi_0(\eta)+c_4\psi_0(\eta)\right)+\int_0^1 F(t,\eta)\left(c_3\phi_0(\eta)+c_4\psi_0(\eta)\right)\,\mathrm{d}\eta,\\ \phi_2(t) &= \int_0^t L(t,\eta)\left(c_1\phi_1(\eta)+c_2\psi_1(\eta)\right)+\int_0^1 F(t,\eta)\left(c_1\phi_1(\eta)+c_2\psi_1(\eta)\right)\,\mathrm{d}\eta,\end{aligned}$$

$$\psi_2(t) = \int_0^t L(t,\eta)\,(c_3\phi_1(\eta) + c_4\psi_1(\eta)) + \int_0^1 F(t,\eta)\,(c_3\phi_1(\eta) + c_4\psi_1(\eta))\;\mathrm{d}\eta,$$
$$\phi_3(t) = \int_0^t L(t,\eta)\,(c_1\phi_2(\eta) + c_2\psi_2(\eta)) + \int_0^1 F(t,\eta)\,(c_1\phi_2(\eta) + c_2\psi_2(\eta))\;\mathrm{d}\eta,$$
$$\psi_3(t) = \int_0^t L(t,\eta)\,(c_3\phi_2(\eta) + c_4\psi_2(\eta)) + \int_0^1 F(t,\eta)\,(c_3\phi_2(\eta) + c_4\psi_2(\eta))\;\mathrm{d}\eta,$$

and so on.

In order to determine the components $\phi_0(t), \phi_1(t),\ \phi_2(t), \cdots$ and $\psi_0(t), \psi_1(t),\ \psi_2(t), \cdots$ of the solution $\phi(t)$ and $\psi(t)$ of Eq.(4.2-4.3) respectively, the above-discussed scheme can be expressed recursively as follows:

$$\phi_0(t) = K(t),$$
$$\phi_{n+1}(t) = \int_0^t L(t,\eta)\,(c_1\phi_n(\eta) + c_2\psi_n(\eta)) + \int_0^1 F(t,\eta)\,(c_1\phi_n(\eta) + c_2\psi_n(\eta))\;\mathrm{d}\eta, \quad n \geqslant 0.$$

And

$$\psi_0(t) = G(t),$$
$$\psi_{n+1}(t) = \int_0^t L(t,\eta)\,(c_3\phi_n(\eta) + c_4\psi_n(\eta)) + \int_0^1 F(t,\eta)\,(c_3\phi_n(\eta) + c_4\psi_n(\eta))\;\mathrm{d}\eta, \quad n \geqslant 0.$$

## 4.5 Illustrating examples

In this section, we give two numerical examples to illustrate the above methods for solve the linear Volterra integral equations of the second kind.

The exact solution is known and used to justify the numerical solution obtained with our method is correct. We used MATLAB to solve these examples.

**Example 4.1.** *Consider the following linear fractional differential equation:*

$$\begin{cases} \mathscr{D}_0^{\frac{3}{2}}\phi(t) = \frac{1}{6}\phi(t) + \frac{1}{4}\psi(t) + 3\cos(t) - 3e^{-t} - \frac{\sin(t)(t-1)}{4} - \frac{(\cos(t)-1)(t-1)}{6} \\ \qquad -t\cos(t) - t\sin(t), \quad t \in \Lambda := [0,1] \\ \mathscr{D}_0^{\frac{3}{2}}\psi(t) = -\frac{1}{3}\phi(t) + -\frac{1}{5}\psi(t) + 3\sin(t) + \frac{\sin(t)(t-1)}{5} + \frac{(\cos(t)-1)(t-1)}{3} + t\cos(t) \\ \qquad -t\sin(t), \quad t \in \Lambda := [0,1] \\ \phi(0) = \phi(1) = 0, \;\; \psi(0) = \psi(1) = 0. \end{cases} \tag{4.12}$$

*The problem (4.12) has an exact solution, which is:*

$$\phi(t) = (t-1)(\cos(t)-1) \quad \text{and} \quad \psi(t) = (t-1)\sin(t).$$

**Algorithm 4.1.** *The following algorithm is coded using MATLAB software. This algorithm is solving problem (4.12) numerically, and plot the absolute error between the exact value and approximated value.*

```
% Solve Volterra System Integral Equations of the Second Kind
%       phi(x)=K(x)+int_a^t L(t,eta)(c_1phi(t)+c_2psi(t))deta +int_a^b ...
    F(t,eta)(c_1phi(t)+c_2psi(t))deta t in [a,b]
%       psi(x)=G(x)+int_a^t L(t,eta)(c_3phi(t)+c_4psi(t))deta +int_a^b ...
    F(t,eta)(c_1phi(t)+c_2psi(t))deta  t in [a,b]
%        by The Adomian Decomposition Method
%        phi_0(t)=K(t)
%        v_0(t)=G(t)
%        phi_n(t)=int_a^t L(t,eta)(c_1phi_(n-1)(eta)+c_2psi_(n-1)(eta))deta
%        +int_a^b F(t,eta)(c_1phi_(n-1)(eta)+c_2psi_(n-1)(eta))deta
%        psi_n(t)==int_a^t ...
    L(t,eta)(c_3phi_(n-1)(eta)+c_4psi_(n-1)(eta))deta
%        +int_a^b F(t,eta)(c_3phi_(n-1)(eta)+c_4psi_(n-1)(eta))deta
%        (phi,psi) Exact solution
% print('-dpsc','Exampleart31.eps')
 clc  ;  clear;
%%%%%%%%%%%%% variables %%%%%%%%%%%%%%%%%%%
a=0;
```

```
b=1;
alpha=0.5;
c1=1/6;
c2=1/4;
c3=-1/3;
c4=-1/5;
N=5;
tic
%%%%%%%%%%%%%% functions %%%%%%%%%%%%%%%%%%
syms t eta ;
phi(t)=(t-1)*(cos(t)-1);
psi(t)=sin(t)*(t-1);
f(t)=3*cos(t) - 3*exp(-eta) - t*cos(eta) - ...
    t*sin(eta)-c1*phi(eta)-c2*psi(eta);
g(t)=3*sin(t) + t*cos(eta) - t*sin(eta)-c3*phi(eta)-c4*psi(eta);
L(t,eta)=alpha*(t-eta)+1-alpha;
F(t,eta)=t*(alpha*eta-1);
K(t)=int(L(t,eta)*f(eta),eta,a,t)+int(F(t,eta)*f(eta),eta,a,1);
G(t)=int(L(t,eta)*g(eta),eta,a,t)+int(F(t,eta)*g(eta),eta,a,1);
phii(t)=K(t);
apphi(t)=K(t);
psii(t)=G(t);
appsi(t)=G(t);
for i=1:N
    i
    phiii(t)=int(L(t,eta)*(c1*phii(eta)+c2*psii(eta)),eta,a,t) ...
        +int(F(t,eta)*(c1*phii(eta)+c2*psii(eta)),eta,a,1);
    psiii(t)=int(L(t,eta)*(c3*phii(eta)+c4*psii(eta)),eta,a,t) ...
        +int(F(t,eta)*(c3*phii(eta)+c4*psii(eta)),eta,a,1);
 phii(t)=phiii(t);
 psii(t)=psiii(t);
 apphi(t)=phii(t)+apphi(t);
 appsi(t)=psii(t)+appsi(t);
end
A(t)=abs(apphi(t)-phi(t));
B(t)=abs(appsi(t)-psi(t));
hold off
```

```
subplot(2,2,1), fplot(phi(t),[a b],'Linewidth',2);
hold on
fplot(apphi(t),[a b],'--or');
title(' Exact and approximate solution at N=5 ')
xlabel('t','FontSize',10)
ylabel('\phi(t)','FontSize',10)
legend('Exact solution','approximate solution')
subplot(2,2,2), fplot(psi(t),[a b],'Linewidth',2);
hold on
fplot(appsi(t),[a b],'--or');
title(' Exact and approximate solution at N=5')
xlabel('t','FontSize',10)
ylabel('\psi(t)','FontSize',10)
legend('Exact solution','approximate solution')
subplot(2,2,[3,4]), fplot(A(t),[a b]);
hold on
subplot(2,2,[3,4]), fplot(B(t),[a b]);
xlabel('t','FontSize',10)
ylabel('Absolute error','FontSize',10)
title(' The Absolute Error with N = 5')
legend('Absolute error of \phi','Absolute error of \psi')
```

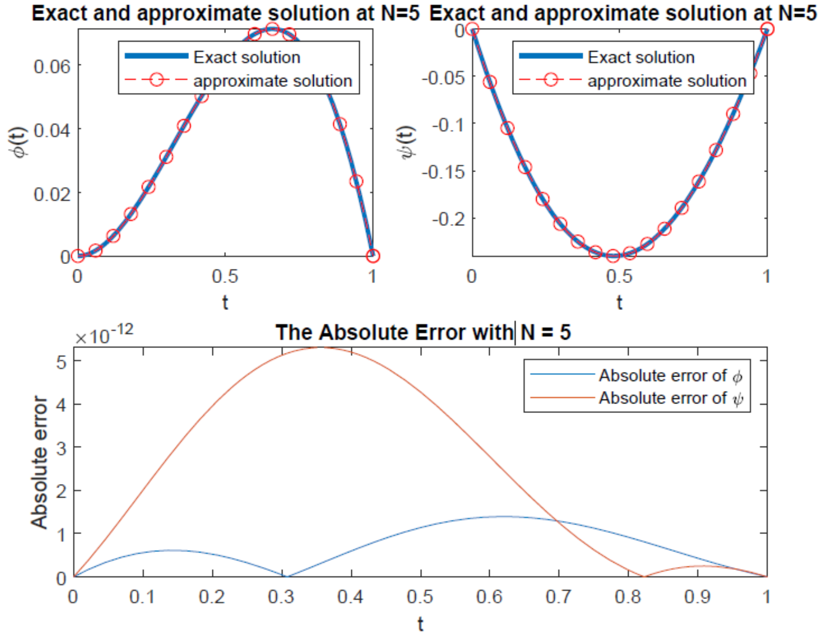


Figure 4.1: A comparison between the exact and approximate solution and the absolute error in Example 4.1.

**Example 4.2.** *Consider the following linear fractional differential equation:*

$$\begin{cases} \mathscr{D}_0^{\frac{7}{4}}\phi(t) = -\frac{1}{5}\phi(t) + \frac{1}{6}\psi(t) + 8 - 8e^{-3t} + \frac{3t(t-1)}{5} - \frac{t(e^t - e^1)}{42}, & t \in \Lambda := [0,1] \\ \mathscr{D}_0^{\frac{7}{4}}\psi(t) = \frac{1}{5}\phi(t) - \frac{1}{3}\psi(t) + \frac{e^t - e^{-3t}}{4} + \frac{t(4e^t - e^1)}{21} - \frac{3t(t-1)}{5}, & t \in \Lambda := [0,1] \\ \phi(0) = \phi(1) = 0, \ \psi(0) = \psi(1) = 0. \end{cases} \tag{4.13}$$

*The problem (4.13) has an exact solution, which is:*

$$\phi(t) = 3t(t-1) \quad \text{and} \quad \psi(t) = \frac{t}{7}(e^t - e^1).$$

**Algorithm 4.2.** *The following algorithm is coded using MATLAB software. This algorithm is solving problem (4.13) numerically, and plot the approximate solution resulting and absolute error between the exact value and approximated value.*

```
% Solve Volterra System Integral Equations of the Second Kind
%      phi(x)=K(x)+int_a^t L(t,eta)(c_1phi(t)+c_2psi(t))deta +int_a^b ... F(t,eta)(c_1phi(t)+c_2psi(t))deta t in [a,b]
%      psi(x)=G(x)+int_a^t L(t,eta)(c_3phi(t)+c_4psi(t))deta +int_a^b ... F(t,eta)(c_1phi(t)+c_2psi(t))deta  t in [a,b]
%       by The Adomian Decomposition Method
%       phi_0(t)=K(t)
%       v_0(t)=G(t)
%       phi_n(t)=int_a^t L(t,eta)(c_1phi_(n-1)(eta)+c_2psi_(n-1)(eta))deta
%       +int_a^b F(t,eta)(c_1phi_(n-1)(eta)+c_2psi_(n-1)(eta))deta
%       psi_n(t)==int_a^t ... L(t,eta)(c_3phi_(n-1)(eta)+c_4psi_(n-1)(eta))deta
%       +int_a^b F(t,eta)(c_3phi_(n-1)(eta)+c_4psi_(n-1)(eta))deta
%       (phi,psi) Exact solution
% print('-dpsc','Exampleart321.eps')
 clc  ;  clear;
%%%%%%%%%%%%% variables %%%%%%%%%%%%%%%%%%%
a=0;
b=1;
alpha=0.75;
c1=-1/5;
c2=1/6;
c3=1/5;
c4=-1/3;
N=7;
tic
%%%%%%%%%%%%%% functions %%%%%%%%%%%%%%%%%%
syms t eta;
phi(t)=3*t*(t-1);
psi(t)=t*(exp(t)-exp(1))/7;
f(t)=8 - 8*exp(-3*t)-c1*phi(t)-c2*psi(t);
g(t)=exp(t)/4 - exp(-3*t)/4 + (t*exp(t))/7-c3*phi(t)-c4*psi(t);
L(t,eta)=alpha*(t-eta)+1-alpha;
F(t,eta)=t*(alpha*eta-1);
K(t)=int(L(t,eta)*f(eta),eta,a,t)+int(F(t,eta)*f(eta),eta,a,1);
G(t)=int(L(t,eta)*g(eta),eta,a,t)+int(F(t,eta)*g(eta),eta,a,1);
phii(t)=K(t);
```

```
apphi(t)=K(t);
psii(t)=G(t);
appsi(t)=G(t);
for i=1:N
    i
    phiii(t)=int(L(t,eta)*(c1*phii(eta)+c2*psii(eta)),eta,a,t) ...
        +int(F(t,eta)*(c1*phii(eta)+c2*psii(eta)),eta,a,1);
    psiii(t)=int(L(t,eta)*(c3*phii(eta)+c4*psii(eta)),eta,a,t) ...
        +int(F(t,eta)*(c3*phii(eta)+c4*psii(eta)),eta,a,1);
 phii(t)=phiii(t);
 psii(t)=psiii(t);
 apphi(t)=phii(t)+apphi(t);
 appsi(t)=psii(t)+appsi(t);
end
A(t)=abs(apphi(t)-phi(t));
B(t)=abs(appsi(t)-psi(t));
hold off
subplot(2,2,1), fplot(phi(t),[a b],'Linewidth',2);
hold on
fplot(apphi(t),[a b],'--or');
title(' Exact and approximate solution at N=7 ')
xlabel('t','FontSize',10)
ylabel('\phi(t)','FontSize',10)
legend('Exact solution','approximate solution')
subplot(2,2,2), fplot(psi(t),[a b],'Linewidth',2);
hold on
fplot(appsi(t),[a b],'--or');
title(' Exact and approximate solution at N=7')
xlabel('t','FontSize',10)
ylabel('\psi(t)','FontSize',10)
legend('Exact solution','approximate solution')
subplot(2,2,[3,4]), fplot(A(t),[a b]);
hold on
subplot(2,2,[3,4]), fplot(B(t),[a b]);
xlabel('t','FontSize',10)
ylabel('Absolute error','FontSize',10)
title(' The Absolute Error with N = 7')
```

```
legend('Absolute error of \phi','Absolute error of \psi')
```

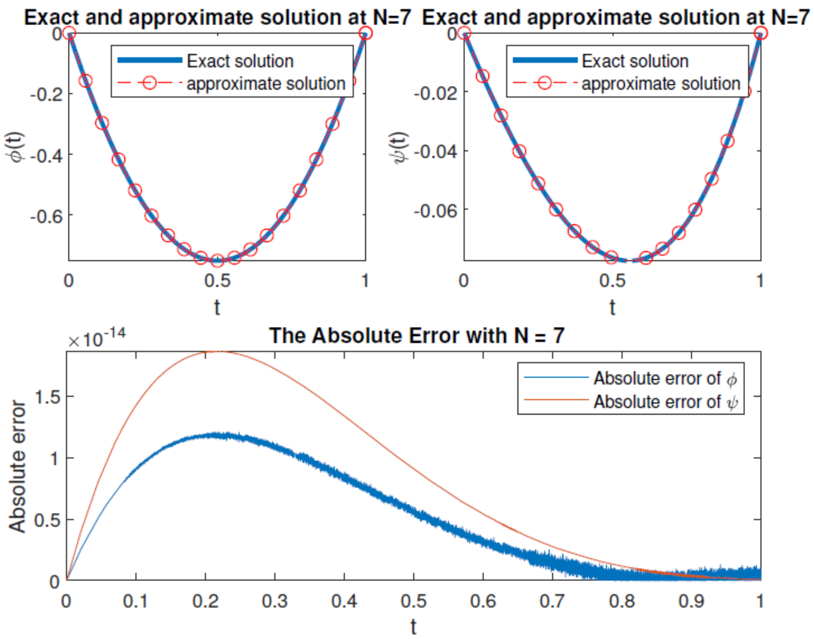


Figure 4.2: A comparison between the exact and approximate solution and the absolute error in Example 4.2.

# Conclusion and perspective

At the end of this thesis, we believe that the results presented will contribute to the development of the study of fractional differential equations, by opening new horizons to scientific research on this emerging theme.

After having presented the preliminary notions useful for the good understanding of the present work, we presented results of existence and uniqueness of certain differential problems of fractional orders relating to the Caputo derivative in Banach spaces. First, we established global existence and uniqueness results of the solution of three problems of Caputo-Fabrizio fractional differential equation. These results were obtained by the application of the fixed point Theory, in particular we used the Picard method, Picard- Lindolf method and the Banach fixed point Theorem.

Other results on convergence of the obtained solution are also justified in order to establish that the formal solutions are analytical solution.

For further research, we can study more general problems. such as studying problems involving fractional partial differential equations with Caputo-Fabrizio fractional derivative. We can also extend the considered nonlinear problems.

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# Annex

## 4.6 MATLAB Operators and Special Characters

Some common commands are listed in this section a full specification of each can be obtained using the help system.

| | |
|---|---|
| $+$ | Addition. |
| $-$ | Subtraction. |
| $*$ | Multiplication. |
| $/$ | Division. |
| **abs** | Absolute value. |
| **sqrt** | Square root function. |
| **clc** | Clear command window. |
| **clear** | Clear variables and functions from memory. |
| $\wedge$ | Exponentiation. |
| **pi** | The mathematical constant $\pi$. |
| $'$ | Transpose. |
| % | To comment out one line in a multiline command. |

| | |
|---|---|
| **clc** | Clean command window. After this function, all previous command written on window will be cleaned. |
| **close all** | Closes all figures, and window. |
| **clear all** | This build in function clear all variable created in work space of matlab. |
| **syms** | Create symbolic variables and functions. |
| **diff(g)** | Differentiates $g$ with respect to $x$, and two are several forms:<br>**diff(g(t))** $= g'(t)$,<br>**diff(g(t),n)** $= g^{(n)}(t)$. |
| **int** | Integrate, and two are several forms:<br>**int(g,x,a,t)** $= \int_a^t g(x)\,\mathrm{d}x$,<br>**int(g,omega)** $= \int_\Omega g(x)\,\mathrm{d}x$. |
| **Da(g,alpha,a)** | The new fractional derivative of Caputo ${}^{CF}\mathscr{D}_0^\alpha\, g(t)$. |
| **floor** | The nearest integer in the direction of negative. |
| **Na** | $N$ the function used in subsection ??. |
| **A(i,j)** | $A_{ij}$ Element of the matrix $A$. |
| **V(i)** | $V_i$ Element of the vector $V$. |
| $==$ | Equal to. |
| $\sim=$ | Not equal to. |
| $>$ | Greater than. |
| $>=$ | Greater than or equal to. |
| $<$ | Less than. |
| $<=$ | Less than or equal to. |
| & | Logical AND. |
| $\vert$ | Logical OR. |
| $\backsim$ | Logical NOT. |