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## Master's Thesis

# ACADEMIC MASTER

**Domain** : Mathematics and Computer Science
**Major** : Mathematics
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## Project

# Introduction to chaotic dynamic systems and applications

Presented by : Hwamdi Ethar and Djmeil Somaya
Under the supervision of: Guedda Lamine

**Supported before the jury:**

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| M: Meneser Bekker | MAB | University El-Oued | President |
| M: Guedda Lamine | PROF | University El-Oued | Rapporteur |
| M : Gabesi Hussein | MAA | University El-Oued | Examiner |

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# الإهداء

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﴿وَآخِرُ دَعْوَاهُمْ أَنِ الْحَمْدُ لِلَّهِ رَبِّ الْعَالَمِينَ﴾

مرحلة الماستر قد شارفت على الانتهاء بالفعل بعد تعب ومشقة دامت سنوات في سبيل العلم حملت في طياتها امنيات الليالي ، ها أنا اليوم أقف على عتبة تخرجي أقطفُ ثمار تعبي وأرفع القبعة فخرًا ،فاللهم لك الحمد قبل أن ترضى ولك الحمد إذا رضيت ولك الحمد بعد الرضا . وبكل حب اهدي ثمرة نجاحيِ وتخرجيِ :

الى الذي زين اسميِ بأجمل الألقاب ،ومن دعمنيِ بلاَ حدود واعطانيِ بلاَ مقابل من علمني أن الدنيا كفاح وسلاحهاَ العلم والمعرفة، داعمي الأول لمسيرتيِ وسندي وقوتي وملاذي بعد الله , فخري واعتزازي : **"والديِ العزيز".**

إلى من جعل الله الجنة تحت اقدامها، وأحتضنني قلبها قبل يدها وسهلت لي الشدائد بدعائها ،إلى القلب الحنون والشمعة التى كانت لي في الليالي المظلمات معلمتي الأولى: **"والدتي العظيمة".**

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إلى الملائكة رزقني الله بهن لأعرف من خلالهن طعم الحياة الجميلة تلك الملائكة اللآتي غيرن مفاهيم الحب والصداقة والسند في حياتي أخواتي الغاليات :" **سميرة، سمية، سعيدة، نادية ".**

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# الإهداء

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أهدي هذا العمل المتواضع الى من كان خير عون لي ، الى من تعب كي ارتاح ،
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سمية

# General notations

| | |
|---|---|
| $\varphi(x,t)$ | A flow. |
| $f(X)$ | A vector field. |
| $\dot{X}$ | The derivative of $X$ with respect $t$. |
| $C^1$ | The space of continuous and differentiable functions. |
| $\Gamma_a$ | the trajectory through $a$. |
| $(a.s)$ | asymptotically stable. |
| $(e.s)$ | exponentially stable. |
| $(m.s)$ | marginally stable. |
| $V(x)$ | The function of Lyapunov . |
| $\mu_c$ | bifurcation point. |
| $h_i$ | the Lyapunov exponent. |
| $H^s$ | s-dimensional Hausdorff measure. |

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# General Introduction

In several domains as mathematics, chemistry or physics, a dynamic system is a system whose evolution over time is described by a law. It can be the evolution of a chemical reaction over time, the movement of the planets in the solar system (governed by Newton's universal law of gravitation) or even the evolution of a computer's memory under the action of a computer program. Formally we distinguish between discrete-time dynamic systems like a computer program and continuous-time dynamic systems like a chemical reaction.

Two important aspects of a dynamic system are that it is:

**causal,** that is to say that its future depends only on phenomena of the past or present; **deterministic**, that is to say that an "initial condition" given at the "present" moment will correspond to each subsequent moment one and only one possible "future" state.

The modern theory of dynamical systems originated at the end of the 19th century with fundamental questions concerning the stability and evolution of the solar system. Attempts to answer those questions led to the development of a rich and powerful field with applications to physics,biology, meteorology,astronomy, economics, and other areas.[17]

Between the late 1960s and the beginning of the 1980s, the wide recognition that simple dynamical laws could give rise to complex behaviors was sometimes hailed as a true scientific revolution impacting several disciplines, for which a striking label was coined chaos. Mathematicians quickly pointed out that the purported revolution was relying on the abstract theory of dynamical systems founded in the late 19th century by Henri Poincaré who had already reached a similar conclusion.[6]

Chaos theory is the study of the behavior of dynamical systems that are highly sensitive to initial conditions (The butterfly effect): A butterfly flapping its wings in China can cause a hurricane in Texas. In other words, small differences in initial conditions yield widely diverging solutions for such dynamical systems and the prediction of their behavior is impossible in general. However, chaos has some patterns such as: Constant feedback loops, repetition, self-similarity, fractals and self-organization. Chaos theory of dynamical systems is very complicated and contains a large number of subjects. This theory has many practical applications, such as secure communications.

We have divided the work into a general introduction and three organized chapters as follows :

- The first chapter gives some basic definitions from the theory of dynamical systems and the mathematical objects necessary for the discussion.
- In the second chapter, we discussed the concept of ”bifurcation”, its diagram, and its various types.
- In the third and final chapter, we discussed chaos theory and its mathematical definition according to Devaney, its characteristics, and the concept of attractors in general, including chaotic attractors , in addition to methods for discovering chaos, and some applied examples of chaotic systems .

# Chapter 1

# Preliminaries on dynamic systems

In this chapter, we are first interested in presenting general concepts about dynamic systems, such as definition, types , equilibrium points , limit cycle , and stability....

## 1.1 General definitions of dynamical system

Dynamics is a time-evolutionary process. It may be deterministic or stochastic. Dynamical systems are generally described by differential or difference equations . Expressed either as a continuous-time or as a discrete-time-evolutionary process.[11]

### 1.1.1 Mathematical representations of dynamic systems

**Definition 1.1.1.** *Let $X$ be a metric space of $\mathbb{R}^n$ , and let $T$ is the set $\mathbb{R}_+$ or $\mathbb{N}$ and*

$$\varphi : U \subset X \times T \to X$$
$$(x,t) \mapsto \varphi(x,t)$$

*a function of class $C^1$.*
*The triplet $(X, T, \varphi)$ is called a dynamic system if it checks :*

*1-* $\forall x \in X, \varphi(x, 0) = x.$

*2-* $\forall x \in X,\ \forall t_1, t_2 \in I\,(x),\ \varphi(\varphi(x, t_1), t_2) = \varphi(x, t_1 + t_2),$

*where* $I(x) = \{t \in T; (x,t) \in U\}$.

$X$ *: is called the phase space ( or state space) and the variable* $x$ *represents the initial state of the system.. In general, the* ***phase space*** *is a structure corresponding to the set of all possible states of the system considered. It can be a vector space, a measurable space.... For a system with* $n$ *degrees of freedom, for example, the phase space* $X$ *of the system has* $n$ *dimensions, such that the complete state* $x(t) \in X$ *of the system at time* $t$ *is in general a vector with* $n$ *components.*

$\varphi$ *: The flow of the dynamic system or (evolution function). It associates a unique image with each point of* $x \in X$*, depending on the variable* $t$*, called the evolution parameter.*

**Definition 1.1.2.** *The dynamical systems* $(X, T, \varphi)$ *is called* ***continuous*** *if* $T = \mathbb{R}_+$ *but in the case* $T = \mathbb{N}$ *it is called* ***discret*** *.*

**Example 1.1.1.** *Let the following Cauchy problem*

$$\begin{cases} y'(t) = y(t) \\ y(0) = x; \ x \in \mathbb{R} \end{cases}$$

*We have:*

$$y' = y \Leftrightarrow \frac{dy}{dt} = y \Rightarrow \frac{dy}{y} = dt \Rightarrow \exp(\ln y) = \exp(t + c)$$

$\Rightarrow y(t) = k \exp(t)$

$y(0) = k \exp(0) = x \Rightarrow k = x$

$y(x,t) = x \exp(t)$

1. $y(x,0) = x \exp(0) = x$.

2. $y(y(x,t_1),t_2) = y(x \exp(t_1), t_2) = x \exp(t_1) \exp(t_2) = x \exp(t_1 + t_2) = y(x, t_1 + t_2)$.

*So our system is a dynamic system.*

According to the above, the study of the evolution of a dynamic system therefore requires knowledge of

1) its initial state at time $t = 0$,

2) its law of evolution.

**Remark 1.1.1.** *In general, the law of evolution of a dynamic system is modeled by a system of first order ordinary differential equations of the form*

$$x'(t) = f(x(t), t) \quad \textit{with } x(0) = x_0, \tag{1.1}$$

*where is $x(t)$ the vector of state variables of dimension $n$ and the function $f : X \longrightarrow \mathbb{R}^n$ is* ***an n-dimensional vector field****.*

### 1.1.2 Trajectory and periodic orbit

**Definition 1.1.3.** *The set $\Gamma_a = \{x \in X; x = \phi(a,t), t \in \mathbb{R}_+\}$*

*where $\phi(a,t)$ solution of (1.1) is called* ***orbit*** *of (1.1) through $a \in X$*

*The function $\phi_a : I(a) \to \mathbb{R}^n$ defined by $\phi_a(t) = \phi(t,a)$ is called* ***the flow through*** *$a$ and its graph is* ***the trajectory*** *of (1.1) through $a$.*

**Definition 1.1.4.** *Cycle or periodic orbit of (1.1) is a closed solution curve of (1.1) which is not an equilibrium point of (1.1).*

## 1.2 Dynamic system classifications

### 1.2.1 Continuous time dynamic systems

In the general case a dynamic system in continuous time can be represented by a differential equation . According to the equation we distinguish several types:

This system is called **non-autonomous** if it depends explicitly on $t$ i.e. when we have $\frac{\partial f}{\partial t} \neq 0$. Otherwise, in the case $\frac{\partial f}{\partial t} = 0$, the system is called **autonomous** takes the form: $x'(t) = f(x(t))$ with $x(0) = x_0$.

### 1.2.2 Discrete-time dynamic system

A discrete dynamic system, denoted $(X, \mathbb{N}, f)$ such that $f \in C^1$ on an open $X \subset \mathbb{R}^n$ , is defined by a recurring sequence $(x_n)$ defined by:

$$x_{n+1} = f(x_n).$$

Thus, we have

$$x_n = f^n(x_0)$$

where
$f^n(x) = fofo...of(x)$ ( $n$ times), $\forall n \in \mathbb{N}$ and $f^0 = Id$.

### 1.2.3 Linear dynamic system

In the continuous case, a linear system is a system described by the equation of state

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t), t > t_0 \\ x(0) = x_0 \end{cases} \tag{1.2}$$

Where $A(t)$ is an n $\times$ n matrix with values in $\mathbb{R}^n$ , $B(t)$ is an n $\times$ m matrix with values in $\mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ .
In the discrete case, a linear system is written in the form of recurring equations

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k), k \geq 0 \\ x(0) = x_0 \end{cases} \tag{1.3}$$

Where $A(k)$ is an n $\times$ n matrix with values in $\mathbb{R}^n$ , B(k) is an n $\times$ m matrix with values in $\mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$ .

### 1.2.4 Nonlinear dynamic system

In the continuous case, a dynamic system can be represented by the equation

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \\ x(0) = x_0. \end{cases} \tag{1.4}$$

where $x : I = [0, h] \longrightarrow \mathbb{R}^n$ and $f : I \times B \longrightarrow \mathbb{R}^n$ where
$B = \{x \in \mathbb{R}^n \text{ such that } \|x\| \leqslant r\}$ with $r > 0$.
A system of the form (1.4) is called a non autonomous system, it is said autonomous if it is written in the form

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = x_0 \end{cases} \tag{1.5}$$

In the discrete case a non autonomous dynamic system is written:

$$\begin{cases} x(k+1) = f(k, x(k)), \\ x(0) = x_0. \end{cases} \tag{1.6}$$

An autonomous discrete system is written in the form

$$\begin{cases} x(k+1) = f(x(k)), \\ x(0) = x_0. \end{cases} \tag{1.7}$$

### 1.2.5 Conservative and Dissipative Dynamical System

The conservative and dissipative systems are defined with respect to the divergence of the corresponding vector field, which in turn refers to the conservation of volume or area in their state space or phase plane, respectively as follows:[11]

**Definition 1.2.1.** *A system is said to be **conservative** if the divergence of its vector field is zero. such that the phase volume in a conservative system is constant under the flow.*

**Definition 1.2.2.** *A system is said to be **dissipative** if its vector field has negative divergence. such that the phase volume in a dissipative system occupied by the system is gradually decreased as the time $t$ increases and tends to zero as $t \longrightarrow \infty$ .*
*When divergence of vector field is positive, the phase volume is gradually expanding.*

**Example 1.2.1.** *(**Conservative plane pendulum**) Consider the classic case of a plane pendulum.*

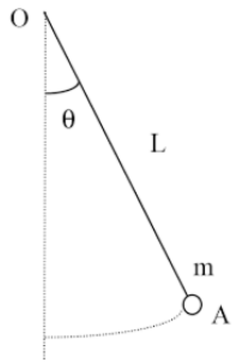


Figure 1.1: Pendulum model.

*It is a heavy bar of mass m and length l, free to rotate around fixed point O. The movement of this system with one degree of freedom can be concluded as follows:*

*as we have kinetic energy* $E_k(t) = \frac{1}{2}\frac{ml^2}{3}\left[\dot{\theta}(t)\right]^2$, *potential energy* $E_p(t) = -\frac{mg}{2} l \cos\theta(t)$ *initial energy* $E_0$ *then,*

$$\begin{aligned} E_k(t) + E_p(t) &= E_0 \Longleftrightarrow \frac{d}{dt}\left[E_k(t) + E_p(t)\right] = 0 \\ &\Longleftrightarrow \frac{ml^2}{3}\dot{\theta}(t)\ddot{\theta}(t) + \frac{mg}{2} l\dot{\theta}(t)\sin\theta(t) = 0 \\ &\Longleftrightarrow \ddot{\theta}(t) + \frac{3g}{2l}\sin\theta(t) = 0 \\ &\Longleftrightarrow \ddot{\theta}(t) + \omega^2\sin\theta(t) = 0 \end{aligned}$$

*where* $\omega = \sqrt{\frac{3g}{2l}}$.

*Taking* $x_1(t) = \theta(t)$ *and* $x_2(t) = \dot{\theta}(t)$, *then in the dynamical system theory this last differential equation can be written as*

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\omega^2 \sin x_1(t) \end{cases} \text{ with the initial condition } \begin{cases} x_1(0) = \theta_0 = \theta(0) \\ x_2(0) = \dot{\theta}_0 = \dot{\theta}(0) \end{cases}.$$

*where the state vector* $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ *is of dimension 2. The phase portrait represents all possible trajectories in phase space* $\left(\text{here}\left( x_1 = \theta(t), x_2 = \dot{\theta}(t)\right)\right)$:

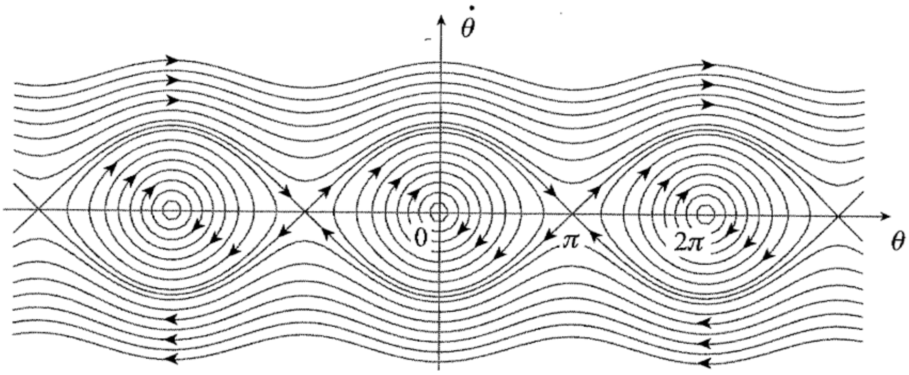


Figure 1.2: The phase portrait.

*It makes it possible to highlight the essential characteristics of the physics of the nonlinear system. According to Cauchy's theorem, each initial condition corresponds to a single trajectory. Different initial conditions can give the same trajectory. The trajectories never cross each other! Here, the trajectories represent the isovalues of the Hamiltonian* $H(t)$ *which is constant and which represents the total energy:*

$$H(t) = E_k(t) + E_p(t) = \frac{1}{2}\frac{mL^2}{3}\dot{\theta}^2(t) - mg\frac{L}{2}\cos(\theta(t)) = E_0$$

*where* $E_0 = \frac{1}{2}\frac{mL^2}{3}\dot{\theta}^2(0) - mg\frac{L}{2}\cos(\theta(0))$ *is the initial energy given to the system.*
*In other words, the initial conditions which give the same* $E_0$ *correspond to the same trajectory in phase space.*
*Two qualitative behaviors of the pendulum can be distinguished depending on the value of the initial energy:*

* *if* $E_0 > mg\frac{L}{2}$ *, the pendulum never stops rotating and never changes direction once thrown . these are the curves which are not closed .*

* *if* $E_0 < mg\frac{L}{2}$ *the pendulum swings around its stable equilibrium positions* $\tilde{x} = (\tilde{\theta}, \dot{\tilde{\theta}})$ *which are such that* $\tilde{\theta}(t) = \pm k\pi$ *and* $\dot{\tilde{\theta}}(t) = 0$ *with* $k = 0, 2, 4, ...$ *The set of initial conditions which give rise to closed trajectories around an equilibrium* $\tilde{x}$ *is called the basin of attraction of this equilibrium.*

## 1.3 Properties of dynamic systems

### 1.3.1 Equilibrium point of a system

If it is possible for the trajectory of a system to correspond, from of a certain moment, only to a point, such a point is called equilibrium point. We will see that the stability problems are naturally formulated in relation to the points of equilibrium.[1]

**Definition 1.3.1.** *The vector $\tilde{x} \in \mathbb{R}^n$ is said to be an equilibrium point of the system (1.4) (respectively (1.6)) if $f(t, \tilde{x}) = 0$ for all $t \geq 0$ (respectively $\tilde{x} = f(k, \tilde{x})$ for all $k \in \mathbb{N}$).*

**Example 1.3.1.** *Consider the system governed by the equation*

$\dot{x}(t) = a\left[1 - \frac{x(t)}{c}\right] x(t)$ *where* $a > 0; c > 0$
*The equilibrium points of this system are solutions of the algebraic equation*

$a\left[1 - \frac{x(t)}{c}\right] x = 0$, *which gives two equilibrium points* $\tilde{x} = 0$ *and* $\tilde{x} = c$.

**Example 1.3.2.** *Consider the following discrete system*

$$\begin{cases} x_1(k+1) = \alpha x_1(k) + x_2(k)^2, \\ x_2(k+1) = x_1(k) + \beta x_2(k). \end{cases} \tag{1.8}$$

*An equilibrium point of (1.8) is a vector $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ solution of system*

$$\begin{cases} x_1 = \alpha x_1 + x_2^2 \\ x_2 = x_1 + \beta x_2 \end{cases}$$

*which gives two points equilibrium*

$\tilde{x} = (0, 0)$ *and* $\tilde{x} = ((1-\alpha)(1-\beta)^2, (1-\alpha)(1-\beta))$.

### 1.3.2 Classification of equilibrium points

**a- Case of linear systems**

Consider the linear system:

$$\dot{x}(t) = Ax. \tag{1.9}$$

where $x = (x_1, x_2, ...., x_n)$ and has an invertible constant matrix. Let $\lambda_1, \lambda_2, ....., \lambda_n$ values properties of A.

**Definition 1.3.2.** *1. If the eigenvalues $\lambda_1, \lambda_2, ....., \lambda_n$ are real and of the same sign,the solution $x = 0$ is called a node.*
*2.If the eigenvalues $\lambda_1, \lambda_2, ....., \lambda_n$ are real, non-zero and of different sign, the solution $x = 0$ is called saddle.*
*3. If the eigenvalues $\lambda_1, \lambda_2, ....., \lambda_n$ are complex with Re $(\lambda_i) \neq 0$, i = 1,..., n, the solution $x = 0$ is called focus.*
*4. If the eigenvalues $\lambda_1, \lambda_2, ....., \lambda_n$ are complex with Re$(\lambda_i) = 0$, i = 1,..., n, the solution $x = 0$ is called center.*
*All these cases are grouped together (Figure (1.3)).*

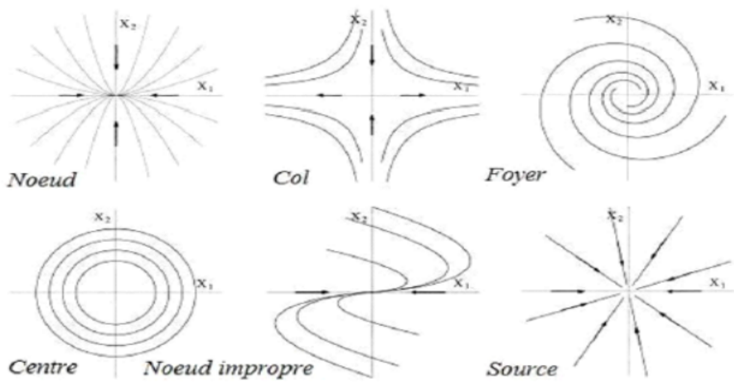


Figure 1.3: Classification of equilibrium points in $\mathbb{R}^2$.

#### b- Case of nonlinear systems

Let us now consider the nonlinear system:

$$\dot{x}(t) = f(x). \tag{1.10}$$

**Definition 1.3.3.** *A critical point of $x$ of (1.10) is called a sink if all values properties of the matrix $A = Df\left(\tilde{x}\right)$ have negative real parts, it is called **sources** if all the eigenvalues of the matrix $A = Df\left(\tilde{x}\right)$ have real positive parts, it is called **saddle** if it is hyperbolic and if $A = Df\left(\tilde{x}\right)$ has at least one eigenvalue with a positive real part or at least an eigenvalue with a negative real part.*

**Theorem 1.3.1.** *Let $\dot{x} = f(x), x \in \mathbb{R}^n, f \in C^1$ a continuous time dynamic system . if $\tilde{x}$ a fixed point (f(x) = 0), then if the eigenvalues of $Df(\tilde{x})$ are from negative real part $\tilde{x}$ is stable, and unstable if one of these eigenvalues of $Df(\tilde{x})$ is of positive real part.*

## 1.4 Stability of dynamic systems

One of the most important concepts in the dynamical system theory is the stability of its points equilibrium . An unstable system is useless and potentially dangerous. Qualitatively a system is stable if each time it is disturbed from its point of equilibrium, it remains around this point of equilibrium afterwards.[1]
The analysis of the stability of a system involves two methods: the linearization method and the direct Lyapunov method. The method linearization allows conclusions to be drawn concerning the local stability of a dynamic system around an equilibrium point, from stability properties of its linear approximation. As for the direct method, introduced in the 19th century by Lyapunov, it is not restricted to a local character and makes it possible to determine the properties of stability of a nonlinear system using an energy function.

### StabilIty of autonomous systems

When analyzing a system, the study of stability is of paramount importance, and although this notion is quite common, it can have several interpretations depending on the

intended application, which leads to appropriate but generally system-related definitions considered.
The systems considered in this section are governed by the equation ( 1.5) for the autonomous continuous system and by the equation ( 1.7) for the autonomous discrete system.

### 1-Definitions- Examples

Let $\tilde{x}$ be an equilibrium point of the system (1.5) (respectively (1.7)).

**Definition 1.4.1.**
***1.*** *$\tilde{x}$ is said to be stable if for all $\xi > 0$, there exists $\eta > 0$ such that,*

$$\|x_0 - \tilde{x}\| \leqslant \eta \Longrightarrow \|x(t) - \tilde{x}\| \leqslant \xi \, , \forall t > 0 \tag{1.11}$$

*(respectively $\|x(k) - \tilde{x}\| \leqslant \xi$ for all $k \in \mathbb{N}$).*
*Otherwise it is said to be unstable.*
***2.*** *$\tilde{x}$ is said to be asymptotically stable (a.s) if it is stable and if*

$$\lim_{t \longrightarrow \infty} \|x(t) - \tilde{x}\| = 0. \tag{1.12}$$

*(respectively $\lim_{k \longrightarrow \infty} \|x(k) - \tilde{x}\| = 0$).*
***3.*** *$\tilde{x}$ is said to be marginally stable (m.s) if it is stable but not (a.s).*
***4.*** *$\tilde{x}$ is said to be exponentially stable (e.s) if there exist two real positive constants $\alpha$ and $\beta$ as*

$$\|x(t) - \tilde{x}\| \leqslant \alpha e^{-\beta t}. \tag{1.13}$$

### 2- Stability of linear systems

We consider a continuous linear system defined by the following equation

$$\begin{cases} \dot{x} = Ax(t), \\ x(0) = x_0 \in \mathbb{R}^n. \end{cases} \tag{1.14}$$

respectively discrete linear systems

$$\begin{cases} x(k+1) = Ax(k) \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{1.15}$$

where A is a square matrix of dimension n $\times$ n.

**Definition 1.4.2.** *The continuous (1.14) or discrete (1.15) system is said to be stable if the origin is stable and it is said to be asymptotically stable if the origin is asymptotically stable. In this case, the matrix of system A is said to be stable or asymptotically stable .*

**Theorem 1.4.1.** *the equilibrium point $\tilde{x} = 0$ of the linear system (1.14) is*

1. *stable, if A is diagonalizable and its eigenvalues have negative or zero real parts,*
2. *asymptotically stable, if all the eigenvalues of A have their strictly negative real part,*
3. *unstable, if at least one of the eigenvalues of A has a strictly positive real part.*

***Preuve.*** Case 2) Let $\lambda_j$ , $j = 1, ..., n$, the distinct eigenvalues of A, the solution of the system (1.14) is written

$$x(t) = \sum_{j=1}^{n} g_j(t) e^{\lambda_j t}$$

where $g_j(t)$ is a polynomial in $t$, so $\lim_{t\to\infty} x(t) = 0$ if and only if $R(\lambda_j) < 0$ , for $j = 1, ..., n$. $\square$

**Remark 1.4.1.** *Case 1) can be expressed differently: the equilibrium point $\tilde{x} = 0$ of the linear system (1.14) is stable if for any eigenvalue $\lambda$ of the matrix A, we have*

1. *$Re(\lambda) \leq 0$ and*
2. *$Re\ (\lambda) = 0 \Longrightarrow \dim\ker(A - \lambda I) = p$,*

*where $p$ is the order of multiplicity of the root $\lambda$ of the polynomial $\det(XI - A)$.*

**3- Stability of nonlinear systems**

Suppose that $\tilde{x}$ is an equilibrium point for the differential system

$$\dot{x}(t) = f(x(t)), \tag{1.16}$$

and that f is differentiable in $\tilde{x}$ . To approximate the function $f(x(t))$ , let us form its Jacobian matrix in the neighborhood of $\tilde{x}$ ,

$$Df(\tilde{x}) = \left(\frac{\partial f_i}{\partial x_j}\right)_{1\le i,j\le m};$$

where the partial derivatives are calculated at the point $\tilde{x}$ , The affine approximation of $f$ is written in the form:

$$f(x) = f(\tilde{x}) + Df(\tilde{x}).(x - \tilde{x}) = Df(\tilde{x}).y$$

where $y = x - \tilde{x}$, which shows that in the neighborhood of $\tilde{x}$ we have

$$\dot{y} = \dot{x} - \dot{\tilde{x}} = f(x) = Df(\tilde{x}).y,$$

because $f(\tilde{x}) = 0$.

**Definition 1.4.3.** *We call the system linearized around $\tilde{x}$ associated with the system non-linear(1.16) the linear system*

$$\dot{y} = Df(\tilde{x}).y \tag{1.17}$$

**Theorem 1.4.2.** *If system (1.16) is linearizable around $\tilde{x}$ , then*

1. *the equilibrium point $\tilde{x}$ is stable if and only if the origin is stable for the linearized system.*

2. *the equilibrium point $\tilde{x}$ is asymptotically stable if and only if the origin is asymptotically stable for the linearized system.*

**4- Stability in the Lyapunov sense**

We therefore cannot say anything about the behavior in the neighborhood of an equilibrium point if all the eigenvalues of the differential have a negative or zero real part, with at least one having a zero real part, even if we can conclude that the linearized system. In this case, we resort to Lyaponov stability.

**a-Case of discrete systems**

Consider the system (1.7), recalled below,

$$\begin{cases} x(k+1) = f(x(k)) \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{1.18}$$

and let $\tilde{x}$ be an equilibrium point of this system, then we have the next definition .

**Definition 1.4.4.** *A function $V$ defined on $\Omega$ , a region of the system state space discrete (1.18) and containing $\tilde{x}$ is Lyapunov if it satisfies the conditions:*

*1 . $V$ is continuous over $\Omega$ .*

*2 . $V$ admits a unique minimum at the point $\tilde{x}$ on $\Omega$ .*

*3 . The function $\Delta\ V(x) = V(f(x)) - V(x) \leqslant 0$ on $\Omega$ .*

**b-Case of continuous systems**

Consider the system again

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{1.19}$$

and let $\tilde{x}$ be an equilibrium point of this system, then we have the next definition .

**Definition 1.4.5.** *A function $V$ defined over a region $\Omega$ which contains $\tilde{x}$ is a function of Lyapunov for the system (1.19) and the equilibrium point $\tilde{x}$ if*

*1 . $V$ is continuous and its partial derivatives are continuous.*

*2 .$V$ admits a unique minimum at $\tilde{x}$ over $\Omega$ .*

*3 . The function $\dot{V}(x) = \nabla\ V(x)f(x)$ satisfies $\dot{V}(x) \leqslant 0$ on $\Omega$.*

**Theorem 1.4.3.** *(from Lyapunov) If there is a Lyapunov function for the equation( 1.19), then the equilibrium point 0 is stable.*
*if $\dot{V}(x) < 0$ , the equilibrium point 0 is asymptotically stable.*

**Example 1.4.1.** *Let system (S) be the following:*

$$\begin{cases} \dfrac{dx}{dt} = x(y^2-1) \\ \frac{dy}{dt} = y(x^2-1) \end{cases}$$

*x = y = 0 is solution of (S) ( (x,y)=(0.0)equilibrium point). The linearized system has as matrix*

$$Df(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

*We calculate the eigenvalues of the matrix by finding the characteristic polynomial det(Df(0.0)-λI)=0 :*
*(-1-λ)(-1-λ)=0 $\Longrightarrow$ λ=-1<0*
*So, according to the theorem (1.4.1), then it is asymptotically stable.*
*The existence of a Lyapounov function will also give us information on the domain of attraction.let* $U(x,y) = \dfrac{1}{2}(x^2+y^2)$ *;* $\dot{U}(x,y) = x^2(y^2-1) + y^2(x^2-1)$
• *We have U(x,y) as a continuous function on* $\mathbb{R}^2$ *(First condition).*
• *Let* $\Omega = x^2 + y^2 < 1$ *,On* $\Omega, \dot{U}(x,y)$ *is defined as negative (Third condition)*
• *We find that 0 is asymptotically stable but in addition that* $\forall(x_0, y_0) \in \Omega$ *the solution resulting from this point has 0 as its limit (The second condition).*

**Remark 1.4.2.** *A non-autonomous equation in* $\mathbb{R}^n$ *can be solved via an auxiliary autonomous equation in* $\mathbb{R}^{n+1}$ *.In reality , let us associate with the non-autonomous equation(1.4) , with* $f : I \times B \to \mathbb{R}^n$ *, the autonomous equation*

$$\dot{y} = g(y(t)). \tag{1.20}$$

*with* $g : I \times B \to \mathbb{R}^{n+1}$ *such that* $g(y) = (1, f(y))$ *. We verify that* $x : I \to \mathbb{R}^n$ *is solution of (1.4) if and only if, the function* $y : I \times B \to \mathbb{R}^{n+1}$ *defined by* $y(t) = (t, x(t))$ *is solution of (1.20). we can therefore deduce the solutions of(1.4) solutions for(1.20) .On the other hand, the fact that it can be done in theory implies that it is enough to demonstrate certain results for the autonomous equations.*

# Chapter 2

# Bifurcation theory

In this chapter, we will discuss the changes in the nature of critical points and the bifurcation of solutions when a parameter passes through a certain value, all of this is called the bifurcation theory.

## 2.1 Introduction

The fundamental aspect of the study of dynamic systems is the notion of **bifurcation**. The term bifurcation was introduced by **Henri Poincaré** at the beginning of the 20th century in this work on differential systems. For certain critical values of the control parameters of the system, the solution of the differential equation changes qualitatively; we say that there is a bifurcation.[8]

Bifurcations are one of the most important aspects of a dynamical system. bifurcations tell you when and how you can expect discrete shifts in behaviour as you change one or more parameters. There are only a few different types of bifurcation that are possible, and we will cover the fundamental ones here. In all cases we will see how changing a particular parameter leads to a change in the stability, or existence of, the equilibria in the model. We will largely do this by looking at bifurcation diagrams.[2]

Let's take the column buckling example . If a low weight is placed on the pole, the "right pole" is a stable equilibrium position. But if the weight exceeds a critical load $\mu = \mu_c$(bifurcation point), the vertical position becomes unstable and the post buckles.

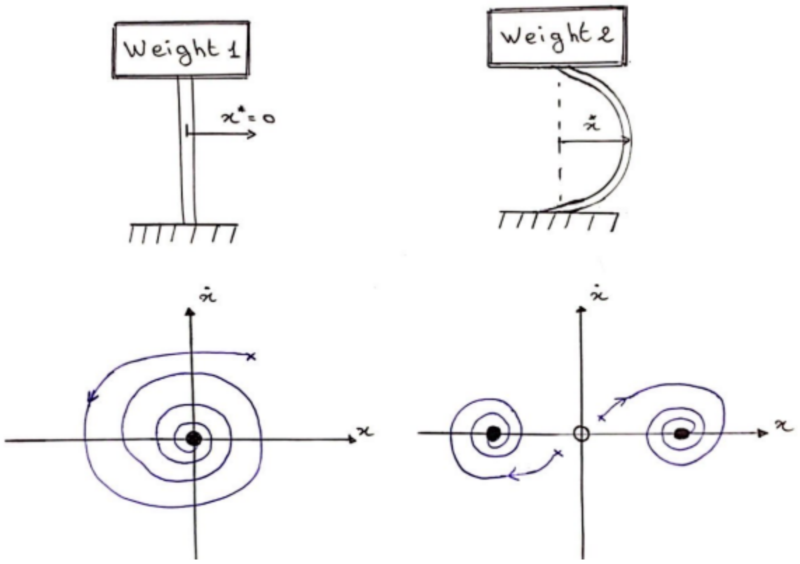


Figure 2.1: the column buckling example

We have phase portrait : In weight1 a single stable equilibrium in $x^\star = 0$ and in weight2 The equilibrium in $x^\star = 0$ unstable and two equilibrium positions are affected.

## 2.2 Definition of bifurcation

Bifurcation means a qualitative change in the properties of a system, such that stability, the number of equilibrium points or the nature of permanent regimes, during a variation quantitative of a system parameter.

**Definition 2.2.1.** ***(Mathematical definition of bifurcation)***
*Consider the following equation:*

$$\dot{x} = f(x, \mu), \tag{2.1}$$

*where $f \in C^1(E \times J), E$ is an open set in $\mathbb{R}^n$ and $J \subset \mathbb{R}$ .*
*We say that a system of differential equations (2.1) has a bifurcation at the value $\mu = \mu_c$, if there is a change in trajectory structure as the parameter $\mu$ passes through the value $\mu_c$. That is, there is a change in the number and /or stability of equilibria of the system to the value of the bifurcation.*
*-The approach to studying dynamic systems is as follows:*
*1. Search for fixed points.*
*2. Study of the stability of fixed points.*
*3. Bifurcation diagram.*

## 2.3 Bifurcation Diagram

The bifurcation diagram is a functional relationship diagram between points of the fixed state of the system and the bifurcation parameters. The graphics that explain these bifurcations are logically called bifurcation diagram.Usually, a variable state is selected and its limit value is plotted based on a only bifurcation parameter [18].

## 2.4 Types of bifurcation

The theory of bifurcations consists of classifying the different types of bifurcations , in this section we will present four most common types of Bifurcations :[22]

$\longrightarrow$ **Saddle-node bifurcation (or noeud-col)**

$\longrightarrow$ **Transcritical bifurcation**

$\longrightarrow$ **Fourche bifurcation (or pitchfork )**

$\longrightarrow$ **Hopf bifurcation**

### 2.4.1 Saddle-Node Bifurcation

The saddle-node bifurcation is the basic mechanism by which fixed points are created and destroyed. As a parameter is varied, two fixed points move toward each other, collide, and mutually annihilate.

The prototypical example of a saddle-node bifurcation is given by the first order system

$$\dot{x} = r + x^2 \tag{2.2}$$

where $r$ is a parameter, which may be positive, negative, or zero. When $r$ is negative, there are two fixed points, one stable and one unstable (Figure 2.2(a)).

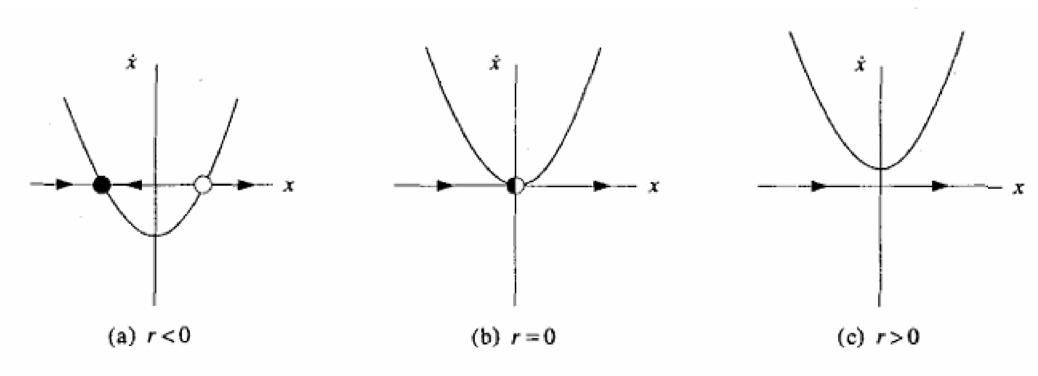


Figure 2.2: Example of the saddle-node bifurcation

As $r$ approaches 0 from below, the parabola moves up and the two fixed points move toward each other. When $r = 0$, the fixed points coalesce into a half-stable fixed point at $x^*= 0$ (Figure 2.2 (b)). This type of fixed point is extremely delicate-it vanishes as soon as $r > 0$ , and now there are no fixed points at all (Figure 2.2(c)). In this example, we say that a bifurcation occurred at $r = 0$, since the vector fields for $r < 0$ and $r > 0$ are qualitatively different.

### 2.4.2 Transcritical Bifurcation

There are certain scientific situations where a fixed point must exist for all values of a parameter and can never be destroyed. For example, in the logistic equation and other simple models for the growth of a single species, there is a fixed point at zero population, regardless of the value of the growth rate. However, such a fixed point may change its stability as the parameter is varied. The transcritical bifurcation is the standard mechanism for such changes in stability. The normal form for a transcritical bifurcation is

$$\dot{x} = rx - x^2. \tag{2.3}$$

We allow x and r to be either positive or negative.
Figure 2.3 shows the vector field as r varies. Note that there is a fixed point at $x^* = 0$ for all values of r.

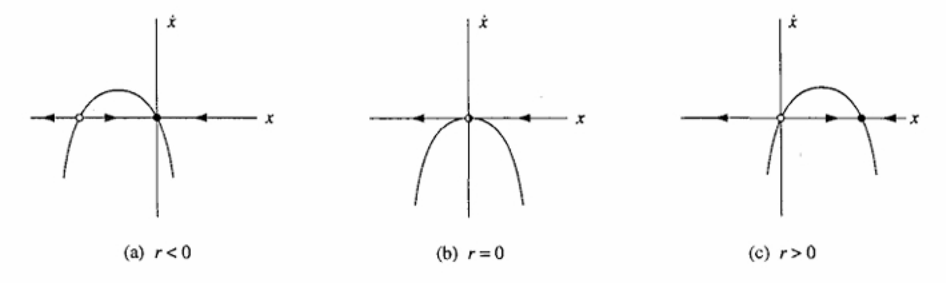


Figure 2.3: the vector field as r varies

For $r < 0$ , there is an unstable fixed point at $x^* = r$ and a stable fixed point at $x^* = 0$. As r increases, the unstable fixed point approaches the origin, and coalesces with it when r = 0. Finally, when $r > 0$ , the origin has become unstable, and $x^* = r$ is now stable. Some people say that an exchange of stabilities has taken place between the two fixed points.
Please note the important difference between the saddle-node and transcritical bifurcations: in the transcritical case, the two fixed points don't disappear after the bifurcation-instead they just switch their stability.
Figure 2.4 shows the bifurcation diagram for the transcritical bifurcation.

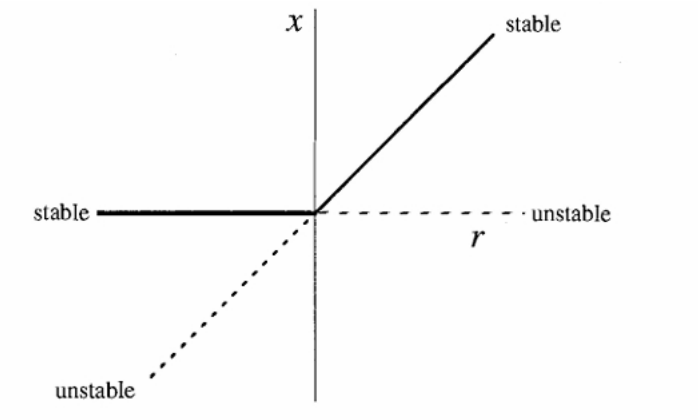


Figure 2.4: the bifurcation diagram for the transcritical bifurcation

### 2.4.3 Pitchfork Bifurcation

We turn now to a third kind of bifurcation, the so-called pitchfork bifurcation. This bifurcation is common in physical problems that have a symmetry. For example, many problems have a spatial symmetry between left and right. In such cases, fixed points tend to appear and disappear in symmetrical pairs.
There are two very different types of pitchfork bifurcation. The simpler type is called supercritical, and will be discussed first.

**Supercritical Pitchfork Bifurcation**

The normal form of the supercritical pitchfork bifurcation is

$$\dot{x} = rx - x^3. \tag{2.4}$$

Note that this equation is invariant under the change of variables $x \to -x$. That is, if we replace $x$ by $-x$ and then cancel the resulting minus signs on both sides of the equation, we get (2.4) back again. This invariance is the mathematical expression of the left-right symmetry mentioned earlier. (More technically, one says that the vector field is equivariant, but we'll use the more familiar language.)
Figure 2.5 shows the vector field for different values of r.

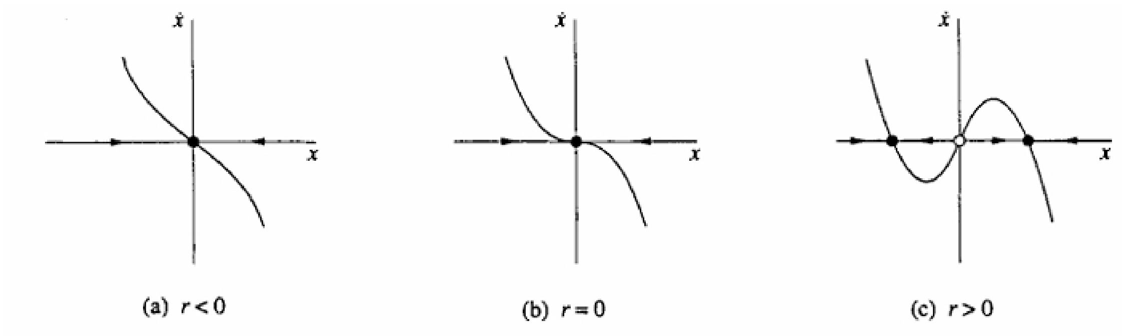


Figure 2.5: the vector field for different values of r

When $r < 0$, the origin is the only fixed point, and it is stable. When $r = 0$, the origin is still stable, but much more weakly so, since the linearization vanishes. Now solutions no longer decay exponentially fast-instead the decay is a much slower algebraic function of time . This lethargic decay is called critical slowing down in the physics literature. Finally, when $r > 0$, the origin has become unstable. Two new stable fixed points appear on either side of the origin, symmetrically located at $x^* = \pm\sqrt{r}$.
The reason for the term "pitchfork" becomes clear when we plot the bifurcation diagram (Figure 2.6).

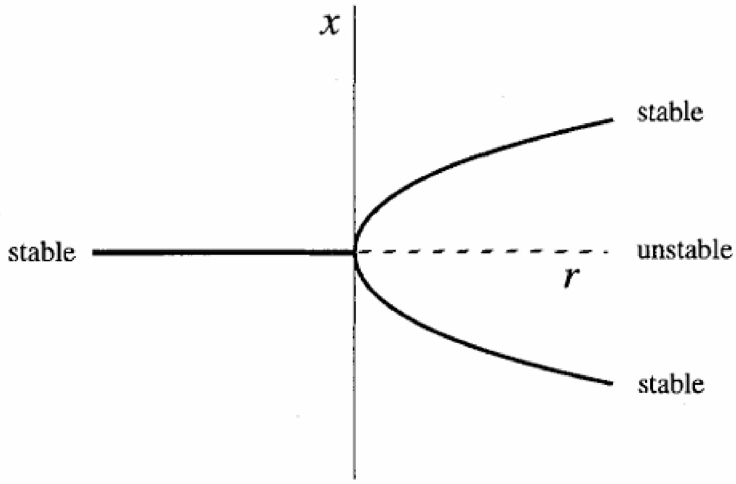


Figure 2.6: the bifurcation diagram of Supercritical pitchfork bifurcation

**Subcritical Pitchfork Bifurcation**

In the supercritical case $\dot{x} = rx - x^3$ discussed above, the cubic term is stabilizing: it acts as a restoring force that pulls $x(t)$ back toward $x = 0$. If instead the cubic term were destabilizing, as in

$$\dot{x} = rx + x^3, \tag{2.5}$$

then we'd have a subcritical pitchfork bifurcation. Figure 2.7 shows the bifurcation diagram.

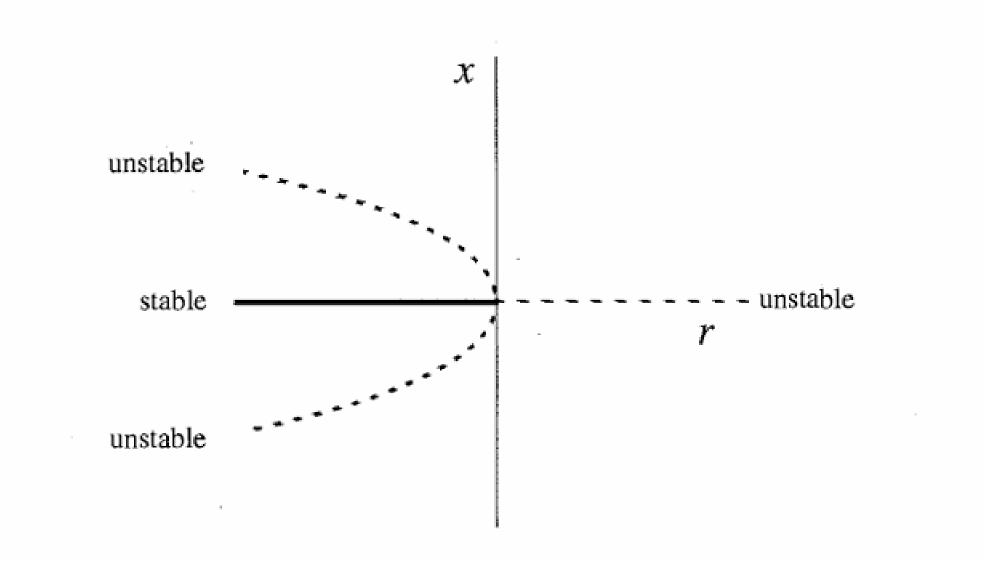


Figure 2.7: the bifurcation diagram of Subcritical Pitchfork Bifurcation

### 2.4.4 Hopf bifurcation

The Hopf bifurcation is the appearance or disappearance of a periodic solution, of an equilibrium point when the parameter passes through a critical value $\mu^{\star}$. This occurs when the pair of conjugate eigenvalues of the linearized system at the equilibrium point become pure imaginary. This implies that the Hopf bifurcation is for systems of dimension $n \geq 2$.

**Theorem 2.4.1.** *Consider the planar system:*

$$\begin{cases} \dot{x} = f_{\mu}(x, y), \\ \dot{y} = g_{\mu}(x, y). \end{cases} \tag{2.6}$$

*Where $\mu$ is a parameter.*
*Suppose $(x_0, y_0)$ is an equilibrium point of (2.6) which can depend on $\mu$ .*
*$\lambda(\mu)$,$\bar{\lambda}(\mu) = \alpha(\mu) \pm i\beta(\mu)$, the eigenvalues of the linearized system at the equilibrium point $(x_0, y_0)$.*
*Suppose that for a certain value of $\mu$ we take$\mu = \mu_0$ the following conditions are satisfied:*

1. $\alpha(\mu_0) = 0$,$\beta(\mu_0) = w \neq 0$ *where sgn(w) = sgn* $[(\frac{\partial g_{\mu}}{\partial x}) \mid_{\mu=\mu_0} (x_0, y_0)]$.

2. $\dfrac{d\alpha(\mu)}{d\mu}\mid_{\mu=\mu_0}= d \neq 0.$ *(transversality condition)*

3. $a \neq 0$ *where*

$$a = \frac{1}{16}[f_{xxx}+f_{xyy}+g_{xxy}+g_{yyy}]+\frac{1}{16w}[(f_{xy}(f_{xx}+f_{yy})-g_{xy}(g_{xx}+g_{yy})-f_{xx}g_{xx}+f_{yy}g_{yy}]$$

*with* $f_{xy} = (\dfrac{\partial^2 f_\mu}{\partial x \partial y}) \mid_{\mu=\mu_0} (x_0, y_0)$, *etc (genericity condition).*
*So*

- *There is a single periodic orbit which branches from the origin for* $\mu > \mu_0$ *if* $ad < 0$ *or* $\mu < \mu_0$ *if* $ad > 0$.
- *The equilibrium point* $(x_0, y_0)$ *is stable for* $\mu > \mu_0$*(respectively* $\mu < \mu_0$*) and unstable equilibrium point for* $\mu < \mu_0$ *(respectively* $\mu > \mu_0$*) if* $d < 0$ *(resp* $d > 0$*).*
- *The periodic solution is stable(respectively unstable)if the equilibrium point is unstable(respectively stable),on the side* $\mu = \mu_0$ *where the periodic solution exists.*
- *The amplitude of the periodic orbit is* $\sqrt{(\mu - \mu_0)}$.
- *The period of the periodic orbit is* $T = \dfrac{2\pi}{|w|}$.

**Example 2.4.1.** *Consider the Van Der Pol equation*

$$\ddot{x} - (\mu - x^2)\dot{x} + x = 0.$$

*We can write the Van Der Pol equation under the following system:*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + (\mu - x^2)y. \end{cases}$$

*The equilibrium point* $(x_0, y_0) = (0, 0)$.
*The Jacobian matrix of the system originally calculated is:*

$$Jf(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}.$$

*Calculate eigenvalues:*

$$P_J(\lambda) = \begin{vmatrix} -\lambda & 1 \\ -1 & \mu - \lambda \end{vmatrix} = \lambda(\lambda-\mu)+1 = \lambda^2 - \mu\lambda + 1 = 0.$$

$$\Delta = \mu^2 - 4 < 0, \qquad \mu \in ]-2,2[\ ,$$
$$\Rightarrow \Delta = i^2(4-\mu^2).$$

*So*

$$\lambda = \frac{\mu + i\sqrt{4-\mu^2}}{2},$$
$$\bar{\lambda} = \frac{\mu - i\sqrt{4-\mu^2}}{2}.$$

*than the eigenvalues of the matrix $Jf(0,0)$ are*

$$\lambda(\mu), \bar{\lambda}(\mu) = \frac{1}{2}(\mu \pm i\sqrt{4-\mu^2}).$$

1. $\alpha(\mu_0) = 0 \Longrightarrow \frac{1}{2}\mu = 0 \Rightarrow \mu_0 = 0, sgn(w) = sgn(\frac{\partial g_\mu}{\partial x} \mid_{\mu=0} (0,0)) = sgn(-1) = -1$ $\Longrightarrow \beta(\mu_0) = -1 = w.$
2. $\frac{d\alpha(\mu)}{d\mu} \mid_\mu = 0 = \frac{1}{2} = d \neq 0.$
3. $a = \frac{1}{16}[f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}] - \frac{1}{16}[(f_{xy}(f_{xx}+f_{yy}) - g_{xy}(g_{xx}+g_{yy})) - f_{xx}g_{xx} + f_{yy}g_{yy}].$

$g_{xxy} = \frac{\partial^3 g}{\partial x^2 \partial y} \mid_{\mu=0} (0,0) = -2$

$g_{xy} = \frac{\partial^2 g}{\partial x \partial y} \mid_{\mu=0} (0,0) = 0$

$g_{xx} = \frac{\partial^2 g}{\partial x^2} \mid_{\mu=0} (0,0) = 0$

*We replace in a we obtain:*

$$a = \frac{1}{16}[0+0-2+0] - \frac{1}{16}[(0(0+0) - 0(0+0)) - 0\times 0 + 0 \times 0] = -\frac{1}{8}.$$

*Since* $ad = -\frac{1}{8}.\frac{1}{2} = -\frac{1}{16} < 0 \Rightarrow$ *there is a periodic solution for* $\mu > 0$.

*since* $a = \dfrac{1}{2} > 0$, *the equilibrium point (0,0) is stable for* $\mu < 0$ *and unstable for* $\mu > 0$.

- *The periodic solution for* $\mu > 0$ *is stable and unstable for* $\mu < 0$.
- *The amplitude* $\sqrt{\mu}$.
- *The periodicity* $T = 2\pi$.

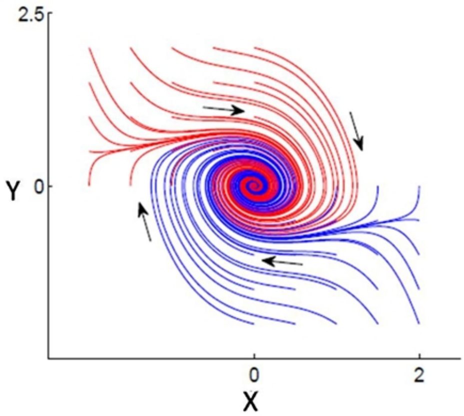


Figure 2.8: Phase portrait for the Van Der Pol equation with $\mu < 0$. The origin is a stable home.

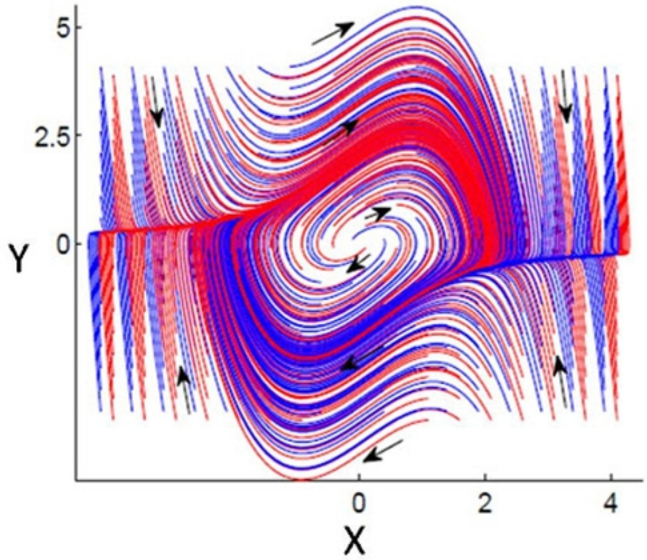


Figure 2.9: Phase portrait for the Van Der Pol equation with $\mu > 0$. The origin is an unstable focus and there is a stable periodic orbit.

**Remark 2.4.1.** *1. There are three possible cases of Hopf bifurcations: Degenerate Hopf Bifurcation, Super-critical Hopf bifurcation and Sub-critical Hopf bifurcation.*

*2. For a super-critical to sub-critical bifurcation, the equilibrium point $(x(\mu^\star), y(\mu^\star))$ is a focus.*

*3. When $\mu^\star$ is a value such that $\alpha(\mu^\star) = 0$ and $\beta(\mu^\star) \neq 0$, the equilibrium point $(x(\mu^\star), y(\mu^\star))$ corresponds to a center.*

**1-Degenerate Hopf Bifurcation:**

If one of the conditions of the Hopf Bifurcation theorem is not satisfied, we can have bifurcation of periodic solutions, we say that the bifurcation is degenerate.

**Example 2.4.2.** *Consider the Van Der Pol equation*

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0.$$

*We can write the Van Der Pol equation as the following system:*

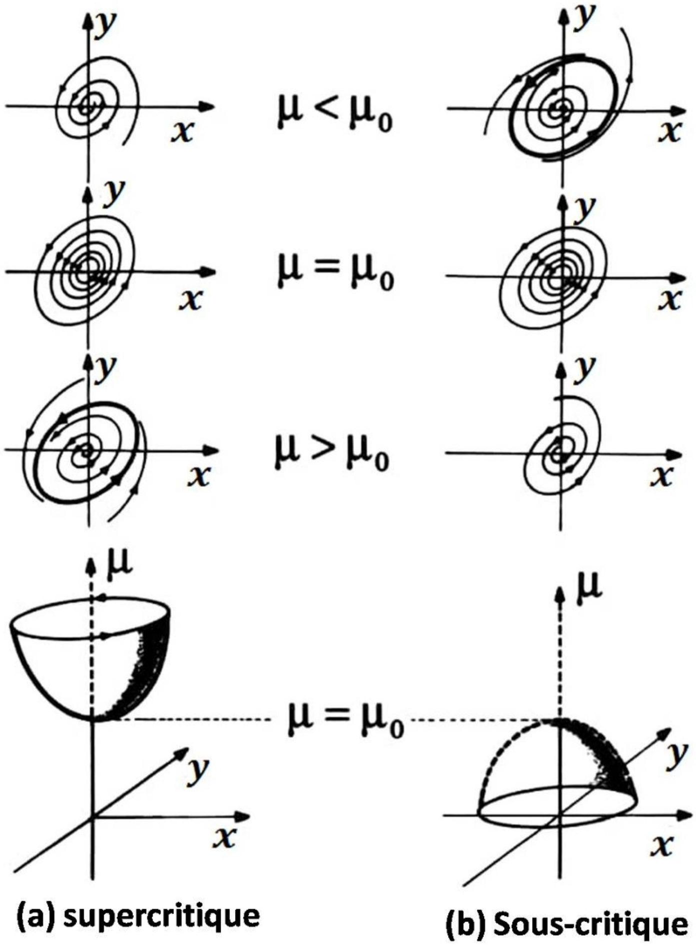


Figure 2.10: Phase portraits and a family of one-parameter periodic orbits for the Hopf bifurcation.

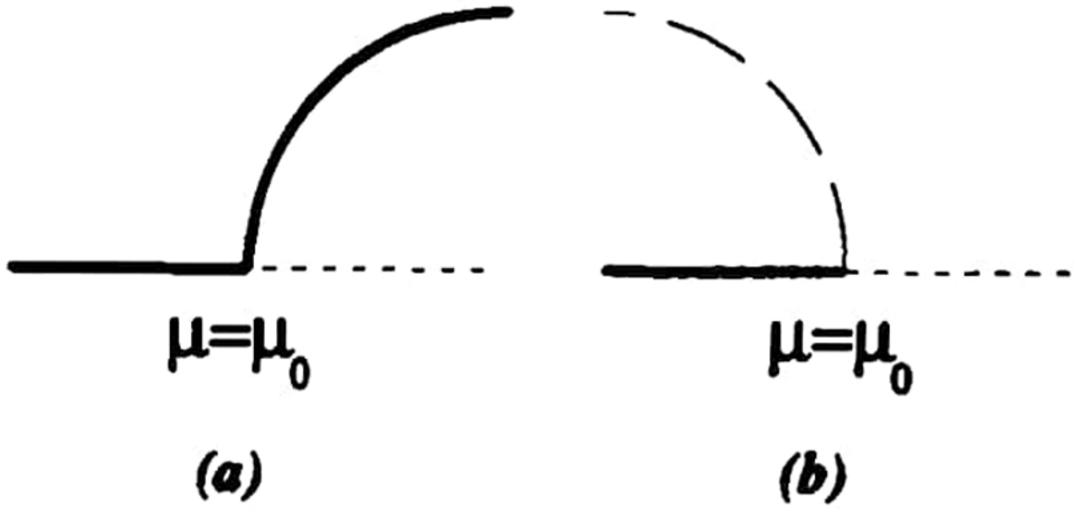


Figure 2.11: Bifurcation diagrams for (a) the supercritical case and (b) the subcritical case.

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \mu(1 - x^2)y. \end{cases}$$

*The system linearized in (0,0) is*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \mu y. \end{cases}$$

*The Jacobian matrix of the system originally calculated is:*

$$Jf(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}.$$

*We have*

$$P_J(\lambda) = \begin{vmatrix} -\lambda & 1 \\ -1 & \mu - \lambda \end{vmatrix} = \lambda(\lambda - \mu) + 1 = \lambda^2 - \mu\lambda + 1 = 0.$$

$$\textit{if } \mu \in ]-2, 2[ \textit{ so } \Delta = \mu^2 - 4 < 0.$$

*The eigenvalues of the matrix J are therefore complex conjugate and equal to*

$$\lambda_{1,2} = \frac{\mu}{2} \pm i\frac{\sqrt{4 - \mu^2}}{2}.$$

1. $sgn(w) = sgn(\frac{\partial g_\mu}{\partial x} \mid_{\mu=0} (0,0)) = -1, w = -1.$

2. $\frac{d\alpha(\mu)}{d\mu} \mid_\mu = 0 = \frac{1}{2} = d > 0.$

3. *a=0.*

*Since $a = 0$ therefore the condition $a \neq 0$ is not verified, we say in this case that the bifurcation is degenerate.*

**2- Super-critical Hopf bifurcation:**

The super-critical Hopf bifurcation is the destabilization of the stable focus part when the complex conjugate eigenvalues associated with this part cross the pure imaginary axis to transform it into an unstable focus part. This destabilization is accompanied by the birth of a stable limit cycle.

The bifurcation is super-critical if the periodic orbit is stable.

If $a < 0$ then the bifurcation is super-critical which means that the periodic orbit is stable.

**Example 2.4.3.** *Consider the following nonlinear dynamic system:*

$$\begin{cases} \frac{dx}{dt} = \mu x - y - x(x^2 + y^2), \\ \frac{dy}{dt} = x + \mu y - y(x^2 + y^2). \end{cases}$$

*We notice that the origin (0,0) is an equilibrium point of the system.*
*The Jacobian matrix of the system originally calculated is:*

$$J = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$$

*We have*

$$P_J(\lambda) = \begin{vmatrix} \mu - \lambda & -1 \\ 1 & \mu - \lambda \end{vmatrix} = (\mu - \lambda)^2 + 1.$$

*The eigenvalues of the matrix J are therefore complex conjugate and equal to $\lambda_{1,2} = \mu \pm i$.*

1. $sgn(w) = sgn(\frac{\partial g_\mu}{\partial x} \mid_{\mu=0} (0,0)) = 1, w = 1.$

2. $\frac{d\alpha(\mu)}{d\mu} \mid_{\mu=0} = 1 = d > 0.$

3. $a = \frac{1}{16}[f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}] + \frac{1}{16}[(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy})) - f_{xx}g_{xx} + f_{yy}g_{yy}].$

$f_{xxx} = \frac{\partial^3 f}{\partial x^3} \mid_{\mu=0} (0,0) = -6$
$f_{xyy} = \frac{\partial^3 f}{\partial x \partial^2 y} \mid_{\mu=0} (0,0) = -2$
$g_{xxy} = \frac{\partial^3 g}{\partial x^2 \partial y} \mid_{\mu=0} (0,0) = -2$
$g_{yyy} = \frac{\partial^3 g}{\partial y^3} \mid_{\mu=0} (0,0) = -6.$
*We replace in a we obtain* :$a = \frac{1}{16}[-6 - 2 - 2 - 6] = -1.$

**3- Sub-critical Hopf bifurcation :**

Subcritical bifurcation is a destabilization of the focal part accompanied by the birth of an unstable limit cycle.

- It is sub-critical if the periodic orbit is unstable.
- If $a > 0$ then the bifurcation is sub-critical which means that the periodic orbit is unstable.

**Example 2.4.4.**
- *Since* $ad = -1.1 = -1 < 0 \Rightarrow$ *there is a periodic solution for* $\mu > 0$.
- *since* $d = 1 > 0$ *, the equilibrium point (0,0) is stable for* $\mu < 0$ *and unstable for* $\mu > 0$.
- *since* $a = -1 < 0$ *then the bifurcation is super-critical which means that the periodic orbit is stable.*
- *The amplitude* $\sqrt{\mu}$.
- *The periodicity* $T = 2\pi$.

*This time we consider the following nonlinear dynamic system:*

$$\begin{cases} \dfrac{dx}{dt} = \mu x - y + x(x^2 + y^2), \\ \dfrac{dy}{dt} = x + \mu y + y(x^2 + y^2). \end{cases}$$

*We notice that the origin (0,0) is an equilibrium point of the system.*

*The Jacobian matrix of the system originally calculated is:*

$$J = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$$

*We have*

$$P_J(\lambda) = \begin{vmatrix} \mu - \lambda & -1 \\ 1 & \mu - \lambda \end{vmatrix} = (\mu - \lambda)^2 + 1.$$

*The eigenvalues of the matrix $J$ are therefore complex conjugate and equal to $\lambda_{1,2} = \mu \pm i$.*

1. $sgn(w) = sgn(\dfrac{\partial g_\mu}{\partial x} \mid_{\mu=0} (0,0)) = 1, w = 1.$
2. $\dfrac{d\alpha(\mu)}{d\mu} \mid_{\mu=0} = 1 = d > 0.$
3. $a = \dfrac{1}{16}[f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}] + \dfrac{1}{16}[(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy})) - f_{xx}g_{xx} + f_{yy}g_{yy}].$

$f_{xxx} = \dfrac{\partial^3 f}{\partial x^3} \mid_{\mu=0} (0,0) = 6$

$f_{xyy} = \dfrac{\partial^3 f}{\partial x \partial^2 y} \mid_{\mu=0} (0,0) = 2$

$g_{xxy} = \dfrac{\partial^3 g}{\partial x^2 \partial y} \mid_{\mu=0} (0,0) = 2$

$g_{yyy} = \dfrac{\partial^3 g}{\partial y^3} \mid_{\mu=0} (0,0) = 6.$

*We replace in a we obtain :*$a = \dfrac{1}{16}[6 + 2 + 2 + 6] = 1.$

- *Since* $ad = 1.1 = 1 > 0 \Rightarrow$ *there is a periodic solution for* $\mu > 0$.
- *since* $d = 1 > 0$ *, the equilibrium point (0,0) is stable for* $\mu < 0$ *and unstable for* $\mu > 0$.
- *since* $a = 1 > 0$ *then the bifurcation is sub-critical which means that the periodic orbit is unstable.*
- *The amplitude* $\sqrt{\mu}$.
- *The periodicity* $T = 2\pi$.

# Chapter 3

# Chaos Theory

In this chapter, we will introduce the concept of chaotic dynamic systems and their characteristics, and we will touch on the concept of attractors, specifically chaotic attractors, as well as methods for detecting chaos, and finally we will provide examples of these systems.

## 3.1 Introduction

One of the most wonderful areas of modern mathematics and physics which provided a new way of seeing the universe is chaos theory, an important tool to understand the behavior of processes in the world.

In 1963, meteorologist **Edward Lorenz** of the Massachusetts Institute of Technology (M.I.T), highlights the chaotic nature of the conditions meteorological and consequently turbulent movements of a fluid like the atmosphere [12]. These forecasts are based on the movement of air masses, and therefore on the application of Newton's theories and differential equations [**?**]. While seeking to determine future weather conditions from initial data on his computer, he noticed that a minimal modification of the data initial tests (of the order of one per thousand) led to radically different results. For improve this discovery, he programmed his computer to obtain a Numerical simulation. At the time, it took a lot of time. One day, so as not to restart the calculations from the beginning, he decided to resume his listing and return to as initial conditions of the values taken during the previous day's simulation. The computer gave it a precision of five digits, but three significant digits

seemed more than sufficient for this type of physical measurements. He therefore truncated these numbers and resumed the calculation. The results that followed were the trigger. First the simulation seemed to give back the same values, but after a while nothing matched, everything happened as if the movement represented by these values changed completely of trajectory and this because of an approximation of the order of $10^{-4}$ [19].
This anecdote [5], was the basis of chaos: a tiny variation in initial conditions of a system completely disrupts its evolution. Lorenz had just highlighted sensitivity to initial conditions. He also explained this notion very nicely using of the following image: the beating of the wings of a few butterflies can cause storms in the antipodes [5].
Thus, chaos is characterized by a complex nature and behavior unpredictable. Because of this complexity, chaotic systems are extremely difficult to control and predict, making them suitable for security, encryption, secure communications, robotics and more [16].

## 3.2 Chaotic dynamic systems

Chaos as the scientist understands it does not mean the absence of order; he attaches himself rather a notion of unpredictability, of the impossibility of predicting long-term developments because the final state depends so significantly on the initial state.
A chaotic dynamic system is a system that depends on several parameters and is characterized by extreme sensitivity to initial conditions. They are not determined or modeled by systems of linear equations nor by the laws of classical mechanics, however they are not necessarily random relating to calculation alone probabilities.
Several mathematical definitions of chaos are found in the literature, but until at present, there is no universal mathematical definition of chaos, before giving the definition of chaos by Devaney , some basic definitions are necessary.

**Definition 3.2.1.** *We say that the function $f : X \longrightarrow X$ has sensitivity to initial conditions if it exists $\delta > 0$ such that, for $x_0 \in X$ and for everyone $\epsilon > 0$ there is a point $y_0 \in X$ and an integer $j \geq 0$ satisfied : $d\left(x_0, y_0\right) < \epsilon \Longrightarrow d\left(f^{(j)}(x_0), f^{(j)}(y_0)\right) > \delta$, or represents the distance and $f^{(j)}(x_0)$ the ith iteration of $f$ .*

**Definition 3.2.2.** *Suppose that $X$ is a set and $Y$ is a subset of $X$ ($Y \subset X$), we say that $Y$ is dense in $X$ if, for all $x \in X$ we can find a sequence $\{y_n\}_{n\in\mathbb{N}}$ of $Y$ which converge towards $x$.*

**Definition 3.2.3.** *$f$ is topologically transitive if $U$ and $V$ being two non empty open sets in $X$, there exists $x_0 \in U$ and an index $j > 0$, such that for $f^{(j)}(x_0) \in V$ or,equivalently, there exists an index $j > 0$ such that for $f^{(j)}(U) \cap V \neq \emptyset$ .*

We are now in a position to state the definition of chaos in Devaney's sense.

**Theorem 3.2.1.** *(Devaney's Theorem) [20] Let a subset $V$ of $X$ $f : X \longrightarrow X$ is said to be chaotic on $V$ if:*

*i ) The function f has sensitivity to initial conditions,*

*ii ) The function f is topologically transitive, in the sense that for any pair of sub open sets $U$; $V \subset X$, there exists $j > 0$ such that $f^{(j)}(U) \cap V \neq \emptyset$,*

*iii ) The sets of periodic points of the function f are dense in X.*

## 3.3 Characteristics of chaos

### 3.3.1 Non-linearity

To predict real phenomena generated by dynamic systems, the approach consists of constructing a mathematical model which establishes a relationship between a set of causes and a set of effects. If this relationship is an operation of proportionality, the phenomenon is linear. In the case of a non-linear phenomenon, the effect is not proportional to the cause. In general, a chaotic system is a non-linear dynamic system, a linear system cannot be chaotic.

### 3.3.2 Determinism

A system is said to be deterministic when it is possible to predict (calculate) its Evolution over time. Exact knowledge of the state of the system at a given moment, the initial instant, allows the precise calculation of the state of the system at any other moment.In random phenomena, it is absolutely impossible to predict the trajectory of a any particle. Therefore, a chaotic system has deterministic and non-probabilistic fundamental rules.It is generally governed by non-linear differential equations which are known,therefore by rigorous and perfectly deterministic laws.

### 3.3.3 Sensitivity to initial conditions

Chaotic systems are extremely sensitive to disturbances , Lorenz illustrated this fact with the butterfly. For two initial conditions that are very close initially, the two trajectories corresponding to these initial data diverge exponentially. Subsequently, the two trajectories are incomparable. A essential properties of chaos is therefore this sensitivity to initial conditions which can be characterized by measuring the trajectory divergence rates (Figure 3.1).
From a mathematical point of view, we say that $f$ shows a sensitive dependence on initial conditions when:

**Definition 3.3.1.** *We say that $f : D \longrightarrow D$ has sensitivity to initial conditions if it exists $\epsilon > 0$ such that*

$$\forall x \in D, \forall \delta > 0, \exists y \in D \quad and \quad n \in \mathbb{N}, d(x, y) < \delta \quad and \quad d(f^n(x), f^n(y)) > \epsilon$$

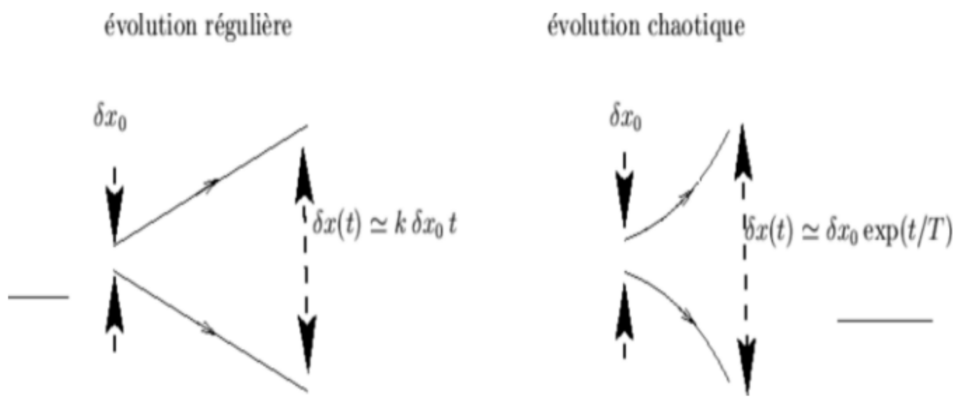


Figure 3.1: Sensitivity to initial conditions

### 3.3.4 Random appearance

The evolution of the chaotic system seems random .

## 3.4 The attractors

### 3.4.1 Definitions of attractor

We can introduce the notion of a chaotic system using a new geometric element, this element is called "**attractor**".
We first give some definitions of attractor:

**Definition 3.4.1.** ***(Ruelle)***[7]
*An attractor is a geometric object towards which all the trajectories of the points of the phase space tend, that is to say a situation or a set of situations towards which evolves a system, whatever its initial conditions. Mathematically, the set $A$ is an attractor if:*
*- For every neighborhood $U$ of $A$, there exists a neighborhood $V$ of $A$ such that any solution $x\left(x_0;t\right) = \varphi_t\left(x_0\right)$ will remain in $U$ if $x_0 \in V$ .*
*- There exists a dense orbit in $A$ such that: $\cap\varphi_t(V) = A, \forall t \geq 0$*

**Definition 3.4.2.** ***(Zeraoulia and Sprott)*** [9]
*Let $A$ be a closed set of the non-wandering set $\Omega(f)$ , $A$ is said to be an attractor if:*

***1.*** *$A$ is invariant under $f$ if it satisfies $f(A) = A$ ,*

***2.*** *There exists a neighborhood $V$ of $A$ in $\mathbb{R}^n$ such that: $\cap_{t\geq 0} f^n(V) = A$,*

***3.*** *$f$ is transitive (there exists $C$; $D \subset A$ such that $f^n(C) \cap D \neq \emptyset$ ; $C$, $D$ open).*

### 3.4.2 Basin of Attraction

**Definition 3.4.3.** *Let $A$ be an attractor of a dynamic system. The basin of attraction of $A$ denoted $B(A)$, is the set of points in the phase space such that: all the trajectories which are derived asymptotically converge towards $A$.*

$$B(A) = \{x \in X / w(x) \subset A\}$$

**Example 3.4.1.** *Let* $f(x) = x^2$

*The fixed points are:* $\{0 \ \textit{attractant}, 1 \ \textit{repellent}\}$

$$B(\{0\}) = \{x \in \mathbb{R}, f_k(x) \longrightarrow_{k\longrightarrow 0} 0\} = \left\{x \in \mathbb{R}, x^{2k} \longrightarrow_{k\longrightarrow 0} 0\right\}$$

### 3.4.3 Types of attractors

There are two types of attractors: regular attractors and strange attractors or chaotic. [12]

**1) Regular attractors:**

Regular attractors characterize the evolution of non-chaotic systems, and can be of three kinds:

**a) fixed point:** it is the simplest attractor, it is represented by a point in space phases.

**b) periodic limit cycle:** It is a closed trajectory which attracts all trajectories relatives.

**c) Quasi-periodic attractor (torus):** It is a trajectory which winds along a torus and fills its surface densely and will end up closing on itself at after an infinite time.

The following figure represents the different types of regular attractors.

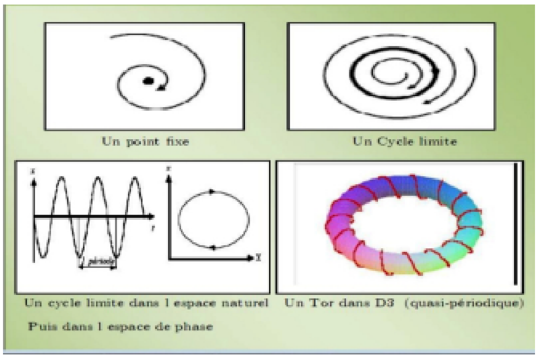


Figure 3.2: Regular attractors

**2) Strange attractors:**

These are complex geometric shapes which characterize the evolution of systems chaotic. - The characteristics of a strange attractor are:

* In phase space the attractor has zero volume.

* The dimension d of the attractor is fractal (not integer), for a continuous system autonomous **2<d<n**, with n the dimension of the phase space.

* Sensitivity to initial conditions (two initially neighboring trajectories end always moving away from each other).

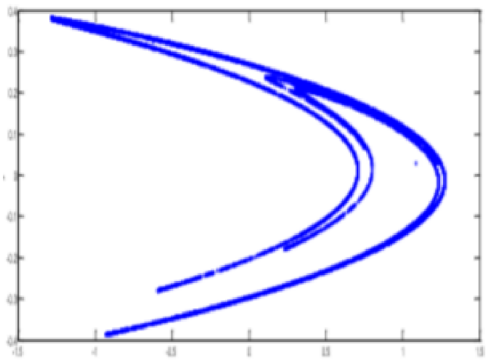

Figure 3.3: Hénon's strange attractors

### 3.4.4 Different Types of Chaotic (Strange) Attractors

Chaotic (strange) attractors can be classified into three main types:

**1. Hyperbolic attractor**

Hyperbolic attractors are structurally stable limit sets. Generally, most known physical systems do not belong to this class of hyperbolic attractor systems.
The Plykin attractor is an example of a hyperbolic attractor [21] (Figure 3.4 ).

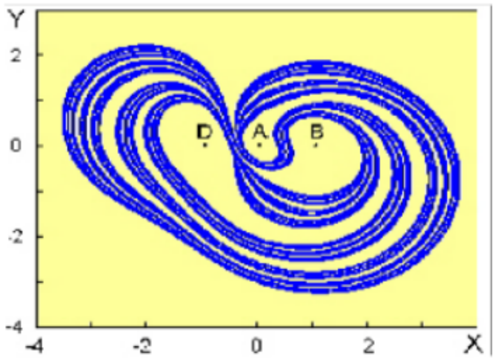


Figure 3.4: Chaotic attractors of Plykin

**2. Lorenz type attractor**

These types of attractors are not structurally stable, but their homoclinic and heteroclinic orbits are structurally stable (hyperbolic) and no unstable periodic orbits appear under small variations of parameters, as for example in the Lorenz system itself [14] .

**3. Quasi-attractors**

These types of attractors are limit sets containing periodic orbits of different topological types and structurally unstable orbits. For example, attractors generated by the Chua circuit [15].

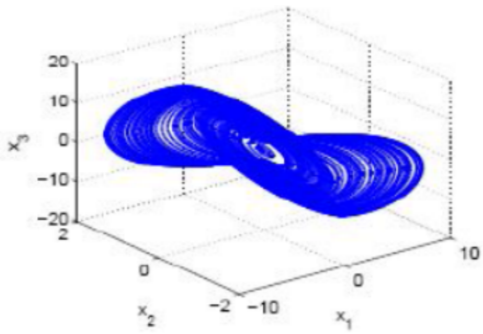


Figure 3.5: Chaotic attractors of Chua.

## 3.5 Chaos Detection

There are several methods for determining whether or not nonlinear systems are chaotic. They are generally not very numerous, nor distributed over a sufficiently long time on the scale of the system studied. We chose to implement two of the most commonly used methods which are called: **the fractal dimension** and **Lyapunov exponents** .

### 3.5.1 Lyapunov exponents

To quantify sensitivity relative to initial conditions, we often use the notion of Lyapunov number or Lyapunov exponent.

Lyapunov exponents are quantities that measure the divergence between different orbits within an attractor.

**One-dimensional case**

Consider the dynamic equation:

$$x_{n+1} = f(x_n), f \in C(\mathbb{R})$$

Let $x_0, \tilde{x}_0$ be two very close initial conditions, that is to say: $\tilde{x}_0 = x_0 + \delta x_0$.

We want to see how the orbits resulting from $x_0, \tilde{x_0}$, behave

$$\tilde{x_1} = x_1 + \delta x_1 = f(x_0 + \delta x_0) = f(x_0) + f^{'}(x_0)\delta x_0.$$

So the distance between the two trajectories after one iteration is given by:

$$|\delta x_1| = |f^{'}(x_0)\delta x_0| = |f^{'}(x_0)||\delta x_0|.$$

Using the rule of chain iterations, after n iterations, we obtain:

$$|\delta x_n| = |\tilde{x_n} - x_n| = \left(\prod_{j=0}^{n}\right)|f^{'}(x_j)||\delta(x_n)|.$$

On the other hand we have:

$$|\tilde{x_n} - x_n| = |f^n(\tilde{x_0}) - f^n(x_0)|.$$

If the two orbits $\tilde{x_n}$ , $x_n$ deviate at an exponential rate after n iterations, then

$$\frac{|\delta x_n|}{|\delta x_0|} = \exp^{nh}, h \in \mathbb{R} \Leftrightarrow \ln\frac{|\delta x_n|}{|\delta x_0|}| = hn \Leftrightarrow h = \frac{1}{n}\ln\prod_{j=0}^{n}|f^{'}(x_j)|$$

For $n \longrightarrow +\infty$ we have

$$h = \lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n}\ln|f^{'}(x_j)|. \tag{3.1}$$

If $h > 0$ the behavior is chaotic (the two orbits diverge).

We call $h$ the Lyapunov exponent.

**Example 3.5.1.** *Let a discrete system be defined by:*

$$x_{n+1} = f(x_n)$$

*and*

$$f(x) = \begin{cases} 4x, if x \leqslant \frac{1}{2} \\ 4(1-x), if x > \frac{1}{2} \end{cases}$$

*this system is chaotic because:*

$$h = \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n} \ln |f^{'}(x_j)| = \ln 4 > 0$$

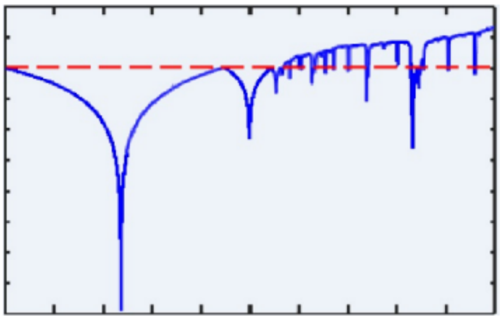

Figure 3.6: Lyapunov exponent of the map $x_{n+1} = ax_n(1-x_n)$as a function of parameter a

**Multidimensional case**

Our previous relation (3.1) generalizes to systems of dimension $p > 1$ , which have $p$ Lyapunov exponent.

Let the dynamic equation be defined by:

$$\begin{cases} x_{n+1} = f(x_n) \\ f : \mathbb{R}^p \longrightarrow \mathbb{R}^p \end{cases}$$

Under the same initial conditions $x_0, \tilde{x_0}$, we have

$$\delta x_1 = J(x_0)\delta x_0.$$

where $J(x_0)$ designates the Jacobian matrix of $f$ evaluated at $x_0$. After $n$ iterations, we obtain:

$$\delta x_n = \prod_{j=0}^{n} J(x_j)\delta x_{0.}$$

We see the chronological product of the Jacobian matrices being formed:

$$\prod_{j=0}^{n} J(x_j) = J(x_0)J(x_1)...J(x_n).$$

Or $J^n(x_0)$ represents the Jacobian matrix of $f^n$ .

If it is diagonalizable, then there exists an invertible matrix $P_p$ such that

$$D_p^n = p_p^{-1} J^n p_p$$

a diagonal matrix of eigenvalues $\lambda_i(f^n(x)), (i = 1, ..., p)$.

So: the Lyapunov exponent:

$$h_i = \lim_{n \to +\infty} \frac{1}{n} \ln |\lambda_i f^n(x_0)|$$

If $h_i > 0$ the attractor is strange.

**Example 3.5.2.** *We give in this figure the two Lyapunov exponents of the application of Hénon for $b = 0.3$ and $0.2 < a < 1.5$.*
*The Lyapunov exponents of a periodic orbit are negative.*

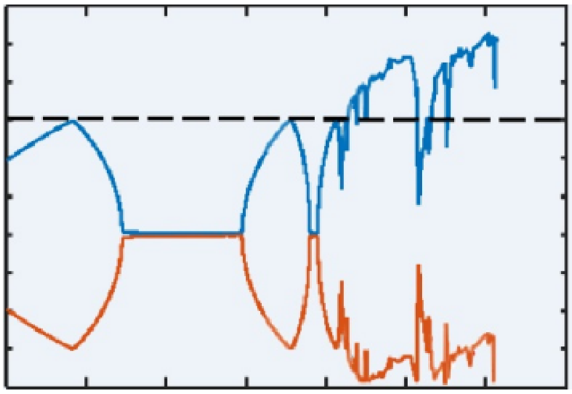

Figure 3.7: Lyapunov exponent of the Hénon application for b = 0.3 and 0.2 < a < 1.5

### 3.5.2 The fractal dimension

If the attractor is a point its dimension is 0, if the attractor is a line or a closed curve its dimension is 1 and so on. But there is another kind of attractor which has an unusual shape, a fractal geometric structure.

**Definition 3.5.1.** *Given space $A$, we cover space $A$ by means of a countable union of parts rated $A_i$, each being of diameter less than $r$ .*
*We enter the quantity:*

$$H_r^s(A) = \inf_{diam(A_i)<r} \left\{ \sum_{i=1}^{\infty} (diam(A_i))^s ; A \subset \bigcup_{i=1}^{\infty} A_i \right\}.$$

*Hence the definition:*

$$H^s(X) = \lim_{r \to 0} H_r^s(X).$$

*$H^s$ is called s-dimensional Hausdorff measure.*

*The Hausdorff dimension of $A \subset \mathbb{R}^n$ is defined by:*

$$D_H = \sup\left\{s, H^s(A) = +\infty\right\} = \inf\left\{s, H^s(A) = 0\right\}.$$

**Definition 3.5.2.** *The set $A$ is fractal if its Hausdorff dimension is not integer.*

## 3.6 Example of chaotic dynamical systems in continuous time

### 3.6.1 Lorenz system

The Lorenz system is a famous example of the differential system with chaotic behavior. This system is defined by the following equations:

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = x(r - z) - y, \\ \dot{z} = xy - bz. \end{cases} \tag{3.2}$$

When the actual parameters $\sigma, r$ and $b$ take the following values: $\sigma = 10$, and $b = \dfrac{8}{3}$ , the behavior of the system (3.2) is chaotic.

The different phase portraits obtained for different values of the control parameters are widely studied.

- **Model equilibrium**

  We look for the equilibrium points $(x, y, z)$ verifying $\dot{x} = \dot{y} = \dot{z} = 0$. A first trivial fixed point is $p_0 = (0, 0, 0)$ for $r < 1$, and for $r > 1$, there are two other points.

  we have:

$$\begin{cases} \dot{x} = \sigma(y - x) = 0 \\ \dot{y} = x(r - z) - y = 0 \\ \dot{z} = xy - bz = 0 \end{cases} \Leftrightarrow \begin{cases} x = y, \\ z = r - 1, \\ y = \pm\sqrt{b(r - 1)}. \end{cases} \tag{3.3}$$

therefore, the two points are:

$$p_{1,2} = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, (r-1)). \tag{3.4}$$

The study of the stability of the equilibrium points is based on the sign of the real part of the eigenvalues of the Jacobian matrix $DJ$ which is written by:

$$DJ = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & -x \\ y & 0 & -b \end{pmatrix} \tag{3.5}$$

- **Zero equilibrium stability (at point (0, 0, 0))**

At the point (0, 0, 0), the eigenvalues of the Jacobian matrix are:

$$DJ = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

So $\mid A - \lambda I_d \mid$ is defined by:

$$\mid A - \lambda I_d \mid = \det \begin{pmatrix} -\sigma-\lambda & \sigma & 0 \\ r & -1-\lambda & 0 \\ 0 & 0 & -b-\lambda \end{pmatrix}$$

Then, the characteristic equation is defined by:

$$P(\lambda) = \det(DJ - \lambda I) = (\lambda + b)(\lambda^2 + (\sigma+1)\lambda + \sigma(1-r)) = 0. \tag{3.6}$$

For the roots, we have:

$$\lambda_1 = \frac{-\sigma - 1 + \sqrt{(\sigma+1)^2 + 4\sigma(r-1)}}{2}, \lambda_3 = \frac{-\sigma - 1 - \sqrt{(\sigma+1)^2 + 4\sigma(r-1)}}{2}, \lambda_2 = -b.$$

We report the values of $\sigma$ and $b$, we obtain:

$$\lambda_1 = \frac{-11 + \sqrt{40r+81}}{2}, \lambda_3 = \frac{-11 - \sqrt{40r+81}}{2}, \lambda_2 = -\frac{8}{3}. \tag{3.7}$$

$*$ For $r < 1$; the three real roots are negative, $\lambda_3 < \lambda_2 < \lambda_1 < 0$ , the equilibrium is therefore a stable node.

* For $r > 1$; one of the eigenvalues is positive $\lambda_1$, the equilibrium is therefore unstable (col).

* There is a bifurcation when r = 1; the equilibrium is said to be marginal.
  So, points $p_1, p_2$ are stable for $\mid r - 1 \mid \ll 1$.

- **Random appearance**

  Figure (3.8) illustrates the random aspect of the state $x$ of the system (3.2).

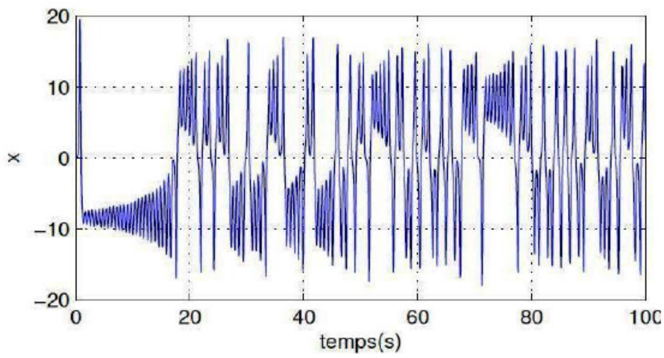


Figure 3.8: Random aspects of the state x of the Lorenz system.

- **Strange attractor**

  The chaotic system (3.2) presents a superb butterfly-shaped strange attractor (Figure 3.9) for values $\sigma = 10$, $b = \dfrac{8}{3}$,$r = 28$.
  We observe that the dynamics of the Lorenz system given by system (3.2) is independent of time t; therefore, this type of system is termed to be self-contained.

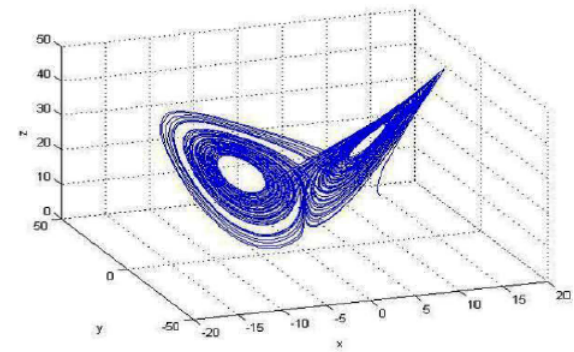


Figure 3.9: Lorenz strange attractor.

- **Lyapunov exponents**

  Calculation of Lyapunov exponents (Figure: 3.10) gives the following values:

$$h_1 == 0.85922, h_2 = -0.0015763, h_3 = -14.5208.$$

We clearly see that there is a positive Lyapunov exponent, which means that the system is chaotic.

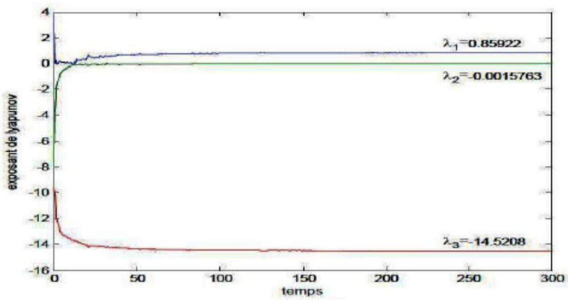


Figure 3.10: Lyapunov exponents of the Lorenz continuous chaotic system.

## 3.7 Example of chaotic dynamical systems in discrete time

### 3.7.1 Application of Hénon

The astronomer Michel Hénon allows us to react quickly and produce a very simple system.
This discrete dynamic system of dimension 2, in discrete time because the points evolve in stages and not continuously. It is defined by the following form:

$$\begin{cases} x_{n+1} = -ax_n^2 + y_n + 1 \\ y_n = bx_n \end{cases} \tag{3.8}$$

$a, b \in \mathbb{R}$ represent bifurcation parameters.
The value of the constant a controls the non-linearity of the iteration, and that of b reflects the role of dissipation. The values usually used for a ; b are a = 1; 4 and b = 0; 3. The Hénon application is invertible, its inverse is:

$$H^{-1}(x, y) = \begin{pmatrix} b^{-1}y \\ x - 1 + \frac{a}{b^2}y^2 \end{pmatrix}. \tag{3.9}$$

The Jacobian matrix $DH_{a,b}$ is :

$$DH_{a,b} = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix} \tag{3.10}$$

The determinant of the jacobiebnne matrix is equal to $|J| = -b$.this application has two hyperbolic fixed points defined by:

$$\begin{cases} P_1 = (x_1, y_1) = \left(\frac{b-1+\sqrt{(1-b)^2+4a}}{2a}, b\frac{b-1-\sqrt{(1-b)^2+4a}}{2a}\right), if \quad 0 < b < 1, \\ P_2 = (x_2, y_2) = \left(\frac{b-1-\sqrt{(1-b)^2+4a}}{2a}, b\frac{b-1-\sqrt{(1-b)^2+4a}}{2a}\right), if \quad 0 < b < 0. \end{cases}$$

- **Stability**

  We can easily determine the local stability of these points by evaluating the eigenvalues of the Jacobian matrix defined in( 3.9 ) the characteristic equation of the Jacobian matrix is:

$$\xi^2 + 2ax\xi - b = 0$$

  and their eigenvalues are:

$$\begin{cases} \xi_1 = -ax + \sqrt{a^2x^2 + b}, For P_1. \\ \xi_2 = -ax - \sqrt{a^2x^2 + b}, For P_2. \end{cases}$$

  If we calculate the absolute values of the eigenvalues, we see that the smallest of the values own is always less than 1, while the largest is less, or equal to or greater to 1 depending on whether $|x|$ is less than, equal to or greater than $\dfrac{(1-b)}{2a}$,we deduce that the fixed point $P_2$ is a saddle point.
  The other fixed point is stable if $a < \dfrac{3(1-b)^2}{4} = 0.3675$ . if $a = \dfrac{3(1-b)^2}{4}$, we have $\xi_1(x_1, y_1) = b$ and $\xi_2(x_2, y_2) = -1$.

- **Sensitivity to initial conditions**

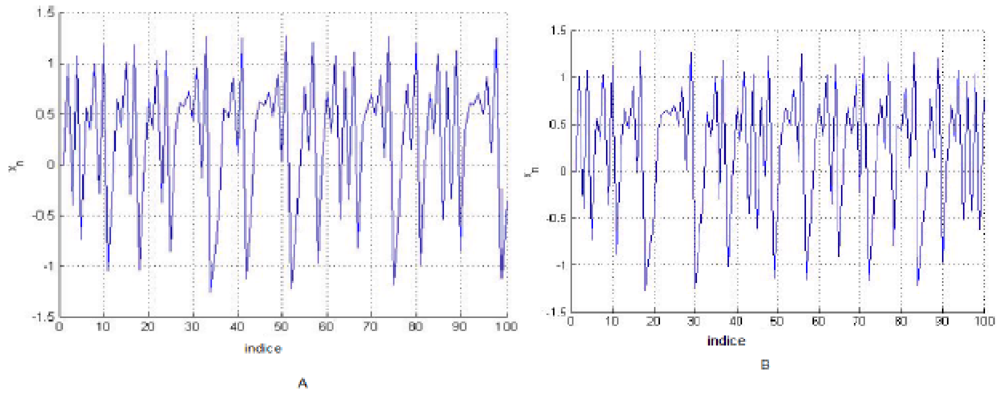


Figure 3.11: The first 100 iterations of $x_n$ with $a = 1.4$ and b $= 0.3$ with initial condition $A = (x_0, y_0) = (0, 0)$ and $B = (x_0, y_0) = (0.001, 0.001)$.

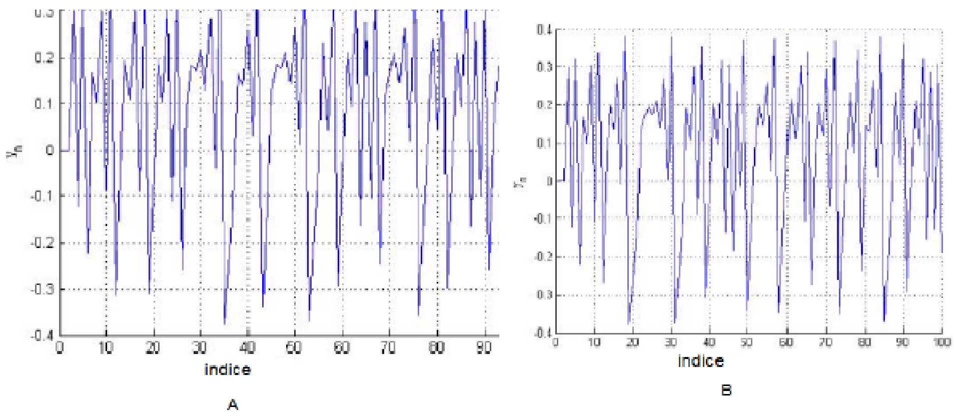


Figure 3.12: The first 100 iterations of $y_n$ with $a = 1.4$ and b $= 0.3$ with initial condition $A = (x_0, y_0) = (0, 0)$ and $B = (x_0, y_0) = (0.001, 0.001)$.

- **Lyapunov exponents**

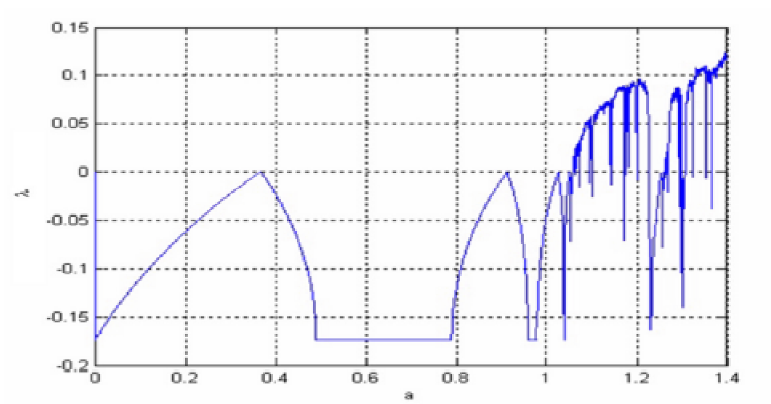


Figure 3.13: The evolution of the Lyapunov exponent $\lambda$ of the system of Hénon as a function of $a$.

We set b $= 0.3$, and we let $a$ vary between 0 and 1.4. From figure (3, 13) we obtain two zones:

∗ a stable zone when $a$ varies in the interval $[0; 1.\ 052]$.

∗ a chaotic zone when $a$ varies in the interval $]1.\ 052; 1.4]$.

- **Bifurcation Diagram**

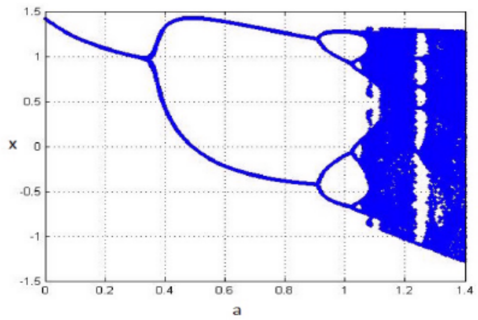


Figure 3.14: Hénon bifurcation diagram.

∗ if $-0.1225 < a < 0.3675$ , the iterations converge towards a point of the plan.

∗ if $0.3675 < a < 0.9$ , the iterations tend to constitute a sequence $(x_n, y_n)$ such as $(x_{2n}, y_{2n})$ converges towards a point and $(x_{2n+1}, y_{2n+1})$ converges towards another point. We therefore have two limit points, we observe a doubling of period.

∗ if $0.9 < a < 1.02$ we are witnessing a new doubling of the period.
The period continues to double until a determined value where the trajectory begins to take a particular shape.
For $a \geq 1.02$ , we no longer distinguish the cycles: the system is chaotic.

- **Hénon attractor**
Hénons chaotic attractor for numerical values $a = 1.4$ and $b = 0.3$.

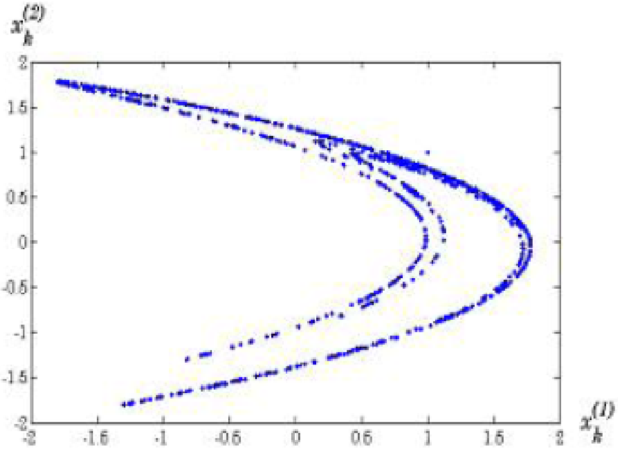


Figure 3.15: Hénon attractor

# Conclusion

The idea of chaos is very useful in many areas. In fact, in meteorology clearly, and also in biology and chemistry, and in physics, and also the detection of chaotic signals can make it difficult to track them, and destroy them if they are infiltrated, which makes chaos also useful in cryptography, and in the world of communications. These signals, which are sensitive to initial conditions and cannot be predicted by Its nature play a role in concealing information signals in order to transmit them in a safe manner.

In this work, we have provided a simple introduction to chaos theory, as our thesis spanned three important and interconnected chapters. The first is devoted to basic concepts, and we touched on bifurcation theory in the second chapter, and finally we touched on chaos and methods for discovering it.

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## ملخص

تهدف هذه المذكرة إلى تقديم الأنظمة الديناميكية، وإعطاء لمحة عامة حول المفاهيم الأساسية لدراسة هذه الأنظمة ، مثل نقاط التوازن، واستقرارها، ثم أنواع التشعبات المختلفة، مما يقودنا إلى اكتشاف السلوك الفوضوي، حيث أن تغير بسيط في الشروط الأولية يسبب اختلافا كبيرا في النتائج طويلة المدى، ومن هنا نشأت فكرة عدم القدرة على التنبؤ . ناقشنا أيضا العديد من الخصائص بما في ذلك الحساسية للشروط الأولية وأسس ليابونوف والجاذب الفوضوي للكشف عن السلوك الفوضوي .

**الكلمات المفتاحية :** النظام الديناميكي، نقطة التوازن ، المدار، المسار، الاستقرار، التشعب، مخطط التشعب ، نقطة التشعب ، الفوضى، الجاذب الفوضوي ، أسس ليابونوف.

## Abstract

The aim of this memory is to present dynamic systems, and give an overview of the basic concepts for studying these systems, such as equilibrium points, their stability, and then the different types of bifurcations, which leads us to discover chaotic behavior, where a small change in initial conditions causes a significant difference in long-term results. Hence the idea of unpredictability arose. We also discussed several properties including sensitivity to initial conditions, Lyapunov exponents, and the chaotic attractor for detecting chaotic behavior.

**Key words** : Dynamical system , equilibrium point , orbit, trajectory, stability, bifurcation, bifurcation diagram , chaos, Chaotic attractor , Lyaponov exponents.

## Résumé

Le but de ce mémoire est de présenter des systèmes dynamiques, et à donner un aperçu des concepts de base pour étudier ces systèmes, comme les points d'équilibre, leur stabilité, puis les différents types de bifurcations, ce qui nous amène à découvrir des comportements chaotiques, où un petit changement dans les conditions initiales entraînent une différence significative dans les résultats à long terme. D'où l'idée d'imprévisibilité. Nous avons également discuté de plusieurs propriétés, notamment la sensibilité aux conditions initiales, les exposants de Lyapunov et l'attracteur chaotique pour détecter un comportement chaotique.

**Mots Clés** : Système dynamique, Point d'équilibre , orbite, trajectoire, stabilité, bifurcation , diagramme de bifurcation , chaos, attracteur chaotique , Les exposants de Lyaponov .