People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research

University of Echahid Hamma Lakhdar - El Oued-
Faculty of Exact Sciences
Department of Mathematics

Academic Master

Final year project
Domain: Mathematics and informatics
Field: Mathematics
Specialization: Fundamental and applied mathematics

Title:

Some minimization theorems and their applications to partial differential equations

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Defended on 6 June 2026, in front of the jury composed of:

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Academic year: 2025 – 2026

Dedication

To our beloved parents,

whose boundless love, endless
sacrifices, and heartfelt prayers
have been the guiding light illuminating our path.

To our dear brothers and sisters,

for their unwavering support
and constant encouragement.

To all our esteemed teachers and professors,

who gracefully shared their knowledge and guided us
through this academic journey.

Acknowledgements

We would like to express our deepest gratitude to our supervisor, Prof. Zaouche El Mehdi, for his invaluable guidance, patience, and encouragement throughout this work. His insights and expertise were fundamental to the completion of this thesis.

We also extend our thanks to the faculty and staff of the Department of Mathematics, University of El-Oued, for providing a stimulating academic environment.

Finally, we are profoundly grateful to our families and friends for their unwavering support and understanding.

Abstract

This thesis aims to explore some minimization theorems and their applications to partial differential equations. We examine three main variational approaches which are the direct method in the calculus of variations, Ekeland's variational principle, and the Mountain Pass Theorem. We then apply these tools to study the existence of solutions for several nonlinear boundary value problems, including a subcritical semilinear elliptic equation, a constrained minimization problem, and a nonlinear eigenvalue problem.

Keywords: Variational Methods, Direct Method, Ekeland Principle, Mountain Pass Theorem, Semilinear Elliptic Equations.

Résumé

Cette thèse vise à explorer certains théorèmes de minimisation et leurs applications aux équations aux dérivées partielles. Nous examinons trois principales approches variationnelles qui sont la méthode directe dans le calcul des variations, le principe variationnel d'Ekeland et le théorème du col. Nous appliquons ensuite ces outils pour étudier l'existence de solutions de plusieurs problèmes aux limites non linéaires, y compris une équation elliptique semi-linéaire sous-critique, un problème de minimisation sous contrainte et un problème de valeurs propres non linéaire.

Mots-clés : Méthodes variationnelles, Méthode directe, Principe d'Ekeland, Théorème du col, Équations elliptiques semi-linéaires.

ملخص

تهدف هذه الأطروحة إلى تناول بعض نظريات التصغير وتطبيقاتها على المعادلات التفاضلية الجزئية. ندرس ثلاثة طرق تغيرية رئيسية وهي الطريقة المباشرة في حساب التغيرات، مبدأ إيكلاند التغيري، ونظرية ممر الجبل. ثم نطبق هذه الأدوات لدراسة وجود حلول لعدة مسائل قيم حدية غير خطية، بما في ذلك معادلة إهليلجية شبه خطية دون حرجة، مسألة تصغير مقيدة، ومسألة قيم ذاتية غير خطية.

الكلمات المفتاحية: الطرق التغيرية، الطريقة المباشرة، مبدأ إيكلاند، نظرية ممر الجبل، المعادلات الإهليلجية شبه الخطية.

Contents

Notation and Symbols

Symbol	**Meaning**
$\mathbb{R}$	Real numbers
$\mathbb{N}$	Natural numbers
Ω	Open subset of $\mathbb{R}^N$
$\partial\Omega$	Boundary of Ω
$C_c^\infty(\Omega)$	Space of smooth functions with compact support
$L^p(\Omega)$	Lebesgue space of p-integrable functions
$W^{k,p}(\Omega)$	Sobolev space of order k with p-integrability
$H^1(\Omega)$	Sobolev space $W^{1,2}(\Omega)$
$H_0^1(\Omega)$	Closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$
X^*	Dual space of X
$\|\cdot\|_X$	Norm on space X
$\langle\cdot,\cdot\rangle$	Inner product or duality pairing
$\rightharpoonup$	Weak convergence
∇	Gradient operator
Δ	Laplacian operator
$J'(x)$	Fréchet derivative at x
$(PS)_c$	Palais–Smale condition at level c
p^*	Sobolev conjugate exponent: $p^* = \frac{Np}{N-p}$
J^c	Sublevel set $\{u : J(u) \leq c\}$
Γ	Set of paths in mountain pass theorem

Introduction

The calculus of variations is a field of mathematical analysis that deals with optimizing functionals, which are mappings from a set of functions to the real numbers. It reformulates complex partial differential equations (PDEs) into minimization tasks, seeking functions that extremize a specific functional (often an integral). This concept serves as a cornerstone of mathematical analysis, deeply linking differential geometry, physics, and optimal control theory. This document explores three primary variational approaches which are the Direct Method, Ekeland's Variational Principle, and the Mountain Pass Theorem. Such frameworks are extensively applied to establish the existence of solutions across a range of nonlinear boundary value problems.

The direct method serves as a traditional procedure for determining minimizers of functionals that exhibit coercivity and weak lower semicontinuity. In scenarios where a functional might not be bounded from below or suffers from a lack of compactness, Ekeland's Variational Principle is employed to identify approximate critical points. Additionally, the Mountain Pass Theorem, originally established by Ambrosetti and Rabinowitz, enables the discovery of critical points exhibiting a saddle-point geometry.

The primary goal of this research is to study these theoretical concepts and demonstrate their application to particular mathematical problems. In brief, the organization of this manuscript is outlined below:

- **Chapter 1** delivers essential background material concerning functional analysis, metric spaces, and Sobolev spaces, used through this memory.
- **Chapter 2** presents the core trio of variational principles—the Direct Method, Ekeland's Variational Principle, and the Mountain Pass Theorem—explaining their rigorous proofs and geometric intuitions.
- **Chapter 3** focuses on practical implementations, illustrating the ways these methods can establish the existence of weak solutions for several nonlinear partial differential equations as well as boundary value problems with constraints.

Chapter 1

Foundations of Analysis

1.1 Metric Spaces

1.1.1 Definition and Basic Concepts

Definition 1.1 (Metric Space [13]). Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a **metric** (or distance) if it satisfies the following properties for all $x, y, z \in X$:

(i) **Positive definiteness:** $d(x, y) = 0$ if and only if $x = y$.

(ii) **Symmetry:** $d(x, y) = d(y, x)$.

(iii) **Triangle inequality:** $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a **metric space**.

Example 1.2 (Standard Metric Spaces [13]). 1. **Real line:** The set $X = \mathbb{R}$ with the metric $d(x, y) = |x - y|$.

2. **Euclidean space:** The space $\mathbb{R}^N$ with $d(x, y) = \sqrt{\sum_{i=1}^{N} (x_i - y_i)^2}$.
3. **Discrete metric:** Any nonempty set X with $d(x, y) = 0$ if $x = y$, and $d(x, y) = 1$ otherwise.
4. **Space of continuous functions:** $C([0, 1])$ with the supremum metric $d(f, g) = \max_{x \in [0,1]} |f(x) - g(x)|$.
5. **Sequence space:** The space ℓ^p with $d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p}$ for $1 \leq p < \infty$.

1.1.2 Convergence, Cauchy Sequences and Completeness

Definition 1.3 (Convergence and Cauchy sequences [13]). Let (X, d) be a metric space.

1. A sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ **converges** to $x \in X$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N$. We write $x_n \to x$.
2. $\{x_n\}$ is a **Cauchy sequence** if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.

Definition 1.4 (Complete Metric Space [13]). A metric space (X, d) is called **complete** if every Cauchy sequence in X converges to a point in X.

Example 1.5 (Complete and Incomplete Metric Spaces [13]). 1. The space $\mathbb{R}$ with the usual metric is complete.
2. $\mathbb{R}^N$ with the Euclidean metric is complete.
3. $C([0,1])$ with the metric $d(f,g) = \max|f-g|$ is complete.
4. $(0,1)$ with the usual metric is not complete (e.g., $x_n = 1/n$ is Cauchy but does not converge in $(0,1)$).
5. $C([0,1])$ with the L^1 metric $d(f,g) = \int_0^1 |f-g|\,dx$ is not complete.

1.2 Normed Spaces and Banach Spaces

1.2.1 Definitions and Basic Examples

Definition 1.6 (Normed Space [13]). Let X be a real vector space. A function $\|\cdot\| : X \to [0,\infty)$ is called a **norm** if it satisfies:

(i) **Positive definiteness:** $\|x\| = 0$ if and only if $x = 0$.
(ii) **Homogeneity:** $\|\lambda x\| = |\lambda|\,\|x\|$ for all $\lambda \in \mathbb{R}$, $x \in X$.
(iii) **Triangle inequality:** $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a **normed space**.

Example 1.7 (Elementary Normed Spaces [13]). 1. **Euclidean space:** $\mathbb{R}^N$ with $\|x\|_2 = \sqrt{\sum_{i=1}^N |x_i|^2}$.
2. **Manhattan norm:** $\mathbb{R}^N$ with $\|x\|_1 = \sum_{i=1}^N |x_i|$.
3. **Supremum norm:** $\mathbb{R}^N$ with $\|x\|_\infty = \max_{1\le i\le N} |x_i|$.
4. **Continuous functions:** $C([0,1])$ with $\|f\|_\infty = \max_{x\in[0,1]} |f(x)|$.

Definition 1.8 (Banach Space [13]). A normed space $(X, \|\cdot\|)$ is called a **Banach space** if it is complete with respect to the metric induced by the norm; i.e., every Cauchy sequence in X converges to an element of X.

Example 1.9 (Banach Spaces [13]). 1. $\mathbb{R}^N$ with any norm is a Banach space.
2. $C([0,1])$ with the maximum norm is a Banach space.
3. $L^p(\Omega)$ for $1 \le p \le \infty$ is a Banach space (Riesz–Fischer theorem, see [3]).
4. ℓ^p for $1 \le p \le \infty$ is a Banach space.
5. $C^k([0,1])$ with the norm $\|f\|_{C^k} = \sum_{j=0}^k \max_{[0,1]} |f^{(j)}|$ is a Banach space.

1.2.2 Hilbert Spaces

Definition 1.10 (Inner Product [3]). Let H be a real vector space. A function $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{R}$ is called an **inner product** if it satisfies:

(i) **Symmetry:** $\langle x, y\rangle = \langle y, x\rangle$.
(ii) **Linearity in the first argument:** $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$.
(iii) **Positive definiteness:** $\langle x, x\rangle \ge 0$, and $\langle x, x\rangle = 0$ iff $x = 0$.

The norm induced by the inner product is $\|x\| = \sqrt{\langle x, x\rangle}$.

Theorem 1.11 (Cauchy–Schwarz Inequality [3]). *For all* $x, y \in H$, $|\langle x, y\rangle| \le \|x\|\,\|y\|$.

Definition 1.12 (Hilbert Space [3]). A **Hilbert space** is a Banach space whose norm is derived from an inner product.

Example 1.13 (Hilbert Spaces [3]). 1. $\mathbb{R}^N$ with the Euclidean inner product $\langle x, y\rangle = \sum_{i=1}^N x_i y_i$.
2. ℓ^2 with $\langle x, y\rangle = \sum_{n=1}^\infty x_n y_n$.
3. $L^2(\Omega)$ with $\langle f, g\rangle = \int_\Omega f(x)g(x)\,\mathrm{d}x$.
4. Sobolev space $H^1(\Omega)$ with $\langle u, v\rangle = \int_\Omega(\nabla u \cdot \nabla v + uv)\,\mathrm{d}x$.

1.2.3 Dual Spaces

Definition 1.14 (Dual Space [3]). Let X be a normed space. The **topological dual** X^* is the space of all bounded linear functionals $f : X \to \mathbb{R}$, with norm

$$\|f\|_{X^*} = \sup_{\|x\|\leq 1} |f(x)|.$$

Example 1.15 (Dual Spaces [3]). 1. For $1 < p < \infty$, $(L^p(\Omega))^* \cong L^{p'}(\Omega)$ where $\frac{1}{p} + \frac{1}{p'} = 1$.
2. $(L^1(\Omega))^* \cong L^\infty(\Omega)$.
3. For a Hilbert space H, $H^* \cong H$ via the Riesz representation theorem.

1.3 Weak Convergence

1.3.1 Definition and Basic Properties

Definition 1.16 (Weak Convergence [3]). A sequence $\{x_n\}$ in a normed space X **converges weakly** to $x \in X$ if $f(x_n) \to f(x)$ for every $f \in X^*$. We write $x_n \rightharpoonup x$.

Definition 1.17 (Strong Convergence [3])**.** $x_n \to x$ strongly if $\|x_n - x\| \to 0$.

Proposition 1.18 (Properties of Weak Convergence [3])**.** *In any normed space X, the following hold:*

1. *Strong convergence implies weak convergence.*
2. *Weak limits are unique.*
3. *If $x_n \rightharpoonup x$, then $\{x_n\}$ is bounded and $\|x\| \leq \liminf \|x_n\|$.*
4. *In finite-dimensional spaces, weak and strong convergence coincide.*

1.3.2 Reflexive Spaces

Definition 1.19 (Reflexive Space [3])**.** A Banach space X is called **reflexive** if the canonical embedding $J : X \to X^{**}$ defined by $J(x)(f) = f(x)$ is surjective (i.e., an isometric isomorphism).

Corollary 1.20 (Weak Sequential Compactness [3])**.** *Let X be a reflexive Banach space. Then every bounded sequence in X has a weakly convergent subsequence.*

Example 1.21 (Reflexive Spaces [3])**.** 1. Every Hilbert space is reflexive.

2. $L^p(\Omega)$ for $1 < p < \infty$ is reflexive.
3. $L^1(\Omega)$ is not reflexive.
4. $C([0, 1])$ is not reflexive.
5. ℓ^1 is not reflexive.

1.4 Sobolev Spaces

1.4.1 Weak Derivatives

Definition 1.22 (Weak Derivative [8]). Let $\Omega \subset \mathbb{R}^N$ be open, $u \in L^1_{\text{loc}}(\Omega)$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_N)$, we say that $v \in L^1_{\text{loc}}(\Omega)$ is the α-th weak derivative of u if

$$\int_\Omega u(x) D^\alpha \varphi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v(x)\varphi(x)\,\mathrm{d}x \quad \forall \varphi \in C_c^\infty(\Omega).$$

We denote $v = D^\alpha u$.

Example 1.23 (Weak derivative of $|x|$ [8]). Let $\Omega = (-1, 1)$ and $u(x) = |x|$. According to Definition 1.22, we compute the weak derivative.

1: Split the integral. For any test function $\varphi \in C_c^\infty(-1, 1)$,

$$\int_{-1}^{1} |x|\varphi'(x)\,\mathrm{d}x = \int_{-1}^{0} (-x)\varphi'(x)\,\mathrm{d}x + \int_0^1 x\varphi'(x)\,\mathrm{d}x.$$

2: Integrate by parts on $(-1, 0)$**.**

$$\int_{-1}^{0} (-x)\varphi'(x)\,\mathrm{d}x = \big[(-x)\varphi(x)\big]_{-1}^{0} - \int_{-1}^{0} (-1)\varphi(x)\,\mathrm{d}x = 0 + \int_{-1}^{0} \varphi(x)\,\mathrm{d}x.$$

3: Integrate by parts on $(0, 1)$**.**

$$\int_0^1 x\varphi'(x)\,\mathrm{d}x = \big[x\varphi(x)\big]_0^1 - \int_0^1 \varphi(x)\,\mathrm{d}x = 0 - \int_0^1 \varphi(x)\,\mathrm{d}x.$$

4: Combine the results.

$$\int_{-1}^{1} |x|\varphi'(x)\,\mathrm{d}x = \int_{-1}^{0} \varphi(x)\,\mathrm{d}x - \int_0^1 \varphi(x)\,\mathrm{d}x = -\int_{-1}^{1} \operatorname{sgn}(x)\varphi(x)\,\mathrm{d}x.$$

5: Identify the weak derivative. By Definition 1.22, the weak derivative of u is $v(x) = \operatorname{sgn}(x)$, which belongs to $L^\infty(-1, 1)$.

1.4.2 Definition of Sobolev Spaces

Definition 1.24 (Sobolev Space $W^{k,p}(\Omega)$ [8]). Let $\Omega \subset \mathbb{R}^N$ be open, $k \in \mathbb{N}$, and $1 \leq p \leq \infty$. The Sobolev space $W^{k,p}(\Omega)$ is defined by

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq k\},$$

equipped with the norm

$$\|u\|_{W^{k,p}} = \begin{cases} \left(\sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^p}^p\right)^{1/p}, & 1 \leq p < \infty, \\ \sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^\infty}, & p = \infty. \end{cases}$$

For $p = 2$, we write $H^k(\Omega) = W^{k,2}(\Omega)$, which is a Hilbert space with inner product

$$\langle u, v\rangle_{H^k} = \sum_{|\alpha|\leq k} \int_\Omega D^\alpha u \, D^\alpha v \, dx.$$

Definition 1.25 ($W_0^{k,p}(\Omega)$ [8]). $W_0^{k,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$. Functions in $W_0^{k,p}(\Omega)$ satisfy homogeneous Dirichlet boundary conditions in the trace sense.

1.4.3 Sobolev Embedding Theorems

Theorem 1.26 (Sobolev Inequality [8]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $1 \leq p < N$. There exists a constant $C = C(N, p, \Omega) > 0$ such that for every $u \in W_0^{1,p}(\Omega)$,*

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)},$$

where $p^* = \frac{Np}{N-p}$ is the Sobolev conjugate exponent.

Theorem 1.27 (Sobolev Embedding Theorem [8]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Then:*

1. *If $1 \leq p < N$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $1 \leq q \leq p^*$, where $p^* = \frac{Np}{N-p}$ is the Sobolev conjugate exponent. Moreover, the embedding is **compact** for $1 \leq q < p^*$.*
2. *If $p = N$, then $W^{1,N}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \in [1, \infty)$, compactly.*
3. *If $p > N$, then $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ with $\alpha = 1 - \frac{N}{p}$, compactly.*

Theorem 1.28 (Trace Theorem [8]). *Let Ω be a bounded open set with C^1 boundary. There exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that $Tu = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. This operator justifies the definition of boundary values for Sobolev functions.

Theorem 1.29 (Rellich–Kondrachov Compactness Theorem [8]). *Under the same hypotheses (i.e., Ω bounded with Lipschitz boundary), for $1 \leq p < N$, the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for every $1 \leq q < p^*$.*

1.4.4 Poincaré Inequality

Theorem 1.30 (Poincaré Inequality [8]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. For every $u \in W_0^{1,p}(\Omega)$ with $1 \leq p < \infty$, there exists a constant $C = C(\Omega, p) > 0$ such that*

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}.$$

Consequently, $\|\nabla u\|_{L^p}$ is an equivalent norm on $W_0^{1,p}(\Omega)$.

1.5 Convexity and Lower Semicontinuity

1.5.1 Convex Sets and Functions

Definition 1.31 (Convex Set [3]). A subset C of a vector space X is called **convex** if for all $x, y \in C$ and $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in C$.

Definition 1.32 (Convex Function [3]). Let $C \subset X$ be convex. A function $J : C \to \mathbb{R} \cup \{+\infty\}$ is called **convex** if for all $u, v \in C$ and $\lambda \in [0,1]$,

$$J(\lambda u + (1-\lambda)v) \leq \lambda J(u) + (1-\lambda)J(v).$$

If X itself is the domain, we write $J : X \to \mathbb{R} \cup \{+\infty\}$ and require the inequality for all $u, v \in X$.

Definition 1.33 (Strictly Convex Function [3]). A convex function J is called **strictly convex** if for all $u \neq v$ and $\lambda \in (0,1)$,

$$J(\lambda u + (1-\lambda)v) < \lambda J(u) + (1-\lambda)J(v).$$

1.5.2 Lower Semicontinuity

Definition 1.34 (Lower Semicontinuity [6]). Let X be a topological space and $J : X \to \mathbb{R} \cup \{+\infty\}$. J is **lower semicontinuous** at u_0 if for every sequence $u_n \to u_0$,

$$\liminf_{n \to \infty} J(u_n) \geq J(u_0).$$

J is **weakly lower semicontinuous** if the same holds for $u_n \rightharpoonup u_0$.

Lemma 1.35 (Mazur's Lemma [3]). *Let* X *be a Banach space and* $\{x_n\} \subset X$ *a sequence such that* $x_n \rightharpoonup x$ *weakly. Then there exists a sequence of convex combinations*

$$y_n = \sum_{k=n}^{N_n} \lambda_k^{(n)} x_k, \quad \lambda_k^{(n)} \geq 0, \; \sum_{k=n}^{N_n} \lambda_k^{(n)} = 1,$$

such that $y_n \to x$ *strongly in* X*.*

Theorem 1.36 (Convexity and Weak Lower Semicontinuity [6]). *Let* X *be a Banach space and* $J : X \to \mathbb{R} \cup \{+\infty\}$ *a convex function. If* J *is lower semicontinuous with respect to strong convergence, then* J *is weakly lower semicontinuous.*

Proof. Assume $u_n \rightharpoonup u$. We need to show $\liminf_{n\to\infty} J(u_n) \geq J(u)$. Consider a subsequence such that $\liminf J(u_n) = \lim_k J(u_{n_k})$. Rename this subsequence as $\{u_n\}$ for simplicity.

Since $u_n \rightharpoonup u$, by Mazur's lemma (Lemma 1.35) there exists a sequence of convex combinations

$$v_m = \sum_{i=m}^{N_m} \lambda_i u_i, \quad \lambda_i \geq 0, \; \sum_{i=m}^{N_m} \lambda_i = 1,$$

such that $v_m \to u$ strongly. By convexity of J,

$$J(v_m) \leq \sum_{i=m}^{N_m} \lambda_i J(u_i).$$

Taking the limit inferior as $m \to \infty$, and using the strong lower semicontinuity of J at u,

$$J(u) \leq \liminf_{m\to\infty} J(v_m) \leq \liminf_{m\to\infty} \sum_{i=m}^{N_m} \lambda_i J(u_i) = \liminf_{n\to\infty} J(u_n).$$

This completes the proof. □

1.6 Fundamental Inequalities

Theorem 1.37 (Hölder's Inequality [3]). *Let* $1 \leq p, q \leq \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$, *then* $fg \in L^1(\Omega)$ *and*

$$\int_\Omega |fg| \, \mathrm{d}x \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Theorem 1.38 (Young's Inequality [3]). *For* $a, b \geq 0$ *and* $1 < p, q < \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

In particular, for $p = q = 2$, $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$.

Chapter 2

The Three Pillars of Variational Analysis

2.1 The Direct Method in the Calculus of Variations

2.1.1 Fundamental Concepts

Definition 2.1 (Minimizing Sequence [6]). Let X be a Banach space and $J : X \to \mathbb{R} \cup \{+\infty\}$ a functional bounded from below. A sequence $\{u_n\} \subset X$ is called a **minimizing sequence** for J if

$$\lim_{n\to\infty} J(u_n) = \inf_{u\in X} J(u).$$

Example 2.2 ([6]). Consider

$$J(u) = \int_0^1 (u')^2\, dx$$

on $H_0^1(0,1)$. Let

$$u_n(x) = \frac{1}{n}\sin(\pi x).$$

Then

$$u_n'(x) = \frac{\pi}{n}\cos(\pi x).$$

Thus

$$J(u_n) = \int_0^1 \frac{\pi^2}{n^2} \cos^2(\pi x)\, dx = \frac{\pi^2}{2n^2}.$$

Hence $J(u_n) \to 0$. Therefore $\{u_n\}$ is a minimizing sequence.

Definition 2.3 (Coercive Functional [6]). A functional $J : X \to \mathbb{R} \cup \{+\infty\}$ is called **coercive** if

$$\lim_{\|u\|_X \to \infty} J(u) = +\infty.$$

Example 2.4 ([6]). Consider

$$J(u) = \int_0^1 (u')^2\, dx$$

on $H_0^1(0,1)$.

By the Poincaré inequality (Theorem 1.30), there exists a constant $C > 0$ such that

$$\|u\|_{L^2} \leq C\|u'\|_{L^2}, \quad \forall u \in H_0^1(0,1).$$

Thus, the seminorm $\|u'\|_{L^2}$ defines a norm equivalent to the standard H_0^1-norm. In particular, there exists $c > 0$ such that

$$\|u'\|_{L^2}^2 \geq c\|u\|_{H_0^1}^2.$$

Therefore,

$$J(u) \geq c\|u\|_{H_0^1}^2.$$

Hence J is coercive.

Example 2.5 ([6]). Consider

$$J(u) = \int_0^1 u^2 \, dx$$

on $H_0^1(0,1)$.

Let

$$v_n(x) = \sin(n\pi x).$$

Then

$$v_n'(x) = n\pi \cos(n\pi x).$$

We compute

$$\|v_n\|_{L^2}^2 = \frac{1}{2}, \qquad \|v_n'\|_{L^2}^2 = \frac{n^2\pi^2}{2}.$$

Hence

$$\|v_n\|_{H^1} \to \infty, \quad \text{while} \quad J(v_n) = \frac{1}{2}.$$

Therefore, J is not coercive.

2.1.2 The Fundamental Existence Theorem

Theorem 2.6 (Direct Method – Existence Theorem [6]). *Let X be a reflexive Banach space and $J : X \to \mathbb{R} \cup \{+\infty\}$ a functional satisfying:*

(H1) ***Properness:*** $\mathrm{dom}(J) \neq \emptyset$.

(H2) ***Coercivity:*** $\lim\limits_{\|u\|_X \to \infty} J(u) = +\infty$.

(H3) ***Weak Lower Semicontinuity:*** *J is sequentially weakly lower semicontinuous.*

Then there exists $u_0 \in X$ such that

$$J(u_0) = \min_{u \in X} J(u).$$

Proof. Let

$$m := \inf_{u \in X} J(u).$$

Since J is proper, we have $m < +\infty$. Moreover, the coercivity of J implies

$$\|u\|_X \to \infty \;\Rightarrow\; J(u) \to +\infty,$$

hence $m > -\infty$.

1. Minimizing sequence. By the definition of the infimum, there exists a sequence $(u_n) \subset X$ such that

$$m \leq J(u_n) \leq m + \frac{1}{n}, \qquad n \in \mathbb{N}.$$

Consequently,

$$J(u_n) \to m,$$

and (u_n) is a minimizing sequence for J.

2. Boundedness. We claim that (u_n) is bounded in X. Indeed, if it were unbounded, there would exist a subsequence (u_{n_k}) such that

$$\|u_{n_k}\|_X \to \infty.$$

By coercivity, this implies

$$J(u_{n_k}) \to +\infty,$$

which contradicts the fact that $J(u_{n_k}) \to m < +\infty$. Hence (u_n) is bounded.

3. Weak convergence. Since X is reflexive and (u_n) is bounded, Corollary 1.20 ensures the existence of $u_0 \in X$ and a subsequence (u_{n_k}) such that

$$u_{n_k} \rightharpoonup u_0 \quad \text{in } X.$$

4. Weak lower semicontinuity. By the sequential weak lower semicontinuity of J,

$$J(u_0) \leq \liminf_{k\to\infty} J(u_{n_k}).$$

Since $J(u_n) \to m$, we obtain

$$J(u_0) \leq m.$$

5. Optimality. From the definition of $m = \inf_{u \in X} J(u)$ we also have $J(u_0) \geq m$. Therefore

$$J(u_0) = m = \inf_{u \in X} J(u),$$

and u_0 is a minimizer of J in X. □

2.1.3 Uniqueness of Minimizers

Theorem 2.7 (Uniqueness via Strict Convexity [6]). *Let X be a normed space and $J : X \to \mathbb{R}$ a strictly convex functional (Definition 1.33). Assume $\exists\, u_0 \in X$ such that $J(u_0) = \inf_{u \in X} J(u)$. Then the minimizer is unique.*

Proof. Let $m := \inf_{u \in X} J(u)$. Assume by contradiction that there exist two distinct minimizers $u_1, u_2 \in X$ with $u_1 \neq u_2$ such that $J(u_1) = J(u_2) = m$. Since J is strictly convex, for every $\lambda \in (0, 1)$ we have

$$J(\lambda u_1 + (1 - \lambda)u_2) < \lambda J(u_1) + (1 - \lambda)J(u_2) = m.$$

Choosing $\lambda = \frac{1}{2}$ gives $J\left(\frac{u_1+u_2}{2}\right) < m$, contradicting the definition of $m = \inf_{u \in X} J(u)$. Therefore $u_1 = u_2$. □

Definition 2.8 (Strong Convexity [6]). Let X be a normed space and let $J : X \to \mathbb{R}$ be differentiable. We say that J is *strongly convex* with parameter $\alpha > 0$ if for all $u, v \in X$,

$$J(v) \geq J(u) + \langle J'(u), v - u \rangle + \frac{\alpha}{2}\|v - u\|^2.$$

Proposition 2.9 ([6]). *If J is strongly convex, then J is strictly convex.*

Proof. Let $u \neq v$ and $\lambda \in (0, 1)$. Set $w = \lambda u + (1 - \lambda)v$. From the definition of strong convexity with parameter $\alpha > 0$, we have

$$J(u) \geq J(w) + \langle J'(w), u - w \rangle + \frac{\alpha}{2}\|u - w\|^2,$$

and

$$J(v) \geq J(w) + \langle J'(w), v - w \rangle + \frac{\alpha}{2}\|v - w\|^2.$$

Multiply the first inequality by λ and the second by $(1 - \lambda)$, then add them:

$$\begin{aligned}\lambda J(u) + (1-\lambda)J(v) \geq J(w) &+ \langle J'(w), \lambda(u - w) + (1-\lambda)(v - w)\rangle \\ &+ \frac{\alpha}{2}\left(\lambda\|u - w\|^2 + (1-\lambda)\|v - w\|^2\right).\end{aligned}$$

Notice that $\lambda(u-w)+(1-\lambda)(v-w) = \lambda u+(1-\lambda)v-w = 0$. Also, $u-w = (1-\lambda)(u-v)$ and $v - w = \lambda(v - u) = -\lambda(u - v)$. Therefore,

$$\begin{aligned}\lambda\|u - w\|^2 + (1-\lambda)\|v - w\|^2 &= \lambda(1-\lambda)^2\|u - v\|^2 + (1-\lambda)\lambda^2\|u - v\|^2 \\ &= \lambda(1-\lambda)\|u - v\|^2.\end{aligned}$$

Substituting, we obtain

$$\lambda J(u) + (1-\lambda)J(v) \geq J(w) + \frac{\alpha}{2}\lambda(1-\lambda)\|u - v\|^2.$$

Since $\|u - v\|^2 > 0$ and $\alpha > 0$, the last term is positive. Hence $J(w) < \lambda J(u) + (1 - \lambda)J(v)$, which proves that J is strictly convex. ☐

2.1.4 Ekeland's Principle in Complete Metric Spaces

Let M be a complete metric space and $J : M \to \mathbb{R}$ a lower semicontinuous functional bounded from below. If $(u_j)_j$ is a minimizing sequence, then for every $\varepsilon > 0$ there exists j_0 such that for $j > j_0$

$$J(u_j) \leq \inf_M J + \varepsilon.$$

We say that u is an ε-minimum point of J if

$$J(u) \leq \inf_M J + \varepsilon.$$

Ekeland's theorem [10] considers the existence of ε-minimum points.

Theorem 2.10 (Ekeland Principle, strong form, 1979 [10]). *Let M be a complete metric space and $J : M \to \mathbb{R}$ be a lower semicontinuous functional which is bounded from below. Let $k > 1$, $\varepsilon > 0$ and $u \in M$ be an ε-minimum point of J. Then there exists $v \in M$ such that*

$$\begin{array}{rcl} J(v) & \leq & J(u), \\ d(u,v) & \leq & \frac{1}{k}, \\ J(v) & < & J(w) + \varepsilon k d(w,v), \quad \forall w \neq v. \end{array} \tag{1.8}$$

Proof. Let (M,d) be a complete metric space and define

$$d_k(u,v) := k\, d(u,v).$$

Define a relation $<$ on M by

$$u < v \iff J(u) \leq J(v) - \varepsilon\, d_k(u,v).$$

It is immediate that:

$$u < u, \quad \forall u \in M,$$

$$u < v \text{ and } v < u \;\Rightarrow\; u = v,$$

$$u < v \text{ and } v < w \;\Rightarrow\; u < w.$$

Transitivity.

Assume $u < v$ and $v < w$. Then

$$J(u) \leq J(v) - \varepsilon d_k(u,v), \quad J(v) \leq J(w) - \varepsilon d_k(v,w).$$

Hence

$$J(u) \le J(w) - \varepsilon\big(d_k(u,v) + d_k(v,w)\big).$$

By the triangle inequality,

$$d_k(u,w) \le d_k(u,v) + d_k(v,w),$$

thus

$$J(u) \le J(w) - \varepsilon d_k(u,w),$$

which proves transitivity.

Construction of the sequence.

Let $u_1 := u$ and define

$$S_n := \{w \in M : w < u_n\}.$$

Choose inductively (u_n) such that

$$u_{n+1} \in S_n, \quad J(u_{n+1}) \le \inf_{w \in S_n} J(w) + \frac{\varepsilon}{2^{n+1}}.$$

Then

$$S_1 \supset S_2 \supset \cdots, \quad u_1 > u_2 > \cdots.$$

Closedness of S_n.

Let $v_j \in S_n$ with $v_j \to v$. Then

$$J(v_j) \le J(u_n) - \varepsilon d_k(v_j, u_n).$$

Passing to the limit using lower semicontinuity of J and continuity of d_k,

$$J(v) \le J(u_n) - \varepsilon d_k(v, u_n),$$

hence $v \in S_n$.

Diameter estimate.

Let $w \in S_n$. Then

$$J(w) \leq J(u_n) - \varepsilon d_k(w, u_n).$$

Also,

$$J(u_n) \leq J(w) + \frac{\varepsilon}{2^n}.$$

Thus

$$\varepsilon d_k(w, u_n) \leq \frac{\varepsilon}{2^n}, \quad d_k(w, u_n) \leq \frac{1}{2^n}.$$

Hence for $w_1, w_2 \in S_n$,

$$d_k(w_1, w_2) \leq d_k(w_1, u_n) + d_k(w_2, u_n) \leq \frac{1}{2^{n-1}}.$$

Thus

$$\operatorname{diam}(S_n) \to 0.$$

Existence of the limit point.

Since (M, d) is complete and (S_n) is a decreasing sequence of nonempty closed sets with vanishing diameter, there exists a unique

$$v \in \bigcap_{n=1}^{\infty} S_n.$$

Verification of the properties.

Since $v \in S_1$, we have

$$J(v) \leq J(u) - \varepsilon d_k(u, v).$$

Let $w \neq v$. If $w < v$, then $w \in S_n$ for all n, hence $w = v$, contradiction. Thus

$$J(w) > J(v) - \varepsilon d_k(w, v).$$

Finally,

$$d_k(u, u_n) \le \sum_{j=1}^{n-1} d_k(u_j, u_{j+1}) \le \sum_{j=1}^{n-1} \frac{1}{2^j} \le 1,$$

and passing to the limit gives

$$d_k(u, v) \le 1.$$

This completes the proof. □

Next, we outline several corollaries that follow from Ekeland's principle.

Corollary 2.11 (Weak form of Ekeland's Variational Principle [10]). *Let* (M, d) *be a complete metric space and let*

$$J : M \to \mathbb{R}$$

be lower semicontinuous and bounded from below. Then for every $\varepsilon > 0$ *there exists* $v \in M$ *such that*

$$J(v) \le J(w) + \varepsilon d(w, v), \qquad \forall w \in M,\ w \neq v.$$

Proof. Let $m := \inf_{u \in M} J(u) > -\infty$.

1. Initial point. Fix $\varepsilon > 0$. By the definition of the infimum, there exists $u_1 \in M$ such that

$$J(u_1) < m + \varepsilon.$$

2. Recursive construction. Assume u_n is given and define

$$S_n := \{w \in M : J(w) + \tfrac{\varepsilon}{2} d(w, u_n) \le J(u_n)\}.$$

Since $u_n \in S_n$, the set S_n is nonempty. Choose $u_{n+1} \in S_n$ such that

$$J(u_{n+1}) \leq \inf_{w \in S_n} J(w) + \frac{1}{n}.$$

In particular,

$$J(u_{n+1}) + \tfrac{\varepsilon}{2} d(u_{n+1}, u_n) \leq J(u_n),$$

hence

$$J(u_{n+1}) \leq J(u_n).$$

Thus the sequence $(J(u_n))$ is decreasing and bounded below by m.

3. Convergence. From the construction we have

$$\frac{\varepsilon}{2} d(u_{n+1}, u_n) \leq J(u_n) - J(u_{n+1}).$$

Summing from $n = 1$ to N gives

$$\frac{\varepsilon}{2} \sum_{n=1}^{N} d(u_{n+1}, u_n) \leq J(u_1) - J(u_{N+1}) \leq J(u_1) - m.$$

Hence

$$\sum_{n=1}^{\infty} d(u_{n+1}, u_n) < \infty,$$

which implies that (u_n) is a Cauchy sequence. Since (M, d) is complete, there exists $v \in M$ such that

$$u_n \to v.$$

4. Passing to the limit. Fix n. Since $u_k \in S_n$ for all $k > n$, we have

$$J(u_k) + \tfrac{\varepsilon}{2} d(u_k, u_n) \leq J(u_n).$$

Letting $k \to \infty$ and using the lower semicontinuity of J, we obtain

$$J(v) + \tfrac{\varepsilon}{2} d(v, u_n) \leq J(u_n).$$

5. **ε-minimality.** Let $w \in M$, $w \neq v$. Using the triangle inequality

$$d(w,u_n) \geq d(w,v) - d(v,u_n),$$

one obtains that if

$$J(w) < J(v) - \varepsilon d(w,v),$$

then for n sufficiently large

$$J(w) + \tfrac{\varepsilon}{2} d(w,u_n) < J(u_n),$$

which contradicts the definition of S_n. Hence

$$J(v) \leq J(w) + \varepsilon d(w,v), \qquad \forall w \in M,\ w \neq v.$$

Thus v is an ε-minimum point of J. □

Theorem 2.12 (Differential Form of Ekeland's Principle [10]). *Let X be a Banach space and let*

$$J \in C^1(X,\mathbb{R})$$

be bounded from below. Define

$$\beta := \inf_{u\in X} J(u).$$

Then for every $\varepsilon > 0$ there exists $u_\varepsilon \in X$ such that

$$J(u_\varepsilon) \leq \beta + \varepsilon \quad \textit{and} \quad \|J'(u_\varepsilon)\| \leq \varepsilon.$$

Proof. **1. Application of Ekeland's principle.** Apply the weak form of Ekeland's variational principle (Corollary 2.11) to the metric space $M = X$, $d(u,v) = \|u-v\|$. Then there exists $v \in X$ such that

$$J(v) \leq \beta + \varepsilon$$

and

$$J(w) \geq J(v) - \varepsilon \|w - v\|, \qquad \forall w \in X.$$

2. Perturbation argument. Let $h \in X$ and $t > 0$. Set $w = v + th$. Substituting into the previous inequality yields

$$J(v + th) \geq J(v) - \varepsilon \|th\|.$$

Hence

$$J(v + th) - J(v) \geq -\varepsilon t \|h\|.$$

Dividing by $t > 0$ gives

$$\frac{J(v + th) - J(v)}{t} \geq -\varepsilon \|h\|.$$

3. Passage to the limit. Since $J \in C^1(X, \mathbb{R})$, letting $t \to 0^+$ yields

$$\langle J'(v), h \rangle \geq -\varepsilon \|h\|.$$

4. Opposite direction. Replacing h by $-h$ gives

$$\langle J'(v), h \rangle \leq \varepsilon \|h\|.$$

5. Gradient estimate. Combining the previous inequalities we obtain

$$-\varepsilon \|h\| \leq \langle J'(v), h \rangle \leq \varepsilon \|h\|, \qquad \forall h \in X.$$

Hence

$$|\langle J'(v), h \rangle| \leq \varepsilon \|h\|.$$

Taking the supremum over all h with $\|h\| = 1$ yields

$$\|J'(v)\| \leq \varepsilon.$$

Finally, setting $u_\varepsilon := v$ gives $J(u_\varepsilon) \leq \beta + \varepsilon$, $\|J'(u_\varepsilon)\| \leq \varepsilon$. □

Corollary 2.13 (Existence of Palais–Smale sequences [10]). *Let X be a Banach space and let $J \in C^1(X, \mathbb{R})$ be bounded from below. Define*

$$m := \inf_{u \in X} J(u).$$

Then there exists a sequence $(u_n) \subset X$ such that

$$J(u_n) \to m \quad \textit{and} \quad \|J'(u_n)\| \to 0.$$

Proof. Apply Theorem 2.12 with $\varepsilon = 1/n$. For each n, there exists u_n such that $J(u_n) \leq m + 1/n$ and $\|J'(u_n)\| \leq 1/n$. Then $J(u_n) \to m$ and $\|J'(u_n)\| \to 0$. □

2.1.5 The Palais–Smale Condition

Definition 2.14 (Palais–Smale condition at level c [1]). Let X be a Banach space and let $J \in C^1(X, \mathbb{R})$. We say that J satisfies the *Palais–Smale condition at level c*, denoted $(PS)_c$, if every sequence $(u_n) \subset X$ such that

$$J(u_n) \to c \quad \text{and} \quad \|J'(u_n)\| \to 0$$

admits a subsequence (u_{n_k}) and $u \in X$ such that

$$u_{n_k} \to u \quad \text{strongly in } X.$$

Example 2.15 (Palais–Smale condition for a nonlinear functional [15]). Let $\Omega \subset \mathbb{R}^N$ be bounded, $1 \leq N \leq 3$, and $2 < p < 2^*$. Define

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{p} \int_\Omega |u|^p \, dx.$$

Then J satisfies the Palais–Smale condition.

Solution. Let $(u_n) \subset H_0^1(\Omega)$ such that

$$J(u_n) \to c, \quad J'(u_n) \to 0 \text{ in } H^{-1}(\Omega).$$

1: Boundedness. We compute

$$J(u_n) = \frac{1}{2}\|u_n\|^2 - \frac{1}{p}\|u_n\|_{L^p}^p,$$

$$\langle J'(u_n), u_n \rangle = \|u_n\|^2 - \|u_n\|_{L^p}^p.$$

Hence

$$J(u_n) - \frac{1}{p}\langle J'(u_n), u_n \rangle = \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2.$$

Since $p > 2$, the coefficient is positive. Moreover,

$$\langle J'(u_n), u_n \rangle \leq \|J'(u_n)\| \, \|u_n\|.$$

Thus

$$\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 \leq C + o(1)\|u_n\|.$$

This implies that (u_n) is bounded in $H_0^1(\Omega)$.

2: Weak convergence. Since $H_0^1(\Omega)$ is reflexive, there exist $u \in H_0^1(\Omega)$ and a subsequence such that

$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega).$$

3: Strong convergence in L^p. Since $p < 2^*$, the embedding

$$H_0^1(\Omega) \hookrightarrow L^p(\Omega)$$

is compact. Hence

$$u_n \to u \quad \text{strongly in } L^p(\Omega).$$

4: Strong convergence in $H_0^1(\Omega)$. We use the identity

$$\|u_n - u\|^2 = \langle J'(u_n) - J'(u),\, u_n - u\rangle + \int_\Omega \big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)(u_n - u)\,dx.$$

We analyze each term:

First term. Since $J'(u_n) \to 0$ and $J'(u)$ is fixed,

$$\langle J'(u_n), u_n - u\rangle \to 0.$$

Second term. The map $t \mapsto |t|^{p-2}t$ is monotone, hence

$$\int_\Omega \big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)(u_n - u)\,dx \geq 0.$$

Moreover, since $u_n \to u$ in $L^p(\Omega)$, this term tends to 0.

Thus

$$\|u_n - u\|^2 \to 0,$$

i.e.

$$u_n \to u \quad \text{strongly in } H_0^1(\Omega).$$

Conclusion. Every Palais–Smale sequence admits a strongly convergent subsequence. Hence J satisfies the $(PS)_c$ condition. □

Example 2.16 (Failure of the Palais–Smale condition [15]). Let $X = H^1(\mathbb{R}^N)$ with $N \geq 3$ and consider the autonomous semilinear functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla u|^2 + |u|^2\big)\,dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p\,dx,$$

where $2 < p < 2^* = \frac{2N}{N-2}$. We demonstrate that J does not satisfy the Palais–Smale condition due to the lack of compact Sobolev embeddings on unbounded domains.

1: Choice of a ground state solution By standard variational theory (e.g., [14, 15]), there exists a non-trivial critical point $u \in H^1(\mathbb{R}^N)$ of J, known as a ground state solution. Since $u \neq 0$ is a critical point, it satisfies

$$J(u) = c > 0 \quad \text{and} \quad J'(u) = 0 \quad \text{in } (H^1(\mathbb{R}^N))^*.$$

2: Construction of the translating sequence Define the sequence of translates

$$u_n(x) = u(x - ne_1), \quad n \in \mathbb{N},$$

where $e_1 = (1, 0, \ldots, 0)$.

3: Boundedness of energy and exact vanishing of derivative By the translation invariance of the Lebesgue measure on $\mathbb{R}^N$, the norm and integral quantities are perfectly preserved under translation. Specifically:

$$\begin{aligned} J(u_n) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u(x - ne_1)|^2 + |u(x - ne_1)|^2\right) dx \\ &\quad - \frac{1}{p}\int_{\mathbb{R}^N} |u(x - ne_1)|^p \, dx \\ &= J(u) = c. \end{aligned}$$

Furthermore, for any test function $v \in H^1(\mathbb{R}^N)$, the Fréchet derivative is given by

$$\langle J'(u_n), v\rangle = \int_{\mathbb{R}^N} \left(\nabla u_n \cdot \nabla v + u_n v - |u_n|^{p-2} u_n v\right) dx.$$

Making the change of variable $y = x - ne_1$, we obtain

$$\begin{aligned} \langle J'(u_n), v\rangle &= \int_{\mathbb{R}^N} \Big(\nabla u(y) \cdot \nabla v(y + ne_1) + u(y) v(y + ne_1) \\ &\qquad\qquad - |u(y)|^{p-2} u(y) v(y + ne_1)\Big) \, dy \\ &= \langle J'(u), v(\cdot + ne_1)\rangle. \end{aligned}$$

Since $J'(u) = 0$, it follows exactly that $\langle J'(u_n), v\rangle = 0$ for all $v \in H^1(\mathbb{R}^N)$. Thus,

$$J'(u_n) = 0 \quad \text{in } (H^1(\mathbb{R}^N))^*.$$

Consequently, (u_n) is exactly a Palais–Smale sequence at level $c > 0$.

4: Weak convergence to zero We show that $u_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$. For any test function $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} u_n(x)\varphi(x)\,dx = \int_{\mathbb{R}^N} u(y)\varphi(y + ne_1)\,dy.$$

Since φ has compact support, $\varphi(y + ne_1) = 0$ for all y when n is sufficiently large. Hence, $\int u_n\varphi \to 0$. By the density of $C_c^\infty(\mathbb{R}^N)$ in $(H^1(\mathbb{R}^N))^*$, we conclude that $u_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$.

5: Lack of strong convergence If (u_n) admitted a strongly convergent subsequence in $H^1(\mathbb{R}^N)$, its strong limit would necessarily coincide with its weak limit 0. However, by translation invariance,

$$\|u_n\|_{H^1} = \|u\|_{H^1} > 0 \quad \text{for all } n \in \mathbb{N}.$$

Therefore, no subsequence of (u_n) can converge strongly to 0.

Conclusion The sequence (u_n) satisfies $J(u_n) \to c > 0$ and $\|J'(u_n)\|_{(H^1)^*} \to 0$, but contains no strongly convergent subsequence. This rigorously demonstrates that J fails the $(PS)_c$ condition due to the non-compactness of translations escaping to infinity.

2.1.6 The Mountain Pass Theorem

2.1.6.1 Geometric Hypotheses

Let X be a real Banach space and $J \in C^1(X, \mathbb{R})$.

$$J(0) = 0, \tag{MP1}$$

$$\exists \rho, \alpha > 0 \ \forall u \in X : \ \|u\| = \rho \implies J(u) \geq \alpha, \tag{MP2}$$

$$\exists e \in X : \ \|e\| > \rho \ \wedge \ J(e) < 0. \tag{MP3}$$

Define the set of continuous paths joining 0 and e:

$$\Gamma := \{\gamma \in C([0,1], X) \mid \gamma(0) = 0, \ \gamma(1) = e\},$$

and the minimax level

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)).$$

Lemma 2.17 (Geometric Hypotheses [15]). *Under (MP1)–(MP3),* $c \geq \alpha > 0$.

Proof. For any $\gamma \in \Gamma$, the map $t \mapsto \|\gamma(t)\|$ is continuous. Since $\|\gamma(0)\| = 0 < \rho$ and $\|\gamma(1)\| > \rho$, there exists $t_0 \in (0,1)$ such that $\|\gamma(t_0)\| = \rho$. By (MP2), $J(\gamma(t_0)) \geq \alpha$. Hence $\max_{t \in [0,1]} J(\gamma(t)) \geq J(\gamma(t_0)) \geq \alpha$. Taking the infimum over $\gamma \in \Gamma$ yields $c \geq \alpha$. □ □

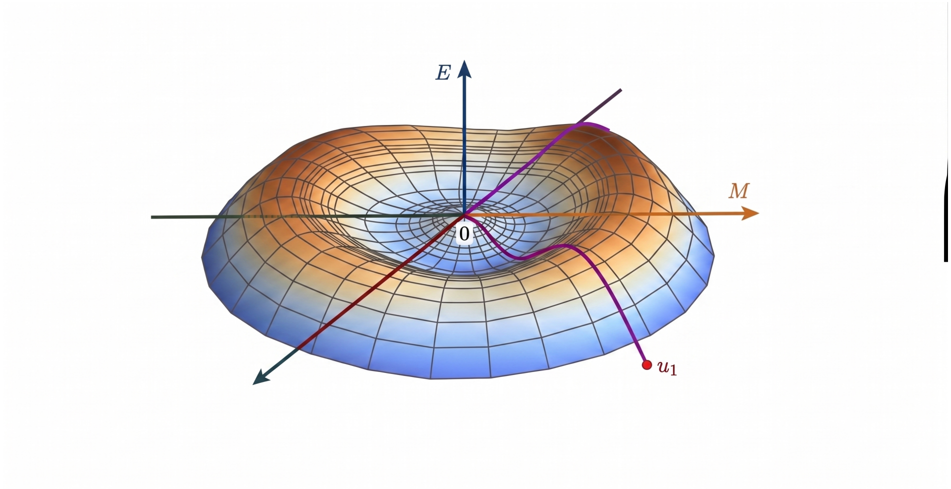

Figure 2.1: On the mountain pass lemma of Ambrosetti and Rabinowitz.

2.1.6.2 Deformation Lemma

Lemma 2.18 (Deformation Lemma [15]). *Let* $J \in C^1(X,\mathbb{R})$ *and* $c \in \mathbb{R}$. *Assume that for some* $\varepsilon, \delta > 0$ *we have*

$$\forall u \in J^{-1}\big([c-\varepsilon, c+\varepsilon]\big) :\ \|J'(u)\| \geq \delta.$$

Then there exists a continuous deformation $\eta : [0,1] \times X \to X$ *such that*

$$\eta(0,u) = u \quad \forall u \in X, \tag{2.1}$$

$$\eta(t,u) = u \quad \forall t \in [0,1]\ \forall u \in X \setminus J^{-1}\big((c-\varepsilon, c+\varepsilon)\big), \tag{2.2}$$

$$\eta\big(1, J^{c+\varepsilon}\big) \subset J^{c-\varepsilon}, \tag{2.3}$$

where $J^a := \{u \in X \mid J(u) \leq a\}$.

2.1.6.3 The Mountain Pass Theorem

Theorem 2.19 (Mountain Pass Theorem [1]). *Assume (MP1)–(MP3) and that J satisfies the Palais–Smale condition at level c (denoted $(PS)_c$). Then there exists $u^* \in X$ such that*

$$J(u^*) = c, \qquad J'(u^*) = 0.$$

Proof. We argue by contradiction. Suppose that c is not a critical value of J, meaning there is no $u \in X$ such that $J(u) = c$ and $J'(u) = 0$.

Since J satisfies the Palais–Smale condition $(PS)_c$, there must exist constants $\varepsilon > 0$ and $\delta > 0$ such that

$$\forall u \in J^{-1}\big([c-\varepsilon, c+\varepsilon]\big) : \ \|J'(u)\| \geq \delta.$$

Indeed, if this were false, we could find a sequence (u_n) such that $J(u_n) \to c$ and $J'(u_n) \to 0$. By the $(PS)_c$ condition, this sequence would have a subsequence converging to a critical point u^* with $J(u^*) = c$ and $J'(u^*) = 0$, contradicting our assumption.

Without loss of generality, we can choose ε small enough such that $0 < \varepsilon < c$ (since $c \geq \alpha > 0$ by Lemma 2.17), so that $c - \varepsilon > 0$.

By the Deformation Lemma (Lemma 2.18), there exists a continuous deformation $\eta : [0,1] \times X \to X$ satisfying:

$$\eta(1,u) = u \quad \forall u \notin J^{-1}\big((c-\varepsilon, c+\varepsilon)\big), \tag{2.4}$$

$$\eta\big(1, J^{c+\varepsilon}\big) \subset J^{c-\varepsilon}. \tag{2.5}$$

By the definition of the minimax level c, there exists a path $\gamma \in \Gamma$ such that

$$\max_{t\in[0,1]} J(\gamma(t)) \leq c + \varepsilon.$$

This means $\gamma(t) \in J^{c+\varepsilon}$ for all $t \in [0,1]$.

Define a new path $\tilde{\gamma}(t) := \eta(1, \gamma(t))$ for $t \in [0,1]$. Since $J(0) = 0 < c - \varepsilon$ and

$J(e) < 0 < c - \varepsilon$, both endpoints 0 and e belong to $X \setminus J^{-1}\big((c-\varepsilon, c+\varepsilon)\big)$. Thus, by property (2.4), the deformation fixes the endpoints:

$$\tilde{\gamma}(0) = \eta(1,0) = 0 \quad \text{and} \quad \tilde{\gamma}(1) = \eta(1,e) = e.$$

Hence, $\tilde{\gamma} \in \Gamma$.

However, for all $t \in [0,1]$, we know $\gamma(t) \in J^{c+\varepsilon}$. Therefore, by property (2.5) of the deformation η,

$$\tilde{\gamma}(t) = \eta(1, \gamma(t)) \in J^{c-\varepsilon} \quad \forall t \in [0,1].$$

Taking the maximum over $t \in [0,1]$, we obtain

$$\max_{t\in[0,1]} J(\tilde{\gamma}(t)) \leq c - \varepsilon < c.$$

This strictly contradicts the definition of $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$.

Therefore, our initial assumption must be false, and there exists a critical point $u^* \in X$ at level c, meaning $J(u^*) = c$ and $J'(u^*) = 0$. □

Chapter 3

Advanced Applications to Partial Differential Equations

3.1 Application of the Direct Method to a Nonlinear Elliptic Problem

Problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary, $N \geq 3$, and let $f \in L^2(\Omega)$. Consider

$$-\Delta u + |u|^{p-2}u = f \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

with $2 < p < 2^*$.

Energy functional

Define

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + \frac{1}{p}\int_\Omega |u|^p \, dx - \int_\Omega f u \, dx.$$

1: Well-definedness

Since $p < 2^*$, the Sobolev embedding

$$H_0^1(\Omega) \hookrightarrow L^p(\Omega)$$

is continuous. Hence all terms are finite. Moreover,

$$\left|\int_\Omega fu\right| \leq \|f\|_{L^2}\|u\|_{L^2} \leq C\|f\|_{L^2}\|u\|_{H_0^1}.$$

2: Differentiability

The functional J is of class $C^1(H_0^1(\Omega))$ and

$$\langle J'(u), v\rangle = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega |u|^{p-2}uv - \int_\Omega fv.$$

This follows from the continuity of the Nemytskii operator

$$u \mapsto |u|^{p-2}u : L^p(\Omega) \to L^{\frac{p}{p-1}}(\Omega).$$

3: Coercivity

Using Sobolev and Young inequalities,

$$\left|\int fu\right| \leq \varepsilon\|u\|_{H_0^1}^2 + C_\varepsilon\|f\|_{L^2}^2.$$

Thus

$$J(u) \geq \left(\frac{1}{2} - \varepsilon\right)\|u\|_{H_0^1}^2 + \frac{1}{p}\|u\|_{L^p}^p - C,$$

which implies

$$J(u) \to +\infty \quad \text{as } \|u\|_{H_0^1} \to \infty.$$

4: Weak lower semicontinuity

- $u \mapsto \int |\nabla u|^2$ is convex and weakly lower semicontinuous,
- $u \mapsto \int |u|^p$ is convex in L^p and weakly lower semicontinuous,
- $u \mapsto \int fu$ is weakly continuous.

Thus J is weakly lower semicontinuous.

5: Existence of a minimizer

Let (u_n) be a minimizing sequence. By coercivity, it is bounded in $H_0^1(\Omega)$, hence

$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega).$$

By weak lower semicontinuity,

$$J(u) \leq \liminf J(u_n) = \inf J,$$

so u is a minimizer.

6: Euler–Lagrange equation and Weak Solution

Since u is a minimizer of J over $H_0^1(\Omega)$ and J is of class C^1, the first variation of J at u must vanish in all directions. That is, for any test function $v \in H_0^1(\Omega)$, we have $\langle J'(u), v\rangle = 0$. Calculating the Gâteaux derivative of J in the direction v, we obtain:

$$\lim_{t\to 0} \frac{J(u+tv) - J(u)}{t} = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega |u|^{p-2} u v \, dx - \int_\Omega f v \, dx = 0.$$

This precisely means that u satisfies the weak formulation of the problem:

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega |u|^{p-2} u v \, dx = \int_\Omega f v \, dx \quad \forall v \in H_0^1(\Omega).$$

To recover the corresponding differential equation, we restrict our choice of test functions to $v \in C_c^\infty(\Omega)$. By the definition of the distributional Laplacian, the first term represents:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \langle -\Delta u, v\rangle_{\mathcal{D}',\mathcal{D}}.$$

Substituting this back into the weak formulation yields:

$$\langle -\Delta u + |u|^{p-2}u - f, v\rangle_{\mathcal{D}',\mathcal{D}} = 0 \quad \forall v \in C_c^\infty(\Omega).$$

This implies that the equation holds in the sense of distributions:

$$-\Delta u + |u|^{p-2}u = f \quad \text{in } \mathcal{D}'(\Omega).$$

Since $f \in L^2(\Omega)$ and $|u|^{p-2}u \in L^{p'}(\Omega)$, we deduce that $\Delta u \in L^{p'}(\Omega) + L^2(\Omega)$. Thus, by elliptic regularity, the equality holds almost everywhere in Ω.

7: Uniqueness

Let u_1, u_2 be two solutions. Then

$$\int |\nabla(u_1 - u_2)|^2 + \int (|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2)(u_1 - u_2) = 0.$$

Since the map $s \mapsto |s|^{p-2}s$ is strictly monotone,

$$(|a|^{p-2}a - |b|^{p-2}b)(a - b) \geq 0,$$

hence $u_1 = u_2$.

Conclusion

There exists a unique $u \in H_0^1(\Omega)$ solving the problem.

3.2 Application of the Direct Method to a Constrained Minimization Problem

Problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and define

$$J(u) = \int_\Omega \left(\frac{1}{2}|\nabla u(x)|^2 + \frac{1}{4}|u(x)|^4 \right) dx$$

on the constraint set

$$\mathcal{M} = \left\{ u \in H_0^1(\Omega) : \|u\|_{L^2(\Omega)} = 1 \right\}.$$

Assume $N \leq 3$. We show that J admits a minimizer $u \in \mathcal{M}$.

1: Non-emptiness and lower bound

The set $\mathcal{M}$ is non-empty since one can take any nonzero $v \in H_0^1(\Omega)$ and normalize it:

$$u = \frac{v}{\|v\|_{L^2(\Omega)}} \in \mathcal{M}.$$

Moreover, since $|\nabla u|^2 \geq 0$ and $|u|^4 \geq 0$, we have

$$J(u) \geq 0 \quad \forall u \in \mathcal{M}.$$

Define

$$c := \inf_{u \in \mathcal{M}} J(u) \geq 0.$$

2: Minimizing sequence

By definition of the infimum, there exists a sequence $(u_n) \subset \mathcal{M}$ such that

$$J(u_n) \to c.$$

3: Boundedness in $H_0^1(\Omega)$

Since

$$J(u_n) = \frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx + \frac{1}{4}\int_\Omega |u_n|^4\,dx \geq \frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx,$$

we deduce

$$\int_\Omega |\nabla u_n|^2\,dx \leq 2J(u_n) \leq C.$$

Thus (∇u_n) is bounded in $L^2(\Omega)$.

Since $u_n \in H_0^1(\Omega)$, we have

$$\|u_n\|_{H_0^1(\Omega)} = \|\nabla u_n\|_{L^2(\Omega)},$$

hence (u_n) is bounded in $H_0^1(\Omega)$.

4: Weak convergence

Since $H_0^1(\Omega)$ is reflexive, there exist $u \in H_0^1(\Omega)$ and a subsequence (u_{n_k}) such that

$$u_{n_k} \rightharpoonup u \quad \text{in } H_0^1(\Omega).$$

5: Strong convergence in L^2

By the Rellich–Kondrachov theorem, the embedding

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega)$$

is compact. Hence

$$u_{n_k} \to u \quad \text{strongly in } L^2(\Omega).$$

6: Preservation of the constraint

Since $\|u_{n_k}\|_{L^2(\Omega)} = 1$ for all k and $u_{n_k} \to u$ in $L^2(\Omega)$, we obtain

$$\|u\|_{L^2(\Omega)} = 1.$$

Thus $u \in \mathcal{M}$.

7: Convergence of the nonlinear term

Since $N \leq 3$, we have the compact embedding

$$H_0^1(\Omega) \hookrightarrow L^4(\Omega).$$

Therefore,

$$u_{n_k} \to u \quad \text{strongly in } L^4(\Omega),$$

which implies

$$\int_\Omega |u_{n_k}|^4 \, dx \to \int_\Omega |u|^4 \, dx.$$

8: Lower semicontinuity and conclusion

By weak lower semicontinuity of the L^2-norm of the gradient,

$$\int_\Omega |\nabla u|^2 \, dx \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_{n_k}|^2 \, dx.$$

Combining this with the strong convergence of the L^4 term, we obtain

$$J(u) \leq \liminf_{k\to\infty} J(u_{n_k}) = c.$$

Since $u \in \mathcal{M}$, we also have $J(u) \geq c$. Therefore,

$$J(u) = c,$$

and u is a minimizer of J on $\mathcal{M}$.

9: Weak Solution via Lagrange Multiplier

Since u minimizes J over the constraint manifold $\mathcal{M} = \left\{v \in H_0^1(\Omega) : I(v) = 0\right\}$ where $I(v) = \frac{1}{2}\|v\|_{L^2}^2 - \frac{1}{2}$, and the derivative $I'(u) \neq 0$ for $u \in \mathcal{M}$, the Lagrange

Multiplier Theorem guarantees the existence of a real number $\lambda \in \mathbb{R}$ such that $J'(u) = \lambda I'(u)$ in $H^{-1}(\Omega)$. Computing the derivatives in an arbitrary direction $v \in H_0^1(\Omega)$, we get:

$$\langle J'(u), v \rangle = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega u^3 v \, dx,$$

and

$$\langle I'(u), v \rangle = \int_\Omega uv \, dx.$$

Thus, the equation $J'(u) = \lambda I'(u)$ translates to the weak formulation:

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega u^3 v \, dx = \lambda \int_\Omega uv \, dx \quad \forall v \in H_0^1(\Omega).$$

To return to the differential equation, we select test functions $v \in C_c^\infty(\Omega)$. By the definition of the distributional Laplacian, we have:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \langle -\Delta u, v \rangle_{\mathcal{D}', \mathcal{D}}.$$

Replacing this in the integral identity gives:

$$\langle -\Delta u + u^3 - \lambda u, v \rangle_{\mathcal{D}', \mathcal{D}} = 0 \quad \forall v \in C_c^\infty(\Omega).$$

This implies that the expression inside the pairing is the zero distribution, which means u solves the equation:

$$-\Delta u + u^3 = \lambda u \quad \text{in } \mathcal{D}'(\Omega).$$

By elliptic regularity, since $u \in H_0^1(\Omega)$, we can deduce further regularity for u making it a strong or classical solution. Combined with $u \in H_0^1(\Omega)$ (which enforces the Dirichlet boundary condition), u is a valid solution to the eigenvalue problem:

$$-\Delta u + u^3 = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

3.3 Application of Ekeland's Variational Principle: Nonlinear Eigenvalue Problem

This section demonstrates the application of Ekeland's variational principle to a constrained minimization scenario that originates from a nonlinear eigenvalue problem. The goal is to show how this principle operates on manifolds, facilitating the derivation of Palais–Smale sequences and subsequently yielding critical points.

3.3.1 Problem Formulation

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary, $N \geq 3$, and let $2 < p < 2^* = \frac{2N}{N-2}$. Consider the following nonlinear eigenvalue problem: find $u \in H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$ such that

$$\begin{cases} -\Delta u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega |u|^p \, dx = 1. \end{cases}$$

This is equivalent to finding a critical point of the Dirichlet integral constrained to the L^p unit sphere.

Define the space $X = H_0^1(\Omega)$ with the norm $\|u\| = \left(\int_\Omega |\nabla u|^2 \, dx \right)^{1/2}$ (which is equivalent to the usual H^1 norm by the Poincaré inequality (Theorem 1.30)). Define the constraint manifold

$$M = \left\{ u \in X : \int_\Omega |u|^p \, dx = 1 \right\}.$$

M is a closed subset of X because the embedding $X \hookrightarrow L^p(\Omega)$ is continuous, so M is a complete metric space with the distance induced by $\| \cdot \|$.

Consider the functional $J : X \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\|u\|^2.$$

J is C^1 and its derivative is given by $\langle J'(u), v\rangle = \int_\Omega \nabla u \cdot \nabla v \, dx$ for all $v \in X$. We are interested in minimizing J on M, i.e., finding

$$m = \inf_{u \in M} J(u).$$

A minimizer $u \in M$ will satisfy, by the Lagrange multiplier rule, the Euler–Lagrange equation

$$J'(u) = \lambda T'(u) \quad \text{for some } \lambda \in \mathbb{R},$$

where $T(u) = \frac{1}{p}\int_\Omega |u|^p \, dx$ is the constraint functional. Since the derivative $T'(u)$ acts as $\langle T'(u), v\rangle = \int_\Omega |u|^{p-2}uv \, dx$, we obtain the weak form

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega |u|^{p-2}uv \, dx \quad \forall v \in X,$$

which is precisely the desired eigenvalue problem with λ as the eigenvalue.

We will prove the existence of a minimizer using Ekeland's variational principle.

3.3.2 Detailed application of Ekeland's principle to a nonlinear eigenvalue problem

1: J is bounded below on M and $m > 0$. By the Sobolev inequality (Theorem 1.26), there exists a constant $C > 0$ such that

$$\|u\|_{L^p} \leq C\|u\| \quad \forall u \in X.$$

On M, $\|u\|_{L^p} = 1$, so $\|u\| \geq 1/C$. Hence

$$J(u) = \frac{1}{2}\|u\|^2 \geq \frac{1}{2C^2} > 0.$$

Thus $m \geq 1/(2C^2) > 0$.

2: Apply Ekeland's principle. Let $\varepsilon_n = 1/n$. By the weak form of Ekeland's variational principle (Corollary 2.11), for each n there exists $u_n \in M$ such that

$$J(u_n) \leq m + \frac{1}{n}, \tag{5.1}$$

and

$$J(v) \geq J(u_n) - \frac{1}{n}\|v - u_n\| \quad \text{for all } v \in M. \tag{5.2}$$

3: Derivation of an approximate Euler–Lagrange equation. To exploit (5.2), we need to consider variations that stay on the manifold M. For any $h \in X$, define a curve $v_t \in M$ for small $|t|$ by

$$v_t = \frac{u_n + th}{\|u_n + th\|_{L^p}}.$$

This curve satisfies $v_0 = u_n$ and $v_t \in M$. Differentiating at $t = 0$, we obtain

$$\left.\frac{d}{dt}\right|_{t=0} v_t = h - u_n \int_\Omega |u_n|^{p-2} u_n h \, dx,$$

because $\frac{d}{dt}\|u_n + th\|_{L^p}^p = p\int |u_n|^{p-2}u_n h$, and since $\|u_n\|_{L^p} = 1$, the derivative of the normalization factor is $-\int |u_n|^{p-2}u_n h$.

Now, using (5.2) with $v = v_t$, we have

$$\frac{J(v_t) - J(u_n)}{t} \geq -\frac{1}{n}\frac{\|v_t - u_n\|}{t}.$$

Taking the limit $t \to 0$, the right-hand side converges to $-\frac{1}{n}\|\dot{v}_0\|$, where $\dot{v}_0 =$

$h - u_n \int |u_n|^{p-2} u_n h$. The left-hand side converges to $\langle J'(u_n), \dot{v}_0 \rangle$. Thus

$$\langle J'(u_n), h \rangle - \left(\int_\Omega |u_n|^{p-2} u_n h \, dx \right) \langle J'(u_n), u_n \rangle \geq -\frac{1}{n} \| h - u_n \int |u_n|^{p-2} u_n h \|.$$

Denote $\lambda_n = \langle J'(u_n), u_n \rangle = \int_\Omega |\nabla u_n|^2 \, dx = 2J(u_n)$. Then we obtain

$$\left| \langle J'(u_n), h \rangle - \lambda_n \int_\Omega |u_n|^{p-2} u_n h \, dx \right| \leq \frac{C}{n} \|h\| \quad \text{for all } h \in X,$$

with a constant C independent of n (since the norm of the projection is bounded). This inequality means that in the dual space X^*,

$$\|J'(u_n) - \lambda_n T'(u_n)\|_{X^*} \leq \frac{C}{n}. \tag{5.3}$$

Thus $\{u_n\}$ is a Palais–Smale sequence for the constrained functional.

4: Boundedness and convergence. From (5.1) we have $J(u_n) = \frac{1}{2} \|u_n\|^2 \leq m + 1$, so $\{u_n\}$ is bounded in X. Since X is reflexive, there exists a subsequence, still denoted $\{u_n\}$, and $u \in X$ such that $u_n \rightharpoonup u$ weakly in X. By the Rellich–Kondrachov theorem (Theorem 1.29), the embedding $X \hookrightarrow L^p(\Omega)$ is compact, so $u_n \to u$ strongly in $L^p(\Omega)$. Hence $\|u\|_{L^p} = \lim \|u_n\|_{L^p} = 1$, so $u \in M$.

Moreover, from (5.3) we have for any $h \in X$,

$$\langle J'(u_n), h \rangle - \lambda_n \int_\Omega |u_n|^{p-2} u_n h \, dx \to 0.$$

Since $u_n \to u$ in L^p, we have $\int |u_n|^{p-2} u_n h \to \int |u|^{p-2} u h$ for each h (by dominated convergence). Also, $\lambda_n = 2J(u_n)$ converges to some $\lambda = 2J(u)$ (because J is weakly lower semicontinuous and we have $J(u) \leq \liminf J(u_n) = m$, but also $J(u) \geq m$ by definition of m, so $J(u) = m$ and $\lambda_n \to \lambda$). Passing to the limit, we obtain

$$\langle J'(u), h \rangle - \lambda \int_\Omega |u|^{p-2} u h \, dx = 0 \quad \forall h \in X.$$

Thus u satisfies the weak formulation of the eigenvalue problem:

$$\langle J'(u), h\rangle - \lambda \int_\Omega |u|^{p-2} u h \, dx = \int_\Omega \nabla u \cdot \nabla h \, dx - \lambda \int_\Omega |u|^{p-2} u h \, dx = 0 \quad \forall h \in X.$$

5: Minimality and Strong Equation. Since $J(u) = m$, u is a minimizer of J on M. The weak formulation obtained above is:

$$\int_\Omega \nabla u \cdot \nabla h \, dx - \lambda \int_\Omega |u|^{p-2} u h \, dx = 0 \quad \forall h \in X.$$

To recover the differential equation, we restrict the test functions to $h \in C_c^\infty(\Omega)$. By the definition of the weak derivative, the first term represents the distributional Laplacian:

$$\int_\Omega \nabla u \cdot \nabla h \, dx = \langle -\Delta u, h\rangle_{\mathcal{D}',\mathcal{D}}.$$

Thus, the equation becomes:

$$\langle -\Delta u - \lambda |u|^{p-2} u, h\rangle_{\mathcal{D}',\mathcal{D}} = 0 \quad \forall h \in C_c^\infty(\Omega).$$

This implies that the equation

$$-\Delta u = \lambda |u|^{p-2} u \quad \text{in } \mathcal{D}'(\Omega)$$

holds in the distributional sense. Therefore, u is a valid weak ground state solution of the nonlinear eigenvalue problem, and standard elliptic regularity can be used to show it is a classical solution.

Conclusion

This application highlights the effectiveness of Ekeland's variational principle in addressing constrained challenges. Utilizing the principle over the manifold M allowed for the construction of a Palais–Smale sequence. Driven by compact embeddings, this sequence ultimately converges to a critical point of the constrained functional. This procedure is highly reliable and may be generalized to accommodate more complex constraints and various functionals.

3.4 Application of the Mountain Pass Theorem: Semilinear Elliptic Equation

Consider the semilinear Dirichlet problem

$$\begin{cases} -\Delta u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is bounded with Lipschitz boundary, $N \geq 3$, and $2 < p < 2^* = \frac{2N}{N-2}$.

3.4.1 Variational Formulation

Define $J : H_0^1(\Omega) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{1}{p}\int_\Omega |u|^p \, dx.$$

Then $J \in C^1(H_0^1(\Omega), \mathbb{R})$ and its critical points are weak solutions.

3.4.2 Verification of the Mountain Pass Geometry

(MP1)

Clearly,

$$J(0) = 0.$$

(MP2): Local positivity

By Sobolev embedding, there exists $C > 0$ such that

$$\|u\|_{L^p} \leq C\|\nabla u\|_{L^2}.$$

Set $t = \|\nabla u\|_{L^2}$. Then

$$J(u) \geq \frac{1}{2}t^2 - \frac{C^p}{p}t^p.$$

Define

$$f(t) = \frac{1}{2}t^2 - \frac{C^p}{p}t^p.$$

Since $p > 2$, $f(t) > 0$ for small t.

Compute

$$f'(t) = t - C^p t^{p-1} = t(1 - C^p t^{p-2}),$$

so

$$t_0 = C^{-\frac{p}{p-2}}.$$

Then

$$f(t_0) = t_0^2 \left(\frac{1}{2} - \frac{1}{p}\right) > 0.$$

Choose $\rho \in (0, t_0)$ and define $\alpha = f(\rho) > 0$. Then

$$\|u\| = \rho \Rightarrow J(u) \geq \alpha.$$

(MP3): Existence of a descending direction

Let $\varphi \in C_c^\infty(\Omega)$, $\varphi \not\equiv 0$. Then

$$J(t\varphi) = \frac{t^2}{2}\int |\nabla\varphi|^2 - \frac{t^p}{p}\int |\varphi|^p.$$

Since $p > 2$,

$$J(t\varphi) \to -\infty \quad \text{as } t \to \infty.$$

Thus there exists e such that $\|e\| > \rho$ and $J(e) < 0$.

3.4.3 Palais–Smale Condition

Let (u_n) be a sequence in $H_0^1(\Omega)$ such that

$$J(u_n) \to c, \qquad J'(u_n) \to 0.$$

1: Boundedness

We compute

$$J(u) - \frac{1}{p}\langle J'(u), u\rangle = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2.$$

Hence

$$\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 = J(u_n) - \frac{1}{p}\langle J'(u_n), u_n\rangle \leq C + o(\|u_n\|).$$

Thus (u_n) is bounded.

2: Weak convergence

Up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega).$$

3: Strong convergence in L^p

By compact embedding,

$$u_n \to u \quad \text{in } L^p(\Omega).$$

4: Strong convergence in H_0^1

We have

$$\langle J'(u_n), u_n\rangle = \|u_n\|^2 - \|u_n\|_p^p \to 0.$$

Since $u_n \rightharpoonup u$ weakly and $J'(u_n) \to 0$ strongly in $H^{-1}(\Omega)$, we can pass to the limit to find $J'(u) = 0$. Consequently, $\langle J'(u), u\rangle = \|u\|^2 - \|u\|_p^p = 0$. Combined with the compact embedding result

$$\|u_n\|_p^p \to \|u\|_p^p,$$

we obtain

$$\lim_{n\to\infty} \|u_n\|^2 = \lim_{n\to\infty} \|u_n\|_p^p = \|u\|_p^p = \|u\|^2.$$

Since $u_n \rightharpoonup u$ and norms converge, we conclude

$$u_n \to u \quad \text{in } H_0^1(\Omega).$$

Thus $(PS)_c$ holds.

3.4.4 Conclusion via Mountain Pass and Weak Solution

All hypotheses are satisfied. Hence there exists a critical point $u \neq 0$ such that $J'(u) = 0$. This means that the Fréchet derivative of J evaluated at u vanishes in all directions $v \in H_0^1(\Omega)$, which translates to the variational identity:

$$\langle J'(u), v\rangle = \int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega |u|^{p-2} uv \, dx = 0 \quad \forall v \in H_0^1(\Omega).$$

By taking $v \in C_c^\infty(\Omega)$, the gradient term gives the distributional Laplacian of u:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \langle -\Delta u, v\rangle_{\mathcal{D}',\mathcal{D}}.$$

Substituting this back, the weak formulation becomes:

$$\langle -\Delta u - |u|^{p-2}u, v\rangle_{\mathcal{D}',\mathcal{D}} = 0 \quad \forall v \in C_c^\infty(\Omega).$$

This precisely means that u solves the partial differential equation

$$-\Delta u = |u|^{p-2}u \quad \text{in } \mathcal{D}'(\Omega),$$

in the sense of distributions. Thus, the mountain pass geometry directly yields a nontrivial weak solution to the semilinear elliptic problem.

3.5 Application of the Mountain Pass Theorem to a Boundary Value Problem

Problem

Consider the boundary value problem

$$-u'' = u^3 \quad \text{in } (0,1)$$

with $u(0) = u(1) = 0$. Define the functional

$$J(u) = \frac{1}{2}\int_0^1 (u')^2\,dx - \frac{1}{4}\int_0^1 u^4\,dx$$

on $H_0^1(0,1)$. We show that this problem admits a nontrivial weak solution using the Mountain Pass Theorem.

1: Mountain Pass geometry

Clearly $J(0) = 0$. Using the Sobolev embedding $H_0^1(0,1) \hookrightarrow L^4(0,1)$ there exists $C > 0$ such that $\|u\|_4 \leq C\|u'\|_2$. Hence

$$J(u) = \frac{1}{2}\|u'\|_2^2 - \frac{1}{4}\|u\|_4^4 \geq \frac{1}{2}\|u'\|_2^2 - C\|u'\|_2^4.$$

Thus $J(u) > 0$ for $\|u\|$ sufficiently small. Therefore there exist $r > 0$ and $\alpha > 0$ such that $\|u\| = r \Rightarrow J(u) \geq \alpha$. Now choose $\varphi \in H_0^1(0,1)$ with $\varphi \neq 0$ and set $u = t\varphi$. Then

$$J(t\varphi) = \frac{t^2}{2}\int_0^1 (\varphi')^2\,dx - \frac{t^4}{4}\int_0^1 \varphi^4\,dx.$$

For t sufficiently large we obtain $J(t\varphi) < 0$. Thus the Mountain Pass geometry holds.

2: Palais–Smale condition

Let $\{u_n\}$ be a sequence such that $J(u_n) \to c$, $J'(u_n) \to 0$. From the definition of J it follows that $\{u_n\}$ is bounded in $H_0^1(0,1)$. Since this space is reflexive, there exists u such that $u_n \rightharpoonup u$ in $H_0^1(0,1)$. By the Rellich–Kondrachov theorem (Theorem 1.29) $H_0^1(0,1) \hookrightarrow L^4(0,1)$ compactly, hence $u_n \to u$ in $L^4(0,1)$. Using this convergence one proves that $u_n \to u$ in $H_0^1(0,1)$. Thus the Palais–Smale condition holds.

3: Mountain Pass theorem

Since the functional J satisfies the Mountain Pass geometry and the Palais–Smale condition, the Mountain Pass Theorem guarantees the existence of a critical point $u \neq 0$ such that $J'(u) = 0$.

4: Weak solution and Differential Equation

For any direction $v \in H_0^1(0,1)$, the condition $J'(u) = 0$ yields:

$$\langle J'(u), v\rangle = \int_0^1 u'v'\,dx - \int_0^1 u^3 v\,dx = 0.$$

This integral equality means that u is a weak solution. To rigorously recover the differential equation, we choose test functions $v \in C_c^\infty(0,1)$. By the definition of the distributional second derivative, we have:

$$\int_0^1 u'v'\,dx = -\langle u'', v\rangle_{\mathcal{D}',\mathcal{D}}.$$

Substituting this back into our equality, we obtain:

$$\langle -u'' - u^3, v\rangle_{\mathcal{D}',\mathcal{D}} = 0 \quad \forall v \in C_c^\infty(0,1).$$

This implies that the equation holds in the sense of distributions:

$$-u'' = u^3 \quad \text{in } \mathcal{D}'(0,1).$$

Since $u \in H_0^1(0,1) \hookrightarrow L^\infty(0,1)$, we have $u^3 \in L^2(0,1)$, which deduces that $u'' \in L^2(0,1)$. Therefore, the equality holds almost everywhere. Combined with the fact that $u \in H_0^1(0,1)$ incorporates the boundary conditions $u(0) = u(1) = 0$ in the trace sense, we conclude that the critical point u is indeed a valid solution to the boundary value problem.

3.6 Comparative Table of Regimes

Problem Type	**Geometry**	**Compactness**	**Method**
Linear (Poisson)	Convex	Full	Direct Method (Theorem 2.6)
Subcritical semilinear	Mountain Pass	Compact embedding	Mountain Pass (Theorem 2.19)
Critical exponent	Mountain Pass	Fails (bubbling)	Concentration–Compactness
Nonlinear Schrödinger (radial)	Mountain Pass	Radial compactness	Mountain Pass (Theorem 2.19)
p-Laplacian ($p < q < p^*$)	Mountain Pass	Compact embedding	Mountain Pass (Theorem 2.19)
Nonlinear eigenvalue	Constraint	Compact embedding	Ekeland Principle (Theorem 2.10)

Table 3.1: Summary of variational regimes.

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