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# Elzaki Transform and its Applications

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# Dedications

To the One who has always been with me, Allah Almighty, Praise be to You for granting me strength, patience, and guidance throughout this journey. Without Your grace, this achievement would never have been possible.

To my beloved parents, Thank you for your endless love, care, sacrifices, and sincere prayers that gave me strength during every difficult moment. This success is dedicated to you with all my gratitude and love.

To my esteemed supervisor, Dr. Khezzani Rimi, I sincerely appreciate your patience, guidance, and continuous support, which greatly contributed to the completion of this work.

And finally, to myself, To the one who endured exhaustion, pressure, and loneliness without giving up, I am proud of you. Today, all your patience and hard work have finally borne fruit.

**Fatma Mennai and Hadda Bouguesba**

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# General introduction

Integral Transforms are regarded as essential tools in applied mathematics, utilized to address ordinary, partial, and integral differential equations more efficiently than conventional methods. The Laplace transform, introduced in 1780, and the Fourier transform, presented in 1822 [4], have been essential in addressing differential equations found in various scientific domains, including physics, engineering, astronomy, and other fields. Due to the significance of these tools, the last twenty years have seen a growing interest in creating new integral transforms and enhancing the ones already in existence. Numerous scholarly articles have been released lately, including the Kamal transform [6] and the Mohand transform [9, 8] based on the Fourier transform, alongside the BA transform, which has been identified as a generalization of various other transforms, such as the Sumudu transform [13] and the Natural transform [7]. These transformations were employed to resolve ordinary and partial differential equations with constant coefficients, as they convert the differential equation into an algebraic equation that is simpler to solve. The Mahgoub transform [1] and the Anuj transform [10] were introduced for solving ordinary linear differential equations with variable coefficients, while the Sawi transform [3] and the Anuj transform [2] were utilized for addressing Volterra integral and differential equations. Additionally, the Gupta transform [5] helped address initial value problems featuring constant coefficients, whereas Emad and Israa introduced the transform [11] and the ”ne” transform [14], which are two-parameter transforms that yielded effective outcomes in solving ordinary differential equations with constant coefficients. A new general integral transform [12] was also derived from the Laplace transform and applied to solve ordinary and partial differential equations, integral equations, and fractional differential equations. All of these transformations prove their efficiency in identifying precise.

Among these modern transformations is the transformation of Elzaki, which has shown great effectiveness in the processing of various types of differential equations, thanks to its distinctive characteristics. This note examines the transformation of Elzaki and its applications in solving ordinary and fractal differential equations. The work aims to provide the theoretical foundations for this transformation, from its definition and basic properties to the calculation of derivative transfers and fractional integrations, and the use of the Mittag-Leffler function in the formulation of solutions. The memorandum also seeks to clarify how to convert the different types of differential equations (homogeneous or heterogeneous, with fixed or variable coefficients) into an algebraic equation in the transformation space, which greatly simplifies the process of solution.

To achieve these objectives, the memorandum is divided as follows: Chapter I: Reviews the basic concepts of the transformation of Elzaki, including its definition and its properties related to derivatives and the wrapping of functions, in addition to dealing with derivatives , fractal integrations , and the Mittag-Leffler function. Chapter II: Is devoted to the application of the transformation of Elzaki to the ordinary linear differential equations of the first and second rank, with the solution of various models with fixed and variable coefficients. Chapter III: Addresses the application of the transformation of Elzaki to homogeneous and heterogeneous fractal differential equations, taking advantage of the theoretical results previously established.

The work concludes with a presentation of the results that confirm the effectiveness of the transformation of Elzaki in simplifying solutions, especially in the case of fractional equations, while offering prospects for expanding the scope of application to include partial equations and fractional equation systems in future studies.

# General Notations

- $\mathcal{E}$: Elzaki transform;
- $*$: convolution;
- $E_{\beta,m}$ : Mittag-Leffler function with the parameters $\beta, m$;
- $\Gamma(m)$ : Gamma function with the parameter $m$;
- $FDE$: Fractional differential equation ;
- $ODE$: Ordinary differential equation ;
- $D^{-m}f(t)$: The Riemann–Liouville fractional integral of order $m$;
- $A$: Set of admissible functions.

# Chapter 1

# Basic concepts about Elzaki transform

This chapter serves as an introduction to the following chapters, as it contains the most important definitions and theories related to Elzaki transformation, which we will use in the subsequent chapters.

## 1.1 Elzaki Transform

**Definition 1.1.1** *[15]*

*For a function $f(t)$ defined for $t \geq 0$, the Elzaki transform is given by:*

$$\mathcal{E}[f(t)] = v \int_0^\infty f(t) e^{-\frac{t}{v}}\, dt = T(v), \qquad v \in (k_1, k_2). \tag{1.1}$$

*The Elzaki transform can also be written in an equivalent form using the substitution $t = vu$:*

$$T(v) = v^2 \int_0^\infty f(tv) e^{-t}\, dt, \qquad k_1 \leq v \leq k_2. \tag{1.2}$$

*This form is sometimes more convenient for certain calculations.*

### Set of admissible functions

*The Elzaki transform applies to functions belonging to the set $A$ defined as:*

$$A = \left\{ f(t) : \exists M, k_1, k_2 > 0,\ |f(t)| < M e^{\frac{t}{k_j}},\ \text{if } t \in\ [0, \infty) \right\}$$

*For any function in this set:*

- *$M$ is a finite constant,*
- *$k_1$ and $k_2$ may be finite or infinite.*

### 1.1.1 Elzaki transforms of basic functions

| $f(t)$ | $\mathcal{E}\{f(t)\} = T(v)$ |
|---|---|
| $1$ | $v^2$ |
| $t$ | $v^3$ |
| $t^n$ | $n!\, v^{n+2}$ |
| $e^{at}$ | $\dfrac{v^2}{1 - av}$ |
| $\sin(at)$ | $\dfrac{av^3}{1 + a^2v^2}$ |
| $\cos(at)$ | $\dfrac{v^2}{1 + a^2v^2}$ |
| $\sinh(at)$ | $\dfrac{av^3}{1 - a^2v^2}$ |
| $\cosh(at)$ | $\dfrac{v^2}{1 - a^2v^2}$ |

**Theorem 1.1.1** *[16] Let $T(v)$ is the Elzaki transform of function $f(t)$:*

$$\mathcal{E}(f(t)) = T(v) \tag{1.3}$$

$$\mathcal{E}\{f'(t)\} = \frac{T(v)}{v} - vf(0) \tag{1.4}$$

$$\mathcal{E}\{f''(t)\} = \frac{T(v)}{v^2} - f(0) - vf'(0) \tag{1.5}$$

$$\mathcal{E}\{f^{(n)}(t)\} = \frac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0) \tag{1.6}$$

### 1.1.2 Derivation of the formula for $\mathcal{E}\{tf(t)\}$

We derive the key formula for the Elzaki transform of $tf(t)$:

$$\mathcal{E}\{tf(t)\} = v\int_0^\infty e^{-t/v} \cdot tf(t)\,dt$$

Observe that:

$$\frac{\partial}{\partial v}e^{-t/v} = e^{-t/v} \cdot \frac{t}{v^2}$$

Therefore:

$$te^{-t/v} = v^2\frac{\partial}{\partial v}e^{-t/v}$$

Substituting into the integral:

$$\mathcal{E}\{tf(t)\} = v\int_0^\infty \left[v^2\frac{\partial}{\partial v}e^{-t/v}\right] f(t)\,dt$$

$$\mathcal{E}\{tf(t)\} = v^3\frac{\partial}{\partial v}\int_0^\infty e^{-t/v}f(t)\,dt$$

From the definition, $\int_0^\infty e^{-t/v}f(t)\,dt = \frac{T(v)}{v}$. Hence:

$$\mathcal{E}\{tf(t)\} = v^3\frac{d}{dv}\left[\frac{T(v)}{v}\right] \tag{1.7}$$

**Theorem 1.1.2** *[17, 18]*

*If $T(v)$ and $G(v)$ are the Elzaki transform of $f(t)$ and $g(t)$ then:*

$$\mathcal{E}\{f*g\}(t) = \mathcal{E}\left[\int_0^t f(t-\epsilon)g(\epsilon)d\epsilon\right] = \frac{1}{v}T(v)G(v) \tag{1.8}$$

*Theorem 1.1.2 states Elzaki transform of the convolution of two functions.*

**Definition 1.1.2** *[19] The Riemann-Liouville fractional integral operator of order $m \geq 0$, of a function $f \in C_\mu, \mu \geq -1$ is defined as:*

$$D^{-m} f(t) = \frac{1}{\Gamma(m)} \int_0^t (t-\epsilon)^{m-1} f(\epsilon) d\epsilon, \quad m > 0, t > 0 \tag{1.9}$$

*Where the gamma function is defined by the relation:*

$$\Gamma(m) = \int_0^\infty e^{-u} u^{m-1} du \quad ; \quad m \in \mathbb{R}, \quad \Gamma(m+1) = m\Gamma(m)$$

*Now if $f$ belongs to class C, its fractional integral of order $m$, defined by* (1.8) *, can be expressed as a convolution integral. consequently, if $f$ is of exponential order then its fractional integral is also of exponential order.*

$$\mathcal{E}\{D^{-m} f(t)\} = \frac{1}{v\Gamma(m)} \mathcal{E}\{t^{m-1}\} \mathcal{E}\{f(t)\} = \frac{1}{v\Gamma(m)} (m-1)! v^{m+1} T(v) = v^m (T(v)) \tag{1.10}$$

*Now we can apply* (1.10) *to find.*

*a)* $\mathcal{E}\{D^{-m} t^n\} = v^m \mathcal{E}\{t^n\} = v^m (n! v^{n+2}) \quad n > 0, v > 0.$

*b)* $\mathcal{E}\{D^{-m} e^{at}\} = v^m \left[\frac{v^2}{1-av}\right] \quad v > 0.$

*c)* $\mathcal{E}\{D^{-m} \sin at\} = v^m \left[\frac{av^3}{1+a^2v^2}\right] \quad v > 0.$

*d)* $\mathcal{E}\{D^{-m} \cos at\} = v^m \left[\frac{v^2}{1+a^2v^2}\right] \quad v > 0.$

## 1.2 Transform of the fractional derivatives

**Theorem 1.2.1** *Let $f$ be continuous on $A$ and let $v > 0$ then $Df$ is continuous on $j$ when $t > 0$, then:*

$$D[D^{-m} f(t)] = D^{-m}[Df(t)] + \frac{f(0)}{\Gamma(m)} t^{m-1} \tag{1.11}$$

*Now to find Elzaki transform of fractional integer of the derivative of the fractional integer, then form* (1.10) *we have:*

$$\mathcal{E}\{D^{-m}[Df(t)]\} = v^m \mathcal{E}\{Df(t)\} = v^m \left[\frac{T(v)}{v} - vf(0)\right] \tag{1.12}$$

*This formula obviously also valid if $v = 0$. Now we consider the problem of find Elzaki transform of derivative of the fractional integer, then form (theorem 1.2.1), we get*

$$\begin{aligned} \mathcal{E}\{D[D^{-m} f(t)]\} &= \mathcal{E}\{D^{-m}[Df(t)]\} + \frac{f(0)}{\Gamma(m)} \mathcal{E}\{t^{m-1}\} \\ &= v^m \left[\frac{T(v)}{v} - vf(0)\right] + \frac{f(0)}{\Gamma(m)} (m-1)! v^{m+1} = v^{m-1} T(v) \end{aligned} \tag{1.13}$$

**Definition 1.2.1** *The fractional derivative can be defined using the definition integer to this end, suppose that $m = n - \mu$ where $0 < \mu < 1$ and $n$ is smallest integer greater than $m$, then, the fractional derivative of $f(x)$ of order $m$ is:*

$$D^m f(t) = D^n[D^{-\mu} f(t)] \tag{1.14}$$

*We can write* (1.14)*:*

$$D^m f(t) = D^n[D^{-(n-m)} f(t)] \tag{1.15}$$

*We assume that Elzaki transform of $f(t)$ exists. Then by using* (1.6) *then:*

$$\begin{aligned}
\mathcal{E}\{D^m f(t)\} &= \mathcal{E}\{D^n[D^{-(n-m)} f(t)]\} \\
&= \frac{1}{v^n}\mathcal{E}\{D^{-(n-m)} f(t)\} - \sum_{k=0}^{n-1} v^{2-n+k} D^k[D^{-(n-m)} f(0)] \\
&= \frac{1}{v^n}[D^{-(n-m)}]T(v) - \sum_{k=0}^{n-1} v^{2-n+k} D^{k-(n-m)} f(0) \\
&= \frac{1}{v^n} v^{n-m} T(v) - \sum_{k=0}^{n-1} v^{2-n+k} D^{k-(n-m)} f(0) \\
&= \frac{1}{v^m} T(v) - \sum_{k=0}^{n-1} v^{2-n+k} D^{k-(n-m)} f(0)
\end{aligned} \tag{1.16}$$

*For $n = 1$ and $2$ we have:*

$$\mathcal{E}\{D^m f(t)\} = \frac{1}{v^m} T(v) - vD^{-(1-m)} f(0) \qquad 0 < m \leq 1$$

$$\mathcal{E}\{D^m f(t)\} = \frac{1}{v^m} T(v) - v^{1-m} f(0) - vD^{-(2-m)} f(0) - vD^{-(1-m)} f(0) \qquad 1 < m \leq 2$$

# Chapter 2

# Applying the Elzaki transform to ordinary differential equations

In this chapter, we will apply some of the results mentioned in the first chapter to the ordinary linear differential equation through several examples.

## 2.1 Solving linear ordinary differential Equations with constant coefficients

### 2.1.1 First order equation

Consider the first-order ordinary differential equation:

$$\frac{dx}{dt} + px = f(t), \quad t > 0 \tag{2.1}$$

With the initial condition:

$$x(0) = a \tag{2.2}$$

Where $p$ and $a$ are constants and $f(t)$ is an external input function.

Applying Elzaki transform:

$$[\frac{1}{v}\,\mathcal{E}\{x(t)\} - vx(0)] + p\,\mathcal{E}\{x(t)\} = \mathcal{E}\{f(t)\}$$

Substituting the initial condition $x(0) = a$:

$$[\frac{1}{v}\,\mathcal{E}\{x(t)\} - va] + p\,\mathcal{E}\{x(t)\} = \mathcal{E}\{f(t)\}$$

Multiplying both sides by $v$:

$$\mathcal{E}\{x(t)\} - v^2 a + pv\,\mathcal{E}\{x(t)\} = v\,\mathcal{E}\{f(t)\}$$

Or: terms:

$$(1 + pv)\,\mathcal{E}\{x(t)\} = v\,\mathcal{E}\{f(t)\} + v^2 a$$

Finally, solving for $\mathcal{E}\{x(t)\}$:

$$\mathcal{E}\{x(t)\} = \frac{v\,\mathcal{E}\{f(t)\} + v^2 a}{1 + pv}$$

### 2.1.2 Second order equation

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + qy = f(x), \quad x > 0 \tag{2.3}$$

Initial conditions:

$$y(0) = a, \quad y'(0) = b \tag{2.4}$$

Applying Elzaki transform:

$$\frac{1}{v^2}\left[\mathcal{E}\{y(x)\} - v^2 y(0) - v^3 y'(0)\right] + 2p\left[\frac{\mathcal{E}\{y(x)\}}{v} - vy(0)\right] + q\,\mathcal{E}\{y(x)\} = \mathcal{E}\{f(x)\}$$

Substituting the initial conditions $y(0) = a$, $y'(0) = b$:

$$\frac{1}{v^2}\left[\mathcal{E}\{y(x)\} - v^2 a - v^3 b\right] + 2p\left[\frac{\mathcal{E}\{y(x)\}}{v} - va\right] + q\,\mathcal{E}\{y(x)\} = \mathcal{E}\{f(x)\}$$

Multiplying by $v^2$:

$$\mathcal{E}\{y(x)\} - v^2 a - v^3 b + 2pv\,\mathcal{E}\{y(x)\} - 2pv^3 a + qv^2\mathcal{E}\{y(x)\} = v^2\mathcal{E}\{f(x)\}$$

Grouping terms:

$$(1 + 2pv + qv^2)\mathcal{E}\{y(x)\} = v^2\mathcal{E}\{f(x)\} + v^2 a + v^3 b + 2pv^3 a$$

Finally:

$$\mathcal{E}\{y(x)\} = \frac{v^2\mathcal{E}\{f(x)\}}{1 + 2pv + qv^2} + \frac{v^2 a}{1 + 2pv + qv^2} + \frac{v^3 b + 2pv^3 a}{1 + 2pv + qv^2}$$

**Example 2.1.1** *Solve the differential equation:*

$$\frac{dy}{dx} + y = 0, \quad y(0) = 1$$

***Solution:***

*Taking the Elzaki transform on both sides:*

$$\mathcal{E}\left\{\frac{dy}{dx}\right\} + \mathcal{E}\{y(x)\} = 0$$

*Using* (1.4)*:*

$$\mathcal{E}\left\{\frac{dy}{dx}\right\} = \frac{1}{v}T(v) - vy(0)$$

*we obtain:*

$$\frac{1}{v}T(v) - v + T(v) = 0$$

*Grouping terms:*

$$\left(\frac{1}{v} + 1\right)T(v) = v.$$

*Then:*

$$T(v) = \frac{v^2}{1+v}$$

*Taking inverse Elzaki transform:*

$$y(x) = e^{-x}$$

***Example 2.1.2*** *Solve:*

$$y'' + y = 0, \quad y(0) = 1, \; y'(0) = 1$$

***Solution:***

*Applying Elzaki transform:*

$$\mathcal{E}\{y''\} + \mathcal{E}\{y\} = 0$$

*Using* (1.5) *find:*

$$\mathcal{E}\{y''\} = \frac{1}{v^2}T(v) - y(0) - vy'(0)$$

*Substitute initial conditions:*

$$\frac{1}{v^2}T(v) - v - 1 + T(v) = 0$$

*Or:*

$$\left(\frac{1}{v^2} + 1\right)T(v) = v + 1$$

$$T(v) = \frac{v^2}{v^2+1} + \frac{v^3}{v^2+1}$$

*Separating:*

$$T(v) = \frac{v^2}{v^2+1} + \frac{v^3}{v^2+1}$$

*Taking inverse Elzaki transform:*

$$y(x) = \sin x + \cos x$$

***Example 2.1.3*** *Solve:*

$$y'' - 3y' + 2y = 0, \quad y(0) = 1,\ y'(0) = 4$$

***Solution:***

*Applying Elzaki transform:*

$$\mathcal{E}\{y''\} - 3\mathcal{E}\{y'\} + 2\mathcal{E}\{y\} = 0$$

*Using* (1.4) *and* (1.5)*:*

$$\mathcal{E}\{y''\} = \frac{1}{v^2}T(v) - y(0) - vy'(0)$$

$$\mathcal{E}\{y'\} = \frac{1}{v}T(v) - vy(0)$$

*Substitute initial conditions:*

$$\frac{1}{v^2}T(v) - 1 - 4v - 3\left(\frac{1}{v}T(v) - v\right) + 2T(v) = 0$$

*Simplify:*

$$\frac{1}{v^2}T(v) - 1 - 4v - \frac{3}{v}T(v) + 3v + 2T(v) = 0$$

$$\left(\frac{1}{v^2} - \frac{3}{v} + 2\right)T(v) = v + 1$$

*Multiply by $v^2$:*

$$(1 - 3v + 2v^2)T(v) = v^3 + v^2$$

$$T(v) = \frac{v^2(v+1)}{(2v-1)(v-1)}$$

*Partial fraction:*

$$T(v) = v^2\left(\frac{2}{v-1} - \frac{3}{2v-1}\right)$$

*Rewriting:*

$$T(v) = \frac{-2v^2}{1-v} + \frac{3v^2}{1-2v}$$

*Taking inverse transform:*

$$y(x) = -2e^x + 3e^{2x}$$

***Example 2.1.4*** *Solve:*

$$y'' + 9y = \cos(2t), \quad y(0) = 1,\ y\left(\frac{\pi}{2}\right) = -1$$

*Let:*

$$y'(0) = c$$

***Solution:***

*Apply Elzaki transform:*

$$\mathcal{E}\{y''\} + 9\mathcal{E}\{y\} = \mathcal{E}\{\cos(2t)\}$$

*Using* (1.5)*:*

$$\mathcal{E}\{y''\} = \frac{1}{v^2}T(v) - y(0) - vy'(0)$$

$$\mathcal{E}\{\cos(at)\} = \frac{v^2}{1+a^2v^2}$$

*Substitute:*

$$\frac{1}{v^2}T(v) - 1 - cv + 9T(v) = \frac{v^2}{1+4v^2}$$

$$\left(\frac{1}{v^2} + 9\right)T(v) = \frac{v^2}{1+4v^2} + 1 + cv$$

*Multiply by $v^2$:*

$$(1+9v^2)T(v) = \frac{v^4}{1+4v^2} + v^2 + cv^3$$

$$T(v) = \frac{v^4}{(4v^2+1)(1+9v^2)} + \frac{cv^3}{(1+9v^2)} + \frac{v^2}{(1+9v^2)}$$

*Thus:*

$$T(v) = \frac{v^4}{(4v^2+1)(1+9v^2)} + \frac{cv^3+v^2}{1+9v^2}$$

$$T(v) = \frac{v^4}{(4v^2+1)(1+9v^2)} + \frac{(cv^3+v^2)(4v^2+1)}{(4v^2+1)(1+9v^2)}$$

$$T(v) = \frac{v^4 + (cv^3+v^2)(4v^2+1)}{(4v^2+1)(1+9v^2)}$$

*Expand the numerator:*

$$(cv^3 + v^2)(4v^2 + 1) = 4cv^5 + cv^3 + 4v^4 + v^2$$

*Add $v^4$:*

$$v^4 + (cv^3 + v^2)(4v^2 + 1) = v^4 + 4cv^5 + cv^3 + 4v^4 + v^2 = 4cv^5 + 5v^4 + cv^3 + v^2$$

*Therefore:*

$$T(v) = \frac{4cv^5 + 5v^4 + cv^3 + v^2}{(4v^2 + 1)(1 + 9v^2)}$$
$$T(v) = \frac{v^2\big(4cv^3 + 5v^2 + cv + 1\big)}{(4v^2 + 1)(1 + 9v^2)}$$

*Partial fractions setup:*

$$\frac{4cv^3 + 5v^2 + cv + 1}{(4v^2 + 1)(1 + 9v^2)} = \frac{Av + B}{4v^2 + 1} + \frac{Cv + D}{1 + 9v^2}$$

*Multiply through:*

$$4cv^3 + 5v^2 + cv + 1 = (Av + B)(1 + 9v^2) + (Cv + D)(4v^2 + 1)$$

*Expand and equate coefficients:*

$$(Av + B)(1 + 9v^2) = 9Av^3 + 9Bv^2 + Av + B$$

$$(Cv + D)(4v^2 + 1) = 4Cv^3 + 4Dv^2 + Cv + D$$

*Adding:*

$$v^3 : \quad 9A + 4C = 4c$$

$$v^2 : \quad 9B + 4D = 5$$

$$v^1 : \quad A + C = c$$

$$v^0 : \quad B + D = 1$$

*Solve the system:*

$$5A = 0 \implies A = 0, \quad C = c$$

$$5B = 1 \implies B = \frac{1}{5}, \quad D = 1 - \frac{1}{5} = \frac{4}{5}$$

*Write the decomposition:*

$$\frac{4cv^3+5v^2+cv+1}{(4v^2+1)(1+9v^2)}=\frac{0\cdot v+\frac{1}{5}}{4v^2+1}+\frac{cv+\frac{4}{5}}{1+9v^2}$$

*Thus:*

$$T(v)=v^2\left[\frac{1/5}{4v^2+1}+\frac{cv+4/5}{1+9v^2}\right]$$

$$T(v)=v^2\left[\frac{1}{5(1+4v^2)}+\frac{4}{5(1+9v^2)}+\frac{cv}{1+9v^2}\right]$$

*Inverse transform:*

$$y(x)=\frac{1}{5}\cos(2x)+\frac{4}{5}\cos(3x)+\frac{c}{3}\sin(3x)$$

*Using condition:*

$$y\left(\frac{\pi}{2}\right)=-1$$

$$\frac{1}{5}\cos(\pi)+\frac{4}{5}\cos\left(\frac{3\pi}{2}\right)+\frac{c}{3}\sin\left(\frac{3\pi}{2}\right)=-1$$

$$-\frac{1}{5}-0-\frac{c}{3}=-1$$

$$c=\frac{12}{5}$$

*Final solution:*

$$y(x)=\frac{1}{5}\cos(2x)+\frac{4}{5}\cos(3x)+\frac{4}{5}\sin(3x)$$

## 2.2 Solving linear ordinary differential Equations with variable coefficients

### 2.2.1 First order equation

***Example 2.2.1*** *Solve:*

$$tu'-2u=0 \tag{2.5}$$

$$u(0)=0 \tag{2.6}$$

*Applying the Elzaki transformation to equation* (2.5)*:*

$$\mathcal{E}\{tu'(t)\} - 2\mathcal{E}\{u(t)\} = 0$$

***Computing*** $\mathcal{E}\{tu'(t)\}$***:***

*Using* (1.4) *and* (2.6) *find:*

$$\mathcal{E}\{u'(t)\} = \frac{T(v)}{v}$$

*Set* $f(t) = u'(t)$ *in the derived formula.*

*Then* $\frac{T_f(v)}{v} = \frac{\mathcal{E}\{u'(t)\}}{v} = \frac{T(v)/v}{v} = \frac{T(v)}{v^2}$.

*By substitution in* (1.7)*, we find:*

$$\mathcal{E}\{tu'(t)\} = v^3 \frac{d}{dv}\left[\frac{T(v)}{v^2}\right]$$

*The transformed ODE becomes:*

$$v^3 \frac{d}{dv}\left[\frac{T(v)}{v^2}\right] - 2T(v) = 0 \tag{2.7}$$

*Let* $W(v) = \dfrac{T(v)}{v^2}$*, so* $T(v) = v^2 W(v)$*.*

*Substitute in* (2.7)*:*

$$v^3 W'(v) - 2v^2 W(v) = 0$$

$$vW'(v) - 2W(v) = 0$$

***Solving the differential equation for*** $W(v)$*:*

$$\frac{W'(v)}{W(v)} = \frac{2}{v}$$

*Integrate both sides:*

$$\ln|W(v)| = 2\ln v + \ln C$$

$$W(v) = Cv^2$$

*Hence:*

$$T(v) = v^2 W(v) = v^2 \cdot Cv^2 = Cv^4$$

***Inverse Elzaki transform***

*Recall the elementary transform:* $\mathcal{E}\{t^2\} = 2!\, v^4 = 2v^4$.

*Thus:*

$$Cv^4 = \frac{C}{2} \cdot 2v^4 = \frac{C}{2}\,\mathcal{E}\{t^2\}$$

*Therefore, inverting:*

$$u(t) = \frac{C}{2}t^2$$

*According to* (2.6), *we obtain:*

$$\boxed{u(t) = t^2}$$

### 2.2.2 Second order equation

***Example 2.2.2*** *Solve:*

$$tu'' + (1-2t)u' - 2u = 0, \tag{2.8}$$

$$u(0) = 1, \tag{2.9}$$

$$u'(0) = 2. \tag{2.10}$$

***Solution:***

*Applying the Elzaki transformation to equation* (2.8)*:*

$$\mathcal{E}\{tu''\} + \mathcal{E}\{u'\} + \mathcal{E}\{-2tu'\} + \mathcal{E}\{-2u\} = 0 \tag{2.11}$$

*Let* $\mathcal{E}\{u(t)\} = T(v)$.

*By using* (1.4), (1.5), (2.9) *and* (2.10)*:*

$$\mathcal{E}\{u'(t)\} = \frac{T(v)}{v} - v \cdot 1 = \frac{T(v)}{v} - v \tag{2.12}$$

$$\mathcal{E}\{u''(t)\} = \frac{T(v)}{v^2} - 1 - 2v$$

*We will compute the Elzaki transform of each term separately.*

***Transform of*** $tu''(t)$

*Let* $f(t) = u''(t)$. *Then* $\mathcal{E}\{f(t)\} = \dfrac{T(v)}{v^2} - 1 - 2v.$

*By using* (1.7) *we find:*

$$\mathcal{E}\{tu''(t)\} = v^3 \frac{d}{dv}\left[\frac{\frac{T(v)}{v^2} - 1 - 2v}{v}\right]$$

*Simplify the expression inside the brackets:*

$$\frac{1}{v}\left(\frac{T(v)}{v^2} - 1 - 2v\right) = \frac{T(v)}{v^3} - \frac{1}{v} - 2$$

*Therefore:*

$$\mathcal{E}\{tu''(t)\} = v^3 \frac{d}{dv}\left[\frac{T(v)}{v^3} - \frac{1}{v} - 2\right]$$

*Now differentiate term by term:*

$$\frac{d}{dv}\left(\frac{T(v)}{v^3}\right) = \frac{T'(v)}{v^3} - \frac{3T(v)}{v^4}$$

$$\frac{d}{dv}\left(-\frac{1}{v}\right) = \frac{1}{v^2}$$

$$\frac{d}{dv}(-2) = 0$$

*Thus:*

$$\mathcal{E}\{tu''(t)\} = v^3\left[\frac{T'(v)}{v^3} - \frac{3T(v)}{v^4} + \frac{1}{v^2}\right]$$

*Simplifying:*

$$\boxed{\mathcal{E}\{tu''(t)\} = T'(v) - \frac{3T(v)}{v} + v} \tag{2.13}$$

***Transform of*** $-2tu'(t)$

*First, find* $\mathcal{E}\{tu'(t)\}$.

*Let* $f(t) = u'(t)$. *By substituting* $(2.12)$ *into* $(1.7)$ *we find :*

$$\mathcal{E}\{tu'(t)\} = v^3 \frac{d}{dv}\left[\frac{\frac{T(v)}{v} - v}{v}\right] = v^3 \frac{d}{dv}\left[\frac{T(v)}{v^2} - 1\right]$$

*Differentiate:*

$$\frac{d}{dv}\left(\frac{T(v)}{v^2}\right) = \frac{T'(v)}{v^2} - \frac{2T(v)}{v^3}$$

$$\frac{d}{dv}(-1) = 0$$

*Thus:*

$$\mathcal{E}\{tu'(t)\} = v^3\left[\frac{T'(v)}{v^2} - \frac{2T(v)}{v^3}\right] = vT'(v) - 2T(v)$$

*Therefore:*

$$\boxed{\mathcal{E}\{-2tu'(t)\} = -2vT'(v) + 4T(v)} \tag{2.14}$$

*Transform of* $-2u(t)$*:*

$$\boxed{\mathcal{E}\{-2u(t)\} = -2T(v)} \tag{2.15}$$

*Substitute* (2.12), (2.13), (2.14), (2.15) *into* (2.11) *we found:*

$$\left[T'(v) - \frac{3T(v)}{v} + v\right] + \left[\frac{T(v)}{v} - v\right] + [-2vT'(v) + 4T(v)] + [-2T(v)] = 0 \tag{2.16}$$

*So the transformed equation simplifies to:*

$$(1-2v)T'(v) - \frac{2T(v)}{v} + 2T(v) = 0$$

*SO:*

$$(1-2v)T'(v) + \left(2 - \frac{2}{v}\right)T(v) = 0$$

*Separate variables:*

$$\frac{T'(v)}{T(v)} = -\frac{2 - \frac{2}{v}}{1-2v}$$

*Thus:*

$$\frac{T'(v)}{T(v)} = -\frac{\frac{2(v-1)}{v}}{1-2v} = -\frac{2(v-1)}{v(1-2v)} \tag{2.17}$$

*We need to decompose:*

$$\frac{2(v-1)}{v(1-2v)} = \frac{A}{v} + \frac{B}{1-2v}$$

*Therefore:*

$$\frac{2(v-1)}{v(1-2v)} = -\frac{2}{v} - \frac{2}{1-2v}$$

*We have:*

$$\frac{T'(v)}{T(v)} = -\left[-\frac{2}{v} - \frac{2}{1-2v}\right] = \frac{2}{v} + \frac{2}{1-2v}$$

*Integrate both sides:*

$$\int \frac{T'(v)}{T(v)}\,dv = \int \frac{2}{v}\,dv + \int \frac{2}{1-2v}\,dv$$

$$\ln|T(v)| = 2\ln|v| - \ln|1-2v| + \ln C$$

*Thus:*

$$\ln|T(v)| = \ln\left(\frac{Cv^2}{|1-2v|}\right)$$

*Therefore:*

$$T(v) = \frac{Cv^2}{1-2v}$$

***Inverse Elzaki transform:***

*We have:*

$$T(v) = C \cdot \frac{v^2}{1-2v}$$

*Recall that* $\mathcal{E}\{e^{2t}\} = \dfrac{v}{1-2v}$.

*We can write:*

$$\frac{v^2}{1-2v} = v \cdot \frac{v}{1-2v} = v \cdot \mathcal{E}\{e^{2t}\}$$

*Using the series expansion method:*

$$\frac{v^2}{1-2v} = v^2 \sum_{n=0}^{\infty} (2v)^n = \sum_{n=0}^{\infty} 2^n v^{n+2}$$

*We know the inverse Elzaki transform:* $\mathcal{E}^{-1}\{v^{n+2}\} = \dfrac{t^n}{n!}$.

*Therefore:*

$$u(t) = C \sum_{n=0}^{\infty} \frac{2^n t^n}{n!} = Ce^{2t} \tag{2.18}$$

*From* (2.9), (2.10) *and* (2.18) *we obtain:*

$$\boxed{u(t) = e^{2t}}$$

***Example 2.2.3*** *Solve:*

$$t^2u'' - tu' + u = 5, \tag{2.19}$$

$$u(0) = 5, \tag{2.20}$$

$$u'(0) = 3. \tag{2.21}$$

***Solution:***

*Applying the Elzaki transformation to equation* (2.19)*:*

$$\mathcal{E}\{t^2u''\} - \mathcal{E}\{tu'\} + \mathcal{E}\{u\} = \mathcal{E}\{5\} \tag{2.22}$$

***Formula for*** $\mathcal{E}\{t^2 f(t)\}$ :

*From property* (1.7)*, we can derive a formula for* $\mathcal{E}\{t^2 f(t)\}$.

*Let* $g(t) = tf(t)$. *Then:*

$$\mathcal{E}\{t^2 f(t)\} = \mathcal{E}\{tg(t)\} = v^3\frac{d}{dv}\left[\frac{\mathcal{E}\{g(t)\}}{v}\right]$$

*But* $\mathcal{E}\{g(t)\} = \mathcal{E}\{tf(t)\} = v^3\frac{d}{dv}\left[\frac{\mathcal{E}\{f(t)\}}{v}\right]$.

*Therefore:*

$$\mathcal{E}\{t^2 f(t)\} = v^3\frac{d}{dv}\left[\frac{v^3\frac{d}{dv}\left(\frac{\mathcal{E}\{f(t)\}}{v}\right)}{v}\right]$$

$$\boxed{\mathcal{E}\{t^2 f(t)\} = v^3\frac{d}{dv}\left[v^2\frac{d}{dv}\left(\frac{\mathcal{E}\{f(t)\}}{v}\right)\right]} \tag{2.23}$$

***Transform of*** $u'(t)$***:***

*Using* (1.4) *and* (2.20) *find:*

$$\mathcal{E}\{u'(t)\} = \frac{T(v)}{v} - v\cdot 5 = \frac{T(v)}{v} - 5v \tag{2.24}$$

***Transform of*** $u''(t)$***:***

*Using* (1.5), (2.20) *and* (2.21) *find:*

$$\mathcal{E}\{u''(t)\} = \frac{T(v)}{v^2} - 5 - v\cdot 3 = \frac{T(v)}{v^2} - 5 - 3v \tag{2.25}$$

***Transform of*** $tu'(t)$**:**

*Using* (1.7), (2.24) *with* $f(t) = u'(t)$ *find:*

$$\mathcal{E}\{tu'(t)\} = v^3 \frac{d}{dv}\left[\frac{\mathcal{E}\{u'(t)\}}{v}\right] = v^3 \frac{d}{dv}\left[\frac{\frac{T(v)}{v} - 5v}{v}\right]$$

$$= v^3 \frac{d}{dv}\left[\frac{T(v)}{v^2} - 5\right]$$

*then:*

$$\frac{d}{dv}\left(\frac{T(v)}{v^2}\right) = \frac{T'(v)}{v^2} - \frac{2T(v)}{v^3}, \qquad \frac{d}{dv}(-5) = 0$$

*Thus:*

$$\mathcal{E}\{tu'(t)\} = v^3\left(\frac{T'(v)}{v^2} - \frac{2T(v)}{v^3}\right) = vT'(v) - 2T(v)$$

*Therefore:*

$$\mathcal{E}\{-tu'(t)\} = -vT'(v) + 2T(v) \tag{2.26}$$

***Transform of*** $t^2u''(t)$**:**

*Let* $f(t) = u''(t)$. *Then:*

$$\mathcal{E}\{f(t)\} = \frac{T(v)}{v^2} - 5 - 3v$$

*First compute* $\frac{\mathcal{E}\{f(t)\}}{v}$*:*

$$\frac{\mathcal{E}\{f(t)\}}{v} = \frac{\frac{T(v)}{v^2} - 5 - 3v}{v} = \frac{T(v)}{v^3} - \frac{5}{v} - 3$$

*Now differentiate with respect to v:*

$$\frac{d}{dv}\left(\frac{T(v)}{v^3}\right) = \frac{T'(v)}{v^3} - \frac{3T(v)}{v^4}$$

$$\frac{d}{dv}\left(-\frac{5}{v}\right) = \frac{5}{v^2}$$

$$\frac{d}{dv}(-3) = 0$$

*So:*

$$\frac{d}{dv}\left(\frac{\mathcal{E}\{f\}}{v}\right) = \frac{T'(v)}{v^3} - \frac{3T(v)}{v^4} + \frac{5}{v^2}$$

*Multiply by $v^2$:*

$$v^2\frac{d}{dv}\left(\frac{\mathcal{E}\{f\}}{v}\right) = \frac{T'(v)}{v} - \frac{3T(v)}{v^2} + 5$$

*Now differentiate again with respect to v:*

$$\frac{d}{dv}\left(\frac{T'(v)}{v}\right) = \frac{T''(v)}{v} - \frac{T'(v)}{v^2}$$

$$\frac{d}{dv}\left(-\frac{3T(v)}{v^2}\right) = -\frac{3T'(v)}{v^2} + \frac{6T(v)}{v^3}$$

$$\frac{d}{dv}(5) = 0$$

*Thus:*

$$\frac{d}{dv}\left[v^2\frac{d}{dv}\left(\frac{\mathcal{E}\{f\}}{v}\right)\right] = \frac{T''(v)}{v} - \frac{T'(v)}{v^2} - \frac{3T'(v)}{v^2} + \frac{6T(v)}{v^3}$$

$$= \frac{T''(v)}{v} - \frac{4T'(v)}{v^2} + \frac{6T(v)}{v^3}$$

*Finally, multiply by $v^3$:*

$$\mathcal{E}\{t^2u''\} = v^3\left(\frac{T''(v)}{v} - \frac{4T'(v)}{v^2} + \frac{6T(v)}{v^3}\right)$$

$$\boxed{\mathcal{E}\{t^2u''\} = v^2T''(v) - 4vT'(v) + 6T(v)} \tag{2.27}$$

*Substitute* (2.26), (2.27) *into* (2.22), *we obtain a Cauchy-Euler equation for $T(v)$:*

$$\boxed{v^2T''(v) - 5vT'(v) + 9T(v) = 5v^2} \tag{2.28}$$

***Solve the ODE for*** $T(v)$***:***

Homogeneous solution

*The homogeneous equation is:*

$$v^2T''(v) - 5vT'(v) + 9T(v) = 0 \tag{2.29}$$

*Assume a solution of the form* $T(v) = v^m$*:*
*Then:*

$$T'(v) = mv^{m-1}, \quad T''(v) = m(m-1)v^{m-2}$$

*Substitute into* (2.29)*, we obtain:*

$$\begin{aligned} v^2 \cdot m(m-1)v^{m-2} - 5v \cdot mv^{m-1} + 9v^m &= 0 \\ m(m-1)v^m - 5mv^m + 9v^m &= 0 \\ [m(m-1) - 5m + 9]v^m &= 0 \\ m^2 - m - 5m + 9 = m^2 - 6m + 9 &= 0 \\ (m-3)^2 = 0 \quad \Rightarrow \quad m = 3 \quad (\textit{double root}). \end{aligned}$$

*Therefore, the homogeneous solution is:*

$$T_h(v) = C_1 v^3 + C_2 v^3 \ln v$$

*Particular solution*

*Since the right-hand side is* $5v^2$*, we try* $T_p = Av^2$ *(since* $v^2$ *is not a solution of the homogeneous equation). Compute derivatives:*

$$T_p' = 2Av, \quad T_p'' = 2A$$

*Substitute into* (2.28)*:*

$$v^2(2A) - 5v(2Av) + 9(Av^2) = 2Av^2 - 10Av^2 + 9Av^2 = Av^2 = 5v^2$$

*SO:*

$$A = 5$$

*Thus* $T_p(v) = 5v^2$*.*

*General solution for* $T(v)$

*The general solution is the sum of the homogeneous and particular solutions:*

$$T(v) = T_h(v) + T_p(v)$$

$$T(v) = C_1 v^3 + C_2 v^3 \ln v + 5v^2$$

***Inverse Elzaki transform***

*We need to find* $u(t) = \mathcal{E}^{-1}\{T(v)\}$.

*Recall the following inverse transforms:*

- $\mathcal{E}^{-1}\{v^{n+2}\} = \dfrac{t^n}{n!}$
- $\mathcal{E}^{-1}\{v^3\} = t$
- $\mathcal{E}^{-1}\{v^3 \ln v\} = t \ln t$ *(known special inverse)*
- $\mathcal{E}^{-1}\{5v^2\} = 5 \cdot 1 = 5$

*Therefore:*

$$u(t) = C_1 t + C_2 t \ln t + 5$$

***Apply initial conditions***

*According to* (2.9)*:*

*As* $t \to 0^+$, *both* $t$ *and* $t \ln t$ *approach* 0*. Thus:*

$$u(0) = 5 \quad \textit{is automatically satisfied.}$$

*According to* (2.10)*:*

*First compute* $u'(t)$*:*

$$u'(t) = C_1 + C_2(\ln t + 1)$$

*As* $t \to 0^+$, $\ln t \to -\infty$. *For* $u'(0)$ *to be finite, we must have* $C_2 = 0$.

*Then:*

$$u'(t) = C_1 \implies u'(0) = C_1 = 3$$

***The solution***

*With* $C_1 = 3$ *and* $C_2 = 0$*:*

$$u(t) = 3t + 5$$

# Chapter 3

# Applying Elzaki transform to fractional differential equations

## 3.1 Fractional ordinary differential equations

*The general formula of fractional linear ordinary differential equation of order m is given by:*

$$[D^m + a_{n-1}D^{m(n-1)} + \cdots + a_0 D^0]x(t) = h(t) \quad t \geq 0 \tag{3.1}$$

*Where $a_0, a_1, a_2, \ldots, a_{n-1}$ are functions of the independent variable t.*
*We can find the solution of eq (3.1), by using Elzaki transform, but before that we introduce the definition of important function, Mittag-Leffler function.*

***Theorem 3.1.1*** *[20, 21] The special function of Mittag-Leffler function is defined as:*

$$E_{\beta,m}(t) = \sum_{k=0}^{\infty} \frac{t^{\beta k}}{\Gamma(\beta k + m)} \tag{3.2}$$

*Where $\beta \in \mathbb{C}$, $Re(m)$, $Re(\beta) > 0$.*
*The function $E(t, m, a)$ is used to solve differential equation of fractional order which is defined by:*

$$E_{\beta,m}(t, m, a) = t^m \sum_{k=0}^{\infty} \frac{(at^{\beta})^k}{\Gamma(\beta k + m + 1)} = t^m E_{\beta,m+1}(at^{\beta}) \tag{3.3}$$

*By using eq (3.3) , and Elzaki transform we prove the following important relations:*
*a)*

$$\mathcal{E}^{-1}\left[\frac{v^{m+2}}{1 - av^{\beta}}\right] = t^m E_{\beta,m+1}(at^{\beta}) \tag{3.4}$$

*b)*

$$\mathcal{E}^{-1}\left[\frac{v^{m+1}}{1-av^{\beta}}\right] = t^{m-1}E_{\beta,m}(at^{\beta}) \tag{3.5}$$

***Proof 3.1.1*** *(a) Using the definition of Elzaki transform:*

$$\mathcal{E}\{t^m E_{\beta,m+1}(at^{\beta})\} = v\int_0^{\infty} e^{-t/v}t^m \sum_{k=0}^{\infty}\frac{a^k t^{\beta k}}{\Gamma(\beta k+m+1)}dt.$$

*Swapping sum and integral:*

$$= \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(\beta k+m+1)}v\int_0^{\infty} e^{-t/v}t^{\beta k+m}dt.$$

*Now* $v\int_0^{\infty} e^{-t/v}t^{\beta k+m}dt = \mathcal{E}\{t^{\beta k+m}\} = (\beta k+m)!\, v^{\beta k+m+2}$.

*Thus:*

$$\mathcal{E}\{t^m E_{\beta,m+1}(at^{\beta})\} = \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(\beta k+m+1)}(\beta k+m)!\, v^{\beta k+m+2}.$$

*Since* $(\beta k+m)! = \Gamma(\beta k+m+1)$*:*

$$\mathcal{E}\{t^m E_{\beta,m+1}(at^{\beta})\} = \sum_{k=0}^{\infty}a^k v^{\beta k+m+2} = v^{m+2}\sum_{k=0}^{\infty}(av^{\beta})^k.$$

*This is a geometric series:* $\sum_{k=0}^{\infty}(av^{\beta})^k = \frac{1}{1-av^{\beta}}$ *(provided* $|av^{\beta}| < 1$*).*

*Therefore:*

$$\mathcal{E}\{t^m E_{\beta,m+1}(at^{\beta})\} = \frac{v^{m+2}}{1-av^{\beta}}.$$

*Taking the inverse Elzaki transform gives* (3.4)*.*

*b)*

$$\begin{aligned}\mathcal{E}\{t^{m-1}E_{\beta,m}(at^{\beta})\} &= \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(\beta k+m)}\,\mathcal{E}[t^{\beta k+m-1}]\\ &= \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(\beta k+m)}\,(\beta k+m-1)!\, v^{\beta k+m+1}.\end{aligned}$$

*Since* $(\beta k+m-1)! = \Gamma(\beta k+m)$*, the denominator cancels completely:*

$$\mathcal{E}\{t^{m-1}E_{\beta,m}(at^{\beta})\} = \sum_{k=0}^{\infty}a^k v^{\beta k+m+1} = v^{m+1}\sum_{k=0}^{\infty}(av^{\beta})^k = \frac{v^{m+1}}{1-av^{\beta}}.$$

*This is exactly the desired result.*

## 3.2 Solving homogeneous fractional differential equation.

***Example 3.2.1*** *Consider the homogeneous fractional ordinary differential equation*

$$D^{\frac{2}{3}}y(t) = ay(t), \qquad 0 < m = \frac{2}{3} \leq 1 \tag{3.6}$$

*We solve this equation using the Elzaki transform .*

***Solution:***

*Apply the Elzaki transform to both sides:*

$$\mathcal{E}\left[D^{\frac{2}{3}}y(t)\right] = \mathcal{E}[ay(t)]$$

*Using the Elzaki transform property for a fractional derivative of order* $0 < m \leq 1$*:*

$$\mathcal{E}\left[D^{m}y(t)\right] = \frac{T(v)}{v^{m}} - vD^{m-1}y(0) \tag{3.7}$$

*where:*

$$T(v) = \mathcal{E}[y(t)]$$

*Substituting* $m = \frac{2}{3}$ *in* (3.7)*, we obtain:*

$$\mathcal{E}\left[D^{\frac{2}{3}}y(t)\right] = \frac{T(v)}{v^{\frac{2}{3}}} - vD^{-\frac{1}{3}}y(0)$$

*Also:*

$$\mathcal{E}[ay(t)] = aT(v)$$

*Hence:*

$$\frac{T(v)}{v^{\frac{2}{3}}} - vD^{-\frac{1}{3}}y(0) = aT(v)$$

*Or:*

$$\frac{T(v)}{v^{\frac{2}{3}}} - aT(v) = vD^{-\frac{1}{3}}y(0)$$

*Then:*

$$T(v)\left(\frac{1}{v^{\frac{2}{3}}}-a\right)=vD^{-\frac{1}{3}}y(0)$$

*Now simplify the term inside parentheses:*

$$\frac{1}{v^{\frac{2}{3}}}-a=\frac{1-av^{\frac{2}{3}}}{v^{\frac{2}{3}}}$$

*Thus:*

$$T(v)\left(\frac{1-av^{\frac{2}{3}}}{v^{\frac{2}{3}}}\right)=vD^{-\frac{1}{3}}y(0)$$

*Multiply both sides by $v^{\frac{2}{3}}$:*

$$T(v)(1-av^{\frac{2}{3}})=v^{\frac{5}{3}}D^{-\frac{1}{3}}y(0)$$

*Therefore:*

$$T(v)=\frac{v^{\frac{5}{3}}D^{-\frac{1}{3}}y(0)}{1-av^{\frac{2}{3}}}$$

*Assume that:*

$$D^{-\frac{1}{3}}y(0)=c$$

*Then:*

$$T(v)=c\left[\frac{v^{\frac{5}{3}}}{1-av^{\frac{2}{3}}}\right] \tag{3.8}$$

*Using the inverse Elzaki transform formula:*

$$\mathcal{E}^{-1}\left[\frac{v^{m+1}}{1-av^{\beta}}\right]=t^{m-1}E_{\beta,m}(at^{\beta})$$

*where:*

$$m=\frac{2}{3},\qquad \beta=\frac{2}{3}$$

*we obtain:*

$$y(t)=ct^{\frac{2}{3}-1}E_{\frac{2}{3},\frac{2}{3}}(at^{\frac{2}{3}})$$

*Hence:*

$$y(t) = ct^{-\frac{1}{3}} E_{\frac{2}{3},\frac{2}{3}}(at^{\frac{2}{3}}) \tag{3.9}$$

*In general, the solution of:*

$$D^m y(t) = ay(t), \qquad y_0 = c, \qquad 0 < m \leq 1$$

*is given by:*

$$y(t) = ct^{m-1} E_{m,m}(at^m)$$

***Example 3.2.2*** *Consider now:*

$$D^{\frac{4}{3}} y(t) = 0, \qquad 1 < m = \frac{4}{3} \leq 2$$

*Applying the Elzaki transform:*

$$\mathcal{E}\left[D^{\frac{4}{3}} y(t)\right] = 0$$

*Using the Elzaki transform property for $1 < m \leq 2$:*

$$\mathcal{E}[D^m y(t)] = \frac{T(v)}{v^m} - vD^{m-1}y(0) - v^2 D^{m-2} y(0)$$

*Substituting $m = \frac{4}{3}$:*

$$\frac{T(v)}{v^{\frac{4}{3}}} - vD^{\frac{1}{3}} y(0) - v^2 D^{-\frac{2}{3}} y(0) = 0$$

*Let*

$$D^{\frac{1}{3}} y(0) = c_1, \qquad D^{-\frac{2}{3}} y(0) = c_2$$

*Then*

$$\frac{T(v)}{v^{\frac{4}{3}}} - vc_1 - v^2 c_2 = 0 \tag{3.10}$$

*Or:*

$$\frac{T(v)}{v^{\frac{4}{3}}} = vc_1 + v^2 c_2$$

*Multiply both sides by $v^{\frac{4}{3}}$:*

$$T(v) = c_1 v^{\frac{7}{3}} + c_2 v^{\frac{10}{3}} \tag{3.11}$$

*Using Inverse Elzaki transform:*

$$\mathcal{E}^{-1}[v^q] = \frac{t^{q-2}}{\Gamma(q-1)}$$

*we compute:*

$$\mathcal{E}^{-1}\left[v^{\frac{7}{3}}\right] = \frac{t^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)}$$

*and*

$$\mathcal{E}^{-1}\left[v^{\frac{10}{3}}\right] = \frac{t^{\frac{4}{3}}}{\Gamma\left(\frac{7}{3}\right)}$$

*Therefore, the final solution is:*

$$y(t) = c_1 \frac{t^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)} + c_2 \frac{t^{\frac{4}{3}}}{\Gamma\left(\frac{7}{3}\right)} \tag{3.12}$$

## 3.3 Solving non-homogeneous fractional differential equation

***Example 3.3.1*** *Consider the non-homogeneous fractional ordinary differential equation*

$$D^{\frac{1}{2}} y(t) + y(t) = t^2 + \frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)} t^{1.5}, \qquad y(0) = 0$$

*We solve this equation using the Elzaki transform.*

*Apply the Elzaki transform to both sides:*

$$\mathcal{E}\left[D^{\frac{1}{2}} y(t)\right] + \mathcal{E}[y(t)] = \mathcal{E}[t^2] + \frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)} \mathcal{E}[t^{1.5}]$$

*Using the Elzaki transform property for a fractional derivative of order $0 < m \leq 1$:*

$$\mathcal{E}\left[D^m y(t)\right] = \frac{T(v)}{v^m} - v D^{m-1} y(0)$$

*where:*

$$T(v) = \mathcal{E}[y(t)]$$

*Substituting $m = \frac{1}{2}$, we obtain:*

$$\mathcal{E}\left[D^{\frac{1}{2}}y(t)\right] = \frac{T(v)}{v^{\frac{1}{2}}} - vD^{-\frac{1}{2}}y(0)$$

*Hence:*

$$\frac{T(v)}{v^{\frac{1}{2}}} - vD^{-\frac{1}{2}}y(0) + T(v) = \mathcal{E}[t^2] + \frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)}\mathcal{E}[t^{1.5}]$$

*Now use the Elzaki transform formula:*

$$\mathcal{E}[t^n] = \Gamma(n+1)v^{n+2}$$

*For the first term:*

$$\mathcal{E}[t^2] = \Gamma(3)v^4$$

*Since:*

$$\Gamma(3) = 2! = 2$$

*then:*

$$\mathcal{E}[t^2] = 2v^4$$

*For the second term:*

$$\mathcal{E}[t^{1.5}] = \Gamma(2.5)v^{3.5}$$

*Thus:*

$$\frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)}\mathcal{E}[t^{1.5}] = \frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)}\Gamma\left(\frac{5}{2}\right)v^{3.5}$$

*we simplify the expression to obtain:*

$$\frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)}\mathcal{E}[t^{1.5}] = 2v^{3.5}$$

*Therefore:*

$$\frac{T(v)}{v^{\frac{1}{2}}} - vD^{-\frac{1}{2}}y(0) + T(v) = 2v^4 + 2v^{3.5}$$

*Using the initial condition:*

$$y(0) = 0$$

*we get:*

$$D^{-\frac{1}{2}}y(0) = 0$$

*Hence:*

$$\frac{T(v)}{v^{\frac{1}{2}}} + T(v) = 2v^4 + 2v^{3.5}$$

*Then:*

$$T(v)\left(\frac{1}{v^{\frac{1}{2}}} + 1\right) = 2v^4 + 2v^{3.5}$$

*Thus:*

$$T(v)\left(\frac{1 + v^{\frac{1}{2}}}{v^{\frac{1}{2}}}\right) = 2v^4 + 2v^{3.5}$$

*Multiply both sides by* $v^{\frac{1}{2}}$*:*

$$T(v)(1 + v^{\frac{1}{2}}) = 2v^{4.5} + 2v^4$$

*Then:*

$$T(v) = \frac{2v^4(v^{\frac{1}{2}} + 1)}{1 + v^{\frac{1}{2}}}$$

*Hence:*

$$T(v) = 2v^4 \tag{3.13}$$

*Now take the inverse Elzaki transform.*

*Using:*

$$\mathcal{E}[t^n] = \Gamma(n+1)v^{n+2}$$

*we have:*

$$2v^4 = \Gamma(3)v^4$$

*Therefore:*

$$\mathcal{E}^{-1}[2v^4] = t^2$$

*Thus, the final solution is:*

$$y(t) = t^2$$

# Conclusion

*This thesis demonstrates the Elzaki transform as an effective and robust integral transform for addressing ordinary and fractional differential equations. The core characteristics of the transform, such as its effect on derivatives, fractional integrals, and the convolution operation, were established and explored. Focused consideration was applied to the handling of fractional derivatives in the Caputo and Riemann-Liouville contexts, utilizing the Mittag - Leffler function to present solutions in closed form. By using various examples, we showed that the Elzaki transform simplifies differential equations-whether linear with constant or varying coefficients, homogeneous or non-homogeneous, ordinary or fractional-into more manageable algebraic equations in the transform domain. This feature greatly eases the solution process and often produces precise, explicit solutions. The technique demonstrated significant benefits for fractional differential equations, where traditional methods often face analytical challenges. In comparison to other integral transforms like Laplace, Sumudu, or natural transforms, the Elzaki transform provides similar or even superior simplicity and computational effectiveness, particularly when addressing initial conditions and fractional orders. The findings demonstrate that the Elzaki transform is a significant enhancement to the resources of applied mathematics, showing promising potential for physics, engineering, and other scientific fields related to fractional dynamics. Ultimately, this study establishes a strong basis for future investigations, encompassing the expansion of the Elzaki transform to partial differential equations, collections of fractional equations, and practical modeling challenges. Future research could additionally investigate numerical applications and hybrid approaches that integrate the Elzaki transform with other analytical methods.*

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