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Stochastic Finite Element Technique For Solving A Class Of SDE

Presented By:
Mouna Sayah
Zohra Kertiou

Youmbai Ahmed El Amine	MCB	President	Univ. El Oued
Touati Brahim Mohammed Said	MCB	Rapporteur	Univ. El Oued
Dehda Bachir	MCA	Exemmer	Univ. El Oued

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Dedicate

To whom God has entrusted with prestige and dignity

To the one who taught me to give without waiting

To whom I proudly carry my name

My dear father

To my angel in life, to the meaning of love and tenderness,

to the smile of life and the secret of existence,

to whom her prayer was the secret of my success,

My beloved mother

To those who are greater than them, rely on those who are with them,

gain strength and love without limits,

My brothers and sisters

To those who are fraternal and distinguished by loyalty and giving

To those with whom I have walked the sweet and sad paths of life To those who

have been with me on the path of success and goodness

My friends.

Thanks

We thank ≪ **Allah** ≫ who gave us the will for the realization of this humble memory.

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As well as all our professors who taught us during our studies at the Faculty of exact Sciences.

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Notations

$(\Omega, \mathcal{F}, P)$	probability space.
P	probability
$\mathbf{R}^n$	n-dimensional Euclidean space.
$corr(X, Y)$	correlation coefficient.
$\mathbb{E}(X) = \mu_X$	the expectation of the random variable$X$.
$\mathbb{E}[X/Y]$	the conditional expectation of a random variables X if Y.
$\{B_s(.), s \leqslant t\}$	the Brownian motion.
$\mathcal{F}_\infty$	the σ-algebra generated by $\bigcap_{t>0} \mathcal{F}_t$.
$\mathcal{F}_t^{(m)}$	the σ-algebra generated by $\{B_s(.), s \leqslant t\}$, $B_s(.)$ is m-dimensional.
$\mathcal{W}_H$	denotes the class of processes $f(t, w) : [0, \infty) \times \Omega \to \mathbb{R}$.
$\mathbf{K}^{-1}$	Neumann series.
eq	equation.
SDE	Stochastic Differential Equation.
FEM	Finite Element Method.
$\mathcal{L}_T^w$	the α-algebra of Lebesgue subsets of $\mathbb{R}^w$.
W_t	the Wiener process.
E	Polish space.
ODE	Ordinary Differential Equation.
a.a, a.e., a.s.	almost all, almost everywhere, almost surely.
w.r.t.	with respect to.
i.e.	id est.
$\mathbf{B}$	matrix that relates the components of strains to the nodal displacements.
$\mathbf{D}$	the matrix describing the constitutive properties of the material.
MPa	Mega Pascal.
$\square$	end of proof.

Introduction general

Before with the impossibility of deterministically describing certain natural phenomena, many authors have the idea of using statistical techniques. The domains of application are of big diversity: fabiality, mechanics,... etc.
To study random phonemes, we need statistical techniques, moreover; when there is no analytical solution, we are using the finite element method, which gives an approximate solution.
We will briefly present the stochastic finite element methods,the stochastic is the calculation of probability applied to the processing of statistical data.
Stochastic finite element methods make it possible to take into account random parameters. They can be divided into three categories:

- The methods of fabialites: (Monte Carlo) makes it possible to obtain the probability of the most probable critical event.

- The methods of sensitivity: (perturbation) allow measurements of the influence of the variability of the input and output parameters.

- The methods quadrature.

The Finite Element Method (FEM) is a widely accepted numerical method for solving problems in science and engineering. The adaptive virtue of this method offers a simple way to solve complex problems in structural analysis, heat transfer, fluid mechanics and electromagnetic fields among other applications. The advantages of the FEM are well known: it can be applied to complex geometries with mixed material and boundary conditions. It is also suitable for time dependent problems and non-linear material behaviour. However, the FEM is deterministic by nature and is therefore limited to describe the general characteristics of a system. In particular, it cannot directly study a system reliably where there exists some degree of uncertainty. The classic FEM has been combined with other methodologies to create a new type of analysis to study systems with random variations and/or uncertainty in parameters. It has been given the names the Stochastic Finite Element Method (SFEM), the Random Finite Element Method (RFEM) and the Probabilistic Finite Element Method (PFEM). To represent the stochastic nature of the system, random fields are introduced to the classic FEM to capture and create different stochastic scenarios. The influence of the random fluctuations is evaluated by calculating the statistical information of the response variables and evaluating the probability of an outcome of the system, such as failure. The SFEM has grown in importance over time. Many articles have been written that cover the mathematical background and extensions of this technique. The concept of the Finite Element Method (FEM) was coined by Clough in the early 1960s in his infamous book entitled “The finite element method in plane stress analysis”. The application of finite elements has been presented for the analysis of aircraft structures and it is referred to as one of the key contributions in the development of the FEM. It is rightly addressed in Hutton (2004) that FEM is a computational technique used to obtain approximate solutions of boundary value problems in engineering.

The Stochastic Finite Element Method (SFEM) is an extension of the FEM that considers the uncertainty of a system that arises through variations in initial conditions, materials or geometry. Systems which display a measurable degree of disorder can be studied efficiently using a probabilistic approach.
Different scenarios can be randomly generated with the SFEM to study the behavior of systems that take into account prior knowledge of the differing variations in properties. This review paper introduces the most commonly used techniques: Direct Monte Carlo Simulation and the Perturbation Method.
It then looks at the currently available software for the SFEM and provides examples from the disciplines of Materials Science, Biomechanics and Engineering to illustrate different procedures by which the SFEM is practically used.
The memory is divided into three Chapter:

Chapter I of this work presents the basic tools necessary for the various theoretical developments.

The second chapter is devoted to the theoretical development of the different sensitivity and reliability methods:
Monte-Carlo method, the quadrature method and perturbation method.
We will give for each of them the solution to the analytical problem defined in the chapter. Finally, we will show the links that exist between the methods of sensitivity and reliability methods. We will show on a simple example the methodology followed.

Chapter three is reserved for applications. We have covered several areas and several types of calculations. Applications relate to: geotechnics and fracture mechanics.

Chapter I

Stochastic Differential Equation

In this chapter, we mention some of the basic concepts related to stochastic finite element technique.

I.1 Probability Space And Random Variable [20]

I.1.1 Probability Space

Definition I.1.1.1: A probability space is an ordered triple $(\Omega, \mathcal{F}, P)$, where Ω is any set, $\mathcal{F}$ a σ-algebra of subsets of Ω, and $P : \mathcal{F} \to [0, 1]$ a probability measure on $\mathcal{F}$ such that

1. $P(\Omega) = 1$.
2. For all $A_1, \ldots, A_n, \ldots \in \mathcal{F}$ with $A_i \cap A_j = \emptyset, i \neq j$

$$P\left(\bigcup A_i\right) = \sum_i P\left(A_i\right). \tag{I.1}$$

The set Ω is called the sample space, $\emptyset$ the empty set, the elements of $\mathcal{F}$ are called events, and every element of Ω is called an elementary event.
Definition I.1.1.2: A probability space $(\Omega, \mathcal{F}, P)$ is finite if Ω has finitely many elementary events.

I.1.2 Random Variable

A (real) random variable x is just a real valued function $X(\omega)$, $\omega \in \Omega$, on a probability space
($\Omega, \mathcal{F}, P$), such that it is possible to talk about its distribution, i.e., the probability

$$P(X \leq a) = P(\{\omega; X(\omega) \leq a\}), \tag{I.2}$$

is defined for all real a. This means that the set (event)

$$A_a = X^{-1}((-\infty, a]) = \{\omega; X(\omega) \leq a\}, \tag{I.3}$$

is a member of the family $\mathcal{F}$, for all $a \in \mathbb{R}$.
This is equivalent to the seemingly more general statement that

$$X^{-1}(B) \in \mathcal{F} \quad \text{for all Borel sets } B \in \mathcal{B}, \tag{I.4}$$

and of course it holds that

$$P\left(X^{-1}(B)\right) = \text{Prob}(X \in B). \tag{I.5}$$

The requirement eq (I.4) is the formal definition of a random variable: a random variable is a Borel measurable function.
If x is a random variable on $(\Omega, \mathcal{F}, P)$, then we write P_X for the probability measure on $(\mathbb{R}, \mathcal{B})$ that is defined by

$$P_X(B) = P\left(X^{-1}(B)\right). \tag{I.6}$$

I.1.3 Expectation, variance, covariance, and correlation

I.1.3.1 Expectation

The ***mean, expected***, or ***expectation*** of a variable X is written as $\mathbb{E}(X)$ or μ_X.
The expectation is defined differently for continuous and discrete random variables.
Definition I.1.3.1.1: Let X be a ***continuous*** random variables with probability density function $f_X(x)$. The expected value of X is

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} x f_X(x) \mathrm{d}x. \tag{I.7}$$

Definition I.1.3.1.2: Let X be a ***discrete*** random variables with probability function $f_X(x)$. The expected value of X is

$$\mathbb{E}(X) = \sum_x x f_X(x) = \sum_x x \mathbb{P}(X = x). \tag{I.8}$$

I.1.3.2 Variance

Definition I.1.3.2.1: Let X by any random variables. The ***variance*** of X is

$$\sigma^2 = Var(X) = \mathbb{E}\left((X - \mu_X)^2\right) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \mathbb{E}(X^2) - (\mu_X)^2. \tag{I.9}$$

Let X be a ***discrete*** random variables

$$\sigma^2 = Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum_x (x - \mathbb{E}(X))^2 \mathbb{P}(X = x). \tag{I.10}$$

Let X be a ***continuous*** random variables

$$\sigma^2 = Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \int_{-\infty}^{+\infty} x^2 f_X(x) \mathrm{d}x - (\mathbb{E}(X))^2. \tag{I.11}$$

Where
$\mathbb{E}(X)$ linear then

$$\begin{aligned} \mathbb{E}\left((X - \mu_X)^2\right) &= \mathbb{E}\left(X^2 - 2\mu_X X + \mu_X^2\right) = \mathbb{E}(X^2) + \mathbb{E}(-2\mu_X X) + \mathbb{E}(\mu_X^2) \\ &= \mathbb{E}(X^2) - 2\mu_X \mathbb{E}(X) + \mu_X^2 = \mathbb{E}(X^2) - 2\mu_X^2 + \mu_X^2 = \mathbb{E}(X^2) - \mu_X^2. \end{aligned} \tag{I.12}$$

the **Standard deviation** is defined as follows

$$\sigma = \sqrt{Var(X)}. \tag{I.13}$$

I.1.3.3 Covariance

Covariance is a measure of the association or dependence between tow random variables X and Y. Covariance can be either positive or negative. (Variance is always positive).
Definition I.1.3.3.1: Let X and Y be any random variables.
The ***covariance*** between X and Y is given by

$$cov(X,Y) = \mathbb{E}\left[(X-\mu_X)\,(Y-\mu_Y)\right] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y). \tag{I.14}$$

Where $\mu_X = \mathbb{E}(X)$, $\mu_Y = \mathbb{E}(Y)$.

I.1.3.4 Correlation (non-examinable)

The correlation coefficient of X and Y is a measure of the linear association between X and Y. It is given by the covariance, scaled by the overall variability in X and Y.
As a result, the correlation coefficient is always between -1 and $+1$, so it is easily compared for different quantities.
Definition I.1.3.4.1: The ***correlation*** between X and Y, also called the ***correlation coefficient***, is given by

$$corr(X,Y) = \frac{cov(X,Y)}{\sqrt{Var(X)Var(Y)}}, \quad with \quad |corr(X,Y)| \leq 1. \tag{I.15}$$

The correlation measures linear association X and Y.
The correlation is ± 1 if and only if there is a perfect linear relationship between X and Y, i.e. $corr(X,Y) = 1 \Leftrightarrow Y = aX + b$ for some constants a and b.
The correlation is 0 if X and Y are independent, but a correlation is 0 does not imply that X and Y are independent.

I.1.4 Conditional expectation and Conditional variance [15]

Definition I.1.4.1: Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, and let $X \in \mathcal{L}^1$ be a random variable. We say that the random variable ζ is (a version of) the conditional expectation of$X$ with respect to $\mathcal{G}$ - and denote it by $\mathbb{E}[X \mid \mathcal{G}]$- if

1. $X \in \mathcal{L}^1$.
2. ζ is $\mathcal{G}$-measurable.
3. $\mathbb{E}[\zeta 1_A] = \mathbb{E}[X 1_A]$ for all $A \in \mathcal{G}$.

Definition I.1.4.2: If X and Y are two discrete random variables that have a common probability mass function $P(X,Y)$, then the conditional expectation of a random variables X if Y or (Y if X) is known and is denoted by the symbol $\mathbb{E}(X/Y)$, or $\mathbb{E}(Y/X)$.

$$\mathbb{E}(X/Y) = \frac{\sum_{\forall X} X\mathbb{P}(X,Y)}{\mathbb{P}(Y)}, / \quad \mathbb{E}(X^2/Y) = \frac{\sum_{\forall X} X^2\mathbb{P}(X,Y)}{\mathbb{P}(Y)}. \tag{I.16}$$

But if X and Y are two continuous random variables that have a common probability density function $\mathbf{f}(X,Y)$ then the conditional expectation of a random variables X if Y or (Y if X) is known and is denoted by the symbol $\mathbb{E}(X/Y)$, or $\mathbb{E}(Y/X)$.

$$\mathbb{E}(X/Y) = \frac{\int_{-\infty}^{+\infty} X\mathbf{f}(X,Y)\mathrm{d}Y}{\mathbf{f}(Y)}, \quad /\mathbb{E}(X^2/Y) = \frac{\int_{-\infty}^{+\infty} X^2\mathbf{f}(X,Y)\mathrm{d}Y}{\mathbf{f}(Y)}. \tag{I.17}$$

As for the conditional variance of the random variables X if Y or (Y if X) is known and is denoted by the symbol $Var(X/Y)$ or $Var(Y/X)$.

$$Var(X/Y) = \mathbb{E}(X^2/Y) - (\mathbb{E}(X/Y))^2, \quad Or \quad Var(Y/X) = \mathbb{E}(Y^2/X) - (\mathbb{E}(Y/X))^2. \quad \text{(I.18)}$$

I.2 Stochastic Processes [10]

A stochastic process with index set T and state space E is a collection of random variables $X = (X_t)_{t\in T}$ (defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in E. We will usually be interested in the case that $T = [0, \infty)$ and E is a Polish space. We interpret $X = (X_t)_{t\in[0,\infty)}$ as a quantity the value of which is determined by chance and that develops in time.

A stochastic process is called measurable if the map $(t, \omega) \mapsto X_t(\omega)$ from $[0, \infty) \times \Omega$ into E is measurable.
The functions $t \mapsto X_t(\omega)$ (with $\omega \in \Omega$) are called the sample paths of the process X.

I.2.1 Autocorrelation and Autocovariance Function [14]

The autocorrelation function $R_X(t_i, t_j)$ of an $1D-1V$ random process $X(t)$ is an important function that quantifies the correlation between the values of $X(t)$ at two different instances t_i and t_j, for $i, j = 1, \ldots, n$ and is in certain cases a function of the difference between the two times $\tau = t_i - t_j$. The autocorrelation function of $X(t)$ is defined as

$$\begin{aligned} R_X(t_i, t_j) &= \mathbb{E}\left[X(t_i)X(t_j)\right] \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_i x_j f_{X(t_i),X(t_j)}(x_i, x_j)\mathrm{d}x_i\mathrm{d}x_j, \end{aligned} \quad \text{(I.19)}$$

where $X(t_i)$ and $X(t_j)$ are the values of the stochastic process at time instances t_i and t_j (i.e., two random variables) with joint pdf $f_{X(t_i),X(t_j)}(x_i, x_j)$. The autocorrelation function satisfies the following properties

- **Symmetry**

$$R_X(t_i, t_j) = R_X(t_j, t_i). \quad \text{(I.20)}$$

- **Cauchy–Schwarz inequality**

$$R_X(t_i, t_j)^2 \leqslant R_X(t_i, t_i)R_X(t_j, t_j). \quad \text{(I.21)}$$

- **Nonnegative definite**

$$\sum_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty} R_X(t_j - t_k)g(t_j)g(t_k) \geq 0, \quad \text{(I.22)}$$

$$\begin{aligned} \lim_{\tau\to 0} R_X(\tau) &= \mathbb{E}[X(t)^2], \\ \lim_{\tau\to\infty} R_X(\tau) &= 0, \end{aligned} \quad \text{(I.23)}$$

for all functions $g : \mathbb{Z} \mapsto \mathbb{C}$ for which the above summation converges. In a similar manner, the autocovariance function can be defined relatively to the autocorrelation function as

$$C_X(ti, tj) = R_X(t_i, t_j) - \mu_X(t_i)\mu_X(t_j). \quad \text{(I.24)}$$

For a zero-mean stochastic process ($\mu_X(t_i) = \mu_X(t_j) = 0$), the covariance is equal to the autocorrelation function. The mean and autocorrelation functions of a stochastic process provide a partial characterization of the process, referred to as second-moment characterization.
It is clear that stochastic processes with the same second-moment characteristics can have very different sample properties:

$$C_X(t_i, t_j) = R_X(t_i, t_j). \tag{I.25}$$

The value of $C_X(t,t)$ on the diagonal $t_i = t_j = t$ is equal to the variance of the stochastic process at timet. Thus, the variance of the stochastic process can be defined as

$$Var(X(t)) = C_X(t,t) = \mathbb{E}[(X(t) - \mu_X(t))^2]. \tag{I.26}$$

Consequently, if we normalize the autocorrelation function by subtracting the mean and dividing by the variance we obtain the autocorrelation coefficient

$$\rho_X(t_i, t_j) = \frac{C_X(t_i, t_j)}{\sqrt{C_X(t_i, t_i)}\sqrt{C_X(t_j, t_j)}} \quad with \quad |\rho(t_i, t_j)| \leq 1. \tag{I.27}$$

In case we want to quantify the dependence between two different stochastic processes $X(t), Y(t)$, we use the so-called cross-correlation function $R_{XY}(t_i, t_j)$, which is a measure of the lag of the one relative to the other, as

$$R_{XY}(t_i, t_j) = \mathbb{E}\left[X(t_i)Y(t_j)\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_i x_j f_{X(t_i),Y(t_j)}(x_i, y_j)\mathrm{d}x_i\mathrm{d}y_j. \tag{I.28}$$

The properties of the autocorrelation also stand for the cross-correlation function. In a similar manner to the autocovariance function, the cross-covariance function $C_{XY}(t_i, t_j)$ is defined if we subtract the product mean value from the cross-correlation function

$$C_{XY}(t_i, t_j) = R_{XY}(t_i, t_j) - \mu X(t_i)\mu Y(t_j). \tag{I.29}$$

I.3 Stationary Stochastic Processes [14]

An important category of stochastic processes is the ones whose complete probabilistic structure is invariant to a shift in the parametric origin

$$f_X(x_1, t_1; \ldots, x_k, t_k) = f_X(x_1, t_1 + a; \ldots, x_k, t_k + a). \tag{I.30}$$

These stochastic processes are called stationary (or homogeneous in the case of random fields). A result of stationarity is that the mean and variance (as well as higher moments) do not change when shifted in time

$$\mu_X(t) = \mu_X(t + a) = \mu_X, \quad Var(X(t)) \to Var(x). \tag{I.31}$$

An important consequence of this time (space) invariance is that the autocorrelation function at times t_i, t_j is independent at the time instants t_i, t_j and depends only on the relative distance $\tau = t_j - t_i$. This leads to the following expression and bounds for R_X and C_X

$$R_X(t_i, t_j) = R_X(t_i + a, t_j + a) = R_X(\tau), \quad C_X(t_i, t_j) = C_X(t_i + a, t_j + a) = C_X(\tau). \tag{I.32}$$

$$|R_X(\tau)| \leqslant R_X(0) \;=\; \mathbb{E}[X(t)^2], \quad \diagup \quad |C_X(\tau)| \leqslant \sigma_X^2 = Variance. \tag{I.33}$$

There are two categories regarding the form of stationarity of a random process

- **Wide-sense stationary (WSS):** The mean value is constant over time and the auto-correlation is a function of the time lag τ.
- **Strictly stationary (SS):** All moments of the stochastic process are constant over time.

I.4 Common Stochastic Processes

I.4.1 Gaussian Processes [12]

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{X} = \{X_1, X_2, X_3, \ldots\}$ denote an index set. A stochastic process is a discrete or real valued function $f(X, \omega)$ which for every fixed $X \in \mathcal{X}$ is a measurable function of $\omega \in \Omega$.
A Gaussian process is a stochastic process where all such finite-dimensional marginal distributions are multivariate normal

$$\begin{aligned} P(\mathbf{f}|\mathbf{X}, \mathbf{\Phi}) &= \mathcal{N}(\mathbf{f}|\mu_{\mathbf{X}}, \mathbf{K}) \\ &= (2\pi)^{-\frac{\mu_{\mathbf{X}}}{2}} |\mathbf{K}|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\mathbf{f} - \mu_{\mathbf{X}})^{\mathsf{T}} \mathbf{K}^{-1} (\mathbf{f} - \mu_{\mathbf{X}})\right). \end{aligned} \tag{I.34}$$

with mean $\mu_{\mathbf{X}}$. and covariance matrix $\mathbf{K}$.

I.4.2 Markov Processes [5], [18]

A Markov Chain is defined as a series of variables with discrete values of $\{X_1, X_2, X_3, \ldots\}$ which represents the conditional distribution of the value X_{n+1} only over the value X_n.

We consider a discrete-time stochastic process$\{X_n, n = 0, 1, 2, \ldots\}$ whose state space $\mathcal{X}$ (the set of all possible states)is countable. We usually go number the states by non-negative integers, and thus assume that $\mathcal{X} = \{0, 1, \ldots, r\}$(finite) or $\mathcal{X} = \{0, 1, 2, \ldots\}$(infnite).

I.4.2.1 A homogeneous Markov Chain

if

$$P\left[X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0\right] = P\left[X_{n+1} = j | X_n = i\right] = P_{i,j}. \tag{I.35}$$

In other words, the probability distribution of the next state X_{n+1} conditional on past history only depends on the current state X_n, and does not depend on n.
The $P_{i,j}$ are the transition probabilities of the chain the adjective homogeneous means that these probabilities do not depend on n.
The $P_{i,j}$ must necessarily satisfy

$$P_{i,j} \geq 0 \; for \; all \; i, j : \sum_{j=1}^{\infty} P_{i,j} = 1 \; for \; all \; i. \tag{I.36}$$

They are the elements of the transition matrix

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & \cdots \\ P_{1,0} & P_{1,1} & P_{1,2} & \cdots \\ P_{2,0} & P_{2,1} & P_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}. \tag{I.37}$$

I.4.2.2 Chapman-Kolmogorov Equation

Let

$$P_{i,j}^{(n)} = P\left[X_n = j | X_0 = i\right], \tag{I.38}$$

the probability of going from i to j in exactly n steps Chapman-Kolmogorov equation

$$P_{i,j}^{(n+m)} = \sum_{k=1}^{\infty} P\left[X_{n+m} = j | X_n = k\right] P\left[X_n = k | X_0 = i\right] = \sum_{k=1}^{\infty} P_{i,k}^{(n)} P_{k,j}^{(m)}. \tag{I.39}$$

In matrix notation, if $P^{(n)}$ is the matrix the containing the $P_{i,j}^{(n)}$, this gives

$$P^{(n+m)} = P^{(n)} \cdot P^{(m)}. \tag{I.40}$$

I.4.2.3 Constraints On The Trajectories [18]

If $i, j \notin \mathcal{A}$ the probability that $X_m = j$ and that the chain has not visited set $\mathcal{A}$ until step m is written

$$\alpha = P\left[X_m = j, N > m - 1 \mid X_0 = i\right]. \tag{I.41}$$

To calculate α, just construct Q and the string $\{W_n, n > 0\}$ as above.
The element $Q_{i,j}^{(m)}$ gives the probability α sought, for the case where $j \in \mathcal{A}$.

I.4.2.4 Classification of states [18]

We say that a state j is **accessible** from state i if there exists an $n \geqslant 0$ such that $\mathbf{P}_{ji}^{(n)} > 0$. This means that if we are in state i, the probabilities of reaching j possibly 'is not zero.
Note that i is always reachable from i.

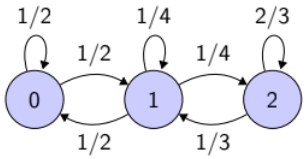
Two states i and j **communicate** if each is reachable from the other. We denote this $i \leftrightarrow j$.

Communication is an **equivalence relation:** it is reflexive and symmetric(derives directly from the definition), and also transitive(derives from the Chaman-Kolmogorov equation).

Equivalence classes form a **partition** of the state space $\mathcal{X}$. The Chain is irreducible if all the states communicate(a single equivalence class).

A discrete state space Markov chain can be represented by a directed graph. The states are the **vertices**, the transitions of positive probability are the **edges**, and one can go from i to j in n steps if there is a path from i to j of length n.

Example I.4.2.4.1



$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/3 & 2/3 \end{pmatrix}. \tag{I.42}$$

I.4.2.5 Equilibrium (Or Steady State) Probabilities [18]

Theorem I.4.2.5.1

For a positive recurrent irreducible chain, the π_j form the unique solution of the following equilibrium equations

$$\pi_j = \sum_{i \in \mathcal{X}} \pi_i P_{i,j} \; for \; all \; j \in \mathcal{X}, \sum_{i \in \mathcal{X}} \pi_i = 1. \tag{I.43}$$

In matrix form: $\pi \mathbf{X} = \pi$ and $\pi \mathbf{1}^t = 1$, Where $\pi = (\pi_0, \pi_1, \ldots)$. If this system has no solution, the chain is zero recurrent.

Proof rationale

The proportion of transitions where one starts from i to go to j is equal to the probability of being at i, then of transiting to j, which is $\pi_i P_{ij}$. By summing over all i, we obtain the proportion of transitions that arrive at j, which is π_j. The sum must equal π_j, which gives the first equation. The second says that the probabilities sum to 1.

This vector π gives the **equilibrium law**, or **stationary law**. If X_0 follows the X_1 and all the X_n also follows the π law (we will come back to this).

I.4.2.6 Limiting Probabilities, Frequencies, And Steady-state Probabilities [18]

1): **Limit Probabilities**

$$\pi_j = \lim_{n \to \infty} P_{i,j}^{(n)}. \tag{I.44}$$

2): **Long-term visit frequency**

$$\pi_j = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P_{i,j}^{(k)}. \ \textit{This is Cesaro's limit.} \tag{I.45}$$

3): **Steady State (or equilibrium) Probabilities:** It is a vector π of probabilities that satisfies

$$\pi = \pi P \quad \pi 1 = 1. \tag{I.46}$$

The first two may depend on the initial state if the string has multiple classes. It may be that the long-term frequencies exist and not the limiting probabilities. But not the other way around.
When limit probabilities exist, and to steady-state probabilities in the irreducible case.

I.4.3 Martingale [9]

Let $(\mathcal{F})_n$ be a filtration. A sequence $(M_n)_{n \geqslant 0}$ of integrable random variables is a martingale (resp. Super-martingale, resp. Sub-martingale) with respect to the filtration $(\mathcal{F}_n)_{n \geqslant 0}$ if,
for all $n \geqslant 0$,

- M_n, is $\mathcal{F}_n$-measurable (one says that $(M_n)_{n \geqslant 0}$, is suitable for filtration).
- $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n$ (resp. $\leqslant M_n$, resp. $\geqslant M_n$).

Most of the time, the filtration is in fact defined from sequence $(M_n)_{n \geqslant 0}$ by

$$\mathcal{F}_n = \sigma(M_0, \ldots, M_n), \tag{I.47}$$

(tribe of the past before instant n). It is said to be filtration associated with the $(M_n)_{n \geqslant 0}$ process, or simply natural filtration. If we do not specify the filtration when we speak of a martingale, it is because we are implicitly considering this one. The first point of the definition is then automatic. And the second point of the definition can be written

$$\mathbb{E}[M_{n+1} | M_0, \ldots, M_n] = M_n \quad (\textit{resp.} \leqslant M_n, \ \textit{resp.} \geqslant M_n). \tag{I.48}$$

A martingale is therefore a sequence of random variables which at each step is centered in relation to the previous value, even if we know its past so far (we speak of $\ll$ *past* $\gg$ when seeing n as a $\ll$ *time* $\gg$). A super-martingale has a tendency to decrease with each step, while

a sub-martingale has a tendency to increase with each step.

The (sub, sur)martingales play a tool role in probabilities that can be compared to that of increasing and decreasing sequences in real analysis: We will see that an increased sub-martingale converges and that a decreased super-martingale converges (and therefore that a minus or plus martingale converges !), and that one can obtain various inequalities on these, so that it is often convenient to introduce a (sub, over) martingale linked to the process that the we study.

Proposition I.4.3.1
If $(M_n)_{n\geqslant 0}$ is a martingale, then $< M, \phi_n >$ is a martingale.
If $(M_n)_{n\geqslant 0}$ sub-martingale (resp. Super-martingale) and $\phi_n \geq 0$ on for all n, then $< M, \phi_n >$ is a sub-martingale (resp. Super-martingale).

Proof
In other words, a strategy, as smart as it is, will not make it possible to obtain anything other that a super-martingale if the game is unfavorable.

One of the simplest strategies consists in stopping betting from a certain time T. This time must be decided according to the values already observed: the fact that $T = n$ or not must depend only on $M_0, \ldots, M_n$. So it must be a downtime.

Theorem I.4.3.1: (Maximum Doob inequality)
Let $\{X_n\}$ be a non-negative submartingale (for example, $X_n = |M_n|$ if $\{M_n\}$ is a martingale, or $X_n = S_n^+$ if $\{S_n\}$ is a sub-martingale), and define $X_n^* := \max_{0\leq k\leq n} X_k$. Then

$$P\left[X_n^* \geq t\right] \leq t^{-1}\mathbb{E}\left[X_{ni} X_n^* \geq b\right] \leq t^{-1}\mathbb{E}\left[X_n\right], \quad \forall t > 0. \tag{I.49}$$

For the proof of this maximal inequality we require the following simple lemma, a hint of better things.

Lemma I.4.3.1: If $\{Y_n\}$ is a submartingale and T is a stopping time bounded above by a positive integer N, then

$$Y_T \leq \mathbb{E}\left[Y_N \mid \mathcal{F}_T\right]. \tag{I.50}$$

Proof If $A \in \mathcal{F}_T$, then

$$\begin{aligned}
\mathbb{E}\left[Y_N,\ A\right] &= \sum_{n=0}^{N} \mathbb{E}\left[Y_N,\ A \cap \{T = n\}\right] \geq \sum_{n=0}^{N} E\left[Y_n,\ A \cap \{T = n\}\right] \\
&= \sum_{n=0}^{N} \mathbb{E}\left[Y_T,\ A \cap \{T = n\}\right] = \mathbb{E}\left[Y_T,\ A\right],
\end{aligned}$$

where the inequality follows from the sub-martingale property of Y because $A \cap \{T = n\} \in \mathcal{F}_n$.□
Proof of the maximal inequality: Fix a positive integer n and define
$T := \min\{k \geq 0\colon\ X_k \geq b\} \wedge n$. Then T is a stopping time bounded above by n and

$$\{X_n^* \geq b\} = \{X_T \geq b\}.$$

Thus

$$\begin{aligned}
P\left[X_n^* \geq t\right] &= P\left[X_T \geq t\right] \leq \mathbb{E}\left[X_T/t,\ X_T \geq t\right], \\
&\leq t^{-1}\mathbb{E}\left[X_n,\ X_T \geq t\right] = t^{-1}\mathbb{E}\left[X_n,\ X_n^* \geq t\right], \\
&\leq t^{-1}\mathbb{E}\left[X_n\right].
\end{aligned}$$

Inequality following from the Lemma. Doob's L^p maximal inequality is a corollary of the sub-martingale maximal inequality, which is a simple consequence of Tonelli's theorem. □

Proposition I.4.3.2

Let $\vartheta : \mathbb{R} \to \mathbb{R}_+$ a convex function, and $(X_n)_n$ a sequence of random variables such that $E[\vartheta(X_n)] < \infty$ for all $n \in \mathbb{N}$.

a) if $(X_n)_{n\geqslant 0}$ is a martingale, then $(\vartheta(X_n))_{n\geqslant 0}$ is a sub-martingale.

b) if $(X_n)_{n\geqslant 0}$ is a sub-martingale and if ϑ is a increasing, then $(\vartheta(X_n))_{n\geqslant 0}$ is a sub-martingale.

For example, if $(M_n)_n$, is a martingale, then $(|M_n|)_n$ is a sub-martingale, likewise for $(M_n)_n^2$.

I.3.4.1 Convergence Theorems [9]

The fundamental theorem is the following

Theorem I.3.4.1.1

Let $(M_n)_{n\geqslant 0}$ be a positive sub-martingale. There exists a random variable M_∞, such that M_n converges to M_∞ palmost surely.

We could extend the theorem (by translation) by saying that a super-martingale bounded by a constant converges almost surely. The same is true, by symmetry, of a majoroe sub-martingale. And so since a martingale and a super-martingale, it converges as soon as it is increased or decreased .

We have a second convergence theorem.

Theorem I.3.4.1.2

Let $(M_n)_{n\geqslant 0}$ be a martingale, and $p > 1$.

If $(M_n)_{n\geqslant 0}$ is bounded in $\mathbb{L}^p$ (i.e.there exists a constant C such that $\mathbb{E}[|M_n|^p] < C$ for $n \geqslant 0$), then there exists a random variable $X_\infty \in \mathbb{L}^p$ such that X_n, converges p.s. and in $\mathbb{L}^p$ towards X_∞.

If $p = 1$, one can show that there is convergence p.s. towards $X_\infty \in \mathbb{L}^1$ but not necessarily convergence in $\mathbb{L}^1$.

I.3.4.2 Link with Markov chains

Some recurrence or transience criteria (Foster criteria) use martingales.

For example, if $(X_n)_n$, is an irreducible Markov chains of infinite state space E and if there exists a function $f : E \to \mathbb{R}_+$, such that f diverges at infinity (i.e. that is, for all $A > 0$, there exists finite $F \subset E$ such that $f(x) \geqslant A$ for all $x \in E \setminus F$) and $Pf(x) \leqslant f(x)$ for all a except a finite number (let us denote $G = \{x | Pf(x) > f(x)\}$,finite), where

$$Pf(x) = \sum_{y\in E} P(x, y)f(y) = \mathbb{E}_x[f(X_1)], \tag{I.51}$$

then $(X_n)_n$ is recurring.

Indeed, the condition $Pf(x) \leqslant f(x)$ is also written $\mathbb{E}_x[f(X_1)] \leqslant f(x)$, from which we deduce $\mathbb{E}[f(X_{n+1}|X_n)] \leqslant f(X_n)$ if $X_n \notin G$, and therefore

$$\mathbb{E}[f(X_{n+1}|X_0, \ldots, X_n)] \leqslant f(X_n) \; if \; X_n \notin G, \tag{I.52}$$

which implies that $f((X_{n\wedge\tau}))_{n\geq 0}$ where $\tau = inf\,\{n \geq 0 | X_n \in G\}$, is a positive super-martingale under $\mathbb{P}_x$ for any initial state $x \in E$. So it converges $\mathbb{P}_x$-p.s. Since f diverges at infinity, $(X_{n\wedge\tau})_n$ therefore remains almost surely in a finite set. As $(X_n)_n$ cannot remain in a finite set indefinitely (because it is irreducible and E is infinite), we deduce that $\tau < \infty$ $\mathbb{P}_x$-p.s. Thus, G is reached p.s. from any starting point x. We easily deduce the recurrence of the Markov chain $((X_n)_n$

visits G infinitely often by what precedes and by the strong Markov property, therefore a vertex of G is visited infinitely often, hence the recurrence).

I.4.4 Brownian Motion [13]

Brownian motion is the random moving of particles suspended in a fluid (a liquid or a gas) resulting from their bombardment by the fast-moving atoms or molecules in the gas or liquid. In 1827, the botanist Robert Brown, looking through a microscope at particles found in pollen grains in water, noted that the particles moved through the water but was not able to determine the mechanisms that caused this motion. Atoms and molecules had long been theorized as the constituents of matter, and many decades later, albert Einstein published a paper in 1905 that explained in precise detail how the motion that Brown had observed was a result of the pollen being moved by individual water molecules. This explanation of Brownian motion served as definitive confirmation that atoms and molecules actually exist, and was further verified experimentally by Jean Perrin in 1908.

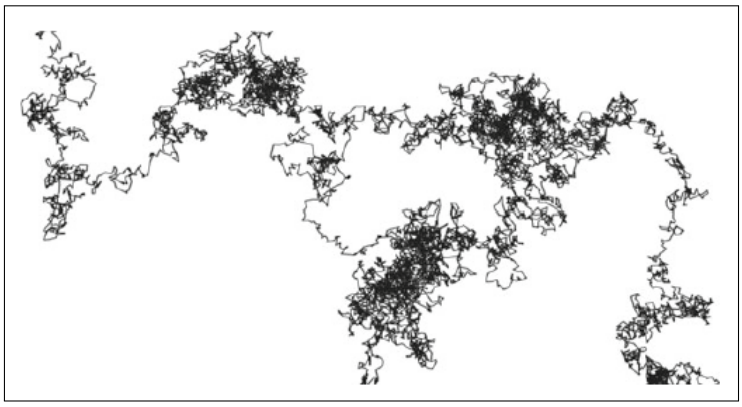


Figure I.1: A typical walk in a Brownian motion.

Definition I.4.4.1 A d-Brownian motion (starting from 0) is a family of $\mathbb{R}^d$-valued random variables $(B_t : t \geqslant 0)$ living on a probability space $(\Omega; \mathcal{F}; P)$ such that

- $B_0 = 0$ almost surely,
- for every $0 = t_0 < t_1 < t_2 < \ldots < t_p$ the variables $B_{t_i} - B_{t_{i-1}}, B_{t_{i-1}} - B_{t_{i-2}}, \ldots, B_{t_2} - B_{t_1}$ for $i \in \{1, \ldots, p\}$ are independent and

$$B_{t_i} - B_{t_{i-1}} = \mathcal{N}\left(0, Id(t_i - t_{i-1})\right). \tag{I.53}$$

- the function $t \to B_t$ is almost surely continuous.

Brownian motion is closely linked to the normal distribution. Recall that a random variable $X$is normally distributed with mean μ and variance σ^2 if

$$\mathbb{P}\{X > x\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_x^\infty \exp -\frac{(s-\mu)^2}{2\sigma^2} \mathrm{d}s, \quad \textit{for all } x \in \mathbb{R}. \tag{I.54}$$

I.4.5 Stationary White Noise [14]

The stationary white noise is a stochastic process that is characterized by complete randomness. In simple terms, the realizations of the stochastic process at $X(t)$ and $X(t+\tau)$ are totally

uncorrelated for any $\tau \neq 0$. The autocorrelation function describing this process is given by

$$R_X(\tau) = 2\pi S_0 \delta_T(\tau), \tag{I.55}$$

where δ_T is the Dirac function defined as

$$\delta_T(\tau) = \begin{cases} 0 & \text{for } \tau \neq 0, \\ \infty & \text{for } \tau = 0, \end{cases} \qquad \int_{-\infty}^{\infty} \delta_T(\tau) \mathrm{d}\tau = 1 \tag{I.56}$$

and S_0 is the corresponding power spectrum which is constant over the frequency domain.

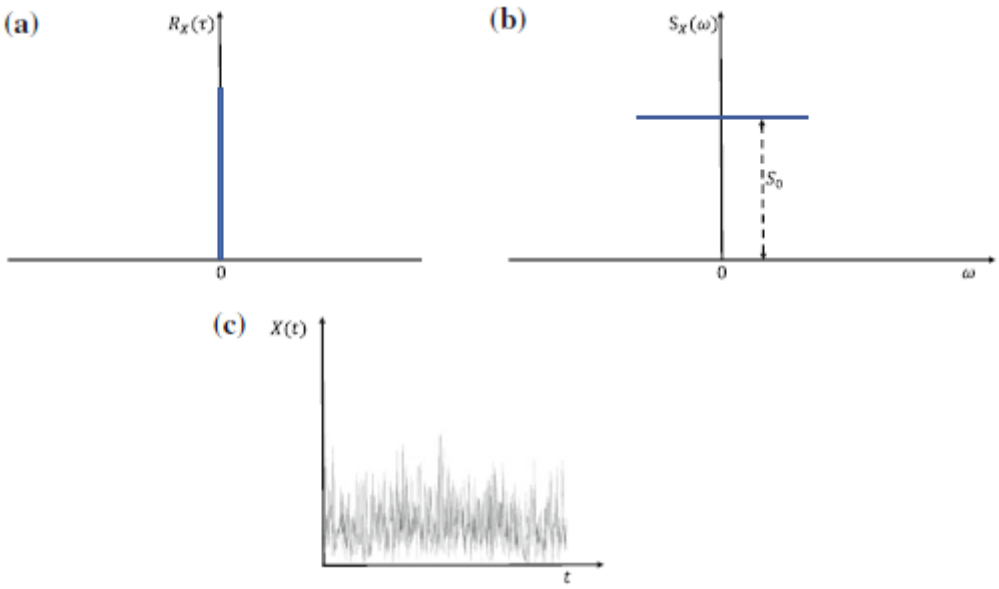


Figure I.2: a The autocorrelation function, b the corresponding power spectrum, and c sample realization of a typical stationary white noise.

A graphical representation of the autocorrelation function, its corresponding power spectrum, and a sample realization of some typical white noise is shown in Fig.I.2. From the sample, it is obvious that white noise is characterized by complete randomness from point to point.
This idealized white noise is not physically realistic since it corresponds to a process with $\mathbb{E}\left[X(t)^2\right] = R_X(0) = \infty$. Instead, if the frequency domain is banded (i.e., $|\omega_1| \leq \omega \leq |\omega_2|$) then we have a band-limited stationary white noise with the following autocorrelation function and power spectrum (Fig. I.3)

$$R_X(\tau) = 2S_0 \frac{\sin(\omega_2 \tau) - \sin(\omega_1 \tau)}{\tau}, \tag{I.57}$$

$$S(\omega) = \begin{cases} S_0 & \text{for } |\omega_1| \leq \omega \leq |\omega_2|, \\ 0 & \text{otherwise .} \end{cases} \tag{I.58}$$

I.4.6 Ergodic Stochastic Processes [14]

In order to apply the theory of random processes, robust estimates of the mean and auto-correlation function are required, based on measurements. This estimation is done by using a sufficient large number of sample realizations of the random process. From the total of sample

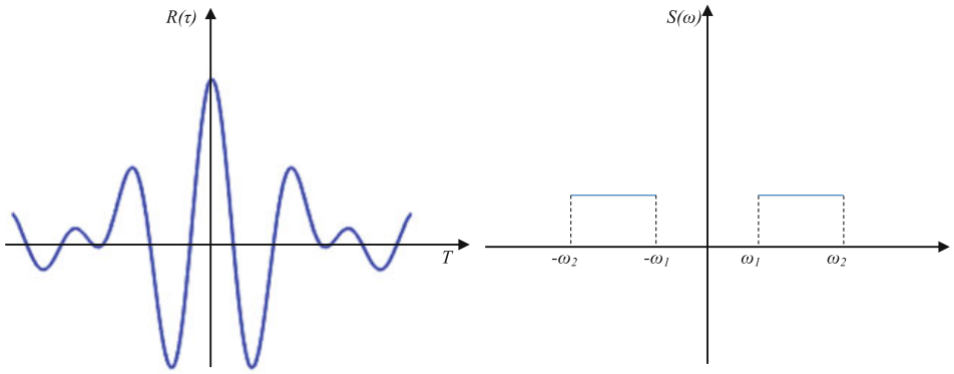


Figure I.3: Autocorrelation function and corresponding power spectrum of a stationary band-limited white noise process

realizations, the ensemble average at each time is defined as the average of the entire population of sample realizations. For n realizations of a sample functions $X(t)$, the ensemble average at time t is defined as

$$\mathbb{E}[X(t)] = \frac{X_1(t) + X_2(t) + \cdots + X_n(t)}{n}. \tag{I.59}$$

In practice, one can have only a very small number of sample functions of the random process and/or a long, single observation of one sample function. So, the question is if statistical averages of the stochastic process can be determined from a single sample function. The answer to this is yes and it comes from the definition of ergodicity which stands only for stationary random processes.
A stationary random process $X(t)$ is called ergodic in the:

(a) mean, if $\mathbb{E}[X(t))]$ equals the time average of sample function

$$\mu_X = \mathbb{E}[X(t)] = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t)\mathrm{d}t, \tag{I.60}$$

where T is the length of the sample function. Necessary and sufficient conditions for $X(t)$ to be ergodic in the mean are

1. $\mathbb{E}[X(t)] = \text{const}$ and.
2. $X(t)$ and $X(t+\tau)$ must become independent as τ reaches ∞.

(b) autocorrelation, if

$$R_X(\tau) = \mathbb{E}[X(t)X(t+\tau)] = \lim_{T\to\infty} \frac{1}{2T} \int_0^T X(t+\tau)X(t)\mathrm{d}t. \tag{I.61}$$

Again, necessary and sufficient conditions for $X(t)$ to be ergodic in the autocorrelation are

1. $\mathbb{E}[X(t+\tau)X(t)]$ is a function of τ only.
2. $\lim_{\tau\to\infty} R_X(\tau) = 0$.

Note that an ergodic field is always stationary, but the reverse does not necessarily hold.
The concept of ergodicity is of great practical importance, since it allows the estimation of the statistics of a random process from a single time or space record of sufficient length. In most

cases though we are only interested a stochastic process to be ergodic in the mean value and in the autocorrelation function.

I.5 Stochastic Differential Equation [8]

A ordinary differential equation (ODE) is

$$\frac{dx(t)}{dt} = f(t,x),\ or \quad dx(t) = f(t,x)dt, \tag{I.62}$$

with initial conditions $x(0) = x_0$ can be written in integral form

$$x(t) = x_0 + \int_0^t f(s, x(s))ds, \tag{I.63}$$

where $x(t) = x\,(t, x_0, t_0)$ is the solution with initial conditions $x\,(t_0) = x_0$.
An example is given as

$$\frac{dx(t)}{dt} = a(t)x(t), \quad x(0) = x_0. \tag{I.64}$$

When we take the ODE eq (I.64) and assume that $a(t)$ is not a deterministic parameter but rather a stochastic parameter, we get a stochastic differential equation (SDE). The stochastic parameter $a(t)$ is given as

$$a(t) = f(t) + h(t)\xi(t), \tag{I.65}$$

where $\xi(t)$ denotes a white noise process. Thus, we obtain

$$\frac{dX(t)}{dt} = f(t)X(t) + h(t)X(t)\xi(t). \tag{I.66}$$

When we write eq (I.66) in the differential form and use $dW(t) = \xi(t)dt$, where $dW(t)$ denotes differential form of the Brownian motion, we obtain

$$dX(t) = f(t)X(t)dt + h(t)X(t)dW(t). \tag{I.67}$$

Definition I.5.1
In general an SDE is given as

$$dX(t,\omega) = f(t, X(t,\omega))dt + g(t, X(t,\omega))dW(t,\omega), \tag{I.68}$$

where ω denotes that $X = X(t,\omega)$ is a random variable and possesses the initial condition $X(0,\omega) = X_0$ with probability one. As an example we have already encountered

$$dY(t,\omega) = \mu(t)dt + \sigma(t)dW(t,\omega). \tag{I.69}$$

Furthermore, $f(t, X(t,\omega)) \in \mathbb{R}$, $g(t, X(t,\omega)) \in \mathbb{R}$, and $W(t,\omega) \in \mathbb{R}$. Similar as in eq (I.63) we may write eq (I.68) as integral equation

$$X(t,\omega) = X_0 + \int_0^t f(s, X(s,\omega))ds + \int_0^t g(s, X(s,\omega))dW(s,\omega), \tag{I.70}$$

where $f(s, X(s,\omega))$: the deterministic part, and the $g(s, X(s,\omega))$: Stochastic part.
where the meaning of the last integral, called an Itô integral.

I.5.1 Itô Integrals and Itô Formula

I.5.1.1 Itô Integrals [19]

The $\int_0^t g(s, X(s,\omega))dW(s,\omega)$ integration is called the Itô integral. To learn how to find this system, we do the following

Let $c = t_0 < t_1 < \ldots < t_{n-1} < t_n = d$ be a grid of points on the interval $[c, d]$.

The Riemann integral is defined as a **limit**

$$\int_c^d f(x)\mathrm{d}x = \lim_{\Delta t\to 0}\sum_{i=1}^{n} f(t_i')\Delta t_i, \tag{I.71}$$

where $\Delta t_i = t_i - t_{i-1}$ and $t_{i-1} \le t_i' \le t_i$. Similarly, the Ito integral is the limit

$$\int_c^d f(x)\mathrm{d}W_t = \lim_{\Delta t\to 0}\sum_{i=1}^{n} f(t_{i-1})\Delta W_i, \tag{I.72}$$

where $\Delta W_i = W_{t_i} - W_{t_{i-1},}$

Because f and W_t are random variables, so is the Ito integral $I = \int_c^d f(x)\mathrm{d}W_t$.

The differential $I\mathrm{d}$ is a notational convenience; thus,

$$I = \int_c^d f(x)\mathrm{d}W_t, \tag{I.73}$$

is expressed in differential form as

$$I\mathrm{d} = f(x)\mathrm{d}W_t. \tag{I.74}$$

The differential $\mathrm{d}W_t$ of Brownian motion W_t is called white noise. A typical solution is a combination of drift and the diffusion of Brownian motion.

I.5.1.2 Itô Formula [16]

Let $u : [0,T]\times\mathbb{R}\to\mathbb{R}$ have continuous partial derivatives $\dfrac{\partial u}{\partial t}$, $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial^2 u}{\partial^2 x}$. Then for any $t, t+\Delta t \in [0,T]$ and $x, x+\Delta x \in \mathbb{R}$ there exist constants $0 \le \alpha \le 1, 0 \le \beta \le 1$ such that

$$u(t+\Delta t, x+\Delta x) - u(t,x) = \frac{\partial u}{\partial t}(t+\alpha\Delta t, x)\Delta t + \frac{\partial u}{\partial x}(t,x)\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial^2 x}(t, x+\beta\Delta x)(\Delta x)^2. \tag{I.75}$$

Then, by a stochastic differential we mean an expression eq (I.68) which is just a symbolical way of writing

$$X_t - X_s = \int_s^t f(u,w)\mathrm{d}u + \int_s^t g(u,w)\mathrm{d}W_t(w), \quad \forall\, 0 \leqslant s \leqslant t \leqslant T. \tag{I.76}$$

Theorem I.5.1.2.1: Let $Y_t = u(t, X_t)$ for $0 \leqslant t \leqslant T$. Where u is as eq (I.75), and X_t satisfies eq (I.76).

With $\sqrt{|f|}$, $g \in \mathcal{L}_T^w$, then

$$Y_t - Y_s = \int_s^t \left\{\frac{\partial u}{\partial t}(u, X_u) + f_u\frac{\partial u}{\partial x}(u, X_u) + \frac{1}{2}g_u^2\frac{\partial^2 u}{\partial^2 x}(u, X_u)\right\}\mathrm{d}u + \int_s^t g_u\mathrm{d}W_t(w), \tag{I.77}$$

with probability 1, for all $0 \leqslant s \leqslant t \leqslant T$.

I.5.2 An Existence and Uniqueness Result [3]

Theorem I.5.2.1 ***(Existence and uniqueness theorem for stochastic differential equations)***
Let $T > 0$ and $b(.,.)$: $[0,T] \times \mathbf{R}^n \to \mathbf{R}^n$; $\sigma(.,.)$: $[0,T] \times \mathbf{R}^n \to \mathbf{R}^{n\times m}$ be measurable functions satisfying

$$|b(t,x)| + |\sigma(t,x)| \leqslant C(1+|x|) \quad x \in \mathbf{R}^n, t \in [0,T], \tag{I.78}$$

for some constant C, (where $|\sigma|^2 = \sum |\sigma_{ij}|^2$) and such that

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leqslant D|x-y| \quad x,y \in \mathbf{R}^n, t \in [0,T], \tag{I.79}$$

for some constant D. Let Z be a random variable which is independent of the σ-algebra $\mathcal{F}_\infty^{(m)}$ generated by $B_s(.)$, $s \geqslant 0$. and such that

$$\mathbb{E}\left[|Z|^2\right] < \infty. \tag{I.80}$$

Then the stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = b(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}B_t, \quad 0 \leqslant t \leqslant T \\ X_0 = Z \end{cases} \tag{I.81}$$

has a unique t-continuous solution $X_t(w)$ with the property that

$$X_t(w) \text{ is adapted to the filtration } \mathcal{F}_t^Z \text{ generated by } Z \text{ and } B_s(.),\ s \leqslant t. \tag{I.82}$$

and

$$\mathbb{E}\left[\int_0^T |X_t|^2 \mathrm{d}t\right] < \infty. \tag{I.83}$$

Proof

The uniqueness follows from the Itô isometry and the Lipschitz property: eq (I.79)
Let $X_1(t,\omega) = X_t(\omega)$ and $X_2(t,\omega) = \widehat{X}_t(\omega)$ be solutions with initial values $Z, \widehat{Z}$ respectively, i.e. $X_1(0,\omega) = Z(\omega), X_2(0,\omega) = \widehat{Z}(\omega), \omega \in \Omega$. For our purposes here we only need the case $Z = \widehat{Z}$, but the following more general estimate will be useful for us later.
Put $a(s,\omega) = b\left(s, X_s\right) - b\left(s, \widehat{X}_s\right)$ and $\gamma(s,\omega) = \sigma\left(s, X_s\right) - \sigma\left(s, \widehat{X}_s\right)$.
Then

$$\begin{aligned}
\mathbb{E}\left[\left|X_t - \widehat{X}_t\right|^2\right] &= \mathbb{E}\left[\left(Z - \widehat{Z} + \int_0^t a\mathrm{d}s + \int_0^t \gamma \mathrm{d}B_s\right)^2\right], \\
&\leq 3\mathbb{E}\left[|Z-\widehat{Z}|^2\right] + 3\mathbb{E}\left[\left(\int_0^t a\mathrm{d}s\right)^2\right] + 3\mathbb{E}\left[\left(\int_0^t \gamma dB_s\right)^2\right], \\
&\leq 3\mathbb{E}\left[|Z-\widehat{Z}|^2\right] + 3t\mathbb{E}\left[\int_0^t a^2 ds\right] + 3\mathbb{E}\left[\int_0^t \gamma^2 ds\right], \\
&\leq 3\mathbb{E}\left[|Z-\widehat{Z}|^2\right] + 3(1+t)D^2 \int_0^t \mathbb{E}\left[\left|X_s - \widehat{X}_s\right|^2\right] ds.
\end{aligned}$$

So the function

$$v(t) = \mathbb{E}\left[\left|X_t - \widehat{X}_t\right|^2\right], \quad 0 \leq t \leq T,$$

satisfies

$$v(t) \leq F + A \int_0^t v(s)ds,$$

$$where\ F = 3\mathbb{E}\left[|Z - \widehat{Z}|^2\right]\ and\ A = 3(1+T)D^2.$$

By the Gronwall inequality we conclude that

$$v(t) \leq F \exp(At).$$

Now assume thst $Z = \widehat{Z}$. Then $F = 0$ and so $v(t) = 0$ for all $t \geq 0$. Hence

$$P\left[\left|X_t - \widehat{X}_{\mathrm{t}}\right| = 0 \quad \text{for all } t \in \mathrm{Q} \cap [0,T]\right] = 1,$$

where Q denotes the rational numbers. By continuity of $t \to \left|X_t - \widehat{X}_{\mathrm{t}}\right|$ it follows that

$$P\left[|X_1(t,\omega) - X_2(t,\omega)| = 0 \quad \text{for all } t \in [0,T]\right] = 1,$$

and the uniqueness is proved.
The proof of the existence is similar to the familiar existence proof for ordinary differential equations: Define $Y_t^{(0)} = X_0$ and $Y_t^{(k)} = Y_{\mathrm{t}}^{(k)}(\omega)$ inductively as follows

$$Y_i^{(k+1)} = X_0 + \int_0^{\mathrm{t}} b\left(s, Y_a^{(k)}\right) ds + \int_0^{\mathrm{t}} \sigma\left(s, Y_a^{(k)}\right) \mathrm{d}B_s.$$

Then, similar computation ss for the uniqueness above gives

$$\mathbb{E}\left[|Y_t^{(k+1)} - Y_t^{(k)}|^2\right] \leq (1+T)3D^2 \int_0^t \mathbb{E}\left[|Y_s^{(k)} - Y_s^{(k-1)}|^2\right] \mathrm{d}s,$$

for $k \geq 1, t \leq T$ and

$$\mathbb{E}\left[\left|Y_t^{(1)} - Y_t^{(0)}\right|^2\right] \leq 2C^2t^2\left(1 + \mathbb{E}\left[|X_0|^2\right]\right) + 2C^2t\left(1 + \mathbb{E}\left[|X_0|^2\right]\right) \leq A_1 t,$$

where the constant A_1 only depends on C, T and $\mathbb{E}\left[|X_0|^2\right]$. So by induction on k we obtain

$$\mathbb{E}\left[\left|Y_t^{(k+1)} - Y_t^{(k)}\right|^2\right] \leq \frac{A_2^{k+1}t^{k+1}}{(k+1)!}, \quad k \geq 0, t \in [0,T], \tag{I.84}$$

for some suitable constant A_2 depending only on C, D, T and $E\left[|X_0|^2\right]$. Now

$$\sup_{0\leq t\leq T}\left|Y_t^{(k+1)} - Y_t^{(k)}\right| \leq \int_0^T \mid b\left(s, Y_s^{(k)}\right) - b\left(s, Y_s^{(k-1)}\right) \mid \mathrm{d}s$$
$$+ \sup_{0\leq t\leq T}\left|\int_0^t \left(\sigma\left(s, Y_s^{(k)}\right) - \sigma\left(s, Y_s^{(k-1)}\right)\right) dB_s\right|.$$

By the martingale inequality we obtsin

$$P\left[\sup_{0\leq t\leq T}\left|Y_{\mathrm{t}}^{(k+1)} - Y_t^{(k)}\right| > 2^{-k}\right],$$

$$\leq P\left[\left(\int_0^T \left|b\left(s,Y_s^{(k)}\right)-b\left(s,Y_s^{(k-1)}\right)\right| ds\right)^2 > 2^{-2k-2}\right],$$

$$+P\left[\sup_{0\leq l\leq T}\left|\int_0^t \left(\sigma\left(s,Y_s^{(k)}\right)-\sigma\left(s,Y_s^{(k-1)}\right)\right) dB_s\right| > 2^{-k-1}\right],$$

$$\leq 2^{2k+2}T\int_0^T \mathbb{E}\left(\left|b\left(s,Y_s^{(k)}\right)-b\left(s,Y_s^{(k-1)}\right)\right|^2\right) \mathrm{d}s,$$

$$+2^{2k+2}\int_0^T \mathbb{E}\left[\left|\sigma\left(s,Y_s^{(k)}\right)-\sigma\left(s,Y_s^{(k-1)}\right)\right|^2\right] \mathrm{d}s,$$

$$\leq 2^{2k+2}D^2(T+1)\int_0^T \frac{A_2^k t^k}{k!}dt \leq \frac{(4A_2T)^{k+1}}{(k+1)!}, if \quad A_2 \geq D^2(T+1).$$

Therefore, by the Borel-Cantelli lemma

$$P\left[\sup_{a\leq t\leq T}\left|Y_t^{(k+1)}-Y_t^{(k)}\right| > 2^{-k} \text{ for infinitely many } k\right] = 0.$$

Thus, for a.a. ω there exists $k_0 = k_0(\omega)$ such that

$$\sup_{0\leq \mathrm{r}\leq T}\left|Y_{\mathrm{t}}^{(k+1)}-Y_{\mathrm{t}}^{(k)}\right| \leq 2^{-k} \quad \text{for } k \geq k_0.$$

Therefore the sequence

$$Y_{\mathrm{t}}^{(n)}(\omega) = Y_{\mathrm{t}}^{(0)}(\omega) + \sum_{k=0}^{n-1}\left(Y_t^{(k+1)}(\omega)-Y_{\mathrm{t}}^{(k)}(\omega)\right),$$

is uniformly convergent in $[0,T]$, for a.a. ω
Denote the limit by $X_{\mathrm{t}} = X_{\mathrm{t}}(\omega)$. Then X_{t} is t-continuous for a.s. ω since $Y_{\mathrm{t}}^{(\mathrm{n})}$ is t-continuous for all n. Moreover, $X_{\mathrm{t}}(\cdot)$ is $\mathcal{F}_{\mathrm{t}}^Z$-measurshle for all t, since $Y_{\mathrm{t}}^{(n)}(.)$ has this property for all n. Next, note that for $m > n \geq 0$ we have by eq (I.84)

$$\left[\mathbb{E}\parallel Y_t^{(m)}-Y_t^{(n)}\parallel^2\right]^{1/2} = \left\|Y_t^{(m)}-Y_t^{(n)}\right\|_{\mathbb{L}^2(P)} = \left\|\sum_{k=n}^{m-1}\left(Y_t^{(k+1)}-Y_t^{(k)}\right)\right\|_{\mathbb{L}^2(P)}$$

$$\leq \sum_{k=n}^{m-1}\left\|Y_t^{(k+1)}-Y_t^{(k)}\right\|_{\mathbb{L}^2(P)} \leq \sum_{k=n}^{\infty}\left[\frac{(A_2t)^{k+1}}{(k+1)!}\right]^{1/2} \to 0 \quad \text{as } n\to\infty. \tag{I.85}$$

So $\left\{Y_t^{(n)}\right\}$ converges in $\mathbb{L}^2(P)$ to a limit Y_t, say. A subsequence of $Y_t^{(n)}(\omega)$ will then converge w-pointwise to $Y_t(\omega)$ and therefore we must have $Y_{\mathrm{t}} = X_t$ a.s.
In particular, X_t satisfies eq (I.82) and eq (I.83).

It remains to show that X_t satisfies eq (I.81). For all n we have

$$Y_t^{(\mathrm{n}+1)} = X_0 + \int_0^t b\left(s, Y_s^{(\mathrm{n})}\right) ds + \int_0^t \sigma\left(s^2 Y_s^{(\mathrm{n})}\right) dB_s. \tag{I.86}$$

Now $Y_t^{(\mathrm{n}+1)} \to X_\mathrm{t}$ ss $n \to \infty$, uniformly in $t \in [0,T]$ for a.a. ω. By eq (I.85) and the Fatou lemma we have

$$\mathbb{E}\left[\int_0^T \left|X_t - Y_t^{(n)}\right|^2 dt\right] \leq \limsup_{m\to\infty} \mathbb{E}\left[\int_0^T \left|Y_t^{(m)} - Y_\mathrm{t}^{(n)}\right|^2 dt\right] \to 0,$$

as $n \to \infty$. It follows by the Itô isometry that

$$\int_0^t \sigma\left(s, Y_s^{(n)}\right) \mathrm{d}B_s \to \int_0^t \sigma\left(s, X_s\right) \mathrm{d}B_s,$$

and by the Hölder inequality that

$$\int_0^t b\left(s, Y_s^{(\mathrm{n})}\right) \mathrm{d}s \to \int_0^t b\left(s, X_n\right) ds,$$

in $\mathbb{L}^2(P)$. Therefore, taking the limit of eq (I.86) as $\pi \to \infty$ we obtain eq (I.81) for X_t. □

I.6 Linear Stochastic Differential Equations [16]

The general form of a scalar linear stochastic differential equation is

$$\mathrm{d}X_t = (a_1(t)X_t + a_2(t))\mathrm{d}t + (b_1(t)X_t + b_2(t))\mathrm{d}W_t, \tag{I.87}$$

where the coefficients a_1, a_2, b_1, b_2 are specified functions of time t or constants. Provided they are Lebesgue measurable and bounded on an interval $0 \leqslant t \leqslant T$, the existence and uniqueness theorem applies, ensuring the existence of a strong solution X_t, on $t_o \leqslant t \leqslant T$ for each $0 \leqslant t_o \leqslant T$.
When $a(t)_2 = 0$ and $b(t)_2 = 0$, eq (I.87) reduces to the homogeneous linear SDE

$$\mathrm{d}X_t = a_1(t)X_t\mathrm{d}t + b_1(t)X_t\mathrm{d}W_t. \tag{I.88}$$

When $b(t)_1 = 0$ in eq (I.87) the SDE has the form

$$\mathrm{d}X_t = (a_1(t)X_t + a_2(t))\mathrm{d}t + b_2(t)\mathrm{d}W_t, \tag{I.89}$$

that is the noise appears additively. In this case we say that the SDE is linear in the narrow-sense. The homogeneous equation obtained from eq (I.89) is then an ordinary differential equation

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = a_1(t)X_t, \tag{I.90}$$

and its fundamental solution is

$$\Phi_{t,t_0} = \exp\left(\int_{t_0}^t a_1(s)\mathrm{d}s\right). \tag{I.91}$$

Applying the Ito formula to the transformation $u(t,x) = \Phi_{t,t_0}^{-1} x$ and the solution X_t, of eq (I.89), we obtain

$$\begin{aligned} \mathrm{d}\left(\Phi_{t,t_0}^{-1} X_{t_0}\right) &= \left(\frac{\mathrm{d}\Phi_{t,t_0}^{-1}}{\mathrm{d}t} X_t + (a_1(t)X_t + a_2(t))\Phi_{t,t_0}^{-1}\right)\mathrm{d}t + b_2(t)\Phi_{t,t_0}^{-1}\mathrm{d}W_t, \\ &= a_2(t)\Phi_{t,t_0}^{-1}\mathrm{d}t + b_2(t)\Phi_{t,t_0}^{-1}\mathrm{d}W_t, \end{aligned} \tag{I.92}$$

since

$$\frac{\mathrm{d}\Phi_{t,t_0}^{-1}}{\mathrm{d}t} = -\Phi_{t,t_0}^{-1} a_1(t). \tag{I.93}$$

The right hand side of

$$\mathrm{d}\left(\Phi_{t,t_0}^{-1} X_{t_0}\right) = a_2(t)\Phi_{t,t_0}^{-1}\mathrm{d}t + b_2(t)\Phi_{t,t_0}^{-1}\mathrm{d}W_t, \tag{I.94}$$

only involves known functions of t and W, so can be integrated to give

$$\Phi_{t,t_0}^{-1} X_{t_0} = \Phi_{t_0,t_0}^{-1} X_{t_0} + \int_{t_0}^{t} a_2(t)\Phi_{s,t_0}^{-1}\mathrm{d}s + \int_{t_0}^{t} b_2(t)\Phi_{s,t_0}^{-1}\mathrm{d}W_s. \tag{I.95}$$

Since if $\Phi_{t_0,t_0} = 1$ this leads to the solution

$$X_t = \Phi_{t,t_0}\left(X_{t_0} + \int_{t_0}^{t} a_2(t)\Phi_{s,t_0}^{-1}\mathrm{d}s + \int_{t_0}^{t} b_2(t)\Phi_{s,t_0}^{-1}\mathrm{d}W_s\right), \tag{I.96}$$

of the narrow-sense linear SDE eq (I.89) where

$$\Phi_{t,t_0} = \exp\left(\int_{t_0}^{t} a_1(s)\mathrm{d}s\right). \tag{I.97}$$

I.7 Some Solution Methods [16]

I.1.7 Linear SDEs: Additive Noise

Constant coefficients: homogeneous

$$\mathrm{d}X_t = -\alpha X_t \mathrm{d}t + \sigma \mathrm{d}W_t. \tag{I.98}$$

$$X_t = \exp^{-\alpha t}\left(X_0 + \sigma\int_0^t \exp^{\alpha s}\,\mathrm{d}W_s\right). \tag{I.99}$$

Constant coefficients: inhomogeneous

$$\mathrm{d}X_t = (a(t)X_t + b)\mathrm{d}t + c\mathrm{d}W_t. \tag{I.100}$$

$$X_t = \exp^{at}\left(X_0 + \frac{b}{a}(1-\exp^{-at}) + c\int_0^t \exp^{-as}\,\mathrm{d}W_s\right). \tag{I.101}$$

Variable coefficients

$$dX_t = (a(t)X_t + b(t))\mathrm{d}t + c(t)\mathrm{d}W_t. \tag{I.102}$$

$$X_t = \Phi_{t,t_0}\left(X_{t_0} + \int_{t_0}^{t} b(s)\Phi_{s,t_0}^{-1}\mathrm{d}s + \int_{t_0}^{t} c(s)\Phi_{s,t_0}^{-1}\mathrm{d}W_s\right), \tag{I.103}$$

with fundamental solution

$$\Phi_{t,t_0} = \exp\left(\int_{t_0}^{t} a(s)\mathrm{d}s\right). \tag{I.104}$$

Example I.7.1.1

$$X_t = \left(\frac{2}{1+t}X_t + b(1+t)^2\right)\mathrm{d}t + b(1+t)^2\mathrm{d}W_t, \tag{I.105}$$

has fundamental solution $\Phi_{t,t_0} = \left(\frac{1+t}{1+t_0}\right)^2$ and general solution

$$X_t = \left(\frac{1+t}{1+t_0}\right)^2 X_0 + b(1+t)^2(W_t - W_{t_0} + t - t_0). \tag{I.106}$$

Usually the Wiener process will appear in an integral, as for

$$\mathrm{d}X_t = \left(\frac{b - X_t}{T-t}\right)\mathrm{d}t + \mathrm{d}W_t, \tag{I.107}$$

which is satisfied by the process

$$X_t = X_0\left(1 - \frac{t}{T}\right) + b\frac{t}{T} + (T-t)\int_0^t \frac{1}{T-s}\mathrm{d}W_t, \tag{I.108}$$

on the interval $0 \leqslant t \leqslant T$.

Chapter II

Basic Concept And Stochastic Finite Element Technique

In this chapter, we study the existence and uniqueness of the solution in deterministic finite elements, we apply the Lax-milligram theorem, and in finite random elements we use several methods, including Monte Carlo, the method of squares and the method of perturbation.

II.1 Finite Element Method

The finite element method (FEM) is a numerical technique for solving problems which are described by partial differential equation, or can be formulated as functional minimization.
A domain of interest is represented as an assembly of finite element. Approximating functions in finite elements are determined in terms of nodal values of a physical field which is sought. A continuous physical problem is transformed into a discretized finite element problem with unknown nodal values. For a linear problem a system of linear algebra equation should be solved. Values inside finite elements can be recovered using nodal values.
Two features of the FEM are worth to be mentioned

1) Piece-wise approximation of physical fields on finite elements we can achieve any precision).

2) Locality of approximation leads to sparse equation systems for a disretizzed problem. This helps to solve problems with very large number of nodal unknowns.

II.1.1 Functional Spaces

Definition II.1.1.1 A functional space is a set of functions with operations.
For example,

$$\mathcal{C}(\Omega) = \mathcal{C}^0(\Omega) = \{u(x); u(x)\ is\ continuous\ on\ \Omega\}. \tag{II.1}$$

The functional space with first-order continuous derivatives in $1D$ is

$$\mathcal{C}^1(\Omega) = \{u(x); u(x); u'(x) are\ continuous\ on\ \Omega\}, \tag{II.2}$$

and similarly

$$\mathcal{C}^k(\Omega) = \{u(x); u(x); u'(x); \ldots; u^{(k)}\ are\ continuous\ on\ \Omega\}. \tag{II.3}$$

Then as $k \to \infty$, we define

$$\mathcal{C}^\infty(\Omega) = \{u(x); u(x)\ is\ indefinitely\ differentiable\ on\ \Omega\}. \tag{II.4}$$

II.1.2 Spaces for Integral Forms, $\mathbb{L}^2(\Omega)$ [22]

Definition II.1.2.1 The square-integrable space $H^0(\Omega) = \mathbb{L}^2(\Omega)$ is defined as

$$\mathbb{L}^2(\Omega) = \left\{ v(x); \int_\Omega |v(x)|^2 \mathrm{d}x < \infty \right\}. \tag{II.5}$$

II.1.3 Sobolev Spaces(The space $H^1(\Omega)$ and $H_0^1(\Omega)$)

Definition II.1.3.1 (the space $H^1(\Omega)$)

Let Ω be an open set of $\mathbb{R}^N$. The Sobolev space $H_1(\Omega)$ is defined by

$$H^1(\Omega) = \left\{ v \in L^2(\Omega) \quad such \quad that \quad \forall i \in \{1, \ldots, N\} \frac{\partial v}{\partial x_i} \in L^2(\Omega) \right\}, \tag{II.6}$$

where $\dfrac{\partial v}{\partial x_i}$is the weak partial derivative of v, the Sobolev space $H^1(\Omega)$ is a Hilbert space.

Definition II.1.3.2: (the space $H_0^1(\Omega)$)

Let us now define another Sobolev space which is a subspace of $H^1(\Omega)$ and which will be very useful for problems with Dirichlet boundary conditions.

Where

$$H_0^1(\Omega) = \left\{ v / v \in H^1(\Omega) \quad v/_\Gamma = 0 \right\}. \tag{II.7}$$

$\Gamma = \partial\Omega$ is its boundary.

II.2 Deterministic Element Method

Suppose $\Omega = [a, b]$ is open and finite set of $\mathbb{R}^N$.
Let the following problem be

$$\begin{cases} \mathbf{F}(x, u, u', \ldots) = f(x) & x \in]a, b[, \\ u(a) = u(b) = 0 & \partial\Omega. \end{cases} \tag{II.8}$$

Where f is finite and continuous, i.e $f \in \mathcal{C}^0(\overline{\Omega})$, and $\mathbf{F}$ continuous functions of a class $\mathcal{C}^\infty(\overline{\Omega})$. This problem eq (II.8) is called boundary value problem. It u solves the problem eq (II.8) defined on Ω to $\mathcal{C}^\infty([a, b])$.

II.2.1 The Finite Element Method [11]

A derivation of the finite element method can be divided into

(1) variational formulation in an infinite dimensional space V,

(2) variational formulation in a finite dimensional subspace, $V_h \in V$,

(3) choice of a basis for V_h, and

(4) solution of the discrete system of equations.

First: Variational formulation in an infinite dimensional space V.
Consider the following Hilbert space eq (II.6).

Let Ω be an open set of $\mathbb{R}^N$, and $\overline{\Omega}$ its closure. We denote by $\mathcal{C}(\Omega)$ (respectively, $\mathcal{C}(\overline{\Omega})$) the space of continuous function in Ω (respectively, in $\overline{\Omega}$.) Let $k \geqslant 0$ be an integer. We denote

by $\mathcal{C}^k(\Omega)$ (respectively, $\mathcal{C}^k(\overline{\Omega})$) the space of functions k times continuously differentiable in Ω (respectively, in $\overline{\Omega}$).

Definition II.2.1.1(Variational Formulation) [1], [17]

Let u be a function of $\mathcal{C}^2(\overline{\Omega})$. Let X be the space defined by

$$X = \left\{\partial\phi \in C^1(\overline{\Omega}) \quad such \quad that \quad \phi = 0 \quad on \quad \partial\Omega\right\}. \tag{II.9}$$

Then u is a solution of the boundary value problem eq (II.8) if and only if u belongs to X and satisfies the equation

$$\int_\Omega \mathbf{F}\cdot v(x)\mathrm{d}x = \int_\Omega f(x)v(x)\mathrm{d}x \quad for\ every \quad v \in X. \tag{II.10}$$

Equation (II.10) is called the variational formulation of the boundary value problem. After converting the partial differential equation to the variational formulation when calculating the integral, we use the integration by part and the Grenn's Formula theorem.

Definition II.2.1.2 (Integration by Part)

Let f and g be two functions of one variable, Which we assume are differentiable, and defined (at least)on an interval $[a, b]$. We have

$$\begin{aligned}\int_a^b f'(x)g(x)\mathrm{d}x &= [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)\mathrm{d}x \\ &= f(b)g(b) - f(a)f(b) - \int_a^b f(x)g'(x)\mathrm{d}x.\end{aligned} \tag{II.11}$$

Theorem II.2.1.1: (Grenn's Formula) [1]

Let Ω be an open bounded regular set of class $\mathcal{C}^1$ If u and v are function of $H^1(\Omega)$, they satisfy

$$\int_\Omega u(x)\frac{\partial v}{\partial x_i}(x)\mathrm{d}x = -\int_\Omega v(x)\frac{\partial u}{\partial x_i}(x)\mathrm{d}x + \int_{\partial\Omega} u(x)v(x)n_i(x)\mathrm{d}s. \tag{II.12}$$

where $n = (n_i)_{1\leqslant i\leqslant N}$ is the outward unit normal to $\partial\Omega$.

Remark II.2.1.1

The variational formulation of the problem eq (II.8) to is to find $u \in V$ such that

$$A(u; v) = L(v) \quad \forall v \in V. \tag{II.13}$$

Where

$$A(u; v) = \int_\Omega \mathbf{F}\cdot v(x)\mathrm{d}x, \quad L(v) = \int_\Omega f(x)v(x)\mathrm{d}x. \tag{II.14}$$

To prove the uniqueness of the solution of a variational formulation eq (II.13) in a Hilbert space V, we use the Lax-Milgram theorem.

Theorem II.2.1.1 (Lax-Milgram) [1], [11]

Let V be a Hilbert space with norm $\|.\|_V$ and scalar product $(.;.)_V$ and assume that A is a bilinear functional and L is a linear functional that satisfy

(1) A is symmetric, i.e. $A(v; w) = A(w; v)\ \forall v\ , w \in V$;

(2) A is V-elliptic, i.e. $\exists\alpha > 0$ such that $A(v; v) \geqslant \alpha\|v\|_V^2$;

(3) A is continuous, i.e. $\exists C \in \mathbb{R}$ such that $|A(v; w)| \leqslant C\|v\|_V\|w\|_V$; and

(4) L is continuous, i.e. $\exists\Lambda \in \mathbb{R}$ such that $|L(v)| \leqslant \Lambda\|v\|_V$.

Then there is a unique function $u \in V$ such that $A(u;v) = L(v) \quad \forall v \in V$; and the stability estimate $\|u\|_V \leqslant \frac{\alpha}{\Lambda}$ holds.

Then the variational formulation eq (II.13) has a unique solution. Further, this solution depends continuously on the linear form L.

Second and Third

Variational formulation in the finite dimensional subspace $V_h \in V$. And choice of a basis for V_h.

the following variational formulation: eq (II.13)

$$To find\ u \in V, such\ that\ A(u,v) = f(v) for\ all\ v \in V. \tag{II.15}$$

LetV_h be a finite-dimensional subspace eq (II.21) of V. Then we may state the following problem: eq (II.16)

$$To\ find\ \ u_h \in V_h\ \ such\ \ that\ \ A(u_h, v_h) = f(v_h) for\ all\ \ v_h \in V_h. \tag{II.16}$$

Where

$$A(u_h, v_h) = \int_\Omega \mathbf{F} \cdot v_h(x)\mathrm{d}x, \quad L(v_h) = \int_\Omega f(x) v_h(x)\mathrm{d}x. \tag{II.17}$$

V_h, being a finite-dimensional subspace, is a Hilbert space for the norm of V. Hence by The Lax-Milgram theorem, u_h exists and is unique. We try to approximate the solution u of eq (II.13) by means of solutions u_h of the problem eq (II.16) for various subspaces V_h. This is known as the internal approximation method.

A) Finite Element En Dimension $N = 1$ [1], [4]

We can now build a mesh of $[a, b]$ by defining a subdivision (not necessarily regular) a

$$a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b. \tag{II.18}$$

The mesh will be called uniform if the points x_j are equidistant, that is

$$x_j = a + jh\ with\ h = \frac{1}{1+n}, 0 \leqslant j \leqslant n+1. \tag{II.19}$$

The points x_j are also called **the vertices** (or nodes) of the mesh.

a) $\mathbb{P}_1$finite elements

The $\mathbb{P}_1$ finite element method uses the discrete space

$$V_{0h} = \left\{ v \in \mathcal{C}([a,b])\ such\ \ that\ v\ |_{[x_j, x_{j+1}]} \in \mathbb{P}_1\ for\ all\ 0 \leqslant j \leqslant n \right\}, \tag{II.20}$$

and its subspace

$$V_h = \left\{ v \in V_{0h}\ such\ that\ v(a) = v(b) = 0 \right\}. \tag{II.21}$$

The $\mathbb{P}_1$ finite element method is then simply the method of internal variational approximation of applied to the spaces V_{0h} or V_h defined by eq (II.20) or eq (II.21). We can represent the functions of V_h or, V_{0h} which are piecewise affine, with the help of very simple basis functions. We introduce the 'hat function' ϕ defined by

$$\phi(x) = \begin{cases} 1 - |x| & if |x| \leqslant 1, \\ 0 & if |x| > 1. \end{cases} \tag{II.22}$$

If the mesh is uniform, for $0 \leq j \leq n+1$ we define the basis functions

$$\phi_{j(x)} = \phi\left(\frac{x - x_j}{h}\right). \tag{II.23}$$

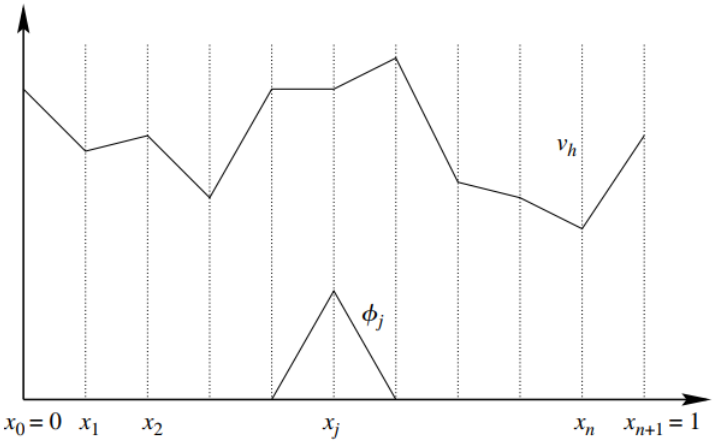


Figure II.1: Mesh of $\Omega =]a, b[$ and $\mathbb{P}_1$ finite element basis functions.

eq (II.16) Let $(\phi_1, \ldots, \phi_{N_h})$ be a basis of V_h. By decomposing u_h on this basis the form

$$u_h = \sum_{i=1}^{N_h} \mu_i \phi_i, \tag{II.24}$$

the problem eq (II.16) becomes find $\mu_1, \ldots, \mu_{N_h}$ such as
$u_h = \sum_{i=1}^{N_h} \mu_i A(\phi_i, v_h) = L(v_h), \forall v_h \in V_h$ linearity of A, L and choosing $v_h = \phi_j(x)$ in eq (II.16).

becomes find $\mu_1, \ldots, \mu_{N_h}$ such as $u_h = \sum_{i=1}^{N_h} \mu_i A(\phi_i, \phi_j) = L(\phi_j), \forall j = 1 \ldots N_h$

solve the linear system

$$\underbrace{\begin{pmatrix} a(\phi_1, \phi_1) & \cdots a(\phi_{N_h}, \phi_1) \\ \vdots & \vdots \\ a(\phi_1, \phi_{N_h}) & \cdots a(\phi_{N_h}, \phi_{N_h}) \end{pmatrix}}_{\mathrm{K}} \underbrace{\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_{N_h} \end{pmatrix}}_{\mathrm{U}} = \underbrace{\begin{pmatrix} L(\phi_1) \\ \vdots \\ L(\phi_{N_h}) \end{pmatrix}}_{b}. \tag{II.25}$$

the variational formulation in V_h reduces to solving in $\mathbb{R}^n$ the linear system

$$KU = b. \tag{II.26}$$

b) $\mathbb{P}_2$finite elements

The $\mathbb{P}_2$ finite element method uses the discrete space

$$V_{0h} = \left\{ v \in \mathcal{C}([a,b]) \; such \quad that \; v \mid_{[x_j, x_{j+1}]} \in \mathbb{P}_2 \; for \; all \; 0 \leqslant j \leqslant n \right\}, \tag{II.27}$$

and its subspace

$$V_h = \{ v \in V_{0h} \; such \; that \; v(a) = v(b) = 0 \}. \tag{II.28}$$

The $\mathbb{P}_2$ finite element method is then simply the method of internal variational approximation of applied to the spaces V_{0h} or V_h. These are composed of continuous, piecewise quadratic functions that we can represent with the help of very simple basis functions.

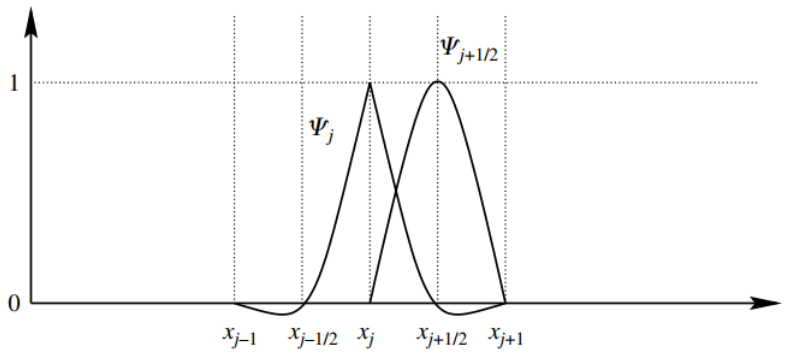


Figure II.2: Basis functions for $\mathbb{P}_2$ finite elements.

Let us introduce first of all the midpoints of the segments $[x_j, x_{j+1}]$ defined by $x_{j+1/2} = x_j + h/2$ for $0 \leqslant j \leqslant n$. We also define two ' reference ' functions

$$\phi(x) = \begin{cases} (1+x)(1+2x) & if \ -1 \leqslant x \leqslant 0, \\ (1-x)(1-2x) & if \ 0 \leqslant x \leqslant 1, \\ 0 & if \ |x| > 1. \end{cases} \tag{II.29}$$

and

$$\psi(x) = \begin{cases} 1 - 4x^2 & if \ |x| \leqslant 1/2, \\ 0 & if \ |x| > 1/2. \end{cases} \tag{II.30}$$

If the mesh is uniform, for $0 \leq j \leq n+1$ we define the basis functions (see (II.2)).

$$\psi_{j(x)} = \phi\left(\frac{x - x_j}{h}\right), \ 0 \leq j \leq n+1, \text{ and } \psi_{j+1/2}(x) = \psi\left(\frac{x - x_{j+1/2}}{h}\right), \ 0 \leq j \leq n.$$

The space V_{0h}, defined by eq (II.27), is a subspace of $H^1(a,b)$ of dimension $2n+3$, and every function $v_h \in V_{0h}$ is defined uniquely by its values at the vertices $(x_j)_{0\leq j\leq n+1}$ and at the midpoints $(x_{j+1/2})_{0\leq j\leq n}$.

$$v_h(x) = \sum_{j=0}^{n+1} v_h(x_j)\psi_j(x) + \sum_{j=0}^{n} v_h(x_{j+1/2})\psi_{j+1/2}(x) \quad \forall x \in [a,b]. \tag{II.31}$$

Likewise, V_h defined by eq (II.28), is a subspace of $H^1(a,b)$, of dimension $2n+1$, and every function $v_h \in V_h$ is defined uniquely by its values at the vertices $(x_j)_{0\leq j\leq n}$ and at the midpoints $(x_{j+1/2})_{0\leq j\leq n}$.

$$v_h(x) = \sum_{j=1}^{n+1} v_h(x_j)\phi_j(x) + \sum_{j=0}^{n} v_h(x_{j+1/2})\psi_{j+1/2}(x) \quad \forall x \in [a,b]. \tag{II.32}$$

The practical solution of problem eq (II.8) by the $\mathbb{P}_2$ finite element method. The variational formulation eq (II.13) of the internal approximation reduces to solving in $\mathbb{R}^{2n+1}$

the linear system

$$K_h U_h = b_h. \tag{II.33}$$

B) Finite Element En Dimension $N \geqslant 2$ [1], [21]
Triangular Finite Elements

We start with the definition of a mesh of the domain Ω by triangles in $N = 2$ dimensions and by tetrahedra in $N = 3$ dimensions. We group the triangles and the tetrahedra in the more general family of N-simplices. We call the N-simplex K of $\mathbb{R}^N$ the convex envelope of $(N + 1)$ points $(a_j)_{1\leqslant j\leqslant N+1}$ of $\mathbb{R}^N$, called the vertices of K. Of course, a 2-simplex is simply a triangle and a 3-simplex a tetrahedron (see Figure (II.4)). We say that the N-simplex K is non degenerate if the points $(a_j)_{1\leqslant j\leqslant N+1}$ do not belong to the same hyperplane of $\mathbb{R}^N$ (the triangle or the tetrahedron is not 'flat'). If we denote by $(ai, j)_{1\leqslant i\leqslant N}$ the coordinates of the vector a_j, the non degeneracy condition on K is that the matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N+1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N+1} \\ 1 & 1 & \cdots & 1 \end{pmatrix}. \tag{II.34}$$

Definition II.2.1 *Let Ω be an open connected polyhedron of $\mathbb{R}^N$. A triangular mesh or a triangulation of Ω is a set Th of (nondegenerate) N-simplices (K_i) $1 \leq i \leq n$ which*

1) $K_i \subset \overline{\Omega}$ and $\overline{\Omega} = \cup_{i=1}^{n} K_i$.

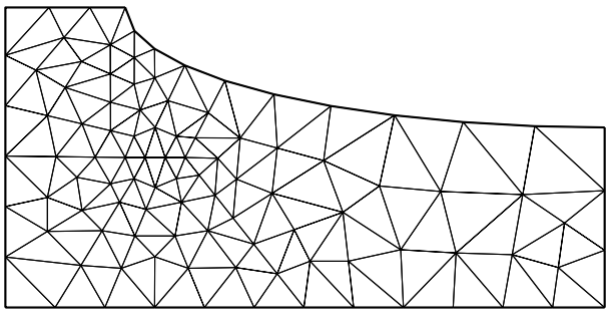


Figure II.3: Example of triangular mesh in $N = 2$ dimensions.

2) The intersection $K_i \cap K_j$ of two distinct N-simplices is an m-simplex, with $0 \leq m \leq N - 1$, whose vertices are also vertices of K_i and K_j.

is invertible (which we shall always assume in what follows). An N-simplex has as many faces as vertices, which are themselves $(N - 1)$-simplices.
In an N-simplex K it is easy to use barycentric coordinates instead of the usual Cartesian coordinates. Recall that, if K is a nondegenerate N-simplex with vertices $(a_j)_{1\leq j\leq N+1}$, the **barycentric coordinates** $(\lambda_j)_{1\leq j\leq N+1}$ of $x \in \mathbb{R}^N$ are defined by

$$\begin{cases} 1 = \sum_{j=1}^{N+1} \lambda_j, \\ x_i = \sum_{j=1}^{N+1} a_{i,j}\lambda_j. \end{cases} \quad for\ 1 \leq j \leq N \tag{II.35}$$

Which have a unique solution because the matrix A, defined by eq (II.34), is invertible. Let us remark that λ_j are affine functions of x. We then verify that

$$K = \left\{ x \in \mathbb{R}^N \; such \; that \; \lambda_j(x) \geqslant 0 \; for \; 1 \leqslant j \leqslant N+1 \right\}, \tag{II.36}$$

and that the $(N+1)$ faces of K are the intersections of K and the hyperplanes $\lambda_j(x) = 0, 1 \leq j \leq N+1$. We can then define a set of points of K which will play a particular role in what follows: for every integer $k \geqslant 1$ we call the lattice of order K the set

$$\sum_k = \left\{ x \in K \; such \; that \; \lambda_{j(x)} \in \left\{ 0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1 \right\} for \; 1 \leqslant j \leqslant N \right\}. \tag{II.37}$$

We now define the set $p \in \mathbb{P}_k$ of polynomials with real coefficients from $p \in \mathbb{R}_N$ into of $\mathbb{R}$ degree less than or equal to k, that is, all $p \in \mathbb{P}_k$ are written in the form

$$p(x) = \sum_{\substack{i_1, \cdots, i_N \geqslant 0 \\ i_1 + \cdots + i_N \leqslant k}} \alpha_{i_1, \ldots, i_N} x_1^{i_1} \cdots x_N^{i_N} \;\; with \;\; x = (x_1, \cdots, x_N). \tag{II.38}$$

The interest in the idea of a lattice $\sum_k$ of an N-simplex K is that it allows us to characterize all the polynomials of $\mathbb{P}_k$(we say that $\sum_k$ is **unisolvant** for $\mathbb{P}_k$)

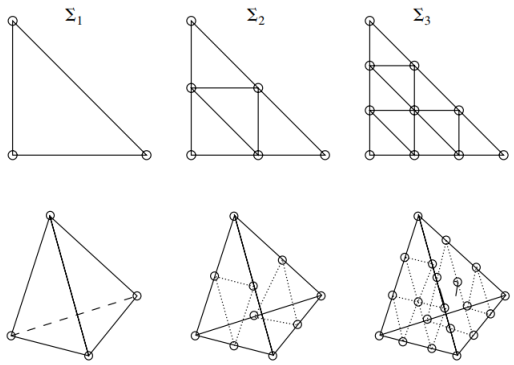


Figure II.4: Lattice of order 1, 2, and 3 for a triangle (top) and a tetrahedron (bottom). The circles represent the points of the lattice.

Definition II.2.2 *Let* $(\tau_h)_{h>0}$*be a sequence of meshes of* Ω*. We say that it is a sequence of* ***regular meshes*** *if*

1): the sequence $h = \max_{K_i \in \tau_h} diam(K_i)$ tends to 0,

2): there exists a constant C such that, for all $h > 0$ and all $K \in \mathcal{T}h$,

$$\frac{diam(K)}{\rho(K)} \leqslant C, \tag{II.39}$$

where:

$$diam(K) = \max_{x,y \in K} \|x - y\|, \quad \rho(K) = \max_{B_r \subset K} (2r). \tag{II.40}$$

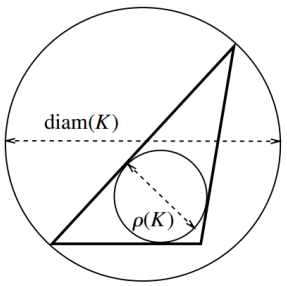


Figure II.5: $diam(K)$ and $\rho(K)$ for a tringle K.

II.2.2 Convergence And Error [1]

Theorem II.2.2.1
Let $(\tau_h)_{h>0}$ be a sequence of regular meshes of Ω. Let $u \in H_0^1(\Omega)$ be the solution of the problem eq (II.8), and $u_h \in V_h$, its internal approximation eq (II.16) by the $\mathbb{P}_k$ finite element method. Then the $\mathbb{P}_k$ finite element method **converges**, that is

$$\lim_{h \longrightarrow 0} \|u - u_h\|_{H^1(\Omega)} = 0. \tag{II.41}$$

Moreover, if $u \in H^{k+1}(\Omega)$ and if $k+1 > N/2$. then we have **the error estimate**

$$\|u - u_h\|_{H^1(\Omega)} \leqslant Ch^k \|u\|_{H^{k+1}(\Omega)}, \tag{II.42}$$

where C is a constant independent of h and of u.
The global **error** is defined by $e_h = u(x) - u_h(x)$, and we seek a sharp upper bound for $\| e_h \|$ using certain norm.

II.2.3 1D Elliptic Example: Solution to the Dirichlet problem [11]

Suppose $\Omega = [0,1]$ is open and finite set of $\mathbb{R}^N$.
Let the following problem be

$$\begin{cases} -u''(x) = f, & x \in]0,1[\\ u(0) = u(1) = 0. \end{cases} \tag{II.43}$$

Where f is finite and continuous, $f \in \mathcal{C}^0([0,1])$, and u continuous functions of a class $\mathcal{C}^2([0,1])$.

i : ***Variational formulation in an infinite dimensional space*** V.
Consider the following Hilbert space

$$V = \left\{ w : (0,1) \to \mathbb{R} : \int_0^1 (w^2(x) + \mid \nabla w(x) \mid^2) \mathrm{d}x < \infty, w(0) = w(1) = 0 \right\}. \tag{II.44}$$

Multiply differential equation eq (II.43) by a so-called test function $w(x) \in V$ which we will assume arbitrary for the moment and integrate over the interval $[0,1]$.
We obtain

$$\int_0^1 -u''(x)w(x)\mathrm{d}x = \int_0^1 f(x)w(x)\mathrm{d}x. \tag{II.45}$$

Let us now integrate by parts (assuming that $w(x)$ is sufficiently regular)using Grenn's Formula section II.2.1to obtain

$$\int_0^1 -u'(x)w'(x)\mathrm{d}x - u'w|_0^1 = \int_0^1 f(x)w(x)\mathrm{d}x. \tag{II.46}$$

$$\int_0^1 -u'(x)w'(x)\mathrm{d}x - (u'(1)w(1) - u'(0)w(0)) = \int_0^1 f(x)w(x)\mathrm{d}x. \tag{II.47}$$

If we now assume that $w(0) = w(1) = 0$ we obtain a variational formulation which consists in determining a function $u(x)$ satisfying $u(0) = u(1) = 0$ and such that

$$\int_0^1 -u'(x)w'(x)\mathrm{d}x = \int_0^1 f(x)w(x)\mathrm{d}x \ \forall \ w(x) | \ w(0) = w(1) = 0. \tag{II.48}$$

Therefore the variational formulation of eq (II.43) is to find $u \in V$ such that

$$A(u,w) = L(w) \qquad \forall w \in V, \tag{II.49}$$

where

$$A(u,w) = \int_0^1 -u'(x)w'(x)\mathrm{d}x, \quad L(w) = \int_0^1 f(x)w(x)\mathrm{d}x. \tag{II.50}$$

To prove the uniqueness of the solution of a variational formulation eq (II.49) in a Hilbert space V,we use the Lax-Milgram eq (II.2.1).

1) The continuity of A is a consequence of

$$|A(v,w)| = \left| \int_0^1 -u'(x)w'(x)\mathrm{d}x \right| = \left| \int_0^1 \nabla u(x)\nabla w(x)\mathrm{d}x \right|. \tag{II.51}$$

We apply Cauchy-Schwarty theorem

$$\leqslant \left(\int_0^1 |\nabla u(x)|^2 \right)^{\frac{1}{2}} \left(\int_0^1 |\nabla w(x)|^2 \right)^{\frac{1}{2}} \leqslant C\|u\|_V \|w\|_V, \tag{II.52}$$

$$|A(v,w)| \leqslant C\|u\|_V\|w\|_V. \textit{ is } A \textit{ continuity.}$$

2) A is symmetric

$$A(u,w) = \int_0^1 -u'(x)w'(x)\mathrm{d}x = \int_0^1 -w'(x)u'(x)\mathrm{d}x = A(w,u). \tag{II.53}$$

where $\quad A(u,w) = A(w,u) \quad$ A symmetric.

3) To verify the V-ellipticity, we use the Poincar´e inequality, i.e. there is a constant C such that

$$w \in V = H_0^1 \Rightarrow \int_0^1 |w(x)|^2\mathrm{d}x \leq C \int_0^1 |\nabla w(x)|^2\mathrm{d}x. \tag{II.54}$$

In one dimension and $(0,1)$, the inequality eq (II.54) takes the form

$$w \in H_0^1 \Rightarrow \int_0^1 w^2\mathrm{d}x \leq \int_0^1 (w'(x))^2\mathrm{d}x, \tag{II.55}$$

provided $w(0) = 0$. Since

$$w(x) = w(0) + \int_0^x w'(s)\mathrm{d}s = \int_0^x w'(s)\mathrm{d}s, \tag{II.56}$$

and by Cauchy's inequality

$$w(x)^2 = \left(\int_0^x w'(s)\mathrm{d}s \right)^2 \leq x \int_0^x w'(s)^2\mathrm{d}s \leq \int_0^1 w'(s)^2\mathrm{d}s, \textit{ Since } x \in (0,1). \tag{II.57}$$

The V-ellipticity of A follows by eq (II.55) and

$$\begin{aligned} A(w,w) = \int_0^1 w'(x)^2 \mathrm{d}x &= \frac{1}{2}\int_0^1 \left((w'(x))^2 \mathrm{d}x + \frac{1}{2}((w'(x))^2\right) \mathrm{d}x, \\ &\geq \frac{1}{2}\int_0^1 \left(w'(x)^2 + w'(x)^2\right) \mathrm{d}x = \frac{1}{2}||w||_V^2 \quad \forall\, w \in V, \\ A(w,w) &\geqslant \frac{1}{2}||w||_V^2. \end{aligned} \tag{II.58}$$

Which means that we may take $\alpha = \frac{1}{2}$.

4) Finally, the functional L is continuous, since

$$|L(w)| \leq ||f||_{\mathbb{L}^2}||w||_{\mathbb{L}^2} \leq ||f||_{\mathbb{L}^2}||w||_V, \tag{II.59}$$

which means that we may take $\Lambda = ||f||_{\mathbb{L}^2}$ provided we assume that $f \in \mathbb{L}^2(0,1)$. Then the variational formulation eq (II.49) has a unique solution. Further, this solution depends continuously on the linear form L.

ii : ***Variational formulation in a finite dimensional subspace*** $V_h \in V$.
First divide the interval $[0,1]$ into

$$x_0 = 0 < x_1 = 0.1 < x_2 = 0.2 <, \ldots, < x_{N+1} = 1. \tag{II.60}$$

i.e. generate the mesh. Then define the space of continuous piecewise linear functions on the mesh with zero boundary conditions

$$V_h = \{w \in V : w(x)|_{(x_j,x_{j+1})} = c_j x + d_j,\ \textit{i.e. } w \textit{ is linear on } (x_j, x_{j+1}), j = 0, \ldots, N$$
$$\textit{and } w \textit{ is continuous on } [0,1]\}. \tag{II.61}$$

The variational formulation in the finite dimensional subspace is to find $u_h \in V_h$ such that

$$A(u_h, w) = L(w) \quad \forall\, w \in V_h. \tag{II.62}$$

The function u_h is a finite element solution of the equation eq (II.43). Other finite element solutions are obtained from alternative finite dimensional subspaces, e.g. Based on piecewise quadratic approximation.

iii : ***Choose a basis for*** V_h.
As in the one dimensional problem, choose the basis $\phi_j \in V_h$ such that

$$\phi_j(x_i) = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases} \quad j = 1, 2, \ldots N, \tag{II.63}$$

where $x_i : i = 0, 1, 2, \ldots, 10$, are the vertices of the triangulation. A function $w \in V_h$ has the representation

$$w(x) = \sum_{j=1}^{N} w_j \phi_j(x), \tag{II.64}$$

where $w_j = w(x_j)$, i.e. each $w \in V_h$ can be written in a unique way as a linear combination of the basis functions ϕ_j.

iv : ***Solve the discrete problem eq (II.62)***

Using the basis functions ϕ_j, for $j = 1, 2, \ldots N$ from (iii), we have

$$u_h(x) = \sum_{j=1}^{N} \xi_j \phi_j(x), \tag{II.65}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^T \in \mathbb{R}^N$, and choosing $w = \phi_j(x)$ in eq (II.62), we obtain

$$\begin{aligned} L(\phi_j) &= A(u_h, \phi_j) \\ &= A(\sum_{j=1}^{N} \phi_j \xi_j, \phi_i) = \sum_{j=1}^{N} \xi_j A(\phi_j, \phi_i), \end{aligned} \tag{II.66}$$

so that $\xi^T \in \mathbb{R}^N$ solves the linear system

$$\tilde{A}\xi = \tilde{L}. \tag{II.67}$$

where

$$\begin{aligned} \tilde{A}_{i,j} &= A(\phi_j, \phi_i), \\ \tilde{L}_j &= L(\phi_j). \end{aligned} \tag{II.68}$$

$$\begin{aligned} \int_0^1 \phi_j'(x)\phi_j'(x)\mathrm{d}x &= \int_{x_{j-1}}^{x_{j+1}} \phi_j'(x)\phi_j'(x)\mathrm{d}x \\ &= \int_{x_{j-1}}^{x_j} \frac{1}{h^2}\mathrm{d}x + \int_{x_j}^{x_{j+1}} \left(\frac{-1}{h}\right)^2 \mathrm{d}x = \frac{h}{h^2} + \frac{h}{h^2} = \frac{2}{h}, \end{aligned} \tag{II.69}$$

$$\int_0^1 \phi_j'(x)\phi_{j+1}'(x)\mathrm{d}x = \int_{x_{j-1}}^{x_{j+1}} \phi_j'(x)\phi_{j+1}'(x)\mathrm{d}x = \frac{-1}{h}, \tag{II.70}$$

$$\int_0^1 \phi_{j-1}'(x)\phi_{j+1}'(x)\mathrm{d}x = \int_{x_{j-1}}^{x_{j+1}} \phi_{j-1}'(x)\phi_{j+1}'(x)\mathrm{d}x = 0, \tag{II.71}$$

$$\tilde{A} = \frac{1}{h}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \tilde{L}_j = (L, \phi_j), \tag{II.72}$$

$$\textit{choose} \quad L = -1 \quad \tilde{L}_j = -1 \int_0^1 \phi_j \mathrm{d}x = -h. \tag{II.73}$$

The $N \times N$ matrix $\tilde{A}$ is called the stiffness matrix and the vector $\tilde{L} \in \mathbb{R}^N$ is called the load vector.

Applying the boundary conditions $u(0) = u(1) = 0$. yields

$$\tilde{A} = \frac{1}{h}\begin{pmatrix} 1 & 0 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & & & 0 & 1 \end{pmatrix}, \quad \tilde{L} = h.(0, 1, 1, \ldots, 1, 0)^T. \tag{II.74}$$

II.3 Stochastic Element Method [6]

In the last section, the dominant trend in the structural analysis has been deterministic. This approach is characterized by the use of predefined values of system properties and external forces, without considering directly the random nature of these variables. A traditional deterministic model can be formulated as

$$\mathbf{L}(x)u(x) = f(x) \qquad x \in D, \tag{II.75}$$

where, $\mathbf{L}(x)$ is a linear deterministic differential operator (i.e. a mapping from one vector space to another), $u(x)$ is the response vector and $f(x)$ is the deterministic system input or excitation. Therefore, when considering the system properties and input excitation as uncertain quantities, a new random differential operator $\Lambda(x, \theta)$ arises, whose coefficients $H_k(x, \theta)$ can be restricted to be second-order random fields (in most of real-life applications).
Thus, a stochastic model analogous to eq (II.75) can be formulated as

$$\Lambda(x, \theta)u(x, \theta) = f(x, \theta) \qquad x \in D, \theta \in \Omega. \tag{II.76}$$

Due to the second-order restriction imposed on each random operator coefficient, they can be divided into a total deterministic and random parts as, $H(\mathrm{x}, \theta) = \mu_H(\mathrm{x}) + H_\sigma(\mathrm{x}, \theta)$. Consequently, $\mathbf{\Lambda}(\mathrm{x}, \theta)$ can be expressed as the sum of a deterministic operator $\mathbf{L}(\mathrm{x})$, with its coefficients being the expectation of the field $H(\mathrm{x}, \theta)$, and a random operator $\Pi(\mathrm{x}, \theta)$, with its coefficients being zero-mean random fields having the same covariance of the original field $H(\mathbf{x}, \theta)$.
Then, eq (II.76) can then be rewritten as

$$[\mathbf{L}(\mathbf{x}) + \mathbf{\Pi}(\mathbf{x}, \theta)]u(\mathbf{x}, \theta) = f(\mathbf{x}, \theta). \tag{II.77}$$

If the explicit dependence on the field $H_\sigma(\mathrm{x}, \theta)$ is highlighted, the expression becomes

$$[\mathbf{L}(\mathbf{x}) + \mathbf{\Pi}\left(\mathbf{x}, H_\sigma(\mathbf{x}, \theta)\right)]\, u\left(\mathbf{x}, H_\sigma(\mathbf{x}, \theta)\right) = f(\mathbf{x}, \theta). \tag{II.78}$$

Furthermore, if it is assumed that only one parameter of the system is uncertain, the field $H_\sigma(\mathbf{x}, \theta)$ becomes a multiplicative factor in the operator $\Pi\left(\mathbf{x}, H_\sigma(\mathbf{x}, \theta)\right)$. Hence, the resulting stochastic model will be given by

$$[\mathbf{L}(\mathbf{x}) + H_\sigma(\mathbf{x}, \theta)\mathbf{R}(\mathbf{x})]\, u(\mathbf{x}, \theta) = f(\mathbf{x}, \theta), \tag{II.79}$$

where, $\mathbf{R}(\mathbf{x})$ is now a deterministic operator. Suppose that the structural system is being affected by uncertain material properties and/or geometrical parameters. Based on the finite element equilibrium equation $\mathbf{Ka} - \mathbf{f} = \mathbf{q}$, both the global stiffness matrix and the response can be expanded using a Taylor series with respect to a set of random variables, as

$$\mathbb{K}(\boldsymbol{\xi}) = \mathbf{K}_0 + \sum_{i=1}^{M} \left.\frac{\partial \mathbf{K}_i(\boldsymbol{\xi})}{\partial \xi_i}\right|_{\xi=0} \xi_i + \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M} \left.\frac{\partial^2 \mathbf{K}_{ij}(\boldsymbol{\xi})}{\partial \xi_i \partial \xi_j}\right|_{\boldsymbol{\xi}=0} \xi_i\xi_j + \cdots \tag{II.80}$$

$$\mathbf{u}(\boldsymbol{\xi}) = \mathbf{u}_0 + \sum_{i=1}^{M} \left.\frac{\partial \mathbf{u}_i(\boldsymbol{\xi})}{\partial \xi_i}\right|_{\xi=0} \xi_i + \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M} \left.\frac{\partial^2 \mathbf{u}_{ij}(\boldsymbol{\xi})}{\partial \xi_i \partial \xi_j}\right|_{\xi=0} \xi_i\xi_j + \cdots \tag{II.81}$$

where, $\boldsymbol{\xi} = [\xi_1, \ldots, \xi M]$ are random variables from the random field discretization, and $\mathbf{K}_0$, $\mathbf{u}_0$ are deterministic mean values of the uncertain parameters.
When the random variables in ξ are independent and standard Gaussian distributed, the second-order derivatives are equal to zero. This is due to the fact that all partial derivatives are evaluated at the mean value, which is zero in that case. Therefore, truncating the series after the second-order term and substituting these expressions in equations $\mathbf{Ka} - \mathbf{f} = \mathbf{q}$, yields to a system of recursive equations,

$$\left(\mathbf{K}_0 + \sum_{i=1}^{M} \frac{\partial \mathbf{K}_i(\boldsymbol{\xi})}{\partial \xi_i}\right)\left(\mathbf{u}_0 + \sum_{i=1}^{M} \frac{\partial \mathbf{u}_i(\boldsymbol{\xi})}{\partial \xi_i}\xi_i + \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M} \frac{\partial^2 \mathbf{u}_{ij}(\boldsymbol{\xi})}{\partial \xi_i \partial \xi_j}\xi_i\xi_j\right) = \mathbf{f}, \tag{II.82}$$

to be solved progressively for the derivatives of the response. Here, each j-th equation is called the $(j-1)$-th perturbation. For instance, considering the zero and first perturbations, the following expressions can be obtained,

$$\mathbf{K}_0\mathbf{u}_0 = \mathbf{f} \tag{II.83}$$

$$\frac{\partial \mathbf{K}_i(\boldsymbol{\xi})}{\partial \xi_i}\mathbf{u}_0 + \mathbf{K}_0\frac{\partial \mathbf{u}_i(\boldsymbol{\xi})}{\partial \xi_i} = \mathbf{f}, \tag{II.84}$$

and the set of recursive equations reads,

$$\mathbf{u}_0 = \mathbf{K}_0^{-1}\mathbf{f}, \tag{II.85}$$

$$\frac{\partial \mathbf{u}_i(\boldsymbol{\xi})}{\partial \xi_i} = -\mathbf{K}_0^{-1}\frac{\partial \mathbf{K}_i(\boldsymbol{\xi})}{\partial \xi_i}\mathbf{u}_0, \tag{II.86}$$

where, the first equation gives the first-order approximation of the expected value of the displacement vector, the second equation gives the first-order perturbation of the displacement vector, and so on as far as more equations are considered. It is seen that high-order perturbations are estimated recursively from low-order ones. From the finite element equilibrium equation, the response solution can be found as $\mathbf{u} = \mathbf{K}^{-1}\mathbf{f}$. Furthermore, under the assumption that the structural system is being affected by random spatial variabilities, the global stiffness matrix can be decomposed as

$$\mathbf{K} = \mathbf{K}_0 + \Delta\mathbf{K}, \tag{II.87}$$

where, $\mathbf{K}_0$ represents the mean stiffness matrix and $\Delta\mathbf{K}$ is constituted by the deviatoric parts of the associated components in $\mathbf{K}$, i.e., $\Delta\mathbf{K} = \mathbf{K} - \mathbf{K}_0$. Since it is clear that $\mathbf{K}$ is close to the mean stiffness matrix $\mathbf{K}_0$ (in the sense that, $\lim_{k\to\infty}\left(\mathbf{I} - \mathbf{K}_0^{-1}\mathbf{K}\right)^k = 0$), then $\mathbf{K}^{-1}$ can be expressed using Neumann series as

$$\begin{aligned}\mathbf{K}^{-1} &= \sum_{k=0}^{\infty}\left(\mathbf{I}-\mathbf{K}_0^{-1}\mathbf{K}\right)^k\mathbf{K}_0^{-1} = \sum_{k=0}^{\infty}\left(\mathbf{K}_0^{-1}\left(\mathbf{K}_0-\mathbf{K}\right)\right)^k\mathbf{K}_0^{-1} = \sum_{k=0}^{\infty}\left(-\mathbf{K}_0^{-1}\Delta\mathbf{K}^k\mathbf{K}_0^{-1}\right)\\ &= \left(\mathbf{I}-\left(\mathbf{K}_0^{-1}\Delta\mathbf{K}\right)+\left(\mathbf{K}_0^{-1}\Delta\mathbf{K}\right)^2-\left(\mathbf{K}_0^{-1}\Delta\mathbf{K}\right)^3+\cdots\right)\mathbf{K}_0^{-1}, \qquad \text{(II.88)}\end{aligned}$$

or, after denoting $\mathbf{P}=\mathbf{K}_0^{-1}\Delta\mathbf{K}$, as

$$\mathbf{K}^{-1}=\left(\mathbf{I}-\mathbf{P}+\mathbf{P}^2-\mathbf{P}^3+\cdots\right)\mathbf{K}_0^{-1}. \qquad \text{(II.89)}$$

Replacing eq (II.89) into $\mathbf{u}=\mathbf{K}^{-1}\mathrm{f}$, the solution can be represented by the series

$$\mathbf{u}=\left(\mathbf{I}-\mathbf{P}+\mathbf{P}^2-\mathbf{P}^3+\cdots\right)\mathbf{K}_0^{-1}\mathbf{f}, \qquad \text{(II.90)}$$

but, since the mean solution $\mathbf{u}_0$ associated to $\mathbf{K}_0$ is given by $\mathbf{u}_0=\mathbf{K}_0^{-1}\mathbf{f}$, the expression becomes

$$\mathbf{u}=\mathbf{u}_0-\mathbf{P}\mathbf{u}_0+\mathbf{P}^2\mathbf{u}_0-\mathbf{P}^3\mathbf{u}_0+\cdots=\mathbf{u}_0-\mathbf{u}_1+\mathbf{u}_2-\mathbf{u}_3+\cdots \qquad \text{(II.91)}$$

This series solution is equivalent to the recursive equation

$$\mathbf{K}_0\mathbf{u}_k=\Delta\mathbf{K}_{k-1} \quad \text{for} \quad k=1,2,\ldots \qquad \text{(II.92)}$$

which is solved for each $\mathbf{u}_k$ up to a desired truncation order. Here, the initial $\mathbf{u}$ is given by the mean solution $\mathbf{u}_0$.

II.4 Monte Carlo Simulation [14]

II.4.1 Neumann Series Expansion Method

Expanding the inverse of the stochastic stiffness matrix in a Neumann series, the solution of the static problem of

$$\mathbf{P}=\left(\mathbf{K}_0+\Delta\mathbf{K}\right)\mathbf{U}, \qquad \text{(II.93)}$$

can be written as

$$\mathbf{U}=\left(\mathbf{K}_0+\Delta\mathbf{K}\right)^{-1}\mathbf{P}=\left(\mathbf{I}+\mathbf{K}_0^{-1}\Delta\mathbf{K}\right)^{-1}\mathbf{K}_0^{-1}\mathbf{P}, \qquad \text{(II.94)}$$

where I is the identity matrix. Introducing $\mathbf{J}=\mathbf{K}_0^{-1}\mid\Delta\mathbf{K}^{-1}$, the inverse of $\left(\mathbf{K}_0+\Delta\mathbf{K}\right)$ can be expanded in Neumann series as

$$\begin{aligned}\left(\mathbf{K}_0+\Delta\mathbf{K}\right)^{-1} &= \left(\mathbf{I}-\mathbf{J}+\mathbf{J}^2-\mathbf{J}^3+\cdots\right)\mathbf{K}_0^{-1}\\ &= \sum_{k=0}^{\infty}\left(-\mathbf{K}_0^{-1}\Delta\mathbf{K}\right)^k\mathbf{K}_0^{-1}. \qquad \text{(II.95)}\end{aligned}$$

Consequently, the response can be expressed with the following series

$$\mathbf{U}=\left(\mathbf{I}-\mathbf{J}+\mathbf{J}^2-\mathbf{J}^3+\cdots\right)\mathbf{U}_0, \qquad \text{(II.96)}$$

or

$$\mathbf{U}=\mathbf{U}_0-\mathbf{U}_1+\mathbf{U}_2-\mathbf{U}_3+\cdots \qquad \text{(II.97)}$$

The solution can also be expressed in the following recursive equation

$$\mathbf{K}_0\mathbf{U}_i = \Delta\mathbf{K}\mathbf{U}_{i-1}, \tag{II.98}$$

where $\mathbf{U}_0 = \mathbf{K}_0^{-1}\mathbf{P}$. The number of terms required in the series can be obtained with the following convergence criterion ‘

$$\frac{\|\mathbf{U}_i\|}{\left\|\sum_{k=0}^{i}(-1)^k\mathbf{U}_k\right\|} \leq \varepsilon_1, \tag{II.99}$$

where ε_1 is the convergence tolerance criterion which is checked after each calculation of an additional term in eq (II.97). The Neumann series converges if and only if all absolute eigenvalues of J are less than unit. This implies that convergence is guaranteed only if parameter variability is small (less than $20-30\%$).

II.4.2 The Weighted Integral Method

This method was developed in the early $90s$ and is claimed not to require any discretization of the stochastic field. Since the elements of matrix $\mathbf{B}$ are obtained by differentiation of the element shape functions, they are going to be polynomials on (x, y, z) of some degree lower than the polynomial approximation of the displacements. Therefore, the stochastic part of the stiffness matrix in equation. $k^e = k_e^0 + \Delta k_e$ can be rewritten as

$$\Delta\mathbf{k}_{ij}^e(\theta) = \int_{\Omega_e} \mathbf{P}_{ij}(\mathbf{x})X(\mathbf{x},\theta)\mathrm{d}\Omega_e, \tag{II.100}$$

where the coefficients of polynomial $\mathbf{P}_{ij}$ are obtained from those of matrices $\mathbf{B}$ and D. We can write P_{ij} as

$$\mathbf{P}_{ij}(\mathbf{x}) = \mathbf{P}_{ij}(x,y,z) = \sum_{l=1}^{N_w} a_{ij}^l x^{a_l} y^{\beta_l} z^h, \tag{II.101}$$

where N_w is the number of monomials in $\mathbf{P}_{ij}$, each of them corresponding to a set of exponents (a_l, β_l, γ_l). Introducing the following weighted integrals for a random field $X(\mathbf{x}, \theta)$

$$\chi_l^e(\theta) = \int_{\Omega_e} x^{a_l} y^{\beta_l} z^{\gamma_l} X(\mathbf{x},\theta)\mathrm{d}\Omega_e, \tag{II.102}$$

it follows that

$$\Delta\mathbf{k}_{ij}^e(\theta) = \sum_{l=1}^{N_w} a_{ij}^l \chi_l^e(\theta). \tag{II.103}$$

Collecting now the coefficients a_{ij}^l in a matrix $\Delta\mathbf{k}_l^e$, the (stochastic) element stiffness matrix can be written as:

$$\mathbf{k}^e(\theta) = \mathbf{k}_0^e + \sum_{l=1}^{N_w} \Delta\mathbf{k}_l^e \chi_l^e(\theta). \tag{II.104}$$

In the above equation, $\mathbf{k}_0^e$ and $\Delta\mathbf{k}_l^e$, $l = 1, \ldots, N_w$ are deterministic matrices and χ_l^e are weighted integrals which are random variables. For example, a 2-node truss element has $N_w = 1$, a 2-node two-dimensional beam $N_w = 2$, and a 4-node plane stress quadrilateral element has $N_w = 3$.

The stochastic part of the global stiffness matrix is given by

$$\mathbf{K}(\theta) = \sum_{e=1}^{N_e} \left(\mathbf{k}_0^e + \sum_{l=1}^{N_w} \Delta \mathbf{k}_l^e \chi_l^e(\theta) \right), \tag{II.105}$$

where N_e is the total number of finite elements.

II.5 Method of perturbation [2]

Like the previous method, this method, which has been the subject of numerous developments and applications over the past twenty years. concerns the problem $G(\boldsymbol{q}) = 0$ in its form $G(\boldsymbol{q}) = \mathrm{k}\boldsymbol{q} - \boldsymbol{f}$. It assumes that M scalar parameters $y_1, \ldots, y_M$, grouped in a vector y and distributed in the expressions of k and $\boldsymbol{f}$, are affected by uncertainty, that this uncertainty is correctly taken into account by a modeling of y by a second-order v.a $\mathbf{Y}$ with values $\mathbb{R}^M$ and that this v.a. is known by its mean $\mu_{\mathbf{Y}} = \mathbf{E}[\mathbf{Y}] \in \mathbb{R}^M$ and its matrix of covariance $C_{\mathbf{Y}} = \mathbf{E}\left[(\mathbf{Y} - \mu_{\mathbf{Y}}) (\mathbf{Y} - \ mu_{\mathbf{Y}})^T \right] \in \mathbb{R}^{M \times M}$. She then tries to provide an approximation of the mean $\mu_{\mathbf{Q}} = \mathbf{E}[\mathbf{Q}] \in \mathbb{R}^m$ and the covariance matrix $C_{\mathbf{Q}} = \mathbf{E}\left[(\mathbf{Q} - \mu_{\mathbf{Q}}) (\mathbf{Q} - \mu_{\mathbf{Q}})^T \right] \in \mathbb{R}^{m \times m}$ of the system's random response i.e. of the v.a. m-dimensional $\mathbf{Q}$ related to $\mathbf{Y}$ by

$$k(\mathbf{Y})\mathbf{Q} = \boldsymbol{f}(\mathbf{Y}), \tag{II.106}$$

and such that, formally, the v.a. Q to be characterized is written

$$\mathbf{Q} = g(\mathbf{Y}) = \mathrm{k}^{-1}(\mathbf{Y})\boldsymbol{f}(\mathbf{Y}). \tag{II.107}$$

For this, it relies on the formulation eq (II.106) in which $\mathbf{Q}$ is replaced by $g(\mathbf{Y})$, in other words

$$k(\mathbf{Y})g(\mathbf{Y}) = \boldsymbol{f}(\mathbf{Y}). \tag{II.108}$$

It further assumes that the random terms $\mathrm{k}(\mathbf{Y})$, $g(\mathbf{Y})$ and $\boldsymbol{f}(\mathbf{Y})$ of eq (II.108) are not very sensitive to the randomness carried by $\mathbf{Y}$. Under these conditions, considering the Taylor expansions in the neighborhood of μ_{Y} of the functions k, g and $\boldsymbol{f}$, we can truncate these expansions to first or second order. In the latter case, the v.a. $\mathrm{k}(\mathbf{Y})$, $g(\mathbf{Y})$ and $\boldsymbol{f}(\mathbf{Y})$ can be approximated by

$$\mathrm{k}(\mathbf{Y}) \simeq \mathrm{k}_0 + \sum_{i=1}^{M} \mathrm{k}_{\mathrm{i}}^I \overset{\circ}{\mathrm{Y}}_{\mathrm{i}} + \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \mathrm{k}_{ij}^{II} \overset{\circ}{\mathrm{Y}}_{\mathrm{i}} \overset{\circ}{\mathrm{Y}}_{\mathrm{j}}, \tag{II.109}$$

$$g(\mathbf{Y}) \simeq g_0 + \sum_{i=1}^{M} g_i^I \overset{\circ}{\mathrm{Y}}_{\mathrm{i}} + \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} g_{ij}^{II} \overset{\circ}{\mathrm{Y}}_{\mathrm{i}} \overset{\circ}{\mathrm{Y}}\ \mathrm{j}, \tag{II.110}$$

$$\boldsymbol{f}(\mathrm{Y}) \simeq \boldsymbol{f}_0 + \sum_{i=1}^{M} \boldsymbol{f}_i^I \overset{\circ}{\mathrm{Y}}_{\mathrm{i}} + \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \boldsymbol{f}_{ij}^{II} \mathrm{Y}_{\mathrm{i}}^{\circ} \mathrm{Y}_{\mathrm{j}}. \tag{II.111}$$

With

$$\mathrm{k}_0 = \mathrm{k}\left(\mu_{\mathbf{Y}}\right), \quad g_0 = g\left(\mu_{\mathbf{Y}}\right) = \mathrm{k}^{-1}\left(\mu_{\mathbf{Y}}\right) \boldsymbol{f}\left(\mu_{\mathbf{Y}}\right), \quad \boldsymbol{f}_0 = \boldsymbol{f}\left(\mu_{\mathbf{Y}}\right), \tag{II.112}$$

$$\mathrm{k}_{\mathbf{i}}^I = \frac{\partial \mathrm{k}}{\partial y_i}\left(\mu_{\mathbf{Y}}\right), \quad g_i^I = \frac{\partial g}{\partial y_i}\left(\mu_{\mathbf{Y}}\right), \quad \boldsymbol{f}_i^I = \frac{\partial \boldsymbol{f}}{\partial y_i}\left(\mu_{\mathbf{Y}}\right), \tag{II.113}$$

$$\mathrm{k}_{ij}^{II} = \frac{\partial^2 \mathrm{k}}{\partial y_i \partial y_j}\left(\mu_{\mathbf{Y}}\right), \quad g_{ij}^{II} = \frac{\partial^2 g}{\partial y_i \partial y_j}\left(\mu_{\mathbf{Y}}\right), \quad \boldsymbol{f}_{ij}^{II} = \frac{\partial^2 \boldsymbol{f}}{\partial y_i \partial y_j}\left(\mu_{\mathbf{Y}}\right), \tag{II.114}$$

$$\overset{\circ}{\mathrm{Y_i}} = \mathrm{Y_i} - \mu_{\mathrm{Y_i}}, \quad \forall i \in \{1, \ldots, M\}, \tag{II.115}$$

where $\mu_{\mathbf{Y}_i}$ is the mean of the coordinate $\mathrm{Y_i}$ of the random vector $\mathbf{Y}$. The coefficients to be determined are g_0, g_i^I and g_{ij}^{II}. By transferring eq (II.109), eq (II.110) and eq (II.111) into eq (II.108) and by identifying term by term, we get

$$g_0 = \mathrm{k}_0^{-1} \boldsymbol{f}_0, \tag{II.116}$$

$$g_i^I = \mathrm{k}_0^{-1} \left(\boldsymbol{f}_i^I - \mathrm{k}_i^I g_0 \right), \tag{II.117}$$

$$g_{ij}^{II} = \mathrm{k}_0^{-1} \left(\boldsymbol{f}_{ij}^{II} - \mathrm{k}_i^I g_j^I - \mathrm{k}_j^I g_i^I - \mathrm{k}_{ij}^{II} g_0 \right). \tag{II.118}$$

As a result, the v.a. Q given by eq (II.107) can be approximated by

$$\tilde{Q} = g_0 + \sum_{i=1}^{M} g_i^I \left(\mathrm{Y_i} - \mu_{\mathrm{Y}_i} \right) + \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} g_{ij}^{II} \left(\mathrm{Y}_i - \mu_{\mathrm{Y}_i} \right) \left(\mathrm{Y_j} - \mu_{\mathrm{Y}_j} \right), \tag{II.119}$$

with g_0, g_i^I and g_{ij}^{II} given by eq (II.116)-eq (II.118). The sought approximations of $\mu_{\mathbf{Q}}$ and $C_{\mathbf{Q}}$ are then taken equal to the corresponding characteristics of $\tilde{\mathbf{Q}}$, which are calculated explicitly. Note that very often, we limit ourselves to the first two terms of the approximation eq (II.119), in other words, we consider it in the form

$$\tilde{\mathbf{Q}} = g_0 + \sum_{i=1}^{M} g_i^I \left(\mathrm{Y_i} - \mu_{\mathbf{Y}_i} \right). \tag{II.120}$$

In this case, we obtain the following approximations for $\mu_{\mathbf{Q}}$ and $C_{\mathbf{Q}}$

$$\begin{aligned} \mu_{\mathbf{Q}} \simeq \mu_{\tilde{\mathbf{Q}}} &= \mathbf{E}[\tilde{\mathbf{Q}}] = g_0. \\ C_{\mathbf{Q}} \simeq C_{\tilde{\mathbf{Q}}} &= \mathbb{E}\left[\left(\tilde{\mathbf{Q}} - \mu_{\tilde{\mathbf{Q}}} \right) \left(\tilde{\mathbf{Q}} - \mu_{\tilde{\mathbf{Q}}} \right)^T \right] = \sum_{i=1}^{M} \sum_{j=1}^{M} g_i^I \left(g_j^I \right)^T C_{\mathrm{Y_i Y_j}}, \end{aligned} \tag{II.121}$$

where $C_{\mathrm{Y_i Y_j}}$ is the rank term (i, j) of the covariance matrix C_{Y}, i.e. the covariance of the coordinates Y_i and Y_j of the random vector Y

$$C_{\mathrm{Y_i Y_j}} = \mathbf{E}\left[\left(\mathrm{Y_i} - \mu_{\mathrm{Y_i}} \right) \left(\mathrm{Y_j} - \mu_{\mathrm{Y_j}} \right) \right]. \tag{II.122}$$

II.6 The quadrature method [2]

The quadrature method makes it possible to estimate the moments of order i of a v.a. $\mathbf{Z} = g(\mathbf{Y})$, this method is in fact a numerical integration method of the diagram type of Gauss. In the simpli e case where the v.a. $\mathbf{Z}$ and $\mathbf{Y}$ are scalar, i.e. $\mathbf{Z} = Z_1 = Z$ and $\mathbf{Y} = Y_1 = Y$, the quadrature method then consists in approaching the moment of order i of the v.a. Z by a sum of the form

$$\mu_i^Z \simeq \sum_{k=1}^{N} g^i(y_k) \omega_k, \tag{II.123}$$

where $(y_k, \omega_k)_{k=1;N}$ are N pairs of points and weights de ned using the recurrence relation next

$$\Phi_{-1}(y) = 0, \;\; \Phi_0(y) = 1, \;\; \Phi_{j+1}(y) = (y - a_j)\Phi_j(y) - b_j(y)\Phi_{j-1}(y). \tag{II.124}$$

This relation is valid for any family of orthogonal polynones Φ_j whose coefficient of monome of highest degree is equal to 1, with

$$a_j = \frac{\mathbb{E}[y\Phi_j|\Phi_j]}{\Phi_j|\Phi_j}, \;\; b_j = \frac{\mathbb{E}[\Phi_j|\Phi_j]}{\Phi_{j-1}|\Phi_{j-1}} \;\; \forall j \in \mathbb{N}. \tag{II.125}$$

The weights $w_k$are deduced from the relation

$$w_k = \frac{\mathbb{E}[\Phi_{N-1}|\Phi_{N-1}]}{\dfrac{\partial \Phi_N}{\partial y_k}(y_k)\Phi_{N-1}(y_k)}. \tag{II.126}$$

The points y_k are the roots of the polynomials Φ_N.
In a nonlinear, static or dynamic framework, for a moderate number of v.a., the quadrature seems an interesting alternative to simulations. A limitation of the approach is that the number of evaluations of the function f then amounts to N^M, where M is the dimension of $\mathbf{Y}$, which makes computation times prohibitive.
We note in n some applications to geotechnical problems, of beam, of a sphere under pressure, of reinforced concrete.Following the estimation of several statistical moments, a probability density e of failure, then a probability of failure are possibly estimated.
If the dimension M is too large, Monte-Carlo simulations and variants are more interesting.

Chapter III

Application of Stochastic Finite Elements in Mechanics

In this chapter, we study an applied work for some finite stochastic element methods, the Monte Carlo method and the quadratics method by applying them to a straight pipe cracked and the reliability of the cracked plate.

III.1 Cracked straight pipe [7]

We propose to study here the case of a cracked straight piping in internal skin subjected to a combined pressure and tension loading (see Figure III.1).
The parameters of the mechanical model are as follows

- R_i: internal radius of the pipe (mm),
- t: pipe thickness (mm),
- L: pipe length (mm),
- a: crack length (mm),
- P: internal pressure (MPa),
- σ_f: tensile stress due to bottom effect (MPa),
- σ_t: applied tensile stress (MPa).

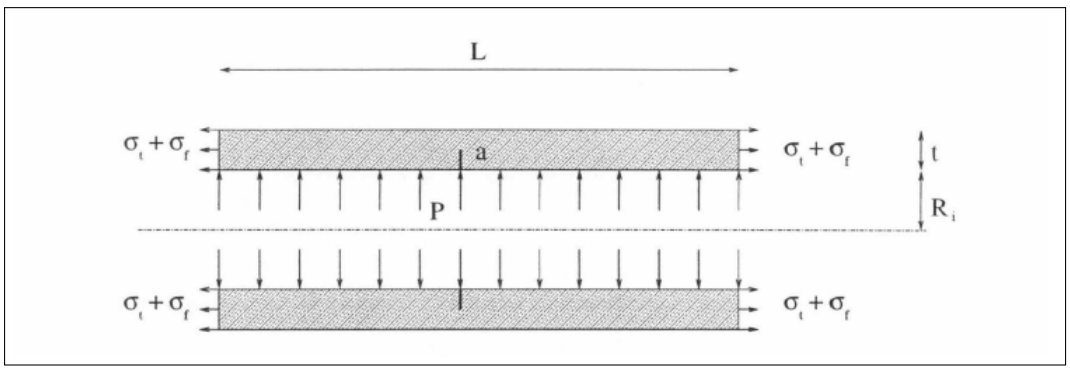


Figure III.1: Cracked straight piping: definition of the circumferential axisymmetric defect subjected to a loading of pressure P and traction σ_t and σ_f. a is the crack length, L the length of the pipe, t its thickness and R_i his internal radius.

The tensile stress due to bottoming is given by

$$\sigma_f = P\left(\frac{R_i^2}{(R_i+t)^2 - R_i^2}\right). \tag{III.1}$$

The behavior of the material is assumed to be described by a Ramberg-Osgood law, whose relation stress-strain in the one-dimensional case can be written

$$\epsilon = \frac{\sigma}{E} + \alpha\frac{\alpha_0}{E}\left(\frac{\alpha}{\alpha_0}\right)^n, \tag{III.2}$$

with

- ϵ: deformation,
- α: stress (MPa),
- α_0: elastic limit v,
- n: hardening coefficient,
- E: Young's modulus (MPa),
- α: smoothing factor.

The traction curve is given in figure III.2.

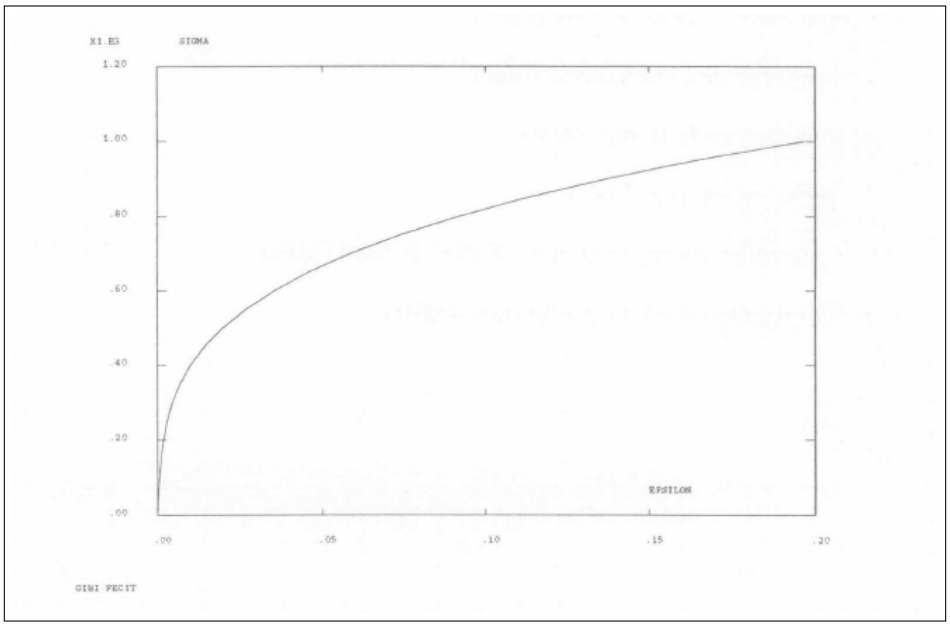


Figure III.2: Cracked straight pipe: Ramberg-Osgood tensile curve (stress versus strain).

III.1.1 Modeling of the tension curve

We will discretize the curve of traction in the following way

- for a stress between 0 and $50MPa$, the deformation is deduced from the linear relation $\epsilon = \frac{\sigma}{E}$.
- for a stress greater than $50MPa$, the strain is deduced from equation (III.2) and the discretization points are chosen every $50MPa$.

III.1.2 Finite element modeling

Taking into account the symmetries of the problem, one modelizes only one half of the piping using axisymmetric elements (see figure III.3).

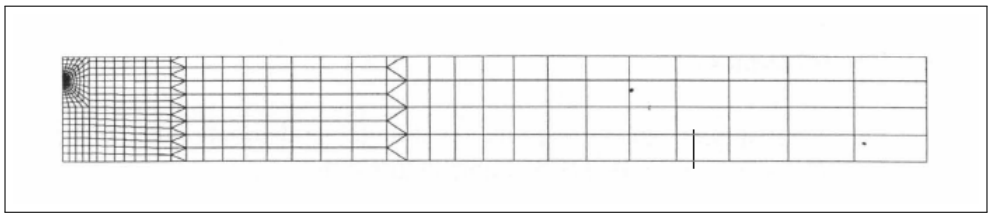


Figure III.3: Cracked straight piping: piping mesh.

III.1.3 Reliability model

The structural failure scenario is the one that corresponds to crack propagation. The limit state is then given by the comparison between the value of the integral J and the toughness $J_{0,2}$ of the material. The reliability of the structure is evaluated for different values of the tensile stress ($100MPa$, $120MPa$ and $140MPa$). The probabilities of failure obtained.
Table III.1 describes all the parameters of the mechanical model as well as the laws they follow, their mean and their standard deviation.

Name	name	Unity	Law	Mean	Ecart-type
Internal radius	R_i	mm	Determinist	393,5	
Thickness	t	mm	Determinist	62,5	
Defect size	a	mm	Determinist	15	
Hose length	L	mm	Determinist	1000	
Internal pressure	P	MPa	Determinist	15,5	
Tensile stress	σ_t	MPa	Determinist	Variabel	
Young's modulus	E	MPa	Log-Normal	175500	10000
Poisson coefficient	v	-	Determinist	0,3	
Elastic limit	σ_0	MPa	Log-Normal	259,5	10
Hardening factor	n	-	Normal	3,5	0,1
Smoothing Parameter	α	-	Normal	1,15	0,15
Material toughness	$J_{0,2}$	MPa.mm	Log-Normal	52	9,5

Table III.1: Cracked straight piping: parameters of the mechanical model Results

Where
If W_e is the elastic strain energy density, we can write

$$\sigma_{ij} = \frac{\partial W_e}{\partial \epsilon^e_{ij}}. \tag{III.3}$$

Then the Rice integral J is defined by

$$J = \int_C \left(W_e n_1 - \sigma_{ij} n_j \frac{\partial u_i}{\partial x_1} \right) ds. \tag{III.4}$$

III.1.4 Results

We used here a quadrature method to determine the statistical parameters of J. We used four integration points for each random variable. This theoretically represents 64 calculations. However, only about thirty of the integration weights are representative: we We have only carried out the calculations associated with the most important weights. A five point method was also used. It gives the same results.
The first four statistical moments are then used with a Johnson method to determine the probability densities associated with J.

Statistical moments

The statistical moments obtained using the quadrature method are given in the table (III.2) for different values of tensile stress.

σ_t	Mean	Standard deviation	Symmetry	Flattening
100 MPa	14,4353	1,095716	0,26650	3,13993
120 MPa	20,6893	1,733331	0,28656	3,16755
140 MPa	28,9024	2,640808	0,30972	3,18493

Table III.2: Cracked Straight Piping: First Four Statistical Moments of J as a Function of Tensile Stress.σ_i

Failure probabilities

The comparison of the different results can be found in table (III.3).

		SRQ	Ryfes	Quadrature
$\sigma_t = 140$ MPa.	P_f	$2,02 \times 10^{-3}$	$2,01 \times 10^{-3}$	$2,01 \times 10^{-3}$
	β	2,87	2,87	2,87
$\sigma_t = 120$ MPa.	P_f	$2,08 \times 10^{-6}$	$2,01 \times 10^{-6}$	$2,35 \times 10^{-6}$
	β	4,60	4,62	4,58
$\sigma_t = 100$ MPa.	P_f	$3,49 \times 10^{-11}$	$3,77 \times 10^{-11}$	$2,53 \times 10^{-15}$
	β	6,52	6,51	7,84

Table III.3: Cracked straight piping: Comparison of failure probabilities P_f and reliability indices β by the SRQ method, $RYFES$ software and quadrature method for different values of tensile stress σ_t.

The representations of the joint probability densities between J and $J_{0,2}$, as well as the isovalues of the probabilities, for the different values of the tensile stress ($\sigma_t = 100$, 120 and $140MPa$), are represented in figures (III.4) to (III.5) and (III.6).
The probabilities were obtained directly with the probability distributions found using the method of Johnson.
Let $\delta_{0,2}$ be the probability density of $J_{0,2}$ and δ the probability density of J. These two variables are independent, their joint probability density is given by

$$F_{J,J_{0,2}} = \delta_{0,2} \times \delta. \tag{III.5}$$

The probability of failure P_f is the probabilistic weight of the part of the space constituted by the domain $D_f = J > J_{o,2}$,

$$P_f = Prob(J > J_{o,2}) = \int_{D_f} \delta_{0,2}(j_{o,2})\delta(j)dj_{o,2}dj. \tag{III.6}$$

The probability of failure is then given by one of the following two expressions

$$P_f = \int_{-\infty}^{\infty} \delta(x)\ \triangle_{o,2}\ (x)dx, \tag{III.7}$$

or

$$P_f = \int_{-\infty}^{\infty} (1 - \triangle\ (x))\delta_{o,2}(x)dx, \tag{III.8}$$

where $\triangle_{o,2}$ is the distribution function of $J_{o,2}$ and $\triangle$ is the distribution function of J.

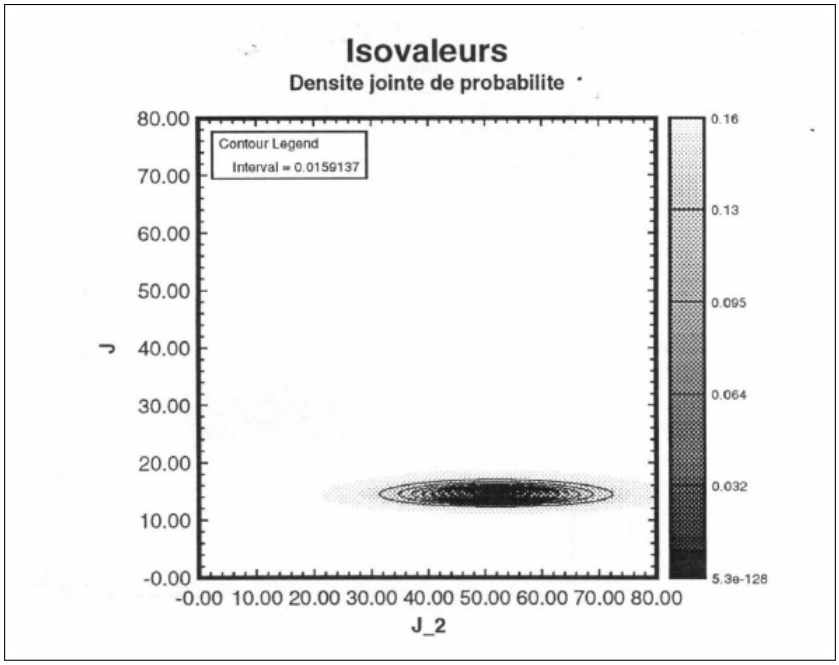


Figure III.4: Cracked Straight Pipe: Probability Isovalues for Tensile Stress $\sigma_t = 100MPa$.

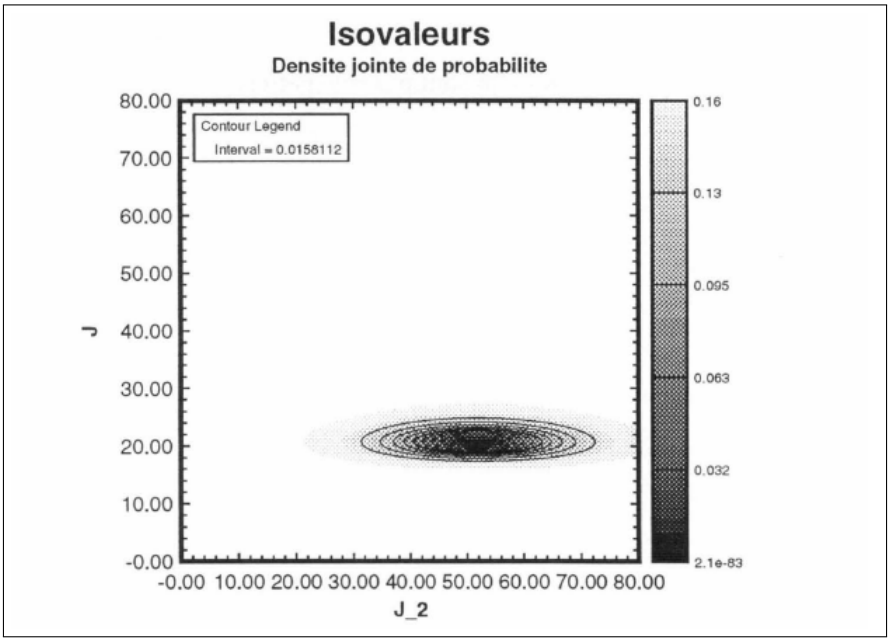


Figure III.5: Cracked Straight Pipe: Probability Isovalues for Tensile Stress $\sigma_t = 120MPa$.

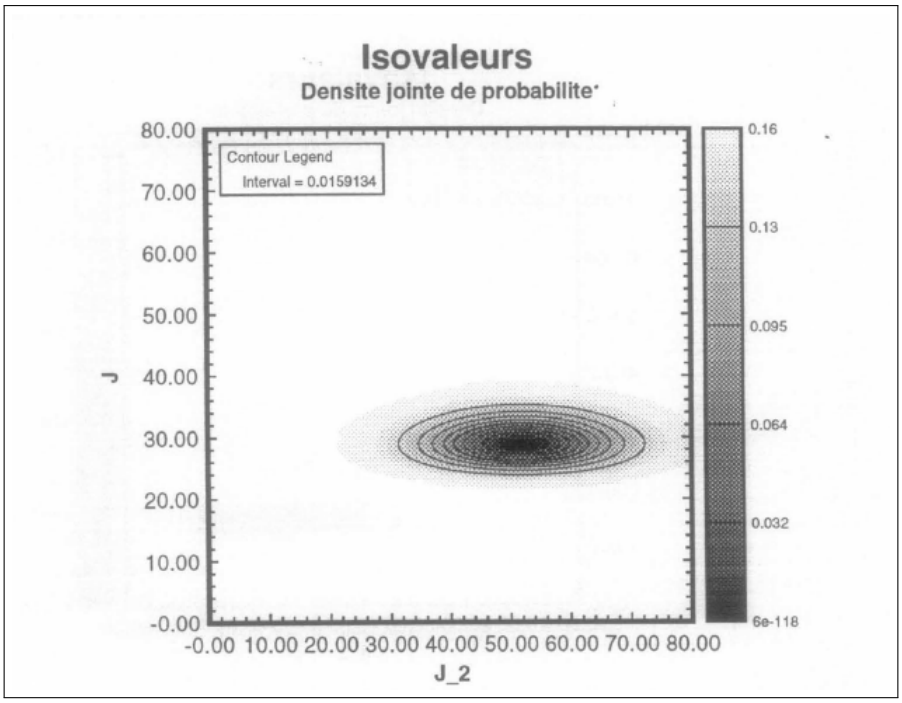


Figure III.6: Cracked Straight Pipe: Probability Isovalues for Tensile Stress $\sigma_t = 140MPa$.

III.1.5 Conclusion

The example that we have just treated shows that the method of quadrature makes it possible to carry out non-linear calculations. Associated with a method of moments, it gives good results on the calculation small probabilities if these are not too small.

III.2 Reliability of a cracked plate [7]

III.2.1 Definition

We will present in this application the evaluation of the reliability, in fracture mechanics linear, of a cracked plate in pure tension. The crack propagation condition is given by the overtaking by J of the tenacity of the material $J_{0,2}$. We will present the reliability results for a plate whose modulus d'Young is a random field.
The notations for the different ratings are defined in Figure III.7.

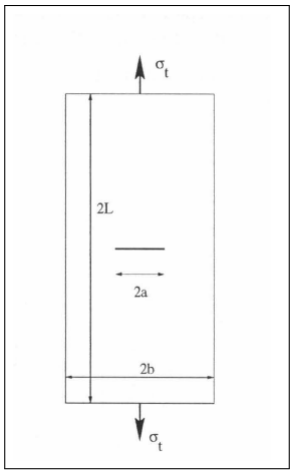


Figure III.7: Cracked plate: definition of parameters. L is the half-height of the plate, b the half-width, a the half crack depth and σ_t the tensile stress.

The values of the various parameters, as well as the average value of the Young's modulus, are given in Table III.4.

Name	name	Unity	Law	Mean	Ecart-type
Tensile stress	σ_t	MPa	Determinist	250	-
crack depth	a	m	Determinist	0,05	-
Width	b	m	Determinist	0,15	-
Height	L	m	Determinist	0,30	-
Poisson coefficient	v	-	Determinist	0,3	-
Young's modulus	E	MPa	Normale	216000	12960

Table III.4: Cracked finished plate: mechanical model parameters.

III.2.2 Validation of finite element modeling

For the validation of the finite element model, we have two approximate formulations of the analytical solution of the stress intensity factor K_I

$$K_I = \sigma_t \sqrt{a\pi} f_{(a,b)}, \tag{III.9}$$

or

$$f_{(a,b)} = \left(1 - 0,025(a/b)^2 + 0,06(a/b)^4\right) / \sqrt{\cos \frac{\pi a}{2b}}, \tag{III.10}$$

$$f_{(a,b)} = \left(1 - 0,5(a/b) + 0,370(a/b)^2 - 0,044(a/b)^3\right) / \sqrt{1 - a/b}. \tag{III.11}$$

The first equation is obtained from Feddersen's formula and gives a result at 0.1%.
The second is obtained from Koiter's formula and gives a result of 0.3%.
For the calculations, we will use the mesh of Figure III.8. To simplify the problem, only one quarter of the plate was meshed.

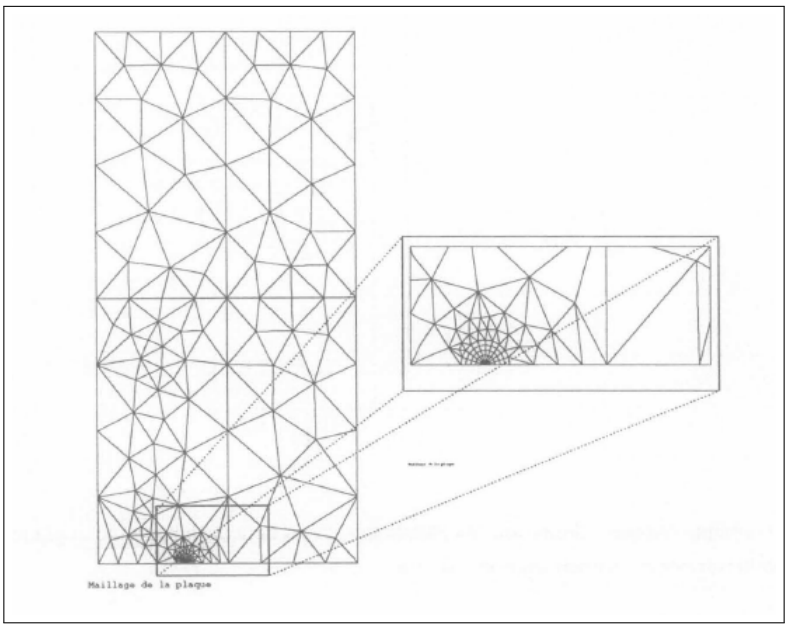


Figure III.8: Cracked plate: mesh of a quarter of the plate.

Numerical evaluation of equations (III.10) and (III.11) give respectively $K_I = 106,46 MPa.m^{\frac{1}{2}}$ and $K_I = 105,92 MPa.m^{\frac{1}{2}}$. The finite element calculation gives $K_I = 105,96 MPa.m^{\frac{1}{2}}$, which validates the choice of mesh.

III.2.3 limit state

The condition for crack propagation in elastic media is given by the overshoot, by the integral of Rice J, of the tenacity $J_{0,2}$ of the material.
The probability of failure is therefore given by

$$P_f = Prob\left[J \geq J_{0,2}\right]. \tag{III.12}$$

The limit state function is then

$$G = J_{0,2} - J. \tag{III.13}$$

III.2.4 Plate reliability with spatial variability of Young's modulus

Definition and discretization of the random field
The modulus of elasticity is considered as a variable exhibiting spatial variability. He will be considered as a Gaussian field. this one is discretized into four domains (figure III.9) by the mean point method.

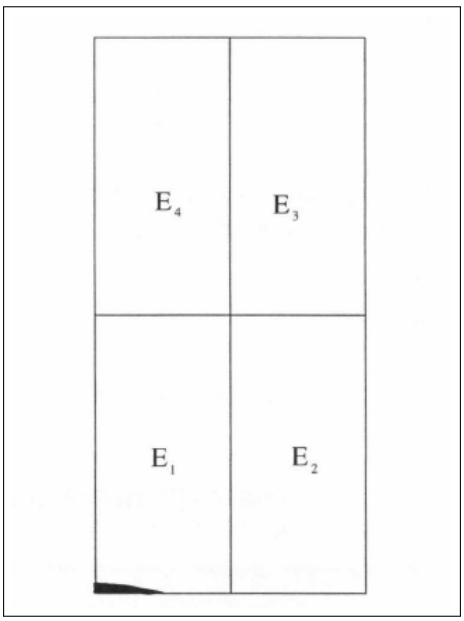


Figure III.9: Cracked finished plate: mesh of discretization of the random field representing the Young's modulus.

We will assume that the correlation coefficient between two random variables E_i and E_j is obtained by

$$\rho_{i,j} = \exp\left[-\left(\frac{\triangle_x}{\lambda L_x}\right)^2 - \left(\frac{\triangle_y}{\lambda L_y}\right)^2\right], \tag{III.14}$$

where $\triangle_x$ and $\triangle_y$ are the x and y coordinate differences between discretized domains i and j, and λ is the correlation distance. Each random variable E_i has the same mean of $216000MPa$ and the same standard deviation of $12960MPa$, which represents a coefficient of variation equal to 6%.
By asking

$$e_\lambda = \exp\left(-\left(\frac{1}{2\lambda}\right)^2\right), \tag{III.15}$$

$$e_{2\lambda} = \exp\left(-2\left(\frac{1}{2\lambda}\right)^2\right), \tag{III.16}$$

the correlation matrix. ($\rho_{i,j}$) can be written

$$(\rho_{i,j}) = \begin{pmatrix} 1 & e_\lambda & e_{2\lambda} & e_\lambda \\ e_\lambda & 1 & e_\lambda & e_{2\lambda} \\ e_{2\lambda} & 1 & e_\lambda & 1 \\ e_\lambda & 1 & e_{2\lambda} & e_\lambda \end{pmatrix}. \tag{III.17}$$

Using a Cholesky decomposition, we will represent this matrix in triangular form.
We thus obtain

$$(\rho_{i,j}) = QQ^T, \tag{III.18}$$

with

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ e_\lambda & \sqrt{1-e_{2\lambda}} & 0 & 0 \\ e_{2\lambda} & e_\lambda\sqrt{1-e_{2\lambda}} & \sqrt{1-e_{2\lambda}} & 0 \\ e_\lambda & 0 & e_\lambda\sqrt{1-e_{2\lambda}} & \sqrt{1-2e_{2\lambda}+e_{2\lambda}^2} \end{pmatrix}. \tag{III.19}$$

The vector of correlated Young's moduli. The Q Matrix will be used to generate the four Young's moduli for the Monte-Carlo method.

Monte Carlo method

After 20000 draws (or calculations), we obtain for the integral J the histogram presented on the Figure III.10.

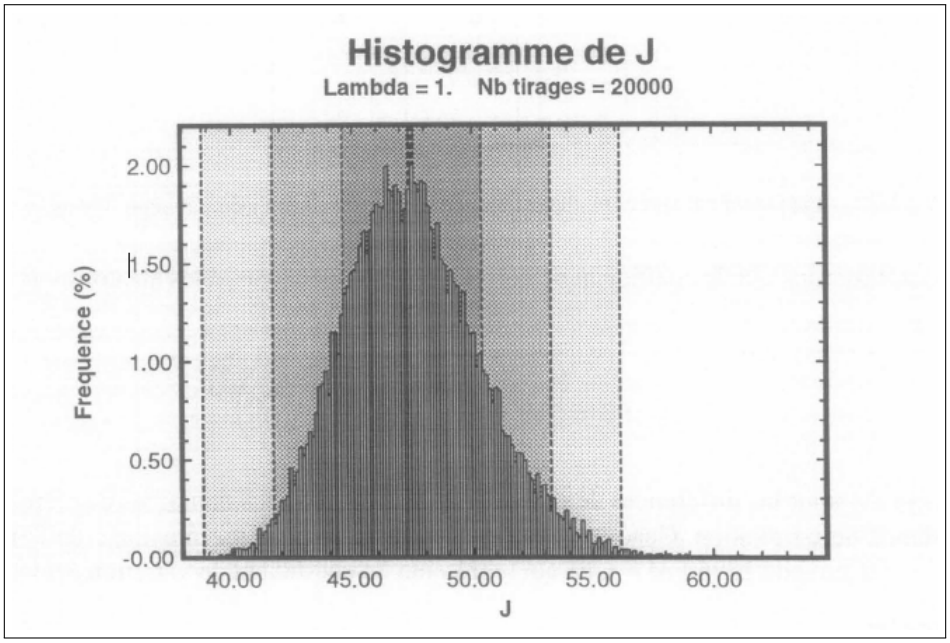


Figure III.10: Cracked finished plate: histogram of J for a random field correlation length λ. equal to 1.

The various statistical parameters calculated for J are given in Table III.5.
The evolutions of the mean, the standard deviation, the symmetry coefficient and the kurtosis coefficient for the first 1000 prints are given in Figures III.11 and III.12

	Mean	Standard deviation	Symmetry	Flattening
J	47,4874	2,8963	0,3777	3,3113

Table III.5: Cracked finished plate: statistical parameters obtained by the Monte-Carlo method.

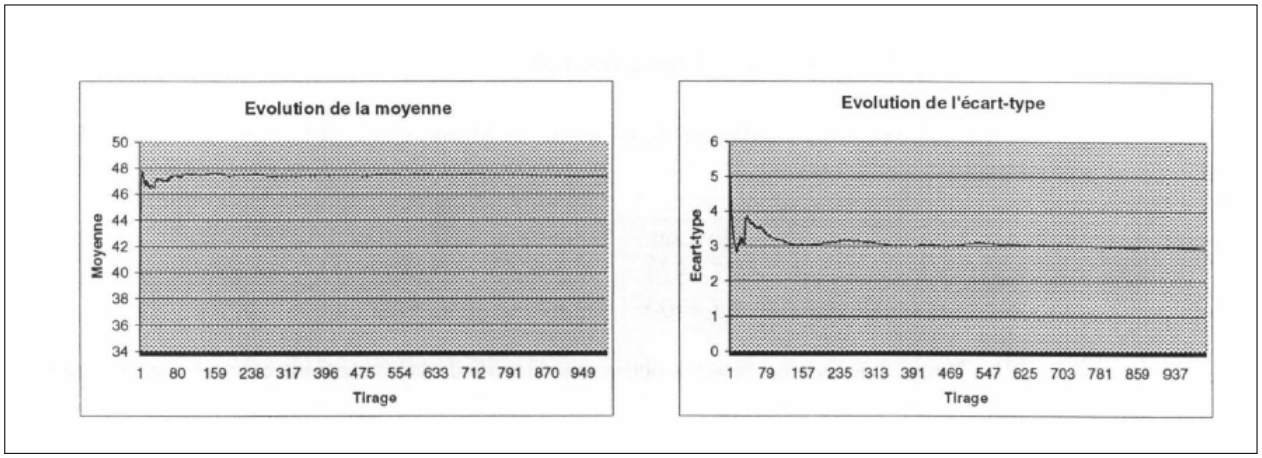


Figure III.11: Cracked finite plate: evolution of the mean and the standard ecat of the J-integral for the first 1000 prints.

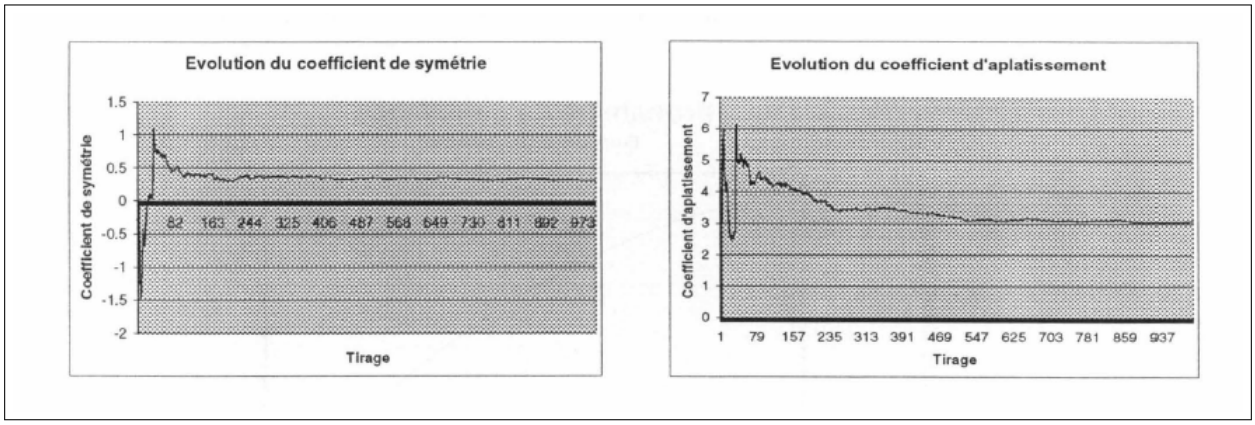


Figure III.12: Cracked finite plate: evolution of the coefficients of symmetry and flattening of the integral $J$for the 1000 first draws.

Figures III.11 and III.12 show how the different statistical parameters vary during the prints. We observe that the higher the statistical moment, the longer the stability takes to obtain. The symmetry and flattening coefficients are not yet fully stabilized after 1000 calculations.

Quadrature Method

To compare the results obtained by the Monte-Carlo simulations, we will use a method 10-point quadrature. The weights and integration points are obtained using the table given in the appendix. The various statistical parameters calculated for J are given in Table III.6.

	Mean	Standard deviation	Symmetry	Flattening
J	47,4700	2,8798	0,3701	3,2774

Table III.6: Cracked finished plate: statistical parameters obtained by the quadrature method.

Comparison of results and calculation of reliability

The statistical parameters calculated by the Monte-Carlo method and the quadrature method are recalled in Table III.7.

Table III.7 shows that the results obtained by the Monte-Carlo method and the method of quadrature are very close for all statistical moments (less than 2% difference), whereas we We

	Mean	Standard deviation	Symmetry	Flattening
Monte-Carlo	47,4874	2,8963	0,3777	3,3113
Quadrature	47,4700	2,8798	0,3701	3,2774

Table III.7: Cracked finished plate: results obtained by the Monte-Carlo method and the quadrature method.

only performed 10 calculations with the quadrature method. The evaluation of the probability of failure, for different values of the toughness $J_{0,2}$ is given in Fig. III.13.

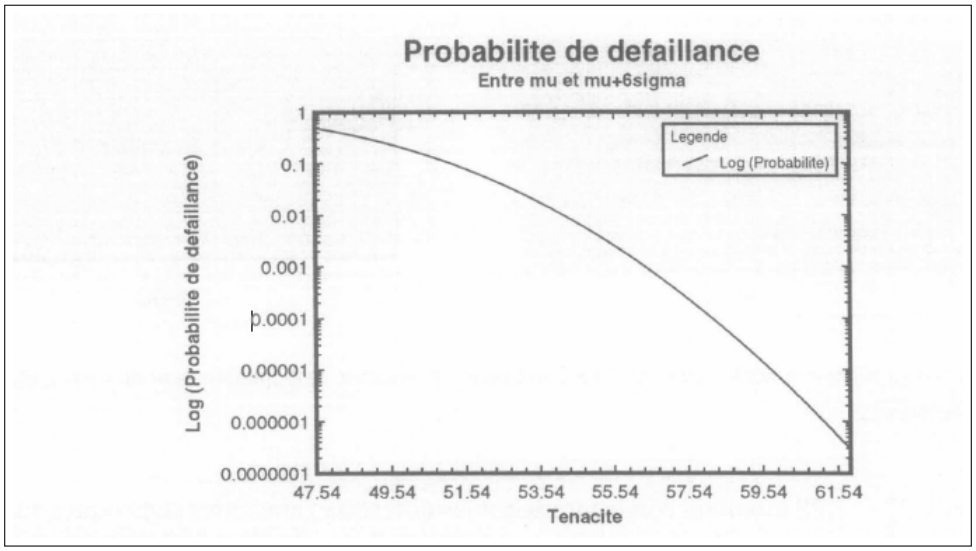


Figure III.13: Cracked finished plate: failure probabilities obtained by the method of moments from the results of the quadrature method, for a toughness between the mean value of J and the mean value of J increased by six standard deviations (logarithmic scale).

III.2.5 Conclusion

The example we have just treated shows how to take into account the spatial variability of a setting. It also shows the power of the quadrature method associated with the method of moments. The probability density obtained allows the immediate calculation of a probability of failure without having to redo all the calculations, which is impossible with reliability methods.

Conclusion general

This work sheds light on the application of finite stochastic elements in mechanics, where we have studied two examples, the first example that has been treated, that the square method makes it possible to perform non-linear arithmetic operations coupled with the method of moments, it gives good results in calculating small probabilities if they are not very small As for the second example In which they dealt with how to take into account the spatial variance of the settings. It also shows the power of the squaring method associated with the moments method. The obtained probability density allows the immediate calculation of the probability of failure without the need to repeat all the calculations, which is impossible with reliability methods.

In the course of preparing this memory, we faced difficulties and obstacles, including the novelty of the topic with the lack of references, as well as the difficulty of the subject itself, because we do not have a background on it due to the lake of time. Nevertheless, we tried with our diligence to facilitate and simplify the subject because we found that it opens a wide future in Bio-mathematics. Also in spectral influences called the spectral method and in other applications such as earthquakes and missiles $\cdots$

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Abstract

In this memory, we presented the method of elements finite stochastic, where we touched on the most important common stochastic processes and also the theory of Lax-Milgram in the deterministic element finite method to prove the existence and uniqueness of the solution. We also mentioned the most important stochastic methods for solving stochastic differential equations (Monte Carlo method, quadrature method, perturbation method), where we applied some of them in mechanics, especially in geo-techniques and fracture mechanics.
Key words: Stochastic processes, the element finite method, elements finites stochastic methods, Monte Carlo method, quadrature method, perturbation method.

Résumé

Dans ce mémoire, nous avons présenté la méthode des éléments finis stochastiques , où nous avons abordé les aspects les plus importants des processus stochastiques communs et aussi la théorie de Lax-Milgram dans le ntionné les méthodes stochastiques les plus importantes pour résoudre les équations différentielles stochastiques, méthode de Monte Carlo, méthode de quadrature, méthode de perturbation, où nous avons appliqué certaines des en mécanique, notamment en géotechniques et en mécanique de la rupture.
Mots clés: Processus stochastiques, des éléments finie méthode, éléments finies stochastiques méthodes (méthode de Monte Carlo, méthode de quadrature, méthode de perturbation).

ملخص

قدمنا في هذه المذكرة طريقة العناصر العشوائية المنتهية، حيث تطرقنا إلى أهم العمليات العشوائية الشائعة و أيضا إلى نظرية لاكس ميلقرام في طريقة العناصر المنتهية الحتمية لإثبات وجود ووحدانية الحل، وكذلك ذكرنا أهم الطرق العشوائية لحل المعادلات التفاضلية العشوائية (طريقة مونت كارلو، طريقة التربيعات، طريقة الإضطراب)، كما طبقنا بعض منها في الميكانيك خاصة في الجيو تقنيات و ميكانيكا الكسر.
الكلمات المفتاحية: العمليات العشوائية. طريقة العناصر المنتهية. طريقة العناصر المنتهية العشوائية. طريقة مونت كارلو. طريقة التربيعات. طريقة الاضطراب.