On the existence of positive solutions of a degenerate reaction diffusion model applied in ecology

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Abstract

The aim of this paper is to show the existence, uniqueness, and asymptotic behavior of time-dependent solutions. Also investigated is the existence of positive maximal and minimal solutions of the corresponding quasilinear elliptic system. The elliptic operators in both systems are allowed to be degenerate in the sense that the density-dependent diffusion coefficients $D_i(u_i)$ may have the property $D_i(0) = 0$ for some or all $i = 1, ..., N$, and the boundary condition is $u_i = 0$. Using the method of upper and lower solutions, we show that a unique global classical time-dependent solution exists and converges to the maximal solution for one class of initial functions and it converges to the minimal solution for another class of initial functions; and if the maximal and minimal solutions coincide then the steady-state solution is unique and the time-dependent solution converges to the unique solution.

Keywords : quasilinear elliptic equations, degenerate reaction diffusion system, method of upper and lower solutions.

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1 Introduction

Degenerate quasilinear parabolic and elliptic equations have received extensive attentions during the past several decades and many topics in the mathematical analysis are well developed and applied to various fields of applied sciences, especially in ecology as in this work.

In this paper, we consider a coupled system of arbitrary number of quasilinear elliptic equations in a bounded domain with Dirichlet boundary condition where the domain is assumed to have the outside sphere property without the usual smoothness condition. The system of equations under consideration is given by

$$\begin{cases} -a_i \nabla . \left(D_i\left(u_i\right) \nabla u_i\right) + b_i . \left(D_i\left(u_i\right) \nabla u_i\right) = f_i\left(x, u\right) \quad , \quad x \in \Omega \\ u_i\left(x\right) = g_i\left(x\right) \quad , \quad x \in \partial\Omega \ , \quad i = 1, ..., N \end{cases} \tag{1}$$

where $u = (u_1, ..., u_N)$, Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, and for each $i = 1, ..., N$, $a_i = a_i(x)$, $b_i = b_i(x)$ are independent of t, $D_i(u_i)$, $f_i(x, u)$ and $g_i(x, u)$ are prescribed functions satisfying the following hypotheses :

(H_1) $a_i(x) \in C^{\frac{\alpha}{2},1}\left(\bar{\Omega}\right)$, $b_i^\ell(x)$ for $\ell = 1, ..., n$ and $f_i(x, .)$ are in $C^{\frac{\alpha}{2},\alpha}\left(\bar{\Omega}\right)$ with $a_i \geq a_i^*$, $g_i(x) \in C^{\frac{\alpha}{2},\alpha}(\partial\Omega)$.

(H_2) $D_i(u_i) \in C^2([0, M_i])$, $D_i(u_i) > 0$ for $u_i \in [0,\ M_i]$, and $D_i(0) \geq 0$, where $M_i = |\tilde{u}|_{C(\bar{\Omega})}$.

(H_3) $f_i(x, .) \in C^\alpha\left(\bar{\Omega}\right)$, $f_i(., u) \in C^1(S^*)$, and

$$\frac{\partial f_i}{\partial u_j} \geq 0 \ , \ for \ j \neq i \ , \ u \in S^*$$

(H_4) There exists a constant $\delta_0 > 0$ such that for any $x_0 \in \partial\Omega$ there exists a ball K outside of Ω with radius $r \geq \delta_0$ such that $K \cap \bar{\Omega} = \{x_0\}$.

In the above system we allow $D_i(0) = 0$ .We also allow $a_i = b_i^\ell = 0$, $\ell = 1, ..., N$, for some i and without the corresponding boundary condition.

The corresponding steady-state system is reduced to (1) for $i = 1, ..., n_0$, and $f(x, u) = 0$ for $i = n_0 + 1, ..., N$. Since the system quasilinear parabolic can be reduced to that in (1) for $(u_1, ..., u_{n_0})$ when the equations $f_i(x, u_1, ..., u_N) = 0$ are solved for $u_{n_0+1}, ..., u_N$ in terms of $u_1, ..., u_{n_0}$ we only consider the system (1).

The positivity of a solution $u(x)$ is in the sense that $u(x) > 0$ in Ω and it may vanish on $\partial\Omega$. In fact, $u(x) = 0$ on $\partial\Omega$ if $g(x) = 0$. The existence result is also based on the method of upper and lower solutions which are defined as follows:

Definition 1 *A pair of functions* $\tilde{u}_s \equiv (\tilde{u}_1, ..., \tilde{u}_N)$, $\hat{u}_s \equiv (\hat{u}_1, ..., \hat{u}_N)$ *in* $C^2(\Omega) \cap C\left(\bar{\Omega}\right)$ *are called ordered upper and lower solutions of* (1) *if* $\tilde{u}_s \leq \hat{u}_s$, *and if*

$$\begin{cases} -a_i \nabla . \left(D_i\left(\hat{u}_i\right) \nabla \hat{u}_i\right) + b_i . \left(D_i\left(\hat{u}_i\right) \nabla u_i\right) \leq f\left(x, \hat{u}_s\right) \quad , \quad x \in \Omega \\ \hat{u}_i\left(x\right) \leq g_i\left(x\right) \quad , \quad x \in \partial\Omega \quad , \quad i = 1, ..., N \end{cases} \tag{2}$$

and $\tilde{u}_i$ *satisfies* (2) *with inequalities reversed.*

For a given pair of ordered upper and lower solutions $\hat{u}_s$ and $\hat{u}_s$, we set

$$S_i^* = \left\{ u_i \in C\left(\bar{\Omega}\right) \mid \hat{u}_i \le u_i \le \tilde{u}_i \right\}$$
$$S^* = \left\{ u \in C\left(\bar{\Omega}\right)^N \mid \hat{u}_s \le u_s \le \tilde{u}_s \right\}$$

when distinction between different pairs of upper and lower solutions is needed we write $S^* = \langle \hat{u}_s, \tilde{u}_s \rangle$. Throughout this and the following sections we make the following hypothesis for each $i = 1, ..., N$.

2 The main result

The main result of this work is the following theorem :

Theorem 2 *Let* $\tilde{u}_s$,$\hat{u}_s$ *be ordered positive upper and lower solutions of* (1), *and let hypotheses* $(H_1)-(H_4)$ *hold. Then problem* (1) *has a minimal solution* $\underline{u}_s$ *and a maximal solution* $\bar{u}_s$ *such that* $\hat{u}_s \le \underline{u}_s \le \bar{u}_s \le \tilde{u}_s$. *If* $u_s = \bar{u}_s \, (\equiv u_s^*)$ *then* u_s^* *is the unique positive solution in* S^*.

References

[1] H. Amann, *Dynamic theory of quasilinear parabolic systems II. Reaction–diffusion systems*, Differential Integral Equations 3 (1990) 13–75.

[2] N. Alaa, S. Mesbahi. *Existence result for triangular Reaction-Diffusion systems with* L^1 *data and critical growth with respect to the gradient*, Mediterr. J. Math. 10 (2013), 255-275.

[3] N. Alaa, S. Mesbahi. *Existence of solutions for quasilinear elliptic degenerate systems with* L^1 data *and nonlinearity in the gradient* . Elect. J. of Diff. Eq., Vol. 2013 (2013), No. 142, pp. 1-13.

[4] S. Mesbahi, *On the existence of positive solutions of a class of parabolic reaction diffusion systems*, to appear in Journal of Nonlinear Analysis and Application, 2019.