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EXISTENCE RESULTS FOR SOME MODELS OF BOUNDARY VALUE PROBLEMS USING VARIOUS DEGREE METHODS

Presented by: **IATIME Khadidja**

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Front of the jury:

BEN ALI.B	MCA	University of ElOued	President
GUEDDA .L	Pr	University of ElOued	Supervisor
GHENDIR AOUN.A	MCA	University of ElOued	Co-supervisor
KOUIDRI .M	MCA	University of Ouargla	Examiner
MENACER.T	Pr	University of Biskra	Examiner
BELOUL.S	Pr	University of ElOued	Examiner

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Thème

RÉSULTATS D'EXISTENCE POUR CERTAINS MODÈLES DE PROBLÈMES DE VALEURS LIMITES UTILISANT DIVERSES MÉTHODES DE DEGRÉ

Présentée par: **IATIME Khadidja**

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Devant le jury:

BEN ALI.B	MCA	Université d'El Oued	Président
GUEDDA. L	Pr.	Université d'El Oued	Encadrant
GENDIR AOUN.A	M.C.A	Université d'El Oued	Co-encadrant
KOUIDRI.M	MCA	Université de Ouargla	Examinateur
MENACER .T	Pr	Université de Biskra	Examinateur
BELOUL .S	Pr	Université d'El Oued	Examinateur

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Dedication

I *dedicate this fruit of my long years of study first of all:*

T*o my dear parents (* ***Adel and Latifa Feroui****), who are the light of my life, who suffered and sacrificed so much for me to be happy, for their advice, their affection and their encouragement.*

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My dear grandmother,
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Abstract

In this work, under appropriate boundary conditions, new existence results for non-trivial solutions of certain classes of nonlinear differential equations are presented. These types of problems can be reformulated to abstract operational equations of the form $Lx = Nx$ proposed in a bounded domain of the Banach space X, where N is a nonlinear operator and L is a Fredholm operator with index 0. such that $kerL \neq \{0\}$, We also discussed another problem in the case of non-compactness concerning the fixed point of the operator $T + F$, where T is an operator condensing with respect to the measure μ and F is $\mu-$ Lipschitz with constant 0; we therefore used the technique of measure of non-compactness and the topological degree to prove the existence and uniqueness of a fixed point.

The first two chapters of this work are devoted to the presentation of prerequisites and preparations, as well as certain results in functional analysis. The axiomatic definition and construction of Mawhin's degree of coincidence are given, along with its main properties and some resulting fixed point theorems, and we have defined the non-compactness measure. The third chapter is entirely devoted to applications of solving boundary problems associated with nonlinear fractional differential equations at the resonance case using Mawhin's coincidence degree theory, and finally, we studied the existence and uniqueness of FBVP solutions using the non-compactness measure technique.

Keywords: topological degree, boundary value problems, Fredholm operator, Mawhin coincidence degree theory, condensing operators, The Kuratowski measure of non-compactnes.

Résumé

Dans ce travail, sous des conditions aux limites appropriées, de nouveaux résultats d'existence des solutions non triviales de certaines classes d'équations différentielles non linéaires sont présentés. Ces types de problèmes peuvent être reformulés sous forme des équations opérationnelles abstraites de la forme $Lx = Nx$ proposées dans un domaine borné de l'espace de Banach X, où N est un opérateur non linéaire et L un opérateur de Fredholm d'indice 0. tel que $kerL \neq \{0\}$, aussi nous avons discuté un autre problème dans cas de non-compacité concernant le point fixe de l'opérateur $T+F$, où T est un opérateur condensant par rapport à la mesure $\mu-$ et F est $\mu-$ Lipschitz avec constant 0 ; nous avons donc utilisé la technique de la mesure de non-compacité et le degré topologique pour prouver l'existence et l'unicité du point fixe.

Les deux premiers chapitres de ce travail sont consacrés à la présentation des prérequis et préparations ainsi que certains résultats de l'analyse fonctionnelle. La définition axiomatique et la construction du degré de coïncidence de Mawhin sont données, ainsi que ses principales propriétés et quelques théorèmes de point fixe qui en résultent, où nous avons défini la mesure de non-compacité. Le troisième chapitre est entièrement consacré aux applications de la résolution des problèmes aux limites associés aux équations différentielles fractionnaires non linéaires au cas de résonance en utilisant la théorie des coïncidences de Mawhin, et dans le dernier nous avons étudié l'existence et l'unicité des solutions de FBVP en utilisant la technique de la mesure de non-compacité.

Mots-clés : degré topologique, problèmes de valeurs aux limites, opérateur de Fredholm, coïncidence de Mawhin théorie, opérateurs de condensation, mesure de Kuratowski des non-compacts.

ملخص

في هذا العمل، وفي ظل شروط حدية مناسبة، تم عرض بعض نتائج الوجود الجديدة للحلول غير البديهية لفئات معينة من المعادلات التفاضلية غير الخطية. يمكن اختزال هذه الأنواع من المسائل إلى معادلة مؤثرات مجردة $Lx = Nx$ في مجال محدد من فضاء بناخ X، حيث N مؤثر غير خطي وL مؤثر فريدهولم بدليل معدوم في حالة التجاوب أي $kerL \neq \{0\}$، وناقشنا كذلك حالة عدم التراص حيث T مؤثر غير متراص في مسألة النقطة الصامدة، لذلك قمنا بإنشاء نقطة صامدة معممة لمجموع المؤثرين $T+F$، حيث T هو مؤثر مكثف بالنسبة للقياس μ وF هو مؤثر يحقق شرط ليبشيتز مع الثابت 0 بالنسبة للقياس μ؛ لذلك قمنا باستخدام تقنية قياس عدم التراص ونظرية النقطة الثابتة ثم الدرجة الطوبولوجية.
تم تخصيص الفصلين الأولين من هذا العمل لعرض الأساسيات المطلوبة بالإضافة إلى نتائج معينة للتحليل التابعي حيث تم تقديم التعريف الأكسيوماتي ثم بناء درجة المصادفة لجون ماوهين انطلاقا من الدرجة الطبولوجية لشاودر، بالإضافة إلى خصائصها الرئيسية وبعض نظريات النقطة الصامدة، كما قمنا بتعريف قياسات عدم التراص. أما الفصل الثالث فقد خصص بالكامل لتطبيقات حل المسائل الحدودية المرتبطة بالمعادلات التفاضلية ذات الرتبة الكسرية غير الخطية لحالة التجاوب باستخدام نظرية ماوهين للمصادفة، وفي الأخير قمنا بدراسة وجود ووجدانية الحل سطص باستخدام تقنية قياس عدم التراص.

الكلمات المفتاحية: الدرجة الطوبولوجية، مسائل القيمة الحدية، معامل فريدهولم، نظرية صدفة ماوهين، مؤثرات مكثفة، قياس كوراتوفسكي لعدم التراص.

Published articles

System of fractional boundary value problems at resonance, Iatime Khadidja, Guedda Lamine, Smaïl Djebali, Fractional Calculus and Applied Analysis, https://doi.org/10.1007/s13540-023-00157-0.

Existence And Uniqueness Result Of Solutions Of A Fractional Differential Boundary Value Problem, Guedda Lamine, Iatime Khadidja, Applied Mathematics E-Notes, https://www.math.nthu.edu.tw/ amen/2024/AMEN-A230709.pdf

Contents

General notations

$\varnothing$	:	The empty set.
λ, ϵ	:	Positive parameters.
$\bigoplus$	:	Direct sum.
$\mathbb{N}$	:	Natural numbers set.
$\mathbb{R}$	:	Real numbers set.
$\mathbb{R}^+$	:	Positive real numbers set.
$\mathbb{R}^n$	:	The Euclidien space of dimension n.
$\mathbb{R}[\mathbb{X}]$	:	The set of polynomials with real coefficients.
$(\mathbb{X}, d)$	:	Metric space.
$\overline{A}$	:	Set of adherent points of A .
∂A	:	Boundary of the set A .
$\mathcal{C}\,[a, b]$	:	Space of all continuous functions on a closed open interval.
$\mathcal{C}^k$	:	Function class k times continuously differentiable over an interval of $\mathbb{R}$.
$\frac{\mathbb{X}}{E}$	:	Quotient space of $\mathbb{X}$ by E.
$\big[a\big]$	:	The integer part of a.
$B(x_j, \delta)$	:	Open ball where $x_j$is its center and with the radius δ.
$det(M)$	:	Determinant of the matrix M .
$G(t, s)$	:	Function of Green .
$I^r_{0+}x$	:	Standard fractional integral of Riemann-Liouville with order $r > 0$ of $x(t)$.
$D^r_{0+}x$	:	Standard fractional derivative of Riemann-Liouville with order r of $x(t)$.
${}^cI^r_{0+}x(t)$	:	Standard fractional integral of Caputo with order $r > 0$ of a function $x(t)$.
${}^cD^r_{0+}x(t)$	:	Standard fractional derivative of Caputo with order $r > 0$ of a function $x(t)$.
deg	:	Topological degree.

$\mathcal{J}_f$	:	Determinant of the Jacobin matrix of f.
$S_f(\Omega)$	:	Set of all critical points of f.
$sgnf$	:	Sign of f.
$dist$	:	Distance.
$Dom(\mathcal{L})$	:	Definition domain of $\mathcal{L}$.
$ker(\mathcal{L})$	:	Set of all elements x of $Dom(\mathcal{L})$ where $\mathcal{L}x = 0$.
$Im(\mathcal{L})$	:	Range of $\mathcal{L}$.
$ind(\mathcal{L})$	:	Index of $\mathcal{L}$ where $\mathcal{L}$ is a Fredholm operator.
$\operatorname{diam} B$	:	Diameter of B, i.e: $\sup\{\lvert x-y\rvert : x \in B, y \in B\}$.
$Conv(X)$	:	The convex hull of the set X.
BVP	:	Value Problems with boundary conditions .
FBVP	:	Fractional Value Problem with boundary conditions.

GENERAL INTRODUCTION

Applied mathematics is a branch of mathematics concerned with the application of mathematical knowledge to other fields. As areas of application of mathematics, we can cite: numerical analysis, engineering mathematics, mathematical physics, linear optimization, dynamic programming, optimization and operations research, biomathematics, information theory, game theory, financial mathematics, cryptography, graph theory as applied to network analysis, as well as a good part of what we call computer science.

The fractional differential equations theory has emerged as an interesting field to explore in recent years. Note that several phenomena of physics and engineering sciences can very well be modeled using differential equations of standard order as well as of non-integer order; for more details see [[12], [21], [39]].

In recent years, the topological degree has proven to be a high-performance tool for the resolution of certain nonlinear problems associated with ordinary and fractional differential equations. An overview of degree theory shows the close connection between this fundamental topological concept and the theory of differential equations. From the year 1883, Poincaré used the Kronecker index [18] in his qualitative study of nonlinear differential equations, and discovered on this occasion some important new consequences of Kronecker's theory, particularly Miranda's theorem.

In fact, some mechanical problems led Bohl in 1904 to formulate and prove, using Kronecker's ideas, results equivalent to the theorems of Brouwer and Poincaré-Bohl for mappings, which are continuously differentiable. Topological degree theory in an infinite-dimensional space was started in a well-known paper that G.D. Birkhoff and O.D. Kellogg published in 1922, expanding on Brouwer's fixed point theorem to include specific function spaces. They showed the existence of at least a fixed point for the definite continuous mappings from a compact convex subset of $C([a,b])$ in itself. The fixed point theorem of Birkhoff-Kellogg was extended by J. Schauder to a continuous operator applying C in itself where C is convex compact subset of a Banach space.

J.Leray and J.Shauder extended the topological degree of Brouwer to a compact perturbation of identity in the Banach space $\mathbb{X}$ as follows; if $B \subset \mathbb{X}$ is a bounded open subset, $K : \overline{B} \to \mathbb{X}$ compact mapping and $y_0 \notin (\mathcal{I}_d - K)(\partial B)$, and K is the uniform limit of $(K_n)_n$ of operators with finite rank, i.e for any $\alpha > 0$, we can find ϵ strictly positive and a compact mapping K in such a way that $ImK_\epsilon \subset \mathbb{X}_\epsilon$ (subspace of $\mathbb{X}$ of finite dimension included y_0) and satisfying

$$\sup_{x\in B} \parallel k_\epsilon(u) - k(u) \parallel < \alpha.$$

For α sufficiently low($\alpha < \frac{1}{2}\,\mathrm{dist}(y_0, (I-K), (\partial B))$), the topological degree of Brouwer $deg((\mathcal{I}_d - K_\epsilon)_{|\mathbb{X}_\epsilon}, B \cap \mathbb{X}_\epsilon, y_0)$ will be associated with a common value defining the degree of Leray-Shauder $deg(\mathcal{I}_d - K, \Omega, y_0)$ If we set $y_0 = 0$, this degree counts in an algebraic manner the number of fixed points of the operator K in B, The topological degree of Leray-Schauder preserves all the essential properties of Brouwer one. The non-nullity of the degree is the most important of these properties, because from which we ensure that equation

$$x - K(x) = y_0$$

admits at least one solution $x \in \overline{\Omega}$. However, we still hope that it is practically easier to prove that degree of leray-Schauder $deg(\mathcal{I}_d - K, \Omega, y_0) \neq 0$ than to solve last equation or otherwise prove that the set of its solutions is not empty.

Subsequently, we will be interested in the study of operational equations of abstract form

$$\mathcal{L}u = Hu, \tag{1}$$

where the operator $\mathcal{L} : Dom(\mathcal{L}) \to \mathbb{Z}$ is linear and $H : \mathbb{X} \to \mathbb{Z}$ is a non-necessarily linear map (here $\mathbb{X}, \mathbb{Z}$ are normal vector spaces). A wide range of issues, including as properties of nonlinear ordinary or partial differential equations, are represented by the equation (1). There is no problem If there is only one zero in the kernel of this equation's linear part, because $\mathcal{L}$ is surjective in this case. The equation (1) can be simplified to a fixed point issue for the operator $\mathcal{L}^{-1}H,$ and to solve it, we will use a suitable fixed point theorem (for example that of Schauder if $\mathcal{L}^{-1}H$ is compact or Banach if $\mathcal{L}^{-1}H$ is contraction and $\mathbb{X}=\mathbb{Z}$ is a Banach space).

In this context, for example in (2015) Shi, H Pei, M, and Wang, L [50] used a continuation theorem of Leray-Schauder, to demonstrate some results of existence and uniqueness of solutions for third-order three-point BVP on a half-line of this following form:

$$v'''(t)=g\left(t,v(t),v'(t),v''(t)\right),\quad t>0$$

with the conditions

$$v(0)=\alpha v(\eta),\quad \lim_{t\to+\infty} v^{(i)}(t)=0,\quad i=1,2.$$

where $\alpha\neq 1$ and $\eta\in(0,+\infty)$, and $g:[0,+\infty)\times\mathbb{R}^3\to\mathbb{R}$ is defined as S^2- Carathéodory function.

In the case where $\mathcal{L}^{-1}$ does not exist (i.e $ker\mathcal{L}\neq\{0\}$) and $\mathbb{X}=\mathbb{Z}$ are Banach spaces. In 1972, Mawhin answered this question in his popular article [23] where he supposed that $\mathcal{L}$ is an operator of Fredholm with index 0.

For example, Xu, N, Liu, W and Xiao, L (2012) in [53] used the coincidence degree theory to demonstrate an existence result for the following BVP at resonance :

$$D^{r}_{0+}u(t)=g\left(t,u(t),D^{r-1}_{0+}u(t),D^{r-2}_{0+}u(t)\right),\quad 0<t<1,$$

subject to the conditions

$$\begin{aligned} &I^{3-r}_{0+}u(t)\big|_{t=0}=0,\quad D^{r-1}_{0+}u(1)=\sum_{i=1}^{m}a_iD^{r-1}_{0+}u\left(\xi_i\right),\\ &D^{r-2}_{0+}u(1)=\sum_{i=1}^{m}b_iD^{\alpha-2}_{0+}u\left(\eta_i\right),\end{aligned}$$

where $2<r\leq 3; 0<\xi_1<\xi_2<\cdots<\xi_m<1; 0<\eta_1<\eta_2<\cdots<\eta_m<1; a_i,b_i\in\mathbb{R}$; the function $g:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ satisfies the Carathéodory conditions; D^{α}_{0+} is the fractional derivative of Riemann-Liouville and I^{α}_{0+} is the standard fractional integral of Riemann-Liouville .

For the purpose of solving operator equations of the form $(I - T)x = y$, where T is a compact map, Leray Schauder's degree theory comes in handy. One may wonder if there is a way to provide an analogy to the Leray-Schauder degree theory in the non-compact case, since T is frequently not a compact map. Leray [184] created an example in 1936 to demonstrate that a map with only one continuity requirement could not have a degree theory defined. Naturally then, the question emerges: for which mappings in infinite-dimensional spaces is it possible to develop a degree theory?
Browder, Nussbaum, Sadovski, Vath, and others demonstrated that for a condensing map can define a full simulation of the Leray-Schauder degree theory. T.
In (2006) isia [32] used The (a priori estimate method)(for μ-condensing perturbations of the identity) to achieve the existence of solutions for the integral equation

$$v(t) = \varphi(t, v(t)) + \int_a^b \psi(t, s, v(s))ds, \quad t \in [a, b]$$

where the continuous functions are $\psi : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ and $\varphi : [a, b] \times \mathbb{R} \to \mathbb{R}$. Rewriting the final integral equation as a fixed point problem for a condensing map $T : C[a, b] \to C[a, b]$ is the fundamental idea. We observe that the Leray-Schauder degree cannot be applied since the assumptions on functions φ and ψ typically do not guarantee the compactness of operator T.

Then there are many fixed point theorems without compactness theorem, that have been solved by using the measure of non-compactness and some fixed point theorem with topological degree, like in [22] the authors investigate an existence and uniqueness results to the next fractional differential equations

$$\begin{cases} {}^cD^r u(t) & = F(t, u(t)) \quad t \in I/\{t_1, t_2, \dots, t_m\}, I := [0, T] \\ \Delta u(t_i) & = I_j(u(t_j)) \qquad j = 1, 2, 3, \dots, m \\ u(0) + \eta(u) = u_0, & \end{cases}$$

where ${}^cD^r$ is the derivative fractional of Caputo of order $0 < r \leq 1$, $F : I \times X \to X$, $u_0 \in X$ and $\eta : PC(J, X) \to X$ is a continuous map. $I_i : X \to X$ are continuous operators as well as t_i satisfying $0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = T$.

Our thesis consists of four chapters. It seemed useful to us to begin it with **Chapter 1** devoted to reminders and fundamental notions. We start in Section 1.1 with algebraic preliminaries which is very important to understanding the basics of work, then in Section 1.2, we present some compactness criteria for functions defined on compact intervals including the Ascoli-Arzelà theorem. In Section 1.3, we will present the definition of the measure of non-compactness and its different properties, then we present some useful concepts (Lebesgue's dominated convergence theorem, Banach's contraction principle). Then, we end this first chapter by presenting some elements from the theory of fractional calculus (Riemann-Liouville type and Caputo type).

In the second chapter, The notion and characteristics of topological degree are covered in this chapter. For example, Brouwer's and Schauder's degrees, and the coincidence degree theorem. J. Mawhin proves this by introducing the concept of compactness of the operators N, which is the usual compactness when studying applications of type $\mathcal{L} + N$ (called $\mathcal{L}-$compact perturbation of $\mathcal{L}$) These are conditions required to replace Schauder's fixed point theorem. Finally, we conclude this chapter by describing the condensing perturbation of identity.

In the third chapter, we have provided the analysis of the following existence result of two-point at resonance (B.V.P) at resonance via coincidence degree:

$$D_{0^+}^{\alpha} v(t) + f\left(t, v(t), D_{0^+}^{\alpha-1} v(t)\right) = \theta,\ 0 < t < 1,$$

$$v\left(0\right) = \theta,$$

$$D_{0^+}^{\alpha-1} v\left(1\right) = \mathcal{B} I_{0^+}^{\alpha} v\left(1\right),$$

where $1 < \alpha \leqslant 2$, $D_{0^+}^{\alpha}$ is the Riemann-Liouville's fractional derivative of and $I_{0^+}^{\alpha}$ is the standard fractional integral of Riemann-Liouville, by applying Mawhin's coincidence degree theory [2.3.1].

$\theta = (0, 0, \ldots, 0)^T$ is the zero vector of $\mathbb{R}^N$, and $\mathcal{B}$ is a square matrix of size N.

We are used this vector notations for $x(t) = (x_1(t), x_2(t), \ldots, x_N(t))^T$, and we denote by

$$D_{0^+}^{\alpha} x(t) = \left(D_{0^+}^{\alpha} x_1(t), D_{0^+}^{\alpha} x_2(t), \ldots, D_{0^+}^{\alpha} x_N(t)\right)^T,$$

to the fractional derivative of x, and its fractional integral by

$I_{0^+}^{\alpha} x(t) = \left(I_{0^+}^{\alpha} x_1(t), I_{0^+}^{\alpha} x_2(t), \ldots, I_{0^+}^{\alpha} x_N(t)\right)^T$.

Also $f:[0,1]\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ is a vector-valued function

$$f\left(t,x(t),D_{0^+}^{\alpha-1}x(t)\right)=$$
$$\left(f_1\left(t,x(t),D_{0^+}^{\alpha-1}x(t)\right),f_2\left(t,x(t),D_{0^+}^{\alpha-1}x(t)\right),\dots f_N\left(t,x(t),D_{0^+}^{\alpha-1}x(t)\right)\right)^T.$$

Finally, to illustrate the theoretical results, an example is given at the end.

The fourth chapter uses topological degree theory to prove the existence and uniqueness of solutions to a class of fractional BVP with fractional derivatives of Caputo , through the a priori method of condensing maps using the topological degree theory.

$$^cD_{0^+}^r x(t)=g(t,x(t)),\ t\in I:=[0,T],$$

$$ax(0)-bx(T)=\Psi\left(x\right),$$
$$^cD^{r-1}x(T)=\phi(x),$$

where $a,b\in\mathbb{R}$ $(a\neq b)$, $r\in(1,2)$, $^cD^r$ is the Caputo derivative, $g:I\times\mathbb{R}\to\mathbb{R}$, $\phi:C(I)\to\mathbb{R}$ and $\Psi:C(I)\to\mathbb{R}$ are given continuous maps.
Lastly, a valid example is provided to show that the findings produced are valid.

CHAPTER

1

REMINDERS AND FUNDAMENTALS

THis chapter will be devoted to elementary definitions and basic notions relating to algebraic, topological, functional, and fractional calculus that will be used in the remainder of this thesis, such as: projections, compactness and finite rank maps, the measure of non-compactness, Banach's contraction principle, fractional calculus, and some definitions, lemmas, and theorems that will be used in subsequent chapters. These definitions and properties are taken from [[8], [9], [10], [13], [17], [25], [32], [37], [38],[44]] to which we refer for more details.

1.1 Algebraic preliminaries

1.1.1 The Direct sum

Definition 1.1.1 *[10]*

Let F and E be two subspaces of the vector space $\mathbb{X}$. $\mathbb{X}$ is said to be the direct sum of F and E,which can be write as $\mathbb{X} = E \oplus F$, if:

i) $F + E := \{x + z | x \in F, z \in E\} = \mathbb{X}$,

ii) $F \cap E = \{0\}$ " F and E are independent."

1.1.2 Quotient space

Definition 1.1.2

Assuming that $\mathbb{X}$ is a normed vector space, if $\mathbb{E}$ is a closed subspace of $\mathbb{X}$, then the quotient space $\mathbb{X}/\mathbb{E}$ has a normed vector space structure whose norm is defined by :

$$\|\pi_{\mathbb{E}}(x)\|_{\mathbb{X}/\mathbb{E}} = \inf_{h \in \mathbb{E}} \|x + h\|_{\mathbb{X}} = d(x, \mathbb{E}),$$

where $\pi_{\mathbb{E}} : X \to \mathbb{X}/\mathbb{E}, \pi_{\mathbb{E}}(x) := x + \mathbb{E} = \{x + h/h \in \mathbb{E}\}$ is the canonical quotient map.

Proposition 1.1.1 *[10]*

Let $\mathbb{X}$ be a Banach space, consequently $\mathbb{X}/\mathbb{E}$ is also a Banach space.

1.1.3 Co-dimension of a vector subspace

Definition 1.1.3 *[10]*

Let the quotient space $\mathbb{X}/\mathbb{E}$ has finite dimension then we may state that the closed vector subspace $\mathbb{E}$ of $\mathbb{X}$ has finite co-dimension belongs to $\mathbb{X}$, then we define

$$\mathrm{codim}(\mathbb{E}) = \dim(\mathbb{X}/\mathbb{E}).$$

Proposition 1.1.2 *[10]*

$\mathrm{codim}(\mathbb{E}) = n < \infty$ equivalent to there is a closed subspace $\mathbb{F}$ belongs to $\mathbb{X}$, such as $\mathbb{X} = \mathbb{F} \oplus \mathbb{E}$ where $dim(\mathbb{F}) = n$.

1.1.4 The projections

Assume that $\mathbb{X}$ is a vector space.

Definition 1.1.4

Let $\mathscr{P} : \mathbb{X} \to \mathbb{X}$ be linear operator and we say that $\mathscr{P}$ is projection if for every x belongs to $\mathbb{X}$, such that $\mathscr{P}(\mathscr{P}(x)) = \mathscr{P}^2 x = \mathscr{P}(x)$.

Proposition 1.1.3

A projection is defined as the linear operator $\mathscr{P} : \mathbb{X} \to \mathbb{X}$ is equivalent to $(\mathcal{I}_d - \mathscr{P})$ is a projection. Furthermore, if the space $\mathbb{X}$ is normalized, then $\mathscr{P}$ is continuous operator if and only if $(\mathcal{I}_d - \mathscr{P})$ is continuous operator.

Proof 1.1.1

Assume that $\mathscr{P}$ is a projection map. then for all x belongs to $\mathbb{X}$:

$$\begin{aligned}
(\mathcal{I}_d - \mathscr{P})^2(x) &= (\mathcal{I}_d - \mathscr{P})((\mathcal{I}_d - \mathscr{P})(x)) \\
&= \mathcal{I}_d(\mathcal{I}_d - \mathscr{P})(x) - \mathscr{P}(\mathcal{I}_d - \mathscr{P})(x) \\
&= \mathcal{I}_d(x - \mathscr{P}x) - \mathscr{P}(x - \mathscr{P}x) \\
&= x - \mathscr{P}(x) - \mathscr{P}(x) + \mathscr{P}^2(x) \\
&= x - 2\mathscr{P}(x) + \mathscr{P}^2(x) \\
&= x - 2\mathscr{P}(x) + \mathscr{P}(x) \\
&= x - \mathscr{P}(x) = (\mathcal{I}_d - \mathscr{P})(x).
\end{aligned}$$

Conversely, if $(\mathcal{I}_d - \mathscr{P})$ is a projection then $(\mathcal{I}_d - (\mathcal{I}_d - \mathscr{P})) = \mathscr{P}$ is also a projection. For topological frameworks, an identity is a continuous map and the sum of two continuous maps is also continuous map, so $\mathscr{P}$ is continuous map equivalent to $(\mathcal{I}_d - \mathscr{P})$ is continuous.

Proposition 1.1.4

Let $\mathscr{P}$ be a projection belongs to $\mathbb{X}$, so

$$ker(\mathscr{P}) = Im(\mathcal{I}_d - \mathscr{P})$$

$$Im(\mathscr{P}) = ker(\mathcal{I}_d - \mathscr{P}).$$

Proof 1.1.2

As in the last proof, it enough to prove that

$$ker\mathscr{P} = Im(\mathcal{I}_d - \mathscr{P})$$

The other result is an immediate corollary, by replacing $\mathscr{P}$ with $(\mathcal{I}_d - \mathscr{P})$ in the proposition former.
If $x \in ker\mathscr{P} \Longrightarrow \mathscr{P}(x) = 0$, then

$$(\mathcal{I}_d - \mathscr{P})(x) = x - 0 = x;$$

hence $x \in Im(\mathcal{I}_d - \mathscr{P})$.
Reciprocally, $\mathscr{P}((\mathcal{I}_d - \mathscr{P})(x)) = \mathscr{P}(x) - \mathscr{P}^2(x) = \mathscr{P}(x) - \mathscr{P}(x) = 0$; so $x \in ker\mathscr{P}$ and $ker\mathscr{P} = Im(\mathcal{I}_d - \mathscr{P})$.

Remark 1.1.1

A topological space $\mathbb{X}$is called a separated space (or Hausdorff) if $\forall x, y \in \mathbb{X}$ such that: $x \neq y$, there exist two opens O_x, O_y such that $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \varnothing$.

Lemma 1.1.1

Assume that $\mathbb{E}$ is a finite-dimensional vector subspace of a normalized vector space $\mathbb{X}$, so there exists a continuous projector $\mathscr{P}$ on $\mathbb{X}$; such as

$$Im(\mathscr{P}) = \mathbb{E}.$$

In general, any normalized vector space can be projected continuously onto a finite-dimensional subspace.

Proof 1.1.3

We choose a base $e_1; e_2; \dots e_n$ of $\mathbb{E}$ and we denote by $(\beta_k)_k$ where $k = \overline{1, N}$ the linear forms coordinated on $\mathbb{E}$ associated respectively with $e_1; e_2; \dots e_n$; we know by definition that for all $i, j \in \overline{1, N}$

$$\beta_j(e_i) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

using the Hahn-Banach theorem to extend $\beta_1, \beta_2, \ldots \beta_n$ into linear forms $f_1, f_2, \ldots, f_n$ continues on $\mathbb{X}$; we obtain that the application $\mathscr{P}$ is defined by;

$$\mathscr{P}x = \sum_{1\leqslant i,j\leqslant n} f_j(x).e_i$$

is a continuous projection of $\mathbb{X}$ onto $\mathbb{E}$ which answers the problem.

Corollary 1.1.1

The images of all continuous projections in a Hausdorff space are closed. Particularly, the images of all continuous projections of Banach space are closed .

Theorem 1.1.1

Suppose that in the topological vector space of Hausdorff, $\mathscr{P}$ is a continuous projection. Accordingly, $\mathbb{X}$ is the direct sum of $Im(\mathscr{P})$ and $ker(\mathscr{P})$; that is, $\mathbb{X} = Im\mathscr{P} \oplus ker\mathscr{P}$.

Proof 1.1.4

According to the previous corollary, $ker\mathscr{P}$ and $Im(\mathscr{P})$ are closed in $\mathbb{X}$; as

$$\mathbb{X} = \mathscr{P}(\mathbb{X}) + (\mathcal{I}_d - \mathscr{P})(\mathbb{X}) = Im(\mathscr{P}) + Ker\mathscr{P}$$

Moreover $Im(\mathscr{P}) \cap Ker\mathscr{P} = \{0\}$, then we have the result.

1.2 Compactness and finite rank maps

This section shows some properties of compact maps and operators with finite rank in topological spaces.

Definition 1.2.1 *[13],[17]**(compact mappings)***

1. Assume that $\mathbb{X}$ is a topological separated space. We say that a subset $S \subset \mathbb{X}$ is compact if all open covers of S have finite covers, i.e., if $S \subset \cup_{i\in I} U_i$, where U_i is an open subset of $\mathbb{X}$ for every $i \in I$, so there exist $i_j \in I$, $j = \overline{1, K}$ such that $S \subset \cup_{j=1}^{k} U_{ij}$.

2. S is said to be relatively compact if $\overline{S}$ is compact.

3. Let $T : \Omega \subset \mathbb{X} \to \mathbb{Z}$ be continuous map is called compact map if $T(\overline{\Omega})$ is relatively compact.

Proposition 1.2.1 *[13]*

Let $(\mathbb{X}, d)$ be a metric space. so, a subset $S \subset \mathbb{X}$ is compact equivalent to for all infinite sequences $(u_n)_{n=1}^{\infty} \subset S$ has a convergent sub-sequence in S.

Proof 1.2.1 *see [13]*

Remark 1.2.1

Any continuous and compact map is completely continuous. The converse is true if Ω is bounded.

1.2.1 Operators with finite rank

Definition 1.2.2 *[25]*

The operator $T : \mathbb{X} \to \mathbb{Z}$ is called operator with finite rank if there exists a finite-dimensional subspace $\mathbb{K}$ such that $\mathrm{Im}(T) \subseteq \mathbb{K} \subseteq \mathbb{Z}$.

Proposition 1.2.2 *[25]*

Let $T : \mathbb{X} \to \mathbb{Z}$ be a finite rank operator. Then T bounded implies that T is completely continuous.

Proof 1.2.2

Indeed for every bounded subset $A \subset \mathbb{X}$, $\overline{T(A)}$ is a closed bounded subset of a finite-dimensional subspace K.

1.2.2 Approximation of compact mappings

Proposition 1.2.3

Let $\mathbb{K}$ be closed bounded subset in a Banach space $\mathbb{X}$ and $f : \mathbb{K} \to \mathbb{X}$ a map. Then, f is compact if and only if f is the uniform limit of a sequence $(f_n)_{n\in\mathbb{N}}$ of compact maps of finite ranks. i.e there exist a sequence $(B_n)_{n\in\mathbb{N}}$ subsets of finite dimensional in $\mathbb{X}$ and a sequence $(f_n)_{n\in\mathbb{N}}$ where $f_n : \mathbb{K} \to \mathbb{X}$ is a compact map with $Im\,(f_n) \subset B_n$ such that

$$\lim_{n\to+\infty} \sup_{x\in\mathbb{K}} \|f_n(x) - f(x)\| = 0$$

Definition 1.2.3 ***(Compact perturbation of the identity)***

An application of the form $f = \mathcal{I}_d - T$ where $\mathcal{I}_d$ is the identity map and T is a compact map (characterized by the proposition 1.2.3 as the uniform limit of a sequence of compact maps of finite ranks $(T_\epsilon)_\epsilon$)is called compact perturbation of identity (or Leray-Schauder map).

Theorem of Ascoli-Arzelà [10]

Let $(\mathbb{Y}, d_\mathbb{Y})$ and $(\mathbb{Z}, d_\mathbb{Z})$ be a complete metric space and a complete metric space, respectively. Denote by $C(\mathbb{Y}, \mathbb{Z})$ the vector space of all continuous functions $f : \mathbb{Y} \to \mathbb{Z}$.

Definition 1.2.4 *(Equicontinuous part)*

The family of maps $T_\lambda : \mathbb{Y} \to \mathbb{Z}$ where λ belongs to a certain set of indices J and $\mathbb{Y}, \mathbb{Z}$ are two Banach spaces is said to be equicontinuous on B ,($B \subset \mathbb{Y}$), if for every $\epsilon > 0$, there exists $\sigma = \sigma(\epsilon)$ such that whatever $y, z \in A$ and whatever $\lambda \in J$ we have :

$$\|z - y\|_\mathbb{Y} < \delta \Rightarrow \|T_\lambda(z) - T_\lambda(y)\|_\mathbb{Z} < \epsilon.$$

Theorem 1.2.1

The family $E \subset C(\mathbb{Y}, \mathbb{Z})$ is relatively compact is equivalent to:

1) E is equicontinuous and

2) for every $t \in \mathbb{Y}$, the set

$$M(t) = \{y(t); y(.) \in E\}.$$

is relatively compact in $\mathbb{Z}$.

Special cases

- If $\mathbb{Z}$ is a Banach space with finite-dimensional, the 2^{nd} condition of Ascoli-Arzela theorem is equivalent to M is uniformly bounded i.e., there exists $N > 0$ such that for all $y \in E$

$$\|y\|_\infty = \sup_{t \in \mathbb{Y}} \|y(t)\| \leq N.$$

  Particularly, for $\mathbb{Z} = \mathbb{R}$ (In effect, in this case, for any $t \in \mathbb{Y}$, $\overline{M(t)}$ is a closed and bounded set of a finite-dimensional space $\mathbb{Z}$.)

- If $\mathbb{Z}$ is a compact metric space, then the theorem of Ascoli-Arzéla may be articulated in this way:: the family $E \subset C(\mathbb{Y}, \mathbb{Z})$ is relatively compact is equivalent to E is equicontinuous.

Example 1.2.1

Assume that $k : ([0,1]) \times \mathbb{R} \to \mathbb{R}$ is a continuous application, consider this integral equation

$$u(t) = \int_0^t k(s, u(s))ds, \quad t \in [0,1],$$

so the operator of Hammerstein

$$K : C([0,1]) \to C([0,1])$$

$$u \to Ku$$

such that

$$Ku(t) = \int_0^t k(s, u(s))ds$$

is compact ?

suppose the set $B = k \in C([0,1]), ||k|| \leqslant N$. As k is continuous and bounded then

$$\begin{aligned}|Ku(t)| &= |\int_0^t k(s,u(s))ds| \\ &\leqslant \int_0^t |k(s,u(s))|ds \\ &\leqslant N\int_0^t ds = Nt \leqslant N, \forall t \in [0,1]\end{aligned}$$

So $K(B)$ is uniformly bounded. We will prove that $K(B)$ is equicontinuous. We have

$$\begin{aligned}|Ku(t_1) - Ku(t_2)| &\leqslant |\int_0^{t_1} k(s,u(s)) - \int_0^{t_2} k(s,u(s))ds| \\ &\leqslant N\int_{t_1}^{t_2} ds \\ &\leqslant N|t_2 - t_1|.\end{aligned}$$

Then for all $\epsilon > 0$, there exists $\delta \leqslant \frac{\epsilon}{M}$ such that, $\forall t_1, t_2 \in [0,1] : |t_2 - t_1| \leqslant \delta$ then

$$|Ku(t_1) - Ku(t_2)| \leqslant N\delta \leqslant \epsilon,$$

hence the equicontinuity of K. According to the Ascoli-Arzela theorem, K is compact in B.

1.3 Some properties of measure of non-compactness and condensing mappings

We first review the definition of the measure of non-compactness for convenience, then we specified by mentioning a measure of Kuratowski of non-compactness and summarize the main properties of this measure.

In the following, we consider $(\mathbb{E}, \|\ .\ \|)$ is a real Banach space and let $\Omega_{\mathbb{E}}$ be the class of all bounded subsets of $\mathbb{E}$.

Definition 1.3.1 *5 (Measure of non-compactness)*

Let $\gamma : \Omega_{\mathbb{E}} \to]0, +\infty[$ be a measure of non-compactness if this following conditions hold:

1) $\gamma(u) = 0 \Longleftrightarrow u$ is relatively compact, $\forall u \in \Omega_{\mathbb{E}}$.

2) $\gamma(u) = \gamma(\overline{u}), \forall u \in \Omega_{\mathbb{E}}$.

3) $\gamma(v \cup u) = \max\{\gamma(v), \gamma(u)\}, \forall v, u \in \Omega_{\mathbb{E}}$.

There exist many measures of non-compactness, in what follows, we will present some of the most used in application. We will focus on Kuratowski measure of non-compactness because it is the one that we will use throughout this thesis.

1.3.1 The measure of non-compactness

Definition 1.3.2 *([9],[17]) **(The Kuratowski measure of non-compactnes)***

Let $\mathbb{E}$ be a Banach space and $\Omega_{\mathbb{E}}$ the set of all bounded subsets $\mathbb{E}$. The map $\mu : \Omega_{\mathbb{E}} \to [0; +\infty)$ is the Krutowski measure, which measures non-compactness as defined by:

$$\mu(B) = \inf \left\{ \epsilon > 0 : B \subset \bigcup_{j=1}^{m} B_j \text{ and } \operatorname{diam}(B_j) \leq \epsilon \right\}, \text{ for } B \in \Omega_{\mathbb{E}}$$

where $\operatorname{diam}(B_j) = \sup\{\| x - y \|_{\mathbb{E}}, x, y \in B_j\}$ is the diameter of B_j.

The properties of The Kuratowski measure of noncompactness

The Kuratowski measure of noncompactness satisfies this following properties (for details, see [9],[17] and [19]).

Property 1.3.1 ***(Regularity)***

$\mu(A) = 0 \Leftrightarrow \overline{A}$ is compact (A is relatively compact).

Proof 1.3.1

A is relatively compact if and only if

$$\forall \epsilon > 0, \exists \{A_j\}_{1 \leqslant j \leqslant n}, \textit{such that } A \subset \bigcup_{j=1}^{n} A_j, \operatorname{diam}(A_j) < \varepsilon, j \in \{1, \cdots, n\}$$

if and only if $\mu(A) = 0$.

Property 1.3.2 ***(Invariance under passage to the closure)***

Let A be a bounded subset of $\mathbb{E}$. Then $\mu(\bar{A}) = \mu(A)$.

Proof 1.3.2

1) Since $A \subset \bar{A}$, then
$$\mu(\bar{A}) \leq \mu(A). \tag{1.1}$$

2) Conversely, we make the following assertion
$$\forall \epsilon > 0, \exists \{A_j\}_{1 \leqslant j \leqslant n}, \textit{such that } A \subset \bigcup_{j=1}^{n} A_j$$
such that
$$A_j \subset E,$$
$$\operatorname{diam}(A_j) < \mu(A) + \varepsilon, j \in \{1, \cdots, n\}.$$
Now, let using that
$$\bar{A} \subset \bigcup_{j=1}^{n} \overline{A_j}$$
and
$$\begin{aligned}\operatorname{diam}\left(\overline{A_j}\right) &= \operatorname{diam}(A_j)\\ &< \mu(A) + \varepsilon, j \in \{1, \cdots, n\},\end{aligned}$$
we find
$$\mu(\bar{A}) < \mu(A) + \varepsilon. \tag{1.2}$$

Considering that $\varepsilon > 0$ was selected at arbitrary, we obtain
$$\mu(\bar{A}) \leq \mu(A).$$

From (1.1) and (1.2), we get
$$\mu(A) = \mu(\bar{A}).$$

This completes the proof.

Property 1.3.3 ***(Monotonicity)***

$\mu(H) \leq \mu(D)$ when $H \subset D$.

Proof 1.3.3

any recovery of the set D is a recovery of H.

Property 1.3.4 ***(Algebraic semi-additivity)***

Let H and D be bounded subsets of $\mathbb{E}$. $\mu(H+D) \leq \mu(H)+\mu(D)$ where

$$H+D=\{x \mid x=y+z ; y \in H ; z \in D\}.$$

Proof 1.3.4

Let $\epsilon>0$, Then there are partitions

$$H \subset \bigcup_{j=1}^{m} H_{j}, D \subset \bigcup_{I=1}^{k} D_{l}$$

such that

$$\begin{aligned}
H_{j} &\subset \mathbb{E}, \\
D_{l} &\subset \mathbb{E}, \\
\operatorname{diam}\left(H_{j}\right) &\leq \mu(H)+\varepsilon, j \in\{1, \cdots, m\} \\
\operatorname{diam}\left(D_{l}\right) &\leq \mu(D)+\varepsilon, l \in\{1, \cdots, k\}.
\end{aligned}$$

Denote

$$V_{jl}=\left\{x+y: x \in H_{j}, y \in D_{j}\right\}, j \in\{1, \cdots, m\}, l \in\{1, \cdots, k\}.$$

we have

$$\begin{aligned}
H+D &\subset \bigcup_{j=1}^{m}\left(\bigcup_{l=1}^{k} V_{jl}\right) \\
&=\bigcup_{i=1}^{k}\left(\bigcup_{j=1}^{m} V_{jl}\right)
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{diam}\left(V_{jl}\right) &\leq \operatorname{diam}\left(H_{j}\right)+\operatorname{diam}\left(D_{l}\right) \\
&\leq \mu(H)+\varepsilon+\mu(B)+\varepsilon \\
&=\mu(H)+\mu(D)+2 \varepsilon, j \in\{1, \cdots, m\}, l \in\{1, \cdots, k\}.
\end{aligned}$$

Consequently

$$\mu(H+D) \leq \mu(H)+\mu(D)+2 \varepsilon.$$

Because $\varepsilon>0$ was selected at arbitrary, we get

$$\mu(H+D) \leq \mu(H)+\mu(D).$$

The proof is now complete.

Property 1.3.5 ***(Invariance under shifting)***

Let H be a bounded subset of $\mathbb{E}$. Then

$$\mu(H+\{x\})=\mu(H) \text{ for any } x\in\mathbb{E}.$$

Proof 1.3.5

Proposition 1.3.4 of Algebraic semi-additivity yields $\mu(H+\{x\}) \leq \mu(A)+\mu(\{x\}) = \mu(H)$. Note that

$$\mu(\{x\}) \leq \operatorname{diam}(\{\mathrm{x}\}) = 0 \Rightarrow \mu(\{\mathrm{x}\}) = 0.$$

Hence

$$H = H+\{x\}+\{-x\} \Rightarrow \mu(H) = \mu(H+\{x\}+\{-x\}) \leq \mu(H+\{x\})+\mu(\{-x\})$$
$$\Rightarrow \mu(H) \leq \mu(H+\{x\}).$$

Then $\mu(H+\{x\})=\mu(H)$.

Property 1.3.6 ***(Semi-homogeneity)***

for any $r\in\mathbb{R}$

$$\mu(rD)=|r|\mu(D).$$

Where

$$rD=\{rx : x\in D\}$$

Proof 1.3.6

1) Let $r=0$. Then

$$rD=\{0\}.$$

Hence,

$$0=\mu(rD)=|r|\mu(D).$$

2) Let $r\neq 0$. Take $\varepsilon>0$ arbitrarily. Then there is a partition

$$D\subset\bigcup_{j=1}^{m} D_j,$$

such that

$$D_j \subset \mathbb{E},$$

$$\operatorname{diam}(D_j) \leq \mu(D) + \varepsilon, j \in \{1, \cdots, m\}.$$

We have

$$\mu D = \bigcup_{j=1}^{m} (rD_j)$$

and

$$\operatorname{diam}(rD_j) \leq |\lambda|(\mu(D) + \varepsilon).$$

Consequently,

$$\mu(\mu D) \leq |r|\mu(D) + |r|\varepsilon.$$

Because $\epsilon > 0$ was arbitrarily chosen, by the last inequality, we get

$$\mu(rD) \leq |r|\mu(D). \tag{1.3}$$

On the other hand, using that

$$D = r^{-1}(r.D),$$

as in above, we obtain

$$\begin{aligned} \mu(D) &= \mu\left(r^{-1}(rD)\right) \\ &\leq |r|^{-1}\mu(rD). \end{aligned}$$

whereupon

$$|r|\mu(D) \leq \mu(rD). \tag{1.4}$$

From (1.3) and (1.4), we get

$$\mu(rD) = |r|\mu(D).$$

Property 1.3.7 ***(Invariant under passage to the convex hull)***

Let D be a bonded subset of $\mathbb{E}$. Then

$$\mu(ConvD) = \mu(D).$$

Proof 1.3.7

The following details support the proof:

1) $\mathrm{diam(D)} = \mathrm{diam(conv\, D)}$,

2) $D \subset \text{Conv } D \Rightarrow \mu(D) \leq \mu(\text{Conv } D)$.

and uses the following Caratheodory's characterization of the convex hull:

$$\text{Conv } D = \left\{ \sum_{i=1}^{n} r_i d_i, d_i \in D, n \in N^*, r_i \geq 0, \sum_{i=1}^{n} r_i = 1 \right\}.$$

1.3.2 Condensing operators

Definition 1.3.3 *[38]*

Let $F : \Omega \to \mathbb{E}$ be a continuous bounded map and let $\Omega \subset \mathbb{E}$.

- It has been stated that F is Lipschitz if there is a value $< k > 0$ such that

$$\|Fx - Fy\| \leq k\|x - y\|, \text{ for every } \quad x, y \in \Omega,$$

- If $k \geq 0$ such that: $\mu(F(M)) \leq k\mu(M), \forall$ bounded $M \subset \Omega$ bounded, (in case $k < 1$, then we call F a strict $\mu-$contraction), then F is said to be $\mu-$Lipschitz.

- We say that F is μ-condensing if $\mu(F(M)) < \mu(M), (\forall) M \subset \Omega$ bounded with $\mu(M) > 0$,and if then $k < 1$, F is a strict contraction. We denote by $C_\mu(\bar{\Omega})$ the set of all $\mu-$condensing maps on Ω, and the class of all strict μ-contractions $F : \Omega \to \mathbb{X}$ is by $SC_\mu(\Omega)$.

Remark 1.3.1 *[32]*

As we can see $SC_\mu(\Omega) \subset C_\mu(\Omega)$ and any $F \in C_\mu(\Omega)$ is μ-Lipschitz with constant $k = 1$.

Proposition 1.3.1 *[32]*

If $F, H : \Omega \to \mathbb{R}$ are μ-Lipschitz maps with constants λ, λ' respectively, then $F + H : \Omega \to \mathbb{X}$ is μ-Lipschitz with constant $\lambda + \lambda'$.

Proposition 1.3.2 *[32]*

Given a compact function $F : \Omega \to \mathbb{X}$, F is μ -Lipschitz with zero constant.

Proposition 1.3.3 *[32]*

Given a Lipschitz function $F : \Omega \to \mathbb{X}$ with constant k, F can be expressed as μ-Lipschitz, where k is the same constant.

1.4 Some useful concepts

1.4.1 L^p Spaces

Definition 1.4.1 *[10]*

Assume that $p \in \mathbb{R}$ with $p \in [1, \infty[$; We decided to

$$L^p(\Omega) = \{g : \Omega \to \mathbb{R}; g \text{ is measurable and } |g|^p \in L^1(\Omega)\}$$

and the norm

$$\|g\|_{L^p} = \|g\|_p = \left[\int_\Omega |g(x)|^p d\mu\right]^{1/p}.$$

1.4.2 Dominated convergence theorem of Lebesgue

Theorem 1.4.1 *[10]*

Let (g_n) be a sequence of measurable real value functions defined of an open $\Omega \subset \mathbb{R}^n$. Suppose that there exists a measurable function g such that :

$$\lim_{n\to+\infty} g_n(x) = f(x), \quad \text{a.e. } x \in \Omega$$

and an integrable function $h : \mathbb{R} \to [0, +\infty]$ satisfying

$$|g_n(x)| \leq h(x), \quad \text{a.e. } x \in \Omega$$

Then f is integrable, and we get

$$\lim_{n\to+\infty} \int_\Omega g_n(x)dx = \int_\Omega \lim_{n\to+\infty} g_n(x)dx = \int_\Omega g(x)dx.$$

1.4.3 Banach's principle of contraction, 1922

The Picard's fixed point theorem or Banach's contraction principle are other names for Banach's fixed point theorem, initially showed up in 1922, in the article [8], in the framework of solving an integral equation.

Theorem 1.4.2

Let $(\mathbb{E}, d)$ be a complete metric space, let $F : \mathbb{E} \to \mathbb{E}$ be a contracting map with contraction constant k. So :

a) F admits a unique fixed point $\alpha \in \mathbb{E}$.

b) For every u belongs to $\mathbb{E}$, $\alpha = \lim_{n\to\infty} F^n(u)$ where $F^0(u) = u$ and $F^n(u) = F(F^{n-1}(u))$.

c) The convergence speed can be estimated by:

$$d(F^n(u), \alpha) \leqslant \frac{k^n}{1-k} d(u, F(u)).$$

Proof 1.4.1

- **Method 1:**

  - The fixed point's uniqueness: If we choose two fixed points of F to be α, α', then:

  $$d(\alpha, \alpha') = d(F(\alpha), F(\alpha')) \leqslant k.d(\alpha, \alpha')$$

  which is impossible unless $d(\alpha, \alpha') = 0$ that's to say $\alpha = \alpha'$.

  - The fixed point's existence: Now let's show that when $u \in \mathbb{E}$, the sequence $(F^n(u))_n \subset \mathbb{E}$ is from Cauchy. $\forall n \in \mathbb{N}$, we have :

  $$d(F^{n+1}(u), F^n(u)) \leqslant kd(F^n(u), F^{n-1}(u)) \leqslant \cdots \leqslant k^n d(F(u), u).$$

  Therefore, for every $n, p \in \mathbb{N}$ we have :

  $$\begin{aligned} d(F^{n+p}(u), F^n ux)) &\leqslant d(F^{n+p}(u), F^{n+p-1}(u)) + \cdots + d(F^{n+1}(u), F^n(u)) \\ &\leqslant (k^n + k^{n+1} + k^{n+2} + \cdots + k^{n+p-1}) d(F(u), u) \\ &\leqslant k^n(1 + k + k^2 + \cdots + k^{p-1}) d(F(u), u) \\ &\leqslant k^n \frac{1-k^p}{1-k} d(F(u), u). \end{aligned} \tag{1.5}$$

Where the last inequality is obtained by calculating the sum of a geometric sequence of reason k. We observe that if $n \to +\infty$, $d(F^{n+p}(u), F^n(u)) \to 0$ and so the sequence $(F^n(u))_n$ is of Cauchy in $\mathbb{E}$.
Therefore, as $\mathbb{E}$ is complete, this sequence converges to a limit that we will call α, which we will show is a fixed point of F.
First, let us remark that

$$\lim_{n\to\infty} F^{n+1}(u) = \lim_{n\to\infty} F^n(u).$$

Then, the fact that F is contracting entails that it is continuous, and therefore

$$\alpha = \lim_{n\to\infty} F^{n+1}(u) = \lim_{n\to\infty} F(F^n(u)) = F(\lim_{n\to\infty} F^n(u)) = F(\alpha).$$

Finally, in the inequality (3.5), if we fix n and in fact tend p towards infinity we will have

$$d(\alpha, F^n(u)) \leqslant \frac{k^n}{1-k} d(u, F(u)).$$

- **Method 2:**

Let $a = \inf d(u, F(u)) : u \in \mathbb{E} \geqslant 0$, suppose that $a > 0$. Then for $\epsilon > 0$, there exists $u \in \mathbb{E}$ such that $d(u, F(u)) < a + \epsilon$, we will therefore have

$$a \leqslant d(F(u), F^2(u)) \leqslant kd(u, F(u)) < k(a + \epsilon),$$

which brings us back to a contradiction when ϵ tends to 0, hence $a = 0$. For $n \in \mathbb{N}^*$, consider the sets

$$D_n = \{u \in E : d(u, F(u)) \leqslant \frac{1}{n}\}$$

- D_n is non-empty, because $a = 0$.
- D_n is closed, because every sequence of D_n convergent in $\mathbb{E}$ has a limit in D_n.
- Moreover, the sets D_n form a decreasing sequence when $n \to \infty$ with $\lim_{n\to\infty} diam(D_n) = 0$. Indeed, $\forall u, v \in D_n$

$$\begin{aligned} d(u, v) &\leqslant d(u, F(u)) + d(F(u), F(v)) + d(F(v), v) \\ &\leqslant \frac{2}{n} + kd(u, v). \end{aligned}$$

Which give

$$d(u,v) \leqslant \frac{2}{n(1-k)} \to 0 \quad \text{when} \quad n \to \infty.$$

Then, Cantor's nested set theorem ensures that the intersection of D_n in which $n \in \mathbb{N}^*$ is reduced to a point which is the unique fixed point of F.

Example 1.4.1

- Let $\mathbb{E} = [a,b]$ and the mapping $F : \mathbb{E} \to \mathbb{E}$ such that F is differentiable in each $u \in]a,b[$ and $|F'(u)| \leqslant k < 1$. Then, we deduce from the main value theorem that:
  $\forall u, v \in \mathbb{E}$, there exists a point ε between u and v such that
  $$F(u) - F(v) = F'(\varepsilon)(u-v),$$
  thus
  $$|F(u) - F(v)| = |F'(\varepsilon)||(u-v)| \leqslant k|u-v|,$$
  Consequently, F is contracting and therefore it admits a unique fixed point.

- Let $\mathbb{E} = \mathbb{R}$ and the application $F : \mathbb{R} \to \mathbb{R}$ defined by:
  $$F(u) = \frac{1}{2}u + 1,$$
  Then F is contracting and it admits a unique fixed point $u = 2$.

1.5 Elements of fractional calculus theory

Fractional calculus is an important field that develops in pure and applied mathematics. The applications of the theory of fractional calculus both in fundamental sciences and in engineering are very diverse. They appear more and more in different fields of research (see [[29], [41]]).

In this section gives some definitions and lemmas from fractional calculus. We start with a simple overview of the definition of the Gamma function and its properties thereafter we introduce fractional derivative and integral of the Riemann-Liouville type, then introduce Caputo's fractional integral and derivative. These definitions have been adopted of ([37], [44],[51]).

1.5.1 Gamma function

The gamma function is described in terms of improper definite integrals and is under the class of special transcendental functions.

Definition 1.5.1

The integral indicates and defines the gamma function:

$$\Gamma(r) = \int_0^{+\infty} t^{r-1} e^{-t} dt, r > 0, t > 0. \tag{1.6}$$

Property 1.5.1

(a) $\Gamma(r+1) = r\Gamma(r), \quad \mathrm{Re}(r) > 0.$

(b) $\Gamma(r+1) = r!, \quad \forall r \in \mathbb{N}.$

(c) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$

Example 1.5.1

Let $s > 1, x > 0$

$$\int_o^{\infty} \frac{x^s}{s^x} dx$$

We pose

$$I = \int_o^{\infty} \frac{x^s}{s^x} dx$$

and $s^x = e^t$ we have

$$\begin{aligned}
(I) &= \int_0^{\infty} \left(\frac{t}{\log s}\right)^s e^{-t} \frac{dt}{\log s} \\
&= \frac{1}{(\log s)^{s+1}} \int_0^{\infty} e^{-t} t^s dt \\
&= \frac{1}{(\log s)^{s+1}} \int_0^{\infty} t^{(s+1)-1} e^{-t} dt \\
&= \frac{1}{(\log s)^{s+1}} \Gamma(s+1).
\end{aligned}$$

1.5.2 Riemann-Liouville' fractional derivative

Definition 1.5.2 *[37]*

For a function x of order $r > 0$, the fractional integral is defined as follows:

$$I_{0+}^{r}x(t) = \frac{1}{\Gamma(r)} \int_{a}^{t} (t-s)^{r-1} x(s) ds,$$

where the gamma function is denoted by $\Gamma(r)$, given that the right side is point-wise defined on $(0, +\infty)$.

Definition 1.5.3 *[37]*

If a function x is defined on the interval $[0, +\infty)$, then the Riemann-Liouville fractional derivative of order $r > 0$ is given by

$$D_{0+}^{r}x(t) = \left(\frac{d}{dt}\right)^{n} I_{0+}^{n-r}x(t) = \frac{1}{\Gamma(n-r)} \left(\frac{d}{dt}\right)^{n} \int_{0}^{t} \frac{x(s)}{(t-s)^{r-n+1}} ds$$

where $n = 1 + [r]$, where $\Gamma(r)$, provided that the right side is point-wisely defined on $(0, +\infty)$.
The main properties of fractional integrals and derivatives are summarized in the following propositions.

Proposition 1.5.1 *[37]*

1- For $r < 0$, we set by convention $D_{0+}^{r}g(t) = I_{0+}^{-r}g(t)$. If $0 \leq q \leq r$, then $D_{0+}^{q}I_{0+}^{r}g(t) = I_{0+}^{r-q}g(t)$.

2- If $r \geq 0, \lambda > -1$, then

$$I_{0+}^{r}t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+r+1)} t^{\lambda+r} \tag{1.7}$$

Also, if $r \geq 0, \lambda > -1$ with $\lambda \neq r-k, k = 1, 2, 3, \ldots, [r]+1$, then

$$D_{0+}^{r}t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-r+1)} t^{\lambda-r} \tag{1.8}$$

3- $D_{0+}^{r}t^{\lambda} = 0$, for every $\lambda = r-k$ with $k = 1, 2, 3, \ldots, [r]+1$.

4- Assume that $g \in L^1(0,+\infty)$ and $r, q \in \mathbb{R}^+$. Then,

$$I_{0+}^r I_{0+}^q g(t) = I_{0+}^{r+q} g(t), \quad D_{0+}^r I_{0+}^r g(t) = g(t) \tag{1.9}$$

If similarly, $D_{0+}^r g(t) \in L^1(0,+\infty)$, then

$$I_{0+}^r D_{0+}^r g(t) = g(t) + \sum_{i=1}^{i=n} c_i t^{r-i} \tag{1.10}$$

for some $c_i \in \mathbb{R}$.

1.5.3 Caputo's fractional derivative

Definition 1.5.4 *[44]*

For a given function F in the closed interval $[a, b]$, the Caputo fractional order derivative of F, is defined by:

$$\left({}^c D_{a+}^r F\right)(t) = \frac{1}{\Gamma(n-r)} \int_a^t (t-s)^{n-r-1} F^{(n)}(s) ds, \tag{1.11}$$

where $n = [r] + 1$.

Example 1.5.2

Assume that $a = 0, r = \frac{1}{2}, (n = 1)$, and $F(t) = t$. Then, applying formula (1.11) we get

$$^C D_{0+}^{\frac{1}{2}} t = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} ds$$

Using substitution $u := (t-s)^{\frac{1}{2}}$ and considering the properties of the Gamma function, the final result for the Caputo fractional derivative of the function $F(t) = t$ is derived as

$$\begin{aligned} ^C D_{0+}^{\frac{1}{2}} t &= \frac{1}{-\sqrt{\pi}} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} d(t-s) \\ &= \frac{1}{-\sqrt{\pi}} \int_{\sqrt{t}}^0 \frac{1}{u} du^2 \\ &= \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{t}} \frac{2u}{u} du \\ &= \frac{2\sqrt{t}}{\sqrt{\pi}} \end{aligned}$$

Consequently, it holds

$$^C D_{0+}^{\frac{1}{2}} t = \frac{2\sqrt{t}}{\sqrt{\pi}}$$

Lemma 1.5.1

Let $n > r > n-1$, then the differential equation ${}^cD^r h(t) = 0$ has solutions

$$h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}, i = overline 0, n-1, n = [r] + 1$.

Lemma 1.5.2 *[51]*

Let $n - 1 < r \leq n$, then

$$I_{a+}^r \left({}^cD_{a+}^r h\right)(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}, \quad i = \overline{0, n-1}, n = [r] + 1$.

Lemma 1.5.3 *[37]*

Let $r > q > 0$ and $h \in L^1(0, +\infty)$, then for the Caputo fractional derivative it holds

1- ${}^cD^r C = 0, \qquad C = const.$

2- $\left({}^cD_{0+}^r h\right)(t) I_{0+}^r h(t) = h(t)$.

3- $\left({}^cD_{0+}^r h\right)(t) I_{0+}^q h(t) = I_{0+}^{r-q} h(t)$.

CHAPTER

2

SOME TOPOLOGICAL DEGREE THEORY

THis chapter goes over the meaning of topological degree and some of its fundamental characteristics in two cases: the Brouwer degree in finite dimensions and the Leray-Schauder degree in infinite dimensions, and then generalize them to Degree of coincidence: For an operator of the abstract form $L + N$, where L is the Fredholm operator at index 0 and N is a generally nonlinear and compact operator, with the remainder pointing to L . Then we propose Mahwin's coincidence theorem and its proof. Finally, we have given definition of condensing perturbation of the identity. For more details, see [[9],[17], [19], [40] and [23]].

2.1 The Brouwer degree

Now we give how to build a Brouwer degree in this section as follows:

Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset, $f : \Omega \to \mathbb{R}^n$ a mapping, $f \in \mathcal{C}^1(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ We consider the problem $(\mathbf{P})$:

$$(\mathrm{P}) \begin{cases} \text{let } y_0 \in \mathbb{R}^n, \\ \text{we look for } x \in \Omega \quad \text{such that,} \\ f(x) = y_0. \end{cases}$$

Example 2.1.1

$(\Omega =]0,1[, n = 1)$ and $f :]0,1[\to \mathbb{R}$, from class $\mathcal{C}^1$ mapping, verifying the hypothesis $(\mathcal{H})$

$$(\mathcal{H}) \text{ for all solution } x \text{ of } (\mathbf{P}), f'(x) \neq 0.$$

We then define the topological degree (which is a relative integer d) of f in y_0 relative to Ω by:

$$d = \deg(f, \Omega, y_0) = \begin{cases} \sum_{i \in I} \operatorname{Sgn}(f'(x_i)), & \text{if } \{x_i, i \in I\} \text{ is the set of solutions of } (\mathbf{P}); \\ 0, & \text{if the problem } (\mathbf{P}) \text{ has not a solution .} \end{cases}$$

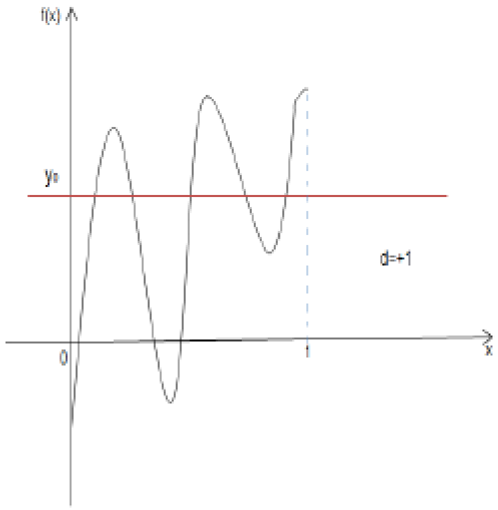
when $y_0 \notin f(\partial\Omega)$ and $I \subset \mathbb{N}$.

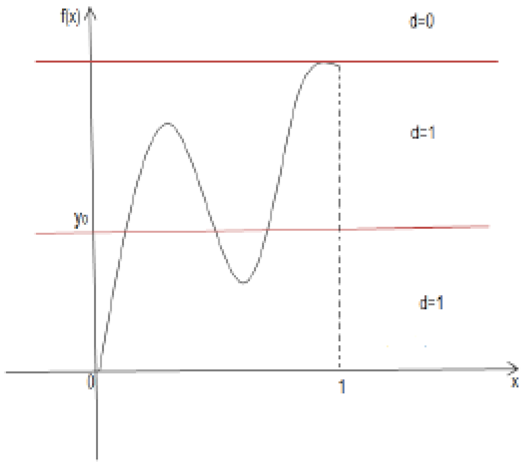
Remark 2.1.1

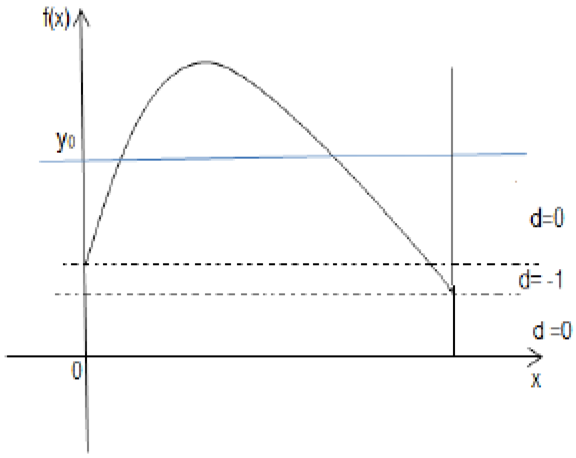
For all $x \in \mathbb{R}$, the number d satisfies the following properties:

- $d \in \mathbb{Z}$
- d depends on $f; \Omega; y_0$.
- The sum involved in the definition of d is finite.

Let us give some illustrative examples:







Remark 2.1.2

The number d verifies some very important properties:

- If $f \in \mathcal{C}^1(\Omega) \cap \mathcal{C}(\bar{\Omega})$, d remains constant over a certain interval.
- If $d \neq 0$ then the problem $(\mathbf{P})$ admits at least one solution.
- If $d = 0$ then the problem $(\mathbf{P})$ may or may not admit a solution.

2.1.1 Definition of Brouwer's topological degree

Assume that Ω an open bounded subset of $\mathbb{R}^n$, $f : \Omega \to \mathbb{R}^n$ an application from $\mathcal{C}^1(\Omega) \cap \mathcal{C}(\bar{\Omega})$ and $y_0 \in \mathbb{R}^n \backslash f(\partial\Omega)$. We take into consideration the following problem:

$$(\mathcal{P}) \quad \text{find } x \in \Omega \text{ such as } f(x) = y_0.$$

♦ ***The regular case:***

For $x_0 \in \Omega$, we note $\mathcal{D}f(x_0) = \left(\frac{\partial f_i}{\partial x_j}\right)_{1 \leq i,j \leq n}(x_0)$, the Jacobian matrix of f in x_0 and $\mathcal{J}_f(x_0) = det\left[\mathcal{D}f(x_0)\right]$ the Jacobian determinant of f in x_0.

Definition 2.1.1

1. A point $x_0 \in \Omega$ is called "regular" if $\mathcal{J}_f(x_0) \neq 0$.
2. A point $x_0 \in \Omega$ is called "singular" or "critical" if it is not regular.
   In this case, the set of singular points of f on Ω is given by
   $$\mathcal{S}_f(\Omega) = \{x_0 \in \Omega : \mathcal{J}_f(x_0) = 0\}.$$
3. A value $y_0 \in f(\bar{\Omega})$ is called "regular" if
   $$f^{-1}(\{y_0\}) \cap \mathcal{S}_f(\Omega) = \varnothing.$$
   In other words, y_0 is regular if: $\forall x \in f^{-1}(\{y_0\}), \mathcal{J}_f(x) \neq 0$. Otherwise, it is called "singular".

Proposition 2.1.1

If $y_0 \notin f(\partial\Omega)$ is a regular value, then the set $f^{-1}(\{y_0\})$ is finite.

Proof 2.1.1

$$\begin{aligned} y_0 \text{ regular } &\Rightarrow \forall x \in f^{-1}(\{y_0\}), \mathcal{J}_f(x) \neq 0 \\ &\Rightarrow \forall x \in f^{-1}(\{y_0\}), \exists U \in \mathcal{V}(x) \text{ tel que } f_{|U} \text{ is a homeomorphism} \\ &\quad \text{(local inversion theorem.)} \\ &\Rightarrow \text{all points of } f^{-1}(\{y_0\}) \text{ are isolated.} \\ &\quad \left(f^{-1}(\{y_0\}) \text{ is a discrete set}\right) \end{aligned}$$

Now, f being continuous, $f^{-1}(\{y_0\})$ is closed and therefore compact because it is included in the bounded Ω. Finally,

$$f^{-1}(\{y_0\}) \text{ is compact and discret } \Leftrightarrow f^{-1}(\{y_0\}) \text{ is finite.}$$

Definition 2.1.2

If $y_0 \notin [f(\partial\Omega) \cup f(\mathcal{S}_f(\Omega))]$, then we define the topological degree of Brouwer of f in y_0 relative to Ω by

$$\deg(f, \Omega, y_0) = \begin{cases} \sum_{x \in \Omega \cap f^{-1}(\{y_0\})} \operatorname{sgn} \mathcal{J}_f(x); \\ 0, \text{ si } \Omega \cap f^{-1}(\{y_0\}) = \varnothing \end{cases}$$

♦ ***The singular case:***

If $y_0 \notin f(\partial\Omega)$ is a singular value, we set

$$deg(f, \Omega, y_0) = deg(f, \Omega, y_1),$$

where y_1 is a regular value close to y_0 and whose existence is ensured by the following lemma:

Lemma 2.1.1 ***(Sard's Lemma)***

Let $\Omega \subset \mathbb{R}^n$ be an open subset and f a map continuously differentiable on Ω. Then $f(\mathcal{S}_f(\Omega))$ is a set of measure zero.

Proof 2.1.2 *[24]*

Example 2.1.2

Let $\Omega = (]-1,1[)^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ the function defined by

$$f(x, y) = (y - x^3, y).$$

We've got $S_f(\Omega) = \{X = (x, y) \in \mathbb{R}^2 : \mathcal{J}_f(X) = 0\}$, $\mathcal{D}_f(x, y) = \begin{pmatrix} -3x^2 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow$ $\mathcal{J}_f(x, y) = -3x^2$.

Therefore, $\mathcal{J}_f(x, y) = 0 \Rightarrow x = 0$ and $y \in \mathbb{R}$. Hence:

$$\mathcal{S}_f(\Omega) = \{(0, y) : y \in \mathbb{R}\} = (Oy).$$

i.e. the singular values are then represented by the y-axis.

$f(x, y) = Y_0 = (0, 0) \Rightarrow (x, y) = (0, 0) \Rightarrow f^{-1}(\{Y_0\}) = \{(0, 0)\}$. So

$$f^{-1}(\{Y_0\}) \cap \mathcal{S}_f(\Omega) = \{(0, 0)\} \neq \varnothing,$$

this shows that Y_0 is a singular value. Let's calculate now $f(\partial\Omega)$.
We have $\partial\Omega = (D_1)\cup(D_2)\cup(D_3)\cup(D_4)$ with

$$(D_1) = \{(-1,y)/-1\leq y\leq 1\},\quad (D_3) = \{(x,+1)/-1\leq x\leq 1\},$$
$$(D_2) = \{(+1,y)/-1\leq y\leq 1\},\quad (D_4) = \{(x,-1)/-1\leq x\leq 1\}.$$

Thus, $f(\partial\Omega) = (\Delta_1)\cup(\Delta_2)\cup(\Delta_3)\cup(\Delta_4)$, $(\Delta_i) = f\left(D_i\right), \forall i = \overline{1,4}$. We find

$(\Delta_1) : \begin{cases} y = x-1 \\ 0\leq x\leq 1 \end{cases}, (\Delta_2) : \begin{cases} y = x+1 \\ -1\leq x\leq 0 \end{cases}, (\Delta_3) : \begin{cases} y = 1 \\ -1\leq x\leq 1 \end{cases}$ et $(\Delta_4) : \begin{cases} y = -1 \\ -1\leq x\leq 1 \end{cases}.$

It's clear that $Y_0\notin f(\partial\Omega)$. So let's put

$$\deg\left(f,\Omega,Y_0\right) = \deg\left(f,\Omega,Y_1\right),$$

with $Y_1 = \left(-\frac{1}{2},0\right)\in\mathcal{B}(0,1)$ for the norm

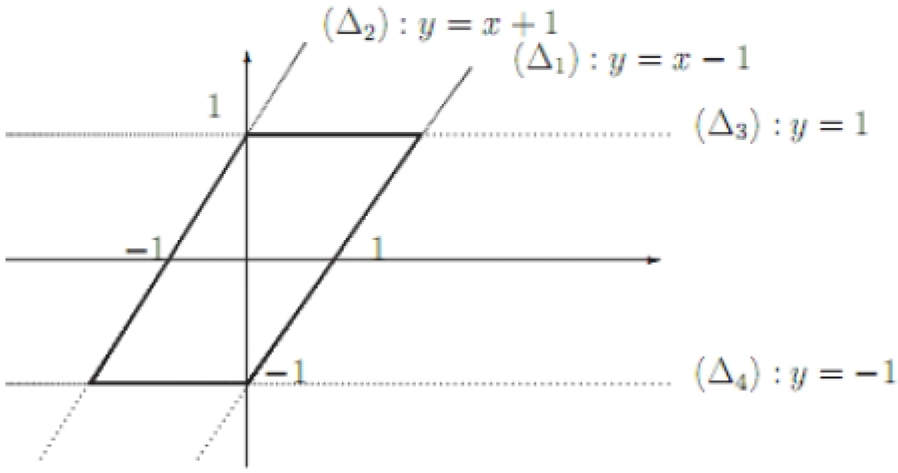
$$\|(x,y)\|_\infty = \max(|x|,|y|).$$

Let's check that Y_1 is a regular value. We have:

$$f(x,y) = \left(-\frac{1}{2},0\right)\Rightarrow(x,y) = \left(\sqrt[3]{\frac{1}{2}},0\right),$$

then $f^{-1}\left(\{Y_1\}\right)\cap\mathcal{S}_f(\Omega) = \varnothing$.
Additionally, $f^{-1}\left(\{Y_1\}\right)\subset\Omega, Y_1\notin f(\partial\Omega)$ and $Y_1\notin f\left(\mathcal{S}_f(\Omega)\right)$, because $f\left(\mathcal{S}_f(\Omega)\right) = f((Oy)) = \{f(x,y) : (x,y)\in(Oy)\} = \{f(0,y) : y\in\mathbb{R}\} = \{(y,y) : y\in\mathbb{R}\}$, so $f\left(\mathcal{S}_f(\Omega)\right)$ is exactly the first bisector. finally, we find:

$$\deg\left(f,\Omega,0_{\mathbb{R}^2}\right) = \operatorname{sgn}\mathcal{J}_f\left(\sqrt[3]{\frac{1}{2}},0\right) = -1.$$



2.1.2 Properties of the Brouwer degree

In all that follows, we will designate by $\mathcal{I}_d$ the identity map on $\mathbb{R}^n$. If $y_0 \notin f(\partial\Omega)$, then an integer $\deg(f, \Omega, y)$ that satisfies the next properties exists:

1) Degree of identity

- $deg\left(\mathcal{I}_d, \Omega, y_0\right) = \begin{cases} 1, & \text{if } y_0 \in \Omega \\ 0, & \text{if } y_0 \notin \bar{\Omega} \end{cases}$
- $\deg\left(-\mathcal{I}_d, \Omega, y_0\right) = \begin{cases} (-1)^n, & \text{if } y_0 \in \Omega \\ 0, & \text{if } y_0 \notin \bar{\Omega} \end{cases}$

2) Validity of the degree

If $\deg(f, \Omega, y) \neq 0$, then $f(x) = y$, for all $x \in \Omega$.

3) Additivity

Let be $y_0 \in \mathbb{R}^n$ and $(\Omega_i)_{i \in I} \subset \Omega$ a family of two-by-two disjoint open sets satisfying the following condition:

$\Omega \subseteq \cup_{i \in I} \Omega_i$ and $y_0 \notin f\left(\bar{\Omega} \setminus \cup_{i \in I} \Omega_i\right)$.

So,

$$\deg\left(f, \Omega, y_0\right) = \sum_{i \in I} \deg\left(f, \Omega_i, y_0\right)$$

where only a finite number of terms in the sum are non-zero.

4) Homotopy invariance

Let be $\{f_t\}_{0 \leq t \leq 1}$ a family of maps from $\mathcal{C}\left(\bar{\Omega}, \mathbb{R}^n\right)$, depending continuously on t and $\{y_0(t)\}_{0 \leq t \leq 1}$ a family of continuous points at t and such that $y_0(t) \notin f_t(\partial\Omega)$, for all $t \in [0, 1]$. Then the degree $\deg\left(f_t, \Omega, y_0(t)\right)$ does not depend on t i.e.$\deg\left(f_t, \Omega, y_0(t)\right) = constant$. In particular, we have

$$\deg\left(f_0, \Omega, y_0(0)\right) = \deg\left(f_1, \Omega, y_0(1)\right).$$

The functions f_t are said to be homotopically connected. More generally, we say that two functions f and g are homotopic, if there exists a continuous function $H : [0, 1] \times \bar{\Omega} \to \mathbb{R}^n$ such as:

$$H(0, x) = f(x), H(1, x) = g(x), \forall x \in \bar{\Omega}.$$

5) Boundary invariance

If $y_0 \notin f(\partial\Omega)$ and $f_{|_{\partial\Omega}}(x) = g_{|_{\partial\Omega}}(x)$, then

$$\deg\left(g, \Omega, y_0\right) = \deg\left(f, \Omega, y_0\right)$$

6) Continuity with respect to y_0

If y_1 is close to $y_0 \notin f(\partial\Omega)$ (in a sense to be specified), then

$$\deg(f, \Omega, y_0) = \deg(f, \Omega, y_1)$$

7) Continuity compared with function

Let be $r = \operatorname{dist}(y_0, f(\partial\Omega)) > 0$ and let be $g \in \mathcal{C}^1(\bar{\Omega})$ such as

$$\sup_{x \in \partial\Omega} \|g(x) - f(x)\| < r.$$

So,

$$\deg(f, \Omega, y_0) = \deg(g, \Omega, y_0)$$

The Brouwer fixed point theorem, 1912

Theorem 2.1.1

Let C be a compact, non-empty convex of $\mathbb{R}^n$ and $f : C \to C$ a continuous map. Then f admits at least one fixed point in C.
In particular, for $C = \overline{\mathcal{B}}$; the closed unit ball of $\mathbb{R}^n$.

Proof 2.1.3

Let us prove this theorem in the case where $C = \overline{\mathcal{B}}(0, R), R > 0$.

- If $f(x_0) = x_0$ for $x_0 \in \partial C$, then the theorem is proved.
- Otherwise, $f(x) \neq x, \forall x \in \partial C$. In this case, we consider the continuous deformation
$$f_t(x) = x - tf(x).$$

For $t \in [0, 1[$ and $x \in \partial C$, we have:

$$\begin{aligned}
\|f_t(x)\| &= \|x - tf(x)\| \\
&\geq |\|x\| - t\|f(x)\|| \\
&\geq |R - t\|f(x)\|| \\
&\geq (1-t)R \\
&> 0
\end{aligned}$$

because: $\forall x \in \overline{\mathcal{B}}(0,R)$, $f(x) \in \overline{\mathcal{B}}(0,R) \Rightarrow \|f(x)\| \leq R$.
So, $f_t(x) \neq 0, \forall x \in \partial C$, hence $y_0 = 0 \notin f_t(\partial C)$. Degree $\deg\left(\mathcal{I}_d - tf, \overset{\circ}{C}, 0\right)$ is therefore well defined and is worth, by homotopy, $\deg\left(f_1, \overset{\circ}{C}, 0\right) = \deg\left(\mathcal{I}_d - f, \overset{\circ}{C}, 0\right) = \deg\left(f_0, \overset{\circ}{C}, 0\right) = \deg\left(\mathcal{I}_d, \overset{\circ}{C}, 0\right) = 1$. Then f admits at least one fixed point in C.

Remark 2.1.3

This theorem has an extension in infinite dimension, Schauder's fixed point theorem.

2.2 The Leray-Schauder degree (in infinite dimension)

Now, we aim to build a topological degree in an infinite dimensional space that results in the same result as Brouwer's. This is a tool that ensures that the equation $f(x) = y_0$ has at least one solution; where f is a continuous map of a Banach space $\mathbb{X}$ on itself. Unlike the case of finite dimension, this degree; known as Leray-Schauder, cannot be defined for all continuous applications from $\mathbb{X}$ to $\mathbb{X}$. It will therefore be necessary to restrict the functions that we consider and for this, we construct the topological degree of Leray-Schauder on the applications which differ by the identity of a compact application which is called compact identity perturbation.

Now, let us state the following proposition which gives the essential difference between the cases in finite and infinite dimensional spaces, namely;

Proposition 2.2.1

Let $\mathcal{B}$ be the open unit ball of a normalized vector space $\mathbb{X}$, Then we have the following equivalences:

$\dim \mathbb{X} < +\infty \Leftrightarrow$ all continues applications $f : \overline{\mathcal{B}} \to \overline{\mathcal{B}}$ admits at least one fixed point.
$\Leftrightarrow$ the closed unit ball $\overline{\mathcal{B}}$ is compact.
$\Leftrightarrow$ the border $\partial\mathcal{B}$ is compact.
$\Leftrightarrow$ from any sequence of $\overline{\mathcal{B}},$ a convergent subsequence can be extracted.

2.2.1 Definition of the Leray-Schauder degree theory, 1934

For more detail see [40]

Definition 2.2.1

Let $\mathbb{X}$ be a Banach space, $\Omega \subset \mathbb{X}$ an open bounded set and

$$f = \mathcal{I}_d - T : \overline{\Omega} \to \mathbb{X} \text{ a compact perturbation of identity.}$$

for $a \in \mathbb{X}\backslash f(\partial\Omega)$, lets put $\delta = dist(a, f(\partial\Omega)) = \inf\limits_{x\in\partial\Omega} \|a - f(x)\|_{\mathbb{X}} > 0$. Let $T_\epsilon : \overline{\Omega} \to \mathbb{X}$ a compact application with value in a space $N_\epsilon \subset \mathbb{X}$ of finite dimension containing b and such that:

$$\sup_{x\in\overline{\Omega}} \|T_\epsilon(x) - T(x)\|_{\mathbb{X}} \leqslant \frac{\delta}{2}$$

Then we define the Leray-Schauder degree by

$$\deg(\mathcal{I}_d - T, \Omega, a) = \deg(\mathcal{I}_d - T_\epsilon, \Omega, a) = \deg(\mathcal{I}_d - T_\epsilon|_{\overline{\Omega}\cap N_\epsilon}, \Omega \cap N_\epsilon, a)$$

The latter is a Brouwer degree, because $\Omega \cap N_\epsilon$ is of finite dimension from the fact that N_ϵ is also.

Proposition 2.2.2

The Brouwer degree $\deg(\mathcal{I}_d - T_\epsilon|_{\overline{\Omega}\bigcap N_\epsilon}, \Omega \cap N_\epsilon, a)$ is well defined.

Proof 2.2.1 *[19]*

2.2.2 Properties of Leray-Schauder degree theory

We will give the important properties of Leray-Schauder degree. So let Ω be an open bounded set in the Banach space $\mathbb{X}$.

1) Normalization

$$\deg(\mathcal{I}_d, \Omega, a) = \begin{cases} 1, & \text{if } a \in \Omega \\ 0, & \text{if } a \notin \bar{\Omega} \end{cases}$$

2) Additivity

Assume that Ω_1 and Ω_2 are open bounded disjoint subsets of Ω. If $a \notin (\mathcal{I}_d - T)(\overline{\Omega}\backslash(\Omega_1 \cup \Omega_2))$, then

$$\deg(\mathcal{I}_d - T, \Omega, a) = \deg(\mathcal{I}_d - T, \Omega_1, a) + \deg(\mathcal{I}_d - T, \Omega_2, a)$$

3) Homotopy invariance

Let $T : [0,1] \times \overline{\Omega} \to \mathbb{X}$ be a compact map, $b : [0,1] \to \mathbb{X}$ is continuous and for all $t \in [0,1]$, $b(t) \notin \deg(\mathcal{I}_d - T(t,.))(\partial\Omega)$, then

$$\deg(\mathcal{I}_d - T(0,.), \Omega, b(0)) = \deg(\mathcal{I}_d - T(1,.), \Omega, b(1))$$

4) Validity of the degree

$$\deg(\mathcal{I}_d - T, \Omega, a) \neq 0 \Longrightarrow \exists x \in \Omega : x - T(x) = a$$

The Schauder fixed point theorem, First version

Theorem 2.2.1

Let F be a compact map of B_r in itself, and let B_r be an open ball of radius r and center 0 in the Banach space $\mathbb{X}$. Then, at least one solution to the equation $Fx = x$ may be found in $\overline{B}_r$.

Proof 2.2.2

$tF : [0,1] \times B_r(0,1) \to \mathbb{X}$ is a compact operator. By the property of homotopic invariance of the degree, we have:

$$\deg(\mathcal{I}_d - tF, B_r, 0) = \deg(\mathcal{I}_d, B_r, 0) = 1$$

provided that $\mathcal{I}_d - tF \neq 0$ on ∂B_r, where $t \in [0,1]$. So $Fx = x$ admits at least one solution.

Let $B_r = B(0,r)$ and suppose $\exists x_0 \in \partial B_r(||x_0|| = r)$ and $F_0 \in [0,1]$ such that $x_0 = t_0 F x_0$

- For $t_0 = 0 : x_0 = 0$ contradiction with $0 \in B_r$
- For $t_0 = 1 : x_0 = Fx_0$ (F admits a fixed point on $\partial B_r \subset \overline{B}_r$)
- For $t_0 \in]0,1[$:

$$\begin{aligned} x_0 = t_0 F x_0 &\Longrightarrow ||x_0|| = t_0 ||Fx_0|| \\ &\Longrightarrow ||x_0|| < ||Fx_0|| \\ &\Longrightarrow ||Fx_0|| > r \end{aligned}$$

contradiction avec $F(B_r) \subset B$.

The Schauder fixed point theorem, 1930, Second version

Theorem 2.2.2

Let C be a convex, closed, bounded, non-empty subset in Banach space $\mathbb{X}$ and $K : C \rightarrow C$ a compact application. Then K admits at least one fixed point.

Proof 2.2.3

- **First step**:
  We assume that $C = B(0, R)$. If there exists $x_0 \in \partial C$ such that $K(x_0) = x_0$, there is nothing to prove.
  Otherwise, $\forall t \in [0, 1]$, the degree $\deg(K_t, C, 0)$, where $K_t = I - tK$, is well defined. Indeed, if there exists $x \in \partial C, tK(x) = x$, then $R = \|x\| = t\|K(x)\| \leq Rt$ because $K(C) \subset C$ and therefore $t = 1$, which leads to a contradiction with $\|K(x)\| = R = \|x\|$. The degree is therefore well defined and is worth, by homotopy, $\deg(I - K, C, 0) = 1$ hence the result.

- **Second step**:
  C is a bounded, closed, convex, non-empty.
  We have B, a ball containing C, and a continuous retraction $R : \mathbb{X} \longrightarrow \mathcal{C}$. Consider the diagram $B \xrightarrow{R} C \xrightarrow{k} B$. Since R is bounded and K is compact, the map $(K \circ R)$ is compact. The map $(K \circ R)$ admits a fixed point $x_0 \in B, x_0 = (K \circ R)(x_0)$, according to the first step. Now, $R(x_0) \in C$ and by hypothesis, $K(C) \subset C$; then $K(R(x_0)) \in C$ and therefore $x_0 \in C$.

Remark 2.2.1

On the other hand, the two fixed-point theorems are similar. Since every continuous map is compact in finite dimension, Brouwer's fixed point theorem is a specific instance of Schauder's fixed point theorem.

Example 2.2.1

Let C be closed, bounded from a Banach space $\mathbb{X}$ and $f : C \to \mathbb{X}$ a compact application such as $\exists x_0 \in \overset{\circ}{C}, \forall x \in \partial C, f(x) - x_0 \neq \lambda(x - x_0)$.
Let us prove that f has a fixed point in C.
we consider the deformation $f_t(x) = (x - x_0) - t(f(x) - x_0)$.
we can assume that $f_t(x) \neq 0, \forall x \in \partial C$ because otherwise the theorem is proved; it is also clear that $f_0(x) \neq 0, \forall x \in \partial C$. moreover, by hypothesis, $\forall t \in]0, 1[$ and $\forall x \in \partial C, \frac{1}{t}(x_0) \neq f(x) - x_0$. the degree of f_t relative to $\overset{\circ}{C}$ and at the origin is well defined and is worth, by homotopy

$$\deg(\mathcal{I}_d - f, \overset{\circ}{C}, 0) = \deg(\mathcal{I}_d - x_0, \overset{\circ}{C}, 0) = \deg(\mathcal{I}_d, \overset{\circ}{C}, 0) = 1$$

the mapping f therefore admits at least one fixed point in $\overset{\circ}{C}$, or on ∂C, so in C.

Theorem 2.2.3 ***(Nonlinear Leray-Schauder alternative)****[[17],[23]]*

Let Ω be a bounded open of a Banach space $\mathbb{X}$ and $K : \Omega \to \mathbb{X}$ a compact map. Then at least one of assertions (i) or (ii) is satisfied:

i) K admits at least one fixed point in Ω.

ii) $\exists x \in \partial\Omega, \exists t \in [0, 1]$ such that $x = tK(x)$.

Proof 2.2.4

If the second criteria (ii) is not met, the following statement is true: $\forall x \in \partial\Omega, \forall t \in [0, 1]$ we have $(\mathcal{I}_d - tK)(x) \neq 0$. The degree $\deg(\mathcal{I}_d - tK, \Omega, 0)$ is therefore well defined and is worth, by homotopy, $\deg(\mathcal{I}_d, \Omega, 0) = 1$. In particular, for $t = 1$, we deduce that K admits at least one fixed point in Ω.
It suffices to apply the nonlinear Leray-Schauder alternative for $\Omega = B$.

Corollary 2.2.1

Let $K : \mathbb{X} \to \mathbb{X}$ be a compact map and $\mathbb{X}$a Banach space and satisfying the following hypothesis

$$(\mathcal{H}) : \{\exists R > 0, \forall t \in [0, 1], tK(x) = x \Longrightarrow x \in B(0, R).\}$$

Then K admits at least one fixed point in $B = B(0, R)$.

Remark 2.2.2

1) $(\mathcal{H})$ is an a priori estimation hypothesis;

2) The fixed point theorem by Schauder is equivalent to this consequence;

3) The Schaefer version of this corollary is as follows as well:

Theorem 2.2.4 ***(Schaefer's Theorem, 1955)*** *[19]*

The Banach space $\mathbb{X}$ and the compact map $K : \mathbb{X} \to \mathbb{X}$ are to be considered. So, the following alternative is available to us:
the equation $tK(x) = x$ admits a solution for every $t \in [0, 1]$.
or else, the set $S = \{x \in \mathbb{X} : \exists t \in [0, 1], tK(x) = x\}$ is unbounded.

Proof 2.2.5

In practice, we show that S is bounded which implies that $tK(x) = x$ admits a solution.

2.3 Coincidence degree for perturbation of Fredholm operators

In this section, we check for existential solutions to operator-form problem:

$$\mathcal{L}x = Nx, \tag{2.1}$$

where $\mathcal{L}$ denotes a linear mapping between $\mathbb{X}$ and certain topological vector spaces $\mathbb{Z}$, and N is a nonlinear operator. First, we start this section by introducing some definitions :

2.3.1 Fredholm operators

Definition 2.3.1

We state that a linear map $\mathcal{L} : D(\mathcal{L}) \subset \mathbb{X} \to \mathbb{Z}$, is a Fredholm operator [13] if it fulfills the subsequent conditions:

1) $\ker(\mathcal{L}) = \mathcal{L}^{-1}(0)$ is finite dimensional space of $\mathbb{X}$,

2) $\mathrm{Im}(\mathcal{L}) = \mathcal{L}(D(\mathcal{L}))$ is closed and has finite codimension.

Definition 2.3.2 (Index of a Fredholm operator)

The integer

$$\mathrm{ind}(\mathcal{L}) = \dim(\ker(\mathcal{L})) - \mathrm{codim}(\mathrm{Im}(\mathcal{L})).$$

is the index of a Fredholm operator L.

Example 2.3.1

- If $\mathbb{Y}$ and $\mathbb{Z}$ are of finite dimensions, then any linear map $\mathcal{L} : \mathbb{Y} \to \mathbb{Z}$ is a Fredholm with

$$\mathrm{ind}(\mathcal{L}) = \dim(\mathbb{Y}) - \dim(\mathbb{Z}).$$

- The identity $\mathcal{I}_d : \mathbb{Y} \to \mathbb{Y}$ is a Fredholm operator with index 0.

2.3.2 Generalized inverse of $\mathcal{L}$

Let $\mathcal{L} : \mathrm{Dom}(\mathcal{L}) \subset \mathbb{X} \to \mathbb{Z}$ be a Fredholm operator of index 0. Let $\mathscr{P}$ and $\mathscr{Q}$ be two continuous projectors, $\mathscr{P} : \mathbb{X} \to \mathbb{X}$ and $\mathscr{Q} : \mathbb{Z} \to \mathbb{Z}$ such that :

$$\mathrm{Im}(\mathcal{L}) = \mathrm{Ker}(\mathscr{Q}) \text{ and } \mathrm{Ker}(\mathcal{L}) = \mathrm{Im}(\mathscr{P}).$$

Let's put

$$X_1 = \mathrm{Im}(\mathcal{I}_d - \mathscr{P}) = \mathrm{Ker}(\mathscr{P}) \text{ and } Z_1 = \mathrm{Im}(\mathscr{Q});$$

then we are able to write

$$\mathbb{X} = \mathrm{Ker}(\mathcal{L}) \oplus X_1; \mathbb{Z} = \mathrm{Im}(\mathcal{L}) \oplus Z_1.$$

Consider an isomorphism,

$$\mathcal{J} : \mathrm{Ker}(\mathcal{L}) \to Z_1.$$

Whose existence is guaranteed by the fact that $\dim \mathrm{Ker}(\mathcal{L}) = \dim(Z_1) = n$.
Noticing

$$\mathrm{Dom}(\mathcal{L}) = ker\mathcal{L} \oplus (\mathrm{Dom}(\mathcal{L}) \cap X_1).$$

Additionally, it is an isomorphism on $\mathrm{Im}(\mathcal{L})$ that restricts $\mathcal{L}$ to $\mathrm{Dom}(\mathcal{L}) \cap X_1$. indicate by $\mathcal{L}_{\mathscr{P}}$ this restriction and by $\mathcal{L}_{\mathscr{P}}^{-1} : \mathrm{Im}(\mathcal{L}) \to \mathrm{Dom}(\mathcal{L}) \cap X_1$ the inverse of $\mathcal{L}_{\mathscr{P}}$. Then the operator

$$\mathcal{J}^{-1} \oplus \mathcal{L}_{\mathscr{P}}^{-1} : \mathbb{Z} = Z_1 \oplus \mathrm{Im}(\mathcal{L}) \to \mathbb{X} = \mathrm{Ker}(\mathcal{L}) \oplus \mathrm{Dom}(\mathcal{L}) \cap X_1,$$

is an isomorphism, and the following operator is its inverse:

$$\mathcal{L} + \mathcal{J}\mathscr{P} : \mathrm{Dom}(\mathcal{L}) \cap \mathrm{Im}(\mathcal{L})(\mathcal{I}_d - \mathscr{P}) \oplus \ker(\mathcal{L}) \to \mathrm{Im}(\mathcal{L}) \oplus Z_1,$$

In fact, for each $x \in \mathrm{Dom}(\mathcal{L}) \cap \mathrm{Im}(\mathcal{L})(\mathcal{I}_d - \mathscr{P}) \oplus \ker(\mathcal{L})$, We fill it out in the form $x = (\mathcal{I}_d - \mathscr{P})x + \mathscr{P}x$
then

$$(\mathcal{L} + \mathcal{J}\mathscr{P})((\mathcal{I}_d - \mathscr{P})x + \mathscr{P}x) = \mathcal{L}(\mathcal{I}_d - \mathscr{P})x + \mathcal{J}\mathscr{P}(\mathscr{P}x) = \mathcal{L}(\mathcal{I}_d - \mathscr{P})x + \mathcal{J}\mathscr{P}x,$$

so

$$\mathcal{J}^{-1} \oplus \mathcal{L}_{\mathscr{P}}^{-1}(\mathcal{L}(\mathcal{I}_d - \mathscr{P})x + \mathcal{J}\mathscr{P}x) = (\mathcal{I}_d - \mathscr{P})x + \mathscr{P}x = x.$$

Conversely, however, for each $z \in Z$ we have

$$(\mathcal{J}^{-1} \oplus \mathcal{L}_{\mathscr{P}}^{-1})z = (\mathcal{J}^{-1} \oplus \mathcal{L}_{\mathscr{P}}^{-1})(\mathscr{Q}z + (\mathcal{I}_d - \mathscr{Q})z) = \mathcal{J}^{-1}\mathscr{Q}z + \mathcal{L}_{\mathscr{P}}^{-1}(\mathcal{I}_d - \mathscr{Q})z,$$

By establishing $K_{\mathscr{P},\mathscr{Q}} = \mathcal{L}_{\mathscr{P}}^{-1}(\mathcal{I}_d - \mathscr{Q})$,($K_{\mathscr{P},\mathscr{Q}}$is the inverse on the right of $\mathcal{L}$ associated with $\mathscr{P}$ and $\mathscr{Q}$ respectively), so we obtain

$$(\mathcal{L} + \mathcal{J}\mathscr{P})^{-1} = \mathcal{J}^{-1}\mathscr{Q} + K_{\mathscr{P},\mathscr{Q}}.$$

2.3.3 $\mathcal{L}$ -compact perturbations of Fredholm operator of zero index

Definition 2.3.3

Let $N : \Omega \subset \mathbb{X} \to \mathbb{Z}$ be nonlinear operator, then, we declare N to be $\mathcal{L}-$compact operator if this operator

$$M = \mathscr{P} + \mathcal{J}^{-1}\mathscr{Q}N + K_{\mathscr{P},\mathscr{Q}}N,$$

is compact on Ω, where the operators presented in the previous subsection are $\mathscr{P}$, $\mathscr{Q}$, and $\mathcal{J}$.

Remark 2.3.1

Assume that Ω an open bounded set of $\mathbb{X}$ such that $\mathrm{Dom}(\mathcal{L}) \cap \Omega \neq \{0\}$.
The map $N : \mathbb{X} \to \mathbb{Z}$ is $\mathcal{L}-$ compact on $\overline{\Omega}$ equivalent to the operator $\mathscr{Q}N(\overline{\Omega})$ is bounded and $K_{\mathscr{P},\mathscr{Q}}N\Omega : \overline{\Omega} \to \mathbb{X}$ is compact.

Example 2.3.2

Let $\mathcal{L} : \mathrm{Dom}(\mathcal{L}) \subset \mathbb{X} \to \mathbb{Z}$ be a bijective zero-index Fredholm operator : $\mathscr{P} = 0 : \mathbb{X} \to \mathbb{X}$, $\mathscr{Q} = 0 : \mathbb{Z} \to \mathbb{Z}$ and $\mathscr{P} = K_{0,0} = \mathcal{L}^{-1}$.

2.3.3.1 Reformulation fixed point problem to $\mathcal{L}x = Nx$

The equation $\mathcal{L}x = z$ may be solved by writing $z = (\mathscr{P}\mathscr{Q} + (\mathcal{I}_d - \mathscr{Q}))z$ and $x = \mathscr{P}x + (\mathcal{I}_d - \mathscr{P})x$.

Substituting x and z into the proceeding equation, we obtain

$$\mathcal{L}(\mathscr{P}x + (\mathcal{I}_d - \mathscr{P})x) = \mathscr{Q}z + (\mathcal{I}_d - \mathscr{Q})z,$$

and since $\mathscr{Q}z = 0$ and $\mathcal{L}\mathscr{P}x = 0$ (since $z \in \mathrm{Im}(\mathcal{L})$ and $\mathscr{P}x \in \ker(\mathcal{L})$), then

$$\mathcal{L}(\mathcal{I}_d - \mathscr{P})x = (\mathcal{I}_d - \mathscr{Q})z.$$

This results in

$$x - \mathscr{P}x = \mathcal{L}_{\mathscr{P}}^{-1}(\mathcal{I}_d - \mathscr{Q})z,$$

consequently

$$x = \mathscr{P}x + \mathcal{J}^{-1}\mathscr{Q}z + \mathcal{L}_{\mathscr{P}}^{-1}(\mathcal{I} - \mathscr{Q})z.$$

Consider the fallowing equation $\mathcal{L}x = Nx$, where the operator $N : G \subset \mathbb{X} \to \mathbb{Y}$ is typically non-linear. Based on the results above, this last equation is equivalent to

$$x = \mathscr{P}x + \mathcal{J}^{-1}\mathscr{Q}Nx + K_{\mathscr{P},\mathscr{Q}}\mathscr{P} = Mx \text{ with } x \in \mathrm{Dom}(\mathcal{L}) \cap G. \tag{2.2}$$

This is a fixed-point problem.

2.3.4 The coincidence degree

Definition 2.3.4

Assume that $\mathcal{L}$ and N are a Fredholm operator of index zero and $\mathcal{L}-$compact operator respectively, on $\overline{\Omega}$, the coincidence degree is the integer

$$\deg[(\mathcal{L}, N), \Omega] = \deg_{LS}[\mathcal{I}_d - M, \Omega, 0],$$

where M is defined in 2.2.

Remark 2.3.2

We note that the Brouwer and Leray-Schauder degrees are two special cases of the coincidence Degree.

2.3.4.1 Properties of coincidence degree

1) Existence

If $\deg L[(\mathcal{L}, N), \Omega] \neq 0$, then $0 \in (\mathcal{L} - N)(\mathrm{Dom}(\mathcal{L}) \cap \Omega)$.

2) Excision

If $\Omega_0 \subset \Omega$ is an open set such that $(\mathcal{L} - N)^{-1}(0) \subset \Omega_0$ then

$$\deg L[(\mathcal{L}, N), \Omega] = \deg L[(\mathcal{L}, N), \Omega].$$

3) Additive

Let Ω_1, Ω_2 be open and $\Omega = \Omega_1 \cup \Omega_2$, then

$$\deg[(\mathcal{L}, N), \Omega] = \deg[(\mathcal{L}, N), \Omega_1] + \deg[(\mathcal{L}, N), \Omega_2].$$

2.3.5 Mawhin continuation theorem

Theorem 2.3.1

With N being $\mathcal{L}$-compact on $\overline{\Omega}$, let $\mathcal{L}$ be a Fredholm operator of index zero. Presume that The requirements listed below are met:

1) $\mathcal{L}x \neq \lambda N x$, for all $(x, \lambda) \in [(\mathrm{Dom}\,\mathcal{L} \setminus \ker \mathcal{L}) \cap \partial\Omega] \times (0, 1)$,

2) $Nx \notin \mathrm{Dom}\,\mathcal{L}$, for all $x \in \mathrm{Ker}\,\mathcal{L} \cap \partial\Omega$,

3) $deg(\mathcal{J}\mathscr{Q}N_{|ker\mathcal{L}}, ker\mathcal{L} \cap \partial\Omega, 0) \neq 0$,

where $\mathscr{Q} : \mathbb{Z} \to \mathbb{Z}$ is a projection as previously mentioned , $J : Im\mathscr{Q} \to ker\mathcal{L}$ is a linear isomorphism with $Im\mathcal{L} = ker\mathscr{Q}$. Then there is at least one solution to the equation $\mathcal{L}x = Nx$ in $dom\mathcal{L} \cap \overline{\Omega}$.

Proof 2.3.1

Review the family of problems for $\lambda \in [0,1]$

$$x \in \mathrm{dom}\,\mathcal{L} \cap \Omega, \mathcal{L}x = \lambda Nx + (1-\lambda)\mathscr{Q}Nx. \tag{2.3}$$

Let $M : [0,1] \times \bar{\Omega} \to \mathbb{Z}$ be a homotopy defined by

$$M(\lambda, x) = \mathscr{P}x + \mathcal{J}^{-1}\mathscr{Q}Nx + K_{\mathscr{P},\mathscr{Q}}Nx.$$

The following fixed point problem can be equated to the problem (2.3)

$$\begin{aligned} x &= \mathscr{P}x + \mathcal{J}^{-1}\mathscr{Q}(\lambda N + (1-\lambda)\mathscr{Q}N)x + K_{\mathscr{P},\mathscr{Q}}(\lambda N + (1-\lambda)\mathscr{Q}N)x \\ &= \mathscr{P}x + \lambda J^{-1}\mathscr{Q}Nx + (1-\lambda)J^{-1}\mathscr{Q}Nx + \lambda K_{\mathscr{P},\mathscr{Q}}Nx + (1-\lambda)K_{\mathscr{P},\mathscr{Q}}\mathscr{Q}Nx \\ &= M(1,x), \end{aligned}$$

Consequently, a fixed point problem is equivalent to the last equation:

$$x = M(1,x), \quad x \in \Omega,$$

In the case that $\mathcal{L}x = Nx$ for every $x \in \partial\Omega$, the proof is considered successful.
Consider now that

$$\mathcal{L}x \neq Nx \text{ for every } x \in \mathrm{dom}\,\mathcal{L} \cap \Omega, \tag{2.4}$$

however, on the other hand

$$\mathcal{L}x \neq \lambda Nx + (1-\lambda)\mathscr{Q}Nx, \tag{2.5}$$

For each $(\lambda, x) \in]0,1[\times(\mathrm{dom}\,\mathcal{L} \cap \Omega)$. If

$$\mathcal{L}x = \lambda Nx + (1-\lambda)\mathscr{Q}Nx,$$

for all $(\lambda, x) \in]0,1[\times(\operatorname{dom}\mathcal{L}\cap\Omega)$. we get by application of $\mathscr{Q}$ to the two participants in the prior equality

$$\mathscr{Q}Nx = 0, \quad \mathcal{L}x = \lambda Nx, \tag{2.6}$$

the condition 2) and the first of these equalities suggest that $x \notin \operatorname{Ker}\mathcal{L}\cap\partial\Omega$, i.e.. for every $x \in (\operatorname{dom}\mathcal{L}\backslash K\text{ er }\mathcal{L})\cap\partial\Omega$ hence the second equality defies the first. Utilizing other times 2), thus, it follows that $\mathcal{L}x \neq \mathscr{Q}Nx, \quad \text{for all} \quad x \in \operatorname{dom}\mathcal{L}\cap\partial\Omega$, by using (2.4), (2.5) and (2.6). We conclude that

$$x \neq M(\lambda, x) \text{ for all } (\lambda, x) \in [0,1]\times\partial\Omega,$$

$M(\lambda, x)$ is compact since N is $\mathcal{L}$-compact.By utilizing the Leray-Schauder degree's homotopy invariance property, we get

$$\deg(\mathcal{I}_d - M(0,.), \Omega, 0) = \deg_(\mathcal{I}_d - M(1,.), \Omega, 0), \tag{2.7}$$

However, we also have

$$\deg(\mathcal{I}_d - M(0,\lambda), \Omega, 0) = \deg\left(\mathcal{I}_d - \left(P + \mathcal{J}^{-1}\mathscr{Q}N\right), \Omega, 0\right), \tag{2.8}$$

because the image of $\mathscr{P} + \mathcal{J}^{-1}\mathscr{Q}N$ contains in $\operatorname{Ker}(\mathcal{L})$, subsequently applying the Leray-Schauder degree's reduction property and the knowledge that $\mathscr{P}|_{\mathrm{KerL}} = \mathcal{I}_d|_{\mathrm{KerL}\,.}$ (since $\operatorname{Ker}(\mathcal{L}) = \operatorname{Im}(\mathscr{P}) = \operatorname{Ker}(\mathcal{I}_d - \mathscr{P})$), we get

$$\begin{aligned}\deg\left(\mathcal{I}_d - \left(\mathscr{P} + \mathcal{J}^{-1}\mathscr{Q}N\right), \Omega, 0\right) &= \deg\left(\mathcal{I}_d - \left(\mathscr{P} + \mathcal{J}^{-1}\mathscr{Q}N\right), \Omega\cap\operatorname{Ker}\mathcal{L}, 0\right) \\ &= \deg\left(\mathcal{J}^{-1}\mathscr{Q}N, \Omega\cap\operatorname{Ker}\mathcal{L}, 0\right),\end{aligned} \tag{2.9}$$

thanks to (2.7), (2.8) and (2.9), it follows that $deg(\mathcal{I}_d - M(1,), .\Omega, 0) \neq 0$, Thus, the Leray-Schauder degree's existence property presupposes the existence of a $x \in \Omega.$ such as $x = M(1,x)$ i.e $x \in \operatorname{dom}\mathcal{L}\cap\Omega, \mathcal{L}x = Nx.$

2.4 Condensing perturbation of the identity

The following theorem states that for μ-condensing perturbations of the identity, the topological degree exists and has the following fundamental characteristics (axiomatic definition).

Definition 2.4.1

Let

$$\mathcal{T} = \left\{ \begin{array}{ll} (\mathcal{I}_d - F, \Omega, y): & \Omega \subset \mathbb{X} \text{ open and bounded,} \\ & F \in C_\mu(\bar{\Omega}), y \in \mathbb{X} \backslash (\mathcal{I}_d - F)(\partial\Omega) \end{array} \right\}$$

be the family of all the admissible triplets (where $\mathbb{X}$ is a Banach space and $C_\mu(\bar{\Omega})$ is the set of $\mu-$condensing maps on Ω).

One degree function that satisfies the following properties exists, $\deg : \mathcal{T} \to \mathbb{Z}$.

Proposition 2.4.1

(P1) **(Normalization)**

$\deg(\mathcal{I}_d, \Omega, x) = 1$ for all $x \in \Omega$.

(P2) **(Additivity)**

For each disjoint, open sets $\Omega_1, \Omega_2 \subset \Omega$ and all $y \notin (\mathcal{I}_d - F)\left(\bar{\Omega} \backslash (\Omega_1 \cup \Omega_2)\right)$ we have

$$\deg(\mathcal{I}_d - F, \Omega, x) = \deg\left(\mathcal{I}_d - F, \Omega_1, x\right) + D\left(\mathcal{I}_d - F, \Omega_2, y\right).$$

(P3) **(Invariance under homotopy)**

$\deg(\mathcal{I}_d - H(t, \cdot), \Omega, y(t))$ is independent of $t \in [0, 1]$ for all continuous,bounded map $H : [0, 1] \times \bar{\Omega} \to \mathbb{X}$ which fulfills

$$\mu(H([0, 1] \times B)) < \mu(B) \quad (\forall) B \subset \bar{\Omega} \text{ with } \mu(B) > 0$$

and each continuous function $y : [0, 1] \to \mathbb{X}$ which satisfies

$$y(t) \neq x - H(t, x) \quad (\forall) t \in [0, 1], (\forall) x \in \partial\Omega$$

(P4) **(Existence)**

$\deg(\mathcal{I}_d - F, \Omega, y) \neq 0$ implies $y \in (\mathcal{I}_d - F)(\Omega)$.

(P5) **(Excision)**

$\deg(\mathcal{I}_d - F, \Omega, y) = \deg\left(\mathcal{I}_d - F, \Omega_1, y\right)$ for all open set $\Omega_1 \subset \Omega$ and all $y \notin (\mathcal{I}_d - F)\left(\bar{\Omega} \backslash \Omega_1\right)$.

Utilizing a degree function constructed on $\mathcal{T}$, we assess the applicability of the (a priori estimate method) by using this degree.

Theorem 2.4.1

Let $F : \mathbb{X} \to \mathbb{X}$ be μ-condensing and $S = \{x \in \mathbb{X} : (\exists)\lambda \in [0,1]\}$ and $x = \lambda F x$. If S is a bounded set in $\mathbb{X}$ such that $r > 0$ exists and $S \subset B_r(0)$, so

$$\deg(\mathcal{I}_d - \lambda F, B_r(0), 0) = 1 \qquad (\forall)\lambda \in [0,1].$$

Therefore, The set of fixed points for F is contained in $B_r(0)$, and F has at least one fixed point.

Proof 2.4.1

Firstly, It is noted that each affine homotopy of μ-condensing maps is a homotopy that can be admitted. So, let $H : [0,1]$ be a bounded open set, and let $Omega \subset \mathbb{X}$ be the maps $F_1, F_2 \in C_\mu(\bar{\Omega})$

and let $H : [0,1] \times \bar{\Omega} \to \mathbb{X}$ is

$$H(t,x) = (1-t)F_1 x + tF_2 x.$$

$\forall B \subset \bar{\Omega}$ and $\mu(B) > 0$ we have

$$H([0,1] \times B) \subset \operatorname{Conv}\left(F_1(B) \cup F_2(B)\right)$$

and, by using 2.4.1

$$\begin{aligned}
\mu(H([0,1] \times B)) &\leq \mu\left(\operatorname{conv}\left(F_1(B) \cup F_2(B)\right)\right) \\
&= \mu\left(F_1(B) \cup F_2(B)\right) \\
&= \max\left\{\alpha\left(F_1(B)\right), \mu\left(F_2(B)\right)\right\} < \mu(B).
\end{aligned}$$

After fixing $\lambda \in [0,1]$ and examine the affine homotopy between the μ-condensing mappings λF and $0 \in C_\mu(\mathbb{X})$.

$$H : [0,1] \times \mathbb{X} \to \mathbb{X}, \quad H(t,x) = (1-t)0x + t\lambda F x = t\lambda F x.$$

Regarding the argument that came before,

$$\mu(H([0,1] \times B)) < \mu(B) \quad (\forall)B \subset \mathbb{X} \text{ bounded with } \mu(B) > 0.$$

Given that $t \in [0,1]$ and $x \in \mathbb{X}$After confirming that $x - H(t,x) = 0$, $x \in S \subset B_r(0)$. As a result, we may apply the degree's properties (P1) and (P3) to get

$$\begin{aligned} D\left(I - \lambda F, B_r(0), 0\right) &= D\left(I - H(1,\cdot), B_r(0), 0\right) \\ &= D\left(I - H(0,\cdot), B_r(0), 0\right) \\ &= D\left(I, B_r(0), 0\right) = 1. \end{aligned}$$

Finally, the application F has at least one fixed point according to the degree's property (P4).

CHAPTER

3

EXISTENCE RESULT OF TWO POINT AT RESONANCE B.V.P VIA COINCIDENCE DEGREE

3.1 Introduction

In this section, We solve the following system of fractional differential equations boundary value problem, BVP for short, by using Mawhin's continuation theorem [2.3.1]:

$$D_{0^+}^{\alpha}x(t) + f\left(t, x(t), D_{0^+}^{\alpha-1}x(t)\right) = \theta,\ t \in [0,1], \tag{3.1}$$

$$x\left(0\right) = \theta, \tag{3.2}$$

$$D_{0^+}^{\alpha-1}x\left(1\right) = \mathcal{B}I_{0^+}^{\alpha}x\left(1\right), \tag{3.3}$$

where $1 < \alpha \leqslant 2, D_{0^+}^{\alpha}$ is the standard fractional derivative of Riemann-Liouville and $I_{0^+}^{\alpha}$ is the standard fractional integral of Riemann-Liouville.
$\theta = (0, 0, \ldots, 0)^T$ is the zero vector of $\mathbb{R}^N$ and $\mathcal{B}$ is a square matrix of size N.
We are used this vector notations $x(t) = (x_1(t), x_2(t), \ldots, x_N(t))^T$, and we denote by

$$D_{0^+}^{\alpha} x(t) = (D_{0^+}^{\alpha} x_1(t), D_{0^+}^{\alpha} x_2(t), \ldots, D_{0^+}^{\alpha} x_N(t))^T ,$$

to the fractional derivative of x, and its fractional integral by
$I_{0^+}^{\alpha} x(t) = \left(I_{0^+}^{\alpha} x_1(t), I_{0^+}^{\alpha} x_2(t), \ldots, I_{0^+}^{\alpha} x_N(t)\right)^T$.
Also $f : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a vector-valued function

$$f\left(t, x(t), D_{0^+}^{\alpha-1} x(t)\right) =$$
$$\left(f_1\left(t, x(t), D_{0^+}^{\alpha-1} x(t)\right), f_2\left(t, x(t), D_{0^+}^{\alpha-1} x(t)\right), \ldots f_N\left(t, x(t), D_{0^+}^{\alpha-1} x(t)\right)\right)^T .$$

We now make the following assumptions:

$(H1)$ The matrix $\frac{1}{\Gamma(2\alpha)}\mathcal{B}$ has a simple eigenvalue denoted by $\lambda = 1$.

$(H2)$ There exists a real constant $\kappa \in \mathbb{R}$ such that:
$\kappa\left(\mathcal{I}_N - \frac{1}{\Gamma(2\alpha)}\mathcal{B}\right)\left(\mathcal{I}_N - \frac{1}{\Gamma(2\alpha)}\mathcal{B}\right) = \mathcal{I}_N - \frac{1}{\Gamma(2\alpha)}\mathcal{B}$, where $\mathcal{I}_N$ is the identity operator in $\mathbb{R}^N$.

$(H3)$ A Carathéodory function denoted by f , i.e.,

1. $f(\cdot, x, z)$ For each (x, z) in $\mathbb{R}^N \times \mathbb{R}^N$ is Lebesgue measurable.
2. $f(t, \cdot, \cdot)$ is continuous for a.e. $t \in [0,1]$,
3. for every compact $K \subset \mathbb{R}^N \times \mathbb{R}^N$, the function

$$\rho_K(t) = \sup_{(x,z)\in K} \|g(t, x, z)\|_{\mathbb{R}^N}$$

is Lebesgue integrable.

Here $\|(x_1, x_2, \ldots, x_N)\|_{\mathbb{R}^N} = \max_{i=\overline{1,N}}(|x_i|)$.

In all that follows, we set $\mathcal{A} = \frac{1}{\Gamma(2\alpha)}\mathcal{B}$.

Remark 3.1.1

i) In the numerical example of Section (5), it is checked that matrix $\mathcal{A} = \begin{pmatrix} 5 & 6 \\ 2 & 4 \end{pmatrix}$ satisfies $(H1)$ and $(H2)$ with $\kappa = -\frac{1}{7}$.

ii) Let $\mathcal{A}$ be an $(N \times N)$ involutory matrix, i.e., $\mathcal{A} = \mathcal{A}^{-1}$ which has $\lambda = 1$ as a simple eigenvalue. Since the eigenvalues of an involutory matrix are ± 1, which implies that $\det \mathcal{A} = (-1)^{N-1}$. Additionally,

$$(\mathcal{I}_d - \mathcal{A})(\mathcal{I}_d - \mathcal{A}) = 2(\mathcal{I}_d - \mathcal{A}).$$

Then, $(H1)$ and $(H2)$ hold with $\kappa = \frac{1}{2}$.

The related homogeneous problem $-D^{\alpha}_{0+}x(t) = 0$ is due to condition $(H1)$ has non-trivial solutions when the boundary conditions are satisfied. $At^{\alpha-1} = x(t)$, where $a = (a_1, a_2, \ldots, a_N)^T \in$ Ker $(\mathcal{I}_N - \mathcal{A}) \neq \{\theta\}$, This implies that the resonance of problem (3.1)-(3.2)-(3.3) is obtained. But the homogeneous problem has only the trivial solution $x = \theta$ (non-resonance case) if $\det (\mathcal{I}_N - \mathcal{A}) \neq 0$..

3.2 Functional framework

We start with some basics from matrix theory.

Lemma 3.2.1

If $\mu > 0$, is provided, let the linear space

$$C^{\mu}[0,1] = \left\{ u : u(t) = I^{\mu}_{0+}x(t) + \sum_{i=1}^{i=[\mu]} c_i t^{\mu-i},\ t \in [0,1] \right\},$$

where $c_i \in \mathbb{R}$ and $x \in C[0,1]$. Equipped with the norm

$$\|x\|_{C^{\mu}} = \|x\|_{\infty} + \sum_{i=0}^{i=[\mu]} \left\| D^{\mu-i}_{0+} x \right\|_{\infty}$$

$C^{\mu}[0,1]$ is a Banach space.

Here $\|u\|_{\infty} = \sup\limits_{0 \le t \le 1} |u(t)|$ is the supremum norm.

Proof 3.2.1 *[7] or [28]*

Definition 3.2.1

$p_{\mathcal{M}}(\lambda) = \det(\lambda\mathcal{I}_N - \mathcal{M})$ is the characteristic polynomial of a square matrix $\mathcal{M}$ of size N, where the identity matrix of order N is represented by $\mathcal{I}_N$.

Definition 3.2.2

The multiplicity of the eigenvalue β as a root of the characteristic polynomial of $\mathcal{M}$ is its algebraic multiplicity, denoted as $\mu_{\mathcal{M}}(\beta)$. Assuming that $\mu_{\mathcal{M}}(\beta) = 1$, We state that β is a **simple eigenvalue** of $\mathcal{M}$; that is, a polynomial Q exits such that $p_{\mathcal{M}}(\lambda) = (\beta - \lambda) Q(\lambda)$, where $Q(\beta) \neq 0$.

According to the Cayley-Hamilton Theorem, a square matrix fulfills the requirements of its own characteristic polynomial.

Theorem 3.2.1 *[43] **(Cayley-Hamilton Theorem)***

$p_{\mathcal{M}}(\mathcal{M}) = O$, i.e., a null matrix.

Condition $(H1)$ implies the existence of a polynomial $H(\lambda)$ such that $p_{\mathcal{A}}(\lambda) = (\mathcal{I} - \lambda) H(\lambda)$ and $H(1) \neq 0$. By Cayley-Hamilton Theorem,

$$(\mathcal{I}_N - \mathcal{A}) H(\mathcal{A}) = H(\mathcal{A})(\mathcal{I}_N - \mathcal{A}) = O. \tag{3.4}$$

The commutativity is due to the fact that $H(\mathcal{A})$ is a linear combination of the powers of $\mathcal{A}$.

Example 3.2.1

$s\mathcal{A}^r m\mathcal{A}^e = sm\mathcal{A}^{r+e} = m\mathcal{A}^e s\mathcal{A}^r$.

Lemma 3.2.2 *[35]**(Bezout Theorem)***

In $\mathbb{R}[\mathbb{X}]$, two polynomials P and Q are coprime if and only if there are polynomials U and V such that $PU + QV = 1$.

Lemma 3.2.3

We can demonstrate this following assertions:

a) $\mathrm{Ker}\ (\mathcal{I}_N - \mathcal{A}) \cap \mathrm{Ker}\, H(\mathcal{A}) = \{\theta\}$

b) $\mathcal{A}H(\mathcal{A}) = H(\mathcal{A})\mathcal{A} = H(\mathcal{A})$,

c) $H(\mathcal{A})^2 = H(1) \cdot H(\mathcal{A})$.

Proof 3.2.2

a) In view of Bezout Theorem, there exist two polynomials U and V such that $U(\lambda)(1-\lambda) + V(\lambda) H(\lambda) = 1$.
Hence, $U(\mathcal{A})(\mathcal{I}_N - \mathcal{A}) + V(\mathcal{A}) H(\mathcal{A}) = \mathcal{I}_N$.Then, we deduce that

$$\mathrm{Ker}\ (\mathcal{I}_N - \mathcal{A}) \cap \mathrm{Ker}\, H(\mathcal{A}) = \{\theta\} \tag{3.5}$$

because for all $u \in \mathrm{Ker}\,(\mathcal{I}_N - \mathcal{A}) \cap \mathrm{Ker}\, H(\mathcal{A})$, we have $(\mathcal{I}_N - \mathcal{A})x = H(\mathcal{A})x = \theta$, therefore $U(\mathcal{A})\theta + V(\mathcal{A})\theta = I_N x = x = \theta$.

b) Notice further that from (3.4), we have immediately

$$\mathcal{A}H(\mathcal{A}) = H(\mathcal{A})\mathcal{A} = H(\mathcal{A}). \tag{3.6}$$

Suppose that $H(\lambda) = \sum_{i=0}^{i=m} \beta_i \lambda^i$, $1 \le m \le N-1$. From (3.6), we can write

$$\begin{gathered} H(\mathcal{A})\beta_0\mathcal{A}^0 = \beta_0 H(\mathcal{A}), \quad H(\mathcal{A})\beta_1\mathcal{A}^1 = \beta_1 H(\mathcal{A}), \\ H(\mathcal{A})\beta_2\mathcal{A}^2 = \beta_2 (H(\mathcal{A})\mathcal{A})\mathcal{A} = \beta_2 H(\mathcal{A})\mathcal{A} = \beta_2 H(\mathcal{A}), \\ \ldots = \ldots \\ H(\mathcal{A})\beta_m\mathcal{A}^m = \beta_m H(\mathcal{A}). \end{gathered}$$

By adding the terms of the equalities, we get

$$H(\mathcal{A})\left(\beta_0\mathcal{A}^0 + \beta_1\mathcal{A}^1 + \beta_2\mathcal{A}^2 + \ldots + \beta_m\mathcal{A}^m\right) = [\beta_0 + \beta_1 + \beta_2 + \ldots + \beta_m] H(\mathcal{A}).$$

c) Consequently,

$$H(\mathcal{A})^2 = H(1) \cdot H(\mathcal{A}) \tag{3.7}$$

Definition 3.2.3

The max norm of a matrix $\mathcal{M} = (d_{ij})_{1\leq i,j\leq n} \in \mathcal{M}_n(\mathbb{R})$ is defined by

$$\|\mathcal{M}\|_* = \max_{1\leq i,j\leq n} |d_{ij}|.$$

If $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, then

$$\|Mu\|_{\mathbb{R}^n} \leq n \|\mathcal{M}\|_* \|u\|_{\mathbb{R}^n},$$

where $\|u\|_{\mathbb{R}^n} = \max_{1\leq i\leq n} |u_i|.$

Consider the spaces

$$\begin{aligned}\mathbb{X} &= C^{\alpha-1}\left([0,1], \mathbb{R}^N\right) \\ &= \left\{x = (x_1, x_2, \ldots, x_N)^T : x_i \in C^{\alpha-1}[0,1], i = 1, 2\ldots, N\right\}\end{aligned}$$

with the norm

$$\|x\|_{\mathbb{X}} = \max_{1\leq i\leq N} (\|x_i\|_{C^{\alpha-1}}) = \max_{1\leq i\leq N}\left(\|x_i\|_\infty + \left\|D_{0^+}^{\alpha-1} x_i\right\|_\infty\right),$$

because $\|x_i\|_{C^{\alpha-1}} = \|x_i\|_\infty + \sum_{j=0}^{j=[\alpha-1]} \left\|D_{0^+}^{\alpha-1-j} x_i\right\|_\infty$ and $[\alpha - 1] = 0.$

Also

$$\begin{aligned}\mathbb{Z} &= L^1\left([0,1] : \mathbb{R}^N\right) \\ &= \left\{z = (z_1, z_2, \ldots, z_N)^T : z_i \in L^1[0,1], i = 1, 2\ldots, N\right\}\end{aligned}$$

with the norm

$$\|z\|_{\mathbb{Z}} = \max_{1\leq i\leq N} (\|z_i\|_1) = \max_{1\leq i\leq N}\left(\int_0^1 |z_i(t)|dt\right).$$

Since $(C^{\alpha-1}[0,1], \|.\|_{C^{\alpha-1}})$ and $(L^1[0,1], \|z_i\|_1)$ are Banach spaces then $(\mathbb{X}, \|x\|_{\mathbb{X}})$ and $(\mathbb{Z}, \|z\|_{\mathbb{Z}})$ are Banach spaces too.

Define the operator $\mathcal{L}$: Dom $(\mathcal{L}) \to \mathbb{Z}$ by

$$\mathcal{L}x(t) = -D_{0^+}^{\alpha} x(t),\ t \in [0,1], \tag{3.8}$$

where

$$\text{Dom } (\mathcal{L}) = \left\{x \in \mathbb{X} : D_{0^+}^{\alpha} x \in \mathbb{Z},\ x(0) = \theta, D_{0^+}^{\alpha-1} x(t)\mid_{t=1} = \mathcal{B} I_{0^+}^{\alpha} x(t)\mid_{t=1}\right\},$$

and the nonlinear Nemytskii operator $N : \mathbb{X} \to \mathbb{Z}$ by

$$Nx(t) = f\left(t, x(t), D_{0^+}^{\alpha-1} x(t)\right). \tag{3.9}$$

It is clear that BVP (3.1)-(3.2)-(3.3) is equivalent to the abstract operator problem

$$\mathcal{L}x = Nx,\ x \in \text{Dom}\,(\mathcal{L}).$$

Define the operator $\psi : \mathbb{Z} \to \mathbb{R}^N$ by

$$\psi\,(z) = I_{0^+}^{1} z\,(1) - \mathcal{B} I_{0^+}^{2\alpha} z\,(1) \tag{3.10}$$

Remark 3.2.1

ψ is a continuous linear operator. Indeed, for all $i = 1, 2, \ldots, N$, we have the estimates

$$\begin{array}{rcl} \left|I_{0^+}^{2\alpha} z_i\,(1)\right| & = & \left|\frac{1}{\Gamma(2\alpha)} \int\limits_0^1 (1-s)^{2\alpha-1} z_i\,(s)\, ds\right| \\ & \leq & \frac{1}{\Gamma(2\alpha)} \int\limits_0^1 (1-s)^{2\alpha-1} \left|z_i\,(s)\right| ds \\ & \leq & \frac{1}{\Gamma(2\alpha)} \int\limits_0^1 \left|z_i\,(s)\right| ds = \frac{1}{\Gamma(2\alpha)} \left\|z_i\right\|_1 \leq \frac{1}{\Gamma(2\alpha)} \left\|z\right\|_{\mathbb{Z}}. \end{array}$$

Then,

$$\left\|I_{0^+}^{2\alpha} z\,(1)\right\|_{\mathbb{R}^N} = \max_{1\leq i\leq n} \left|I_{0^+}^{2\alpha} z_i\,(1)\right| \leq \frac{1}{\Gamma\,(2\alpha)} \left\|z\right\|_{\mathbb{Z}}.$$

Similarly, we prove that

$$\left\|I_{0^+}^{1} z\,(1)\right\|_{\mathbb{R}^N} \leq \left\|z\right\|_{\mathbb{Z}}.$$

Therefore,

$$\begin{array}{rcl} \left\|\psi\,(z)\right\|_{\mathbb{R}^N} & \leq & \left\|I_{0^+}^{1} z\,(1)\right\|_{\mathbb{R}^N} + \left\|\mathcal{B} I_{0^+}^{2\alpha} z\,(1)\right\|_{\mathbb{R}^N} \\ & \leq & \left\|I_{0^+}^{1} z\,(1)\right\|_{\mathbb{R}^N} + N \left\|\mathcal{B}\right\|_* \left\|I_{0^+}^{2\alpha} z\,(1)\right\|_{\mathbb{R}^N} \\ & \leq & \left\|z\right\|_{\mathbb{Z}} + \frac{N}{\Gamma(2\alpha)} \left\|\mathcal{B}\right\|_* \left\|z\right\|_{\mathbb{Z}} \\ & = & \left(1 + N \left\|\mathcal{A}\right\|_*\right) \left\|z\right\|_{\mathbb{Z}}. \end{array}$$

3.3 Some auxiliary Lemmas

Lemma 3.3.1

Suppose that $\mathcal{L}$ the operator where defined in (3.8), so

$$(1)\ \text{Ker}\,(\mathcal{L}) = \left\{\omega t^{\alpha-1} :\ \omega \in \text{Ker}\ (\mathcal{I}_N - \mathcal{A})\right\}$$
$$(2)\ \text{Im}\ (\mathcal{L}) = \left\{z \in \mathbb{Z}\,\psi\,(z) \in \text{Im}\ (\mathcal{I}_N - \mathcal{A})\right\}.$$

Proof 3.3.1

1) $\forall x \in \operatorname{Ker}(\mathcal{L})$, $D_{0^+}^{\alpha} x(t) = \theta$. So, $x(t) = at^{\alpha-1} + bt^{\alpha-2}$, where $a, b \in \mathbb{R}^N$. Taking in account condition (3.2), we obtain $b = \theta$. Using (1.7), and the boundary condition at $t = 1$, we find $a\Gamma(\alpha) = \mathcal{B}a\frac{\Gamma(\alpha)}{\Gamma(2\alpha)} = \mathcal{A}a\Gamma(\alpha)$. Hence $(\mathcal{I}_N - \mathcal{A})\, a\Gamma(\alpha) = \theta$ and $x(t) = at^{\alpha-1}$ with $a \in \operatorname{Ker}(\mathcal{I}_N - \mathcal{A})$.
2) Suppose that $z \in \operatorname{Im}(\mathcal{L})$. Then, $\exists x \in \operatorname{Dom}(\mathcal{L})$ such that $-D_{0^+}^{\alpha} x(t) = z(t), t \in [0,1]$. By (1.10), $x(t) = -I_{0^+}^{\alpha} z(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$, where $c_1, c_2 \in \mathbb{R}^N$. From (3.2), $c_2 = \theta$ and from (3.3), we get

$$\begin{aligned} -I_{0^+}^{1} z(1) + c_1\Gamma(\alpha) &= \mathcal{B} I_{0^+}^{\alpha} \left(-I_{0^+}^{\alpha} z(t) + c_1 t^{\alpha-1}\right)|_{t=1} \\ &= -\mathcal{B} I_{0^+}^{2\alpha} z(1) + \mathcal{B} c_1 \tfrac{\Gamma(\alpha)}{\Gamma(2\alpha)} 1^{2\alpha-1} \\ &= -\mathcal{B} I_{0^+}^{2\alpha} z(1) + \mathcal{A} c_1 \Gamma(\alpha). \end{aligned}$$

Therefore, $\psi(z) = (\mathcal{I}_N - \mathcal{A})\, c_1\Gamma(\alpha)$. Conversely, if $z \in \mathbb{Z}$ is such that $\psi(z) \in \operatorname{Im}(\mathcal{I}_N - \mathcal{A})$ so, $\exists a \in \mathbb{R}^N$ such that $\psi(z) = (\mathcal{I}_N - \mathcal{A})\, a$. Then it suffices to take $x(t) = -I_{0^+}^{\alpha} z(t) + \frac{a}{\Gamma(\alpha)} t^{\alpha-1}$ which satisfies $x \in \operatorname{dom}(\mathcal{L})$ and $-D_{0^+}^{\alpha} x(t) = z(t) = \mathcal{L}x(t)$.

Which complete this proof.

Lemma 3.3.2

The operators $\mathscr{P} : \mathbb{X} \to \mathbb{X}$ and $\mathscr{Q} : \mathbb{Z} \to \mathbb{Z}$ defined by

$$\mathscr{P}x(t) = \frac{H(\mathcal{A})}{H(1)} x(1)\, t^{\alpha-1} \text{ and } \mathscr{Q}z(t) = \frac{2\alpha}{2\alpha-1} \frac{H(\mathcal{A})}{H(1)} \psi(z)$$

are continuous linear projectors.

The function H was introduced after Theorem 3.2.1.

Proof 3.3.2

By (3.7), for each $x \in \mathbb{X}$, we get

$$\begin{aligned} \mathscr{P}^2 x(t) = \mathscr{P}(\mathscr{P}x)(t) &= \tfrac{H(\mathcal{A})}{H(1)} \mathscr{P}x(1)\, t^{\alpha-1} \\ &= \tfrac{H(\mathcal{A})}{H(1)} \tfrac{H(\mathcal{A})}{H(1)} x(1)\, t^{\alpha-1} \\ &= \tfrac{H(1)}{H(1)} \tfrac{H(\mathcal{A})}{H(1)} x(1)\, t^{\alpha-1} \\ &= \mathscr{P}x(t). \end{aligned}$$

Assuming that $H(\mathcal{A}) = (h_{ij})_{1\le i,j\le N}$. For any $i = \overline{1,N}$ the ith component of the vector $\mathscr{P}x(t)$ is $(\mathscr{P}x)_i(t) = \frac{t^{\alpha-1}}{H(1)} \sum\limits_{j=1}^{j=N} h_{ij}x_j(1)$.

Then,

$$|(\mathscr{P}x)_i(t)| \le \frac{1}{|H(1)|} \sum_{j=1}^{j=N} |h_{ij}|\,|x_j(1)| \le \frac{N}{|H(1)|} \|H(\mathcal{A})\|_* \|x\|_{\mathbb{X}}$$

and

$$\left|D_{0^+}^{\alpha-1}(\mathscr{P}x)_i(t)\right| = \left|\frac{\Gamma(\alpha)}{H(1)} \sum_{j=1}^{j=N} h_{ij}x_j(1)\right| \le \frac{N\Gamma(\alpha)}{|H(1)|} \|H(\mathcal{A})\|_* \|x\|_{\mathbb{X}}.$$

As a consequence,

$$\|(\mathscr{P}x)_i\|_\infty + \left\|D_{0^+}^{\alpha-1}(\mathscr{P}x)_i\right\|_\infty \le (1+\Gamma(\alpha)) \frac{N}{|H(1)|} \|H(\mathcal{A})\|_* \|x\|_{\mathbb{X}}.$$

Thus, $\|\mathscr{P}x\|_{\mathbb{X}} \le C_1 \|x\|_{\mathbb{X}}$, where

$$C_1 = (1+\Gamma(\alpha)) \frac{N}{|H(1)|} \|H(\mathcal{A})\|_*. \tag{3.11}$$

Since $\mathscr{Q}z(t) = c$ is independent of t, we can calculate

$$I_{0^+}^{2\alpha}c\,|_{t=1} = \frac{1}{\Gamma(2\alpha)} \int_0^1 (1-s)^{2\alpha-1} c\,ds = \frac{c}{2\alpha\Gamma(2\alpha)}.$$

So,

$$\psi(\mathscr{Q}z) = \mathscr{Q}z - \frac{1}{2\alpha\Gamma(2\alpha)}\mathcal{B}\mathscr{Q}z = \left(\mathcal{I}_N - \frac{1}{2\alpha}\mathcal{A}\right)\mathscr{Q}z.$$

This with (3.7) allow us to compute

$$\begin{aligned}
\mathscr{Q}^2 z = \tfrac{2\alpha}{2\alpha-1}\tfrac{H(\mathcal{A})}{H(1)}\psi(\mathscr{Q}z) &= \tfrac{2\alpha}{2\alpha-1}\tfrac{H(\mathcal{A})}{H(1)}\left(\mathcal{I}_N - \tfrac{1}{2\alpha}\mathcal{A}\right)\mathscr{Q}z \\
&= \tfrac{2\alpha}{2\alpha-1}\tfrac{1}{H(1)}\left(H(\mathcal{A}) - \tfrac{1}{2\alpha}H(\mathcal{A})\mathcal{A}\right)\mathscr{Q}z \\
&= \tfrac{2\alpha}{2\alpha-1}\tfrac{1}{H(1)}\left(H(\mathcal{A}) - \tfrac{1}{2\alpha}H(\mathcal{A})\right)\mathscr{Q}z \\
&= \tfrac{2\alpha}{2\alpha-1}\tfrac{2\alpha-1}{2\alpha}\tfrac{H(\mathcal{A})}{H(1)}\mathscr{Q}z = \tfrac{H(\mathcal{A})}{H(1)}\mathscr{Q}z \\
&= \tfrac{H(\mathcal{A})}{H(1)}\left(\tfrac{2\alpha}{2\alpha-1}\right)\tfrac{H(\mathcal{A})}{H(1)}\psi(z) = \tfrac{2\alpha}{2\alpha-1}\tfrac{H(\mathcal{A})}{H(1)}\psi(z) \\
&= \mathscr{Q}z.
\end{aligned}$$

Then, $\mathscr{Q}$ is a projector. Moreover,

$$\begin{aligned}
\|\mathscr{Q}z\|_{\mathbb{Z}} = \max\nolimits_{1\le i\le N}\|(\mathscr{Q}z)_i\|_1 &\le \tfrac{2\alpha N}{2\alpha-1}\tfrac{1}{|H(1)|}\|H(\mathcal{A})\|_* \|\psi(z)\|_{\mathbb{R}^N} \\
&\le \tfrac{2\alpha N}{2\alpha-1}\tfrac{1}{|H(1)|}\|H(\mathcal{A})\|_* (1+N\|\mathcal{A}\|_*)\|z\|_{\mathbb{Z}}.
\end{aligned}$$

Therefore, $\|\mathscr{Q}z\|_{\mathbb{Z}} \leq C_2 \|z\|_{\mathbb{Z}}$, where

$$C_2 = \frac{2\alpha N}{2\alpha - 1} \frac{\|H(\mathcal{A})\|_*}{|H(1)|} (1 + N \|\mathcal{A}\|_*), \tag{3.12}$$

proving the continuity of $\mathscr{Q}$.

Which complete the proof.

Lemma 3.3.3

The operator $\mathcal{L}$ defined in (3.8) is a Fredholm operator with index 0.

Proof 3.3.3

Since Im $(\mathcal{I}_N - \mathcal{A})$ is closed in $\mathbb{R}^N$ and ψ is continuous, the range Im $(\mathcal{L}) = \psi^{-1}(\text{Im }(\mathcal{I}_N - \mathcal{A}))$ is closed in $\mathbb{Z}$. By using definition of the kernel,

$$0 < \dim \text{Ker } (\mathcal{L}) = \dim \text{Ker } (\mathcal{I}_N - \mathcal{A}) \leq N < \infty.$$

Moreover, $z = (\mathcal{I} - \mathscr{Q}) z + \mathscr{Q}z$ implies $\mathbb{Z} = \text{Ker } (\mathscr{Q}) + \text{Im } (\mathscr{Q})$. Since

$$z \in \text{Im } (\mathcal{L}) \Longleftrightarrow \psi(z) \in \text{Im } (\mathcal{I}_N - \mathcal{A}) = \text{Ker } H(\mathcal{A}) \Longleftrightarrow \mathscr{Q}z = \theta,$$

$\text{Im}(\mathcal{L}) = \text{Ker } (\mathscr{Q})$. For each $z \in \text{Im}(\mathcal{L}) \cap \text{Im } (\mathscr{Q})$, there exists $h \in \mathbb{Z}$ satisfies that

$$z = \mathscr{Q}h \text{ and } \mathscr{Q}z = \theta = \mathscr{Q}^2 h = \mathscr{Q}h = z.$$

Then

$$\mathbb{Z} = \text{Im } (\mathcal{L}) \oplus \text{Im } (\mathscr{Q}).$$

It remains to demonstrate that $\dim \text{Im } (\mathscr{Q}) = \dim \text{Ker } (\mathcal{L})$. It is clear that Im $(\mathscr{Q}) \subset$ Ker $(\mathcal{I}_N - \mathcal{A})$. If $z \in$ Ker $(\mathcal{I}_N - \mathcal{A})$, then we may write $z = \mathcal{A}z = \mathcal{A}^2 z = \ldots = \mathcal{A}^k z$ with $k \in \mathbb{N}$. so, $H(\mathcal{A}) z = H(1) z$ and

$$\begin{aligned} \mathscr{Q}z &= \tfrac{2\alpha}{2\alpha-1} \tfrac{H(\mathcal{A})}{H(1)} \psi(z) \\ &= \tfrac{2\alpha}{2\alpha-1} \tfrac{H(\mathcal{A})}{H(1)} \left(\mathcal{I}_N - \tfrac{\mathcal{A}}{2\alpha}\right) z \\ &= \tfrac{H(\mathcal{A})}{H(1)} z = \tfrac{H(1)}{H(1)} z = z. \end{aligned}$$

So $z \in$ Im $(\mathscr{Q})$ which proves that Im $(\mathscr{Q}) =$ Ker $(\mathcal{I}_N - \mathcal{A})$. Then,

$$\dim \text{Im } (\mathscr{Q}) = \dim \text{Ker } (\mathcal{I}_N - \mathcal{A}) = \dim \text{Ker } (\mathcal{L}).$$

Which complete the proof.

Lemma 3.3.4

The operator $K_{\mathscr{P}}$: Im $(\mathcal{L})$ $\rightarrow$ Ker $\mathscr{P}$ $\cap$ Dom $(\mathcal{L})$ is the inverse of the operator $\mathcal{L}_{\mathscr{P}}$ $= \mathcal{L}\mid_{\text{Ker }\mathscr{P}\cap\text{Dom }(\mathcal{L})}$ is defined by

$$(K_{\mathscr{P}}z)(t) = t^{\alpha-1}\left(\frac{H(\mathcal{A})}{H(1)}I_{0^+}^{\alpha}z(1) + \frac{\kappa}{\Gamma(\alpha)}\psi(z)\right) - \mathcal{I}_{0^+}^{\alpha}z(t).$$

Moreover, it satisfies

$$\|K_{\mathscr{P}}z\|_{\mathbb{X}} \leq C_3\|z\|_{\mathbb{Z}},$$

where

$$C_3 = \frac{1+\Gamma(\alpha)}{\Gamma(\alpha)}\left(N\frac{\|H(\mathcal{A})\|_*}{|H(1)|} + |\kappa|\left(N\|\mathcal{A}\|_* + 1\right) + 1\right). \tag{3.13}$$

Proof 3.3.4

For $z \in$ Im $(\mathcal{L})$, $(K_{\mathscr{P}}z)(0) = \theta$. Since $\psi(z) = (\mathcal{I}_N - \mathcal{A})\,a$, and $a \in \mathbb{R}^N$, in view of (3.7), we find

$$\begin{aligned}
\mathscr{P}(K_{\mathscr{P}}z)(t) &= \tfrac{H(\mathcal{A})}{H(1)}\left[(K_{\mathscr{P}}z)(1)\right]t^{\alpha-1} \\
&= \tfrac{H(\mathcal{A})}{H(1)}\left[\tfrac{H(\mathcal{A})}{H(1)}I_{0^+}^{\alpha}z(1) + \tfrac{\kappa}{\Gamma(\alpha)}\psi(z) - I_{0^+}^{\alpha}z(1)\right]t^{\alpha-1} \\
&= \left[\tfrac{H(\mathcal{A})}{H(1)}I_{0^+}^{\alpha}z(1) + \tfrac{\kappa}{\Gamma(\alpha)}\theta - \tfrac{H(\mathcal{A})}{H(1)}I_{0^+}^{\alpha}z(1)\right]t^{\alpha-1} = \theta.
\end{aligned}$$

On the other hand, using (1.7), (1.8), (1.9), (3.6) and $(H2)$, we find

$$D_{0^+}^{\alpha-1}(K_{\mathscr{P}}z)(1) = \Gamma(\alpha)\frac{H(\mathcal{A})}{H(1)}I_{0^+}^{\alpha}z(1) + \kappa\psi(z) - I_{0^+}^{1}z(1),$$

$$I_{0^+}^{\alpha}(K_{\mathscr{P}}z)(t) = \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}t^{2\alpha-1}\left(\frac{H(\mathcal{A})}{H(1)}I_{0^+}^{\alpha}z(1) + \frac{\kappa}{\Gamma(\alpha)}\psi(z)\right) - I_{0^+}^{2\alpha}z(t),$$

and

$$\begin{aligned}
\mathcal{B}I_{0^+}^{\alpha}(K_{\mathscr{P}}z)(1) &= \mathcal{A}\left(\Gamma(\alpha)\tfrac{H(\mathcal{A})}{H(1)}I_{0^+}^{\alpha}z(1) + \kappa\psi(z)\right) - \mathcal{B}I_{0^+}^{2\alpha}z(1) \\
&= \left(\Gamma(\alpha)\tfrac{H(\mathcal{A})}{H(1)}I_{0^+}^{\alpha}z(1) + \kappa\mathcal{A}\psi(z)\right) - \mathcal{B}I_{0^+}^{2\alpha}z(1).
\end{aligned}$$

Then,

$$\begin{aligned}
D_{0^+}^{\alpha-1}(K_{\mathscr{P}}z)(1) - \mathcal{B}I_{0^+}^{\alpha}(K_{\mathscr{P}}z)(1) &= \kappa(\mathcal{I}_N - \mathcal{A})\psi(z) - \psi(z) \\
&= \kappa(\mathcal{I}_N - \mathcal{A})(\mathcal{I}_N - \mathcal{A})\,a - (\mathcal{I}_N - \mathcal{A})\,a \\
&= (\mathcal{I}_N - \mathcal{A})\,a - (\mathcal{I}_N - \mathcal{A})\,a = \theta.
\end{aligned}$$

So, $K_{\mathscr{P}}z \in \mathbb{X}$ because we can write

$$(K_{\mathscr{P}}z)(t) = I_{0^+}^{\alpha-1}\left(\left(\frac{H(\mathcal{A})}{H(1)}I_{0^+}^{\alpha}z(1) + \frac{\kappa}{\Gamma(\alpha)}\psi(z)\right)\Gamma(\alpha) - I_{0^+}^{1}z(t)\right),$$

the proof of operator $K_{\mathscr{P}}$ is well defined. Now let us demonstrate that $K_{\mathscr{P}} = \mathcal{L}_{\mathscr{P}}^{-1}$.

For each $z \in \mathrm{Im}\ (\mathcal{L})$, we have

$$(\mathcal{L}_{\mathscr{P}}K_{\mathscr{P}}z)(t) = -D_{0^+}^{\alpha}(K_{\mathscr{P}}z(t)) =$$

$$\left(\frac{H(\mathcal{A})}{H(1)}I_{0^+}^{\alpha}z(1) + \frac{\kappa}{\Gamma(\alpha)}\psi(z)\right)0 + z(t) = z(t).$$

If $x \in \mathrm{dom}\ (\mathcal{L}) \cap \mathrm{Ker}\ \mathscr{P} \subset \mathbb{X}$, then there exists $x \in C\left([0,1], \mathbb{R}^N\right)$ such that $x(t) = I_{0^+}^{\alpha-1}x(t)$, it enables us to write write $D_{0^+}^{\alpha-1}x(t) = x(t)$ and $D_{0^+}^{\alpha}x(t) = x'(t)$.

Then,

$$I_{0^+}^{\alpha}(D_{0^+}^{\alpha}x(t)) = I_{0^+}^{\alpha-1}\left[I_{0^+}^{1}x'(t)\right] = I_{0^+}^{\alpha-1}(x(t) - x(0)) = x(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}D_{0^+}^{\alpha-1}x(0)$$

$$I_{0^+}^{2\alpha}(D_{0^+}^{\alpha}x(t)) = I_{0^+}^{\alpha}\left(x(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}D_{0^+}^{\alpha-1}x(0)\right) = I_{0^+}^{\alpha}x(t) - \frac{t^{2\alpha-1}}{\Gamma(2\alpha)}D_{0^+}^{\alpha-1}x(0).$$

$$I_{0^+}^{1}(D_{0^+}^{\alpha}x(t)) = I_{0^+}^{1}\left(\frac{d\left(D_{0^+}^{\alpha-1}x(t)\right)}{dt}\right) = D_{0^+}^{\alpha-1}x(t) - D_{0^+}^{\alpha-1}x(0).$$

Based on (3.3), the boundary condition, we have

$$\begin{aligned}
\psi\left(D_{0^+}^{\alpha}x\right) &= I_{0^+}^{1}D_{0^+}^{\alpha}x(1) - \mathcal{B}I_{0^+}^{2\alpha}D_{0^+}^{\alpha}x(1) \\
&= D_{0^+}^{\alpha-1}x(1) - D_{0^+}^{\alpha-1}x(0) - \mathcal{B}\left(I_{0^+}^{\alpha}x(1) - \tfrac{1}{\Gamma(2\alpha)}D_{0^+}^{\alpha-1}x(0)\right) \\
&= D_{0^+}^{\alpha-1}x(1) - D_{0^+}^{\alpha-1}x(0) - \mathcal{B}I_{0^+}^{\alpha}x(1) + \mathcal{A}D_{0^+}^{\alpha-1}x(0) \\
&= -(\mathcal{I}_N - \mathcal{A})D_{0^+}^{\alpha-1}x(0).
\end{aligned}$$

By replacement, we can reduce the equation

$$\begin{aligned}
&(K_{\mathscr{P}}\mathcal{L}_{\mathscr{P}}x)(t) \\
= \ & t^{\alpha-1}\left(\tfrac{H(\mathcal{A})}{H(1)}I_{0^+}^{\alpha}\left(-D_{0^+}^{\alpha}x\right)(1) + \tfrac{\kappa}{\Gamma(\alpha)}\psi\left(-D_{0^+}^{\alpha}x\right)\right) - I_{0^+}^{\alpha}\left(-D_{0^+}^{\alpha}x\right)(t) \\
= \ & t^{\alpha-1}\left(-\tfrac{H(\mathcal{A})}{H(1)}\left(x(1) - \tfrac{1}{\Gamma(\alpha)}D_{0^+}^{\alpha-1}x(0)\right) + \tfrac{\kappa}{\Gamma(\alpha)}(\mathcal{I}_N - \mathcal{A})D_{0^+}^{\alpha-1}x(0)\right) \\
& + x(t) - \tfrac{t^{\alpha-1}}{\Gamma(\alpha)}D_{0^+}^{\alpha-1}x(0) \\
= \ & -\mathscr{P}x(t) + \tfrac{t^{\alpha-1}}{\Gamma(\alpha)}\left(\tfrac{H(\mathcal{A})}{H(1)} - \mathcal{I}_N + \kappa(\mathcal{I}_N - \mathcal{A})\right)D_{0^+}^{\alpha-1}x(0) + x(t).
\end{aligned}$$

Since $x \in \operatorname{Ker} \mathscr{P}$, $\mathscr{P}x(t) = \theta$. Let

$$c = \left(\frac{H(\mathcal{A})}{H(1)} - \mathcal{I}_N + \kappa (\mathcal{I}_N - \mathcal{A}) \right) D_{0^+}^{\alpha-1} x(0).$$

According to $(H2)$ and (3.4), (3.5), (3.7), we have

$$\begin{aligned}
(\mathcal{I}_N - \mathcal{A}) c &= \left((\mathcal{I}_N - \mathcal{A}) \tfrac{H(\mathcal{A})}{H(1)} - (\mathcal{I}_N - \mathcal{A}) + \kappa (\mathcal{I}_N - \mathscr{A}) (\mathcal{I}_N - \mathcal{A}) \right) D_{0^+}^{\alpha-1} x(0) \\
&= (O - (\mathcal{I}_N - \mathcal{A}) + (\mathcal{I}_N - \mathcal{A})) D_{0^+}^{\alpha-1} x(0) = \theta
\end{aligned}$$

and

$$\begin{aligned}
H(\mathcal{A}) c &= \left(H(\mathcal{A}) \tfrac{H(\mathcal{A})}{H(1)} - H(\mathcal{A}) + \kappa H(\mathcal{A}) (\mathcal{I}_N - \mathcal{A}) \right) D_{0^+}^{\alpha-1} x(0) \\
&= \left(H(1) \tfrac{H(\mathcal{A})}{H(1)} - H(\mathcal{A}) + \kappa H(\mathcal{A}) (\mathcal{I}_N - \mathcal{A}) \right) D_{0^+}^{\alpha-1} x(0) = \theta.
\end{aligned}$$

Then $c \in \operatorname{Ker} (\mathscr{I}_N - \mathcal{A}) \cap \operatorname{Ker} H(\mathcal{A}) = \{\theta\}$, this demonstrates that $(K_{\mathscr{P}} \mathcal{L}_{\mathscr{P}} x)(t) = x(t)$. For every $i = \overline{1, N}$ and each $t \in [0,1]$, easily observe that

$$\begin{aligned}
|(K_{\mathscr{P}} z)_i (t)| &\leq \left\| t^{\alpha-1} \left(\tfrac{H(\mathcal{A})}{H(1)} I_{0^+}^{\alpha} z(1) \right) \right\|_{\mathbb{R}^N} + \left\| t^{\alpha-1} \tfrac{\kappa}{\Gamma(\alpha)} \psi(z) \right\|_{\mathbb{R}^N} + \left\| -I_{0^+}^{\alpha} z(t) \right\|_{\mathbb{R}^N} \\
&\leq \tfrac{N}{|H(1)|\Gamma(\alpha)} \|H(\mathcal{A})\|_* \|z\|_{\mathbb{Z}} + \tfrac{|\kappa|}{\Gamma(\alpha)} (1 + N \|\mathcal{A}\|_*) \|z\|_{\mathbb{Z}} + \tfrac{1}{\Gamma(\alpha)} \|z\|_{\mathbb{Z}}
\end{aligned}$$

and

$$\begin{aligned}
\left| D_{0^+}^{\alpha-1} (K_{\mathscr{P}} z)_i (t) \right| &\leq \left\| \Gamma(\alpha) \left(\tfrac{H(\mathcal{A})}{H(1)} I_{0^+}^{\alpha} z(1) \right) \right\|_{\mathbb{R}^N} + \|\kappa \psi(z)\|_{\mathbb{R}^N} + \left\| -I_{0^+}^{1} z(t) \right\|_{\mathbb{R}^N} \\
&\leq \tfrac{N}{|H(1)|} \|H(\mathcal{A})\|_* \|z\|_{\mathbb{Z}} + |\kappa| (1 + N \|\mathcal{A}\|_*) \|z\|_{\mathbb{Z}} + \|z\|_{\mathbb{Z}}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\|(K_{\mathscr{P}} z)_i\|_\infty + \left\| D_{0^+}^{\alpha-1} (K_{\mathscr{P}} z)_i \right\|_\infty \\
&\leq \tfrac{\Gamma(\alpha)+1}{\Gamma(\alpha)} \left(\tfrac{N}{|H(1)|} \|H(\mathcal{A})\|_* + |\kappa| (1 + N \|\mathcal{A}\|_*) + 1 \right) \|z\|_{\mathbb{Z}}.
\end{aligned}$$

Finally,

$$\|K_{\mathscr{P}} z\|_X = \max_{1=1,2,\ldots,N} \left(\|(K_{\mathscr{P}} z)_i\|_\infty + \left\| D_{0^+}^{\alpha-1} (K_{\mathscr{P}} z)_i \right\|_\infty \right) \leq C_3 \|z\|_{\mathbb{Z}}.$$

Which complete the proof.

Lemma 3.3.5

The operator N defined by (3.9) under condition $(H3)$ is $\mathcal{L}$-completely continuous.

Proof 3.3.5

For some $R > 0$, let $S \subset \overline{B}(0, R)$ be a bounded subset from X with dom $(L) \cap \overline{S} \neq \emptyset$. According to the condition $(H3)$, the function

$$\rho_R(t) = \sup_{(x,z)\in K} \|g(t, x, z)\|_{\mathbb{R}^N}$$

is Lebesgue integrable (K is any compact in $\mathbb{R}^N \times \mathbb{R}^N$). For every $x \in \overline{S}$, it is simple to observe that

$$\|Nx(t)\|_{\mathbb{R}^N} = \left\|f\left(t, x(t), D_{0^+}^{\alpha-1}x(t)\right)\right\|_{\mathbb{R}^N} \leq \rho_R(t), \quad \text{for a.e. } t \in [0, 1].$$

This means that for each $i = \overline{1, N}$, $|(Nx)_i(t)| \leq \rho_R(t)$. So, $\|(Nx)_i\|_1 \leq \|\rho_R\|_1$. Hence,

$$\|Nx\|_{\mathbb{Z}} = \max_{i=1,2,\ldots,N} \|(Nx)_i\|_1 \leq \|\rho_R\|_1$$

and

$$\|\mathscr{Q}Nx\|_{\mathbb{Z}} \leq C_2 \|Nx\|_{\mathbb{Z}} \leq C_2 \|\rho_R\|_1,$$

where the constant C_2 is found in (3.12). This demonstrates the boundness of $\mathscr{Q}N\left(\overline{S}\right)$. We can verify that $\mathscr{Q}N$ and $K_{\mathscr{P},\mathscr{Q}}N$ are continuous using Lebesgue dominated convergence.

Let's now demonstrate that $K_{\mathscr{P},\mathscr{Q}}N\left(\overline{S}\right)$ is compact. For ease of use $K_{\mathscr{P},\mathscr{Q}}Nx = K_{\mathscr{P}}(\mathcal{I} - \mathscr{Q}) Nx = K_P Nx - K_{\mathscr{P}}\mathscr{Q}Nx$. By (3.13), we have

$$\begin{aligned}
\|K_{\mathscr{P},\mathscr{Q}}Nx\|_{\mathbb{X}} &\leq \|K_{\mathscr{P}}Nx\|_{\mathbb{X}} + \|K_{\mathscr{P}}\mathscr{Q}Nx\|_{\mathbb{X}} \\
&\leq C_3 \|Nx\|_{\mathbb{Z}} + C_3 \|\mathscr{Q}Nx\|_{\mathbb{Z}} \\
&\leq C_3 \left(\|\rho_R\|_1 + C_2 \|\rho_R\|_1\right) \\
&= C_3 (1 + C_2) \|\rho_R\|_1 .
\end{aligned}$$

proving that $K_{\mathscr{P},\mathscr{Q}}N\left(\overline{S}\right)$ is uniformly bounded. It remains to show that the family $K_{\mathscr{P},\mathscr{Q}}N\left(\overline{S}\right)$ is equicontinuous. Let $E = (\mathcal{I} - \mathscr{Q}) Nx$. For $0 \leq t_1 < t_2 \leq 1$, we can formulate

$$\begin{aligned}
I_{0^+}^{\alpha} E(t_2) - I_{0^+}^{\alpha} E(t_1) &= \tfrac{1}{\Gamma(\alpha)} \left(\int_0^{t_1} \left((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right) E(s)\, ds \right. \\
&\quad \left. + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} E(s)\, ds \right).
\end{aligned}$$

Then,

$$\begin{array}{rl} & \left\|K_{\mathscr{P}}E\left(t_2\right)-K_{\mathscr{P}}E\left(t_1\right)\right\|_{\mathbb{R}^N} \\ = & \left\|\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)\left(\frac{H(\mathcal{A})}{H(1)}I_{0^+}^{\alpha}E\left(1\right)+\frac{\kappa}{\Gamma(\alpha)}\psi\left(E\right)\right)-\left(I_{0^+}^{\alpha}E\left(t_2\right)-I_{0^+}^{\alpha}E\left(t_1\right)\right)\right\|_{\mathbb{R}^N} \\ \leq & \frac{\left|t_2^{\alpha-1}-t_1^{\alpha-1}\right|}{\Gamma(\alpha)}\left(\frac{N}{|H(1)|}\left\|H\left(\mathcal{A}\right)\right\|_*+|\kappa|\left(1+N\left\|\mathcal{A}\right\|_*\right)\right)\|E\|_{\mathbb{Z}} \\ & +\frac{1}{\Gamma(\alpha)}\left(\int\limits_0^{t_1}\left(\left(t_2-s\right)^{\alpha-1}-\left(t_1-s\right)^{\alpha-1}\right)ds+\int\limits_{t_1}^{t_2}\left(t_2-s\right)^{\alpha-1}ds\right)\|E\|_{\mathbb{Z}} \\ \leq & \frac{1}{\Gamma(\alpha)}\left(\left(\frac{N}{|H(1)|}\left\|H\left(\mathcal{A}\right)\right\|_*+|\kappa|\left(1+N\left\|\mathcal{A}\right\|_*\right)\right)\left|t_2^{\alpha-1}-t_1^{\alpha-1}\right|\right. \\ & \left.+\frac{1}{\alpha}\left(t_2^{\alpha}-t_1^{\alpha}\right)\right)\|E\|_{\mathbb{Z}}. \end{array}$$

Similarly, we have

$$\begin{array}{rcl} \left\|D_{0^+}^{\alpha-1}K_{\mathscr{P}}E\left(t_2\right)-D_{0^+}^{\alpha-1}K_{\mathscr{P}}E\left(t_1\right)\right\|_{\mathbb{R}^N} & = & \left\|I_{0^+}^{1}E\left(t_2\right)-I_{0^+}^{1}E\left(t_1\right)\right\|_{\mathbb{R}^N} \\ & = & \left\|\int\limits_{t_1}^{t_2}E\left(s\right)ds\right\|_{\mathbb{R}^N} \\ & \leq & \left|t_2-t_1\right|\|E\|_{Z}. \end{array}$$

Moreover, from (3.12), we have

$$\|E\|_{\mathbb{Z}}\leq\|Nx\|_{\mathbb{Z}}+\|\mathscr{Q}Nx\|_{\mathbb{Z}}\leq\left(1+C_2\right)\|\rho_R\|_1.$$

Hence,

$$\left\|K_{\mathscr{P}}E\left(t_2\right)-K_{\mathscr{P}}E\left(t_1\right)\right\|_{\mathbb{R}^N}\to 0 \text{ and } \left\|D_{0^+}^{\alpha-1}K_{\mathscr{P}}E\left(t_2\right)-D_{0^+}^{\alpha-1}K_{\mathscr{P}}E\left(t_1\right)\right\|_{\mathbb{R}^N}\to 0,$$

As $|t_2-t_1|\to 0$. Consistently $K_{\mathscr{P},\mathscr{Q}}N\left(\overline{S}\right)$ is equicontinuous. In view of Ascoli-Arzela Theorem, we conclude that $K_{\mathscr{P},\mathscr{Q}}N\left(\overline{S}\right)$ is relatively compact in $\mathbb{X}$.
Which complete this proof.

3.4 Existence theorem

Our demonstration of the main existence result for BVP (3.1) - (3.2) - (3.3) will be aided by the description of the projection operators gained in the previous section. We are unable to apply Theorem 2.3.1 unless we have a priori estimates of the desired solution and a sufficient set Ω. We impose the following criteria with respect to the nonlinear source term:

$(H4)$ There are three functions. In $L^1([0,1],\mathbb{R}_+)$ β,γ,δ so that each and every $x,z\in\mathbb{R}^N$and $t\in[0,1]$.

$$\|g(t,x,z)\|_{\mathbb{R}^N}\leq\beta(t)\|x\|_{\mathbb{R}^N}+\gamma(t)\|z\|_{\mathbb{R}^N}+\delta(t), \tag{3.14}$$

$(H5)$ There exists $M_0>0$ such that for each $x\in$ Dom (L) if there exists at least $i=\overline{1,N}$ such that $\left|D_{0^+}^{\alpha-1}x_i(t)\right|>M_0$ for each $t\in[0,1]$, then $\psi(Nx)\notin$ Im $(\mathcal{I}_N-\mathcal{A})$.

$(H6)$ There exists $M_1>0$ such that, for all $\xi\in$ Ker $(\mathcal{I}_N-\mathcal{A})$ fulfills $\|\xi\|_{\mathbb{R}^N}>M_1$, we have

$$\langle\xi,\mathscr{Q}Nx\rangle<0, \tag{3.15}$$

or

$$\langle\xi,\mathscr{Q}Nx\rangle>0, \tag{3.16}$$

where $x=\xi t^{\alpha-1}$.

Theorem 3.4.1

Let Assumptions $(H1)-(H6)$ hold. Then, BVP (3.1) - (3.2) - (3.3) acknowledges at least one fix provided by $x\in Dom(\mathcal{L})$

$$\|\beta\|_1+\|\gamma\|_1<P_c=\frac{\Gamma(\alpha)|H(1)|}{(\Gamma(\alpha)+1)\left(3N\|H(\mathcal{A})\|_*+|H(1)||\kappa|(1+N\|\mathcal{A}\|_*)+|H(1)|\right)}. \tag{3.17}$$

Proof 3.4.1

Step 1. Let

$$S_1=\{x\in\text{Dom }(\mathcal{L})\setminus\text{Dom Ker }(\mathcal{L}):\ \mathcal{L}x=\sigma Nx,\ \sigma\in[0,1]\}$$

$\forall x\in S_1$, we remark that $\sigma\neq0$ because $x\notin$ Ker $(\mathcal{L})$. Then $Nx=\mathcal{L}\left(\frac{1}{\sigma}x\right)$ i.e., $Nx\in$ Im $(\mathcal{L})$. Hence, $\psi(Nx)\in$ Im $(\mathcal{I}_N-\mathcal{A})$. In view of condition $(H5)$, for all $i=1,2,3,...,N$ there exists $t_i\in[0,1]$ such that $\left|D_{0^+}^{\alpha-1}x_i(t_i)\right|\leq M_0$. Since $\frac{d}{dt}\left(D_{0^+}^{\alpha-1}x_i(t)\right)=D_{0^+}^{\alpha}x_i(t)$, we have

$$D_{0^+}^{\alpha-1}x_i(t)=D_{0^+}^{\alpha-1}x_i(t_i)+\int_{t_i}^{t}D_{0^+}^{\alpha}x_i(s)\,ds.$$

From (3.14), we have $x_i(t) = I_{0^+}^{\alpha}\left(D_{0^+}^{\alpha} x_i(t)\right) + \frac{t^{\alpha-1}}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} x_i(0)$.So

$$\begin{aligned} \left| D_{0^+}^{\alpha-1} x_i(t) \right| &\leq \left| D_{0^+}^{\alpha-1} x_i(t_i) \right| + \int_{t_i}^{t} \left| D_{0^+}^{\alpha} x_i(s) \right| ds \\ &\leq M_0 + \left\| D_{0^+}^{\alpha} x_i \right\|_1 = M_0 + \left\| \sigma N_i x \right\|_1 \\ &\leq M_0 + \left\| Nx \right\|_{\mathbb{Z}} \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} |x_i(t)| &\leq \left| I_{0^+}^{\alpha}\left(D_{0^+}^{\alpha} x_i(t)\right) \right| + \left| \frac{t^{\alpha-1}}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} x_i(0) \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \left\| D_{0^+}^{\alpha} x_i \right\|_1 + \frac{1}{\Gamma(\alpha)} \left| D_{0^+}^{\alpha-1} x_i(0) \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \left\| Nx \right\|_{\mathbb{Z}} + \frac{1}{\Gamma(\alpha)} \left(M_0 + \left\| Nx \right\|_{\mathbb{Z}} \right) \\ &= \frac{1}{\Gamma(\alpha)} \left(M_0 + 2 \left\| Nx \right\|_{\mathbb{Z}} \right). \end{aligned} \tag{3.19}$$

Then,

$$\|x\|_{\mathbb{X}} = \max_{1 \leq i \leq N} \left(|x_i(t)| + \left| D_{0^+}^{\alpha-1} x_i(t) \right| \right) \leq \frac{1+\Gamma(\alpha)}{\Gamma(\alpha)} M_0 + \frac{2+\Gamma(\alpha)}{\Gamma(\alpha)} \|Nx\|_{\mathbb{Z}}.$$

According to the condition $(H4)$, for any $i = \overline{1, N}$,

$$\left| f_i\left(s, x(s), D_{0^+}^{\alpha-1} x(s)\right) \right| \leq \beta(s) \|x(s)\|_{\mathbb{R}^N} + \gamma(s) \left\| D_{0^+}^{\alpha-1} x(s) \right\|_{\mathbb{R}^N} + \delta(s).$$

Hence,

$$\int_0^1 \left| f_i\left(s, x(s), D_{0^+}^{\alpha-1} x(s)\right) \right| ds \leq \|\beta\|_1 \|x\|_{\mathbb{X}} + \|\gamma\|_1 \|x\|_{\mathbb{X}} + \|\delta\|_1.$$

Therefore,

$$\|Nx\|_{\mathbb{Z}} = \max_{1 \leq i \leq N} \int_0^1 \left| f_i\left(s, x(s), D_{0^+}^{\alpha-1} x(s)\right) \right| ds \leq \left(\|\beta\|_1 + \|\gamma\|_1 \right) \|x\|_X + \|\delta\|_1. \tag{3.20}$$

Now, we apply these final three results to the estimation of any element x of S_1. For any $x \in \text{Dom}(\mathcal{L})$, $x = \mathscr{P}x + (\mathcal{I} - \mathscr{P})x$. So,

$$\begin{aligned} \|\mathscr{P}x\|_{\mathbb{X}} &= \max\nolimits_{1 \leq i \leq N} \left(\left\| (\mathscr{P}x)_i \right\|_{\infty} + \left\| D_{0^+}^{\alpha-1} (\mathscr{P}x)_i \right\|_{\infty} \right) \\ &\leq \tfrac{N}{|H(1)|} \|H(\mathcal{A})\|_* \|x(1)\|_{\mathbb{R}^N} + \Gamma(\alpha) \tfrac{N}{|H(1)|} \|H(\mathcal{A})\|_* \|x(1)\|_{\mathbb{R}^N} \\ &= (1 + \Gamma(\alpha)) \tfrac{N}{|H(1)|} \|H(\mathcal{A})\|_* \|x(1)\|_{\mathbb{R}^N} \\ &\leq \left(\tfrac{1+\Gamma(\alpha)}{\Gamma(\alpha)} \right) \tfrac{N}{|H(1)|} \|H(\mathcal{A})\|_* \left(M_0 + 2 \|Gx\|_{\mathcal{Z}} \right). \end{aligned}$$

Similarly,

$$\begin{aligned}\|(\mathcal{I}-\mathscr{P})\,x\|_{\mathcal{X}} &= \|K_{\mathscr{P}}\mathcal{L}\,(\mathcal{I}-\mathscr{P})\,x\|_{\mathbb{X}}\\ &\leq C_3\,\|\mathcal{L}\,(\mathcal{I}-\mathscr{P})\,x\|_{\mathbb{Z}}\\ &= C_3\,\|\mathcal{L}x\|_{\mathbb{Z}} \leq C_3\,\|Nx\|_{\mathbb{Z}}\,,\end{aligned}$$

where (3.13) defines the constant C_3. Next,

$$\begin{aligned}\|x\|_{\mathbb{X}} &\leq \left(\tfrac{1+\Gamma(\alpha)}{\Gamma(\alpha)}\right)\tfrac{N}{|H(1)|}\,\|H\,(A)\|_{*}\,(M_0+2\,\|Nx\|_{\mathbb{Z}})+C_3\,\|Nx\|_{\mathbb{Z}}\\ &= \left(\tfrac{1+\Gamma(\alpha)}{\Gamma(\alpha)}\right)\tfrac{N}{|H(1)|}\,\|H\,(\mathcal{A})\|_{*}\,M_0+\left[\left(\tfrac{1+\Gamma(\alpha)}{\Gamma(\alpha)}\right)\tfrac{2N}{|H(1)|}\,\|H\,(\mathcal{A})\|_{*}+C_3\right]\|Nx\|_{\mathbb{Z}}\\ &\leq \left(\tfrac{1+\Gamma(\alpha)}{\Gamma(\alpha)}\right)\tfrac{N}{|H(1)|}\,\|H\,(\mathcal{A})\|_{*}\,M_0\\ &\quad+\left[\left(\tfrac{1+\Gamma(\alpha)}{\Gamma(\alpha)}\right)\tfrac{2N}{|H(1)|}\,\|H\,(\mathcal{A})\|_{*}+C_3\right]\left[(\|\beta\|_1\;+\|\gamma\|_1)\,\|x\|_{\mathbb{X}}\;+\|\delta\|_1\right]\\ &= \left(\tfrac{1+\Gamma(\alpha)}{\Gamma(\alpha)}\right)\tfrac{N}{|H(1)|}\,\|H\,(A)\|_{*}\,M_0+\left[\left(\tfrac{1+\Gamma(\alpha)}{\Gamma(\alpha)}\right)\tfrac{2N}{|H(1)|}\,\|H\,(\mathcal{A})\|_{*}+C_3\right]\|\delta\|_1\\ &\quad+\left[\left(\tfrac{1+\Gamma(\alpha)}{\Gamma(\alpha)}\right)\tfrac{2N}{|H(1)|}\,\|H\,(A)\|_{*}+C_3\right](\|\beta\|_1\;+\|\gamma\|_1)\,\|x\|_{\mathbf{X}}\,.\end{aligned}$$

In view of the condition (3.17) by simple computation, we obtain that

$$1-\left(\left(\frac{1+\Gamma\,(\alpha)}{\Gamma\,(\alpha)}\right)\frac{2N}{|H\,(1)|}\,\|H\,(\mathcal{A})\|_{*}+C_3\right)(\|\beta\|_1\;+\|\gamma\|_1)>0.$$

Then,

$$\|x\|_{\mathbb{X}} \leq \frac{\left(\tfrac{1+\Gamma(\alpha)}{\Gamma(\alpha)}\right)\tfrac{N}{|H(1)|}\,\|H\,(\mathcal{A})\|_{*}\,M_0+\left[\left(\tfrac{1+\Gamma(\alpha)}{\Gamma(\alpha)}\right)\tfrac{2N}{|H(1)|}\,\|H\,(\mathcal{A})\|_{*}+C_3\right]\|\delta\|_1}{1-\left[\left(\tfrac{1+\Gamma(\alpha)}{\Gamma(\alpha)}\right)\tfrac{2N}{|H(1)|}\,\|H\,(\mathcal{A})\|_{*}+C_3\right](\|\beta\|_1\;+\|\gamma\|_1)}=C,$$

proving that S_1 is bounded in $\mathbb{X}$.

Step 2. Let

$$S_2=\{x\in \text{Ker }(\mathcal{L}):\;Nx\in \text{Im }(\mathcal{L})\}$$

For all $x\in S_2$, there is $x(t)=\xi t^{\alpha-1}$ such that $\xi\in$ Ker $(\mathcal{I}_N-\mathcal{A})$ and $Nx\in$ Im $(\mathcal{L})$. Thus $\psi(Nx)\in$ Im $(\mathcal{I}_N-\mathcal{A})$. From $(H5)$, for any $i=\overline{1,N}$ there exists $t_i\in[0,1]$ such that

$$\left|D_{0^+}^{\alpha-1}x_i\,(t_i)\right|=\Gamma\,(\alpha)\,|\xi_i|\leq M_0.$$

Consequently, $|\xi_i|\leq\frac{M_0}{\Gamma(\alpha)}$ and

$$\begin{aligned}\|x\|_{\mathcal{X}} &= \max\nolimits_{1\leq i\leq N}\left(\|\xi_i t^{\alpha-1}\|_{\infty}+\|\xi_i\Gamma\,(\alpha)\|_{\infty}\right)\\ &= \max\nolimits_{1\leq i\leq N}\left(|\xi_i|+|\xi_i\Gamma\,(\alpha)|\right)\\ &\leq \tfrac{1+\Gamma(\alpha)}{\Gamma(\alpha)}M_0.\end{aligned}$$

Hence S_2 is bounded.

Step 3.

Define the isomorphism $\mathcal{J}$: Ker $(\mathcal{L}) \to$ Im $(\mathcal{Q})$ by $\mathcal{J}\left(\xi t^{\alpha-1}\right) = \xi$. Let

$$S_3 = \{x \in \text{Ker } (\mathcal{L}) : \ \sigma \mathcal{J} x + (1-\sigma)\, \mathcal{Q} N x = \theta, \ \sigma \in [0,1]\}$$

We demonstrate that S_3 is bounded. For $x = \xi t^{\alpha-1} \in S_3$, we have

$$\sigma \mathcal{J} x + (1-\sigma)\, \mathcal{Q} N x = \sigma \xi + (1-\sigma)\, \mathcal{Q} N x = \theta,$$

where

$$\mathcal{Q} N x = \frac{2\alpha}{2\alpha - 1} \frac{H\left(\mathcal{A}\right)}{H\left(1\right)} \psi\left(f\left(t, \xi t^{\alpha-1}, \xi \Gamma\left(\alpha\right)\right)\right).$$

For $\sigma = 0$, $\mathcal{Q} N x = \theta$. So $Nx \in$ Im $\mathcal{L}$ which means that $x \in S_2$ and if $\sigma = 1$, then $x = \theta$.

We use condition (3.16) of $(H6)$ in case $0 < \sigma < 1$,. Let us assume that $M_1 < left\|\xi_{\mathbb{R}^N}$. Next,

$$\langle \xi, \sigma \xi + (1-\sigma)\, \mathcal{Q} N x \rangle = \sigma \langle \xi, \xi \rangle + (1-\sigma) \langle \xi, \mathcal{Q} N x \rangle = 0.$$

contradicting

$$0 < \sigma \langle \xi, \xi \rangle = -\left(1-\sigma\right) \langle \xi, \mathcal{Q} N x \rangle \leq 0.$$

Thus $\|\xi\|_{\mathbb{R}^N} \leq M_1$ and then $\|x\|_{\mathbb{X}} \leq \left(1 + \Gamma\left(\alpha\right)\right) M_1$. The proof is finished.

Alternatively, if condition (3.15) of $(H6)$ holds, we demonstrate that the set using the same technique

$$S_3 = \{x \in \text{Ker } (\mathcal{L}) : \ -\sigma \mathcal{J} x + (1-\sigma)\, \mathcal{Q} N x = \theta, \ \sigma \in [0,1]\}$$

is bounded.

Step 4.

Let Ω be an open bounded subset of $\mathbb{X}$ such that $S_1 \cup S_2 \cup S_3 \subset \Omega$. Since S_1, S_2 and S_3 are bounded sets, we obtain

(1) $\mathcal{L}x \neq \sigma N x$, for every $(x, \sigma) \in$ (Dom $(\mathcal{L}) \backslash$Ker $\mathcal{L}) \cap \partial\Omega \times (0,1)$;

(2) $\mathcal{Q} N x \neq \theta$, for every $x \in$ Ker $\mathcal{L} \cap \partial\Omega$.

(3) Assume for instance that $(H6) - (3.16)$ holds and define the operator

$$N(x, \sigma) = \sigma \mathcal{J} x + (1-\sigma) \mathcal{Q} N x.$$

Since S_3 is bounded, $N(x,\sigma) \neq \theta$, for all $(x,\sigma) \in (\operatorname{Ker}\mathcal{L} \cap \partial\Omega) \times (0,1)$. Based on the homotopy of the degree and the invariance property, we obtain

$$\begin{aligned}
\deg(\mathscr{Q}N_{|\operatorname{Ker}\mathcal{L}}, \Omega \cap \operatorname{Ker}\mathcal{L}, 0) &= \deg(N(.,0), \Omega \cap \operatorname{Ker}\,(\mathcal{L})\,, 0) \\
&= \deg(N(.,1), \Omega \cap \operatorname{Ker}\,(\mathcal{L})\,, 0) \\
&= \deg(\mathcal{J}, \Omega \cap \operatorname{Ker}\,(\mathcal{L})\,, 0) \neq 0.
\end{aligned}$$

In the end, every requirement stated in Theorem (2.3.1) is fulfilled. Thus, there is at least one solution to the equation $\mathcal{L}x = Nx$ in $\operatorname{Dom}(\mathcal{L}) \subset X$ that is BVP (3.1)- (3.2)- (3.3) has at least one solution in the space $\mathbb{X}$.

3.5 A numerical example

Consider the system of BVP for Riemann-Liouville fractional differential equations of the form :

$$\begin{cases}
\begin{array}{c} D_{0^+}^{\frac{3}{2}} x_1(t) + \frac{1}{125}\left((t^2+1)\left(|x_1(t)| + |x_2(t)|\right) + \frac{4}{3\sqrt{\pi}}\,(t+2)\, D_{0^+}^{\frac{1}{2}} x_2(t)\right) \\ + \sin D_{0^+}^{\frac{1}{2}} x_2(t) + 2t + 1 = 0 \end{array}, \\
\begin{array}{c} D_{0^+}^{\frac{3}{2}} x_2(t) + \frac{1}{125}\left(\frac{1}{2}\,(t^2+1)\left(|x_1(t)| + |x_2(t)|\right) - \frac{2}{\sqrt{\pi}}\,(2-t)\, D_{0^+}^{\frac{1}{2}} x_2(t)\right) \\ + t - \frac{1}{2} = 0. \end{array} \\
x_1(0) = x_2(0) = 0, \\
D_{0^+}^{\frac{1}{2}} x_1\,(1) = 2\left(5 I_{0^+}^{\frac{3}{2}} x_1\,(1) + 6 I_{0^+}^{\frac{3}{2}} x_2\,(1)\right), \\
D_{0^+}^{\frac{1}{2}} x_2\,(1) = 2\left(2 I_{0^+}^{\frac{3}{2}} x_1\,(1) + 4 I_{0^+}^{\frac{3}{2}} x_2\,(1)\right),
\end{cases} \tag{3.21}$$

Here, $\alpha = \frac{3}{2}$, $\Gamma(\alpha) = \frac{\sqrt{\pi}}{2}$, $N = 2$, matrix $\mathcal{A} = \begin{pmatrix} 5 & 6 \\ 2 & 4 \end{pmatrix}$ with characteristic polynomial $p_{\mathcal{A}}(\lambda) = (1-\lambda)(8-\lambda)$.

So, $H(\mathcal{A}) = 8\mathcal{I}_2 - \mathcal{A} = \begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix}$, $H(1) = 8 - 1 = 7$, $\|H(\mathcal{A})\|_* = 6$, $\kappa = -\frac{1}{7}$, $P_c = \frac{7\sqrt{\pi}}{56\left(\sqrt{\pi}+2\right)} \simeq 0.058\,73$ and the nonlinearity is $f = (f_1, f_2)^T$, where $f_i : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ for $i = 1, 2$:

$$f_1(t,x,z) = \frac{1}{125}\left(\left(t^2+1\right)\left(|x_1| + |x_2|\right) + \frac{4}{3\sqrt{\pi}}\,(t+2)\, z_2\right) + \sin z_2 + 2t + 1,$$

$$f_2(t,x,z) = \frac{1}{125}\left(\frac{1}{2}\left(t^2+1\right)\left(|x_1|+|x_2|\right) - \frac{2}{\sqrt{\pi}}(2-t)\,z_2\right) + t - \frac{1}{2},$$

where $x=(x_1,x_2)^T$, $z=(z_1,z_2)^T$.

It is easy checked that conditions $(H1)-(H3)$ are fulfilled. Furthermore,

$$\|f(t,x,z)\|_{\mathbb{R}^2} \le \frac{1}{125}\left(2\left(t^2+1\right)\|x\|_{\mathbb{R}^2} + \frac{8}{3\sqrt{\pi}}(t+2)\|z\|_{\mathbb{R}^2}\right) + 2t + 2,$$

where

$$\|\beta\|_1 + \|\gamma\|_1 = .\frac{2.67+3.76}{125} \simeq 0.05144 < 0.058\,73.$$

Then, conditions (H_4) and (3.17) are satisfied. Let us compute $\psi(Nx)$. By definition, we have

$$\begin{aligned}
\psi(Nx) &= I_{0^+}^1 Nx(1) - \Gamma(3)\,\mathcal{A} I_{0^+}^3 Nx(1) \\
&= \begin{pmatrix} I_{0^+}^1 (Nx)_1(1) - \Gamma(3)\left(5I_{0^+}^3(Nx)_1(1) + 6I_{0^+}^3(Nx)_2(1)\right) \\ I_{0^+}^1 (Nx)_2(1) - \Gamma(3)\left(2I_{0^+}^3(Nx)_1(1) + 4I_{0^+}^3(Nx)_2(1)\right) \end{pmatrix} \\
&= \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.
\end{aligned} \tag{3.22}$$

Since Ker $(H(\mathcal{A})) =$ Ker $(8I_2 - \mathcal{A}) = \{(2b,b):\ b\in\mathbb{R}\}$, we have

$$\psi(Nx) \in \text{Im } (\mathcal{I}_2 - \mathcal{A}) \Longleftrightarrow 2\omega_2 - \omega_1 = 0.$$

In addition,

$$\begin{aligned}
&2\omega_2 - \omega_1 = \\
&\int_0^1 \left(1-(1-s)^2\right)\left(f_1\left(s, x(s), D_{0^+}^{\frac{1}{2}}x(s)\right) - 2f_2\left(s, x(s), D_{0^+}^{\frac{1}{2}}x(s)\right)\right) ds \\
= \;& \tfrac{4}{375\sqrt{\pi}}\int_0^1 \left(1-(1-s)^2\right)(8-2s)\, D_{0^+}^{\frac{1}{2}}x_2(s)\, ds \\
&+ \int_0^1 \left(1-(1-s)^2\right)\left(\sin D_{0^+}^{\frac{1}{2}}x_2(s) + 2\right) ds.
\end{aligned}$$

Set $M_0 = 75$. If $\left|D_{0^+}^{\frac{1}{2}}x_2(t)\right| \ge M_0$, for all $t\in[0,1]$, then by simple calculations, we reach the following discussion:

(1) in case $D_{0^+}^{\frac{1}{2}}x_2(s) \le -M_0$, we have

$$\begin{aligned}
2\omega_2 - \omega_1 &\le \tfrac{-4\times 75}{375\sqrt{\pi}}\int_0^1\left(1-(1-s)^2\right)(8-2s)\,ds + 3\int_0^1\left(1-(1-s)^2\right)ds \\
&= -\tfrac{300}{375\sqrt{\pi}} \times \tfrac{9}{2} + 3\times\tfrac{2}{3} = -3.\,108\,3\times 10^{-2} < 0.
\end{aligned}$$

(2) In case $D_{0^+}^{\frac{1}{2}}x_2(s) \ge M_0$, we find $2\omega_2 - \omega_1 > 0$, showing in both cases that condition (H_5) is fulfilled.

We note that the result in the first two examples holds true regardless of the value of $D_{0^+}^{\frac{1}{2}}x_1(s)$, whether $D_{0^+}^{\frac{1}{2}}x_1(s) \leq -M_0$, or $-M_0 \leq D_{0^+}^{\frac{1}{2}}x_1(s) \leq M_0$, or $D_{0^+}^{\frac{1}{2}}x_1(s) \geq M_0$ instead.

Finally, we check condition (H_6). For $x(t) = \xi\sqrt{t}$ with $\xi \in$ Ker $(\mathcal{I}_2 - \mathcal{A}) = \left\{\left(-\frac{3}{2}b, b\right)^T : b \in \mathbb{R}\right\}$, we have

$$\begin{aligned}(Nx)_1(t) &= f_1\left(t, \xi\sqrt{t}, \xi\tfrac{\sqrt{\pi}}{2}\right) = \tfrac{1}{125}\left((t^2+1)\left(\tfrac{3}{2}|b|\sqrt{t} + |b|\sqrt{t}\right) + \tfrac{2}{3}(t+2)\,b\right) \\ &\quad + \sin\left(\tfrac{1}{2}b\sqrt{\pi}\right) + 2t + 1\end{aligned}$$

and

$$\begin{aligned}(Nx)_2(t) &= f_2\left(t, \xi\sqrt{t}, \xi\tfrac{\sqrt{\pi}}{2}\right) = \tfrac{1}{125}\left(\tfrac{1}{2}(t^2+1)\left(\tfrac{3}{2}|b|\sqrt{t} + |b|\sqrt{t}\right) - (2-t)\,b\right) \\ &\quad + t - \tfrac{1}{2}.\end{aligned}$$

After computing $\psi(Nx)$ as previously,

$$\mathscr{Q}Nx = \frac{3}{14}\begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \frac{3}{14}\begin{pmatrix} 3\omega_1 - 6\omega_2 \\ -2\omega_1 + 4\omega_2 \end{pmatrix} = \frac{3}{14}(2\omega_2 - \omega_1)\begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

After doing the calculations and simplifications, we get the following

$$\mathscr{Q}Nx = \frac{3}{14}\left(\frac{3}{125}b + h\right)\begin{pmatrix} -3 \\ 2 \end{pmatrix},$$

where $h = \int_0^1 \left(1-(1-s)^2\right)\left(\sin\frac{b\sqrt{\pi}}{2} + 2\right)ds$.

So, for $\xi = b\begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$ with $|b| \geq M_1 = 85$, we have

$$\begin{aligned}\langle \xi, \mathscr{Q}Nx\rangle &= \tfrac{3}{14}b\left(\tfrac{3}{125}b + h\right)\left(\tfrac{9}{2} + 2\right) \\ &= \tfrac{39}{28}b\left(\tfrac{3}{125}b + h\right) > 0,\end{aligned}$$

It demonstrates the validity of condition (H_6). We deduce that there is at least one solution to Boundary Value Problem (3.21) in $\mathbb{X}$.

CHAPTER

4

EXISTENCE AND UNIQUENESS OF FBVP IN THE NON-COMPACTNESS CASE

THis chapter studies the uniqueness and existence of solutions to certain types of boundary value problems related to the fractional Caputo derivative. Our approach is based on reducing a given problem to an equivalent fixed-point problem. We used the coincidence degree theory approach for condensing maps (neither Leray-Schauder degree nor Brouwer degree) to provide sufficient conditions for the existence and uniqueness of solutions. We chose an example in which all conditions are met, requires great precision. This is how the following part of the chapter is structured: In the first Section, we introduce the fractional boundary value problem. In Sections 2 and 3, we created necessary conditions for the existence and uniqueness of solutions to our problem, Lastly, we provide an example, to show how to achieve result in Section 4.

4.1 Introduction

This work is concerned with the existence and uniqueness of solution to the following fractional boundary value problems (FBVP for short)

$$^{c}D_{0^+}^{r}x(t) = g(t, x(t)),\ t \in I := [0, T], \tag{4.1}$$

$$ax(0) - bx(T) = \Psi\left(x\right), \tag{4.2}$$

$$^{c}D^{r-1}x(T) = \phi(x), \tag{4.3}$$

where $a, b \in \mathbb{R}$ $(a \neq b)$, $r \in (1, 2)$, ${}^{c}D^{r}$ is the Caputo derivative, $g : I \times \mathbb{R} \to \mathbb{R}$, $\phi : C(I) \to \mathbb{R}$ and $\Psi : C(I) \to \mathbb{R}$ are given continuous maps.

4.2 Existence result

In order to discuss existence and uniqueness of solution to the FBVP (4.1), (4.2), (4.3) we require the following assumptions:
there exists

[H1] two constants λ, $\lambda' > 0$ such that for all $x, v \in C(I)$

$$\begin{aligned} |\phi(x) - \phi(v)| &\leq \lambda\|x - v\|_{\infty}, \\ |\Psi(x) - \Psi(v)| &\leq \lambda'\|x - v\|_{\infty}. \end{aligned}$$

[H2] two functions $\beta', \beta \in L^1\left([0; T], \mathbb{R}_+\right)$ and a constant $r_1 \in [0, 1)$ such that

$$\begin{aligned} |g(t, x)| &\leq \beta'\left(t\right)|x|^{r_1} + \beta\left(t\right), \\ \text{for arbitrary } (t, x) &\in I \times \mathbb{R}. \end{aligned}$$

[H3] six real constants $\gamma, \delta, \gamma', \delta' > 0, r_2, r_3 \in [0, 1)$ such that for each $x \in C(I)$

$$\begin{aligned} |\phi(x)| &\leq \gamma\|x\|_{\infty}^{r_2} + \delta, \\ |\Psi(x)| &\leq \gamma'\|x\|_{\infty}^{r_3} + \delta'. \end{aligned}$$

[H4] a constant $\lambda'' > 0,$ such that for all $x, v \in \mathbb{R}, t \in I$

$$|g(t,x) - g(t,v)| \leq \lambda'' |x - v|.$$

Lemma 4.2.1

Let $1 < r < 2$, the fractional integral equation (FIE for short)

$$x(t) = \frac{1}{a-b}\int_0^T G(t,s)g(s,x(s))ds + \frac{1}{a-b}\Psi(x) + (\frac{b}{a-b}T^{r-1} + tT^{r-2})\Gamma(3-r)\phi(x), \quad (4.4)$$

where

$$G(t,s) = \begin{cases} \frac{a-b}{\Gamma(r)}(t-s)^{r-1} + \frac{b}{\Gamma(r)}(T-s)^{r-1} - & \\ & 0 \leq s \leq t \leq T \\ ((a-b)t + bT)T^{r-2}\Gamma(3-r), & \\ \frac{b}{\Gamma(r)}(T-s)^{r-1} - ((a-b)t + bT)T^{r-2}\Gamma(3-r), & 0 \leq t \leq s \leq T \end{cases}$$

has a solution $x \in C(I)$ if and only if x is a solution of the FBVP (4.1), (4.2), (4.3).

Proof 4.2.1

Assuming that x is a solution of FBVP (4.1), (4.2), (4.3). We need to demonstrate that x is also a solution of FIE (4.4). By the lemmas (1.5.2, 1.5.3) we have

$$x(t) = I_{0+}^r g(t,x(t)) + c_0 + c_1 t. \quad (4.5)$$

And by introducing the Caputo derivative of order $r-1$ on both sides of equation (4.5) and taking $t = T$, we get

$$\begin{aligned} {}^cD_{0+}^{r-1}x(t) &= {}^c D_{0+}^{r-1}(I_{0+}^r g(t,x(t)) + {}^c D_{0+}^{r-1}(c_0) + {}^c D_{0+}^{r-1}(c_1 t) \\ &= I_{0+}^1 g(t,x(t)) + \frac{c_1}{\Gamma(3-r)}t^{2-r}, \end{aligned}$$

therefore

$${}^cD_{0+}^{r-1}x(T) = \int_0^T g(s,x(s))ds + \frac{c_1}{\Gamma(3-r)}T^{2-r} = \phi(x).$$

So

$$c_1 = T^{r-2}\Gamma(3-r)\left[\phi(x) - \int_0^T g(s,x(s))ds\right].$$

Then from (4.2) and (4.5) we have

$$\begin{aligned}
&ac_0 - b\Big(\frac{1}{\Gamma(r)}\int_0^T (T-s)^{r-1} g(s,x(s))ds + c_0 + c_1 T\Big) = \\
&(a-b)c_0 - bc_1 T - \frac{b}{\Gamma(r)}\int_0^T (T-s)^{r-1} g(s,x(s))ds = \Psi(x).
\end{aligned}$$

Substituting c_1 by its value, we obtain $c_0 = \dfrac{1}{a-b}\left\{\begin{array}{l} \Psi(x) + b\frac{1}{\Gamma(r)}\int_0^T (T-s)^{r-1} g(s,x(s))ds + \\ bT^{r-1}\Gamma(3-r)\left[\phi(x) - \int_0^T g(s,x(s))ds\right] \end{array}\right\}.$

Consequently

$$\begin{aligned}
x(t) =& \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} g(s,x(s))ds \\
&+\frac{1}{a-b}\left(\Psi(x) + \frac{b}{\Gamma(r)}\int_0^T (T-s)^{r-1} g(s,x(s))ds\right) \\
&+\Big(\frac{bT}{a-b}+t\Big)T^{r-2}\Gamma(3-r)\left(\phi(x) - \int_0^T g(s,x(s))ds\right).
\end{aligned}$$

By simple calculations, we get

$$\begin{aligned}
x(t) =& \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} g(s,x(s))ds + \frac{b}{(a-b)\Gamma(r)}\int_0^T (T-s)^{r-1} g(s,x(s))ds \\
&-\Big(\frac{bT}{a-b}+t\Big)T^{r-2}\Gamma(3-r)\int_0^T g(s,x(s))ds \\
&+\frac{1}{a-b}\Psi(x) + \Big(\frac{bT}{a-b}+t\Big)T^{r-2}\Gamma(3-r)\phi(x).
\end{aligned} \tag{4.6}$$

Finally, we conclude that

$$x(t) = \frac{1}{a-b}\int_0^T G(t,s) g(s,x(s))ds + \frac{1}{a-b}\Psi(x) + \Big(\frac{bT}{a-b}+t\Big)T^{r-2}\Gamma(3-r)\phi(x),$$

where

$$G(t,s) = \begin{cases} \frac{a-b}{\Gamma(r)}(t-s)^{r-1} + \frac{b}{\Gamma(r)}(T-s)^{r-1} - \\ ((a-b)t + bT)T^{r-2}\Gamma(3-r), & 0 \le s \le t \le T \\ \frac{b}{\Gamma(r)}(T-s)^{r-1} - ((a-b)t + bT)T^{r-2}\Gamma(3-r), & 0 \le t \le s \le T. \end{cases}$$

Now, assuming that x is a solution of FIE (4.4). By applying ${}^cD^r_{0+}$ on both sides of the equation (4.6) with taking account the definitions (1.5.2), (1.11) and the properties of Caputo derivatives, it yields

$${}^cD^r_{0+}x(t) = g(t,x(t))$$

because ${}^{c}D_{0+}^{r}\left[\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} g(s,x(s))ds\right] ={}^{c} D_{0+}^{r}\left(I_{0+}^{r} g(t,x(t))\right) = g(t,x(t))$
and ${}^{c}D_{0+}^{r}(c_2+c_3 t) ={}^{c} D_{0+}^{r}(C) = 0$, where c_2, c_3 and C are arbitrary constants. However, we also have

$x(0) = \frac{b}{(a-b)\Gamma(r)}\int_0^T (T-s)^{r-1} g(s,x(s))ds - \dfrac{bT^{r-1}}{a-b}\Gamma(3-r)\int_0^T g(s,x(s))ds + \dfrac{1}{a-b}\Psi(x) + \dfrac{bT^{r-1}}{a-b}\Gamma(3-r)\phi(x)$

and

$x(T) = \dfrac{a}{a-b}\int_0^T (T-s)^{r-1} g(s,x(s))ds - \dfrac{a}{a-b}T^{r-1}\Gamma(3-r)\int_0^T g(s,x(s))ds + \dfrac{1}{a-b}\Psi(x) + \dfrac{a}{a-b}T^{r-1}\Gamma(3-r)\phi(x).$

Then,

$$ax(0) - bx(T) = \frac{a-b}{a-b}\Psi(x) = \Psi(x).$$

${}^{c}D_{0+}^{r-1}x(t) = \int_0^t g(s,x(s))ds - \frac{t^{2-r}}{\Gamma(3-r)}T^{r-2}\Gamma(3-r)\int_0^T g(s,x(s))ds + \frac{t^{2-r}}{\Gamma(3-r)}T^{r-2}\Gamma(3-r)\phi(x)$

hence,

$${}^{c}D_{0+}^{r-1}x(T) = \int_0^T g(s,x(s))ds - \int_0^T g(s,x(s))ds + \phi(x) = \phi(x),$$

which complete the proof.

From the expression (4.4), we can also define the following operators

$$\begin{aligned}
F_1 : C(I) &\to C(I) \\
(F_1 x)(t) &= \frac{1}{a-b}\int_0^T G(t,s) g(s,x(s))ds, && t \in I, \\
F_2 &: C(I) \to C(I) \\
(F_2 x)(t) &= \frac{1}{a-b}\Psi(x) + \left(\frac{b}{a-b}T^{r-1} + tT^{r-2}\right)\Gamma(3-r)\phi(x), && t \in I, \\
P &: C(I) \to C(I), \quad Px = F_1 x + F_2 x.
\end{aligned}$$

By definition of operators ϕ, Ψ and the continuity of g, the operator P able to be written and the FIE (4.4) can be written as the following operator equation

$$x = Px = F_1 x + F_2 x. \tag{4.7}$$

Therefore, each solution of the fractional FBVP (4.1), (4.2), (4.3) is a fixed point of the operator P.

Remark 4.2.1

For every $(t,s) \in I \times I$;

$$|G(t,s)| \leq (|a-b|+|b|)\left(\frac{1}{\Gamma(r)}+\Gamma(3-r)\right)T^{r-1} := R_1. \tag{4.8}$$

Lemma 4.2.2

Under the assumption $[H2]$, the operator F_1 is completely continuous and fulfills the subsequent growth condition

$$\|F_1 x\|_\infty \leq \frac{R_1}{|a-b|}\|\beta'\|_1 \|x\|_\infty^{r_1} + \frac{R_1}{|a-b|}\|\beta\|_1. \tag{4.9}$$

Proof 4.2.2

Let $\{x_n\}$ be a sequence of the bounded set $B(\kappa) = \{x \in C(I); \|x\|_\infty \leq \kappa\}$ such that $||x_n - x||_\infty \to 0$ as $n \to \infty$. We have to show that $||F_1 x_n - F_1 x||_\infty \to 0$ as $n \to \infty$. Due to the continuity of g, it is obvious that $||g(s, x_n(s)) - g(s, x(s))|| \to 0$ as $n \to \infty$. Using the triangular inequality and condition $[H2]$, we get for each $t \in I$,

$$\begin{aligned}
|g(t,x_n(t)) - g(t,x(t))| &\leq |g(t,x_n(t)| + |g(t,x(t))| \\
&\leq \beta'(t)\left[|x_n(t)|^{r_1} + |x(t)|^{r_1}\right] + 2\beta(t) \\
&\leq 2\left(\kappa^{r_1}\beta'(t) + \beta(t)\right).
\end{aligned}$$

Then, $|G(t,s)|\,|g(s,x_n(s)) - g(s,x(s))| \leq 2R_1\left(\kappa^{r_1}\beta'(s) + \beta(s)\right)$.
As $t \mapsto 2R_1\left(\kappa^{r_1}\beta'(t) + \beta(t)\right)$ is Lebesgue integrable function on $[0,T]$. by means of the Lebesgue-dominated convergence theorem

$$\int_0^T |G(t,s)|\,|g(s,x_n(s)) - g(s,x(s))|\,ds \text{ tends to } 0 \text{ as } n \to \infty,$$

then,

$$||F_1 x_n - F_1 x||_\infty \leq \frac{1}{|a-b|}\int_0^T |G(t,s)|\,|g(s,x_n(s)) - g(s,x(s))| \text{ tends to } 0 \text{ as } n \to \infty.$$

Which means that F_1 is continuous.
Let B be a bounded subset of $C(I)$ i.e there exists $\kappa > 0$ such that $B \subseteq B_\kappa(0)$.

As the function g is a continuous function, then there exists $M > 0$ such that for all $(t, x) \in [0, T] \times B$

$$|g(s, x(s))| \leq M,$$

because in this case $(s, x(s)) \in [0, T] \times [-\kappa, \kappa]$. Therefore, for every $x \in B$ and $0 \leq t \leq T$, we have

$$\begin{aligned} |(F_1 x)(t)| &= \left| \frac{1}{a-b} \int_0^T G(t,s) g(s, x(s)) ds \right| \\ &\leq \frac{R_1}{|a-b|} \int_0^T |g(s, x(s))| \, ds \\ &\leq \frac{R_1}{|a-b|} \int_0^T M ds = \frac{R_1 M T}{|a-b|.} \end{aligned}$$

Consequently, $F_1(B)$ is uniformly bounded in $C(I)$.

For $0 \leq t_1 \leq t_2 \leq T$, we have

$$\begin{aligned} |(F_1x)(t_2) - (F_1x)(t_1)| &= \left| \begin{array}{c} \frac{1}{\Gamma(r)} \left(\int_0^{t_2} (t_2 - s)^{r-1} g(s, x(s)) ds - \int_0^{t_1} (t_1 - s)^{r-1} g(s, x(s)) ds \right) \\ -T^{r-2}\Gamma(3-r)(t_2 - t_1) \int_0^T g(s, x(s)) ds \end{array} \right| \\ &= \left| \begin{array}{c} \frac{1}{\Gamma(r)} \int_0^{t_1} \left((t_2 - s)^{r-1} - (t_1 - s)^{r-1} \right) g(s, x(s)) ds + \\ \frac{1}{\Gamma(r)} \int_{t_1}^{t_2} (t_2 - s)^{r-1} g(s, x(s)) ds - T^{r-2}\Gamma(3-r)(t_2 - t_1) \int_0^T g(s, x(s)) ds \end{array} \right| \\ &\leq \frac{M}{\Gamma(r)} \left[\begin{array}{c} \int_0^{t_1} \left((t_2 - s)^{r-1} - (t_1 - s)^{r-1} \right) ds + \int_{t_1}^{t_2} (t_2 - s)^{r-1} ds \\ +T^{r-1}\Gamma(r)\Gamma(3-r)(t_2 - t_1) \end{array} \right] \\ &= \frac{M}{\Gamma(r+1)} \left[(t_2^r - t_1^r) \right] + T^{r-1} M \Gamma(3-r)(t_2 - t_1) \to 0 \text{ uniformly as } t_1 \to t_2. \end{aligned}$$

Therefore, $F_1(B)$ is equicontinuous. Hence, as a result of Ascoli-Arzela theorem, the operator F_1 is completely continuous.

On the other hand, in view of the remark (4.2.1) and the condition $[H1]$, we have for every $x \in C(I)$ and all $t \in I$

$$\begin{aligned} |F_1 x(t)| &= \left| \frac{1}{a-b} \int_0^T G(t,s) g(s, x(s)) ds \right| \\ &\leq \frac{1}{|a-b|} \int_0^T |G(t,s)| \, |g(s, x(s))| \, ds \\ &\leq \frac{R_1}{|a-b|} \int_0^T \left(\beta'(t) |x(s)|^{r_1} + \beta(t) \right) ds \\ &\leq \frac{R_1}{|a-b|} \left(\|\beta'\|_1 \|x\|_\infty^{r_1} + \|\beta\|_1 \right) = \frac{R_1}{|a-b|} \|\beta'\|_1 \|x\|_\infty^{r_1} + \frac{R_1}{|a-b|} \|\beta\|_1 . \end{aligned}$$

Then, $\|F_1x\|_\infty \leq \frac{R_1}{|a-b|}\|\beta'\|_1\|x\|_\infty^{r_1} + \frac{R_1}{|a-b|}\|\beta\|_1$.
Which complete the proof.

Lemma 4.2.3

Under the assumption $[H1]$ and $[H3]$, the operator F_2 fulfills the Lipschitz condition with constant $l = \dfrac{\lambda'}{|a-b|} + \left(\dfrac{|b|}{|a-b|} + 1\right) T^{r-1}\Gamma(3-r)\lambda$, consequently F_2 is $\alpha-$lipschitz with the same constant and fulfills the growth condition as follows:

$$\begin{aligned}\|F_2x\|_\infty &\leq \frac{\gamma'}{|a-b|}\|x\|_\infty^{r_3} + \left(\frac{|b|}{|a-b|} + 1\right) T^{r-1}\Gamma(3-r)\gamma\|x\|_\infty^{r_2} \\ &\quad + \left(\frac{|b|}{|a-b|} + 1\right) T^{r-1}\Gamma(3-r)\delta + \frac{\delta'}{|a-b|}.\end{aligned} \tag{4.10}$$

Proof 4.2.3

By the triangular inequality and the assumption $[H1]$, we have for every $x, v \in C(I)$,

$$\begin{aligned}|F_2u(t) - F_2v(t)| &= \left|\frac{1}{a-b}(\Psi(u) - \Psi(v)) + (\frac{b}{a-b}T^{r-1} + tT^{r-2})\Gamma(3-r)(\phi(u) - \phi(v))\right| \\ &\leq \frac{1}{|a-b|}|\Psi(u) - \Psi(v)| + \left(\frac{|b|}{|a-b|} + 1\right) T^{r-1}\Gamma(3-r)|\phi(u) - \phi(v)| \\ &\leq \frac{\lambda'}{|a-b|}\|u-v\|_\infty + \left(\frac{|b|}{|a-b|} + 1\right) T^{r-1}\Gamma(3-r)\lambda\|u-v\|_\infty \\ &= \left[\frac{\lambda'}{|a-b|} + \left(\frac{|b|}{|a-b|} + 1\right) T^{r-1}\Gamma(3-r)\lambda\right]\|u-v\|_\infty.\end{aligned}$$

In view of proposition (1.3.2), we conclude that F_2 is $\alpha-$lipschitz with constant l.

Further, using condition $[H3]$, we get for all $x \in C(I)$, $t \in I$

$$\begin{aligned}|F_2x(t)| &= \left|\frac{1}{a-b}\Psi(x) + (\frac{b}{a-b}T^{r-1} + tT^{r-2})\Gamma(3-r)\phi(x)\right| \\ &\leq \frac{1}{|a-b|}|\Psi(x)| + \left|(\frac{b}{a-b}T^{r-1} + tT^{r-2})\Gamma(3-r)\right||\phi(x)| \\ &\leq \frac{1}{|a-b|}(\gamma'\|x\|_\infty^{r_3} + \delta') + \left|(\frac{b}{a-b}T^{r-1} + tT^{r-2})\Gamma(3-r)\right|(\gamma\|x\|_\infty^{r_2} + \delta).\end{aligned}$$

Then,

$$\begin{aligned}\|F_2x\|_\infty &\leq \frac{\gamma'}{|a-b|}\|x\|_\infty^{r_3} + \left(\frac{|b|}{|a-b|} + 1\right) T^{r-1}\Gamma(3-r)\gamma\|x\|_\infty^{r_2} \\ &\quad + \left(\frac{|b|}{|a-b|} + 1\right) T^{r-1}\Gamma(3-r)\delta + \frac{\delta'}{|a-b|},\end{aligned}$$

which complete the proof.

Theorem 4.2.1

If the conditions $[H1]-[H3]$ hold, the fractional integral equation (4.4) and then the FBVP (4.1), (4.2), (4.3) admits at least one solution in $C\left(I\right)$ provided that

$$l=\frac{\lambda'}{|a-b|}+\left(\frac{|b|}{|a-b|}+1\right)T^{r-1}\Gamma(3-r)\lambda<1.$$

Moreover, in this case, the set of solutions of this problem is bounded.

Proof 4.2.4

In view of propositions (1.3.2), (1.3.2) the operator F_1 is $\alpha-$Lipschitz with constant 0 and F_2 is $\alpha-$Lipschitz with constant l then, $P=F_1+F_2$ is $\alpha-$Lipschitz with constant $0+l=l<1$. Therefore, the operator P is a strict α-contraction i.e P is α-condensing operator.

Now, by using the theorem (2.4.1), we will prove that $S=\{x\in C\left(I\right);\exists\theta\in[0,1]$ such that $x=\theta P(x)\}$ is a bounded set in $C\left(I\right)$.

Consider $x\in S$ (arbitrary), in view of results (4.9) and (4.10), we get

$$\begin{aligned}
\|x\|_\infty &= \|\theta P(x)\|_\infty=\theta\left\|Px\right\|_\infty=\theta\left\|F_1x+F_2x\right\|_\infty\\
&\leq \|F_1x\|_\infty+\|F_2x\|_\infty\\
&\leq \frac{R_1}{|a-b|}\left\|\beta'\right\|_1\left\|x\right\|_\infty^{r_1}+\frac{\gamma'}{|a-b|}\left\|x\right\|_\infty^{r_3}+\left(\frac{|b|}{|a-b|}+1\right)T^{r-1}\Gamma(3-r)\gamma\|x\|_\infty^{r_2}\\
&\quad+\frac{R_1}{|a-b|}\left\|\beta\right\|_1+\left(\frac{|b|}{|a-b|}+1\right)T^{r-1}\Gamma(3-r)\delta+\frac{\delta'}{|a-b|}.
\end{aligned}$$

Then,

$$\|x\|_\infty\leq A_1\left\|x\right\|_\infty^{r_1}+A_2\|x\|_\infty^{r_2}+A_3\left\|x\right\|_\infty^{r_3}+A, \tag{4.11}$$

where

$$A_1:=\frac{R_1}{|a-b|}\left\|\beta'\right\|_1,$$

$$A_2:=\left(\frac{|b|}{|a-b|}+1\right)T^{r-1}\Gamma(3-r)\gamma,$$

$$A_3:=\frac{\gamma'}{|a-b|},$$

$$A:=\frac{R_1}{|a-b|}\left\|\beta\right\|_1+\left(\frac{|b|}{|a-b|}+1\right)T^{r-1}\Gamma(3-r)\delta+\frac{\delta'}{|a-b|}.$$

The inequality (4.11), together with $0 < r_i < 1$ and $A_i > 0$ for $i = 1, 2, 3$ implies that there exists a constant $C > 0$ such that $\|x\|_\infty \leq C$ i.e. S is bounded in $C(I)$. If it is not the case, assume by contradiction that $\|x\|_\infty \to +\infty$ and dividing both sides of inequality (4.11) by $\|x\|_\infty$, we find $1 \leq \frac{A_1}{\|x\|_\infty^{1-r_1}} + \frac{A_2}{\|x\|_\infty^{1-r_2}} + \frac{A_3}{\|x\|_\infty^{1-r_3}} + \frac{A}{\|x\|_\infty} \to 0$. Thus, The set of all fixed points of the operator P is bounded, and there is at least one fixed point for P in $C(I)$.

Remark 4.2.2

In the conditions $[H2]$, $[H3]$, if there exists $i \in \{1, 2, 3\}$ such that $r_i = 1$, the above result remains true provided that $A_i < 1$.
Particular case: For $\Psi(x) = \int_0^T x(s)\, ds$ with $0 < T < 1$, we have $|\Psi(x) - \Psi(v)| \leq T \|x - v\|_\infty$ and $|\Psi(x)| \leq T \|x\|_\infty^1 + \delta'$ for all $x, v \in C(I)$ and arbitrary $\delta' > 0$.

4.3 Uniqueness result

Theorem 4.3.1

In addition of condtions of theorem (4.2.1), if the conditions $[H4]$ hold, the BVP (4.1), (4.2), (4.3) admits a unique solution in $C(I)$ provided that $\frac{\lambda'' R_1 T}{|a-b|} + l < 1$.

Proof 4.3.1

For $u, v \in C(I)$, by $[H1]$ we have $|(F_2 u)(t) - (F_2 v)(t)| \leq l \|u - v\|_\infty$ and by $[H4]$,

$$\begin{aligned} |(F_1 u)(t) - (F_1 v)(t)| &= \left| \frac{1}{a-b} \int_0^T G(t,s) \left(g(s, u(s)) - g(s, v(s))\right) ds \right| \\ &\leq \frac{\lambda'' \|x - v\|_\infty}{|a-b|} \int_0^T |G(t,s)|\, ds \leq \frac{\lambda'' R_1 T}{|a-b|} \|u - v\|_\infty . \end{aligned}$$

Consequently,

$$\begin{aligned} |(Pu)(t) - (Pv)(t)| &\leq |(F_1 u)(t) - (F_1 v)(t)| + (F_2 u)(t) - (F_2 v)(t) \\ &\leq \left[\frac{\lambda'' R_1 T}{|a-b|} + l\right] \|u - v\|_\infty . \end{aligned}$$

As $\frac{\lambda'' R_1 T}{|a-b|} + l < 1$, the operator $P : C(I) \to C(I)$ is a contraction, then by Banach contraction principle it admits a unique fixed point in $C(I)$.

4.4 Example

Consider the following BVP

$$\begin{cases} {}^{c}D_{0^{+}}^{\frac{3}{2}}x(t)=\frac{1}{90}\left(2+t+3\left(t^{2}+1\right)\sin^{2}x\left(t\right)\right), \\ 2x(0)-x(\frac{\pi}{2})=\frac{1}{\pi^{2}}\int_{0}^{\frac{\pi}{4}}x\left(s\right)ds, \\ {}^{c}D_{0^{+}}^{\frac{1}{2}}x(\frac{\pi}{2})=\frac{1}{8}\cos^{2}x\left(0\right). \end{cases} \tag{4.12}$$

In this example, $r=\frac{3}{2},T=\frac{\pi}{4},a=2,\ b=1,\ g\left(t,x\right)=\frac{1}{90}\left(2+t+3\left(t^{2}+1\right)\sin^{2}x\right)$, $\phi\left(x\right)=\frac{1}{8}\cos^{2}x\left(0\right),\ \Psi\left(x\right)=\frac{1}{\pi^{2}}\int_{0}^{\frac{\pi}{4}}x\left(s\right)ds.$
By simple combutation, we have $R_{1}=2\left(\frac{1}{\Gamma\left(\frac{3}{2}\right)}+\Gamma\left(\frac{3}{2}\right)\right)\sqrt{\frac{\pi}{4}}=2+\frac{\pi}{2}\simeq 3.57.$

Also, for each $x,v\in C\left(\left[0,\frac{\pi}{4}\right]\right)$;

$$\begin{aligned} \left|\phi\left(x\right)-\phi\left(v\right)\right| &= \frac{1}{8}\left|\cos^{2}\left(x\left(0\right)\right)-\cos^{2}\left(v\left(0\right)\right)\right| \\ &= \frac{1}{8}\left|\cos\left(x\left(0\right)\right)+\cos\left(v\left(0\right)\right)\right|\left|\cos\left(x\left(0\right)\right)-\cos\left(v\left(0\right)\right)\right| \\ &\leq \frac{2}{8}.\left|2\sin\left(\frac{x\left(0\right)+v\left(0\right)}{2}\right)\right|\left|\sin\frac{x\left(0\right)-v\left(0\right)}{2}\right| \\ &\leq \frac{2}{8}.2.1\left|\frac{x\left(0\right)-v\left(0\right)}{2}\right| \\ &\leq \frac{1}{4}\left|x\left(0\right)-v\left(0\right)\right|\leq\frac{1}{4}\left\|x-v\right\|_{\infty}, \end{aligned}$$

and

$$\begin{aligned} \left|\phi\left(x\right)\right| &= \frac{1}{8}\left|1-\sin^{2}x\left(0\right)\right|\leq\frac{1+\sin^{2}x\left(0\right)}{8}\leq\frac{1+\sqrt{\left|\sin x\left(0\right)\right|}}{8} \\ &\leq \frac{\sqrt{\left|x\left(0\right)\right|}+1}{8}\leq\frac{1}{8}\left\|x\right\|_{\infty}^{\frac{1}{2}}+\frac{1}{8}. \end{aligned}$$

For each $t\in\left[0,\frac{\pi}{4}\right]$ and all $x,v\in\mathbb{R}$, we have

$$\begin{aligned} \left|g\left(t,x\right)-g\left(t,v\right)\right| &= \left|\frac{1}{30}\left(t^{2}+1\right)\right|\left|\sin^{2}x-\sin^{2}v\right| \\ &= \left|\frac{1}{30}\left(t^{2}+1\right)\right|\left|\sin x+\sin v\right|\left|\sin x-\sin v\right| \\ &\leq \frac{\left(t^{2}+1\right)}{15}\left|2\cos\left(\frac{x+v}{2}\right)\right|\left|\sin\left(\frac{x-v}{2}\right)\right| \\ &\leq \frac{2}{15}\left|x-v\right|. \end{aligned}$$

It is clear to observe that $\left|\Psi\left(x\right)-\Psi\left(v\right)\right|\leq\frac{1}{4\pi}\left\|x-v\right\|_{\infty},\ \left|\Psi\left(x\right)\right|\leq\frac{1}{4\pi}\left\|x\right\|_{\infty}$ with $\frac{1}{4\pi}\simeq 7.957\,7\times 10^{-2}<1$ and $g\left(t,x\right)\leq\frac{\left(t^{2}+1\right)}{30}\left|x\right|^{\frac{1}{2}}+\frac{2+t}{90}.$

Then, we conclude that, $\lambda = \frac{1}{4}, \lambda' = \frac{1}{4\pi}, \gamma = \frac{1}{8} = \delta, r_2 = \frac{1}{2} \quad \gamma' = \frac{1}{4\pi}, \delta'$(arbitrary) $r_3 = 1, \beta'(t) = \frac{\left(t^2+1\right)}{30}, \beta(t) = \frac{2+t}{90}, r_1 = \frac{1}{2}$, $\lambda'' = \frac{2}{15}$

$$l = \frac{\lambda'}{|a-b|} + \left(\frac{|b|}{|a-b|} + 1\right) T^{r-1}\Gamma(3-r)\lambda = \frac{1}{4\pi} + 2\sqrt{\frac{\pi}{4}}\Gamma(\tfrac{3}{2})\frac{1}{4} \simeq 0.392\,70 < 1,$$

$\frac{\lambda'' R_1 T}{|a-b|} + l \simeq 0.373\,85 + 0.392\,70 = 0.766\,55 < 1.$

Consequently, Each of the conditions of theorem (4.3.1) is satisfied, demonstrating that there is only one solution to the problem (4.12) in $C\left(\left[0, \frac{\pi}{4}\right]\right)$.

Conclusion

In the present thesis, we have considered some boundary value problems of fractional order differential equations (FBVPs) posed on bounded intervals and associated with boundary conditions. Most of these problems stem from in the some modeling of many physical, mechanical, and biological phenomena.

Our contribution in this thesis was mainly to establish new existence results for some boundary problems associated with nonlinear fractional differential equations to the resonance case using Mawhin's theory of coincidences and in the last we studied the existence and uniqueness of FBVPs solutions using non-compactness measure technique.

The main difficulty in dealing with such types of FBVPs was the possible is related to the construction of the projections which requires the realization of several technical conditions. In order to overcome these difficulties, we have made use of some special Banach spaces adapted to each situation and we stated some definitions and lemmas about the fractional calculus and the linear algebra, in addition to some sufficient conditions on the nonlinear term are assumed throughout which was helpful in proving our main results.

All of our existence results are supported by examples of applications.

This work has clarified certain methods for the existence and uniqueness of solutions to some nonlinear fractional problems. Its aim is to provide the desired benefit to researchers in this field of study or to use its results for applications in different fields.

We hope that this research will benefit researchers studying in the specialty of differential equations, and that they will have support in other research that is an extension of this research.

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