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Rothberger's property in uniform spaces

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Abstract.

In this work, we introduce several equivalent concepts of uniform spaces then give the notion of selection principles in each one (Specifically Rothberger-type property), and study some properties.

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AMS (2000) Subject Classifications: 54E15, 54D20.

1 Introduction

In this section, we give some important definitions which explain this topic and help to understand it. Firstly, let us speak about necessary concept of selection principles in topological spaces

- Let X a topological space, $\mathcal{U}$ is a family of subset of X and $A \subset X$, then the star of A with respect to $\mathcal{U}$, written $St(A,\mathcal{U})$, is the union of all members of $\mathcal{U}$ that intersect A:

$$St(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} | A \cap U \neq \emptyset\}$$

- The symbol $S_1(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis:
  for each sequence $(\mathcal{U}_n|n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(V_n|n \in \mathbb{N})$ such that for each n, $V_n \in \mathcal{U}_n$ and $\{V_n|n \in \mathbb{N}\} \in \mathcal{B}$.

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- The symbol $S_1^*(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis:
  For each sequence $(\mathcal{U}_n|n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(V_n|n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $V_n \in \mathcal{U}_n$ and $\{St(V_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

- The symbol $SS_{\mathcal{K}}^*(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis:
  for every sequence $(\mathcal{U}_n|n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(K_n|n \in \mathbb{N})$ of elements of $\mathcal{K}$ such that $\{St(Kn, Un) : n \in \mathbb{N}\} \in \mathcal{B}$.

Let $\mathcal{O}$ denotes the collection of all open covers of X, then a space X is said to be Rothberger-star Rothberger-strongly star Rothberger if it satisfies the selection hypothesis Respectively: $S_1(\mathcal{O},\mathcal{O}) - S_1^*(\mathcal{O},\mathcal{O}) - SS_1^*(\mathcal{O},\mathcal{O})$.

A uniform space is a set X together with a structure called a uniformity defined on it, this structure can be defined in three different but equivalent ways:

- (1) covering uniformity, by using a family $\mathbb{C}$ of covers.
- (2) diagonal uniformity, by using a family $\mathbb{U}$ of entourages of the diagonal.
- (3) pseudometric uniformity, by using a family $\mathbb{D}$ of pseudometrics.

1.1 Covering Uniformities

- Let $\mathcal{U}$ and $\mathcal{U}'$ be covers of X. $\mathcal{U}'$ is said to refine $\mathcal{U}$ if every U' in $\mathcal{U}'$ is contained in some U in $\mathcal{U}$.
- Let $\mathcal{U}$ and $\mathcal{V}$ be covers of X. One says that $\mathcal{U}$ star-refines $\mathcal{V}$ (is a star-refinement of $\mathcal{V}$), if for each $U \in \mathcal{U}$, there is some $V \in \mathcal{V}$ such that $St(U,\mathcal{U}) \subset V$.

Definition 1.1.1. *A covering uniformity on a set X is a family $\mathbb{C}$ of covers of X such that:*

(C.1) If $\beta, \gamma \in \mathbb{C}$ then there exists $\alpha \in \mathbb{C}$ that refines both β and γ.

(C.2) If α refines β and $\alpha \in \mathbb{C}$, then $\beta \in \mathbb{C}$.

(C.3) Every element of $\mathbb{C}$ has a star-refinement in $\mathbb{C}$.

The covers from $\mathbb{C}$ are called uniform covers, and the pair $(X, \mathbb{C})$ is a uniform space.

1.2 Diagonal Uniformities

Definition 1.2.1. *A uniformity on a set X is a filter $\mathbb{U}$ of subsets of $X \times X$ with the following properties:*

(U.1) each $U \in \mathbb{U}$ contains the diagonal $\Delta_X = (x, x) : x \in X$ of X.

(U.2) *for each* $U \in \mathbb{U}$ *there is* $V \in \mathbb{U}$ *with* $V \circ V := \{(x, y) \in X \times X : \exists z \in V$ *such that* $(x, z) \in V, (z, y) \in V\} \subset U$.

(U.3) *for each* $U \in \mathbb{U}$, $U^{-1} := \{(x, y) \in X \times X : (y, x) \in U\} \in \mathbb{U}$.

The pair $(X, \mathbb{U})$ is called a uniform space.
Each member of $\mathbb{U}$ is called an entourage of the diagonal.

For any entourage $U \in \mathbb{U}$, a point $x \in X$ and a subset A of X one defines the set $U[x] := \{y \in X : (x, y) \in U\}$ called the U-ball with the center x, and the set : $U[A] := \{y \in X : (x, y) \in U$ for some $x \in A\}$ called the U-neighborhood of A.

1.3 Pseudometric Uniformities

Definition 1.3.1. *(Pseudometric)*
Let X *is a set, athen a function* $d : X \times X \to \mathbb{R}$ *is called a Pseudometric if for all* $x, y, z \in X$ *:*
(PM1) $d(x, y) \geq 0$.
(PM2) $d(x, x) = 0$
(PM3) $d(x, y) = d(y, x)$.
(PM4) $d(x, y) + d(y, z) \geq d(x, z)$.
If d *is a metric, then* (X, d) *is called a pseudometric space.*

Definition 1.3.2. *Let* $\mathbb{D}$ *be a family of Pseudometrics on* X *that satisfies the following properties:*

(P.1) *if* $d_1, d_2 \in \mathbb{D}$ *then* $d_1 \vee d_2 \in \mathbb{D}$, *where* $d_1 \vee d_2 = sup(d_1, d_2)$.

(P.2) *if* d_1 *is a Pseudometric and for every* $\epsilon > 0$ *there are* $d_2 \in \mathbb{D}$ *and* $\delta > 0$ *such that always* $d_2(p, q) < \delta$ *implies* $d_1(p, q) < \epsilon$ *then* $d_1 \in \mathbb{D}$.

A family $\mathbb{D}$ *of Pseudometrics with these properties is called a Pseudometric uniformity and the pair* $(X, \mathbb{D})$ *is a uniform space.*
Pseudometric uniformity $\mathbb{D}$ *provides* $\{B_d(x, r) | d \in \mathbb{D}\}$, *where* $B_d(x, r) = \{y | d(x, y) < r\}$.

2 Selection principles in uniform space

2.1 Definitions

Definition 2.1.1. *A uniform space* $(X, \mathbb{C})$ *is said to be:*

1. *Rothberger-bounded (or uniformly Rothberger) if for each sequence* $(\alpha_n | n \in \mathbb{N})$ *of elements of* $\mathbb{C}$ *there is a sequence* $(U_n | n \in \mathbb{N})$ *such that for each* $n \in \mathbb{N}$, $U_n \in \alpha_n$ *and* $X = \bigcup_{n \in \mathbb{N}} U_n$

2. *Weakly Rothberger-bounded if for each sequence* $(\alpha_n | n \in \mathbb{N})$ *of elements of* $\mathbb{C}$ *there is a sequence* $(U_n | n \in \mathbb{N})$ *such that for each* $n \in \mathbb{N}$, $U_n \in \alpha_n$ *and* $X = \overline{\bigcup_{n \in \mathbb{N}} U_n}$

Definition 2.1.2. *A uniform space $(X, \mathbb{U})$ is called:*

1. *Rothberger-bounded if for every sequence $(U_n|n \in \mathbb{N})$ of entourages of the diagonal of X there exists a sequence $(x_n|n \in \mathbb{N})$ of points in X, such that $X = \bigcup_n U_n[x_n]$.*

2. *Weakly Rothberger-bounded if for every sequence $(U_n|n \in \mathbb{N})$ of entourages of the diagonal of X there exists a sequence $(x_n|n \in \mathbb{N})$ of points in X, such that $X = \overline{\bigcup_n U_n[x_n]}$.*

Definition 2.1.3. *A uniform space $(X, \mathbb{D})$ is called:*

1. *Rothberger-bounded if for every sequence $(r_n|n \in \mathbb{N})$ of positive real numbers there exists a sequence $(x_n|n \in \mathbb{N})$ of elements of X such that $X = \bigcup_n B_d(x_n, r_n)$.*

2. *Weakly Rothberger-bounded if for every sequence $(r_n|n \in \mathbb{N})$ of positive real numbers there exists a sequence $(x_n|n \in \mathbb{N})$ of elements of X such that $X = \overline{\bigcup_n B_d(x_n, r_n)}$.*

2.2 Some properties of Rothberger bounded

Theorem 2.2.1. *For a uniform space $(X, \mathbb{D})$ the following are equivalent:*

- *(a) X is Rothberger-bounded.*
- *(b) for every sequence $(r_n|n \in \mathbb{N})$ of positive real numbers there exists a sequence $x_n \subset X$ such that $U_n = \{B_d(x_n, r_n)|n \in \mathbb{N}\}$ is a sequence of elements of $\alpha_n = \{B_d(x, r_n)|n \in \mathbb{N}\}$, and $\bigcup_n St(U_n, \alpha_n) = X$.*
- *(c) for every sequence $(r_n|n \in \mathbb{N})$ of positive real numbers there exists a sequence $x_n \subset X$ such that $A_n = \{B_d(x_n, r_n)|n \in \mathbb{N}\}$ is a subset of X and $\bigcup_n St(A_n, \alpha_n) = X$.*

Proof. □

Theorem 2.2.2. *For a uniform space $(X, \mathbb{D})$ the following are equivalent:*

- *(a) X is weakly Rothberger-bounded.*
- *(b) for every sequence $(r_n|n \in \mathbb{N})$ of positive real numbers there exists a sequence $x_n \subset X$ such that $U_n = \{B_d(x_n, r_n)|n \in \mathbb{N}\}$ is a sequence of elements of $\alpha_n = \{B_d(x, r_n)|n \in \mathbb{N}\}$, and $\overline{\bigcup_n St(\cup U_n, \alpha_n)} = X$.*
- *(c) for every sequence $(r_n|n \in \mathbb{N})$ of positive real numbers there exists a sequence $x_n \subset X$ such that $A_n = \{B_d(x_n, r_n)|n \in \mathbb{N}\}$ is a subset of X and $\overline{\bigcup_n St(A_n, \alpha_n)} = X$.*

Theorem 2.2.3. *Every subspace of a Rothberger-bounded uniform space $(X, \mathbb{D})$ is Rothberger-bounded.*

Recall that a mapping $f : (X, \mathbb{D}) \Rightarrow (Y, \mathbb{S})$ is called uniformly continuous, if $\forall \epsilon > 0\ \exists \delta > 0$ such that if $\{d(x, y) < \delta, d \in \mathbb{D}\}$ then $\{s(f(x), f(y)) < \epsilon, s \in \mathbb{S}\}$.

Theorem 2.2.4. *If a uniform space Y is a uniformly continuous image of a Rothberger-bounded space X, then Y is also Rothberger-bounded space.*

Theorem 2.2.5. *If a uniform space Y is a uniformly continuous image of a weakly Rothberger-bounded space X, then Y is also weakly Rothberger-bounded space.*

Proof. Let $f : X \to Y$ be a uniformly continuous mapping. Let $s \in \mathbb{S}$, then $\mathcal{C}_n = \{B_s(y, \epsilon_n) | y \in Y\}$ represent a sequence of uniform cover of Y and $\mathcal{C}_n^* = \{f^{-1}(C) : C \in \mathcal{C}_n\}$ is a covering uniformities on a set X where C is an open ball generated by $d \in \mathbb{D}$. Since X is a Rothberger bounded, there exists a sequence $(x_n | n \in \mathbb{N})$ of elements of X and $\mathcal{V}_n = B_d(x_n, r_n)$ is a finite subsets of $\mathcal{C}_n^*$ such that we have $X = \overline{\bigcup_n \mathcal{V}_n}$. The set $y_n = f(x_n)$ is a sequence of elemenys of Y, then $\mathcal{W}_n = B_s(y_n, \epsilon_n)$ is a subsets of $\mathcal{C}_n$. Therefore $Y = f(X) = f(\overline{\bigcup_n \mathcal{V}_n}) = \overline{f(\bigcup_n \mathcal{V}_n)} = \overline{\bigcup_n \mathcal{W}_n} = \overline{\bigcup_n B_s(y_n, \epsilon_n)}$, thus Y is Rothberger-bounded space. □

Theorem 2.2.6. *The product $(X \times Y, \mathbb{D} \times \mathbb{S})$ of two Rothberger uniform spaces $(X, \mathbb{D})$ and $(Y, \mathbb{S})$ is also Rothberger.*

3 Conclusion

Uniform spaces are a good fields for study selection principles, many properties in topological spaces remain valid in uniform cases, and can be demonstrated in different ways. This means that uniform spaces give broad and multiple concepts of selection principles, better than topological spaces.

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