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Theme

Existence results for nonlinear differential equation of fractional order of the Riemann-Liouville type.

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TABLE DES MATIÈRES

شكر وتقدير

الحمد لله السميع العليم ذي العزة والفضل العظيم و الصلاة والسلام على المصطفى الكريم وعلى آله وصحبه أجمعين وبعد

مصداقا لقوله تعالى « **وَإِذْ تأذن ربكم لَئِن شكرتم لأزيدنكُمْ وَلَئِن كَفَرْتُمْ إِنَّ عَذَابِي الشَّدِيدُ** » . سورة إبراهيم الآية 07 .

نشكر الله العلى القدير الذي أنار لنا درب العلم والمعرفة ووفقنا لاتمام هذا العمل المتواضع.

وعملا بقوله صلى الله عليه و سلم « من لم يشكر الناس لم يشكر الله»

نتقدم بجزيل الشكر والامتنان إلى أساتذة قسم الرياضيات بجامعة حمة لخضر بالوادي .

الشكر الخاص إلى الأستاذ المشرف « غندير عون عبد اللطيف »الذي لم يبخل علينا بنصائحه وتوجيهاته جزاه الله خيرا.

إلى العائلة الكريمة وكل من ساهم في إنجاز هذا العمل من قريب أو بعيد ولو بكلمة طيبة.

إلى كل من ساهم في تكويننا طيلة مشوارنا الدراسي.

إهداء

بسم خالقي ومُيسر أموري، وعِصمتُ أمري، لك كُل الحمد والإمتنان !!
تعبنا، تعلمنا، فَفُزنا، وحققنا.
لستُ بمفردي
لم أحمل على عاتقي كثيراً، لقد بادلني العناء، خفف عني الشقاء، ساعدني على الصمود والبقاء"أبي" حفظهُ الله وزادهُ من العمر والصحة والسعادة، كما رافقتني، علمتني ، شجعتني، أنكَسر فتجبرُني، أذبل فتسقِيني، أسقُط فترفعني "أمي" أطال الله في عمرها وجعلها لي السند للأبد، وأختي التي كانت عوناً وظهراً وسندًا ، لن أنجح من غير دعواتهم، إنهم نجاحي قبل النجاح، هُم الحلم وهُم تحقيقه، هُم بداية مشواري وخواتمه، إخوتي ،وصديقاتي وكل أساتذتي وكل من علمني حرفا .وكل من ساندني بكل حب وقت ضعفي أطلب من الله لكُم دوام العافية والتيسير في الأمور.
كما أهدي أيضاً تخرجي ونجاحي إلى عمتي وجدتي لكم كُل الشُكر والإمتنان، وأدامكم الله بصحة وعافية.
والحمدلله حتى يرضى وعند الرضى وبعد الرضى واللهم صلِ على خاتم الأنبياء ومن ختمت به كلامي (محمد صلى عليك الله)

امباركة قاضي

NOTATIONS

We will use the following notations throughout this :

General notations

$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of whole numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$[a, b]$	the closed interval $a \leq x \leq b$
(a, b)	the open interval $a < x < b$
$(.)' = \frac{d(.)}{dt}$	the ordinary derivative with respect to t
a.e.	almost everywhere
i.e.	that is to say.

Spaces

$\Omega \subset \mathbb{R}^n$:	open set in $\mathbb{R}^n$
$\overline{\Omega}$:	closure of Ω
$\partial\Omega$:	boundary of Ω
$L^p(\Omega)$	$= \{$u measurable on Ω and $\int_\Omega |u|^p dx < \infty\}$, $1 \leq p < \infty$
$C(\overline{\Omega})$:	Space of continuous functions on Ω
$C^n(\overline{\Omega})$:	Space of n times continuously differentiable functions on Ω

In this memory, we will specifically use the following spaces :

• $C([0,1],\mathbb{R})$: Banach space of continuous functions $u : [0,1] \to \mathbb{R}$, equipped with the norm

$$\|u\|_\infty = \sup\{|u(t)|,\, t \in [0,1]\}.$$

• $C^1([0,1],\mathbb{R})$: Space of functions $u : [0,1] \to \mathbb{R}$ such that u, u' are continuous functions on [0,1], equipped with the norm

$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}.$$

• $L^1([0,1],\mathbb{R})$: Space of Lebesgue integrable functions $u : [0,1] \to \mathbb{R}$ equipped with the norm

$$\|u\|_1 := \|u\|_{L^1} = \int_0^1 |u(t)|dt.$$

INTRODUCTION

Students of mathematics early encounter the differential operators

$$\frac{d.}{dx}, \frac{d^2.}{dx^2}, \frac{d^3.}{dx^3},$$

etc..., and some doubtless ponder whether it is necessary for the order of differentiation to be an integer. Why should there not be a $\frac{d^{\frac{1}{2}}.}{dx^{\frac{1}{2}}}$ operator, for instance ? Or $\frac{d^{-1}.}{dx^{-1}}$ or even $\frac{d^{\sqrt{2}}.}{dx^{\sqrt{2}}}$? It is to these and related questions that the present work is addressed. It will come as no surprise to one versed in the calculus that the operator $\frac{d^{-1}.}{dx^{-1}}$ is nothing but an indefinite integral in disguise, but fractional orders of differentiation are more mysterious because they have no obvious geometric interpretation along the lines of the customary introduction to derivatives and integrals as slopes and areas. The reader who is prepared to dispense with a pictorial representation, however, will soon find that fractional order derivatives and integrals are just as tangible as those of integer order and that a new dimension in mathematics opens to him when the order α of the operator $\frac{d^\alpha}{dx^\alpha}$ becomes an arbitrary parameter. Nor is this a sterile exercise in pure mathematics many problems in the physical sciences can be expressed and solved succinctly by recourse to the fractional calculus. Our interest in this subject began in 1968 with the realization that the use of half-order derivatives and integrals leads to a formulation of certain electro- chemical problems which is more economical and useful than the classical approach in terms of Fick's laws of diffusion. This discovery stimulated our interest, not only in the applications of the notions of the derivative and integral to arbitrary order, but also in the basic mathematical properties of these fascinating operators. Our collaboration since 1968 has taken us far beyond the original motivation and has produced a wealth of material, some of which we believe to be original. As befits a cooperative effort between a mathematician [J. S.] and a chemist [K. B. O.], our work attempts to expose not only the theory underlying the properties of the generalized operator, but also to illustrate the wide variety of fields to which these ideas may be applied with profit. We do not presume to present an exhaustive survey of the subject, but our aim has been to introduce as many readers as possible to the beauty and utility of this material. Accordingly, we have made a deliberate attempt to keep the mathematical discussions as simple as possible.

The concept of differentiation and integration to noninteger order is by no means new. Interest in this subject was evident almost as soon as the ideas of the classical calculus were known Leibniz (1859) mentions it in a letter to L'Hospital in 1695. The earliest more or less systematic studies seem to have been made in the beginning and middle of the 19th century by Liouville (1832), Riemann (1953), and Holmgren (1864), although Euler (1730), Lagrange (1772), and others made contributions even earlier.

It was Liouville (1832) who expanded functions in series of exponentials and defined the qth derivative of such a series by operating term-by-term as though q were a positive integer. Riemann (1953) proposed a different definition that involved a definite integral and was applicable to power series with noninteger exponents. Evidently it was Griinwald and Krug who first unified the results of Liouville and Riemann. Griinwald (1867), disturbed by the restrictions of Liouville's approach, adopted as his starting point the definition of a derivative as the limit of a difference quotient and arrived at definite-integral formulas for the qth derivative. Krug (1890), working through Cauchy's integral formula for ordinary derivatives, showed that Riemann's definite integral had to be interpreted as having a finite lower limit while Liouville's definition, in which no distinguishable lower limit appeared, corresponded to a lower limit $-\infty$.

Parallel to these theoretical beginnings was a development of the appli- cations of the fractional calculus to various problems. In a sense, the first of these was the discovery by Abel (1823, 1825) in 1823 that the solution of the integral equation for the tautochrone could be accomplished via an integral transform, which, as we shall see, benefits from being written as a semi- derivative. A powerful stimulus to the use of fractional calculus to solve prob- lems was provided by the development by Boole (1844) of symbolic methods for solving linear differential equations with constant coefficients. The essence of Boole's idea is the formal expansion of an arbitrary function $f(D)$ of the differential operator as a power series and the solution of differential equations by formal inversion of such series. Boole's methods have subsequently been made rigorous for certain classes of functions and extended in many directions.

The operational calculus of Heaviside (1892, 1893, 1920), developed by him to solve certain problems of electromagnetic theory, was an important next step in the application of generalized derivatives. Heaviside (1920) introduced fractional differentiation in his investigation of transmission line theory ; this concept has been extended by Gemant (1936) for use in problems of elasticity. While Heaviside seemed to scorn the "wet blankets of rigorists," at least some theorists recognized the merit of his techniques, and attempted to justify them by acceptable mathematical standards.

In the present century notable contributions have been made to both the theory and application of the fractional calculus. Weyl (1917), Hardy (1917), Hardy and Littlewood (1925, 1928, 1932), Kober (1940), and Kuttner (1953) examined some rather special, but natural, properties of differintegrals of functions belonging to Lebesgue and Lipschitz classes. Erdelyi (1939, 1940, 1954) and Osier (1970) have given definitions of differintegrals with respect to arbitrary functions, and Post (1930) used difference quotients to define generalized differentiation for operators $f(D)$, where D denotes differentiation and/is a suitably restricted function. Riesz (1949) has developed a theory of fractional integration for functions of more than one variable. Erdelyi (1964, 1965) has applied the fractional calculus to integral equations and Higgins (1967) has used fractional integral operators to solve differential equations. Other applications include those to rheology (Scott Blair et ah, 1947 ; Sher- mergor, 1966 ; Scott Blair, 1947, 1950 ; Scott Blair and Caffyn, 1949 ; Graham, 1961), to electrochemistry (Belavine, 1964 ; Oldham, 1969 ; Oldham and Spanier, 1970 ; Grenness and Oldham, 1972), to chemical physics (Somorjai and Bishop, 1970), and to general transport problems (Oldham, 1973 ; Oldham and Spanier, 1972). The developments are far too numerous to give an exhaustive survey here, nor is this our purpose. The readers inter- ested in further references to the literature may consult the chronological bibliography which ends this section. Virtually no area of classical analysis has been left untouched by the fractional calculus. Indeed, could one expect less from the natural extension of perhaps the two most basic operations of mathematics differentiation and integration ?. And in the following additional historical perspective to our subject for fractional calculus We sum it up in the following list.

In 1772, Lagrange's contribution in this direction is the law of exponents (indices) for operators of integer order :

$$\frac{d^m}{dx^m}\frac{d^n}{dx^n}y = \frac{d^{m+n}}{dx^{m+n}}y.$$

Later, when the theory of fractional calculus started, it became important to know whether this law held true if m and n were fractions.

Lacroix (1819), develops a formula for fractional differentiation for the nth derivative of v^n by induction. Then, he formally replaces n with the fraction $\frac{1}{2}$, and together with the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, he obtains

$$\frac{d^{\frac{1}{2}}}{dv^{\frac{1}{2}}}v = \frac{2\sqrt{v}}{\sqrt{\pi}}.$$

Abel (1823) was probably the first to give an application of fractional calculus. He used derivatives of arbitrary order to solve the tautochrone (isochrone) problem. The integral he worked with $\int_0^t (t-s)^{-\frac{1}{2}} f(s)ds$ is precisely of the same form that Riemann used to define fractional operations.

The first major study of fractional calculus starts with Liouville (1832). Where considered $\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}e^{2x}$. In his work, some problems in mechanics and geometry are solved by the use of fractional operations.

Liouville (1834) continues work on the complementary function. He argues that if the differential equation $\frac{d^n y}{dx^n} = 0$ has a complementary solution, why shouldn't $\frac{d^u y}{dx^u} = 0$ have a complementary solution when u is arbitrary ?.

Liouville (1835) gives the definition of a fractional derivative, as an infinite series : $\frac{d^u y}{dx^u} = \sum a_n e^{nx} n^u$, where u is any number, integer or fractional, positive or negative, real or imaginary.

Riemann (1847) sought a generalization of a Taylor's series expansion and derived the following definition for fractional integration :

$$\frac{d^{-r}}{dt^{-r}}u(t) = \frac{1}{\Gamma(r)}\int_c^t (t-s)^{r-1}u(s)ds.$$

However, he saw fit to add a complementary function to the above definition. Today, this definition is in common use as a definition for fractional integration but with the complementary function taken to be identically zero, and the lower limit of integration c is usually zero.

Kober (1940) extended some results over a wider range where he deals with Mellin transforms, and also a uniqueness theorem for a solution to the equation

$$g(t) = \int_a^t (t-s)^{\alpha-1}f(s)ds.$$

M. Riesz (1949) used in his memore (The Riemann-Liouville integral and the Cauchy Problem) various aspects of the fractional integral

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds.$$

B. Kuttner (1953) considers in his work (Some Theorems on Fractional Derivatives) relation between the integrals :

$$\frac{d^n}{dt^n}\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f(s)ds$$

and

$$(-1)^n\frac{d^n}{dt^n}\frac{1}{\Gamma(n-\alpha)}\int_t^1 (s-t)^{n-\alpha-1}f(s)ds.$$

In addition to all of this research, in recent years we have noticed a major revolution in research on this topic by studying many types of differential equations with fractional orders, and this research is still ongoing to this day.

Fractional equations are natural generalization of the classical integer-order differential equations. They turn out to be very adequate for modeling dynamics of many processes involving complex systems that can found in science, engineering, aerodynamics, etc. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical models of systems and processes in the fields of physics, chemistry, electrical circuits, biology, and so on, and involves derivatives of fractional order.

Fractional derivatives turn out to be an excellent tool for the description of memory and hereditary properties of various materials and processes. They involve derivatives of fractional order which provide an excellent tool to describe memory and hereditary properties of various materials and processes.

This memory is organized as follows.

The first chapter is devoted to presenting some preliminaries and general notions used throughout. Some basic tools from functional analysis and some classical fixed point theorems : Banach contraction theorem, Schaefer's fixed point theorem, Schauder's fixed point theorem, Leray-Schauder's nonlinear alternative fixed point theorem. We also collect some definitions and basic lemmas from fractional calculus (see [13], [15] for more details).

In **the second chapter**, we present some results on the existence and uniqueness of solutions to fractional differential equations. We studied in the first section Cauchy problem of order $0 < \alpha < 1$. As we included in section two boudary value problem of Dirichlet type, afer then we study a Dirichlet-Newmann type problem in the case $1 < \alpha < 2$. According to using the properties of Green's function and fixed point theorem to prove the result listed in this chapter.

In **the third chapter**, we discuss the fractional differential equations with nonlocal conditions. This chapter presents some existence and uniqueness results of solutions for some problems of Cauchy with nonlocal conditions for differential equations of fractional order. In other hand a boudary value problems with nonlocal conditions of frational orde $1 < \alpha \leq 2$. For the importance of nonlocal conditions in different fields, we let $g(u) = \sum\limits_{i=1}^{i=p} c_i u(t_i)$, where $c_1, c_2, \ldots, c_n$ are given constants with $n \in \mathbb{N}^*$ and $0 < t_1 < t_2 < \ldots < t_n \leq b$.

Each time, and in confirmation of the results of the existence and uniqueness of solutions that have been obtained to fractional differential equationsin our work, we give some examples which give applications to our results.

CHAPITRE 1

PRELIMINARIES

(Main references for this chapter : [1, 5, 6, 11, 13, 15, 17])

basic tools

In the remainder of this chapter we introduce the notations, definitions preliminaries and theorems necessary for this studdy.

1.1 Useful function

1.1.1 the Gamma function :

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt, \quad \alpha > 0$$

Properties 1.1.1.

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$$

Proof : we have

$$\Gamma(\alpha + 1) = \int_0^{+\infty} t^{\alpha} e^{-t} dt.$$

An integration by parts aplied to the definition of gamma function

$$u = t^{\alpha} \longrightarrow u' = \alpha t^{\alpha-1}$$

$$v' = e^{-t} \longrightarrow v = -e^{-t}.$$

$$\Gamma(\alpha+1) = [-t^{\alpha}e^{-t}]_0^{+\infty} + \alpha \int_0^{+\infty} t^{\alpha-1}e^{-t}dt = 0 + \alpha\Gamma(\alpha) = \alpha\Gamma(\alpha).$$

Properties 1.1.2. *Since* $\Gamma(1) = \int_0^{+\infty} e^{-t}dt = 1,$

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)\cdots \times 2 \times 1 \times \Gamma(1) = n!.$$

$$\Gamma(n+1) = n\Gamma(n) = n! \quad n \in \mathbb{N}^*$$

we accept that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\forall n \in \mathbb{N}, \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2n)!}{4^n n!}$$

In fact, we use proof by recurrent :

for $n = 0$ *:* $\quad \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}(0)!}{4^0 0!} = \sqrt{\pi}$

Suppose that $\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2n)!}{4^n n!}$ *and let's prove*

$\Gamma\left(n+1+\frac{1}{2}\right) = \Gamma\left(n+\frac{3}{2}\right) = \frac{\sqrt{\pi}(2n+2)!}{4^{n+1}(n+1)!}$

$$\begin{aligned}
\Gamma\left(n+1+\frac{1}{2}\right) &= \Gamma\left(n+\frac{1}{2}+1\right) \\
&= \left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right) \\
&= \left(n+\frac{1}{2}\right)\frac{\sqrt{\pi}(2n)!}{4^n n!} \\
&= \frac{(2n+1)\sqrt{\pi}(2n)!}{2 \times 4^n n!} \\
&= \frac{(2n+1)(2n+2)\sqrt{\pi}(2n)!}{2(2n+2) \times 4^n n!} \\
&= \frac{\sqrt{\pi}(2n+2)!}{2 \times 2(n+1) \times 4^n (n+1)n!} \\
&= \frac{\sqrt{\pi}(2n+2)!}{4^{n+1}(n+1)!}.
\end{aligned}$$

Definition 1.1. *For $\alpha > 0, \gamma > 0$, the Euler Beta function is defined by*

$$\beta(\alpha,\gamma) = \int_0^1 t^{\alpha-1}(1-t)^{\gamma-1}dt.$$

the Beta function is linked to the Gamma function by the following relationship :

$$\beta(\alpha,\gamma) = \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}$$

Then,we have

$$\beta(\alpha,\gamma) = \beta(\gamma,\alpha)$$

Definition 1.2. *(**convex sets**) : Let X be a real vector space. Asubset C of X is said to the be convex if for all $x, y \in C$ and $\lambda \in [0,1]$, then $\lambda x + (1-\lambda)y \in C$.*

Definition 1.3. *Let X be an ordered Banach space. It is said that a mapping T is increasing (resp.decreasing) if $T(x) \leq T(y)$(resp.$T(x) \geq T(y)$), for all $x, y \in X$ with $x < y$.*

Definition 1.4. *Let X be Banach space with the norme $\| \, . \, \|$ and $T : X \to Y$ a mapping. we say that T is a contraction on X if there exists a real $k \in]0,1[$ such that, for all elements $x, y \in X$ we have $\| Tx - Ty \| \leq k \| x - y \|$.*

1.2 Fractional integral and derivative of Riemann-liouville :

1.2.1 The fractional integral on an interval :

Let h be a continuous function on the intervale $[a, b]$.
We consider the integral

$$I_{a^+}^1 h(t) = \int_a^t h(s)ds.$$

$$I_{a^+}^2 h(t) = \int_a^t ds \int_a^s h(\tau)d\tau.$$

An integration by parts :

$$u = \int_a^s h(\tau)d\tau \longrightarrow u' = h(s)$$

$$v' = ds \longrightarrow v = s$$

$$I_{a^+}^2 h(t) = \left[s \int_a^s h(\tau) d\tau \right]_a^t - \int_a^t s \times h(s) ds$$
$$= t \int_a^t h(\tau) d\tau - a \int_a^a h(\tau) d\tau - \int_a^t s \times h(s) ds = \int_a^t (t-s) h(s) ds$$
$$I_{a^+}^2 h(t) = \int_a^t (t-s) h(s) ds.$$

More generally the n^{th} iteration of the operator I can be written

$$I_{a^+}^n h(t) = \int_a^t ds_1 \int_a^{s_1} ds_2 ... \int_a^{s_{n-1}} h(s_n) ds_n$$
$$= \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} h(t) dt, \quad n \in \mathbb{N}^*$$

Since the generalization of the factorial by the Gamma function : $\Gamma(n) = (n-1)!$, Riemann's account could make sense even when n takes a non-integer value, it was natural to define fractional integration as follows.

Definition 1.5. *If $h \in C([a,b]), \alpha \in \mathbb{R}_+^*$ the integral*

$$I_{a^+}^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds,$$

is called fractional integral of Riemann-Liouville of order α.

Example :

Calculate $I_{0^+}^\alpha t^\mu$, for $\mu > -1$

$$I_{0^+}^\alpha t^\mu = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^\mu ds \quad \text{for} \quad \alpha > 0, \mu > -1.$$

Let's make the change $s = \tau t, ds = t d\tau$, we have

$$I_{0^+}^\alpha t^\mu = \frac{1}{\Gamma(\alpha)} \int_0^1 (t-\tau t)^{\alpha-1} (\tau t)^\mu t d\tau$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} (1-\tau)^{\alpha-1} (\tau)^\mu (t)^\mu t d\tau$$
$$= \frac{1}{\Gamma(\alpha)} t^{\alpha-1+\mu+1} \int_0^1 (1-\tau)^{\alpha-1} \tau^\mu d\tau$$
$$= \frac{1}{\Gamma(\alpha)} t^{\alpha+\mu} B(\mu+1, \alpha)$$
$$= \frac{1}{\Gamma(\alpha)} t^{\alpha+\mu} \frac{\Gamma(\mu+1)\Gamma(\alpha)}{\Gamma(\alpha+\mu+1)}.$$

So, $I_{0^+}^\alpha t^\mu = \dfrac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} t^{\alpha+\mu}, \quad \alpha > 0, \mu > -1$

Properties 1.2.1. *([13]) we have the following properties :*

(P_1) $I^0_{0^+}h(t) = h(t)$.

(P_2) $I^{\alpha}_{0^+}I^{(\beta)}_{0^+} = I^{(\beta)}_{0^+}I^{\alpha}_{0^+} = I^{(\alpha+\beta)}_{0^+}, \quad \alpha > 0, \beta > .0$

(P_3) $I^{\alpha}_{0^+}$ *is a linear operator.*

1.3 Fractional derivative

1.3.1 Riemann-Liouville fractional derivative

Definition 1.6. *For a given function f defined on the interval $[a, b]$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by*

$$^{RL}D^{\alpha}_{a^+}h(t) = \frac{d^n}{dt^n}I^{n-\alpha}_{a^+}h(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{h(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n = [\alpha] + 1$

($[\alpha]$ is the integer part of α and $n - 1 \leq \alpha < n$).

In particular :

- If $\alpha = 0$, so

$$^{RL}D^0_{a^+}h(t) = I^0_{a^+}h(t) = h(t).$$

- If $\alpha = n \in \mathbb{N}$, so

$$^{RL}D^n_{a^+}h(t) = h^{(n)}(t).$$

- If $0 < \alpha < 1$, so $n = 1$ hence

$$^{RL}D^{\alpha}_{a^+}h(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{-\alpha}h(s)ds.$$

The derivative of $f(t) = t^{\mu}$ in the Riemann-Liouville sense :

Let $\alpha > 0, \mu > -1$ and $f(t) = t^{\mu}$, so $^{RL}D^{\alpha}_{0^+}t^{\mu} = \dfrac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}$.

In fact, $\quad ^{RL}D^{\alpha}_{0^+}t^{\mu} = \dfrac{1}{\Gamma(n-\alpha)}\dfrac{d^n}{dt^n}\displaystyle\int_0^t (t-s)^{n-\alpha-1}s^{\mu}ds, \quad n = [\alpha] + 1.$

By changing the variable $s = \tau t$, we will have :

$$\begin{aligned}
{}^{RL}D_{0^+}^{\alpha}t^{\mu} &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}t^{n+\mu-\alpha}\int_0^1(1-\tau)^{n-\alpha-1}\tau^{\mu}d\tau\\
&= \frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(n-\alpha+\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}B(n-\alpha,\mu+1)\\
&= \frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(n-\alpha+\mu+1)}{\Gamma(\mu-\alpha+1)}\times\frac{\Gamma(n-\alpha)\Gamma(\mu+1)}{\Gamma(n-\alpha+\mu+1)}t^{\mu-\alpha}\\
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}.
\end{aligned}$$

In particular, if $\mu = 0$ and $\alpha > 0$, then the Riemann-Liouville fractional derivative of a constant is in general non-zero. In fact, we have

$$\begin{aligned}
{}^{RL}D_{a^+}^{\alpha}C &= {}^{RL}D_{a^+}^{\alpha}Ct^0\\
&= C\frac{\Gamma(1)}{\Gamma(1-\alpha)}t^{-\alpha}\\
&= \frac{C}{\Gamma(1-\alpha)}t^{-\alpha}.
\end{aligned}$$

Lemma 1.1. ***(Composition with the fractional integral) :*** *For $\alpha > 0$ and $h \in L^1[a,b]$, we have*

$$ {}^{RL}D_{a^+}^{\alpha}I_{a^+}^{\alpha}h(t) = h(t).$$

This property means that the fractional derivation operator in the sense of Riemann-Liouville is a left inverse of the fractional integration operator in the sense of Riemann-Liouville of the same order.

Proof : We have for $n = [\alpha] + 1$,

$$\begin{aligned}
{}^{RL}D_{0^+}^{\alpha}I_{a^+}^{\alpha}h(t) &= \frac{d^n}{dt^n}(I_{a^+}^{n-\alpha}I_{a^+}^{\alpha}h(t))\\
&= \frac{d^n}{dt^n}(I_{a^+}^{n}h(t))\\
&= h(t).
\end{aligned}$$

In general, ${}^{RL}D_{a^+}^{\alpha}I_{a^+}^{\gamma}h(t) = I_{a^+}^{\gamma-\alpha}h(t), \quad \gamma > \alpha > 0, h \in L^1[a,b]$.

Lemma 1.2. *([13]) Let $\alpha > 0, h \in L^1[0,b]$,then*

$I_{0^+}^{\alpha}\,{}^{RL}D^{\alpha}h(t) = h(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + ... + c_nt^{\alpha-n}$,

for some $c_i \in \mathbb{R}, i = 1, 2, ..., n$ and $n = [\alpha] + 1$.

Corollary 1.1. *Let $\alpha > 0$ and $h \in L^1[a,b]$,then the equation ${}^{RL}D_{0^+}^{\alpha}h(t) = 0$ has the solutions $h(t) = c_1t^{\alpha-1} + c_2t^{\alpha-2} + ... + c_nt^{\alpha-n}$,*

such that $n = [\alpha] + 1$ and $c_1, c_2, ..., c_n$ are constant.

- *If* $0 < \alpha < 1$, $D^{\alpha}_{0^+}h(t) = 0 \Longrightarrow h(t) = ct^{\alpha-1}, \quad c \in \mathbb{R}$.

- *If* $1 < \alpha < 2$, $D^{\alpha}_{0^+}h(t) = 0 \Longrightarrow h(t) = ct^{\alpha-1} + dt^{\alpha-2}, \quad c, d \in \mathbb{R}$.

1.3.2 Caputo fractional derivative

The definition of the Riemann-Liouville type fractional derivation played an important role in the development of the theory of fractional derivatives and integrals because of their applications in pure mathematics (solution of integer order differential equations,definition of new classes function,summation of series,etc.).However,modern technology requires some revision of the well-known pure mathematical approach. Much work has appeared, especially on the theory of viscoelasticity and solid mechanics, where fractional derivatives are used for a good description of material properties. Mathematical modeling based on rheological models naturally leads to differential equations of fractional order, and to the need to formulate the initial conditions of such equations.The applied problems require definitions of fractional derivatives authorizing the use of physically interpretable initial conditions, which contain f(a) ;f(b),etc...Despite the fact that initial value problems with such initial conditions can be solved mathematically , the solution of this problem was proposed by M.Caputo(in the sixties) in his definition which he adapted with Mainardi in the structure of the theory of viscoelastics :Therefore we introduces a fractional derivative which is more restrictive than that of Riemann-Liouvile.
Let [a,b] be a finite interval of $\mathbb{R}$.

Definition 1.7. *The fractional derivative of order $\alpha > 0$ of Caputo of a function h defined on $[a, b]$ is given by*

$$^{C}D^{\alpha}_{a^+}h(t) = ^{RL}D^{\alpha}_{a^+}\left(h(t) - \sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{k!}(t-a)^k\right),$$

such that $n = [\alpha] + 1$.

Remark 1.1. - *If $\alpha = 0$, then*

$$^{C}D^{0}_{a^+}h(t) = h(t) - h(a).$$

- *If $0 < \alpha < 1$, then*

$$^{C}D^{\alpha}_{a^+}h(t) = ^{RL}D^{\alpha}_{a^+}\left(h(t) - h(a)\right).$$

- $^{C}D^{\alpha}_{a^+}h(t) =^{RL} D^{\alpha}_{a^+}h(t) - \sum_{k=0}^{n-1} \frac{h^{(k)}(a)}{\Gamma(n-\alpha+1)}(t-a)^{k-\alpha}, \quad n = [\alpha]+1.$

According to the previous relationship :

$^{C}D^{\alpha}_{a^+}h(t) =^{RL} D^{\alpha}_{a^+}h(t) \Leftrightarrow h(a) = h'(a) = ... = h^{(n-1)}(a) = 0.$

Theorem 1.1. *Let* $\alpha > 0$ *and* $n = [\alpha]+1$*,If* $f \in AC^n[a,b]$*,then :*

(i) $If \alpha \notin \mathbb{N} \quad {}^{C}D^{\alpha}_{a^+}h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$

$$= I^{n-\alpha}_{a^+} D^{\alpha}_{a^+}h(t).$$

(ii) If $\alpha = n \in \mathbb{N}, \quad {}^{C}D^{\alpha}_{a^+}h(t) = h^n(t).$

Proof : According to the definition, we have

$$\begin{aligned} {}^{C}D^{\alpha}_{a^+}h(t) &=^{RL} D^{\alpha}_{a^+}\left(h(t) - \sum_{k=0}^{n-1} \frac{h^k(a)}{k!}(t-a)^k\right) \\ &= \frac{d^n}{dt^n} I^{n-\alpha}_{a^+}\left(h(t) - \sum_{k=0}^{n-1} \frac{h^k(a)}{k!}(t-a)^k\right) \\ &= \frac{d^n}{dt^n}\int_a^t \frac{1}{\Gamma(n-\alpha)}(t-s)^{n-\alpha-1}\left(h(s) - \sum_{k=0}^{n-1} \frac{h^k(a)}{k!}(s-a)^k\right) ds. \end{aligned}$$

Integrating by parts, we will have

$$u = h(s) - \sum_{k=0}^{n-1} \frac{h^k(a)}{k!}(s-a)^k \longrightarrow u' = \frac{d}{ds}\left(h(s) - \sum_{k=0}^{n-1} \frac{h^k(a)}{k!}(s-a)^k\right)$$

$$v' = \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} \longrightarrow v = -\frac{(t-s)^{n-\alpha}}{(n-\alpha)\Gamma(n-\alpha)} = -\frac{(t-s)^{n-\alpha}}{\Gamma(n-\alpha+1)}$$

$$\int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}\left(h(s) - \sum_{k=0}^{n-1} \frac{h^k(a)}{k!}(s-a)^k\right) ds = -\left[\frac{(t-s)^{n-\alpha}}{\Gamma(n-\alpha+1)}\left(h(s) - \sum_{k=0}^{n-1} \frac{h^k(a)}{k!}(s-a)^k\right)\right]_a^t$$
$$+\frac{1}{\Gamma(n-\alpha+1)}\int_a^t (t-s)^{n-\alpha}\frac{d}{ds}\left(h(s) - \sum_{k=0}^{n-1} \frac{h^k(a)}{k!}(s-a)^k\right) ds$$

$$= I_{a^+}^{n-\alpha+1} \frac{d}{dt} \left(h(t) - \sum_{k=0}^{n-1} \frac{h^k(a)}{k!}(t-a)^k \right)$$
$$= I_{a^+}^{n-\alpha+n} \frac{d^n}{dt^n} \left(h(t) - \sum_{k=0}^{n-1} \frac{h^k(a)}{k!}(t-a)^k \right)$$
$$= I_{a^+}^{n} I_{a^+}^{n-\alpha} \frac{d^n}{dt^n} \left(h(t) - \sum_{k=0}^{n-1} \frac{h^k(a)}{k!}(t-a)^k \right)$$
$$= I_{a^+}^{n} I_{a^+}^{n-\alpha} \frac{d^n}{dt^n} h(t),$$

because $\frac{d^n}{dt^n} \left(\sum_{k=0}^{n-1} \frac{h^k(a)}{k!}(t-a)^k \right) = 0$

Ainsi,

$$\begin{aligned} {}^C D_{a^+}^{\alpha} h(t) &= \frac{d^n}{dt^n} I_{a^+}^{n} I_{a^+}^{n-\alpha} \frac{d^n}{dt^n} h(t) \\ &= I_{a^+}^{n-\alpha} \frac{d^n}{dt^n} h(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h(s)^n ds. \end{aligned}$$

hence the result.

Example 1.1. *Let* $0 < \alpha < 1$ *and* $h(t) = t^{\mu}, \mu > -1$ *then* ${}^C D_{0^+}^{\alpha} h(t) = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}$

In fact,

$$\begin{aligned} {}^C D_{0^+}^{\alpha} t^{\mu} &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} (s^{\mu})' ds. \\ &= \frac{\mu}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} s^{\mu-1} ds. \end{aligned}$$

let's make the change $s = \tau t, ds = t d\tau$, *we have*

$$\begin{aligned} {}^C D_{0^+}^{\alpha} t^{\mu} &= \frac{\mu}{\Gamma(1-\alpha)} \int_0^1 t^{-\alpha} (1-\tau)^{-\alpha} \tau^{\mu-1} t^{\mu-1} t d\tau \\ &= \frac{\mu}{\Gamma(1-\alpha)} t^{-\alpha+\mu} \int_0^1 (1-\tau)^{-\alpha} \tau^{\mu-1} d\tau \\ &= \frac{\mu}{\Gamma(1-\alpha)} t^{\mu-\alpha} B(1-\alpha, \mu) \\ &= \frac{\mu}{\Gamma(1-\alpha)} t^{\mu-\alpha} \frac{\Gamma(1-\alpha)\Gamma(\mu)}{\Gamma(1-\alpha+\mu)} \\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}. \end{aligned}$$

In particular, the caputo derivative of a constant is zero :

${}^C D_{0^+}^{\alpha} C = 0, \quad C$ *a constant.*

$$\left({}^C D_{a^+}^{\alpha} C = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} c^n ds = 0 \right).$$

Lemma 1.3. *([17]) Let $\beta > \alpha > 0$ and $h \in L^1[a,b]$, then*

$$^{C}D^{\alpha}_{a^+} I^{\beta}_{a^+} h(t) = I^{\beta-\alpha}_{a^+} h(t).$$

- *If* $\beta = \alpha$

$$^{C}D^{\alpha}_{a^+} I^{\alpha}_{a^+} h(t) = h(t).$$

Lemma 1.4. *([17]) Let $\alpha > 0, h \in L^1[a,b]$, then*

$I^{\alpha}_{0^+} {}^{C}D^{\alpha}_{0^+} h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, \quad n = [\alpha] + 1.$

Corollary 1.2. *Let $\alpha > 0$ and $h \in L^1[a,b]$, then the equation $^{C}D^{\alpha}_{0^+} h(t) = 0$ has the solutions* $h(t) = c_0 + c_1 t + ... + c_{n-1} t^{n-1}$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, \quad n = [\alpha] + 1$

- *If* $0 < \alpha < 1,\ {}^{c}D^{\alpha}_{0^+} h(t) \Rightarrow h(t) = c, \quad c \in \mathbb{R}$

- *If* $1 < \alpha < 2, {}^{c}\ D^{\alpha}_{0^+} h(t) \Rightarrow h(t) = c + dt, \quad c, d \in \mathbb{R}$

1.3.3 Comparison between the fractional derivative in the sense of Caputo and that of Riemann-liouville

- The main advantage of the Caputo approach is that the initial conditions of fractional differential equations with Caputo derivatives accept the same form as for integer differential equations, i.e., contain the limiting values of the derivatives of integer order of unknown functions in lower bound $X = a$

- Another difference between the definition of Riemann-liouville and that of Caputo is that derivative of a constant is zero by Caputo on the other hand by Riemann-liouville it is $\dfrac{c}{\Gamma(1-\alpha)} t^{-\alpha}$.

1.4 Some fixed point theorems

In this section, we will cite in particular four fixed point theorems(Banach contraction, nonlinear alternative of leray-schauder type schaefer, chauder type)used in the resolution of differential equations by the approach of fixed point.

1.4.1 Banach fixed point theorem

Theorem 1.2. *([1]) Let X be a Banach space and $T : X \to X$ an mapping. If T is a contraction on X then T admits a unique fixed point in X.*

1.4.2 Schaefer's fixed point theorem

Theorem 1.3. *([1]) Let X be a Banach space and U be convex set such that $0 \in U$. Let T be an operator defined in X such that $T : U \to U$ is completely continuous. If the set $\Omega = \{x \in U : \lambda Tx = x,$ for some $\lambda \in]0, 1[\}$ is bounded, then T has at least one fixed point.*

1.4.3 Shauder's fixed point theorem

Theorem 1.4. *([1]) Let C be a convex, closed, bounded, non-empty subset of a Banach space X and $T : C \to C$ it's a compact mapping. Then T admits at least one fixed point.*

1.4.4 Leray-Schauder nonlinear alternative theorem

Theorem 1.5. *([1]) Let C be a convex subset of a Banach space and Ω an open subset of C with $0 \in \Omega$. Then every completely continuous map $N : \overline{\Omega} \to C$ satisfies at least one of the following two properties :*
$(A1)$ N has a fixed point in $\overline{\Omega}$, or
$(A2)$ There is an $x \in \partial\Omega$ and $\lambda \in (0,\ 1)$ with $x = \lambda Nx$.

Remark 1.2. *Hypothesis $(A2)$ indicates that the set $S = \{x \in \Omega : x = \lambda \mathrm{T}x; \lambda \in [0, 1]\}$ is not bounded. So to apply the nonlinear alternative of Leray-Schauder, one can just show that T is completely continuous by checking the following a priori estimate :*

$$\exists R > 0, \forall x \in E, \forall \lambda \in [0, 1] : (x = \lambda \mathrm{T}x \Longrightarrow \|x\| < R)$$

to assert that T admits at least one fixed point in $B(0,\ R)$.

1.4.5 The Green Function

Let $p,\ q,\ f\in C([a,b])$ where $p\in C^1([a,b])$, $(a<b)$ and $(\alpha_i,\beta_i)\in\mathbb{R}^2$ such that $\forall\, i=1,2,\ |\alpha_1|+|\alpha_2|,\ |\beta_1|+|\beta_2|\neq 0$. Consider the ordinary differential equations :

$$(H)\quad (py')'+qy=0$$
$$(NH)\quad (py')'+qy=f$$

and the associated edge conditions :

$$(CB)_h\quad \begin{cases}\alpha_1 y(a)+\alpha_2 y'(a)=0\\ \beta_1 y(b)+\beta_2 y'(b)=0\end{cases}$$

$$(CB)_{nh}\begin{cases}\alpha_1 y(a)+\alpha_2 y'(a)=\gamma\\ \beta_1 y(b)+\beta_2 y'(b)=\delta.\end{cases}$$

Definition 1.8. *We call the Green's function associated with the homogeneous problem $(H)-(CB)_h$ a function $G\colon\ [a,b]\times[a,b]\to\mathbb{R}$ satisfied The following properties :*

(a) *G is continuous on $[a,b]\times[a,b]$;*

(b) *G is symmetric : $G(x,y)=G(y,x),\ \forall\,(x,y)\in[a,b]^2$;*

(c) *$\dfrac{\partial G}{\partial x}(x,y)$ is continuous for all $x\neq y$;*

(d) *$\dfrac{\partial G}{\partial x}(y^+,y)-\dfrac{\partial G}{\partial x}(y^-,y)=\dfrac{1}{p(y)}$ for all $y\in[a,b]$;*

(e) *Partial Function $x\mapsto G(x,y)$ is the solution of the equation (H) for all $x\neq y$;*

(f) *Partial Function $x\mapsto G(x,y)$ satisfied the conditions $(CB)_h$ for all $y\in[a,b]$.*

Theorem 1.6. *(**(Existence and uniqueness of the Green function)**) Suppose that the homogeneous problem $(H)-(CB)_h$ does not admit of a non-trivial solution. Then there exists a (and only one) function G that does not depend on f, and is called Green's function, such that, for any function f, the solution y of the non-homogeneous problem $(NH)-(CB)_h$ is uniquely written as ::*

$$y(x)=\int_a^b G(x,s)f(s)\,ds.$$

Proof :

(i) Existence of the function G : Let ϕ_1 et ϕ_2 the respective solutions of the problems with initial conditions.

$$(H) + \begin{cases} \phi_1(a) = \alpha_2 \\ \phi_1'(a) = -\alpha_1, \end{cases} \quad \text{et} \quad (H) + \begin{cases} \phi_2(b) = \beta_2 \\ \phi_2'(b) = -\beta_1. \end{cases}$$

Then, ϕ_1, $\varphi_2 \not\equiv 0$ are linearly independent because otherwise ϕ_1 (and also ϕ_2)would be the solution to the problem $(P_0) : (H) + (CB)_h$ a contradiction hypotheses . So be it $W = \phi_1\phi_2' - \phi_1'\phi_2 \neq 0$ their Vronskian and G the Green function defined by

$$G(x,s) = \begin{cases} \frac{\phi_1(s)\phi_2(x)}{p(s)W(s)}, & a \leqslant s \leqslant x, \\ \frac{\phi_1(x)\phi_2(s)}{p(s)W(s)}, & x \leqslant s \leqslant b. \end{cases}$$

Because, for any solution y of the non-homogeneous problem $(NH) - (CB)_h$ is written in the form :

$$\begin{aligned} y(x) &= \int_a^x \frac{\phi_1(s)\phi_2(x)}{p(s)W(s)} f(s)\,ds + \int_x^b \frac{\phi_1(x)\phi_2(s)}{p(s)W(s)} f(s)\,ds \\ &= \int_a^b G(x,s)f(s)\,ds. \end{aligned}$$

Note that the product pW is constant because $p(x)W(x) = p(s)W(s) = p(a)W(a) = p(b)W(b) \neq 0$, $\forall\, x,s \in [a,b]$. Indeed, according to Lagrange's formula, we have

$$\begin{aligned} \varphi_2(p\varphi_1')' - \varphi_1(p\varphi_1')' = 0 &\Leftrightarrow (p\varphi_2\varphi_2')' - (p\varphi_1\varphi_2')' = 0 \\ &\Leftrightarrow [p(\varphi_2\varphi_1' - \varphi_1\varphi_2')]' = 0 \\ &\Leftrightarrow pW = cte. \end{aligned}$$

Finally, G satisfies the hypotheses of the Green function.

(ii) Uniqueness of the function G Let G, H two Green functions ; then $\int_a^b [G(x,s) - H(x,s)]f(s)ds = 0$, $\forall\, x \in [a,b]$ et $\forall\, f\colon [a,b] \to \mathbb{R}$ continuous function. Four x fixed, let's put down $f(s) = G(x,s) - H(x,s)$; We have $\int_a^b [G(x,s) - H(x,s)]^2 ds = 0$, $\forall\, x \in [a,b]$. As G and H are continuous , $G \equiv H$, that is $(G(x,.) = H(x,.), \forall\, x \in [a,b])$.

(iii) Existence and uniqueness of a solution : The function F defined by,

$$F(x) = \int_a^b G(x,s)f(s)ds = \frac{\phi_2(x)}{pW}\int_a^x \phi_1(s)f(s)ds + \frac{\phi_1(x)}{pW}\int_x^b \phi_2(s)f(s)ds$$

is the solution to the problem $(NH)+(CB)_h$.Indeed ,

$$F'(x)=\int_a^b \frac{\partial G}{\partial x}(x,s)f(s)\,ds$$

$$\text{and}\quad F''(x)=\int_a^b \frac{\partial^2 G}{\partial x^2}(x,s)f(s)\,ds+f(x)[\frac{\partial G}{\partial x}(x,x^-)-\frac{\partial G}{\partial x}(x,x^+)].$$

Or,

$$\begin{aligned}\frac{\partial G}{\partial x}(x^+,x) &= \frac{\partial G}{\partial x}(x,x^-)\\ \frac{\partial G}{\partial x}(x^-,x) &= \frac{\partial G}{\partial x}(x,x^+)\\ \frac{\partial G}{\partial x}(x^+,x) &- \frac{\partial G}{\partial x}(x^-,x)=\frac{1}{p}.\end{aligned}$$

From this we deduce the expression :

$$F''(x)=\int_a^b \frac{\partial^2 G}{\partial x^2}(x,s)f(s)\,ds+\frac{f(x)}{p(x)}$$

as well as :

$$(pF')'=\int_a^b \left(p\frac{\partial G}{\partial x}\right)'(x,s)f(s)\,ds+f(x).$$

As a result,

$$\begin{aligned}(pF')'+qF &= \int_a^b \left[\left(p\frac{\partial G}{\partial x}\right)'+qG\right]f(s)\,ds+f(x)\\ (\text{by definition of } G) &= -\int_a^b \varphi_0(x)\varphi_0(s)f(s)\,ds+f(x)\\ (\text{by definition of } \varphi_0) &= f(x).\end{aligned}$$

The uniqueness of the solution y results from the hypothesis on the homogeneous problem as well as from Fredholm's Alternative.

Example 1.2. *Consider the two-point problem posed over an interval* $[a,b]$.

$$\begin{cases} y''=f(x), & a<x<b\\ y(a)=0, & y(b)=0.\end{cases}\tag{1.1}$$

Let's build the functions φ_1 *et* φ_2 *solution of Cauchy problems :*

$$\begin{cases}\varphi_1''=0\\ \varphi_1(a)=0\\ \varphi_1'(a)=-1\end{cases}\qquad\begin{cases}\varphi_2''=0\\ \varphi_2(b)=0\\ \varphi_2'(b)=-1.\end{cases}$$

Then, $\varphi_1(x) = (a - x)$, $\varphi_2(x) = (b - x)$ *et* $W(\varphi_1, \varphi_2) = b - a$.

Hence the function of Green :

$$G(x,s) = \begin{cases} \frac{(x-b)(s-a)}{(b-a)}, & si\ a \leqslant s \leqslant x; \\ \frac{(x-a)(s-b)}{(b-a)}, & si\ x \leqslant s \leqslant b. \end{cases}$$

The unique solution of the problem (1.1) *is therefore given by*

$$y(x) = \int_a^b G(x,s)f(s)ds = \frac{x-b}{b-a}\int_a^x (s-a)f(s)\,ds + \frac{x-a}{b-a}\int_x^b (s-b)f(s)\,ds.$$

1.5 Lebesgue dominated convergence theorem

Theorem 1.7. *([6]) Let* (f_n) *be a sequence of functions in* $L^1(\Omega)$ *that satisfies*
(i) $f_n(t) \to f(t)$ *for almost every* $t \in \Omega$,
(ii) there is a function $g \in L^1(\Omega)$ *such that* $|f_n(t)| \leq g(t)$ *for all* $n \in \mathbb{N}$ *and for almost every* $t \in \Omega$.
Then $f \in L^1(\Omega)$ *and* $\|f_n - f\|_1 \to 0$.

1.6 Ascoli-Arzela theorem

Theorem 1.8. *([6]) Let* E *a compact metric space,* F *a complete metric space. A subset* $\mathfrak{F} \subset C(E, F)$ *is relatively compact, is relatively compact, if and only if it satisfies the following two conditions :*
1.For all $t \in E$, *the set*

$$\mathfrak{F}(t) = \{T(t);\, T(.) \in \mathfrak{F}\}$$

is relatively compact in$F$.
2. $\mathfrak{F}$ *is equicontinuous i.e :*
For all $\epsilon > 0$, *there exists* $\delta > 0$, *such that for any,* $T(.) \in \mathfrak{F}$ *and for any,* $t, s \in E$, *with* $d_E(t,\, s) \leqslant \delta$ *we have* $d_F(T(t),\, T(s)) \leqslant \epsilon$.

Theorem 1.9. *(A special case of the Ascoli-Arzela theorem) Let* $\mathfrak{F} \subset C(J, \mathbb{R}^n)$ *and* J *a bounded interval of* $\mathbb{R}$. $\mathfrak{F}$ *is relatively compact if :*
1. $\mathfrak{F}$ *is uniformly bounded i.e :*

$$\exists M > 0 \text{ such that } |T(t)| \leqslant M, \text{ for any } T(.) \in \mathfrak{F} \text{ et for any } t \in J.$$

2. $\mathfrak{F}$ *is equicontinuous, i.e :*

$\forall \epsilon > 0, \exists \delta > 0,$ *such that for any,* $T(.) \in \mathfrak{F}$ *and for any,* $t, s \in J,$ *with*$|t - s| \leqslant \delta$ *we have* $\|T(t) - T(s)\| \leqslant \epsilon$.

1.7 Compact application

Let X and Y two spaces of Banach and $\Omega \subset X$ an open.

Definition 1.9. *A continuous map* $f : \Omega \longrightarrow Y$ *is said to be compact if* $f(\overline{\Omega})$ *is compact. It is said to be completely continuous if the image of any bounded is relatively compact.*

Definition 1.10. *(compact operator) A linear operator* T *is said to be compact if it transforms any bounded part of* X *in a relatively compact part of* Y*. In other words,* T *is compact if and only if for any suite* $(x_n)_n$ *bounded in* X *the suite* $(Tx_n)_n$ *admits a convergent subsequence.*

Remark 1.3. **(a)** *Any compact application is completely continuous ; The reverse is true if* Ω *is bounded.*

(b) *Any compact linear map is continuous ; The converse is true if this map is of finite rank. (Rank is the dimension of the image space).*

Proposition 1.7.1. *([6]) A map* $f : X \longrightarrow Y$ *is compact if and only if for any suite* $(x_n)_{n\in\mathbb{N}}$ *of* X*, we can extract a subsequence* $(x_{n_k})_{k\in\mathbb{N}}$ *such as the suite* $(f(x_{n_k}))_{k\in\mathbb{N}}$ *converges in* Y*.*

Definition 1.11. *(* ***convex sets*** *) Let* X *be a real vector space. A subset* C *of* X *is said to the be convex if for all* $x, y \in C$ *and* $\lambda \in [0, 1]$*, then* $\lambda x + (1 - \lambda)y \in C$*.*

Definition 1.12. *Let* X *be an ordered Banach space. It is said that a mapping* T *is increasing (resp.decreasing) if* $T(x) \leq T(y)$ *(resp.*$T(x) \geq T(y)$*), for all* $x, y \in X$ *with* $x < y$*.*

Definition 1.13. *Let* X *be Banach space with the norme* $\| \, . \, \|$ *and* $T : X \to Y$ *a mapping. we say that* T *is a contraction on* X *if there exists a real* $k \in]0, 1[$ *such that, for all elements* $x, y \in X$ *we have* $\| Tx - Ty \| \leq k \| x - y \|$*.*

CHAPITRE 2

SOLVABILITY FOR SOME DIFFIRENTUAL EQUATIONS OF FRACTIONAL ORDER

(Main references for this chapter : [2, 7, 8, 9, 16, 18, 19])

Introduction

We present in this chapter some results on the existence and uniqueness of solutions to fractional differential equations. We studied in the first section Cauchy problem of order $0 < \alpha < 1$. In the second section we discuss the boudary value problem of Dirichlet type, then we study a Dirichlet-Newmann type problem in the case $1 < \alpha < 2$. On the construction of Green's function associated with the proposed problem, and finally on the call to the appropriate fixed point theorem to prove the result listed in this chapter.

Fractional differential equations of order $\alpha \in]0;1[$

2.1 Cauchy problem of fractional order $\alpha \in]0;1[$

Consider the following problem

$$ {}^{RL}D^{\alpha}_{0^+}u(t) = f(t,u(t)) \quad , \quad t \in [0,b], (b>0) \tag{2.1} $$

$$ \lim_{t \overset{>}{\to} 0} t^{1-\alpha}u(t) = u_0 \tag{2.2} $$

such that $0<\alpha<1$ and $f:[0,b]\times\mathbb{R}\to\mathbb{R}$ a given fonction, $u_0\in\mathbb{R}$.
Suppose space $\widetilde{C}([0,b],\mathbb{R}) = \left\{ u \in C((0,b],\mathbb{R}), \lim_{t \overset{>}{\to} 0} t^{1-\alpha}u(t) \text{ exists} \right\}$ it's a Banach space with the norm :

$$ \| u \|_{\widetilde{C}} = \sup_{t\in[0,b]} t^{1-\alpha}|u(t)|. $$

We say that $u \in \widetilde{C}([0,b],\mathbb{R})$ is said solution (2.1)-(2.2) if u satisfied (2.1) and (2.2).

Lemma 1.
Let $h:[0,b]\to\mathbb{R}$ a continuous fonction.
Then the linear problem

$$ \begin{cases} {}^{RL}D^{\alpha}_{0^+}u(t) = h(t), & t\in[0,b] \\ \lim_{t \overset{>}{\to} 0} t^{1-\alpha}u(t) = u_0 \end{cases} $$

admits only one solution given by

$$ u(t) = u_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds. $$

Proof 1.

By lemma 1.2 and from ${}^{RL}D^{\alpha}_{0^+}u(t) = h(t)$, we have

$$ I^{\alpha}_{0^+}\left({}^{RL}D^{\alpha}_{0^+}u(t)\right) = I^{\alpha}_{0^+}h(t) $$

it's who gives,

$$ u(t) = ct^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds $$

and by the condition (2.2) we have $c = u_0$.
So

$$ u(t) = u_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds $$

Remark 1.

$u \in \widetilde{C}([0,b],\mathbb{R})$ is solution of problem (2.1)-(2.2) if and only if u is in $\widetilde{C}([0,b],\mathbb{R})$ and satisfies the integral equation :

$$u(t) = u_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s))ds \tag{2.3}$$

Our first result on the problem (2.1)-(2.2) is based on the Banach contraction principale.

We introduce the following conditions :

(H1) The fonction $f : [0,b] \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) $\exists k > 0$ such that

$$\begin{aligned} |t^{\alpha-1} f(t,u_1(t)) - t^{\alpha-1} f(t,u_2(t))| &\leq k|u_1(t) - u_2(t)| \\ &\forall t \in (0,b], \quad \forall u_1, u_2 \in C([0,b],\mathbb{R}). \end{aligned}$$

Theorem 1.

Suppose (H1), (H2) are satisfied and if $0 < k < \dfrac{\Gamma(\alpha+1)}{b}$, then the problem (2.1)-(2.2) admits only one solution.

Proof 2.

Considering the operator T

$$\begin{aligned} \widetilde{C}([0,b],\mathbb{R}) &\ \widetilde{C}([0,b],\mathbb{R}), \\ T : (Tu) &= u_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s))ds. \end{aligned}$$

The problem (2.1)-(2.2) has a solution u if and only if u solves the operator equation $u = Tu$.

We will prove the existence and uniqueness of a fixed point of T. For this we verify that the operator T satisfies the Theorem (Banach). To do this we show that T is contradiction with respect to norm $\| \, . \, \|_{\widetilde{C}}$.

Let $u_1, u_2 \in \widetilde{C}([0,b],\mathbb{R})$ *and* $t \in [0,b]$,

$$\begin{aligned}
|(Tu_1)(t)+(Tu_2)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u_1(s))-f(s,u_2(s)|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{1-\alpha}s^{\alpha-1}|f(s,u_1(s))-f(s,u_2(s)|ds \\
&\leq \frac{k}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{1-\alpha}|u_1(s)-u_2(s)|ds \\
\| Tu_1 - Tu_2 \|_{\widetilde{C}} &\leq \frac{bk}{\Gamma(\alpha+1)} \| u_1 - u_2 \|_{\widetilde{C}} \\
&\leq \frac{k}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \| u_1 - u_2 \|_{\widetilde{C}} ds \\
&\leq \frac{k}{\Gamma(\alpha)} \| u_1 - u_2 \|_{\widetilde{C}} \int_0^t (t-s)^{\alpha-1}ds \\
&\leq \frac{kb^\alpha}{\alpha\Gamma(\alpha)} \| u_1 - u_2 \|_{\widetilde{C}} \\
&= \frac{kb^\alpha}{\Gamma(\alpha+1)} \| u_1 - u_2 \|_{\widetilde{C}}
\end{aligned}$$

this is implies, $t^{1-\alpha}|(Tu_1)(t)-(Tu_2)(t)| \leq \dfrac{bk}{\Gamma(\alpha+1)} \| u_1 - u_2 \|_{\widetilde{C}}$
As $k < \dfrac{\Gamma(\alpha+1)}{b}$ *i.e.* $0 < \dfrac{kb}{\Gamma(\alpha+1)} < 1$ *then* T *is contraction.*
According to Theorem (Banach), we conclude that the problem (2.1)-(2.2) has only one solution.

Exemple 1.

Consider the following problem :

$$^{RL}D_{0^+}^{\frac{1}{2}}u(t) = \frac{t^{\frac{1}{2}}e^t|u(t)|}{(2e^t+2)(1+|u(t)|)}, t \in [0,1] \qquad \text{(E}\times\text{1)}$$

$$\lim_{t \overset{>}{\to} 0} t^{\frac{1}{2}}u(t) = 0 \qquad \text{(E}\times\text{2)}$$

Let $f(t, u(t)) = \dfrac{t^{\frac{1}{2}} e^t |u(t)|}{(2e^t + 2)(1 + |u(t)|)}$.
So we have

(H1) $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(H2) Let $u_1, u_2 \in \widetilde{C}([0, 1], \mathbb{R}), t \in [0, 1]$,

$$\begin{aligned} |t^{-\frac{1}{2}} f(t, u_1(t)) - t^{-\frac{1}{2}} f(t, u_2(t))| &= \frac{t^{-\frac{1}{2}} t^{\frac{1}{2}} e^t}{2e^t + 2} \left| \frac{|u_1(t)|}{1 + |u_1(t)|} - \frac{|u_2(t)|}{1 + |u_2(t)|} \right| \\ &= \frac{e^t}{2(e^t + 1)} \frac{||u_1(t)| - |u_2(t)||}{(1 + |u_1(t)|)(1 + |u_2(t)|)} \\ &\leq \frac{1}{2} |u_1(t) - u_2(t)|. \end{aligned}$$

In this case, we choose $k = \frac{1}{2}$ *as achieved*

$$0 < k < \frac{\Gamma(\alpha + 1)}{b} \qquad \left(0 < \frac{1}{2} < \frac{\Gamma\left(\frac{3}{2}\right)}{1} \text{ with } \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \right).$$

By Theorem 1, we deduce that the problem has only one solution.

Our second result id based on Schawder's fixed point theorem.
Insufficient the following hypotheses :

$$(H3) \qquad \exists \phi \in C([a, b], \mathbb{R}_+) \text{ and } \psi : [0, +\infty) \to [0, +\infty)$$

continuous and increasing functions such that

$$|f(t, t^{\alpha-1} u(t))| \leq \phi(t) \psi(|u(t)|) \forall t \in [0, b], \forall u \in C([a, b], \mathbb{R})$$

$(H4)$ $\exists R > 0$ such that

$$|u_0| + \frac{\phi^* b \psi(R)}{\Gamma(\alpha + 1)} \leq R$$

Theorem 2.

If $(H1)$, $(H3)$, $(H4)$ *are satisfied. Then the problem (2.1)-(2.2) admits at least one solution.*

Proof 3.

To be the operator T *defined in the previous Theorem* 1.

Let $C = \left\{ u \in \widetilde{C}([a,b],\mathbb{R}), \| u \|_{\widetilde{C}} \leq R \right\}$ *such that* R *it's the constant given by* $(H4)$.

C *is a closed, bounded, convex of* $\widetilde{C}([a,b],\mathbb{R})$

- $TC \subset C$:

  Let $u \in C$ *and* $t \in [0,b]$ *we have*

$$\begin{aligned} |(Tu)(t)| &\leq t^{\alpha-1}|u_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u(s))|ds \\ &\leq b^{\alpha-1}|u_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,s^{\alpha-1}s^{1-\alpha}u(s))|ds \\ &\leq b^{\alpha-1}|u_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi(s)\psi(s^{1-\alpha}|u(s)|)ds \\ &\leq b^{\alpha-1}|u_0| + \frac{1}{\Gamma(\alpha)}\phi^*\psi(R)\int_0^t (t-s)^{\alpha-1}ds \\ &\leq b^{\alpha-1}|u_0| + \frac{1}{\Gamma(\alpha+1)}\phi^*\psi(R)b^{\alpha} \end{aligned}$$

  this is implies

$$\begin{aligned} t^{1-\alpha}|Tu(t)| &\leq b^{1-\alpha}b^{\alpha-1}|u_0| + \frac{b^{1-\alpha}}{\Gamma(\alpha+1)}\phi^*\psi(R)b^{\alpha} \\ &\leq |u_0| + \frac{b\phi^*\psi(R)}{\Gamma(\alpha+1)} \\ &\leq R \end{aligned}$$

  So $Tu \in C$.

- T *is continuous* :

  Let $(u_n) \subset C$ *be a sequence converges to* u^*.

  So

$$\begin{aligned} |(Tu_n)(t) - (Tu^*)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u_n(s)) - f(s,u^*(s))|ds \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,s^{\alpha-1}s^{1-\alpha}u_n(s)) - f(s,s^{\alpha-1}s^{1-\alpha}u^*(s))|ds. \end{aligned}$$

  this is implies

$$|(Tu_n)(t)-(Tu^*)(t)| \leq \frac{b^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,s^{\alpha-1}s^{1-\alpha}u_n(s))-f(s,s^{\alpha-1}s^{1-\alpha}u^*(s))|ds.$$

  From Lebesgue dominated convergence theorem and from the continuity of f,

  we have

$$|(Tu_n)(t) - (Tu^*)(t)| \to 0 \text{ as } n \to +\infty$$

  Hence the continuity of T.

- *T is compact :*

  To do this just show that TC is relatively compt.

  (a) TC is bounded, because $TC \subset C$ and C is bounded.

  (b) TC is equicontinuous :

  Let $u \in C$ and $t_1, t_2 \in [0, b]$ with $t_1 \leq t_2$,

$$
\begin{aligned}
|(Tu)(t_1) - (Tu)(t_2)| &= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, u(s)) ds - \int_0^{t_2} (t_2 - s)^{\alpha-1} f(s, u(s)) ds \right| \\
&= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, u(s)) ds - \int_0^{t_1} (t_2 - s)^{\alpha-1} f(s, u(s)) ds \right. \\
&\quad \left. + \int_0^{t_1} (t_2 - s)^{\alpha-1} f(s, u(s)) ds - \int_0^{t_2} (t_2 - s)^{\alpha-1} f(s, u(s)) ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}| |f(s, s^{\alpha-1} s^{1-\alpha} u(s)| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} |f(s, s^{\alpha-1} s^{1-\alpha} u(s)| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}| \phi(s) \psi(s^{1-\alpha} |u(s)|) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \phi(s) \psi(s^{1-\alpha} |u(s)|) ds \\
&\leq \frac{\psi(R)\phi^*}{\Gamma(\alpha)} \left(\int_0^{t_1} |(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}| ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \right) \\
&\quad \to 0 \text{ as } t_1 \to t_2.
\end{aligned}
$$

  So TC is relatively compact.

  According to theorem (Schauder), we conclude that the problem (2.1)-(2.2) has at least one solution.

The following résultat is based on the fixed point theorem of Schaefer.
We introduce the following condition :

$$
(H5) \qquad \lim_{r \to +\infty} \frac{r}{\psi(r)} = +\infty
$$

Theorem 3.

If $(H1)$*,* $(H3)$*,* $(H5)$ *are satisfied, then the problem (2.1)-(2.2) has at least one solution.*

Proof 4.

Let to be the operator T *defined in the previous Theorem* 1*.*
As in the proof of the previous Theorem 2*, we got that the operator* T *is continuous and compact, so it's completely continuous.*
Now consider the subsets :

$$\begin{aligned} B &= \left\{ u \in \widetilde{C}([0,b],\mathbb{R}), \| u \|_{\widetilde{C}} \leq \rho \right\} \\ &\quad \textit{and} \\ \Omega &= \{u \in B, \lambda T u = u, \ \textit{for some} \ \lambda \in]0,1[\} \end{aligned}$$

Let $u \in \Omega$ *and* $t \in [0,b]$ *we have*

$$\begin{aligned} |u(t)| &= \lambda |(Tu)(t)| \\ &\leq |(Tu)(t)| \\ &\leq b^{\alpha-1}|u_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s) \psi(s^{1-\alpha}|u(s)|) ds \end{aligned}$$

this is implies

$$t^{1-\alpha}|u(t)| \leq |u_0| + \frac{\phi^* b \psi(\| u \|_{\widetilde{C}})}{\Gamma(\alpha+1)}$$

So

$$\| u \|_{\widetilde{C}} \leq |u_0| + \frac{\phi^* b \psi(\| u \|_{\widetilde{C}})}{\Gamma(\alpha+1)}$$

then

$$\frac{\| u \|_{\widetilde{C}}}{|u_0| + \dfrac{\phi^* b \psi(\| u \|_{\widetilde{C}})}{\Gamma(\alpha+1)}} \leq 1$$

If $\| u \|_{\widetilde{C}} \to +\infty$*, so we will have a contradiction with* $(H5)$*.*
Then it exists $\overline{\rho} > 0$ *such that* $\| u \|_{\widetilde{C}} \leq \overline{\rho}$ *it's who gives* Ω *is bounded.*
From Theorem (Schaefer) the operator T *has at least one fixed point which represents a solution to the problem (2.1)-(2.2).*

The following résultat is based on the nonlinear alternative of Learay-Schauder we introduce the following condition :

$$(H6) \qquad \exists M > 0 \text{ such that } \frac{M}{|u_0| + \dfrac{\phi^* b\psi(\| u \|_{\widetilde{C}})}{\Gamma(\alpha+1)}} > 1.$$

Theorem 4.

If $(H1)$, $(H3)$, $(H6)$ are satisfied, then the problem (2.1)-(2.2) has at least one solution.

Proof 5.

Let to be operator T defined in the previous Theorem 1.
Assume that the open set

$$\nu = \left\{ u \in \widetilde{C}([0,b],\mathbb{R}), \| u \|_{\widetilde{C}} < M \right\}.$$

As in the proof of the previous Theorem 2 we got that the operator T is continuous.
If it exists $u \in \partial\nu$ and $\lambda \in [0,1]$ such that $u = \lambda T u$ then

$$\begin{aligned} |u(t)| &= \lambda |(Tu)(t)|, \quad t \in [0,b] \\ &\leq |(Tu)(t)| \\ &\leq b^{\alpha-1}|u_0| + \frac{\phi^* b^\alpha \psi(\| u \|_{\widetilde{C}})}{\Gamma(\alpha+1)} \end{aligned}$$

this is implies

$$t^{1-\alpha}|u(t)| \leq |u_0| + \frac{\phi^* b^\alpha \psi(\| u \|_{\widetilde{C}})}{\Gamma(\alpha+1)}$$

So

$$\| u \|_{\widetilde{C}} \leq |u_0| + \frac{\phi^* b^\alpha \psi(\| u \|_{\widetilde{C}})}{\Gamma(\alpha+1)}$$

then

$$\frac{\| u \|_{\widetilde{C}}}{\dfrac{\phi^* b^\alpha \psi(\| u \|_{\widetilde{C}})}{\Gamma(\alpha+1)}} \leq 1.$$

Which means this from $(H6)$, $\| u \|_{\widetilde{C}} \neq M$ but $\| u \|_{\widetilde{C}} = M$ because $u \in \partial\nu$ this is a contradiction.
From Theorem (Learay-Schauder) the operator T has at least one fixed point which represents a solution to the problem (2.1)-(2.2).

2.2 Boundary value problem with Dirichlet condition :

2.2.1 Proposed Dirichlet type problem :

Consider the following problem :

$$^{RL}D^{\alpha}_{0+}u(t) = f(t, u(t)); t \in [a, b], (b > 0) \tag{2.4}$$

$$\lim_{t \overset{>}{\to} 0} t^{2-\alpha}u(t) = u_0, \lim_{t \overset{<}{\to} b} t^{2-\alpha}u(t) = u_b \tag{2.5}$$

such that $1 < \alpha < 2$ and $f : [a; b] \times \mathbb{R} \to \mathbb{R}$ a given function $u_0, u_b \in \mathbb{R}$.
Suppose space

$$\tilde{C}^2([a, b], \mathbb{R}) = \left\{ u \in C^2([a, b], \mathbb{R}), \lim_{t \overset{>}{\to} 0} t^{2-\alpha}u(t), \lim_{t \overset{<}{\to} b} t^{2-\alpha}u(t) \text{ exists} \right\}$$

We say that $u \in \tilde{C}^2([a, b], \mathbb{R})$ is said solution of (2.4)-(2.5) if u satisfied (2.4) and (2.5).

Lemma 2.

let $h : [0, b] \to \mathbb{R}$ a continuous function.
Then the liner problem

$$\begin{cases} ^{RL}D^{\alpha}_{0+}u(t) &= h(t); t \in [a, b] \\ \lim_{t \overset{>}{\to} 0} t^{2-\alpha}u(t) &= u_0, \lim_{t \overset{<}{\to} b} t^{2-\alpha}u(t) = u_b \end{cases}$$

has only one solution given by

$$u(t) = u_0 t^{\alpha-1} + \frac{u_b - u_0}{h} t^{\alpha-1} + \int_0^b G(t, s)h(s)ds$$

such that G is the Green function defined by

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (t-s)^{\alpha-1} - t^{\alpha-1}b^{1-\alpha}(b-s)^{\alpha-1} & if \quad 0 \le s \le t \le b \\ -t^{\alpha-1}b^{1-\alpha}(b-s)^{\alpha-1} & if \quad 0 \le t \le s \le b \end{cases}$$

Proof 6. *By lemma 1.2 and from $^{RL}D^{\alpha}_{0+}u(t) = h(t)$ we have*

$$I^{\alpha}_{0+}(^{RL}D^{\alpha}_{0+}u(t)) = I^{\alpha}_{0+}h(t)$$

i.e.

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds$$

and

$$t^{2-\alpha}u(t) = c_1 t + c_2 + \frac{1}{\Gamma(\alpha)} t^{2-\alpha} \int_0^t (t-s)^{\alpha-1} h(s)ds$$

with c_1, c_2 *are constants.*

From $\lim\limits_{t \overset{>}{\to} b} t^{2-\alpha}u(t) = u_0$ *we get* $c_2 = u_0$ *and from* $\lim\limits_{t \overset{<}{\to} b} t^{2-\alpha}u(t) = u_b$ *we obtain*

$$c_1 b + c_2 + \frac{1}{\Gamma(\alpha)} b^{2-\alpha} \int_0^b (b-s)^{\alpha-1} h(s)ds = u_b$$

This gives

$$c_1 = \frac{u_b}{b} - \frac{u_0}{b} - \frac{1}{\Gamma(\alpha)} b^{1-\alpha} b^{2-\alpha} \int_0^b (b-s)^{\alpha-1} h(s)ds$$

so

$$\begin{aligned} u(t) &= \frac{u_b}{b} t^{\alpha-1} - \frac{u_0}{b} t^{\alpha-1} - \frac{t^{\alpha-1}}{\Gamma(\alpha)} b^{1-\alpha} \textstyle\int_0^b (b-s)^{\alpha-1} h(s)ds \\ &+ u_0 t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \textstyle\int_0^t (t-s)^{\alpha-1} h(s)ds \end{aligned}$$

which can be written in the from

$$\begin{aligned} u(t) &= u_0 t^{\alpha-2} + \frac{u_b - u_0}{b} t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \textstyle\int_0^t t^{\alpha-1} b^{\alpha-1} (b-s)^{\alpha-1} h(s)ds \\ &- \frac{1}{\Gamma(\alpha)} \textstyle\int_t^b t^{\alpha-1} b^{\alpha-1} (b-s)^{\alpha-1} h(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \textstyle\int_0^t (t-s)^{\alpha-1} h(s)ds \\ &= u_0 t^{\alpha-2} + \frac{u_b - u_0}{b} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \textstyle\int_0^t t^{\alpha-1} - b^{\alpha-1}(b-1)^{\alpha-1} h(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \textstyle\int_t^b -t^{\alpha-1} b^{1-\alpha} (b-s)^{\alpha-1} h(s)ds \\ &= u_0 t^{\alpha-2} + \frac{u_b - u_0}{b} t^{\alpha-1} + \textstyle\int_0^b G(t,s)h(s)ds \end{aligned}$$

2.2.2 Existence of solutions :

Our first result for the problem (2.4)-(2.5) is based on the principle of Banach contraction.

Theorem 5. *Let's put* $G = \sup\limits_{(t,s)\in[0,b]\times[a,b]} |G(t,s)|$ *suppose are satisfied and if then the problem (2.4)-(2.5) admits only one solution.*

Proof 7. *Considering the operator* T

$$\begin{aligned} T : \widetilde{C}([a,b],\mathbb{R}) &\to \widetilde{C}([a,b],\mathbb{R}) \\ (Tu)(t) &= u_0 t^{\alpha-2} + \frac{u_b - u_0}{b} t^{\alpha-1} + \textstyle\int_0^b G(t,s) f(s,u(s))ds \end{aligned}$$

We show that T is contraction :

Let $u, v \in \widetilde{C}([a,b],\mathbb{R})$ and $t \in [0,b]$.

$$\begin{aligned} |(Tu)(t)-(Tv)(t)| &\leq \int_0^b |G(t,s)||f(s,u(s))-f(s,v(s))|ds \\ &\leq \int_0^b |G(t,s)|s^{1-\alpha}s^{\alpha-1}|f(s,u(s))-f(s,v(s))|ds \\ &\leq G^*k\int_0^b s^{\alpha-1}|u(s)-v(s)|ds \\ &\leq G^*kb \parallel u-v \parallel_{\widetilde{C}} \end{aligned}$$

This implies

$$t^{1-\alpha}|(Tu)(t)-(Tv)(t)| \leq G^*kb^{2-\alpha} \parallel u-v \parallel_{\widetilde{C}}$$

i.e.

$$\parallel Tu - Tv \parallel_{\widetilde{C}} \leq G^*kb^{2-\alpha} \parallel u-v \parallel_{\widetilde{C}}$$

As $G^*kb^{2-\alpha} < 1$ so T is contraction according to theorem (Banach), we conclude that the problem has only one solution.

Exemple 2. *consider the following problem*

$$^{RL}D^{\frac{3}{2}}u(t) = \frac{t^{\frac{1}{2}}}{(e^t+e^{-t})(1+|u(t)|)}, t \in [0,1] \tag{2.6}$$

$$\lim_{t \overset{>}{\to} 0} t^{\frac{1}{2}} = 0 \quad , \quad \lim_{t \overset{<}{\to} 0} t^{\frac{1}{2}} = \frac{1}{2} \tag{2.7}$$

with $1 < \alpha = \dfrac{3}{2} < 2$

Putting $f(t,u(t)) = \dfrac{t^{\frac{1}{2}}}{(e^t+e^{-t})(1+|u(t)|)}$.

We have

(H_1) *$f : [u,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.*

(H_2) *Let $u, v \in \widetilde{C}([0,1],\mathbb{R}), t \in [0,1]$*

$$\begin{aligned} |t^{-\frac{1}{2}}f(t,u(t)) - t^{-\frac{1}{2}}f(t,v(t))| &= \frac{1}{e^t+e^{-t}}\left|\frac{1}{1+|u(t)|} - \frac{1}{1+|v(t)|}\right| \\ &= \frac{1}{e^t+e^{-t}}\frac{||u(t)|-|v(t)||}{(1+|u(t)|)(1+|v(t)|)} \\ &\leq ||u(t)|-|v(t)|| \end{aligned}$$

In this case we have $k = 1$, $b^{1-1} = 1$.

On the other hand

$$G(t,s) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\begin{cases} (t-s)^{\frac{1}{2}} - t^{\frac{1}{2}}(1-s)^{\frac{1}{2}} & if \quad 0 \leq s \leq t \leq 1 \\ -t^{\frac{1}{2}}(1-s)^{\frac{1}{2}} & if \quad 0 \leq t \leq s \leq 1 \end{cases}$$

Then $|G(t,s)| \leq \dfrac{2}{\Gamma\left(\dfrac{3}{2}\right)}$ *for all* $(t,s) \in [0,1] \times [0,1]$ *and* $G^* \leq \dfrac{2}{\Gamma\left(\dfrac{3}{2}\right)}$.

So $G^* k b^{1-\alpha} \leq \dfrac{1}{\Gamma\left(\dfrac{3}{2}\right)} \times 1 \times 1 < 1, \left(\Gamma\left(\dfrac{3}{2}\right) = \dfrac{\sqrt{\pi}}{2}\right)$.

Then the problem (2.6)-(2.7) has only one solution.

2.3 Boundary value problem with Dirichlet-Neumann type

2.3.1 Proposed Dirichlet-Neumann type problem

We impose now the following problem

$$^{RL}D^{\alpha}_{t+}u(t) = f(t, u(t)), t \in [a, b], (b > 0) \tag{2.8}$$

$$u(0) = 0, u'(b) = 0 \tag{2.9}$$

such that $1 < \alpha < 2$ and $f : [a; b] \times \mathbb{R} \to \mathbb{R}$ a given function.

Lemma 3. *Let* $h : [0, b] \to \mathbb{R}$ *a continuous function then the linear problem*

$$\begin{cases} ^{RL}D^{\alpha}_{0+}u(t) = h(t), t \in [0, b] \\ u(0) = 0, u'(b) = 0 \end{cases}$$

has only one solution given by

$$u(t) = -\frac{1}{b^{\alpha-1}\Gamma(\alpha)} t^{\alpha-1} \int_0^b (b-s)^{\alpha-2} h(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds$$

such that G is the green function defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (t-s)^{\alpha-1} - \dfrac{1}{b^{\alpha-2}} t^{\alpha-1} (b-s)^{\alpha-2} & if \quad 0 \leq s \leq t \leq b \\ -\dfrac{1}{b^{\alpha-2}} t^{\alpha-1} (b-s)^{\alpha-2} & if \quad 0 \leq t \leq s \leq b \end{cases}$$

Proof 8.

By lemma 1.2 and from $^{RL}D^{\alpha}_{0+}u(t) = h(t)$, *we have* $I^{\alpha}\left(^{RL}D^{\alpha}_{0+}u(t)\right) = I^{\alpha}_{0+}h(t) = 0$ *which gives*

$$u(t) = c_1 t^{\alpha-1} + c_1 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds$$

and

$$u'(t) = (\alpha-1)c_1 t^{\alpha-2} + (\alpha-2)c_1 t^{\alpha-3} + \frac{\alpha-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}h(s)ds$$

with c_1, c_2 *are constants.*

From $u(0) = 0$ *we have* $c_1 = 0$ *and from* $u'(t) = 0$ *we get* $c_1 = -\frac{1}{b^{\alpha-1}\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-2}h(s)ds$.

So

$$u(t) = -\frac{1}{b^{\alpha-1}\Gamma(\alpha)} t^{\alpha-1}\int_0^b (b-s)^{\alpha-2}h(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds$$

Which can be written in the form

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \left((t-s)^{\alpha-1} - \frac{1}{b^{\alpha-2}}t^{\alpha-1}(b-s)^{\alpha-1}\right)h(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t -\frac{1}{b^{\alpha-2}}t^{\alpha-1}(b-s)^{\alpha-1}h(s)ds$$

the latter is written in the form

$$u(t) = \int_0^b G(t,s)h(s)ds$$

Remark 2.

The function G *is not continuous on* $[0,b]\times[0,b]$ *but the function* $t \mapsto \int_0^b G(t,s)ds$ *is continuous on* $[0,b]$ *so we'll put* $\widetilde{G} = \sup_{t\in[0,b]}\int_0^b |G(t,s)|ds$

2.3.2 Existence of solutions

Consider the condition :

$(H'2)$ $\exists k > 0$ such that

$$|f(t,u_1(t)) - f(t,u_2(t))| \leq k|u_1(t) - u_2(t)| \quad \forall t\in[a,b] \quad \forall u_1,u_2 \in C([a,b],\mathbb{R})$$

Theorem 6. *:*

Suppose (H1),(H'2) are satisfied and if $0 < \widetilde{G}k < 1$ *then the problem (2.8)-(2.9) has only one solution.*

Proof 9. *:*

Considering the operator T

$$\begin{aligned} T \quad &: C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R}) \\ (Tu)(t) \quad &= \int_0^b G(t,s)f(s,u(s))ds \end{aligned}$$

We show that T is contraction :

Let $u, v \in C([0,b],\mathbb{R})$ and $t \in [0,b]$,

$$\begin{aligned}
|(Tu)(t) - (Tv)(t)| &\leq \int_0^b |G(t,s)||f(s,u(s)) - f(s,v(s))|ds \\
&\leq k \int_0^b |G(t,s)||u(s) - v(s)|ds \\
&\leq k\widetilde{G} \parallel u - v \parallel_\infty
\end{aligned}$$

this implies $\parallel Tu - Tv \parallel_\infty \leq \parallel u - v \parallel_\infty$

Since $k\widetilde{G} < 1$ so T is contraction.

According to theorem (Banach), we conclution that the problem (2.8)-(2.9) has only one solution.

2.4 Cauchy problem of fractional order $\alpha \in]1, 2[$

Consider the Cauchy problem :

$$^{RL}D^{\alpha}_{t+}u(t) = f(t, u(t), {^{RL}D^{\alpha+1}}u(t)), t \in [a,b] \tag{2.10}$$

$$u(0) = 0, \quad u'(0) = 0 \tag{2.11}$$

such that $1 < \alpha \leq 2$ and $f : [0,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ a given function $u_0 \in \mathbb{R}$.
Suppose the espace

$$\widetilde{C}([a,b],\mathbb{R}) = \left\{u \in C([a,b],\mathbb{R}), {^{RL}D^{\alpha-1}_{0+}}u \in C([a,b],\mathbb{R})\right\}$$

with the norm $\|u\|_{\widetilde{C}} = \max\left\{\parallel u \parallel_\infty, \parallel {^{RL}D^{\alpha-1}_{0+}}u \parallel_\infty\right\}$

Lemma 4.
let $h : [0,b] \to \mathbb{R}$ a continuous function, then the linear problem

$$\begin{cases} {^{RL}D^{\alpha-1}_{0+}}u(t) = h(t), & t \in [0,b] \\ u(0) = 0, \quad u'(0) = 0 \end{cases}$$

has only one solution given by

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds$$

Proof 10.

By lemma 1.2 and from ${}^{RL}D_{0^+}^{\alpha-1}u(t) = h(t)$, *we have* $I^{\alpha}(D_{0^+}^{\alpha-1}u(t) = I^{\alpha}(h(t))$

Which gives

$$u(t) = c_1 t^{\alpha-2} + c_2 t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds$$

and

$$u(t) = (\alpha-1)c_1 t^{\alpha-2} + (\alpha-2)c_2 t^{\alpha-3} + \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} h(s) ds$$

with c_1, c_2 *are constants.*

From $u(0) = 0$ *we have* $c_2 = 0$ *and from* $u'(0) = 0$ *we get* $c_1 = 0$

So $u(t) = \dfrac{\alpha-1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-2} h(s) ds$

Now consider the following hypotheses :

(C_1) $f : [0,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuos.*

(C_2) $\exists k > 0$, *such that*

$$|f(t, y(t), {}^{RL}D_{0^+}^{\alpha}u_1(t)) - f(t, y(t), {}^{RL}D_{0^+}^{\alpha}u_2(t))| \leq k(|u_1(t) - u_2(t)| + |{}^{RL}D_{0^+}^{\alpha}u_1(t) - {}^{RL}D_{0^+}^{\alpha}u_2(t)|)$$

for all $u_1, u_2 \in \widetilde{C}([a,b], \mathbb{R})$ *and* $t \in [0,b]$.

We have the following result.

Theorem 7.

Suppose $(C_1), (C_2)$ *are satisfied and if*

$$\max\left\{\frac{2kb^{\alpha}}{\Gamma(\alpha+1)}, 2kb\right\} < 1$$

then the problem (2.10)-(2.11) has only one solution.

Proof 11. *:*

Considering the operator T

$$(Tu)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), {}^{RL}D_{0^+}^{\alpha}u(s)) ds$$

We show that T *is contraction.*

Let $u, v \in \widetilde{C}([a,b], \mathbb{R})$ *and* $t \in [a,b]$, *we have*

$$\begin{aligned}
|(Tu)(t) - (Tv)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, u(s), {}^{RL}D_{0^+}^{\alpha}u(s)) - f(s, v(s), {}^{RL}D_{0^+}^{\alpha}u(s))| ds \\
&\leq \frac{k}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |u(s) - v(s)| + |{}^{RL}D_{0^+}^{\alpha} - {}^{RL}D_{0^+}^{\alpha})| ds \\
&\leq \frac{2k}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \parallel u - v \parallel_{\widetilde{C}} ds \\
&\leq \frac{2kb^{*}}{\Gamma(\alpha+1)} \parallel u - v \parallel_{\widetilde{C}}
\end{aligned}$$

this implies $\| Tu - Tv \|_{\infty} \leq \dfrac{2kb^*}{\Gamma(\alpha+1)} \| u - v \|_{\widetilde{C}}$ *on the other hand*

$$\begin{aligned} {}^{RL}D_{0^+}^{\alpha-1}(Tu)(t) &= {}^{RL}D^{\alpha-1} I_{0^+}^{\alpha}(f) \\ &= I_{0^+}^{1}(f) \\ &= \int_0^t (t-s)^{\alpha-1} |f(s, u(s), {}^{RL}D_{0^+}^{\alpha} u(s)) ds \end{aligned}$$

and we obtain

$$\begin{aligned} |{}^{RL}D_{0^+}^{\alpha-1}(Tu)(t) - {}^{RL}D_{0^+}^{\alpha-1}(Tv)(t)| &\leq k \int_0^t (|u(s) - v(s)| + |{}^{RL}D_{0^+}^{\alpha} u(s) - {}^{RL}D_{0^+}^{\alpha} v(s))|) ds \\ &\leq 2k \| u - v \|_{\widetilde{C}} \end{aligned}$$

this implies $\| {}^{RL}D_{0^+}^{\alpha} u - {}^{RL}D_{0^+}^{\alpha} v \|_{\infty} \leq 2kb \| u - v \|_{\widetilde{C}}$.
Then, $\| Tu - Tv \|_{\widetilde{C}} \leq \max \left\{ \dfrac{2kb^{\alpha}}{\Gamma(\alpha+1)}, 2kb \right\} \| u - v \|_{\widetilde{C}}$.
Since $0 < \max \left\{ \dfrac{2kb^{\alpha}}{\Gamma(\alpha+1)}, 2kb \right\} < 1$ *so* T *is contraction.*
According to theorem (Banach), we conclude that the problem (2.10)-(2.11) has one solution.
Let's give another existence result of problem (2.10)-(2.11) by using the Schauder theorem.
First, we present the following conditions

(C_3) $\exists \phi \in C([a,b], \mathbb{R}^+)$ *and* $\psi : [0; +\infty) \to [0; +\infty)$ *continuous and increasing functions such that*

$$|f(t, u(t), v(t)| \leq \phi(t) \psi(|u(t)| + |v(t)|)$$
$$\forall t \in [0,b], \forall u, v \in \widetilde{C}([a,b], \mathbb{R})$$

(C_4) $\exists R > 0$ *such that*

$$\max \left\{ \frac{\phi^* \psi(2R) b^{\alpha}}{\Gamma(\alpha+1)}, \phi^* \psi(2R) b \right\} \leq R, \ \textit{with} \ \phi^* = \sup_{t \in [0,b]} \phi(t)$$

Theorem 8.

If $(C_1), (C_3), (C_4)$ *are satisfied then the problem (2.10)-(2.11) has at least one solution.*

Proof 12.

Let T *the operator defined in the provious Theorem 3.*
Consider $C = \left\{ u \in \widetilde{C}([a,b], \mathbb{R}), \| u \|_{\widetilde{C}} \leq R \right\}$ *such that* R *it is the constant given by*

(C_4).

C is a closed, bounded, convex of $\widetilde{C}([a,b],\mathbb{R})$.

- Setep 1 : $TC \subset C$

Let $u \in C$ *and* $t \in [0,b]$ *we have*

$$\begin{aligned} |(Tu)(t)| & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u(s),{}^{RL}D^{\alpha}_{0^+}u(s))|ds \\ & \leq \frac{\phi^*\psi(2\alpha)b^{\alpha}}{\Gamma(\alpha+1)} \end{aligned}$$

Also

$$\begin{aligned} |{}^{RL}D^{\alpha-1}_{0^+}(Tu)(s)| & \leq \int_0^t |f(s,u(s),{}^{RL}D^{\alpha-1}_{0^+}u(s))|ds \\ & \leq \phi^*\psi(2R)b \end{aligned}$$

this implies $\| {}^{RL}D^{\alpha-1}_{0^+}Tu \|_{\infty} \leq \phi^*\psi(2R)b$.

Then, we obtain

$$\| Tu \|_{\widetilde{C}} \leq \max\left\{\frac{\phi^*\psi(2R)b^{\alpha}}{\Gamma(\alpha+1)}, \phi^*\psi(2R)b\right\} \leq R$$

So $Tu \in C$.

- Setep 2 : T is continuous.

Let $(u_n) \subset C$ *be a sequence converge to* u^* *and* ${}^{RL}D^{\alpha-1}_{0^+}u_n$ *converge to* ${}^{RL}D^{\alpha-1}_{0^+}u^*$.

So

$$|(Tu_n)(t)-(Tu^*)(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u_n(s),{}^{RL}D^{\alpha}_{0^+}u_n(s)) - |f(s,u^*(s),{}^{RL}D^{\alpha}_{0^+}u^*(s))|ds.$$

from lebesgue dominated convergence theorem and from the continuty of f we have $|(Tu_n)(t)-(Tu^*)(t)| \to 0$ *as* $n \to +\infty$.

In addition,

$$|{}^{RL}D^{\alpha}_{0^+}(Tu_n)(t) - {}^{RL}D^{\alpha}_{0^+}(Tu^*)(t)| \leq \int_0^t |f(s,u_n(s),{}^{RL}D^{\alpha}_{0^+}u_n(s)) - |f(s,u^*(s),{}^{RL}D^{\alpha}_{0^+}u^*(s))|ds.$$

and we have ${}^{RL}D^{\alpha}_{0^+}|(Tu_n)(t) - {}^{RL}D^{\alpha}_{0^+}(Tu^*)(t)| \to 0$ *as* $n \to +\infty$.

So $\|Tu_n - Tu^*\|_{\widetilde{C}} \to 0$ *as* $n \to +\infty$. *hence the contininty of T.*

- Setep 3 : T is compact.

To do this just show that TC is relatively compact.

(a) *TC is bounded; because* $TC \subset C$ *and C is bounded.*

(b) TC is equicontinuous :

Let $u \in C$ and $t_1, t_2 \in [0,b]$ with $t_1 \leq t_2$,

$$\begin{aligned}
|(Tu)(t_1) - (Tu)(t_2)| &\leq \frac{1}{\Gamma(\alpha)} \bigg| \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, u(s), {}^{RL}D_{0^+}^{\alpha} u(s)) \\
&- \int_0^{t_2} (t_2 - s)^{\alpha-1} f(s, u(s), {}^{RL}D_{0^+}^{\alpha} u(s)) ds \bigg| \\
&\leq \frac{1}{\Gamma(\alpha)} \bigg| \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, u(s), {}^{RL}D_{0^+}^{\alpha} u(s)) \\
&- \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, u(s), {}^{RL}D_{0^+}^{\alpha} u(s)) ds \\
&+ \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, u(s), {}^{RL}D_{0^+}^{\alpha} u(s)) ds \bigg| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |(t_1 - s)^{\alpha-1} (t_2 - s)^{\alpha-1}| |f(s, u(s), {}^{RL}D_{0^+}^{\alpha} u(s)) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} |(t_2 - s)^{\alpha-1}| f(s, u(s), {}^{RL}D_{0^+}^{\alpha} u(s))| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}| \phi(s) \psi((u(s) + |{}^{RL}D_{0^+}^{\alpha} u(s)|) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} |(t_2 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}| \phi(s) \psi((u(s)(t) + |{}^{RL}D_{0^+}^{\alpha} u(s)|) ds \\
&\leq \frac{\phi^* \psi(2R) b^{\alpha}}{\Gamma(\alpha)} \bigg(\int_0^{t_1} |(t_1 - s)^{\alpha-1} (t_2 - s)^{\alpha-1} ds \\
&+ \int_{t_1}^{t_2} |(t_2 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} ds \bigg) \\
&\to 0 \quad as \quad t_1 \to t_2
\end{aligned}$$

Moreover

$$\begin{aligned}
|{}^{RL}D_{0^+}^{\alpha-1}(Tu)(t_1) - {}^{RL}D_{0^+}^{\alpha-1}(Tu)(t_2)| &\leq \int_{t_1}^{t_2} |f(s, u(s), {}^{RL}D_{0^+}^{\alpha-1} u(s))| ds \\
&\leq \phi^* \psi(2R) |t_1 - t_2| \\
&\to 0 \quad as \quad t_1 \to t_2.
\end{aligned}$$

So TC is relatively compact.

According to theorem (Schauder), we conclude that the problem (2.10)-(2.11) has at least one solution.

CHAPITRE 3

BOUNDARY VALUS PROBLEMS WITH NON LOCAL CONDITIONS.

(Main references for this chapter : [4, 10, 12, 14])

Introduction :

This chapter presents some existence and uniqueness results of solutions for some problems of Cauchy with nonlocal conditions for differential equations of fractional order. This results will be based on the banach fixed point theorem and the schacfer fixed poins theorem. This type of nonlocal Cauchy problem was introduced by the poloni mathematicians L.Byzewshi, He noticed that the nonlocal condition is very suitable as local condition (intial) to correctly describe some physical phenomena [3]. He proved the existence and uniqueness of the solutions for this kind of problems on nonlocal conditions as follows :

$$g(u) = \sum_{i=1}^{n} c_i u(t_i)$$

such that $c_i, i = 1, ..., n$ are constants and $0 < t_1 < ... < t_n \leqslant b$.

3.1 Boundrry value problem with nonlocal condition for $\alpha \in]0;1[$

We consider the following nonlocal problem

$$^{RL}D^{\alpha}_{a^+}u(t) = f(t,u(t)), t \in [0,b] \tag{3.1}$$

$$\lim_{t \overset{>}{\to} 0} t^{1-\alpha}u(t) + g(u) = u_0 \tag{3.2}$$

such that $0 < \alpha \leq 1$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : C([a,b],\mathbb{R}) \to \mathbb{R}$ are given functions $u_0 \in \mathbb{R}$.

3.1.1 Existence of solutions :

We introduce the following hypotheses :

(C1) There exists a constant $k > 0$ such that :

$$|t^{\alpha-1}f(t,u_1)(t) - t^{\alpha-1}f(t,u_2)(t)| \leq k|u_1 - u_2|$$

for all $t \in [0,b]$ and $u_1, u_2 \in \mathbb{R}$

(C2) There exists a constant $M > 0$ such that

$$|f(t,u)| \leq M \text{ for all } t \in [0,b], \text{ and } u \in \mathbb{R}$$

(C3) There exists a constant $M' > 0$ such that

$$|g(u)| \leq M' \text{ for all } u \in C([0,b],\mathbb{R})$$

(C4) There exists a constant $k' > 0$ such that $|g(u_1) - g(u_2)| \leq k'|u_1 - u_2|$ for all $u_1, u_2 \in C([0,b],\mathbb{R})$.

Lemma 5.

Let $h : [a,b] \to \mathbb{R}$ a continuous function then the linear problem

$$\begin{cases} ^{RL}D^{\alpha}_{0^+}u(t) = h(t) \quad t \in [a,b] \\ \lim_{t \overset{>}{\to} 0} t^{1-\alpha}u(t) + g(u) = u_0 \end{cases}$$

admets only one solution given by

$$u(t) = u_0 t^{\alpha-1} - g(u)t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds$$

Proof 13. *By lemma 1.2 and from* ${}^{RL}D^{\alpha}_{0^+}u(t) = h(t)$ *have*

$$I^{\alpha}_{0^+}\left({}^{RL}D^{\alpha}_{0^+}u(t)\right) = I^{\alpha}_{0^+}h(t)$$

it's who gives,

$$u(t) = ct^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds$$

and by condition (3.2) we have $c = u_0 + g(u)$ *so*

$$u(t) = u_0 t^{\alpha-1} - g(u)t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds$$

Our first result on the problem (3.1)-(3.2) is based on the Banach contraction principle.

Theorem 9. *Assume that the assumptions (C1), (C2) are satisfied.*
If $k'b_+^{\alpha-1}\dfrac{kb}{\Gamma(\alpha+1)}$ *then the problem (3.1)-(3.2) has only one solution on* $[a, b]$.

Proof 14. *We transform the problem (3.1)-(3.2) into problem of fixed point.*
We define the operator :

$$T : \widetilde{C}([a, b], \mathbb{R}) \to \widetilde{C}([a, b], \mathbb{R})$$

by

$$(Tu)(t) = u_0 t^{\alpha-1} - g(u)t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, u(s))ds,$$

we're going to show that T *is a contraction.*
In first, let $u, v \in \widetilde{C}([a, b], \mathbb{R})$, *then for all* $t \in [a, b]$

$$\begin{aligned}
|(Tu)(t) - (Tv)(t)| &\leq t^{\alpha-1}|g(u) - g(v)| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s, u(s)) - f(s, v(s))|ds \\
&\leq t^{\alpha-1}|g(u) - g(v)| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}|s^{\alpha-1}f(s, u(s)) - s^{\alpha-1}f(s, v(s))|ds \\
&\leq k't^{\alpha-1}\sup_{t\in[a,b]}|u(t) - v(t)| + \frac{k}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \parallel u - v \parallel_{\widetilde{C}} ds \\
&\leq k't^{\alpha-1}\sup_{t\in[a,b]}|u(t) - v(t)| + \frac{kb^{\alpha}}{\Gamma(\alpha+1)} \parallel u - v \parallel_{\widetilde{C}} ds
\end{aligned}$$

this implies,

$$t^{1-\alpha}|(Tu)(t) - (Tv)(t)| \leq k'b^{\alpha-1} \parallel u - v \parallel_{\widetilde{C}} + \frac{kb}{\Gamma(\alpha+1)} \parallel u - v \parallel_{\widetilde{C}}$$

So

$$\parallel Tu - Tv \parallel_{\widetilde{C}} \leq \left(k'b^{\alpha+1} + \frac{kb}{\Gamma(\alpha+1)}\right) \parallel u - v \parallel_{\widetilde{C}} .$$

Consequently, T *admitted only one fixed point which is the solution to the problem (3.1)-(3.2).*

The second result on the problem (3.1)-(3.2) is bused on the schacfer fixed point theorem.

Theorem 10. *Suppose that $f \in C([a,b],\mathbb{R})$ and the hypotheses (C2), (C3) are satisfied. Then the problem (3.1)-(3.2) has at least one solution on $[a,b]$*

Proof 15. *Let to be the operator T defined in the previous Theorem 1.*
Now consider the subsets :

$$C = \left\{ u \in \widetilde{C}([a,b],\mathbb{R}), \| u \|_{\widetilde{C}} \leq \rho \right\}$$

with $\rho = |u_0| + M' + \dfrac{Mb}{\Gamma(\alpha+1)}$ *and*

$$\Omega = \{ u \in C, \lambda T u = u, 0 < \lambda < 1 \}.$$

C is a closed, bounded, convex of $\widetilde{C}([a,b],\mathbb{R})$ we will show that $T : C \to C$ is compmetly continuous.
***Step 1** : $TC \subset C$.*
Let $u \in C$ and $t \in [a,b]$ we have

$$\begin{aligned} |(Tu)(t)| &\leq t^{\alpha-1}|u_0| + t^{\alpha-1}|g(u)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|f(s,u(s))|ds \\ &\leq b^{\alpha-1}|u_0| + b_{\alpha-1}M' + \frac{Mb^\alpha}{\Gamma(\alpha+1)} \end{aligned}$$

this implies,

$$\begin{aligned} t^{1-\alpha}|Tu(t)| &\leq b^{1-\alpha}b^{\alpha-1}|u_0| + b^{1-\alpha}b^{\alpha-1}M' + \frac{b^{1-\alpha}Mb^\alpha}{\Gamma(\alpha+1)} \\ \| Tu \|_{\widetilde{C}} &\leq |u_0| + M' + \frac{Mb}{\Gamma(\alpha+1)} = \rho \end{aligned}$$

So $Tu \in C$.
***Step2 :** T is a continuous.*
Let $(u_n) \subset C$ be a sequense converge to u^*.
So

$$|(Tu_n)(t) - (Tu^*)(t)| \leq t^{\alpha-1}||g(u_n) - g(u^*)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u_n(s))ds$$

this imples

$$t^{1-\alpha}|(Tu_n)(t) - (Tu^*)(t)| \leq ||g(u_n) - g(u^*)| + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-1} f(s,u_n(s))ds$$

from lebesgue dominated convergence theorem and from the contininty of f and g we have

$$\| Tu_n - Tu^* \|_{\widetilde{C}} \to 0 \quad as \quad n \to +\infty$$

Hence the continuity of T.

Step 3 : *T is a compact.*

To do this just show that TC is relatively compact.

(a) *TC is bounded, because $TC \subset C$ and C is bounded.*

(b) *TC is equicontinuous.*

Let $u \in C$ and $t_1, t_2 \in [0, b]$ with $t_1 \leq t_2$

$$\begin{aligned}
|(Tu)(t_1) - (Tu)(t_2)| &\leq |u_0||t_1^{\alpha-1} - t_2^{\alpha-1}| + |g(u)||t_1^{\alpha-1} - t_2^{\alpha-2}| \\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1} |(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}||f(s, u_n(s))|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} |(t_2 - s)^{\alpha-1}||f(s, u_n(s))|ds \\
\\
&\leq (|u_0| + M')|t_1^{\alpha-1} - t_2^{\alpha-1}| + \\
&\frac{M}{\Gamma(\alpha)}\int_0^{t_1} |(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}|ds \\
&+\frac{M}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}ds \\
&\to 0 \quad as \quad t_1 \to t_2.
\end{aligned}$$

So TC relacively compact .

Then T is completely continuous.

let us now prove that the subset Ω is bounded.

Let $u \in \Omega$ and $t \in [0, b]$ we have

$$\begin{aligned}
|u(t)| &\leq \lambda|(Tu)(t)| \\
&\leq |(Tu)(t)| \\
&\leq |u_0|t^{\alpha-1} + |g(u)|t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1} f(s, u_n(s))ds
\end{aligned}$$

this implies :

$$t^{1-\alpha}|u(t)| \leq |u_0| + M' + \frac{Mb}{\Gamma(\alpha + 1)} = \rho$$

so $\| u \|_{\widetilde{C}} \leq \rho$.

It's whe gives Ω is bounded from theorem (schaefer), the operator T has at least one fixed point which is the solution to the problem (3.1)-(3.2).

3.2 Boundary value problems with non local condition for $\alpha \in]1,2]$

We consider the following nonlocal problem

$$^{RL}D^{\alpha}_{0+}u(t) = f(t,u(t)), \quad t \in [0,b] \tag{3.3}$$

$$\lim_{t \overset{>}{\to} 0} t^{2-\alpha}u(t) = g(u), \qquad u(b) = u_b \tag{3.4}$$

such that $1 < \alpha < 2, u_b \in \mathbb{R}$, suppose the space

$$C([0,b],\alpha) = \{u \in C([a,b],\mathbb{R}, \lim_{t \overset{>}{\to} 0} t^{2-\alpha}u(t) \text{ existe } \}$$

with the norm $\| u \|_{\tilde{C}} = \sup_{t\in[a,b]} t^{\alpha-1}|u(t)|$

3.2.1 Existence of solutions :

Lemma 6. *:*

Let $h : [0,b] \to \mathbb{R}$ a continuous function. Then the linear problem

$$\begin{cases} ^{RL}D^{\alpha}_{0+}u(t) = h(t), & t \in [0,b] \\ \lim_{t \overset{>}{\to} 0} t^{1-\alpha}u(t) = g(u) \end{cases}$$

admits only one solution given by

$$\begin{aligned} u(t) &= \frac{u_b}{t^{\alpha+1}}t^{\alpha+1} - \frac{1}{b}g(u)t^{\alpha-1} + \frac{t^{\alpha-1}}{\Gamma(\alpha)}b^{1-\alpha}\int_0^t |(t-s)^{\alpha-1}h(s)ds \\ &+ g(u)t^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \end{aligned}$$

Proof 16. *By lemma 1.2 and from $^{RL}D^{\alpha}_{0+}u(t) = h(t)$, we have $I^{\alpha}_{0+}(^{RL}D^{\alpha}_{0+}u(t)) = I^{\alpha}_{0+}h(t))$, i.e. :*

$$u(t) = c_1t^{\alpha-1} + c_2t_{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_0^t |(t-s)^{\alpha-1}h(s)ds$$

and

$$t^{2-\alpha}u(t) = c_1t + c_2 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds$$

with c_1, c_2 are constants.

From $\lim_{t \overset{>}{\to} 0} t^{2-\alpha}u(t) = g(u)$ we have $c_2 = g(u)$ and from $u(b) = u_b$ we obtain

$$c_1b^{\alpha-1} + g(u)b_{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}h(s)ds = u_b$$

So

$$c_1 = \frac{u_b}{b^{\alpha-1}} - \frac{1}{b}g(u) - \frac{1}{\Gamma(\alpha)}b^{1-\alpha}\int_0^b (b-s)^{\alpha-1}h(s)ds.$$

Then

$$\begin{aligned} u(t) &= \frac{u_b}{b^{\alpha-1}}t^{\alpha-1} - \frac{1}{b}g(u)t^{\alpha-1} + \frac{t^{\alpha-1}}{\Gamma(\alpha)}b^{1-\alpha}\int_0^b (b-s)^{\alpha-1}h(s)ds \\ &+g(u)t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \end{aligned}$$

Theorem 11. *Assume that the assumptions (C1), (C4) are satisfied. If*

$$2k'b^{\alpha-2} + \frac{k(b^{\alpha} + b^{2\alpha-1})}{\Gamma(\alpha+1)} < 1,$$

then the problem (3.3)-(3.4) has only solution on $[0, b]$.

Proof 17.

We consider the operator T *defined from* $\widetilde{C}([a, b], \mathbb{R})$ *in* $\widetilde{C}([a, b], \mathbb{R})$ *by*

$$\begin{aligned} (Tu)(t) &= \frac{u_b}{b^{\alpha-1}}t^{\alpha-1} - \frac{1}{b}g(u)t^{\alpha-1} + \frac{t^{\alpha-1}}{\Gamma(\alpha)}b^{1-\alpha}\int_0^t (b-s)^{\alpha-1}f(s, u(s))ds \\ &+g(u)t^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, u(s))ds \end{aligned}$$

We show that T *is contraction.*

Let $u, v \in \widetilde{C}([a, b], \mathbb{R})$ *then for all* $t \in [a, b]$, *we have*

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| &\leq \frac{1}{b}|g(u) - g(v)|t^{\alpha-1} + \frac{b^{1-\alpha}}{\Gamma(\alpha)}t^{\alpha-1}\int_0^t (t-s)^{\alpha-1}|f(s, u(s))|ds \\ &+|g(u) - g(v)|t^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s, u(s))|ds \\ &\\ &\leq \frac{k'}{b}|u - v|t^{\alpha-1} + \frac{b^{1-\alpha}}{\Gamma(\alpha)}t^{\alpha-1}\int_0^b (b-s)^{\alpha-1}k|u - v|ds \\ &+k'|u - v|t^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}k|u - v|ds \end{aligned}$$

his implies

$$\begin{aligned}|t^{2-\alpha}(Tu)(t)-t^{2-\alpha}(Tv)(t)| &\leq \frac{k'}{b}t|u-v|+\frac{b^{1-\alpha}}{\Gamma(\alpha)}t\int_0^b (b-s)^{\alpha-1}k|u-v|ds\\ &+k'|u-v|+\frac{1}{\Gamma(\alpha)}t\int_0^t (t-s)^{\alpha-1}k|u-v|ds\\ \\ &\leq \frac{2k'}{b}t^{\alpha-2}t^{2-\alpha}|u-v|+\frac{b^{1-\alpha}}{\alpha\Gamma(\alpha)}bkt^{\alpha-2}t^{2-\alpha}|u-v|b^{\alpha}\\ &+\frac{k}{\alpha\Gamma(\alpha)}b^{\alpha}t^{\alpha-2}t^{2-\alpha}|u-v|\\ \\ &\leq 2k'b^{\alpha-2}\parallel u-v\parallel_{\widetilde{C}}+\frac{kb^2}{\Gamma(\alpha+1)}b^{\alpha-2}\parallel u-v\parallel_{\widetilde{C}}\\ &+\frac{kb^{\alpha}}{\Gamma(\alpha+1)}b^{\alpha-2}\parallel u-v\parallel_{\widetilde{C}}\\ \\ &\leq \left(2k'b^{\alpha-2}+\frac{k(b^{\alpha}+b^{2\alpha-2})}{\Gamma(\alpha+1)}\right)\parallel u-v\parallel_{\widetilde{C}}\end{aligned}$$

So

$$\parallel Tu-Tv\parallel_{\widetilde{C}}\leq \left(2k'b^{\alpha-2}+\frac{k(b^{\alpha}+b^{2\alpha-2})}{\Gamma(\alpha+1)}\right)\parallel u-v\parallel_{\widetilde{C}}$$

Hence the problem (3.3)-(3.4) has only one solution on $[0,b]$.

Theorem 12.

Suppose that $f\in C([a,b],\mathbb{R})$ *and the hypotheses (C2), (C3) are satisfied. Then the problem (3.3)-(3.4) has at least one solution.*

Proof 18. *:*

Let to be the operator T *defined in the previous Theorem 3. Now consider the subsets :*

$$C=\left\{u\in\widetilde{C}([0,b],\mathbb{R}),\parallel u\parallel_{\widetilde{C}}\leq\rho\right\}$$

with $\rho=|u_b|b^{2-\alpha}+2M'+\dfrac{2Mb^2}{\Gamma(\alpha+1)}$ *and*

$$\Omega=\{u\in C,u=\lambda Tu,0<\lambda<1\}$$

We will show that the operator T *is completely continuous from* C *in* C *:*

•$TC\subset C$ *:*

Let $u\in C$ *and* $t\in[0,b]$ *we have*

$$\begin{aligned}|(Tu)(t)| &\leq |u_b|\frac{t^{\alpha-1}}{b^{\alpha-1}}+\frac{1}{b}|g(u)|t^{\alpha-1}+\frac{b^{2-\alpha}}{\Gamma(\alpha)}t^{\alpha-1}\int_0^b |(b-s)^{\alpha-1}|f(s)|ds\\ &+|g(u)|t^{\alpha-2}+\frac{1}{\Gamma(\alpha)}\int_0^t |(t-s)^{\alpha-1}|f(s)|ds\end{aligned}$$

this implies

$$\begin{aligned} t^{2-\alpha}|(Tu)(t)| &\leq |u_b|\frac{t}{t^{\alpha-1}} + \frac{t}{b}M' + \frac{b^{1-\alpha}tM}{\alpha\Gamma(\alpha)}b^{\alpha} + M' + \frac{b^{2-\alpha}Mb^{\alpha}}{\alpha\Gamma(\alpha)} \\ &\leq |u_b|b^{2-\alpha} + 2M' + 2\frac{Mb^2}{\Gamma(\alpha+1)} \end{aligned}$$

So

$$\| Tu \|_{\tilde{C}} \leq |u_b|b^{2-\alpha} + 2M' + 2\frac{Mb^2}{\Gamma(\alpha+1)}$$

then $Tu \in C$.

• T *is continuous :*

Let $(u_n) \subset C$ *be a sequence converge to* u^*.

So

$$\begin{aligned} |(Tu_n)(t) - (Tu^*)(t)| &\leq 2b^{\alpha-2}|g(u_n) - g(u^*)| + \frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}|f(s,u_n(s)) - f(s,u^*(s))|ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u_n(s)) - f(s,u^*(s))|ds \end{aligned}$$

From the continuity of f *and* g *we have*

$$\| (Tu_n)(t) - (Tu^*)(t)| \to 0 \quad as \quad n \to +\infty$$

So

$$\| Tu_n - Tu^* \|_{\tilde{C}} \to 0 \quad as \quad n \to +\infty$$

Hence the continuity of T.

• T *is compact :*

it's sufficient to show to show that TC *is relatively compact.*

(a) *It's clear thar* TC *is bouded.*

(b) *We show that* TC *is equicontinuous.*

Let $u \in C$ *and* $t_1, t_2 \in [0,b]$ *with* $t_1 \leq t_2$

$$\begin{aligned} |(Tu)(t_1) - (Tu)(t_2)| &\leq \frac{|u(s)|}{b^{\alpha-1}}|t_1^{\alpha-1} - t_2^{\alpha-1}| + 2|g(u)||t_1^{\alpha-2} - t_2^{\alpha-2}| + \frac{Mb}{\Gamma(\alpha)}|t_1^{\alpha-1} - t_2^{\alpha-1} \\ &+ \frac{M}{\Gamma(\alpha)}\left(\int_0^{t_1} |(t_1-s)^{\alpha-1} - (t_1-s)^{\alpha-1}|ds + \int_{t_1}^{t_2} |(t_2-s)^{\alpha-1}ds\right) \\ &\to 0 \qquad as \qquad |t_1 - t_2| \to 0 \end{aligned}$$

So TC *is relatively compact.*

Now we show that Ω is bounded :

Let $u \in \Omega$ and $t \in [0, b]$,we have

$$\begin{aligned}|u(t)| &= \lambda|(Tu_n)(t)| \\ &\leq |(Tu)(t)| \\ &\leq |u_n|b^{2-\alpha} + 2M' + \frac{2Mb^2}{\Gamma(\alpha+1)} = \rho\end{aligned}$$

this implies $t^{2-\alpha}|u(t)| \leq b^{2-\alpha}\rho$ i.e $\parallel u \parallel_{\tilde{C}} \leq b^{1-\alpha}\rho$, it's who gives Ω is bounded.
From theorem (Schaefer) the operator has at least one fixed point which is the solution to the problem (3.3)-(3.4) .

As first application, we consider the function g on the from

$$g(u) = \sum_{i=1}^{n} c_i u(t_i)$$

where $c_1, c_2, ..., c_n$ are given constants with $n \in \mathbb{N}^*$ and $0 < t_1 < t_2 < ... < t_n \leq b$.
Consider the multi-point fractional boundary value problem :

$$^{RL}D^{\alpha}_{0^+}u(t) = f(t, u(t)), t \in [0, b] \tag{3.5}$$

$$\lim_{t \overset{>}{\to} 0} t^{2-\alpha}u(t) = \sum_{i=1}^{n} c_i u(t_i), \qquad u(b) = u_b \tag{3.6}$$

with $1 < \alpha \leq 2$ and $u_b \in \mathbb{R}$ then the following result is a direct consequence of Theorem 3.

Corollary 1.
assume that (C1) , (C4) hold and suppose that

$$0 < 2b^{\alpha-1}\sum_{i=1}^{n} c_i + \frac{k(b^{\alpha} + b^{2\alpha-2})}{\Gamma(\alpha+1)} < 1.$$

Then problem (3.5)-(3.6) has only one solution on $[0, b]$.

Remark 3.
In this Corollary, we have

$$|g(u) - g(v)| \leq \sum_{i=1}^{n} c_i|u - v|$$

Indeed, if we put k' instead of $\sum_{i=1}^{n} c_i$ that is (C4), we get that the operator T identifier in the previous theorem 3, it satisfies the following inequality :

$$\| Tu - Tv \|_{\widetilde{C}} \leq \left(2\sum_{i=1}^{n} c_i + \frac{k(b^{\alpha} + b^{2\alpha-2})}{\Gamma(\alpha+1)} \right) \| u - v \|_{\widetilde{C}}$$

and with $0 < 2b^{\alpha-2}\sum_{i=1}^{n} c_i + \frac{k(b^{\alpha} + b^{2\alpha-2})}{\Gamma(\alpha+1)} < 1$, T is a contraction.
So the problem (3.5)-(3.6) has only one solution on $[a, b]$.

Exemple 3. *Consider the nonlocal boundary value problem :*

$$^{RL}D_{0^+}^{\alpha} u(t) = \frac{t^{1-\alpha}e^{-t}|u(t)|}{(9+e^t)(1+|u(t)|)}, t \in [0,1] \tag{3.7}$$

$$\lim_{t \overset{>}{\to} 0} t^{2-\alpha} u(0) = \sum_{i=1}^{n} c_i u(t_i), \qquad u(1) = 0 \tag{3.8}$$

such that $1 < \alpha \leq 2$ *and* $0 < t_1 < t_2 < ... < t_n < 1, c_i, i = 1, 2, ..., n$ *are positives constants with*

$$\sum_{i=1}^{n} c_i < \frac{2}{5}$$

In this case, let's choose $f(t, x) = \frac{t^{1-\alpha}e^{-t}x}{(9+e^t)(1+x)}$ *for all* $(t, x) \in [0, 1] \times \mathbb{R}_+$, *and* $g(u) = \sum_{i=1}^{n} c_i u(t_i)$.
Let $x, y \in \mathbb{R}_+$ *and* $t \in [0, 1]$ *then, we have :*

$$\begin{aligned} |t^{\alpha-1}f(t,x) - t^{\alpha-1}f(t,y)| &= \frac{e^{-t}}{9+e^t}\left|\frac{x}{1+x} - \frac{y}{1+y}\right| \\ &= \frac{e^{-t}}{9+e^t}\frac{|x-y|}{(1+x)(1+y)} \\ &\leq \frac{e^{-t}}{9+e^t}|x-y| \\ &\leq \frac{1}{10}|x-y|. \end{aligned}$$

Therefore the condition (C1) satisfied with $k = \frac{1}{10}$ *and we have,*

$$|g(x) - g(y)| \leq \sum_{i=1}^{n} c_i |x - y|$$

thus (C4) satisfied with $k' = \sum_{i=1}^{n} c_i$.
On the other hand, we get

$$2b^{\alpha-1}\sum_{i=1}^{n} c_i + \frac{k(b^{\alpha} + b^{2\alpha-2})}{\Gamma(\alpha+1)} = 2\sum_{i=1}^{n} c_i + \frac{1}{5\Gamma(\alpha+1)} < 1,$$

(because $\Gamma(\alpha+1) > 1$ *with* $1 < \alpha \leq 2$ *).*

According to previous corollary, the problem (3.7)-(3.8) has only one solution on $[0,1]$*.*

Exemple 4.

Consider the following boundary value problem :

$$^{RL}D^{\alpha}_{0^+}u(t) = \frac{|u(t)|}{\sqrt{t}(10+t)^2(1+|u(t)|)}, t \in]0,1] \tag{3.9}$$

$$\lim_{t \xrightarrow{>} 0} t^{\frac{1}{2}}u(0) = \frac{1}{5}u\left(\frac{1}{4}\right) + \frac{1}{20}u\left(\frac{1}{2}\right), \qquad u(1) = 0 \tag{3.10}$$

In this case, $\alpha = \frac{3}{2}, \Gamma(\alpha+1) = \Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4} > 1$ *and* $g(u) = \frac{1}{5}u\left(\frac{1}{4}\right) + \frac{1}{20}u\left(\frac{1}{2}\right)$.
We apply previous Corollary to show that the problem (3.9)-(3.10) has only one solution on $[0,1]$*.*
Let $f(t,u(t)) = \dfrac{|u(t)|}{\sqrt{t}(10+t)^2(1+|u(t)|)}, \quad t \in]0,1]$ *then*
(C1) Let $x, y \in \mathbb{R}_+$ *and* $t \in]0,1]$ *then, we have*

$$\begin{aligned} |t^{\frac{1}{2}}f(t,x) - t^{\frac{1}{2}}f(t,y)| &= \frac{1}{(10+t)^2}\left|\frac{x}{1+x} - \frac{y}{1+y}\right| \\ &\leq \frac{1}{100}|x-y|. \end{aligned}$$

This the condition (C1) satisfied with $k = \frac{1}{100}$

$$\begin{aligned} (C4)|g(u) - g(v)| &\leq \left(\frac{1}{5} + \frac{1}{20}\right)|u-v| \\ &\leq \frac{1}{4}|u-v|. \end{aligned}$$

Then the condition (C4) satisfied with $k' = \frac{1}{4}$ *and as*

$$2b^{\alpha-1}\sum_{1}^{2} c_i + \frac{b(b^{\alpha} + b^{2\alpha-2})}{\Gamma(\alpha+1)} = 2\left(\frac{1}{4}\right) + \frac{1}{50\Gamma\left(\frac{5}{2}\right)} < 1.$$

We conclude that the problem (3.9)-(3.10) has only one solution on $[0,1]$*.*

CONCLUSION AND PERSPECTIVES

In this work, we dealt with the existence and uniqueness of solutions to fractional differential equations associated with local and non-local conditions.

Initially, we introduced the concepts of fractional calculus and some basic tools from functional analysis and some classic fixed point theorems.

In the different chapters of this research, to study the proposed problems, we transformed each problem into an operator equation, in this case we sometimes use the Green function associated with the imposed problem with applying the fixed point theory to prove the results provided.Other times we choose sufficient conditions so that the operators associated with the proposed problems satisfy the appropriate fixed point theory to demonstrate our results.

This work has clarified certain methods for the existence and uniqueness of solutions to some nonlinear fractional problems.Its aim is to provide the desired benefit to researchers in this field of study or to use its results for applications in different fields.

We hope that this research will benefit researchers studying in the specialty of differential equations, and that they will have support in other research that is an extension of this research.

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Résumé

Dans ce mémoire, nous avons étudié l'existence et l'unicité des solutions aux équations différentielles non linéaires d'ordre fractionnaire du type Riemann-Liouville associés des conditions initiales et des conditions aux limites locaux et non locaux posés sur des intervalles bornés. Nous avons utilisé quelques méthodes. D'un côté, pour étudier ces équations, nous les avons transformés en une équation opérationnelle et appliquer la théorie de point fixe pour trouver les solutions. D'autre part, nous avons choisi des conditions suffisantes pour démontrer nos résultats. Dans toutes les phases de notre travail nous avons soumis quelques exemples qui donnent des applications pour les résultats obtenus.

Mots clés: Équation différentielle fractionnaire ; Existence ; Unicité ; Problème aux limites ; Conditions non locales ; Théorème du point fixe.

Abstract

In this memoire, we studied the existence and uniqueness of solutions to nonlinear differential equations of fractional order of the Riemann-Liouville type associated with initial conditions, local and non-local boundary condition placed on bounded intervals. We used a few methods. On the one hand, to study these equations, we transformed them into an operational equation and applied the fixed point theory to find the solutions. On the other hand, we chose sufficient conditions to demonstrate our results. In all phases of our work we have submitted some examples that give applications for the results obtained. **key words:** Fractional differential equation; Existence; Uniqueness; Boundary value problem; Nonlocal conditions; Fixed point theorem.

ملخص

درسنا في هذا البحث وجود ووحدانية الحلول للمعادلات التفاضلية الكسرية لريمان-ليوفيل المرتبطة بالشروط الإبتدائية والشروط الحدية المحلية وغير المحلية الموضوعة على المجالات المنتهية. استخدمنا بعض الأساليب، من جهة لدراسة هذه المعادلات نقوم بتحويلها إلى معادلات تكاملية ونطبق نظرية النقطة الصامدة لإيجاد الحلول. ومن ناحية أخرى، اخترنا الشروط الكافية لبرهان نتائجنا. في جميع مراحل عملنا قدمنا بعض الأمثلة التي تعطي تطبيقات للنتائج المحصل عليها.

الكلمات المفتاحية: المعادلة التفاضلية الكسرية، الوجود، الوحدانية، الجملة الحدية، الشروط غير المحلية، نظرية النقطة الصامدة.