**République Algérienne Démocratique et Populaire**

**Ministère de l'Enseignement Supérieur et de la Recherche Scientifique**

**Université Echahide Hamma Lakhdar – El-oued**

**Faculté des Sciences Exactes**

**Département de Mathématiques**


**Thèse en vue de l'obtention du diplôme de doctorat LMD en mathématiques**

**Spécialité : Equations différentielles et applications**

**SUJET DE LA THESE :**

# Sur le degré topologique et ses applications à quelques problèmes aux limits non-linéaires

**Présentée Par :**


**Azouzi Mounira**


**Devant le jury composé de:**

| | | | |
|---|---|---|---|
| Nisse Lamine | Prof. | Univ. Eloued | Président |
| Guedda Lamine | MCA. | Univ. Eloued | Encadreur |
| Akrout Kamel | Prof. | Univ. Tebessa | Co-Encadreur |
| Beloul Said | Prof. | Univ. Eloued | Examinateur |
| Abdelkader Amara | MCA. | Univ. Ouargla | Examinateur |
| Tedjani Menacer | Prof. | Univ. Biskra | Examinateur |

**Année universitaire : 2022/2023**

People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

Echahide Hamma Lakhdar University – El-oued

Faculty of Exact Sciences

Mathematics Department

**Thesis with a view to obtaining the LMD doctorate degree in mathematics**

**Speciality: Differential equation and applications**

**SUBJECT OF THE THESIS:**

# On the topological degree and its applications for some nonlinear boundary value problems

**Presented by :**

**Azouzi Mounira**

**In front of the jury composed of :**

| | | | |
|---|---|---|---|
| Nisse Lamine | Prof. | Univ. Eloued | President |
| Guedda Lamine | MCA. | Univ. Eloued | Supervisor |
| Akrout Kamel | Prof. | Univ. Tebessa | Co-Supervisor |
| Beloul Said | Prof. | Univ. Eloued | Examiner |
| Abdelkader Amara | MCA. | Univ. Ouargla | Examiner |
| Tedjani Menacer | Prof. | Univ. Biskra | Examiner |

**University year : 2022/2023**

# ملخص

لقد درسنا في هذه الأطروحة بعض الخصائص الأساسية للدرجة الطوبولوجية للتذبذبات المتراصة بالنسبة لمؤثرالوحدة أوالتى تتعلق بمؤثر فريدهولم ذات الدليل صفر.

بداية مع الجزء النظري الذي سنقدم فيه بعض المفاهيم الأساسية للتحليل الدالي والتذكير بالدرجة الطوبولوجية لبراور و لوري شودار ثم درجة المصادفة لموان وأهم الخصائص والنظريات المتعلقة بها .

الجزء الثاني مخصص بالكامل لتطبيق هذه الأداة لحل المشاكل ذات الشروط الحدية المرتبطة بالمعادلات التفاضلية الكسرية الغير الخطية في حالة التجاوب باستخدام نظرية المصادفة لموان.

**الكلمات المفتاحية :** الدرجة الطبولوجية، مؤثر فريدهولم، حالة التجاوب، نظرية المصادفة لموان.

# Résumé

Cette thèse porte sur l'étude de certaine propriétés essentielles du degré topologique pour les perturbations compactes de l'identité, ou les perturbations compactes relativement à un opérateur de Fredholm d'indice zéro.
On commencé la partie théorique par quelques préliminaires d'analyse fonctionnelle et des rappels sur le degré topologique de Brouwer et de Leray-Schauder puis degré de coincidence de Mawhin avec ses propriétés et les theorémes fondamentales.
Dans la deuxième partie de ce manuscrit on a appliqué cette outile sur quelques problèmes aux limites associés des équations différentielles fractionnaire non linéaires dans le cas de resonanse en utilisant la théorie de coincidence de Mawhin.

**Mots clés**: Degré topologique, Opérateur de Fredholm, Cas de résonance, Théorie de coincidence de Mawhin.

# Absract

In this thesis, we have studied some essential properties of the topological degree for compact perturbations of the identity or respect to a Fredholm operator of index zero. This theoretical part occupies the first part of the thesis, which begins with some preliminaries about functional analysis. Next we presented preliminaries about the topological degree of Brouwer and Leray-Schauder, Mawhin's coincidence degree and some fundamental properties and theories related to it .

The second part is entirely devoted to the applications of this tool to the resolution of boundary problems associated with nonlinear fractional differential equations at resonance case by using the Mawhin coincidence theory.

**Keywords**: Topological degree, Fredholm operator, Resonance case, Mawhin coincidence theory.

# Acknowledgement

First of all, I would like to thank **ALLAH** for his blessing and success that made me complete this work

In these lines, I would like to extend a special thanks to the people who helped me enter the doctoral phase first and then formulate my thesis, and the first of them is the doctor **Guedda Lamine** who suggested this topic to me and helped me a lot in completing the thesis.

I would also like to thank him for his kindness, availability and time spent on my work. I would also like to thank Professors:**Lamine Nisse, Kamel Akrout, Said Biloul, Abdelkader Amara** and **Tedjani Menacer**.

I also thank the members of the Department of Mathematics for allowing me to work in good conditions during the performance of my work.

Thanks also to all the teachers who helped me during my studies, not forgetting their valuable advice.

I also thank everyone from near and far contributed to the finalization of this work.

A big thank you to all of my family who were always there when I needed them.

# Dedication

I dedicate this thesis

To my father **Hessan and my mother Lazaar Fadda**, without them I wouldn't have had this degree and such joy, all words cannot express my thanks and gratitude to them.

A special feeling of gratitude to who I will complete my life with him **Achab Abderahmane**.

To my brothers **Eussama, Billal, Souhaib and Hamza**

To my friends for encouraging me.

# General notations

| | |
|---|---|
| $\|.\|$ | The norm. |
| $\oplus$ | Direct sum. |
| $\mathbb{R}$ | The set of real numbers. |
| $\dim(X)$ | Dimension of X. |
| $\text{codim}(X)$ | Codimension of X. |
| $\bar{\Omega}$ | The closure of $\Omega$ . |
| $\partial\Omega$ | Its boundary . |
| $\frac{X}{E}$ | Quotient space of X by E. |
| dist | Distance . |
| $(X, d)$ | Metric space. |
| $B(0, R)$ | The open ball of center 0 and radius R . |
| $C[a, b]$ | The space of continuous functions. |
| $\Phi_p$ | Laplacian operator. |
| $I_{0^+}^{\beta} f(t)$ | The Riemann-Liouville fractional integral of order $\beta > 0$ of a function $f(t)$. |
| $D_{0^+}^{\beta} f(t)$ | The Riemann-Liouville fractional derivative of order $\beta > 0$ of a function $f(t)$. |
| ${}^{C}D_{0^+}^{\beta} f(t)$ | The Caputo fractional derivative of order $\beta > 0$ of a function $f(t)$. |
| $\mathcal{J}_a^{\alpha,\rho} f(t)$ | The left generalized proportional fractional integral of f(t). |

| | |
|---|---|
| ${}^{C}\mathfrak{D}_{a}^{\alpha,\rho} f\,(t)$ | The left generalized proportional fractional derivative of Caputo of f(t) . |
| $\gamma(\alpha, t)$ | The lower incomplete Gamma function. |
| $\mathfrak{P}(\alpha, t)$ | The lower regularized incomplete Gamma function. |
| max | Maximum. |
| $C^{\theta}[0, 1]$ | The space of all functions $\theta$ times continuously differentiable on [0,1]. |
| $\deg_{\mathrm{B}}$ | Brouwer degree. |
| $\deg_{LS}$ | Leray-Schuder degree. |
| $\deg_{L}$ | The coincidence degree with respect to a Fredholm operator L of index 0. |
| $J_f$ | The Jacobian of f. |
| $S_f(\Omega)$ | The set of critical points of f. |
| $\operatorname{sgn} f$ | Sign of f. |
| $BVPs$ | Boundary value problem . |
| $\operatorname{dom} L$ | Domaine of definition of $L$. |
| $\ker L$ | Kernel of $L$. |
| $\operatorname{Im} L$ | Image of $L$. |
| $\operatorname{ind}(L)$ | The index of a Fredholm operator $L$. |
| $l_2$ | The space of sequences of complex numbers $x = (x_1, x_2, ...)$ with $(\sum_{i=1}^{\infty} \lvert x_i\rvert^2 < +\infty)$. |

# Contents

# General Introduction

Our aim in this thesis is to present some new existence results for some resonant problems with fractional differential equations associated with boundary conditions of multipoint and integral type with linear or nonlinear differential operators with one-dimensional kernel and then expand to a two-dimensional kernal.

Topological methods are used in nonlinear analysis in both finite and infinite dimensions. This include fixed point theory, and coincidence degree theory. These two methods are indispensable in the study of qualitative properties of solutions to various types of analytical problems. In particular, those pertaining to questions of existence and multiplicity of solutions, see [20, 21].

The topological degree is a tool that describes the number of solutions for the equation $f(x) = y_0$ in a given open bounded set $\Omega \subset X$, where $f : \Omega \subset X \to X$ is a continuous function $y_0 \notin f(\partial\Omega)$, $X$ is a topological space, and be used to prove existence of a second nontrivial solution, and is a generalization of the mean values theory as follows :

Let $\Omega \subset R^N$ be an open bounded subset and $f : \bar{\Omega} \to R^N$ in $C^1(\Omega) \cap C^0(\bar{\Omega})$ and $y_0 \in R^N$. We consider the problem:

$$\text{find} \quad x \in \Omega, f(x) = y_0. \tag{Pr}$$

**Particular cases**$(N = 1)$

$\Omega = ]0, 1[$ and $f : \Omega \to \mathbb{R}$ is a differentiable mapping verifying this hypothesis:

$$\text{For any solution} \quad x \quad \text{of} \quad (P_r), \quad f'(x) \neq 0. \tag{H}$$

we introduce the integer :

$$\deg(y_0) = \begin{cases} \sum_{i\in I} \operatorname{sgn}(f'(x_i)), & \text{if } \{x_i, i \in I\} \text{ is the set of solutions of } (P_r) \text{ in } \Omega, \\ 0, & \text{if the problem } (P_r) \text{ has no solution.} \end{cases}$$

Let us give an example:

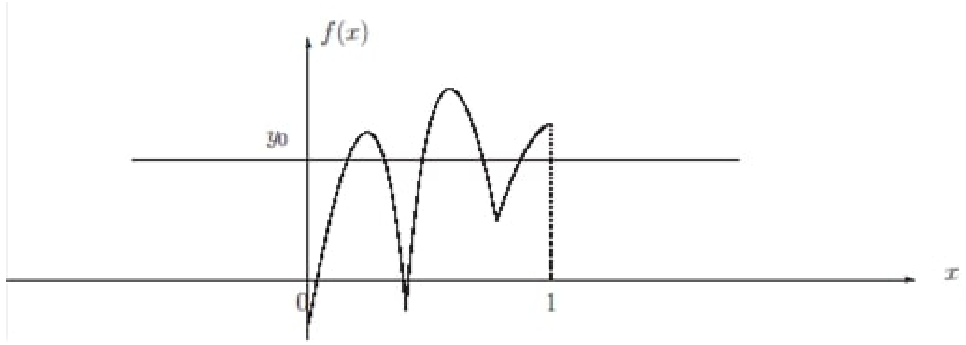


Figure 1: $\deg(y_0) = +1$

There are different types of degree for different types of maps: e.g. for continuous maps in fnite-dimensional spaces there is the **Brouwer degree** in the space $E$ such that $\dim E < \infty$ the **Leray-Schauder degree** for compact mappings in Banach spaces, the **coincidence degree** and various other types, see [5].

The Mawhin's degree theory permits the use of an approach of topological degree type to problems which can be written as an abstract operator equation of the form:

$$Lx = Nx, \tag{1}$$

where $L$ is a Fredholm operator of index zero and $N$ is a nonlinear operator. When the homogeneous problem associated to (1) admits nontrivial solutions, i.e., $\ker L \neq \{0\}$, the operator equation is called resonant problem, else it is nonresonant problem $(\ker L = \{0\})$. This theory was first established by Brouwer in 1912 in finite dimensional spaces. In 1934 Leray and Schauder generalized Brouwer degree theory to compact perturbations of identity in infinite Banach space and established the so called Leray Schauder degree, as follows: if $\Omega$ is a bounded open subset in $X$, $T : \Omega \to X$ is a compact operator and $y_0 \notin (I - T)(\partial\Omega)$, according to proposition 8.1 of [10], $T$ is the uniform limit of the sequence $(T_n)_{n\in\mathbb{N}}$ of finite rank operators, which means that for every $\delta > 0$ there exist $\varepsilon > 0$ and a compact operator $T_\varepsilon$ such that $\operatorname{Im} T_\varepsilon \subset X_\varepsilon$, where $X_\varepsilon$ is a subspace of finite dimension in $X$ containing $y_0$ and satisfies

$$\sup_{x\in\Omega} \|T_\varepsilon(x) - T(x)\| < \delta,$$

for $\delta$ sufficiently small $(\delta < \frac{1}{2}dist(y_0, (I-T)(\partial\Omega))$, the Brouwer degree

$$\deg_B((I-T_\varepsilon)_{|X_\varepsilon}, \Omega\cap X_\varepsilon, y_0),$$

allows to defined the degree of Leray-Schauder $\deg_{LS}(I-T,\Omega,y_0)$.
In 1972, J. Mawhin has developed a method to solve (1) in his famous paper "Topological degree and boundary value problems for nonlinear differential equations", see [5] and established a new topological degree theory for couples $(L,N)$ in $\Omega$, called the coincidence degree and defined as follows :

$$\deg_L(L+N,\Omega) = \deg_{LS}(I-T,\Omega,0).$$

In [3, 11, 12, 17, 23, 24, 25], the authors have investigated resonant problems with linear differential operators and one-dimensional kernels. As we can see the situation becomes more problematic when dealing with non-linear two-dimensional operators like the case of $p$-Laplacian boundary value problems. See [8, 16, 18, 19, 28], for more details.
Most natural phenomena have recently been described by some type of boundary value problems for differential equations like physical phenomena, chemistry, engineering, and control of dynamical systems, etc. See [2, 4, 6, 22, 26, 27], that's why their study is an important research area despite it's difficult as long as there is no general method to apply.
Let us give a brief outline of each chapter of the thesis:
The **first chapter** presents the fundamental results about projections in finite-dimensional subspaces, compact operators, and some auxiliary and basic tools which we need in this dissertation including versions of Ascoli-Arzela theorem and lemmas from fractional calculus theory.
The **second chapter** is devoted to presenting the concept of the topological degree and its properties as well as coincidence degree theorem. J.Mawhin with it's proof.

The **third chapter** is devoted to the study the following multi-point boundary value problem for a nonlinear fractional differential equation with a p-Laplacian operator at resonance

$$\left(\phi_p\left(D_{0^+}^{\alpha}x\left(t\right)\right)\right)' = f\left(t, x\left(t\right), D_{0^+}^{\alpha-1}x\left(t\right)\right), \quad t \in [0,1],$$

$$x\left(0\right) = D_{0^+}^{\alpha}x\left(1\right) = 0,$$

$$D_{0^+}^{\alpha-1}x\left(1\right) = \sum_{i=1}^{i=m-2} \beta_i D_{0^+}^{\alpha-1}x\left(\eta_i\right),$$

where $1 < \alpha < 2$, $0 < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < 1$, $\beta_i \in \mathbb{R}_+$, for $i = 1, 2, 3, \ldots, m-2$ $(m \geq 3)$, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and the $p$-Laplacian is defined by $\phi_p\left(s\right) = |s|^{p-2}s$ , $(p > 1)$. Recall that $\phi_p : \mathbb{R} \to \mathbb{R}$ is an odd continuous, strictly increasing operator with $\phi_p^{-1} = \phi_q\left(s\right)$ $(\frac{1}{p} + \frac{1}{q} = 1)$. The proofs are based on Mahwin's theory of coincidence.

In the **forth chapter** we have presented a result of existence of solutions for the following generalized proportional fractional differential equation, with multi-point boundary conditions at resonance cas of the form:

$${}^{c}\mathfrak{D}_0^{\alpha,\rho}u(t) = f(t, u(t), {}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(t)), \qquad 0 < t < 1,$$

$$u(0) = 0,$$

$${}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(1) = \sum_{i=1}^{i=m} \sigma_i \, {}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(\eta_i),$$

where ${}^{c}\mathfrak{D}_0^{\alpha,\rho}$ denote the generalized proportional fractional derivative of Caputo type of order $\alpha$, with $\alpha \in (1,2]$, $\rho \in (0,1]$, $0 < \eta_i < 1$, $\sigma_i \in \mathbb{R}$, $\sum_{i=1}^{i=m} \sigma_i = 1$, $m \in \mathbb{N}^*$,and $f : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a given continuous function, where we have used the Mawhin coincidence degree theory.

The **fifth chapter** using the Mawhin coincidence degree theory, we have established sufficient conditions for the existence of solutions for the following system :

$$\begin{cases} D_{0^+}^{\beta}u_1(t) = \phi_q\left(u_2(t)\right), \\ u_2'(t) = -g(t)f\left(t, u_1(t), \phi_q u_2(t)\right), \\ u_2(0) = u_2(T) = \int_0^T g(t)u_2(t)dt, \end{cases}$$

where $0 \leq \beta < 1$, $D_{0^+}^{\beta}$ is the Riemann-Liouville fractional derivative of order $\beta$, $g \in L^1[0,T]$ with $g(t) > 0$ and $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is a $g$-Caratheodory function.

# Chapter 1

# Preliminaries

In this chapter we provide the basic notions and results that will be used in the sequel such the compact operators, projections, and some properties of the fractional derivative.

## 1.1 Direct sum and projections

Let $X$ and $Y$ be two normed vector spaces, $\Omega$ be an open set of $X$.

### 1.1.1 Direct sum

**Definition 1.1.1.** *[7] Let $E, F \subset X$ be subspaces of vector space $X$. We say that $X$ is the direct sum of $E$ and $F$, or $X = E \oplus F$ if the following two conditions holds:*
*(1) $E + F := \{x + y \mid x \in E, y \in F\} = X$,*
*(2) $E \cap F = 0$ " $E$ and $F$ are independent."*

**Example 1.1.1.** *Let $E, F \subset \mathbb{R}^3$ be subspaces of vector space $\mathbb{R}^3$, such that :*

$$\begin{aligned} E = & \quad \langle (1,1,1) \rangle, \\ F = & \quad \{(x,y,0)/x,y \in \mathbb{R}\}, \end{aligned}$$

*so $\mathbb{R}^3 = E \oplus F$. Car*
*$E + F = \mathbb{R}^3$,*
*$E \cap F = (0,0,0)$.*

### 1.1.2 Quotient space

If $F$ is a closed subspace of $X$, then the quotient space $\frac{X}{F}$ has a normed vector space structure whose norm is defined by :

$$\|\pi_F(x)\|_{\frac{X}{F}} = \inf_{h \in F} \|x+h\|_X = d(x,F),$$

where $\pi_F : X \to \frac{X}{F}$, $\pi_F(x) := x+F = \{x+h \ / \ h \in F\}$ is the canonical quotient mapping.

**Proposition 1.1.1.** *[21] If $X$ is a Banach space, then so is $\frac{X}{F}$.*

### 1.1.3 Codimension of a vector subspace

**Definition 1.1.2.** *[7] If the quotient space $\frac{X}{F}$ has finite dimension, we say that the closed vector subspace $F$ of $X$ is of finite codimension in $X$ and we write*

$$\operatorname{codim}(F) = \dim(\frac{X}{F}).$$

**Proposition 1.1.2.** *[7] $\operatorname{codim}(F) = n < \infty$ if and only if there exists a closed subspace $E$ of $X$, such that*

$$X = F \oplus E \text{ and } \dim(E) = n.$$

### 1.1.4 Projections

Let $X$ be a vector space.

**Definition 1.1.3.** *[7] We say that a linear operator $P : X \to X$ is a projection if for all $x \in X$, we have $P(P(x)) = P^2x = P(x)$.*

**Proposition 1.1.3.** *[7] A linear operator $P : X \to X$ is a projection if and only if $(I-P)$ is a projection. Moreover, if the space $X$ is normed, then $P$ is continuous if and only if $(I-P)$ is continuous.*

***Proof.*** Let $P$ be a projection. So for all $x \in X$

$$\begin{aligned}(I-P)^2(x) &= (I-P)((I-P)(x)) \\ &= I(I-P)(x) - P(I-P)(x) \\ &= I(x-Px) - P(x-Px) \\ &= x - p(x) - p(x) + p^2(x) \\ &= x - 2p(x) + p^2(x) \\ &= x - p(x) = (I-P)(x),\end{aligned}$$

Reciprocally, if $(I-P)$ is a projection, $(I-(I-P)) = P$ is too. For the topological framework, as the identity is a continuous mapping and the sum of two continuous mappings is also continuous, then $P$ is continuous if and only if $(I-P)$ is. □

**Proposition 1.1.4.** *[7] If $P$ is a projection in $X$, then :*

$$\ker P = \operatorname{Im}(I-P) \quad \textit{and} \quad \operatorname{Im} P = \ker(I-P).$$

***Proof.*** We prove that $\ker P = \operatorname{Im}(I-P)$. If $x \in \ker P \Longrightarrow P(x) = 0$ then

$$(I-P)(x) = x - P(x) = x \Longrightarrow x \in \operatorname{Im}(I-P),$$

which implies

$$\operatorname{Ker} P \subset \operatorname{Im}(I-P),$$

next, if $x \in \operatorname{Im}(I-P)$, then

$$\begin{aligned}P((I-P)(x)) &= P(x) - P^2(x) \\ &= P(x) - P(x) = 0 \Longrightarrow (I-P)x \in \operatorname{Ker}(P),\end{aligned}$$

hence $\operatorname{Im}(I-P) \subset \ker P$, and then

$$\operatorname{Ker} P = \operatorname{Im}(I-P).$$

□

**Lemma 1.1.1.** *[7] If $E$ is a finite dimensional subspace of $X$, then there exists a continuous projection $P : X \to X$ such that $\operatorname{Im} P = E$.*

**Remark 1.1.** *[7] A topological space $X$ is separated (or Hausdorff) if for all $x, y \in X : x \neq y$, there exists $Vx, Vy$ open as $Vx \cap Vy = \phi$.*

**Lemma 1.1.2.** *[7] The image of any continuous projection in a Hausdorff space is closed. In particular, the images of continuous projections of the Banach spaces are closed.*

**Theorem 1.1.1.** *[7] If $P$ is a continuous projection in a topological vector space of Hausdorff $X$, then $X$ is the direct sum of $\operatorname{Im} P$ and $\ker P$, ( ie $X = \operatorname{Im} P \oplus \ker P$).*

## 1.2 Compact operators

Let $X, Y$ be two Banach spaces, $\Omega$ be an open set of $X$.

**Definition 1.2.1.** *[10] A mapping $T : \Omega \subset X \to Y$ is called compact if $\overline{T(\Omega)}$ is compact in $Y$*

**Proposition 1.2.1.** *[10] Let $(X, d)$ be a metric space. Then, a subset $M \subset X$ is compact if and only if every infinite sequence $(x)_n^\infty \subset M$ has a convergent subsequence in $M$.*

**Lemma 1.2.1.** *[10] A continuous mapping $T : \Omega \subset X \to Y$ is said to be completely continuous, if the image of any bounded subset $B$ of $\Omega$ is relatively compact.*

### 1.2.1 Finite rank operator

**Definition 1.2.2.** *[10] The operator $T : \Omega \subset X \to Y$ is said to be of finite rank, if there exists a finite dimensional subspace $F$ such that $\operatorname{Im}(T) \subseteq F \subseteq Y$.*

Let $T : \Omega \subset X \to Y$ be a finite rank operator. Then $T$ bounded implies that $T$ is completely continuous. (Because for all bounded subset $B \subset \Omega$, $\overline{T(B)}$ is a closed bounded subset of a finite dimensional subspace $F$.

**Remark 1.2.** *[7]*

1. *Any compact mapping is completely continuous. (Because for every bounded $B \subset \Omega$ we have $T(B) \subset T(\overline{\Omega})$). The converse is true if $\Omega$ is bounded.*

2. *Let $X, Y$ be two Banach spaces, $T : X \to Y$ be a linear mapping. Then it is sufficient that $T(B(0;1))$ is precompact, for $T$ to be compact. If at least one of spaces $X$ or $Y$ has finite dimension, then $T$ is compact if and only if $T$ is continuous.*

**Lemma 1.2.2.** *[7] Let $T$ compact continuous mappings on $\bar{\Omega}$, then for any $\epsilon > 0$, there exists a finite rank map $T^{\epsilon}$ on $\bar{\Omega}$ such that $\|T - T^{\epsilon}\| < \epsilon$.*

### 1.2.2 Ascoli-Arzela theorem

Let $(X, d_X)$ be a compact metric space and $(Y, d_Y)$ be a complete metric space. By $C(X, Y)$ we denote the vector space consisting of all continuous functions $f : X \to Y$.

**Definition 1.2.3.** *[7] The family $M \subset C(X, Y)$ is called equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ for all $x, y \in X$ satisfying $d_X(x, y) < \delta$ and all $f \in M$.*

**Theorem 1.2.1.** *[7] The family $M \subset C(X, Y)$ is relatively compact if and only if*

1. *$M$ is equicontinuous ,*

2. *for all $t \in X$, the set*

$$M(t) = \{x(t); x(.) \in M\},$$

*is relatively compact in $Y$.*

In particulary, If $Y$ is a finite dimensional Banach space, the second condition of Ascoli-Arzela theorem is equivalent to $M$ is uniformly bounded i.e., there exists $A > 0$ such that for all $x \in M$

$$\|x\|_{\infty} = \sup_{t \in X} \|x(t)\| \leq A.$$

**Definition 1.2.4.** *[7] (**Compact perturbations of the identity**)*
*A map of the form $f = I - T$, where $I$ is the identity operator and $T$ is a compact operator is said to be a compact perturbation of the identity or application of Leray Schauder.*

## 1.3 Fractional calculus

In this section we give some definitions and lemmas from the theory of fractional calculus. We begin with fractional derivative of Riemann-Liouville type, and next we present the Generalized Proportional fractional integrals and derivatives. These definitions are adopted from [1, 30].

### 1.3.1 Fractional derivative of Riemann-Liouville type

**Definition 1.3.1.** *[1] The Riemann-Liouville fractional integral of order $\beta > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by*

$$I_{0^+}^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds,$$

*where $\Gamma(\beta)$ represents the gamma function, provided that the right side is pointwisely defined on $(0, +\infty)$.*

**Definition 1.3.2.** *[1] The Riemann-Liouville fractional derivative of order $\beta > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by*

$$D_{0^+}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\beta-n+1}} ds,$$

*where $n = [\beta] + 1$, provided that the right-hand side is defined pointwise on $(0, +\infty)$. Here $[\beta]$ denotes the integer part of the real number $\beta$.*

Given these definitions, it can be checked that the Riemann-Liouville fractional integration and fractional diferentiation operators of the power functions $t^{\gamma}$ yield power functions of the same form :

$$I_{0^+}^{\beta} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\beta+1)} t^{\gamma+\beta},$$

also

$$D_{0^+}^{\beta} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\beta+1)} t^{\gamma-\beta},$$

Notice that $D_{0+}^{\beta}t^{\gamma}=0$, for all $\gamma=\beta-i$ with $i=1,2,3,\ldots,n$ and $n$ is the smallest integer greater than or equal to $\beta$. In this case

$$\frac{d^n t^{\gamma-\beta+n}}{dt^n}=\frac{d^n t^{n-i}}{dt^n}=0.$$

We can also prove the following auxiliary lemmas.

**Lemma 1.3.1.** *[1] Suppose that $u\in L^1(0,+\infty)$ and $\alpha,\beta$ are positive real numbers, Then*

$$I_{0+}^{\alpha}I_{0+}^{\beta}u(t)=I_{0+}^{\alpha+\beta}u(t),\quad D_{0+}^{\alpha}I_{0+}^{\alpha}u(t)=u(t).$$

**Lemma 1.3.2.** *[1] Let $\beta>0$. If we assume $u\in C(0,1)\cap L(0,1)$, then the fractional differential equation*

$$D_{0+}^{\beta}u(t)=0,$$

*has $u(t)=c_1t^{\beta-1}+c_2t^{\beta-2}+\cdots+c_nt^{\beta-n}, c_i\in\mathbb{R}, i=1,2,\ldots,n$ as unique solutions, where $n$ is the smallest integer greater than or equal to $\beta$.*

**Lemma 1.3.3.** *[1] Given $u\in C(0,1)\cap L(0,1)$ with a fractional derivative of order $\beta>0$. Then*

$$I_{0+}^{\beta}D_{0+}^{\beta}u(t)=u(t)+c_1t^{\beta-1}+c_2t^{\beta-2}+\cdots+c_nt^{\beta-n},$$

*for some $c_i\in\mathbb{R}, i=1,2,\ldots,n$, where $n$ is the smallest integer greater than or equal to $\beta$.*

### 1.3.2 Generalized proportional fractional derivative of Caputo type

**Definition 1.3.3.** *[30] Let $\beta\geq 0$. The left fractional derivative of Caputo type of the function $f\in C^{(n)}[a,b]$. is defined by ${}^{C}\mathfrak{D}_a^{0}f(t)=f(t)$ and*

$$\begin{aligned}{}^{C}\mathfrak{D}_a^{\beta}f(t) &= I_a^{n-\beta}f^{(n)}(t)\\ &= \frac{1}{\Gamma(n-\beta)}\int_a^t (t-\tau)^{n-\beta-1}f^{(n)}(\tau)\,d\tau \text{ for } \beta>0,\end{aligned}$$

*where $n-1<\beta\leq n$, $n\in\mathbb{N}$.*

**Definition 1.3.4.** *[30] For $\rho\in(0,1]$ and $\beta>0$. The left generalized proportional fractional integral of $f$ is defined by*

$$\left(\mathcal{J}_a^{\beta,\rho}f\right)(t)=\frac{1}{\rho^{\beta}\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}e^{\delta(t-s)}f(s)\,ds,$$

*where $t\in[a,b]$, and $\delta=\frac{\rho-1}{\rho}$.*

**Definition 1.3.5.** *[30] For $\rho \in (0,1]$, where $\delta = \frac{\rho-1}{\rho}$, and $\beta > 0$. The left generalized proportional fractional derivative of Caputo type of the function $f \in C^{(n)}[a,b]$ is given by :*

$$\begin{aligned} {}^{C}\mathfrak{D}_a^{\beta,\rho} f(t) &= \mathcal{J}_a^{n-\beta,\rho} \left( D^{n,\rho} f \right)(t) \\ &= \frac{1}{\rho^{\beta}\Gamma(n-\beta)} \int_a^t (t-s)^{n-\beta-1} e^{\delta(t-s)} \left( D^{n,\rho} f \right)(s)\, ds, \end{aligned}$$

*where $n-1 < \beta \leq n$, $n \in \mathbb{N}$, and $(D^{1,\rho} f)(t) = (D^{\rho} f)(t) = (1-\rho)f(t) + \rho f'(t)$, and*

$$(D^{n,\rho} f)(t) = (\underbrace{D^{\rho} D^{\rho} \cdots D^{\rho}}_{n \text{ times}} f)(t), \quad \text{for } n \geq 1. \tag{1.1}$$

**Remark 1.3.** *[30] In the case $\rho = 1$, the definitions (1.3.4) and (1.3.5) reduce to a left Riemann-Liouville fractional integral and left Caputo fractional derivative, respectively.*

**Remark 1.4.** *[30] We can writeen the formula (1.1) for $\rho \in (0,1]$, as follows*

$$(D^{n,\rho} f)(t) = \rho^n f^{(n)}(t) + \sum_{k=0}^{n-1} C_n^k \rho^k (1-\rho)^{n-k} f^{(k)}(t),$$

*where $C_n^k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.*

**Proposition 1.3.1.** *[30] For $\rho \in (0,1]$, and $\alpha, \beta \in \mathbb{C}$, such that $\alpha > 0$, $\beta > 0$ and $n$ is the integer part of $\alpha$ then for $f \in L^1[0,1]$ we have :*

$$\mathcal{J}_0^{\alpha,\rho} \mathcal{J}_0^{\beta,\rho} f(t) = \mathcal{J}_0^{\beta,\rho} \mathcal{J}_0^{\alpha,\rho} f(t) = \mathcal{J}_0^{\alpha+\beta,\rho} f(t), \tag{1.2}$$

$$(\mathcal{J}_0^{\alpha,\rho} t^{\beta-1} e^{\delta t})(x) = \frac{\Gamma(\beta)}{\rho^{\alpha}\Gamma(\alpha+\beta)} x^{\alpha+\beta-1} e^{\delta x}, \tag{1.3}$$

$$({}^{C}\mathfrak{D}_0^{\alpha,\rho} t^{\beta-1} e^{\delta t})(x) = \frac{\rho^{\alpha}\Gamma(\beta)}{\Gamma(\beta-\alpha)} x^{\beta-\alpha-1} e^{\delta x}, \quad \Re(\beta) > n, \tag{1.4}$$

$${}^{C}\mathfrak{D}_0^{\alpha,\rho} \mathcal{J}_a^{\alpha,\rho} f(t) = f(t), \tag{1.5}$$

$$\mathcal{J}_0^{\alpha,\rho} \left( {}^{C}\mathfrak{D}_0^{\alpha,\rho} f \right)(t) = f(t) - \sum_{k=0}^{k=n-1} c_k t^k e^{\delta t}, \;\; f \in C^{(n)}[0,1], \tag{1.6}$$

*where $c_k = \frac{(D^{k,\rho} f)(a)}{\rho^k k!}$, and $\delta = \frac{\rho-1}{\rho}$.*

**Definition 1.3.6.** *[30] Let $\alpha \in \mathbb{C}$ $(\Re(\alpha) > 0)$, and $t > 0$. The lower incomplete gamma function is defined by :*

$$\gamma(\alpha, t) = \int_0^t y^{\alpha-1} e^{-y} dy.$$

*Also, the lower regularized incomplete gamma function is defined by :*

$$\mathfrak{P}(\alpha, t) = \frac{\gamma(\alpha, t)}{\Gamma(\alpha)}.$$

**Remark 1.5.** *[30] The function* $\mathfrak{P}$ *is also called "Incomplete gamma function".*

**Lemma 1.3.4.** *[30] Let* $\alpha, \eta \in \mathbb{R}^{+}, \alpha \geq 0$ *It is clear that* $\mathfrak{P}(\alpha, t)$ *is a non-decreasing function with respect to* $t \in [0,1]$. *And moreover*

$$\int_{t_1}^{t_2} y^{\alpha-1} e^{-y} dy = \gamma(\alpha, t_2) - \gamma(\alpha, t_1), t_2 \geq t_1 > 0,$$

$$\mathfrak{P}(\alpha, t) \in [0,1] \text{ for all } t \in [0,1],$$

$$\max_{t \in [0,1]} \mathfrak{P}(\alpha, t)|_{t=1} = \mathfrak{P}(\alpha, 1),$$

$$\min_{t \in [0,1]} \mathfrak{P}(\alpha, \eta(t-a)) = \mathfrak{P}(\alpha, t)|_{t=0} = 0.$$

**Lemma 1.3.5.** *[30] Let* $\rho \in (0,1]$, $\delta = \frac{\rho-1}{\rho}$, $t_1, t_2 \in [0,1]$ $(t_1 \leq t_2)$, *and* $\alpha > 0$. *Then*

$$\int_{t_1}^{t_2} (1-s)^{\alpha-1} e^{\delta(b-s)}) ds = \frac{\rho^{\alpha}\Gamma(\alpha)}{(1-\rho)^{\alpha}} \left[\mathfrak{P}\left(\alpha, -\delta(1-t_1)\right) - \mathfrak{P}\left(\alpha, -\delta(1-t_2)\right)\right].$$

**Lemma 1.3.6.** *[30] Let* $\rho \in (0,1]$, *and* $0 \leq t_1 \leq t_2 \leq 1$. *For all* $0 < \alpha$, *then*

$$\lim_{t_2 \to t_1} \int_0^{t_1} \left|(t_2-s)^{\alpha-1}\ e^{\delta(t_2-s)} - (t_1-s)^{\alpha-1} e^{\delta(t_1-s)}\right| ds = 0.$$

**Lemma 1.3.7.** *[30] Let* $\rho \in (0,1]$, $\beta > 0$, *and* $g_{\beta}(t) = e^{\delta t} t^{\beta}, t \in [0,1]$, *then*

$$\max_{t \in [0,1]} g_{\beta}(t) = \begin{cases} (\frac{-\beta}{\delta e})^{\beta}, & \text{if} \quad -\frac{\beta}{\delta} \in [0,1], \\ \\ e^{\delta}, & \text{if} \quad -\frac{\beta}{\delta} \notin [0,1] \text{ or } \rho = 1. \end{cases}$$

## 1.4 Some useful concepts

### 1.4.1 $L^p$ Spaces

**Definition 1.4.1.** *[7] Let* $p \in \mathbb{R}$ *with* $1 < p < \infty$, *we set*

$$\mathrm{L}^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}, f \text{ is measurable and } |f|^p \in \mathbb{L}^1(\Omega) \right\},$$

*with this norm*

$$\|f\|_{L^p} = \left[\int_{\Omega} |f(x)|^p \, \mathrm{d}x\right]^{1/p}.$$

### 1.4.2 $\Phi_p$-Laplacian operator

**Lemma 1.4.1.** *[21] $\phi_p : \mathbb{R} \to \mathbb{R}, \phi_p(u) = |u|^{p-2} u$ is an odd continuous, increasing operator and $\phi_p^{-1} = \phi_q$, where $\frac{1}{p} + \frac{1}{q} = 1$ and satisfies the following properties:*
*(i) $\phi_p(u+v) \leq \phi_p(u) + \phi_p(v)$, if $1 < p < 2$,*
*(ii) $\phi_p(u+v) \leq 2^{p-2}(\phi_p(u) + \phi_p(v))$, if $p \geq 2$,*
*for all $u, v \geq 0$.*

### 1.4.3 Lebesgue dominated convergence theorem

**Theorem 1.4.1.** *[7] Let $(f_n)$ be a sequence of $L^p(\Omega)$ such that*

*(i) $f_n(t) \to f(t)$ for almost every $t \in \Omega$,*

*(ii) there exists $g \in L^p(\Omega)$ such that $|f_n(t)| \leq g(t)$ for all $n \in \mathbb{N}$ and almost every $t \in \Omega$.*

*Then, $f \in L^p(\Omega)$ and $\|f_n - f\|_p \to 0$.*

# Chapter 2

# Topological degree theory

In this chapter, we review the definition of the topological degree and its basic properties in each of the two cases: the Brouwer degree in the finite dimension and the Leray-Schauder degree in the infinite dimension, then extend it to the coincidence degree for the operators of the form $L + N$, where $L$ is a Fredholm operator of index 0 and $N$ is an operator that is generally nonlinear and compact with respect to $L$. Finally, we introduce the Mawhin coincidence degree theorem with its proof. For more detail we can refer to [5].

## 2.1 Degree theory in finite-dimensional spaces

### 2.1.1 Definition of the Brouwer degree

Let $\Omega \subset R^N$ be an open bounded subset and $f : \bar{\Omega} \to R^N$, if $f$ is differentiable at $x_0$. We denote the Jacobian of $f$ at $x_0$, by $J_f(x_0) = \det f'(x_0)$, if $J_f(x_0) = 0$, then $x_0$ is said to be a critical point of $f$ and we use $S_f(\Omega) = \{x \in \Omega : J_f(x) = 0\}$ to denote the set of critical points of $f$ in $\Omega$. If $f^{-1}(y) \cap S_f(\Omega) = \emptyset$, then $y$ is said to be a regular value of $f$. Otherwise $y$ is said to be a singular value of $f$.

**Definition 2.1.1.** *For a regular value $y \notin f(\partial\Omega)$ the $C^1$-mapping degree is defined by :*

$$\deg_B(f, \Omega, y) = \begin{cases} \sum_{x \in f^{-1}(y) \cap \Omega} \operatorname{sgn} J_f(x), & f^{-1}(y) \cap \Omega \neq \emptyset. \\ 0, & f^{-1}(y) \cap \bar{\Omega} = \emptyset. \end{cases}$$

**Example 2.1.1.** *Let* $\Omega = B(0,R), Y_0 = (1,0)$*, and* $f(x,y) = (x^3 - 3xy^2, -y^3 + 3x^2y)$*, then* $f(x,y) = (0,1)$*, thus* $(x,y) = (1,0) \vee (-\frac{1}{2}, \frac{\sqrt{3}}{2}) \vee (-\frac{1}{2}, -\frac{\sqrt{3}}{2}) \in \partial\Omega$*. Note that at least the point* $(1,0)$ *is on the border of the unit ball whatever the standard usual norm we consider the* $\mathbb{R}^2$*. Therefore, the degree is not defined if* $R = 1$*.*
*If* $0 < R < 1$*, then* $B(0,R) \bigcap f^{-1}(y) = \emptyset$*, thus* $\deg_B(f, B(0,R), y) = 0$*.*
*Finally, if* $R > 1$*, then the degree is defined, and we have*

$$Df(x,y) = \begin{pmatrix} -3y^2 & -6xy \\ 6xy & -3y^2 + 3x^2 \end{pmatrix},$$

*we conclude that*

$$J_f(x,y) = (3x^2 - 3y^2)^2 + 36x^2y^2 = 0 \Leftrightarrow (x,y) = (0,0).$$

*The three points are then regular and as* $\operatorname{sgn}(J_f(x,y)) > 0, \forall (x,y) \neq (0,0)$ *then*

$$\deg_B(f, B(0,R), y) = 3.$$

### 2.1.2 Properties of the Brouwer degree

If $y \notin f(\partial\Omega)$, then there exists an integer $\deg_B(f, \Omega, y)$ satisfying the following properties:

1. **Normality**

$$\deg_B(I, \Omega, y) = \begin{cases} 1, & y \in \Omega, \\ 0, & y \notin \bar{\Omega}, \end{cases}$$

   where $I$ denotes the identity mapping.

2. **Validity of the degree**
   If $\deg_B(f, \Omega, y) \neq 0$, then there exists $x \in \Omega$, such that $f(x) = y$.

3. **Homotopy**
   Let $f, g : X \to Y$ be maps, $f$ homotopoic to $g$ if there exists a map $H : X \times I \to Y$ sit $H(x,0) = f(x)$ and $H(x,1) = g(x)$, $\forall x \in X, I = [0,1]$.
   If $f_t(x) : [0,1] \times \bar{\Omega} \to R^N$ is continuous and $y \notin \cup_{t \in [0,1]} f_t(\partial\Omega)$, then $\deg_B(f_t, \Omega, y)$ does not depend on $t \in [0,1]$.

**Example 2.1.2.** *Let* $f, g : [-1,1] \to \mathbb{R}$ *be given by* $f(x) = x^2$ *and* $g(x) = 2$. *As we can see the map* $h : [0,1] \times [-1,1] \to \mathbb{R}$ *given by :*

$$h(t,x) = (1-t)f(x) + \mathrm{tg}(x),$$

*is a valid homotopy joining* $f$ *and* $g$.

4. **Additivity**

   For $\Omega_1, \Omega_2$ are two disjoint open subsets of $\Omega$, and $y \notin f\left(\bar{\Omega} - \Omega_1 \cup \Omega_2\right)$, it holds that

$$\deg_B(f, \Omega, y) = \deg_B(f, \Omega_1, y) + \deg_B(f, \Omega_2, y).$$

5. **Continuity**

   The degree $\deg_B(f, \Omega, y)$ is continuous in $f$, i.e. there exists $\delta = \delta(f, y) > 0$, such that for all $g$ satisfying $\|f - g\|_{C^0} < \delta$, it holds that $y \notin g(\partial\Omega)$, and $\deg_B(g, \Omega, y) = \deg_B(f, \Omega, y)$.

6. **Translation property**

   For any $\tilde{y} \in \mathbb{R}^n$ it holds that

$$\deg_B(f - \tilde{y}, \Omega, y - \tilde{y}) = \deg_B(f, \Omega, y).$$

As consequences of this properties, we have the following results:

**Theorem 2.1.1.** ***(Brouwer fixed point theorem)***
*Let* $f : \overline{B(0,R)} \subset R^n \to \overline{B(0,R)}$ *be a continuous mapping. If* $|f(x)| \leq R$ *for all* $x \in \partial B(0,R)$, *then* $f$ *has a fixed point in* $\overline{B(0,R)}$.

***Proof.*** We may assume that $x \neq f(x)$ for all $x \in \partial B(0,R)$, put :

$$H(t,x) = x - tf(x),$$

for all $(t,x) \in [0,1] \times B(0,R)$. Then $0 \neq H(t,x)$ for all $[0,1]\times\ \partial B(0,R)$. Therefore, we have

$$\deg(I - f, B(0,R), 0) = \deg_B(I, B(0,R), 0) = 1.$$

Hence $f$ has a fixed point in $\overline{B(0,R)}$. This completes the proof. □

## 2.2 Degree theory in infinite-dimensional spaces

### 2.2.1 Definition of the Leray-Schauder degree

**Example 2.2.1.** *Let $X = l_2$ and $B$ be the closed unit ball in $l_2$. and $f : B \longrightarrow B$ be defined by :*

$$f(x) = (\sqrt{1 - \|x\|^2}, x_1, x_2, ...).$$

*The map $f(x)$ is continuous but has no fixed points. In fact, if $x = (x_1, x_2, ...)$ were a fixed point of $f$, then $\|x\| = 1$ since $\|f(x)\| = 1$ for all $\|x\| \leq 1$.*
*On the other hand, $x = (\sqrt{1 - \|x\|^2}, x_1, x_2, ...)$ implies $x_1 = 0, x_2 = x_1, x_3 = x_2$, etc. Hence $x = (0, 0, ...)$, wich contradicts the fact that $\|x\| = 1$.*

We see, therefore, that in infinite-dimensional spaces we must require more on $f$ than continuity. We shall require compactness.
Let $X$ be a real Banach space and $X_1 \subset X$, let $\Omega$ be a bounded, open subset of $X$ and let $f = I - T$, where $I$ is the identity map of $\bar{\Omega}$ into $X$ and $T : \bar{\Omega} \longrightarrow X$ is operator.
If $b \notin f(\partial\Omega)$, then there exists a map of finite range $T_1 : \bar{\Omega} \longrightarrow X_1$ (finite range means that $\dim X_1 < \infty$) such that

$$\sup_{u \in \bar{\Omega}} \|T_1 u - Tu\| < \text{dist}(b, f(\partial\Omega)).$$

In addition, the integer given by the Brouwer degree $\deg_B \left( (I - T_1)|_{\Omega \cap X_1}, \Omega_1, b \right)$ is independent on $T_1$. Therefore we can define the topological degree of Leray-Schauder

$$\deg_{LS}(f, \Omega, b) = \deg_B \left( (I - T_1)|_{\Omega \cap X_1}, \Omega_1, b \right),$$

it satisfies also the following basic properties.

**i) Normalization property.**

$$\deg_{LS}(I, \Omega, b) = 1, \quad \text{if } b \in \Omega.$$

**ii) Additivity property.**

Assume that $\Omega_1$ and $\Omega_2$ are open bounded disjoint subsets of $\Omega$. If $b \notin f \left( \bar{\Omega} \backslash (\Omega_1 \cup \Omega_2) \right)$ then

$$\deg_{LS}(f, \Omega, b) = \deg_{LS}(f, \Omega_1, b) + \deg_{LS}(f, \Omega_2, b).$$

**iii) Homotopy property.**

Let $S \in C([0,1]\times\bar{\Omega}, X)$ be a compact map and define $H(t,u) = u - S(t,u)$. If $b : [0,1] \longrightarrow X$ is continuous and $b(t) \notin H([0,1] \times \partial\Omega)$, then

$$\deg_{LS}(H(t,.), \Omega, b(t)) = \text{ const.} \quad \forall t \in [0,1].$$

**v) Existence property.**

If $\deg_{LS}(f, \Omega, b) \neq 0$, then there exists $u \in \Omega$ such that $f(u) = b$.

### 2.2.2 The Schauder fixed point theorem

In this part we give an extension of the Brouwer fixed point theorem to Banach spaces due to Schauder, a open set $\Omega \subset X$ is convex if $(1-t)x + ty \in \bar{\Omega}$ for all $x, y \in \bar{\Omega}$ and all $t \in [0,1]$. For convex sets we have the following analogue of the Brouwer fixed point theorem.

**Theorem 2.2.1.** *Let $\Omega \subset X$ be a bounded, open and convex set and let $T : \bar{\Omega} \longrightarrow X$ be compact mapping, with $0 \in \Omega$. If $T(\bar{\Omega}) \subset \bar{\Omega}$, then $T$ has at least one fixed point.*

***Proof.*** We argue by contradiction, i.e. Suppose $T(x) \neq x$ for all $x \in \bar{\Omega}$. Consider the homotopy

$$h_t(x) = x - tT(x),$$

in particular, $T(x) \neq x$ for $x \in \partial\Omega$. This implies that $0 \notin h_1(x)$.
Observe, since $\Omega$ is open, contains 0 and is convex, that $tK(x) \in \Omega$ for all $0 \leq t < 1$ and for all $x \in \bar{\Omega}$.
Consequently, if $x \in \partial\Omega$, then $h_t(x) \neq 0$ for all $0 \leq t \leq 1$. The Leray-Schauder degree $\deg_{LS}(h_t, \Omega, 0)$ is well-defined and independent of $t \in [0,1]$.
For $t = 0$, $\deg_{LS}(h_0, \Omega, 0) = \deg_{LS}(I, \Omega, 0) = 1$. On the other hand, since $T(x) \neq x$ for all $x \in \bar{\Omega}$, we have that $h_1^{-1}(0) = \emptyset$ which implies that $\deg_{LS}(h_1, \Omega, 0) = 0$, a contradiction. □

## 2.3 Coincidence degree for perturbations of Fredholm operators

In this section we study the existance solutions of problems which written in operator form :

$$Lx = Nx, \tag{2.1}$$

with $L$ (resp. $N$) a linear (resp. nonlinear) mapping between some topological vector spaces $X$ and $Y$, and with $L$ non invertible. Firstly, we begin this section by present some definitions of Fredholm mappings :

### 2.3.1 Fredholm operators

**Definition 2.3.1.** *[5] Let $L : \operatorname{dom} L \subset X \to Y$ be a linear operator. Then one says that $L$ is a Fredholm operator provided that*
*(i) $\ker L$ is finite dimensional set,*
*(ii) $\operatorname{Im} L$ is closed and has finite codimension.*

**Definition 2.3.2.** *[5] The index of a Fredholm operator $L$ is the integer*

$$ind\ L = \dim \ker L - \operatorname{codim} \operatorname{Im} L.$$

**Example 2.3.1.**
1. *If $X$ and $Y$ are Banach spaces and $L : X \to Y$ is a linear mapping bijective, then $L$ is a Fredholm operator of index 0, in fact*

$$\dim(\ker(L)) = \dim \operatorname{co}(\operatorname{Im}(L) = 0.$$

2. *The identity is a Fredholm operator of index 0 .*

### 2.3.2 Generalized inverse

Let $L : D(L) \subset X \to Y$ be a Fredholm operator of index 0. Let $P$ and $Q$ be two continuous projectors, $P : X \to X$ and $Q : Y \to Y$ such that :

$$\operatorname{Im}(P) = \ker L \text{ and } \ker Q = \operatorname{Im}(L),$$

set

$$X_1 = \mathrm{Im}(I-P) = \ker P \text{ and } Y_1 = \mathrm{Im}(Q),$$

so we can write

$$X = \ker L \oplus X_1, Y = \mathrm{Im}(L) \oplus Y_1.$$

Consider an isomorphism,

$$J : \ker L \to Y_1.$$

Whose existence is ensured by the fact that dim ker $L = \dim Y_1 = n$. Note that

$$\mathrm{dom}\, L = \ker L \oplus (\mathrm{dom}\, L \cap X_1).$$

And that the restriction of $L$ to $\mathrm{dom}\, L \cap X_1$ is an isomorphism on $\mathrm{Im}(L)$. Denote by $Lp$ this restriction and by $L_p^{-1} : \mathrm{Im}(L) \to \mathrm{dom}\, L \cap X_1$ the inverse of $L_p$. So the operator

$$J^{-1} \oplus L_p^{-1} : Y = Y_1 \oplus \mathrm{Im}\, L \to X = \ker L \oplus \mathrm{dom}\, L \cap X_1,$$

is an isomorphism whose inverse is the operator

$$L + JP : \mathrm{dom}\, L \cap \mathrm{Im}(I-P) \oplus \ker L \to \mathrm{Im}\, L \oplus Y_1,$$

indeed, for every $x \in \mathrm{dom}\, L \cap \mathrm{Im}(I-P) \oplus \ker L$, we write it in the form $x = (I-P)x + Px$ so

$$(L+JP)((I-P)x + Px) = L(I-P)x + JP(Px) = L(I-P)x + JPx,$$

consequently

$$\left(J^{-1} \oplus L_p^{-1}\right)(L(I-P)x + JPx) = (I-P)x + Px = x.$$

On the other hand, for all $y \in Y$ we have

$$\left(J^{-1} \oplus L_p^{-1}\right) y = \left(J^{-1} \oplus L_p^{-1}\right)(Qy + (I-Q)y) = J^{-1}Qy + L_p^{-1}(I-Q)y,$$

by setting $K_{P,Q} = L_p^{-1}(I-Q)$, ($K_{P,Q}$ is the inverse on the right of $L$ associated with $P$ and $Q$ respectively), then we get $(L+JP)^{-1} = J^{-1}Q + K_{P,Q}$.

### 2.3.3 $L$ - compact operator

**Definition 2.3.3.** *For the operator $N : \Omega \subset X \to Y$ to be $L-$compact on $\Omega$, it is necessary and sufficient that the operator*

$$M = P + J^{-1}QN + K_{P,Q}N,$$

*is compact on $\Omega$, where $P, Q$, and $J$ are the operators introduced in the above subsection.*

**Corollary 2.3.1.** *Let $\Omega$ an open bounded set of $X$ such that $\operatorname{dom} L \cap \Omega \neq 0$.*
*The map $N : X \to Y$ is $L$ - compact on $\bar{\Omega}$ if only if the operator $QN(\bar{\Omega})$ is bounded and $K_{P,Q}N\Omega : \bar{\Omega} \to X$ is compact.*

**Example 2.3.2.** *1. If $\dim X = \dim Y = n < \infty$ and $L = 0$ we can take*

$$P = I : X \to X, Q = I : Y \to Y \text{ et } K_{P;Q} = K_{I,I} = 0 : Y \to X.$$

*2. If $L = I : X \to X$, then*

$$P = 0 : X \to X; Q = 0 : Y \to Y \text{ et } K_{P;Q} = K_{0,0} = I : Y \to X.$$

*3. In the case where $L : \operatorname{dom} L \subset X \to Y$ is a bijective zero-index Fredholm operator :*

$$P = 0 : X \to X, Q = 0 : Z \to Y \text{ et } K_{P;Q} = K_{0,0} = L^{-1}.$$

### 2.3.4 Equivalent fixed point problem to Lx=Nx

To solve the equation $Lx = y$, we can write $x = Px + (I - P)x$ and $y = Qy + (I - Q)y$ and by substitution of $x$ and $y$ in the previous equation, we obtain

$$L(Px + (I - P)x) = Qy + (I - Q)y,$$

and since $Qy = 0$ and $LPx = 0$ (because $y \in \operatorname{Im}(L)$ and $Px \in \ker L$ ), then

$$L(I - P)x = (I - Q)y,$$

which leads to

$$x - Px = Lp^{-1}(I - Q)y,$$

and thus

$$x = Px + J^{-1}Qy + L_p^{-1}(I - Q)y.$$

Now consider the equation $Lx = Nx$, where $N : G \subset X \to Y$ is an operator (usually nonlinear) according to the above result, this last equation with $x \in \operatorname{dom} L \cap G$ is equivalent to

$$x = Px + J^{-1}QNx + K_{P,Q}P = Mx. \tag{2.2}$$

Which is a fixed point problem.

### 2.3.5 Coincidence degree and properties

**Definition 2.3.4.** *Let $L$ be a Fredholm operator of index zero and let $N$ be $L$-compact on $\bar{\Omega}$, the coincidence degree is the integer*

$$\deg_L[(L, N), \Omega] = d_{LS}[I - M, \Omega, 0],$$

*where $d_{LS}$ is the Laury Schauder degree, and $M$ is defined in (2.2)*

Now we can give the basic properties of coincidence degree :

**property 2.3.1.** ***(Existence property)***
*If $\deg_L[(L, N), \Omega] \neq 0$, then $0 \in (L - N)($ dom $L \cap \Omega)$.*

**property 2.3.2.** ***(Excision property)***
*If $\Omega_0 \subset \Omega$ is an open set such that $(L - \mathbb{N})^{-1}(0) \subset \Omega_0$ then*

$$\deg_L[(\mathrm{L}, \mathrm{N}), \Omega] = \deg_L\left[(\mathrm{L}, \mathrm{N}), \Omega_0\right].$$

**property 2.3.3.** ***(Additivity property)***
*If $\Omega = \Omega_1 \cup \Omega_2$, with $\Omega_1, \Omega_2$ open and such that $\Omega_1 \cap \Omega_2 = \emptyset$, then*

$$\deg_L[(\mathrm{L}, \mathrm{N}), \Omega] = \deg_L\left[(\mathrm{L}, \mathrm{N}), \Omega_1\right] + \deg_L\left[(\mathrm{L}, \mathrm{N}), \Omega_2\right].$$

### 2.3.6 Mawhin continuation theorem

In the 1970 s, Mawhin systematically studied a class of mappings of the form $L + N$ where $L$ is a Fredholm mapping of index zero and $N$ is a nonlinear mapping, which he called a $L$ - compact mapping.

**Theorem 2.3.1.** *Let $L$ be a Fredholm operator of index zero and let $N$ be $L$-compact on $\bar{\Omega}$. Assume that the following conditions are satisfied.*
*(i) $Lx \neq \lambda Nx$, for every $(x, \lambda) \in [(\operatorname{dom} L \backslash KerL) \cap \partial\Omega] \times (0, 1)$,*
*(ii) $Nx \notin \mathrm{ImL}$, for every $x \in \operatorname{Ker} L \cap \partial\Omega$,*
*(iii) $\deg\left(JQN_{|\ker L}, \ ker \ L \cap \partial\Omega, 0\right) \neq 0$,*
*where $J : \operatorname{Im} Q \to \ker L$ is a linear isomorphism, $Q : Y \to Y$ is a projection as above with $\operatorname{Im} L = \ker Q$. Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$.*

***Proof***. For $\lambda \in [0, 1]$, consider the family of problems

$$x \in \operatorname{dom} L \cap \bar{\Omega}, \quad Lx = \lambda Nx + (1 - \lambda)QNx. \tag{2.3}$$

Let $M : [0, 1] \times \bar{\Omega} \to Y$ be a homotopy defined by

$$M(\lambda, x) = Px + J^{-1}QNx + \lambda K_{P,Q}Nx.$$

The problem (2.3) is equivalent to a fixed point problem :

$$\begin{aligned} x &= Px + J^{-1}Q(\lambda N + (1 - \lambda)QN)x + K_{P,Q}(\lambda N + (1 - \lambda)QN)x \\ &= Px + \lambda J^{-1}QNx + (1 - \lambda)J^{-1}QNx + \lambda K_{P,Q}Nx + (1 - \lambda)K_{P,Q}QNx \\ &= M(1, x), \end{aligned}$$

so this last equation is equivalent to a fixed point problem :

$$x = M(1, x), \quad x \in \bar{\Omega}, \tag{2.4}$$

if there exists an $x \in \partial\Omega$ such that $Lx = Nx$, then the proof is completed. Now suppose that

$$Lx \neq Nx \quad \text{for all} \quad x \in \operatorname{dom} L \cap \Omega, \tag{2.5}$$

and on the other hand

$$Lx \neq \lambda Nx + (1 - \lambda)QNx, \tag{2.6}$$

For all $(\lambda, x) \in ]0, 1[\times(\operatorname{dom} L \cap \Omega)$. If

$$Lx = \lambda Nx + (1 - \lambda)QNx,$$

for all $(\lambda, x) \in ]0,1[\times(\operatorname{dom} L \cap \Omega)$, we obtain by application of $Q$ to both members of the previous equality

$$QNx = 0, \quad Lx = \lambda Nx,$$

the first of these equalities and the condition (ii) imply that $x \notin KerL \cap \partial\Omega$, i.e., for all $x \in (\operatorname{dom} L \backslash K\,\mathrm{er}\, L) \cap \partial\Omega$ and therefore the second equality contradicts (i). By using other times (ii), it follows that

$$Lx \neq QNx, \quad \text{for every} \quad x \in \operatorname{dom} L \cap \partial\Omega, \tag{2.7}$$

using (2.5), (2.6) and (2.7), we deduce that

$$x \neq M(\lambda, x) \text{ for all } (\lambda, x) \in [0,1] \times \partial\Omega, \tag{2.8}$$

since $N$ is $L$-compact then $M(\lambda, x)$ is compact because. Using the homotopy invariance property of the Leray-Schauder degree, we obtain

$$\deg_{LS}(I - M(0,.), \Omega, 0) = \deg_{LS}(I - M(1,.), \Omega, 0), \tag{2.9}$$

on the other hand we have

$$\deg_{LS}(I - M(0,\lambda), \Omega, 0) = \deg_{LS}\left(I - \left(P + J^{-1}QN\right), \Omega, 0\right), \tag{2.10}$$

since the image of $P + J^{-1}QN$ is contained in $\operatorname{Ker}(L)$, then using the property of reduction of the Leray-Schauder degree and the fact that $P|_{KerL} = I|_{KerL}$, (since $\operatorname{Ker}(L) = \operatorname{Im}(P)$ $= \operatorname{Ker}(I - P)$), we obtain

$$\begin{aligned}\deg_{LS}\left(I - \left(P + J^{-1}QN\right), \Omega, 0\right) &= \deg\left(I - \left(P + J^{-1}QN\right), \Omega \cap KerL, 0\right) \\ &= \deg\left(J^{-1}QN, \Omega \cap \operatorname{Ker} L, 0\right),\end{aligned} \tag{2.11}$$

thanks to (2.9), (2.10) and (2.11), it follows that $\deg_{LS}(I - M(1,.), \Omega, 0) \neq 0$, and so the existence property of the Leray-Schauder degree implies the existence of an $x \in \Omega$ such as $x = M(1, x)$ i.e $x \in \operatorname{dom} L \cap \Omega, Lx = Nx$. □

# Chapter 3

# Solvability of a resonant fractional problem with one-dimensional kernel

## 3.1 Introduction

In this chapter, we are concerned the multi-point BVPs for a nonlinear fractional differential equation with a $p$-Laplacian operator [13] :

$$\left(\phi_p\left(D_{0^+}^{\alpha}x\left(t\right)\right)\right)' = f\left(t, x\left(t\right), D_{0^+}^{\alpha-1}x\left(t\right)\right), \quad t\in[0,1], \tag{3.1}$$

$$x\left(0\right) = D_{0^+}^{\alpha}x\left(1\right) = 0, \tag{3.2}$$

$$D_{0^+}^{\alpha-1}x\left(1\right) = \sum_{i=1}^{i=m-2}\beta_i D_{0^+}^{\alpha-1}x\left(\eta_i\right), \tag{3.3}$$

where $1<\alpha<2$, $0<\eta_1<\eta_2<\ldots<\eta_{m-2}<1$, $\beta_i\in\mathbb{R}_+$, for $i=1,2,3,\ldots,m-2$ ($m\geq 3$), $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative, $f:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a given continuous function and $\phi_p\left(s\right)=\left|s\right|^{p-2}s$ is the $p$-Laplacian ($p>1$). Recall that $\phi_p:\mathbb{R}\to\mathbb{R}$ is an odd continuous, strictly increasing operator with $\phi_p^{-1}=\phi_q\left(s\right)$ ($\frac{1}{p}+\frac{1}{q}=1$).

Assume also that

$$\sum_{i=1}^{i=m-2}\beta_i = 1. \tag{3.4}$$

However, there are only few results for boundary value problems at resonance when the $p$-Laplacian differentiation operator is involved. Indeed, the basic difficulty in solving such problems resides in the fact that since $\phi_p$ (in case $p \neq 2$) is a nonlinear operator, then the coincidence degree method cannot be applied in a direct way. For example, in the recent work [24], the authors proved the existence of at least one solution of the same problem at the resonance case by using an alternative approach based on the decomposition $u - Lu = Nu$ Motived by [11], we will establish an existence result of solution for the (3.1) - (3.2) - (3.3) by mean of the coincidence degree theory applied to an equivalent semi linear problem, it's easy to check that our problem is equivalent to following boundary value probem :

$$D_{0^+}^{\alpha} x(t) = \phi_q \left( \int_1^t f\left(s, x(s), D_{0^+}^{\alpha-1} x(s)\right) ds \right), \tag{3.5}$$

$$x(0) = 0, \tag{3.6}$$

$$D_{0^+}^{\alpha-1} x(1) = \sum_{i=1}^{i=m-2} \beta_i D_{0^+}^{\alpha-1} x(\eta_i). \tag{3.7}$$

## 3.2 Main result

### 3.2.1 Functional framework

**Lemma 3.2.1.** *[22] For a given $\theta > 0$, the linear space defined by :*

$$C^{\theta}[0,1] = \left\{ x(t) : x(t) = I_{0^+}^{\theta} z(t) + \sum_{i=1}^{i=[\theta]} c_i t^{\theta - i},\ z \in C[0,1] \right\},$$

*where $[\theta]$ is the integer part of $\theta$ and $c_i \in \mathbb{R}$ with the norm*

$$\|x\|_{C^{\theta}} = \|x\|_{\infty} + \sum_{i=0}^{i=[\theta]} \left\| D_{0^+}^{\theta - i} x \right\|_{\infty},$$

*is a Banach space.*

**Lemma 3.2.2.** *[22] $M \subset C^{\theta}[0,1]$ is relatively compact set if and only if :*

1. *$M$ is uniformly bounded: there exists $m > 0$, such that $\|u\|_{C^{\theta}} \leq m$, for every $u \in M$,*

2. *M is equicontinuous : for every $\varepsilon > 0$, there exists $\delta > 0$, such that for all $t_1, t_2 \in [0, 1]$ : $|t_2 - t_1| < \delta$, we have*

$$|u(t_2) - u(t_1)| < \varepsilon \text{ and } \left|{}^c\mathfrak{D}_0^{\theta-k} u(t_2) - {}^c\mathfrak{D}_0^{\theta-k} u(t_1)\right| < \varepsilon,$$

*for all $u \in M$ with $k = 0, 1, 2, \ldots, [\theta]$.*

Let $X = C^{\alpha-1}[0,1] = \{x(t) = I^{\alpha-1}z(t) ; z \in C[0,1]\}$, with the norm

$$||x||_{C^{\alpha-1}} = ||x||_\infty + \left|\left|D_{0^+}^{\alpha-1}x\right|\right|_\infty .$$

And $Y = \{y \in C[0,1] \ / \ y(1) = 0\}$, with the norm :

$$||y||_Y = ||y||_\infty = \max_{t\in[0;1]} |y(t)| ,$$

by analysis theory we can prove that $(X, ||.||_X)$ and $(Y, ||.||_Y)$ are both Banach spaces (see also [7]). Define the operators $L : dom(L) \subset X \to Y$ and $N : dom(L) \subset X \to Y$ as follows

$$Lx(.) = D_{0_+}^{\alpha}x(.) \ , \quad x \in dom(L), \tag{3.8}$$

$$N : X \to Y; \ Nx(t) = \phi_q\left(\int_1^t f\left(s, x(s), D_{0^+}^{\alpha-1}x(s)\right) ds\right), \tag{3.9}$$

where

$$dom(L) = \left\{x \in X \ / \ D_{0_+}^{\alpha}x(t) \in L^1[0,1], \ x(0) = 0, \ D_{0_+}^{\alpha-1}x(1) = \sum_{i=1}^{m-2} \beta_i D_{0_+}^{\alpha-1}x(\eta_i)\right\}.$$

Notice that problem (3.5)-(3.6)-(3.7) can be converted to the abstract operator equation $Lx = Nx$, $x \in dom(L)$.

### 3.2.2 Auxiliary lemmas

In this part, we present some auxiliary lemmas which illustrating linear part of this problem.

**Lemma 3.2.3.** *Let $L$ be the operator defined by (3.8) then :* $\ker L = \{at^{\alpha-1}; a \in \mathbb{R}, t \in [0;1]\}$ *and* $\operatorname{Im} L = \left\{y \in Y; \sum\limits_{i=1}^{i=m-2} \beta_i \int\limits_{\eta_i}^{1} y(s)\, ds = 0\right\}$.

***Proof.*** We have for each $x \in \ker L$, $Lx(t) = D_{0+}^{\alpha} x(t) = 0$ with $t \in [0;1]$. By applying Lemma 1.3.1, we get

$$x(t) = at^{\alpha-1} + bt^{\alpha-2},$$

as $x(0) = 0$, thus $b = 0$. On the other hand, for $x(t) = at^{\alpha-1}$ we have

$$Lx(t) = D_{0+}^{\alpha} x(t) = a.D_{0+}^{\alpha} t^{\alpha-1} = a.0 = 0.$$

Now for all $y \in \operatorname{Im} L$, there exits $x \in dom(L)$ such that

$$D_{0+}^{\alpha} x(t) = y(t),$$

again Lemma 1.3.1 leads us to

$$x(t) = I_{0+}^{\alpha} y(t) + at^{\alpha-1} + bt^{\alpha-2},$$

with $(a,b) \in \mathbb{R}^2$. From the condition $x(0) = 0$, we find $b = 0$, and in view of boundary condition ( 3.7), we obtain that

$$\int_0^1 y(s)\,ds + a\Gamma(\alpha) = a\Gamma(\alpha) + \sum_{i=1}^{i=m-2} \beta_i \int_0^{\eta_i} y(s)\,ds,$$

which is equivalent to :

$$\sum_{i=1}^{i=m-2} \beta_i \int_{\eta_i}^1 y(s)\,ds = 0. \tag{3.10}$$

Conversely, if $y$ satisfies (3.10). Let

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds,$$

we have $x(t) = I_{0+}^{\alpha-1}\left(I_{0+}^{1} y\right)$ with $I_{0+}^{1} y \in C[0;1]$ and $x(0) = 0$, $D_{0+}^{\alpha} x(1) = y(1) = 0$ also

$$\sum_{i=1}^{i=m-2} \beta_i D_{0+}^{\alpha-1} x(\eta_i) = \sum_{i=1}^{i=m-2} \beta_i \int_0^{\eta_i} y(s)\,ds = \int_0^1 y(s)\,ds = D_{0+}^{\alpha-1} x(1).$$

□

**Remark 3.1.** *It easy to show that* $1 - \sum_{i=1}^{i=m-2} \beta_i \eta_i > 0$.

*In fact, for all* $i = 1, 2, ..., m-2$, $\beta_i \geq 0, \eta_i \in ]0;1[$ *we have* $\beta_i (1-\eta_i) \geq 0$. *By the resonance condition (3.4), there exists at least* $i_0 \in \{1, 2, ..., m-2\}$ *such that* $\beta_{i_0} \neq 0$ *and hence* $\beta_{i_0}(1-\eta_{i_0}) > 0$, *which prove that*

$$\sum_{i=1}^{i=m-2} \beta_i (1-\eta_i) = \sum_{i=1}^{i=m-2} \beta_i - \sum_{i=1}^{i=m-2} \beta_i \eta_i = 1 - \sum_{i=1}^{i=m-2} \beta_i \eta_i > 0.$$

**Lemma 3.2.4.** *We can define two linear continuous projectors $P$ and $Q$ as follow :*

$P : X \to X;\ Px(t) = \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} x(0)\, t^{\alpha-1},$

$Q : Y \to Y;\ Qy(t) = \frac{1}{1-\sum\limits_{i=1}^{i=m-2} \beta_i \eta_i} \sum\limits_{i=1}^{i=m-2} \beta_i \int_{\eta_i}^{1} y(s)\, ds.$

*The inverse of the operator $L_P = L_{|dom(L)\cap \ker P}$ is the operator $K_P : \operatorname{Im} L \to dom(L) \cap \ker P$ defined by $K_P y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds$ and checking :*

$$\|K_P y\|_X \leq \left(1 + \frac{1}{\Gamma(\alpha+1)}\right) \|y\|_\infty . \tag{3.11}$$

***Proof.*** For each $x \in X$, putting $Px(t) = \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} x(0)\, t^{\alpha-1} = u(t)$ then

$$P^2 x(t) = P(Px)(t) = Pu(t) = \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} u(0)\, t^{\alpha-1} = Px(t),$$

because $D_{0^+}^{\alpha-1} u(t) = \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} x(0) .\Gamma(\alpha) = D_{0^+}^{\alpha-1} x(0)$ thus $P^2 = P$.
We note that $\operatorname{Im} P = \ker L$, and $\ker P = \left\{x \in X; D_{0^+}^{\alpha-1} x(0) = 0\right\}$, by simple calculation we have :

$$\begin{aligned} \|Px\|_X &= \|Px\|_\infty + \left\|D_{0^+}^{\alpha-1} Px\right\|_\infty = \frac{1}{\Gamma(\alpha)} \left|D_{0^+}^{\alpha-1} x(0)\right| + \left|D_{0^+}^{\alpha-1} x(0)\right| \\ &= \left(1 + \frac{1}{\Gamma(\alpha)}\right) \left|D_{0^+}^{\alpha-1} x(0)\right| \leq \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|x\|_X . \end{aligned}$$

On the other hand, for all $y \in Y$, taking $\frac{1}{1-\sum\limits_{i=1}^{i=m-2} \beta_i \eta_i} \sum\limits_{i=1}^{i=m-2} \beta_i \int_{\eta_i}^{1} y(s)\, ds = r$. Then

$Q^2 y = \frac{1}{1-\sum\limits_{i=1}^{i=m-2} \beta_i \eta_i} \sum\limits_{i=1}^{i=m-2} \beta_i \int_{\eta_i}^{1} r ds = \frac{r}{1-\sum\limits_{i=1}^{i=m-2} \beta_i \eta_i} \sum\limits_{i=1}^{i=m-2} \beta_i (1-\eta_i) = r$, so $Q^2 = Q$.
Furthermore, we have

$$\|Qy\|_Y \leq \|y\|_Y .$$

For any $x \in dom(L) \cap \ker P$, by lemma 1.3.1 we can write

$$K_P L x(t) = I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) = x(t) + a t^{\alpha-1} + b t^{\alpha-2},$$

with $t \in [0;1]$ and $a, b$ are two real constants. As $K_P L x \in dom(L) \cap \ker P$, then $b = 0$ and $D_{0^+}^{\alpha-1} \left(x(t) + a t^{\alpha-1}\right) |_{t=0} = D_{0^+}^{\alpha-1} x(0) + a\Gamma(\alpha) = a\Gamma(\alpha) = 0$ which impy that $a = 0$ therefore $K_P L x = x$. If $y \in \operatorname{Im} L$, we get $L K_P y(t) = D_{0^+}^{\alpha} I_{0^+}^{\alpha} y(t) = y(t)$ which show that $K_P = (L_P)^{-1}$. We have also

$$\|K_P y\|_X = \|I_{0^+}^{\alpha} y\|_\infty + \left\|I_{0^+}^{1} y\right\|_\infty \leq \frac{1}{\Gamma(\alpha+1)} \|y\|_\infty + \|y\|_\infty .$$

Which complete the proof. □

**Lemma 3.2.5.** *$L$ defined in (3.8) is a Fredholm operator of index 0.*

***Proof.*** For any $y \in Y$, we can write $y = (I - Q)\,y + Qy$, where $(I - Q)\,y \in \ker Q$, with $\ker Q = \operatorname{Im} L$, $Qy \in \operatorname{Im} Q$ then $y \in \operatorname{Im} L + \operatorname{Im} Q$. Assume that $y \in \operatorname{Im} L \cap \operatorname{Im} Q$ thus $y = c \in \mathbb{R}$ and $\sum\limits_{i=1}^{i=m-2} \beta_i \int_{\eta_i}^{1} c ds = c \left(1 - \sum\limits_{i=1}^{i=m-2} \beta_i \eta_i\right) = 0$ i.e $c = 0$ because $1 - \sum\limits_{i=1}^{i=m-2} \beta_i \eta_i \neq 0$ therefore $\operatorname{Im} L \cap \operatorname{Im} Q = \phi$ and then $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. Finaly, as $\dim \operatorname{Im} Q = \dim \ker L = 1$ then $L$ is a Fredholm operator of index 0. □

Now let's go to the proof the compactness of nonlinear part of this problem

**Lemma 3.2.6.** *Assume that $B$ is an open bounded subset in $X$ such that $dom\,(L) \cap B \neq \phi$. The operator $N$ defined by (3.9) is $L$-compact on $\overline{B}$.*

***Proof.*** The boundness of $B$ imply that there exists $R > 0$ such that for all $x \in B$, we have $\|x\|_X = \|x\|_\infty + \left\|D_{0^+}^{\alpha-1} x\right\|_\infty \leq R$. By the continuity of $\phi_q$ and $f$, there exists $D > 0$ such that for all $x \in B$, we get $\left|f\left(s, x\,(s)\,, D_{0^+}^{\alpha-1} x\,(s)\right)\right| \leq D$ then

$$\|QNx\|_\infty \leq \|Nx\|_\infty \leq D^{q-1},$$

because $\left|\phi_q \left(\int\limits_1^t f\left(s, x\,(s)\,, D_{0^+}^{\alpha-1} x\,(s)\right) ds\right)\right| = \left|\int\limits_t^1 f\left(s, x\,(s)\,, D_{0^+}^{\alpha-1} x\,(s)\right) ds\right|^{q-1} \leq D^{q-1}$. Also we have

$$\|(I - Q)\, Nx\|_\infty \leq \|Nx\|_\infty + \|QNx\|_\infty \leq 2D^{q-1},$$

from (3.11), we conclude that

$$\|K_P\,(I - Q)\, Nx\|_x \leq \left(1 + \frac{1}{\Gamma\,(\alpha)}\right) \|(I - Q)\, Nx\|_\infty \leq \left(1 + \frac{1}{\Gamma\,(\alpha)}\right) 2D^{q-1}.$$

Then $QN\,(B)$ and $K_{P,Q}N\,(B)$ are bounded. It remains to prove that $K_{P,Q}N\,(B)$ is equicontinuous. Putting $0 \leq t_1 \leq t_2 \leq 1$

$$
\begin{aligned}
& \left|K_P (I-Q) Nx(t_2) - K_P (I-Q) Nx(t_1)\right| \\
= & \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2-s)^{\alpha-1} (I-Q) Nx(s)\, ds - \int_0^{t_1} (t_1-s)^{\alpha-1} (I-Q) Nx(s)\, ds \right| \\
= & \frac{1}{\Gamma(\alpha)} \left| \begin{array}{c} \int_0^{t_1} (t_2-s)^{\alpha-1} (I-Q) Nx(s)\, ds - \int_0^{t_1} (t_1-s)^{\alpha-1} (I-Q) Nx(s)\, ds \\ + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} (I-Q) Nx(s)\, ds \end{array} \right| \\
= & \frac{1}{\Gamma(\alpha)} \left| \begin{array}{c} \int_0^{t_1} (t_2-s)^{\alpha-1} Nx(s)\, ds - \int_0^{t_1} (t_1-s)^{\alpha-1} Nx(s)\, ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} Nx(s)\, ds \\ + QNx \left( - \int_0^{t_2} (t_2-s)^{\alpha-1} ds + \int_0^{t_1} (t_1-s)^{\alpha-1} ds \right) \end{array} \right| \\
= & \frac{1}{\Gamma(\alpha)} \left| \begin{array}{c} \int_0^{t_1} (t_2-s)^{\alpha-1} Nx(s)\, ds - \int_0^{t_1} (t_1-s)^{\alpha-1} Nx(s)\, ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} Nx(s)\, ds \\ - QNx \left( \frac{t_2^\alpha - t_1^\alpha}{\alpha} \right) \end{array} \right| \\
\leq & \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \left( (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right) Nx(s)\, ds \right| + \left| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} Nx(s)\, ds \right| \\
+ & |QNx| \left| \frac{t_2^\alpha - t_1^\alpha}{\alpha} \right| \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left( (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right) |Nx(s)|\, ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} |Nx(s)|\, ds \\
+ & |QNx| \frac{t_2^\alpha - t_1^\alpha}{\alpha} \\
\leq & \frac{1}{\Gamma(\alpha)} D^{q-1} \left( \int_0^{t_1} \left( (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right) ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds + \frac{t_2^\alpha - t_1^\alpha}{\alpha} \right) \\
= & \frac{2D^{q-1}}{\Gamma(\alpha+1)} \left( t_2^\alpha - t_1^\alpha \right) \quad \to 0 \text{ as } \quad t_1 \to t_2 \text{ uniformly.}
\end{aligned}
$$

Similarly, we have $D_{0^+}^{\alpha-1} K_P (I-Q) Nx(t) = I_{0^+}^1 (I-Q) Nx = \int_0^t (I-Q) Nx(s)\, ds.$

Then

$$\begin{aligned}
&\left|D_{0^+}^{\alpha-1}K_P\left(I-Q\right)Nx\left(t_2\right)-D_{0^+}^{\alpha-1}K_P\left(I-Q\right)Nx\left(t_1\right)\right| \\
=&\left|\int_0^{t_2}\left(I-Q\right)Nx\left(s\right)ds-\int_0^{t_1}\left(I-Q\right)Nx\left(s\right)ds\right| \\
=&\left|\int_{t_1}^{t_2}Nx\left(s\right)ds-\left(t_2-t_1\right)QNx\right| \\
\leq&\int_{t_1}^{t_2}\left|Nx\left(s\right)\right|ds+\left|QNx\right|\left(t_2-t_1\right) \\
\leq&\ 2D^{q-1}\left(t_2-t_1\right)\to 0 \text{ as } t_1\to t_2.
\end{aligned}$$

According to the lemma 3.2.2, $K_P\left(I-Q\right)N\left(B\right)$ is compact, which show that $N$ is $L$-compact on $B$. □

### 3.2.3 Existence theorem

**Theorem 3.2.1.** *Suppose that there exist*

$(H1)$ *A function $\Psi:[0;1]\times\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ which is $L^1-$Caratheodory and non decreasing with respect to the last two variables such that*

$$\left|f\left(t,x,y\right)\right|\leq\Psi\left(t,\left|x\right|,\left|y\right|\right),$$

*for all $(x;y)\in\mathbb{R}^2$ and a.e $t\in[0;1]$,*

$(H2)$ *A real $M_0>0$, such that if we have $\left|D_{0^+}^{\alpha-1}x\left(t\right)\right|>M_0$ for all $t\in[0;1]$, then*

$$\sum_{i=1}^{i=m-2}\beta_i\int_{\eta_i}^{1}\phi_q\left(\int_1^t f\left(s,x\left(s\right),D_{0^+}^{\alpha-1}x\left(s\right)\right)ds\right)dt\neq 0,$$

$(H3)$ *A real $M_1>0$, such that for $|a|>M_1$, we get else*

$$af\left(s,as^{\alpha-1},a\Gamma\left(\alpha\right)\right)ds<0, \tag{3.12}$$

*or*

$$af\left(s,as^{\alpha-1},a\Gamma\left(\alpha\right)\right)ds>0. \tag{3.13}$$

*Then the fractional B.V.Ps (3.5)-(3.6)-(3.7) has at least one solution in dom(L) ⊂ X provided that*

$$\int_0^1 \Psi(t,m,m)\,dt \le \left(\frac{\Gamma(\alpha)}{8}m\right)^{\frac{1}{q-1}}. \tag{3.14}$$

***Proof.*** **Step1**: Let

$$\Omega_1 = \{x \in dom(L) - \ker L : Lx = \lambda Nx, \lambda \in [0;1]\},$$

we will show that it is a bounded set. If $x \in \Omega_1$ then $\lambda \in ]0;1]$ because $\Omega_1 \cap \ker L = \phi$ which allows us to write $Nx = L\frac{1}{\lambda}x \in \operatorname{Im} L = \ker Q$, then

$$QNx = \sum_{i=1}^{i=m-2} \beta_i \int_{\eta_i}^1 \phi_q\left(\int_1^t f\left(s, x(s), D_{0^+}^{\alpha-1}x(s)\right) ds\right) dt = 0,$$

by the condition $(H_2)$, there exists $t_0 \in [0;1]$ such that $\left|D_{0^+}^{\alpha-1}x(t_0)\right| \le M_0$. On the other hand, we have $\frac{d}{dt}\left(D_{0^+}^{\alpha-1}x(t)\right) = D_{0^+}^{\alpha}x(t)$ so

$$D_{0^+}^{\alpha-1}x(t) = D_{0^+}^{\alpha-1}x(t_0) + \int_{t_0}^t D_{0^+}^{\alpha}x(s)\,ds = D_{0^+}^{\alpha-1}x(t_0) + \lambda \int_{t_0}^t \phi_q\left(\int_1^r f\left(s, x(s), D_{0^+}^{\alpha-1}x(s)\right) ds\right) dr,$$

then

$$\begin{aligned} \left|D_{0^+}^{\alpha-1}x(t)\right| &\le \left|D_{0^+}^{\alpha-1}x(t_0)\right| + \left|\int_{t_0}^t \phi_q\left(\int_1^r f\left(s, x(s), D_{0^+}^{\alpha-1}x(s)\right) ds\right) dr\right| \\ &\le M_0 + \|Nx\|_\infty. \end{aligned} \tag{3.15}$$

Furthermore, we can write

$$\begin{aligned} x = (I-P)x + Px &= K_P L(I-P)x + Px \\ &= K_P Lx + Px, \end{aligned}$$

then

$$\|x\|_X \le \|K_P Lx\|_X + \|Px\|_X,$$

by using (3.15), we obtain

$$|Px(t)| = \frac{1}{\Gamma(\alpha)}\left|D_{0^+}^{\alpha-1}x(0)\right| t^{\alpha-1} \le \frac{1}{\Gamma(\alpha)}\left|D_{0^+}^{\alpha-1}x(0)\right| \le \frac{1}{\Gamma(\alpha)}\left(M_0 + \|Nx\|_\infty\right),$$

and $\left|D_{0^+}^{\alpha-1}Px(t)\right| = \left|D_{0^+}^{\alpha-1}x(0)\right| \le M_0 + \|Nx\|_\infty$, then

$$\|Px\|_X \le \left(1 + \frac{1}{\Gamma(\alpha)}\right)\left(M_0 + \|Nx\|_\infty\right),$$

in view of (3.11), we have $\|K_P Lx\|_X \leq \left(1+\frac{1}{\Gamma(\alpha+1)}\right)\|Lx\|_\infty \leq \left(1+\frac{1}{\Gamma(\alpha+1)}\right)\|Nx\|_\infty$ which give

$$\|x\|_X \leq \left(1+\frac{1}{\Gamma(\alpha)}\right)M_0+\left(2+\frac{\alpha+1}{\Gamma(\alpha+1)}\right)\|Nx\|_\infty, \tag{3.16}$$

it is easy to see that

$$\begin{aligned}|Nx(t)| &= \left|\phi_q\left(\int_1^t f\left(s,x(s),D_{0^+}^{\alpha-1}x(s)\right)ds\right)\right| \\ &= \left|\int_1^t f\left(s,x(s),D_{0^+}^{\alpha-1}x(s)\right)ds\right|^{q-1} \\ &\leq \left(\int_0^1 \left|f\left(s,x(s),D_{0^+}^{\alpha-1}x(s)\right)\right|ds\right)^{q-1},\end{aligned} \tag{3.17}$$

according to conditions $(H_1)$ and ( 3.14), we obtain

$$\begin{aligned}\int_0^1 \left|f\left(s,x(s),D_{0^+}^{\alpha-1}x(s)\right)\right|ds &\leq \int_0^1 \Psi\left(s,|x(s)|,\left|D_{0^+}^{\alpha-1}x(s)\right|\right)ds \\ &\leq \int_0^1 \Psi\left(s,\|x\|_X,\|x\|_X\right)ds \\ &\leq \left(\frac{\Gamma(\alpha)}{8}\|x\|_X\right)^{\frac{1}{q-1}},\end{aligned} \tag{3.18}$$

by using (3.17) and (3.18) we get

$$\|Nx\|_\infty \leq \left(\int_0^1 \left|f\left(s,x(s),D_{0^+}^{\alpha-1}x(s)\right)\right|ds\right)^{q-1} \leq \frac{\Gamma(\alpha)}{8}\|x\|_X,$$

substituting this result in (3.16), we conclude that

$$\|x\|_X \leq \left(1+\frac{1}{\Gamma(\alpha)}\right)M_0+\left(2+\frac{\alpha+1}{\Gamma(\alpha+1)}\right)\frac{\Gamma(\alpha)}{8}\|x\|_X,$$

hence

$$\|x\|_X \leq \frac{1+\frac{1}{\Gamma(\alpha)}}{1-\left(2+\frac{\alpha+1}{\Gamma(\alpha+1)}\right)\frac{\Gamma(\alpha)}{8}}M_0,$$

where $1-\left(2+\frac{\alpha+1}{\Gamma(\alpha+1)}\right)\frac{\Gamma(\alpha)}{8}=1-\left(\frac{2\alpha\Gamma(\alpha)+\alpha+1}{8\alpha}\right)>0$, because $1<\alpha<2$ then $0<\Gamma(\alpha)<1$ consequently $0<2\alpha\Gamma(\alpha)+\alpha+1<7$ and $8\alpha>8$. Thus $\Omega_1$ is bounded.

**Step 2**: Let

$$\Omega_2=\{x\in\ker L : Nx\in\operatorname{Im} L\},$$

for all $x\in\Omega_2$ there exists a real constant $a$ such that $x(t)=at^{\alpha-1}$, for all $t\in[0;1]$ and as $Nx\in\operatorname{Im} L$ then

$$QNx=0.$$

In view of $(H_2)$, there exists $t_1\in[0;1]$ satisfying $\left|D_{0^+}^{\alpha-1}x(t_1)\right|=|a\Gamma(\alpha)|\leq M_0$, i.e $|a|\leq\frac{M_0}{\Gamma(\alpha)}$ which yields that

$$\|x\|_X=|a|+|a\Gamma(\alpha)|=|a|(1+\Gamma(\alpha))\leq\frac{M_0}{\Gamma(\alpha)}(1+\Gamma(\alpha)).$$

Then $\Omega_2$ is bounded.

**Step 3**: Assume that condition $(H_3)-(3.12)$ holds. Let

$$\Omega_3=\{x\in\ker L : \lambda Jx+(1-\lambda)QNx=0, \lambda\in[0;1]\},$$

where $J$ is the isomorphism defined by $J : KerL\to\operatorname{Im} Q$, $J(at^{\alpha-1})=a$. For $x=at^{\alpha-1}\in\Omega_3$ we have

$$\lambda Jx+(1-\lambda)QNx=\lambda a+\frac{1-\lambda}{1-\sum\limits_{i=1}^{i=m-2}\beta_i\eta_i}\sum_{i=1}^{i=m-2}\beta_i\int_{\eta_i}^{1}\phi_q\left(\int_1^t f\left(s,as^{\alpha-1},a\Gamma(\alpha)\right)ds\right)dt=0. \tag{3.19}$$

If $\lambda=0$, we get $QNx=0$ so by the condition $(H_2)$ there exists $t_2\in[0;1]$ such that $\left|D_{0^+}^{\alpha-1}x(t_2)\right|=|a\Gamma(\alpha)|\leq M_0$ so $|a|\leq\frac{M_0}{\Gamma(\alpha)}$ and hence

$$\|x\|_X=|a|+|a\Gamma(\alpha)|=|a|(1+\Gamma(\alpha))\leq\frac{M_0}{\Gamma(\alpha)}(1+\Gamma(\alpha)).$$

In the case $\lambda\neq0$, multiplying both sides of (3.19) by $\phi_q(a)$ and in view of the condition $(H_3)-(3.12)$ we get

$$-\lambda a^2|a|^{q-2}=\frac{1-\lambda}{1-\sum\limits_{i=1}^{i=m-2}\beta_i\eta_i}\sum_{i=1}^{i=m-2}\beta_i\int_{\eta_i}^{1}\phi_q\left(\int_1^t af\left(s,as^{\alpha-1},a\Gamma(\alpha)\right)ds\right)dt>0,$$

which contradict (3.12). Then $|a| \leq M_1$ which show that $\Omega_3$ is bounded.
If $(H_3) - (3.13)$ holds, we prove by the same method that

$$\Omega_3 = \{x \in \ker L : -\lambda Jx + (1-\lambda) QNx = 0, \lambda \in [0;1]\},$$

is bounded set. It remains to check that all conditions of Theorem 2.3.1 are fulfilled.
Let $\Omega$ be a bounded open set containing $\Omega_1 \cup \Omega_2 \cup \Omega_3$. As $\Omega_1, \Omega_2, \Omega_3$ are bounded sets then

- $Lx \neq \lambda Nx$ for all $(x, \lambda) \in [dom(L) \setminus \ker L \cap \partial\Omega] \times (0;1)$,
- $QNx \neq 0$ for all $x \in \ker L \cap \partial\Omega$.
- Without loss of generality, assume that $(H_3) - (3.12)$ holds and define the operator

  $$F(x, \lambda) = \lambda Jx + (1-\lambda)QNx,$$

  as $\Omega_3$ is bounded then, $F(\lambda, x) \neq 0$ for all $(x, \lambda) \in (\ker L \cap \partial\Omega) \times (0;1)$. Thus, by the homotopy property of degree, we have

  $$\begin{aligned} \deg(QN_{|\ker L}, \Omega \cap \ker L, 0) &= \deg(F(.,0), \Omega \cap \ker L, 0) \\ &= \deg(F(.,1), \Omega \cap \ker L, 0) \\ &= \deg(J, \Omega \cap \ker L, 0) \\ &\neq 0. \end{aligned}$$

  Consequently, the equation $Lx = Nx$ has at least one solution in $dom(L) \subset X$. Namely, BVPs ( 3.5)- (3.6)-(3.7 ) have at least one solution in the space $X$.

□

## 3.3 Example

Consider the following fractional differential boundary value problem with $p$-laplacian

$$\begin{cases} \left[\phi_p\left(D_{0^+}^{\frac{5}{4}} u(t)\right)\right]' = \frac{p}{2^p(2^p-1)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}\left(\sin u(t) + D_{0^+}^{\frac{1}{4}} u(t)\right)(t+1)\right], t \in [0;1], \\ u(0) = 0 = D_{0^+}^{\frac{5}{4}} u(1), \\ D_{0^+}^{\frac{1}{4}} u(1) = \frac{1}{2} D_{0^+}^{\frac{1}{4}} u\left(\frac{2}{5}\right) + \frac{1}{3} D_{0^+}^{\frac{1}{4}} u\left(\frac{3}{7}\right) + \frac{1}{6} D_{0^+}^{\frac{1}{4}} u\left(\frac{4}{9}\right), \end{cases} \tag{Ex.1}$$

where $\alpha=\frac{5}{4},\beta_1=\frac{1}{2},\beta_2=\frac{1}{3},\beta_3=\frac{1}{6},\eta_1=\frac{2}{5},\eta_2=\frac{3}{7},\eta_3=\frac{4}{9}$, and

$$f\left(t.u.v\right)=\frac{p}{2^p\left(2^p-1\right)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}\left(\sin u+v\right)\left(t+1\right)\right],$$

then, we have

$$\begin{aligned}\left|f\left(t.u.v\right)\right| &= \left|\frac{p}{2^p\left(2^p-1\right)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}\left(\sin u+v\right)\left(t+1\right)\right]\right| \\ &= \frac{p}{2^p\left(2^p-1\right)}\left(\frac{\Gamma\left(\frac{5}{4}\right)}{8}\left(t+1\right)\right)^{p-1}\left|\left(\sin u+v\right)\right|^{p-1} \\ &\leq \frac{p}{2^p\left(2^p-1\right)}\left(\frac{\Gamma\left(\frac{5}{4}\right)}{8}\left(t+1\right)\right)^{p-1}\left(\left|u\right|+\left|v\right|\right)^{p-1}=\Psi\left(t,\left|u\right|,\left|v\right|\right).\end{aligned}$$

It's clear that $\Psi:[0;1]\times\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+,\Psi\left(t,x,y\right)=\frac{p}{2^p(2^p-1)}\left(\frac{\Gamma\left(\frac{5}{4}\right)}{8}\left(t+1\right)\right)^{p-1}\left(x+y\right)^{p-1}$ is a function $L^1-$Caratheodory, non decreasing with respect to each of the variables $x$ or $y$ and

$$\begin{aligned}\int_0^1\Psi\left(t,m,m\right)dt &= \frac{p}{2^p\left(2^p-1\right)}\left(\frac{\Gamma\left(\frac{5}{4}\right)}{8}\right)^{p-1}\left(2m\right)^{p-1}\int_0^1\left(t+1\right)^{p-1}dt \\ &= \frac{p}{2^p\left(2^p-1\right)}\left(\frac{\Gamma\left(\frac{5}{4}\right)}{8}\right)^{p-1}\left(2m\right)^{p-1}\frac{2^p-1}{p} \\ &= \frac{1}{2}\left(\frac{\Gamma\left(\frac{5}{4}\right)}{8}m\right)^{\frac{1}{q-1}}\leq\left(\frac{\Gamma\left(\frac{5}{4}\right)}{8}m\right)^{\frac{1}{q-1}}.\end{aligned}$$

Taken $M_0=3$, if we have $D_{0^+}^{\frac{1}{4}}u\left(t\right)>3$ for all $t\in[0;1]$, then $\sin u\left(s\right)+D_{0^+}^{\frac{1}{4}}u\left(s\right)>2$ so

$$\frac{\Gamma\left(\frac{1}{4}\right)}{32}\left(s+1\right)\left(\sin u\left(s\right)+D_{0^+}^{\frac{1}{4}}u\left(s\right)\right)>\frac{\Gamma\left(\frac{1}{4}\right)}{16}\left(s+1\right),$$

as $\phi_p,\phi_q$ are nondecreasing, with $p,q>1$, $0<\frac{t+1}{2}\leq 1$ for all $t\in[0;1]$ and $\eta_i\in\,]0;1[$ $\left(i=1,2,3\right)$, therefore

$$\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}\left(s+1\right)\left(\sin u\left(s\right)+D_{0^+}^{\frac{1}{4}}u\left(s\right)\right)\right]>\left(\frac{\Gamma\left(\frac{1}{4}\right)}{16}\left(s+1\right)\right)^{p-1},$$

which leads us to

$$\begin{aligned}\int_t^1 \frac{p}{2^p(2^p-1)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s)\right)\right]ds &> \int_t^1 \frac{p}{2^p(2^p-1)}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{16}(s+1)\right)^{p-1}ds\\ &= \frac{1}{2^p-1}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{16}\right)^{p-1}\left(1-\left(\frac{t+1}{2}\right)^p\right),\end{aligned}$$

then

$$\begin{aligned}\int_1^t \frac{p}{2^p(2^p-1)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s)\right)\right]ds &< \frac{1}{2^p-1}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{16}\right)^{p-1}\left(\left(\frac{t+1}{2}\right)^p-1\right)\\ &\leq \frac{1}{2^p-1}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{16}\right)^{p-1}\left(\frac{t+1}{2}-1\right),\end{aligned}$$

thus

$$\phi_q\left(\int_1^t \frac{p}{2^p(2^p-1)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s)\right)\right]ds\right) \leq \frac{1}{(2^p-1)^{q-1}}\frac{\Gamma\left(\frac{1}{4}\right)}{16}\left(-\left(\frac{1-t}{2}\right)^{q-1}\right),$$

so

$$\int_{\eta_i}^1 \phi_q\left(\int_1^t \frac{p}{2^p}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s)\right)\right]ds\right)dt \leq \frac{1}{(2^p-1)^{q-1}}\frac{\Gamma\left(\frac{1}{4}\right)}{16}\left(-\frac{2}{q}\left(\frac{1-\eta_i}{2}\right)^q\right) < 0,$$

which proves that

$$\sum_{i=1}^{i=3}\beta_i\int_{\eta_i}^1 \phi_q\left(\int_1^t \frac{p}{2^p}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}s+1\left(\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s)\right)\right]\right)ds < 0,$$

because $\beta_i > 0$ for $i = 1,2,3$. Now, if we assume that $D_{0^+}^{\frac{1}{4}}u(t) < -3$ for all $t \in [0;1]$ similarly we get :

$$\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s) < -2,$$

thus

$$\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s)\right) < -\frac{\Gamma\left(\frac{1}{4}\right)}{16}(s+1),$$

consequontly

$$\frac{p}{2^p(2^p-1)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}s\left(\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s)\right)\right] < -\frac{p}{2^p(2^p-1)}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{16}(s+1)\right)^{p-1},$$

and

$$\int_t^1 \frac{p}{2^p(2^p-1)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s)\right)\right]ds < -\int_t^1 \frac{p}{2^p(2^p-1)}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{16}(s+1)\right)^{p-1}ds$$
$$= \frac{1}{2^p(2^p-1)}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{16}\right)^{p-1}\left((t+1)^p-2^p\right),$$

by multiplying by -1 the two members, we find

$$\int_1^t \frac{p}{2^p(2^p-1)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}s\left(\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s)\right)\right]ds > \frac{1}{2^p(2^p-1)}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{16}\right)^{p-1}\left(2^p-(t+1)^p\right)$$
$$= \frac{1}{2^p-1}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{16}\right)^{p-1}\left(1-\left(\frac{t+1}{2}\right)^p\right)$$
$$\geq \frac{1}{2^p-1}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{16}\right)^{p-1}\left(1-\frac{t+1}{2}\right),$$

then

$$\phi_q\left(\int_1^t \frac{p}{2^p(2^p-1)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s)\right)\right]ds\right) \geq \frac{1}{(2^p-1)^{q-1}}\frac{\Gamma\left(\frac{1}{4}\right)}{16}\left(\frac{1-t}{2}\right)^{q-1},$$

so

$$\int_{\eta_i}^1 \phi_q\left(\int_1^t \frac{p}{2^p}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s)\right)\right]ds\right)dt \geq \frac{1}{(2^p-1)^{q-1}}\frac{\Gamma\left(\frac{1}{4}\right)}{16}\frac{2}{q}\left(\frac{1-\eta_i}{2}\right)^q > 0,$$

therefore

$$\sum_{i=1}^{i=3}\beta_i\int_{\eta_i}^1 \phi_q\left(\int_1^t \frac{p}{2^p}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}s+1\left(\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s)\right)\right]\right)ds > 0 \text{ (i.e. } \neq 0).$$

Let $M_1=\frac{2}{\Gamma\left(\frac{5}{4}\right)}$. For all $a\in\mathbb{R}$ such that $|a|>M_1$

$$af\left(s,as^{\frac{1}{4}},a\Gamma\left(\frac{5}{4}\right)\right) = \frac{\phi_p(a)}{|a|^{p-2}}\frac{p}{2^p(2^p-1)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}\left(\sin as^{\frac{1}{4}}+a\Gamma\left(\frac{5}{4}\right)\right)(s+1)\right]$$
$$= \frac{1}{|a|^{p-2}}\frac{p}{2^p(2^p-1)}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\right)^{p-1}\phi_p\left[a\left(\sin as^{\frac{1}{4}}+a\Gamma\left(\frac{5}{4}\right)\right)\right].$$

If $a>\frac{2}{\Gamma\left(\frac{5}{4}\right)}$, we have $a>0$ and $\sin as^{\frac{1}{4}}+a\Gamma\left(\frac{5}{4}\right)\geq a\Gamma\left(\frac{5}{4}\right)-1>2-1=1>0$, then $af\left(s,as^{\frac{1}{4}},a\Gamma\left(\frac{5}{4}\right)\right)>0$. If $a<-\frac{2}{\Gamma\left(\frac{5}{4}\right)}$, we get $a<0$ and

$$\sin as^{\frac{1}{4}}+a\Gamma\left(\frac{5}{4}\right)\leq 1+a\Gamma\left(\frac{5}{4}\right)<1-2=-1<0,$$

thus $af\left(s,as^{\frac{1}{4}},a\Gamma\left(\frac{5}{4}\right)\right)>0$. Then the problem (Ex.1) satisfies all conditions of theorem 3.2.1, therefore it admeats at least one solution in $C^{\frac{1}{4}}[0,1]$.

# Chapter 4

# Generalized proportional fractional differential equations with multi-point boundary conditions

## 4.1 Introduction

In this chapter we study the existance of solutions for a class of fractional differential equations by using the extension of Mawhin's continuation theorem [15], more specifically, we consider the following generalized proportional fractional differential equation, with multi-point boundary conditions of the form:

$$ {}^{c}\mathfrak{D}_0^{\alpha,\rho}u(t) = f(t, u(t), {}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(t)), \qquad 0 < t < 1, \tag{4.1}$$

$$ u(0) = 0, \tag{4.2}$$

$$ {}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(1) = \sum_{i=1}^{i=m} \sigma_i \; {}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(\eta_i), \tag{4.3}$$

where ${}^{c}\mathfrak{D}_0^{\alpha,\rho}$ denote the generalized proportional fractional derivative of Caputo type of order $\alpha \in (1, 2], \rho \in (0, 1]$, $0 < \eta_i < 1$, $\sigma_i \in \mathbb{R}$, $\sum_{i=1}^{i=m} \sigma_i = 1, m \in \mathbb{N}^*$, and $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function to investigate the problem, we use the condition

$$ \sum_{i=1}^{i=m} \sigma_i \eta_i^{2-\alpha} e^{-\delta(1-\eta_i)} = 1, \tag{4.4}$$

where $\delta = \frac{\rho-1}{\rho}$.

Firstly, we consider the following constants:

$$\begin{aligned}
\Delta &= \int_0^1 e^{\delta(1-s)}\, ds - \sum_{i=1}^{i=m} \sigma_i \int_0^{\eta_i} e^{\delta(\eta_i - s)} ds, && (4.5)\\
L_1 &= \max_{0<t\leq 1} \left| t e^{\delta t} \right| = \max\{\frac{-1}{\delta e}, e^{\delta}\}, && (4.6)\\
L_2 &= \max_{0<t\leq 1} \left| t^{2-\alpha} e^{\delta t} \right| = \max\{(\frac{\alpha-2}{\delta e})^{2-\alpha}, e^{\delta}\}, && (4.7)\\
\kappa &= \frac{\rho^{\alpha}\Gamma(\alpha) e^{2\delta}}{\rho\Gamma(\alpha)(L_1+L_2) + e^{\delta}(1+\rho^{\alpha-1}\Gamma(\alpha))}. && (4.8)
\end{aligned}$$

And the function

$$\Lambda(t) = \frac{\rho^{\alpha-1}}{\Gamma(3-\alpha)} t^{2-\alpha} e^{\delta t} t \in [0,1]. \tag{4.9}$$

Also, we define the two linear operators $I_1, I_2 : Y \to Y$ by

$$I_1 y = \int_0^1 e^{\delta(1-s)} y(s) ds, \tag{4.10}$$

and

$$I_2 y = \sum_{i=1}^{i=m} \sigma_i \int_0^{\eta_i} e^{\delta(\eta_i - s)} y\,(s)\, ds. \tag{4.11}$$

We will consider the Banach spaces

$$X = C^{\alpha-1}[0,1] = \left\{ u : u(t) = \mathcal{J}^{\alpha-1,\rho} z(t); z(t) \in C[0,1] \right\},$$

with the norm

$$||u||_X = ||u||_{\infty} + \left\| {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} u \right\|_{\infty},$$

where $||u||_{\infty} = \max_{t\in[0,1]} |u\,(t)|$ and $Y = L^1[0,1]$ with the norm $||y||_Y = ||y||_1$.

Define the two operators $L, N : dom(L) \subset X \to Y$ as follows:

$$Lu(.) =^{c} \mathfrak{D}_0^{\alpha,\rho} u(t)\ , \quad u \in dom(L), \tag{4.12}$$

and

$$Nu\,(t) = f(t, u(t), {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} u(t)), \tag{4.13}$$

where

$$dom(L) = \left\{ u \in X \ \text{ s.t. } \mathfrak{D}_0^{\alpha,\rho} u(t) \in L^1[0,1], u(0) = 0, {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} u(1) = \sum_{i=1}^{i=m} \sigma_i\ {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} u(\eta_i) \right\}.$$

Notice that problem(4.1)-(4.2)-(4.3) can be converted to the abstract operator equation $Lu = Nu$, $u \in dom(L)$.

**Lemma 4.1.1.** *[15] $M \subset C^{\theta}[0,1]$ is relatively compact set if and only if*

1. *$M$ is uniformly bounded: there exists $m > 0$, such that $\|u\|_{C^{\theta}} \leq m$, for every $u \in M$.*
2. *$M$ is equicontinuous: for every $\varepsilon > 0$, there exists $\delta > 0$, such that for all $t_1, t_2 \in [0,1]$, $|t_2 - t_1| < \delta$, we have*

$$|u(t_2) - u(t_1)| < \varepsilon \text{ and } \left|{}^{c}\mathfrak{D}_0^{\theta-k,\rho}u(t_2) - {}^{c}\mathfrak{D}_0^{\theta-k,\rho}u(t_1)\right| < \varepsilon,$$

*for all $u \in M$ with $k = 0, 1, 2, \ldots, [\theta]$.*

## 4.2 Main result

### 4.2.1 Some auxiliary lemmas

**Lemma 4.2.1.** *Let $L$ be the operator defined by (4.12), then*

$$\ker L = \left\{cte^{\delta t} : c \in \mathbb{R}\right\} \quad \text{and} \quad \operatorname{Im} L = \{y \in Y : \ I_1 y - I_2 y = 0\}.$$

***Proof.*** For each $u \in \ker L$, we have $Lu(t) = {}^{c}\mathfrak{D}_0^{\alpha,\rho}u(t) = 0$. So, it's equivalent to

$$u(t) = c_0 e^{\delta t} + c_1 t e^{\delta t}, \ t \in [0,1],$$

as condition (4.2), imply $c_0 = 0$. So $u(t) = c_1 t e^{\delta t}$.

Now for all $y \in \operatorname{Im} L$, there exist $u \in dom(L)$ such that

$${}^{c}\mathfrak{D}_0^{\alpha,\rho}u(t) = y(t),$$

by (1.6) we get

$$u(t) = \mathcal{J}_0^{\alpha,\rho}y(t) + c_0 e^{\delta t} + c_1 t e^{\delta t},$$

from the condition (4.2) we find $c_0 = 0$, and in view of conditions (4.3)-(4.4), we obtain that

$$u(t) = c_1 t e^{\delta t} + \mathcal{J}_0^{\alpha,\rho}y(t),$$

thus

$$ {}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(t) = c_1\left({}^{c}\mathfrak{D}_0^{\alpha-1,\rho}te^{\delta t}\right) + \frac{1}{\rho}\int_0^t e^{\delta(t-s)}y(s)ds, $$

applying (1.4) we get

$$ {}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(t) = \frac{c_1\rho^{\alpha-1}}{\Gamma(3-\alpha)}t^{2-\alpha}e^{\delta t} + \frac{1}{\rho}\int_0^t e^{\delta(t-s)}y(s)ds, $$

from ${}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(1) = \sum_{i=1}^{i=m}\sigma_i\ {}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(\eta_i)$, we obtain

$$ \frac{c_1\rho^{\alpha-1}}{\Gamma(3-\alpha)}e^{\delta} + \frac{1}{\rho}\int_0^1 e^{\delta(1-s)}y(s)ds = \sum_{i=1}^{i=m}\sigma_i\frac{c_1\rho^{\alpha-1}}{\Gamma(3-\alpha)}\eta_i^{2-\alpha}e^{\delta\eta_i} + \frac{1}{\rho}\sum_{i=1}^{i=m}\sigma_i\int_0^{\eta_i}e^{\delta(\eta_i-s)}y(s)ds, $$

also

$$ \int_0^1 e^{\delta(1-s)}y(s)ds = \sum_{i=1}^{i=m}\sigma_i\int_0^{\eta_i}e^{\delta(\eta_i-s)}y(s)ds, $$

we conclude that

$$ I_1y - I_2y = 0. \tag{4.14} $$

On other hand, suppose that $y \in Y$ satisfies (4.14). If $u(t) = \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{\delta(t-s)}y(s)ds$ then $u \in dom(L)$,indeed $u(t) = \mathcal{J}_0^{\alpha-1,\rho}\mathcal{J}_0^{1,\rho}y(t)$
and we can easily show the boundary conditions(4.2)-(4.3) hold, which means that ${}^{c}\mathfrak{D}_0^{\alpha,\rho}u(t) = y(t)$ so $y \in \mathrm{Im}(L)$. □

**Remark 4.1.** *It easy to show that* $\Delta \neq 0$.

***Proof.*** As $\frac{1}{\delta} < 0$, suffices to proof that $\sum_{i=1}^{i=m}\sigma_i\left(1-e^{\delta\eta_i}\right) + e^{\delta} - 1 < 0$.
Or $\sum_{i=1}^{i=m}\sigma_i\left(1-e^{\delta\eta_i}\right) + e^{\delta} - 1 > 0$, by the resonance condition (4.4), we have

$$ \sum_{i=1}^{i=m}\sigma_i\ \left(1-e^{\delta\eta_i}\right) + e^{\delta} - 1 = \sum_{i=1}^{i=m}\sigma_i\ e^{\delta\eta_i}\left(\eta_i^{2-\alpha}-1\right), $$

for all $i = 1,2,...,m$, $\sigma_i \geq 0, \eta_i \in ]0;1[$, $0 < e^{\delta\eta_i} < 1$, $\sigma_i e^{\delta\eta_i} \geq 0$, and $\left(\eta_i^{2-\alpha}-1\right) < 0$, we have $\sigma_i e^{\delta\eta_i}\left(\eta_i^{2-\alpha}-1\right) \leq 0$, by the condition $\sum_{i=1}^{i=m}\sigma_i = 1$, there exists at least $i_0 \in \{1,2,...,m\}$ such that $\sigma_{i_0} \neq 0$ and hence $\sigma_{i_0}e^{\delta\eta_{i_0}}\left(\eta_{i_0}^{2-\alpha}-1\right) < 0$ which prove that

$$ \sum_{i=1}^{i=m}\sigma_i\ e^{\delta\eta_i}\left(\eta_i^{2-\alpha}-1\right) < 0. $$

Therefore $\Delta \neq 0$. □

**Lemma 4.2.2.** *We can define two linear continuous projectors $P$ and $Q$ as follow*

$$P : X \to X \text{ such that} Pu\,(t) = \frac{\Gamma(3-\alpha)}{\rho^{\alpha-1}e^{\delta}} te^{\delta t}\ {}^{c}\mathfrak{D}_{0}^{\alpha-1,\rho}u(1),$$

*and*

$$Q : Y \to Y \text{ such that } Qy(t) = \frac{1}{\Delta}(I_1 y - I_2 y),$$

*where* $\Delta \neq 0$.

*The inverse of the operator* $L_P = L_{|dom(L)\cap \ker P}$ *is the operator* $K_P : \operatorname{Im} L \to dom\,(L) \cap \ker P$ *defined by*

$$K_P y(t) = \mathcal{J}_{0}^{\alpha,\rho} y\,(t) = \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} e^{\delta(t-s)} y(s) ds,$$

*and checking*

$$\|K_P y\|_X \leq C' \|y\|_1\,, \tag{4.15}$$

*where* $C' = \frac{1+\rho^{\alpha-1}\Gamma(\alpha)}{\rho^{\alpha}\Gamma(\alpha)}$.

***Proof.*** For all $u \in X$, we get

$${}^{c}\mathfrak{D}_{0}^{\alpha-1,\rho} Pu\,(1) = {}^{c}\,\mathfrak{D}_{0}^{\alpha-1,\rho} u\,(1)\,,$$

we have

$${}^{c}\mathfrak{D}_{0}^{\alpha-1,\rho} Pu\,(t) = \frac{1}{e^{\delta}}\,{}^{c}\mathfrak{D}_{0}^{\alpha-1,\rho} u\,(1) \left[t^{2-\alpha} e^{\delta t}\right],$$

then

$$P\,(Pu(t)) = \frac{\Gamma(3-\alpha)}{\rho^{\alpha-1}e^{\delta}} te^{\delta t}\ {}^{c}\mathfrak{D}_{0}^{\alpha-1,\rho} Pu(1) = Pu(t),$$

for $t \in [0,1]$. We note that $\operatorname{Im} P = \ker L, \ker P = \left\{u \in X; {}^{c}\mathfrak{D}_{0}^{\alpha-1,\rho} u\,(1) = 0\right\}$. From Lemma 1.3.7 we have

$$\begin{aligned} |Pu(t)| &= \frac{\Gamma(3-\alpha)}{\rho^{\alpha-1}e^{\delta}} \left|te^{\delta t}\ {}^{c}\mathfrak{D}_{0}^{\alpha-1,\rho} u(1)\right| \\ &\leq \frac{L_1\Gamma(3-\alpha)}{\rho^{\alpha-1}e^{\delta}} \left|{}^{c}\mathfrak{D}_{0}^{\alpha-1,\rho} u(1)\right| \\ &\leq \frac{L_1}{\rho^{\alpha-1}e^{\delta}} \|u\|_X\,, \end{aligned}$$

on other hand

$$\left|{}^{c}\mathfrak{D}_{0}^{\alpha-1,\rho} Pu(t)\right| \leq \frac{L_2}{\rho^{\alpha-1}e^{\delta}} \left|{}^{c}\mathfrak{D}_{0}^{\alpha-1,\rho} u(1)\right| \leq \frac{L_2}{\rho^{\alpha-1}e^{\delta}} \|u\|_X\,, \tag{4.16}$$

then

$$\|Pu\|_X = \|Pu\|_\infty + \left\|{}^c\mathfrak{D}_0^{\alpha-1,\rho}Pu\right\|_\infty \leq \left(\frac{L_1+L_2}{\rho^{\alpha-1}e^\delta}\right)\|u\|_X . \tag{4.17}$$

For all $y \in Y$, taking $\frac{1}{\Delta}(I_1y - I_2y) = \upsilon$, thus

$$\begin{aligned} Q^2y &= Q(Qy) \\ &= \frac{1}{\Delta}\left(\int_0^1 e^{\delta(1-s)}\ \upsilon ds - \sum_{i=1}^{i=m}\sigma_i\int_0^{\eta_i} e^{\delta(\eta_i-s)}\upsilon ds\right) \\ &= \frac{\upsilon}{\Delta}\Delta = \upsilon, \end{aligned}$$

so $Q^2 = Q$. Furthermore, we have

$$\|Qy\|_1 \leq C\|y\|_1 , \tag{4.18}$$

where $C = \frac{1+\Delta}{\Delta}$. For any $u \in dom(L) \cap \ker P$, by proposition 1.3.1-(1.6) we can write

$$K_PLu(t) = \mathcal{J}_0^{\alpha,\rho}\ {}^c\mathfrak{D}_0^{\alpha,\rho}u(t) = u(t) + c_0e^{\delta t} + c_1te^{\delta t},$$

with $t \in (0,1]$ and $c_0, c_1$ are two real constants. As $K_PLx \in dom(L) \cap \ker P$, then $c_0 = 0$ and

$$\begin{aligned} {}^c\mathfrak{D}_0^{\alpha-1,\rho}\left(u(t) + c_1te^{\delta t}\right)\Big|_{t=1} &= {}^c\mathfrak{D}_0^{\alpha-1,\rho}u(1) + c_1\frac{\rho^{\alpha-1}}{\Gamma(3-\alpha)}e^\delta \\ &= c_1\frac{\rho^{\alpha-1}}{\Gamma(3-\alpha)}e^\delta = 0, \end{aligned}$$

which imply that $c_1 = 0$, therefore $K_PLu = u$.
If $y \in \operatorname{Im} L$, we get $LK_Py(t) ={}^c \mathfrak{D}_0^{\alpha,\rho}\mathcal{J}_0^{\alpha,\rho}y(t) = y(t)$ which show that $K_P = (L_P)^{-1}$, and the other hand

$$\begin{aligned} \|K_Py\|_X &= \|\mathcal{J}_0^{\alpha,\rho}y\|_\infty + \left\|\mathcal{J}_0^{1,\rho}y\right\|_\infty \\ &\leq C'\|y\|_1 . \end{aligned}$$

Which complete the proof. □

**Lemma 4.2.3.** *L is a Fredholm operator of index 0.*

***Proof.*** For any $y \in Y$, we can write $y = (I-Q)y + Qy$, with $(I-Q)y \in \ker Q = \operatorname{Im} L$ and $Qy \in \operatorname{Im} Q$ then $y \in \operatorname{Im} L + \operatorname{Im} Q$. Assume that $y \in \operatorname{Im} L \cap \operatorname{Im} Q$ thus $y = c \in \mathbb{R}$ such that

$I_1c - I_2c = 0 = c\left(\int_0^1 e^{\delta(1-s)}ds - \sum_{i=1}^{i=m} \sigma_i \int_0^{\eta_i} e^{\delta(\eta_i - s)}ds\right) = \Delta c = 0$, i.e., $c = 0$, which imply that $\operatorname{Im} L \cap \operatorname{Im} Q = \phi$ an hence $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. As $\dim \operatorname{Im} Q = \dim \ker L = 1$, then $L$ is a Fredholm operator of index 0. □

**Lemma 4.2.4.** *Assume that $M$ is an open bounded subset in $X$ with $dom\,(L) \cap M \neq \phi$. The operator $N$ L-compact on $\overline{M}$.*

***Proof***. The boundness of $M$ imply that there exists $R > 0$ such that for all $u \in M$, we have $\|u\|_X = \|u\|_\infty + \left\|{}^c\mathfrak{D}_0^{\alpha-1,\rho}u\right\|_\infty \leq R$. By the continuity of $f$ there exists $A > 0$ such that $|f\,(s, u(s))| \leq A$ for all $u \in M$, we get

$$\|QNu\|_1 \leq C\,\|Nu\|_1 \leq CA,$$

and

$$\|(I-Q)\,Nu\|_1 \leq \|Nu\|_1 + \|QNu\|_1 \leq (C+1)A, \tag{4.19}$$

we conclude that

$$\|K_P\,(I-Q)\,Nu\|_X \leq C'\,\|(I-Q)\,Nu\|_\infty \leq C'(C+1)A,$$

then $QN\,(M)$ and $K_{P,Q}N\,(M)$ are bounded, we only need to prove that $K_{P,Q}N\,(M)$ is equicontinuous. Putting $0 \leq t_1 \leq t_2 \leq 1$

$$\begin{aligned}
& |K_P\,(I-Q)\,Nu\,(t_2) - K_P\,(I-Q)\,Nu\,(t_1)| \\
= & \frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_0^{t_2}(t_2-s)^{\alpha-1}e^{\delta(t_2-s)}\,(I-Q)\,Nu(s)ds - \int_0^{t_1}(t_1-s)^{\alpha-1}e^{\delta(t_1-s)}\,(I-Q)\,Nu(s)ds\right| \\
= & \frac{1}{\rho^\alpha\Gamma(\alpha)}\left(\left|\int_0^{t_1}(t_2-s)^{\alpha-1}e^{\delta(t_2-s)}\,(I-Q)\,Nu(s)ds - \int_0^{t_1}(t_1-s)^{\alpha-1}e^{\delta(t_1-s)}\,(I-Q)\,Nu(s)ds\right.\right. \\
+ & \left.\left.\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}e^{\delta(t_2-s)}\,(I-Q)\,Nu(s)ds\right|\right) \\
\leq & \frac{1}{\rho^\alpha\Gamma(\alpha)}\left(\int_0^{t_1}\left|(t_2-s)^{\alpha-1}e^{\delta(t_2-s)} - (t_1-s)^{\alpha-1}e^{\delta(t_1-s)}\right|\left|(I-Q)\,Nu(s)\right|ds\right. \\
+ & \left.\int_{t_1}^{t_2}\left|(t_2-s)^{\alpha-1}e^{\delta(t_2-s)}\right|\left|(I-Q)\,Nu(s)\right|ds\right) \\
\leq & \frac{\|(I-Q)\,Nu\|_1}{\rho^\alpha\Gamma(\alpha)}\left(\int_0^{t_1}\left|(t_2-s)^{\alpha-1}e^{\delta(t_2-s)} - (t_1-s)^{\alpha-1}e^{\delta(t_1-s)}\right|ds\right. \\
+ & \left.\int_{t_1}^{t_2}\left|(t_2-s)^{\alpha-1}e^{\delta(t_2-s)}\right|ds\right),
\end{aligned}$$

using Lemma 1.3.5 and the inequality (4.19) we get

$$\begin{aligned}
& \left|K_P\left(I-Q\right)Nu\left(t_2\right)-K_P\left(I-Q\right)Nu\left(t_1\right)\right| \\
\leq\ & \frac{(C+1)A}{\rho^{\alpha}\Gamma(\alpha)}\left(\int_0^{t_1}\left|(t_2-s)^{\alpha-1}e^{\delta(t_2-s)}-(t_1-s)^{\alpha-1}e^{\delta(t_1-s)}\right|ds\right. \\
+\ & \frac{(C+1)A}{(1-\rho)^{\alpha}}\left[\mathfrak{P}(\alpha,-\delta(t_2-t_1))-0\right],
\end{aligned}$$

from Lemma 1.3.6 we obtain

$$\left|K_P\left(I-Q\right)Nu\left(t_2\right)-K_P\left(I-Q\right)Nu\left(t_1\right)\right|\to 0 \text{ as } t_1\to t_2.$$

On other hand, we have

$$\begin{aligned}
{}^{c}\mathfrak{D}_0^{\alpha-1,\rho}K_P\left(I-Q\right)Nu\left(t\right) &= \mathcal{J}_0^{1,\rho}\left(I-Q\right)Nu(t) \\
&= \frac{1}{\rho}\int_0^t e^{\delta(t-s)}\left(I-Q\right)Nu\left(s\right)ds.
\end{aligned}$$

Similarity

$$\begin{aligned}
& \left|{}^{c}\mathfrak{D}_0^{\alpha-1,\rho}K_P\left(I-Q\right)Nu\left(t_2\right)-{}^{c}\mathfrak{D}_0^{\alpha-1,\rho}K_P\left(I-Q\right)Nu\left(t_1\right)\right| \\
=\ & \frac{1}{\rho}\left|\int_0^{t_2}e^{\delta(t_2-s)}\left(I-Q\right)Nu(s)ds-\int_0^{t_1}e^{\delta(t_1-s)}\left(I-Q\right)Nu(s)ds\right| \\
\leq\ & \frac{(C+1)A}{\rho}\left(\int_0^{t_1}\left|e^{\delta(t_2-s)}-e^{\delta(t_1-s)}\right|ds+\int_{t_1}^{t_2}e^{\delta(t_2-s)}ds\right),
\end{aligned}$$

from Lemma 1.3.6 (with $\alpha=1$) we get

$$\left|{}^{c}\mathfrak{D}_0^{\alpha-1,\rho}K_P\left(I-Q\right)Nu\left(t_2\right)-{}^{c}\mathfrak{D}_0^{\alpha-1,\rho}K_P\left(I-Q\right)Nu\left(t_1\right)\right|\to 0 \text{ as } t_1\to t_2.$$

According to the Lemma 4.1.1, $K_P\left(I-Q\right)N\left(M\right)$ is compact, which show that $N$ is $L$-compact on $M$. □

### 4.2.2 An existence theorem

**Theorem 4.2.1.** *Suppose that there exists:*

$(C_1)$ *$L^1-$Carathéodory function $\Phi:[0,1]\times\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ which is non decreasing with respect to the last two variables such that*

$$\left|f\left(t,x,y\right)\right|\leq\Phi\left(t,\left|x\right|,\left|y\right|\right),$$

*for all $(x;y)\in\mathbb{R}^2$ and $t\in\left[0,1\right]$.*

$(C_2)$ *A real* $M_0 > 0$, *such that if we have* $\left|{}^c\mathfrak{D}_0^{\alpha-1,\rho}u(t)\right| > M_0$ *for all* $t \in [0,1]$, *then*

$$I_1 f(t,u(t),{}^c\mathfrak{D}_0^{\frac{1}{4},\frac{1}{2}}u(t)) - I_2 f(t,u(t),{}^c\mathfrak{D}_0^{\alpha-1,\rho}u(t)) \neq 0.$$

$(C_3)$ *A real* $M_1 > 0$, *such that for* $|c| > M_1$, *then either*

$$c\left(I_1 N(cte^{\delta t}) - I_2 N(cte^{\delta t}\right) > 0, \tag{4.20}$$

*or*

$$c(I_1 N(cte^{\delta t}) - I_2 N(cte^{\delta t})) < 0, \tag{4.21}$$

*then the fractional BVPs (4.1)-(4.2)-(4.3) has at least one solution in* $dom\,(L) \subset X$ *provided that*

$$\int_0^1 \Phi(t,r,r)\,dt \leq \frac{\rho^\alpha \Gamma(\alpha) e^{2\delta}}{\rho\Gamma(\alpha)(L_1+L_2)+e^\delta(1+\rho^{\alpha-1}\Gamma(\alpha))} r + \beta. \tag{4.22}$$

*Where* $\beta$ *is a positive constant.*

***Proof.*** **Step1**: Let

$$\Omega_1 = \{u \in dom\,(L) \setminus \ker L : Lu = \lambda N u, \lambda \in [0,1]\},$$

we will show that it is a bounded set.

Notice that if $u \in \Omega_1$ then $\lambda \in (0,1]$, because $\Omega_1 \cap \ker L = \phi$, which allows us to write $Nu = L\frac{1}{\lambda}u \in \operatorname{Im} L = \ker Q$, then

$$QNu = \int_0^1 e^{\delta(1-s)} f(s,u(s),{}^c\mathfrak{D}_0^{\alpha-1,\rho}u(s))ds - \sum_{i=1}^{i=m}\sigma_i \int_0^{\eta_i} e^{\delta(\eta_i-s)} f(s,u(s),{}^c\mathfrak{D}_0^{\alpha-1,\rho}u(s))ds = 0,$$

by the condition $(C_2)$, there exists $t_0 \in [0,1]$ such that $\left|\mathfrak{D}_0^{\alpha-1,\rho}u(t_0)\right| \leq M_0$. On the other hand, we have

$$\begin{aligned} {}^c\mathfrak{D}_0^{\alpha-1,\rho}u(t) &= {}^c\mathfrak{D}_0^{\alpha-1,\rho}u(t_0) + \int_{t_0}^t e^{\delta(t-s)}\ {}^c\mathfrak{D}_0^{\alpha,\rho}u(s)\,ds \\ &= {}^c\mathfrak{D}_0^{\alpha-1,\rho}u(t_0) + \int_{t_0}^t e^{\delta(t-s)} f\left(s,u(s),{}^c\mathfrak{D}_0^{\alpha-1,\rho}u(s)\right)ds, \end{aligned}$$

then

$$\begin{aligned} \left|{}^c\mathfrak{D}_0^{\alpha-1,\rho}u(t)\right| &\leq \left|{}^c\mathfrak{D}_0^{\alpha-1,\rho}u(t_0)\right| + \left|\int_{t_0}^t e^{\delta(t-s)} f\left(s,u(s),{}^c\mathfrak{D}_0^{\alpha-1,\rho}u(s)\right)ds\right| \\ &\leq M_0 + \|Nu\|_1, \end{aligned} \tag{4.23}$$

furthermore, we can write

$$u=(I-P)\,u+Pu \quad = K_P L\,(I-P)\,u+Pu$$
$$= K_P Lu+Pu,$$

then

$$\|u\|_X \le \|K_P Lu\|_X + \|Pu\|_X\,,$$

by using (4.23), we obtain

$$\begin{aligned}
|Pu\,(t)| &= \frac{\Gamma(3-\alpha)}{\rho^{\alpha-1}e^{\delta}}\left|te^{\delta t}\;{}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(1)\right| \\
&\le \frac{L_1}{\rho^{\alpha-1}e^{\delta}}\left|{}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u\,(1)\right| \\
&\le \frac{L_1}{\rho^{\alpha-1}e^{\delta}}\left(M_0+\|Nu\|_1\right),
\end{aligned}$$

and

$$\begin{aligned}
\left|{}^{c}\mathfrak{D}_0^{\alpha-1,\rho}Pu\,(t)\right| &= \frac{L_2}{e^{\delta}}\left|{}^{c}\mathfrak{D}_0^{\alpha-1}u\,(1)\right| \\
&\le \frac{L_2}{\rho^{\alpha-1}e^{\delta}}(M_0+\|Nx\|_1),
\end{aligned}$$

then

$$\|Pu\|_X \le \frac{L_1+L_2}{\rho^{\alpha-1}e^{\delta}}\left(M_0+\|Nu\|_1\right),$$

by simple calculations, we have

$$\|K_P Lu\|_X \;\le\; \frac{1+\rho^{\alpha-1}\Gamma(\alpha)}{\rho^{\alpha}\Gamma(\alpha)}\,\|Nu\|_1\,,$$

which give

$$\|u\|_X \le \left(\frac{L_1+L_2}{\rho^{\alpha-1}e^{\delta}}\right)M_0+\left(\frac{\rho\Gamma(\alpha)\,(L_1+L_2)+e^{\delta}\,(1+\rho^{\alpha-1}\Gamma(\alpha))}{\rho^{\alpha}\Gamma(\alpha)e^{\delta}}\right)\|Nu\|_1, \tag{4.24}$$

it is easy to see that

$$|Nu\,(t)| = \left|f\left(s,u\,(s)\,,{}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u\,(s)\right)ds\right|, \tag{4.25}$$

according to conditions $(C_1)$ and (4.22), we obtain

$$\begin{aligned}
\int_0^1 \left|f\left(s,u\,(s)\,,{}^{c}\mathfrak{D}_0^{\alpha-1}u\,(s)\right)\right|ds &\le \int_0^1 \Phi\left(s,|u\,(s)|\,,\left|{}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u\,(s)\right|\right)ds \\
&\le \int_0^1 \Phi\left(s,\|u\|_X\,,\|u\|_X\right)ds \\
&\le k\,\|u\|_X+\beta,
\end{aligned} \tag{4.26}$$

then $\|Nu\|_1 \leq \kappa.\|u\|_X + \beta$ substitude this result in (4.26) we get

$$\begin{aligned}\|u\|_X &\leq \left(\frac{L_1+L_2}{\rho^{\alpha-1}e^{\delta}}\right) M_0 + \left(\frac{\rho\Gamma(\alpha)\left(L_1+L_2\right)+e^{\delta}\left(1+\rho^{\alpha-1}\Gamma(\alpha)\right)}{\rho^{\alpha}\Gamma(\alpha)e^{\delta}}\right)\|Nu\|_1 \\ &\leq \left(\frac{L_1+L_2}{\rho^{\alpha-1}e^{\delta}}\right) M_0 + \left(\frac{\rho\Gamma(\alpha)\left(L_1+L_2\right)+e^{\delta}\left(1+\rho^{\alpha-1}\Gamma(\alpha)\right)}{\rho^{\alpha}\Gamma(\alpha)e^{\delta}}\right)(\kappa.\|u\|_X+\beta) \\ &= \left(\frac{L_1+L_2}{\rho^{\alpha-1}e^{\delta}}\right) M_0 + e^{\delta}\|u\|_X + \left(\frac{\rho\Gamma(\alpha)\left(L_1+L_2\right)+e^{\delta}\left(1+\rho^{\alpha-1}\Gamma(\alpha)\right)}{\rho^{\alpha}\Gamma(\alpha)e^{\delta}}\right)\beta,\end{aligned}$$

we conclude that

$$\|u\|_X \leq \frac{\left(\frac{L_1+L_2}{\rho^{\alpha-1}e^{\delta}}\right) M_0 + \left(\frac{\rho\Gamma(\alpha)(L_1+L_2)+e^{\delta}\left(1+\rho^{\alpha-1}\Gamma(\alpha)\right)}{\rho^{\alpha}\Gamma(\alpha)e^{\delta}}\right)\beta}{1-e^{\delta}}.$$

Thus $\Omega_1$ is bounded.

**Step 2** Let :

$$\Omega_2 = \{u \in \ker L : Nu \in \operatorname{Im} L\},$$

for all $u \in \Omega_2$ there exists a real constant $c$ such that $u(t) = cte^{\delta t}, t \in [0,1]$ and as $Nu \in \operatorname{Im} L$ then

$$QNu = 0.$$

In view of $(C_3)$, there exists $t_1 \in [0,1]$ satisfying $\left|{}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(t_1)\right| = \left|\frac{c\rho^{\alpha-1}}{\Gamma(3-\alpha)}t_1^{2-\alpha}e^{\delta t_1}\right| \leq M_0$ i.e $|c| \leq \frac{M_0}{L_2\rho^{\alpha-1}}$, which yields that

$$\begin{aligned}\|u\|_X &= |cL_1| + \left|c\frac{L_2\rho^{\alpha-1}}{\Gamma(3-\alpha)}\right| \\ &= |c|\left(L_1 + \frac{L_2\rho^{\alpha-1}}{\Gamma(3-\alpha)}\right) \\ &\leq \frac{M_0}{\rho^{\alpha-1}L_2}(L_1+1).\end{aligned}$$

Then $\Omega_2$ is bounded.

**Step 3**: Assume that condition $(C_3)-(4.20)$ holds. Let :

$$\Omega_3 = \{u \in \ker L : \lambda Ju + (1-\lambda)QNu = 0, \lambda \in [0,1]\},$$

where $J$ is the isomorphism defined by $J : \ker L \to \operatorname{Im} Q, J\left(cte^{\delta t}\right) = c$. For $u = c\,te^{\delta t} \in \Omega_3$ we have

$$\lambda Ju + (1-\lambda)QNu = \lambda c + \frac{1-\lambda}{\Delta}\left(I_1 f(s, cse^{\delta s}, \Lambda(s)) - I_2 f(s, cse^{\delta s}, \Lambda(s))\right) = 0, \quad (4.27)$$

if $\lambda = 0$, we get $QNu = 0$ so by the condition $(C_3)$ there exists $t_2 \in [0,1]$ such that

$$\left|{}^{c}\mathfrak{D}_0^{\alpha-1,\rho} u\,(t_2)\right| = \left|\frac{c\rho^{\alpha-1}}{\Gamma(3-\alpha)} t_2^{2-\alpha} e^{\delta t_2}\right| \leq M_0,$$

so $|c| \leq \frac{M_0}{L_2 \rho^{\alpha-1}}$ and hence

$$\begin{aligned} \|u\|_X &= |cL_1| + \left|c\frac{L_2\rho^{\alpha-1}}{\Gamma(3-\alpha)}\right| \\ &= |c|\left(L_1 + \frac{L_2\rho^{\alpha-1}}{\Gamma(3-\alpha)}\right) \\ &\leq \frac{M_0}{\rho^{\alpha-1}L_2}\left(L_1+1\right). \end{aligned}$$

In the case $\lambda \neq 0$, in view of the condition $(C_3) - (4.20)$ we get

$$-\lambda c^2 = \frac{(1-\lambda)c}{\Delta}(I_1 f(s, cse^{\delta s}, \Lambda(s)) - I_2 f(s, cse^{\delta s}, \Lambda(s))) > 0,$$

which contradict (4.20). Then $|c| \leq M_1$ which show that $\Omega_3$ is bounded.

If $(C_3) - (4.20)$ holds, we prove by the same method that

$$\Omega_3 = \{u \in \ker L : -\lambda Ju + (1-\lambda)\,QNu = 0, \lambda \in [0,1]\},$$

is bounded set. It remains to check that all conditions of Theorem 2.3.1 are fulfilled.

Let $\Omega_1 \cup \Omega_2 \cup \Omega_3 \subset \Omega$. As $\Omega_1, \Omega_2$ and $\Omega_3$ are bounded sets, then

(1) $Lu \neq \lambda Nu$ for all $(u,\lambda) \in (dom(L)\backslash \ker L) \cap \partial\Omega \times (0,1)$,

(2) $QNu \neq 0$for all $x \in \ker L \cap \partial\Omega$,

(3) Without loss of generality, assume that $(C_3) - (4.20)$ holds and define the operator

$$F(u,\lambda) = \lambda Ju + (1-\lambda)QNu,$$

as $\Omega_3$ is bounded then, $F(\lambda, u) \neq 0$ for all $(u,\lambda) \in (\ker L \cap \partial\Omega) \times (0,1)$. Thus, by the homotopy property of degree, we have

$$\begin{aligned} \deg(QN_{|\ker L}, \Omega \cap \ker L, 0) &= \deg(F(.,0), \Omega \cap \ker L, 0) \\ &= \deg(F(.,1), \Omega \cap \ker L, 0) \\ &= \deg(J, \Omega \cap \ker L, 0) \\ &\neq 0. \end{aligned}$$

Consequently, the equation $Lu = Nu$ has at least one solution in $dom(L) \subset X$. Namely BVPs (4.1)-(4.2)-(4.3 ) has at least one solution in the space $X$. □

## 4.3 A numerical example

Consider the boundary value problem :

$$\begin{cases} {}^cD_0^{\frac{3}{2},\frac{1}{2}}u(t) = f\left(t, u(t), {}^cD_0^{\frac{1}{2},\frac{1}{2}}u(t)\right), t \in [0;1], \\ u(0) = 0, \\ {}^cD_0^{\frac{1}{2},\frac{1}{2}}u(1) = \sigma_1 {}^cD_0^{\frac{1}{2},\frac{1}{2}}u(\eta_1) + \sigma_2 {}^cD_0^{\frac{1}{2},\frac{1}{2}}u(\eta_2), \end{cases} \tag{Ex.2}$$

where

$$\alpha = \frac{3}{2}, \rho = \frac{1}{2}, \delta = \frac{\rho - 1}{\rho} = -1, \sigma_1 = \frac{6 - 2e^{\frac{8}{9}}}{3e^{\frac{3}{4}} - 2e^{\frac{8}{9}}} \simeq 0.7638.., \sigma_2 = \frac{3e^{\frac{3}{4}} - 6}{3e^{\frac{3}{4}} - 2e^{\frac{8}{9}}} \simeq 0.23$$

$\eta_1 = \frac{1}{4}, \eta_2 = \frac{1}{9}$, and

$$f\left(t, u(t), {}^cD_0^{\frac{1}{2},\frac{1}{2}}u(t)\right) = \begin{cases} 0 & \text{if} \quad t \in \left[0; \frac{1}{4}\right] \\ \frac{\kappa}{4e^{\frac{3}{4}} - 7} \left[\sin u(t) + {}^c D_0^{\frac{1}{2},\frac{1}{2}}u(t) - 1\right]\left(t - \frac{1}{4}\right)e^{1-t} & \text{if} \quad t \in \left[\frac{1}{4}; 1\right] \end{cases}$$

notice that obviously, condition $(A_1)$ holds with

$$\Phi(t, x, y) = \begin{cases} 0 & \text{if} \quad t \in \left[0; \frac{1}{4}\right], \\ \frac{\kappa}{4e^{\frac{3}{4}} - 7}[x + y + 1]\left(t - \frac{1}{4}\right)e^{1-t} & \text{if } t \in \left[\frac{1}{4}; 1\right], \end{cases}$$

and

$$\begin{aligned} \int_0^1 \Phi(t, r, r)dt &= \int_{\frac{1}{4}}^1 \Phi(t, r, r)dt = \frac{(2r+1)\kappa}{4e^{\frac{3}{4}} - 7} e \int_{\frac{1}{4}}^1 \left(t - \frac{1}{4}\right)e^{-t}dt \\ &= \frac{(2r+1)\kappa}{4e^{\frac{3}{4}} - 7} \cdot \left(\frac{4e^{\frac{3}{4}} - 7}{4}\right) = \frac{\kappa}{2}r + \frac{\kappa}{4} \leq \kappa r + \frac{\kappa}{4}. \end{aligned}$$

Choosing $M_0 = 3$. Assume that $|{}^c D_0^{\frac{1}{2},\frac{1}{2}}u(t)| > M_0$ for each $t \in [0;1]$. if $|{}^c D_0^{\frac{1}{2},\frac{1}{2}}u(t)| > M_0$ for all $t \in [0;1]$ we have

$$f\left(t, u(t), {}^c D_0^{\frac{1}{2},\frac{1}{2}}u(t)\right) > \frac{\kappa}{4e^{\frac{3}{4}} - 7}[-1 + M_0 - 1]\left(t - \frac{1}{4}\right)e^{1-t},$$

then

$$\begin{aligned} I_1 Nu &= \int_{\frac{1}{4}}^1 e^{-(1-s)} f\left(s, u(s), {}^c D_0^{\frac{1}{2},\frac{1}{2}}u(s)\right)ds \\ &> \frac{\kappa}{4e^{\frac{3}{4}} - 7}[-2 + M_0]\int_{\frac{1}{4}}^1 \left(s - \frac{1}{4}\right)ds \\ &= \frac{9\kappa}{32\left(4e^{\frac{3}{4}} - 7\right)}(M_0 - 2) > 0 \end{aligned}$$

$$\begin{aligned} I_2Nu &= \sigma_1 \cdot \int_0^{\frac{1}{4}} e^{-\left(\frac{1}{4}-s\right)} f\left(s, u(s), {}^c D_0^{\frac{1}{2},\frac{1}{2}} u(s)\right) ds \\ &+ \sigma_2 \cdot \int_0^{\frac{1}{9}} e^{-\left(\frac{1}{9}-s\right)} f\left(s, u(s), {}^c D_0^{\frac{1}{2},\frac{1}{2}} u(s)\right) ds = 0. \end{aligned}$$

On the other hand, if ${}^cD_0^{\frac{1}{2},\frac{1}{2}}u(t) \leq -M_0$ for all $t \in \left[\frac{1}{4}; 1\right]$ we get

$$f\left(t, u(t), {}^c D_0^{\frac{1}{2},\frac{1}{2}} u(t)\right) < \frac{\kappa}{4e^{\frac{3}{4}} - 7}\left[1 - M_0 - 1\right]\left(t - \frac{1}{4}\right) e^{1-t},$$

so

$$\begin{aligned} I_1Nu &= \int_{\frac{1}{4}}^1 e^{-(1-s)} f\left(s, u(s), {}^c D_0^{\frac{1}{2},\frac{1}{2}} u(s)\right) ds \\ &< \frac{\kappa}{4e^{\frac{k}{4}} - 7}(-M_0) \int_{\frac{1}{4}}^1 \left(s - \frac{1}{4}\right) ds \\ &= -\frac{9\kappa}{32\left(4e^{\frac{2}{4}} - 7\right)} M_0 < 0, \end{aligned}$$

then $I_2Nu = 0$. Which assure that the condition $(A_2)$ is satisfied.

Taking $M_1 = 7$

$$\begin{aligned} & c\left(I_1N\left(cte^{-t}\right) - I_2N\left(cte^{-t}\right)\right) = cI_1N\left(cte^{-t}\right) \\ =\ & c\int_{\frac{1}{4}}^1 \frac{\kappa}{4e^{\frac{3}{4}} - 7}\left[\sin\left(\text{cte}^{-t}\right) + \frac{c}{\sqrt{2}} \frac{\sqrt{s}e^{-s}}{\Gamma\left(\frac{3}{2}\right)} - 1\right]\left(s - \frac{1}{4}\right) ds \\ =\ & \frac{\kappa}{4e^{\frac{3}{4}} - 7} c[Ac + B], \end{aligned}$$

where

$$\begin{aligned} A &= \frac{1}{\sqrt{2} \cdot \Gamma\left(\frac{3}{2}\right)} \int_{\frac{1}{4}}^1 \left(s - \frac{1}{4}\right) \sqrt{s} e^{-s} ds, \\ B &= \int_{\frac{1}{4}}^1 \sin\left(\text{ cte }^{-t} - 1\right)\left(s - \frac{1}{4}\right) ds, \end{aligned}$$

we have

$$\begin{aligned} A &= \sqrt{\frac{2}{\pi}} \int_{\frac{1}{4}}^1 \left(s - \frac{1}{4}\right) \sqrt{s} e^{-s} ds, \simeq 9.0401 \times 10^{-2} > 0, \\ |B| &\leq \int_{\frac{1}{4}}^1 2\left(s - \frac{1}{4}\right) ds = \frac{9}{16}, \end{aligned}$$

thus

$$\left|\frac{B}{A}\right| \leq \frac{\frac{9}{16}}{0.09} = \frac{100}{16} = 6.25.$$

If $c > 7$, we have $-c < -7 < -6.25 \leq \frac{B}{A}$ then $Ac + B > 0$ which imply that

$$c\left(I_1 N\left(cte^{-t}\right) + I_2 N\left(cte^{-t}\right)\right) > 0.$$

If $c < -7$, we get $\frac{B}{A} \leq 6.25 < 7 < -c$ thus $Ac + B < 0$, therefore

$$c\left(I_1 N\left(cte^{-t}\right) + I_2 N\left(cte^{-t}\right)\right) > 0.$$

Then condition is fufilled and the problem (Ex.2) has at least one solution in $C^{\frac{1}{2}}[0;1]$.

# Chapter 5

# Existence results for the boundary value problem with two-dimensional kernel

## 5.1 Introduction

In this chapter we shall study the existence of solutions for the following fractional-order $p$-Laplacian BVP at resonance with integral boundary conditions [14] :

$$\begin{cases} (\phi_p(D^{\beta}_{0+}u(t)))' + g(t)f(t,u(t),D^{\beta}_{0+}u(t)) = 0, \quad t \in [0,T], 0 \leq \beta < 1, \\ \phi_p(D^{\beta}_{0+}u(0)) = \int_0^T g(t)\phi_p(D^{\beta}_{0+}u(t))dt, \\ \phi_p(D^{\beta}_{0+}u(T)) = \int_0^T g(t)\phi_p(D^{\beta}_{0+}u(t))dt, \end{cases} \tag{5.1}$$

where $D^{\beta}_{0+}$ is the Riemann-Liouville fractional derivative of order $\beta$, $g \in L^1[0,T]$ with $g(t) > 0$ and $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is a $g$-Caratheodory function, that is, (i) for each $(x,y) \in \mathbb{R}^2$, the mapping $t \to f(t,x,y)$ is Lebesgue measurable, (ii) for a.e. $t \in [0,T]$ the mapping $(x,y) \to f(t,x,y)$ is continuous on $\mathbb{R}^2$ and (iii) for each $r > 0$, there exists $\omega_r(t) : [0,T] \to [0,+\infty)$ satisfying $\int_0^T g(t)\,|\omega_r(t)| < +\infty$ such that, for a.e. $t \in [0,T)$ and every $(x,y) \in [-r,r] \times [-r,r]$, we have

$$|f(t,x,y)| \leq \omega_r(t).$$

Recall also that $\phi_p : \mathbb{R} \to \mathbb{R}$ is an odd continuous, increasing operator, and $\phi_p^{-1} = \phi_q$. The problem (5.1) is said to be *at resonance* if $\int_0^T g(t)dt = 1$.

Moreover, because $(\phi_p(D^{\beta}_{0_+}u(t)))'$ is a nonlinear operator, the coincidence degree theory for linear differential operators with resonant boundary value conditions fails to apply to it directly. However, rewriting (5.1) as a semilinear system allows us to apply the continuation theorem to the problem (5.2) and obtain the existence of some solutions.

## 5.2 The functional framework

Now, we consider the following system :

$$\begin{cases} D^{\beta}_{0+}u_1(t) = \phi_q\left(u_2(t)\right), \\ u_2'(t) = -g(t)f\left(t, u_1(t), \phi_q u_2(t)\right), \\ u_2(0) = u_2(T) = \int_0^T g(t)u_2(t)dt, \end{cases} \tag{5.2}$$

where $0 \leq \beta < 1$ and we introduce the spaces

$$X_1 = C^{1-\beta}[0,T] = \left\{u \in C[0,T] \mid \text{ such that } t^{1-\beta}u(t) \in C[0,T]\right\},$$

with the norm

$$\|u\|_{X_1} = \|u\|_{C^{1-\beta}} = \max_{t\in[0,T]}\left|t^{1-\beta}u(t)\right|,$$

and

$$X_2 = C[0,T] = \{u \mid u(t) \text{ is continuous on the interval } [0,T]\},$$

equipped by the norm $\|u\|_{X_2} = \max_{t\in[0,T]}|u(t)|$, taking the space

$$X = \left\{u = (u_1, u_2)^{\top} \mid u_1 \in C^{1-\beta}[0,T], u_2 \in C[0,T]\right\},$$

with the norm

$$\|u\|_X = \max\left\{\|u_1\|_{C^{1-\beta}}, \|u_2\|_C\right\},$$

and $Y_1 = C[0,T]$, $Y_2 = L^1[0,T]$ where $\|y\|_{Y_1} = \max_{t\in[0,T]}|y(t)|$, $\|y\|_{Y_2} = \int_0^T |y(t)|dt$. Define the space $Y$ as follows:

$$Y = \left\{y = (y_1, y_2)^{\top} \mid y_1 \in C[0,T], y_2 \in L^1[0,T]\right\},$$

with the norm

$$\|y\|_Y = \max\left\{\|y_1\|_{C[0,T]}, \|y_2\|_{L^1[0,T]}\right\}.$$

Obviously, $(Y, ||.||_Y)$ is a Banach space, and $||.||_{C^{1-\beta}}$ is a norme in $C^{1-\beta}$ because

$$||u||_{C^{1-\beta}} = ||u||_{\infty}.$$

Now, let's prove that $(X, ||.||_X)$ is also a Banach space. Suppose $(u_n)$ is Cauchy sequence in $C^{1-\beta}$, that's for ever $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $m > n > n_0$, we have

$$||u_m - u_n||_{C^{1-\beta}} < \varepsilon,$$

which implies that

$$\sup_{t\in[0,T]} t^{1-\beta} \left| u_m(t) - u_n(t) \right|_{C^{1-\beta}} < \varepsilon,$$

then,

$$\sup_{t\in[0,T]} \left| t^{1-\beta} u_m(t) - t^{1-\beta} u_n(t) \right|_{C^{1-\beta}} < \varepsilon,$$

finally, we obtain

$$||(u_m)_{1-\beta} - (u_n)_{1-\beta}||_{\infty} < \varepsilon,$$

which shows that $(u_n)_{1-\beta}$ is a Cauchy sequence in $C[0,T]$ (real Banach space), then $(u_n)_{1-\beta}$ is convergent to a function $u \in C[0,T]$. Therefore, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, if $n > n_0$, we have

$$\sup_{t\in[0,T]} \left| t^{1-\beta} u_n(t) - u(t) \right| < \varepsilon,$$

hence

$$\sup_{t\in[0,T]} t^{1-\beta} \left| u_n(t) - \frac{u(t)}{t^{1-\beta}} \right| < \varepsilon,$$

which means that $(u_n)$ converges to the function $v \in C^{1-\beta}$, defined as $v(t) = \frac{u(t)}{t^{1-\beta}}$.
$0 < t \leq T$, which shows that $X_1$ is Banach space.
On the other hande $X_2 = C[0,T]$ with the norm $\|u\|_{X_2} = \max_{t\in[0,T]} |u(t)|$ is Banach space then $X = X_1 \times X_2$, with the norm $\|u\|_X = \max\left\{\|u_1\|_{C^{1-\beta}}, \|u_2\|_C\right\}$, is Banach space.
It is clear that, $(u_1(\cdot), u_2(\cdot))^{\top}$ is a solution of the problem (5.2), if and only if $u_1(\cdot)$ is a solution of the problem (5.1). Define the operator $L : \operatorname{dom} L \subset X \to Y$ by

$$Lu(t) := \begin{pmatrix} (Lu)_1(t) \\ (Lu)_2(t) \end{pmatrix} = \begin{pmatrix} D_{0^+}^{\beta} u_1(t) \\ u_2'(t) \end{pmatrix}, \quad \forall t \in [0,T], \tag{5.3}$$

where

$$\operatorname{dom} L = \left\{ u \in X, (D^{\beta}_{0+}u_1, u_2') \in Y, u_2(0) = u_2(T) = \int_0^T g(t)u_2(t)dt \right\},$$

$(Lu)_1(.)$ is the first component and $(Lu)_2(.)$ is the second. Let the operator $N : X \to Y$ be defined by

$$Nu(t) := \begin{pmatrix} (Nu)_1(t) \\ (Nu)_2(t) \end{pmatrix} = \begin{pmatrix} \phi_q\left(u_2(t)\right) \\ -g(t)f\left(t, u_1(t), \phi_q u_2(t)\right) \end{pmatrix}, \quad \forall t \in [0, T]. \tag{5.4}$$

It is easy to see that problem (5.2) can be converted to the operator equation

$$Lu = Nu, \quad u \in \operatorname{dom} L.$$

Throughout this paper we will use the following notations: $D_1, D_2 : Y_2 \to Y_2$ are two linear operators defined by the following relations

$$D_1 y_2 := \int_0^T y_2(s)ds \text{ and } D_2 y_2 := \int_0^T g(t) \int_0^t y_2(s)dsdt,$$

where $Y_2 = L^1[0, T]$, and for all $\beta \in [0, 1)$ denote by For all $\beta \in [0, 1)$ denote by

$$\Delta = \delta_{11}\delta_{22} - \delta_{12}\delta_{21},$$

where

$$\delta_{11} = T,\ \delta_{22} = \frac{1}{\beta}\int_0^T t^{\beta} g(t)dt,\ \delta_{12} = \frac{T^{\beta}}{\beta} \text{ and } \delta_{21} = \int_0^T tg(t)dt,$$

and the operators $R_1, R_2 : Y_2 \to Y_2$ as

$$\begin{cases} R_1 y_2 := \frac{1}{\Delta}\left(\delta_{22}D_1 y_2 - \delta_{12}D_2 y_2\right), \\ \\ R_2 y_2 := \frac{1}{\Delta}\left(\delta_{11}D_2 y_2 - \delta_{21}D_1 y_2\right). \end{cases} \tag{5.5}$$

**Proposition 5.2.1.** *(Proposition, p-219, [10] If the continuous function $f \geq 0$ in $[a, b]$ then :*

$$\int_a^b f(x)dx = 0 \Longrightarrow \forall x \in [a, b]; f(x) = 0.$$

***Proof.*** Indeed, if $f(x_0) > 0, x_0 \in [a, b]$, then by continuity, there exist an interval $[\alpha, \beta] \subset [a, b]$, where $f > \frac{f(x_0)}{2}$, which imply

$$\int_a^b f dx \geq \int_\alpha^\beta f dx \geq \frac{f(x_0)(\beta - \alpha)}{2}.$$

Contradiction. □

**Remark 5.1.** *In view of the precedent proposition* $\Delta = \delta_{11}\delta_{22} - \delta_{12}\delta_{21} \neq 0$.
*Indeed, for each* $0 \leq \beta < 1$, *the function* $F$ *defined by* $F(t) = Tt^{\beta} - T^{\beta}t$ *is positive continuous so* $F(t)g(t) \geq 0$ *(because* $g(t) > 0$*), and as* $g\left(\frac{T}{2}\right) F\left(\frac{T}{2}\right) \neq 0$, *then the function* $Fg$ *is not identically zero in* $[0, T]$ *which prove that*

$$\Delta = \frac{1}{\beta} \int_0^T F(t)g(t)dt \neq 0.$$

## 5.3 Existence results

### 5.3.1 Linear Part

In this part, we needed three lemmas to prove the existence of solutions of our problem by applying Mawhin's coincidence degree theory.

**Lemma 5.3.1.** *We have the following results:*

$$\operatorname{Ker} L = \left\{c_1(t^{\beta-1}, 0)^\top + c_2(0, 1)^\top, \forall t \in [0, T], c_1, c_2 \in \mathbb{R}\right\}, \tag{5.6}$$

$$\operatorname{Im} L = \left\{y = (y_1, y_2)^\top \in Y : D_1 y_2 = D_2 y_2 = 0\right\}. \tag{5.7}$$

***Proof.*** On one hand, for each $u = (u_1, u_2)^\top \in \ker L$, we have $Lu(t) = 0$ for all $t \in [0; 1]$, so

$$\begin{cases} D_{0^+}^{\beta} u_1(t) = 0 \\ u_2'(t) = 0 \end{cases} \Rightarrow \begin{cases} u_1(t) = c_1 t^{\beta-1} \\ u_2(t) = c_2 \end{cases}$$

then

$$\operatorname{Ker} L = \left\{c_1(t^{\beta-1}, 0)^\top + c_2(0, 1)^\top, \forall t \in [0, T], c_1, c_2 \in \mathbb{R}\right\}.$$

If $y = (y_1, y_2)^\top \in \operatorname{Im} L$, then there exists $u = (u_1, u_2)^\top \in \operatorname{dom} L$ such that $y = Lu$, i.e., $y_1(t) = D_{0^+}^{\beta} u_1(t), y_2(t) = u_2'(t)$, which yields

$$u_2(t) = c_2 + \int_0^t y_2(s)ds, \quad c_2 \in \mathbb{R},$$

with consideration of the boundary conditions $u_2(0) = u_2(T) = \int_0^T g(t)u_2(t)dt$, we conclude that

$$c_2 = c_2 + \int_0^T y_2(s)ds = c_2 + \int_0^T g(t) \int_0^t y_2(s)ds,$$

so, we have

$$\int_0^T y_2(s)ds = \int_0^T g(t)\int_0^t y_2(s)ds = 0,$$

then,

$$D_1y_2 = D_2y_2 = 0, \tag{5.8}$$

thus

$$\operatorname{Im} L \subset \left\{ y = (y_1, y_2)^\top \in Y : D_1y_2 = D_2y_2 = 0 \right\}. \tag{5.9}$$

Now, suppose that $y = (y_1, y_2)^\top \in Y$, and satisfies (5.8). Let

$$\begin{cases} u_1(t) = I_{0^+}^{\beta} y_1(t), \\ u_2(t) = I_{0^+}^{1} y_2(t). \end{cases}$$

Since $\int_0^T g(t)dt = 1$, we get

$$u_2(0) = u_2(T) = \int_0^T g(t)u_2(t)dt,$$

then $u = (u_1, u_2)^\top \in \operatorname{dom} L$ and $Lu = y$ i.e., $y \in \operatorname{Im} L$. Hence,

$$\left\{ y = (y_1, y_2)^\top \in Y : D_1y_2 = D_2y_2 = 0 \right\} \subset \operatorname{Im} L, \tag{5.10}$$

from (5.9) and (5.10), we conclude that

$$\operatorname{Im} L = \left\{ y = (y_1, y_2)^\top \in Y : D_1y_2 = D_2y_2 = 0 \right\}.$$

The proof is completed. □

**Lemma 5.3.2.** *Under the assumption $\int_0^T g(t)dt = 1$, the following conditions hold:*

*(i)* $L : \operatorname{dom} L \subset \Omega \to X$ *is a Fredholm operator of index zero. Furthermore, the linear continuous projectors $P : X \to X$ and $Q : Y \to Y$ satisfy*

$$Pu(t) := \begin{pmatrix} (Pu)_1(t) \\ (Pu)_2(t) \end{pmatrix} = \begin{pmatrix} T^{1-\beta} u_1(T)\, t^{\beta-1} \\ \int_0^T g(s)u_2(s)ds \end{pmatrix}, \quad \forall t \in [0, T],$$

*where the first and the second component of $P$ are independent of each other, and*

$$Qy(t) := \begin{pmatrix} (Qy)_1(t) \\ (Qy)_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ R_1y_2 + R_2y_2t^{\beta-1} \end{pmatrix}, \quad \forall t \in [0, T],$$

*where $R_1, R_2$ are defined in (5.5).*

*(ii) The inverse $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ of $L_P$ can be written as*

$$K_P y := \begin{pmatrix} (K_P y)_1 \\ (K_P y)_2 \end{pmatrix} = \begin{pmatrix} I_{0+}^{\beta} y_1 \\ I_{0+}^{1} y_2 \end{pmatrix},$$

*and satisfy*

$$\|K_P y\|_X \leq L \|y\|_Y,$$

*where $L = \max\{\frac{T}{\Gamma(\beta+1)}, 1\}$.*

***Proof.*** (i) For all $u \in X$, and $t \in [0, T]$, we have

$$\begin{aligned} P^2 u(t) = P(Pu)(t) &= P \begin{pmatrix} (Pu)_1(t) \\ (Pu)_2(t) \end{pmatrix} = P \begin{pmatrix} T^{1-\beta} u_1(T) t^{\beta-1} \\ \int_0^T g(s) u_2(s) ds \end{pmatrix} \\ &= \begin{pmatrix} T^{1-\beta} (Pu)_1(T) t^{\beta-1} \\ \int_0^T g(s) (Pu)_2(s) ds \end{pmatrix} = \begin{pmatrix} T^{1-\beta} (T^{1-\beta} u_1(T) T^{\beta-1}) t^{\beta-1} \\ (Pu)_2(t) \int_0^T g(s) ds \end{pmatrix} \\ &= Pu(t), \end{aligned}$$

because $\int_0^T g(t) dt = 1$.

For all $y \in Y$, and $t \in [0, T]$, we get

$$\begin{aligned} R_1(R_1 y_2) &= \frac{1}{\Delta} \left[ \delta_{22} D_1(R_1 y_2) - \delta_{12} D_2(R_1 y_2) \right] \\ &= \frac{1}{\Delta} \left[ \delta_{22} \delta_{11} - \delta_{12} \delta_{21} \right] R_1 y_2 \\ &= R_1 y_2, \end{aligned}$$

and similarly we can derive that

$$R_1\left(R_2 y_2 t^{\beta-1}\right) = 0,$$

$$R_2(R_1 y_2) = 0,$$

and

$$R_2\left(R_2 y_2 t^{\beta-1}\right) = R_2 y_2.$$

So, for $y=(y_1,y_2)^\top \in Y$, it follows from the four relations above that

$$\begin{aligned}
Q^2y_2 = Q(Qy_2) &= R_1[R_1y_2+R_2y_2t^{\beta-1}]+R_2[R_1y_2+R_2y_2t^{\beta-1}]t^{\beta-1}\\
&= R_1\left(R_1y_2\right)+R_1\left(R_2y_2t^{\beta-1}\right)+R_2\left(R_1y_2\right)+R_2\left(R_2y_2t^{\beta-1}\right)\\
&= R_1y_2+R_2y_2t^{\beta-1}=Qy_2.
\end{aligned}$$

Thus, we get

$$Q^2y(t)=Q(Qy)(t)=Q\begin{pmatrix}(Qy)_1\\(Qy)_2\end{pmatrix}=\begin{pmatrix}0\\Q(Qy_2)\end{pmatrix}=Qy,\quad \forall t\in[0,T].$$

We have also

$$\begin{aligned}
\|(Pu)_1\|_{C^{1-\beta}} &= \max_{t\in[0,T]}\left|t^{1-\beta}T^{1-\beta}u_1(T)\,t^{\beta-1}\right| = \left|T^{1-\beta}u_1(T)\right|\\
&\le \max_{t\in[0,T]}\left|t^{1-\beta}u_1(t)\right| = \|u_1\|_{C^{1-\beta}},\\
\|(Pu)_2\|_\infty &= \max_{t\in[0,T]}\left|\int_0^T g(s)u_2(s)ds\right|\\
&\le \int_0^T g(s)\left|u_2(s)\right|ds = \|u_2\|_\infty,
\end{aligned}$$

then

$$\begin{aligned}
\|Pu\|_X &= \max\{\|(Pu)_1\|_{C^{1-\beta}},\|(Pu)_2\|_\infty\}\\
&\le \max\{\|u_1\|_{C^{1-\beta}},\|u_2\|_\infty\}=\|u\|_X.
\end{aligned}$$

And we have also

$$\begin{aligned}
\|(Qy)_1\|_\infty &= \max_{t\in[0,T]}|(Qy)_1(t)|=0\le\|y_1\|_\infty,\\
\|(Qy)_2\|_{L^1[0,T]} &= \int_0^T|(Qy)_2(s)|\,ds\le C\|y_2\|_{L^1[0,T]},
\end{aligned}$$

where $C=\frac{|\delta_{11}\delta_{22}|+2|\delta_{12}\delta_{11}|+|\delta_{12}\delta_{21}|}{|\Delta|}$, then

$$\|Qy\|_Y=\max\{\|(Qy)_1\|_\infty,\|(Qy)_2\|_{L^1[0,T]}\}\le\max\{\|y_1\|_\infty,C\|y_2\|_{L^1[0,T]}\}\le C\|y\|_Y. \quad (5.11)$$

Now we prove that $Y=\operatorname{Im}L\oplus\operatorname{Im}Q$.

For each $y=(y_1,y_2)^\top\in Y$, can be write as $y=((I-Q)y+Qy)=(y_1,(I-Q)y_2+Qy_2))^\top$

where $y - Qy \in \operatorname{Im} L = \operatorname{Ker} Q$ and $Qy \in \operatorname{Im} Q$, thus we have $Y = \operatorname{Im} L + \operatorname{Im} Q$.
Let $y \in \operatorname{Im} L \bigcap \operatorname{Im} Q$ so $y(t) = (y_1, y_2)^\top = (0, a + bt^{\beta-1})^\top$ where $a, b \in \mathbb{R}$, and since $y \in \operatorname{Im} L$ then $D_1 y_2 = \int_0^T (a + bs^{\beta-1}) ds = 0$ and $D_2 y_2 = \int_0^T g(t) \int_0^T (a + bs^{\beta-1}) ds dt = 0$, we derive $a = b = 0$, thus $\operatorname{Im} L \bigcap \operatorname{Im} Q = \{0\}$, which implies that $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$, and as

$$\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L = 2.$$

So $L$ is a Fredholm operator of index zero.
Furthemore, for all $u$ in $X$, we can write $u = (u - Pu) + Pu$ and since $\operatorname{Im} P = \operatorname{Ker} L$, then $P^2 u = Pu$, so

$$X = \operatorname{Ker} P + \operatorname{Ker} L$$

By simple calculation, we can get that $\operatorname{Ker} P \cap \operatorname{Ker} L = \{0\}$ which prove that

$$X = \operatorname{Ker} L \oplus \operatorname{Ker} P.$$

(ii) From the definition of $K_P$, for $y \in \operatorname{Im} L$, we have

$$LK_P y = \begin{pmatrix} D_{0+}^\beta I_{0+}^\beta y_1 \\ \frac{d}{dt} I_{0+}^1 y_2 \end{pmatrix} = y.$$

For $u \in \operatorname{dom} L \cap \operatorname{Ker} P$, and by using Lemma 1.3.3 we get

$$K_P L u = \begin{pmatrix} u_1 + c_1 t^{\beta-1} \\ u_2 + c_2 \end{pmatrix},$$

where $c_1, c_2$ are real constants. Since $u \in \operatorname{dom} L \cap \operatorname{Ker} P$, it is easily to show that $c_1 = c_2 = 0$. So $K_P$ is the inverse of $L_P$. We have also

$$\begin{aligned} \|K_P y\|_X &= \{\|(K_P y)_1\|_{C^{1-\beta}}, \|(K_P y)_2\|_\infty\} \\ &= \max\{\left\|I_{0+}^\beta y_1\right\|_{C^{1-\beta}}, \left\|I_{0+}^1 y_2\right\|_\infty\} \\ &\le \max\{\frac{T}{\Gamma(\beta+1)} \|y_1\|_{Y_1}, \|y_2\|_{Y_2}\} \\ &\le L \|y\|_Y, \end{aligned}$$

where $L = \max\{\frac{T}{\Gamma(\beta+1)}, 1\}$, which completes the proof. □

### 5.3.2 Nonlinear Part

**Lemma 5.3.3.** *Let* $\Omega \subset X$ *be open and bounded subset with* $\operatorname{dom} L \cap \bar{\Omega} \neq \emptyset$. *If* $f$ *is* $g$*-Caratheodory, then* $N$ *is* $L$*-compact on* $\bar{\Omega}$.

***Proof.*** Let $\Omega = B(0,r)$, then for $u \in \bar{\Omega}, \|u\| \leq r$. Since $f$ is a $g$-Caratheodory function then, there exists $\omega_r : [0,T] \to [0,+\infty)$ satisfying $\int_0^T g(t)\,|\omega_r(t)| < +\infty$,
for a.e $t \in [0,T]$,

$$\left| f\left(t, u(t), D_{0+}^{\beta} u(t)\right)\right| \leq \omega_r(t),$$

then

$$\|QNu\|_Y \leq C\,\|Nu\|_Y \leq C_1,$$

where $C_1 = C(r^{q-1} + \|\omega_r\|_{L^1[0,T]})$.
We will use the following two steps to prove that $K_P(I-Q)N(\bar{\Omega})$ is compact.
**Step 1**: Let $u \in \bar{\Omega}$, then

$$\|K_P(I-Q)Nu\|_X \leq L\,\|(I-Q)Nu\|_Y \leq L\left(\|Nu\|_Y + \|QNu\|_Y\right) \leq C_2,$$

where $C_2 = L(C_1 + r^{q-1} + \|\omega_r\|_{L^1[0,T]})$ .
**Step 2**: Let $u \in \bar{\Omega}$ and $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ then

$$\begin{aligned}
& \left|t_2^{1-\beta}(K_P(I-Q)Nu)_1(t_2) - t_1^{1-\beta}(K_P(I-Q)Nu)_1(t_1)\right| \\
= & \left|\frac{t_2^{1-\beta}}{\Gamma(\beta)}\int_0^{t_2}(t_2-s)^{\beta-1}\left((I-Q)Nu\right)_1(s)ds\right. \\
- & \left.\frac{t_1^{1-\beta}}{\Gamma(\beta)}\int_0^{t_1}(t_1-s)^{\beta-1}\left((I-Q)Nu\right)_1(s)ds\right| \\
\leq & \left|\frac{t_2^{1-\beta}}{\Gamma(\beta)}\int_{t_1}^{t_2}(t_2-s)^{\beta-1}\left((I-Q)Nu\right)_1(s)ds\right| \\
+ & \left|\frac{t_2^{1-\beta}-t_1^{1-\beta}}{\Gamma(\beta)}\int_0^{t_1}(t_2-s)^{\beta-1}\;\left((I-Q)Nu\right)_1(s)ds\right| \\
+ & \left|\frac{t_1^{1-\beta}}{\Gamma(\beta)}\int_0^{t_1}\left((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\right)\left((I-Q)Nu\right)_1(s)ds\right| \\
\leq & \frac{(C+1)(r^{q-1}+\|\omega_r\|_{L^1[0,T]})}{\Gamma(\beta+1)}\,|t_2-t_1|\,\|\omega_r\|_{L^1[0,T]} \;\to 0
\end{aligned}$$

As $t_1 \to t_2$ uniformly. Similarly we can derive that

$$\begin{aligned}
& |(K_P(I-Q)Nu)_2(t_2) - (K_P(I-Q)Nu)_2(t_1)| \\
= & \left| \int_0^{t_2} ((I-Q)Nu)_2(s)ds - \int_0^{t_1} ((I-Q)Nu)_2(s)ds \right| \\
= & \left| \int_0^{t_1} ((I-Q)Nu)_2(s)ds + \int_{t_1}^{t_2} ((I-Q)Nu)_2(s)ds \right. \\
- & \left. \int_0^{t_1} ((I-Q)Nu)_2(s)ds \right| \\
= & \left| \int_{t_1}^{t_2} ((I-Q)Nu)_2(s)ds \right| \le (t_2 - t_1) \left| ((I-Q)Nu)_2 \right| \\
\le & (t_2 - t_1)(C+1)(r^{q-1} + \|\omega_r\|_{L^1[0,T]}) \ \to 0
\end{aligned}$$

As $t_1 \to t_2$ uniformly. Thus, $K_P(I-Q)Nu(\bar{\Omega})$ is compact, therefore, the nonlinear operator $N$ is $L$ -compact on $\bar{\Omega}$. □

### 5.3.3 Existence theorem for the fractional-order $p$-Laplacian boundary value problem

**Theorem 5.3.1.** *Assume the following condition holds.*

$(H_1)$ *There exist functions $a_1(t) > 0, a_2(t) > 0, a_3(t) > 0$, in $L^1[0,T]$ such that*

$$g(t)\,|f(t,u,v)| \le a_1(t) + a_2(t)|t^{1-\beta}u|^{p-1} + a_3(t)|v|^{p-1},$$

*for all $(u,v) \in \mathbb{R}^2$ and $t \in [0,T]$, where $g(t) \in L^1[0,T], g(t) > 0$.*

$(H_2)$ *There exists a constant $A > 0$ such that if $|u| > A$ or $|v| > A$, then either*

$$uD_1(N(u,v)^\top)_2 + vD_2(N(u,v)^\top)_2 > 0, \tag{5.12}$$

*or*

$$uD_1(N(u,v)^\top)_2 + vD_2(N(u,v)^\top)_2 < 0. \tag{5.13}$$

*Then problem (5.2) has at least one solution, provided that*

$$\frac{T^{p-1}}{\Gamma(\beta+1)^{p-1}}\|a_2\|_{L^1[0;T]} + \|a_3\|_{L^1[0;T]} < 1, \ \text{if } 1 < p < 2, \tag{5.14}$$

$$\frac{2^{p-2}T^{p-1}}{\Gamma(\beta+1)^{p-1}}\|a_2\|_{L^1[0;T]} + \|a_3\|_{L^1[0;T]} < 1, \ \text{if } p \ge 2. \tag{5.15}$$

***Proof.*** Step 1. Consider the set

$$\Omega_1 = \{u \in \operatorname{dom} L \backslash \operatorname{Ker} L \mid Lu = \rho N u, \rho \in (0,1)\}.$$

For $u \in \Omega_1$, and $\rho \neq 0$, we get $Nu \in \operatorname{Im} L = \operatorname{Ker} Q$, hence

$$\int_0^T g(t) f\left(t, u_1(t), \phi_q(u_2(t))\right) dt = \int_0^T g(t) \int_0^t g(s) f\left(s, u_1(s), \phi_q(u_2(s))\right) ds dt = 0.$$

From the integral mean value theorem there exist $t_0 \in (0,T)$, such that

$$f\left(t_0, u_1(t_0), \phi_q(u_2(t_0))\right) = 0,$$

according to condition $(H_2)$, we get $|u_2(t_0)| \leq A^{p-1}$. Since

$$u_2(t) = u_2(t_0) + \int_{t_0}^{t_1} u_2'(s)\, ds.$$

We have

$$\|u_2\|_C \leq A^{p-1} + \|u_2'\|_{L^1[0,T]}, \tag{5.16}$$

by Lemma 1.3.3 we can write, $t^{1-\beta} u_1(t) = t^{1-\beta} I_{0+}^{\beta} D_{0+}^{\beta} u_1(t) + c_1$, then

$$\left|t^{1-\beta} u_1(t)\right| \leq \frac{T}{\Gamma(\beta+1)} \left\|D_{0+}^{\beta} u_1\right\|_C + |c_1|, \forall t \in [0,T],$$

which gives

$$\|u_1\|_{X_1} \leq \frac{T}{\Gamma(\beta+1)} \left\|D_{0+}^{\beta} u_1\right\|_C + |c_1|. \tag{5.17}$$

Now, $Lu = \rho N u$ is equivalent to

$$\begin{cases} D_{0+}^{\beta} u_1(t) = \rho \phi_q\left(u_2(t)\right), \\ u_2'(t) = -\rho g(t) f\left(t, u_1(t), \phi_q\left(u_2(t)\right)\right). \end{cases} \tag{5.18}$$

Using (5.18), we get $\left\|D_{0+}^{\beta} u_1\right\|_C \leq \|u_2\|_C^{\frac{1}{p-1}}$. Substitute this inequality in (5.17) we conclude that

$$\|u_1\|_{X_1} \leq \frac{T}{\Gamma(\beta+1)} \|u_2\|_C^{\frac{1}{p-1}} + |c_1|. \tag{5.19}$$

By the second equation of (5.18), and $(H_1)$, we get

$$\begin{aligned}
\|u_2'\|_{L^1[0,T]} &= \|g(t)f\left(t,u_1(t),\phi_q\left(u_2(t)\right)\right)\|_{L^1[0,T]} \\
&= \int_0^T g(s)\left|f\left(s,u_1(s),\phi_q\left(u_2(s)\right)\right)\right| ds \\
&\leq \int_0^T \left[a_1(s)+a_2(s)|t^{1-\beta}u|^{p-1}+a_3(s)|v|^{p-1}\right] ds \\
&\leq \|a_1\|_{L^1[0,T]}+\|a_2\|_{L^1[0,T]}\left|t^{1-\beta}u_1\right|^{p-1}+\|a_3\|_{L^1[0,T]}\|\phi_q(u_2)\|_C^{p-1} \\
&= \|a_1\|_{L^1[0,T]}+\|a_2\|_{L^1[0,T]}\|u_1\|_{X_1}^{p-1}+\|a_3\|_{L^1[0,T]}\|u_2\|_C,
\end{aligned}$$

where the functions $a_1(t)>0, a_2(t)>0, a_3(t)>0$, in $L^1[0,T]$.

If $1<p<2$, then from the above inequalities and Lemma 1.4.1 we obtain

$$\begin{aligned}
\|u_2'\|_{L^1[0,T]} &\leq \|a_1\|_{L^1[0,T]}+\|a_2\|_{L^1[0,T]}\left(\frac{T^{p-1}}{\Gamma(\beta+1)^{p-1}}\|u_2\|_{L^1[0,T]}+|c_1|^{p-1}\right) \\
&+\|a_3\|_{L^1[0,T]}\|u_2\|_C \\
&\leq \|a_1\|_{L^1[0,T]}+\|a_2\|_{L^1[0,T]}|c_1|^{p-1}+A^{p-1}\left(\frac{T^{p-1}}{\Gamma(\beta+1)^{p-1}}\|a_2\|_{L^1[0,T]}\right. \\
&\left.+\|a_3\|_{L^1[0,T]}\right)+\left(\frac{T^{p-1}}{\Gamma(\beta+1)^{p-1}}\|a_2\|_{L^1[0,T]}+\|a_3\|_{L^1[0,T]}\right)\|u_2'\|_{L^1[0,T]}.
\end{aligned}$$

Similarly, if $p\geq 2$, then

$$\begin{aligned}
\|u_2'\|_{L^1[0,T]} &\leq \|a_1\|_{L^1[0,T]}+\frac{2^{p-2}T^{p-1}}{\Gamma(\beta+1)^{p-1}}\|a_2\|_{L^1[0,T]}+|c_1|^{p-1} \\
&+A^{p-1}\left(\frac{2^{p-2}T^{p-1}}{\Gamma(\beta+1)^{p-1}}\|a_2\|_{L^1[0,T]}+\|a_3\|_{L^1[0,T]}\right) \\
&+\left(\frac{2^{p-2}T^{p-1}}{\Gamma(\beta+1)^{p-1}}\|a_2\|_{L^1[0,T]}+\|a_3\|_{L^1[0,T]}\right)\|u_2'\|_{L^1[0,T]}.
\end{aligned}$$

Where $A$ is positive constant, using (5.14) or (5.15), we have

$$\|u_2'\|_{L^1[0;T]} \leq K_0.$$

By (5.16), we get

$$\|u_2\|_C \leq A^{p-1}+K_0=K_1,$$

and by (5.19)

$$\|u_1\|_{X_1} \leq \frac{T}{\Gamma(\beta+1)}\|u_2\|_C^{\frac{1}{p-1}}+|c_1| \leq |c_1|+\frac{T}{\Gamma(\beta+1)}K_1^{\frac{1}{p-1}}=K_2,$$

as a consequence, we obtain

$$\|u\|_X = \max\left\{\|u_1\|_{X_1}, \|u_2\|_{X_2}\right\} = \max\{K_1, K_2\} = k.$$

Then the set $\Omega_1$ is bounded.

**Step 2.** Let

$$\Omega_2 = \{u \in \operatorname{Ker} L \mid QNu = 0\}.$$

For $u \in \Omega_2$, with $u = (u_1, u_2)^\top = (c_1 t^{\beta-1}, c_2)^\top, \forall (c_1, c_2) \in \mathbb{R}^2$.
We have $D_1 (Nu)_2 = D_2 (Nu)_2 = 0$, from $(H_2)$ there exist $t_1 \in [0, T]$, such that $\left|c_1 t^{\beta-1}\right| \leq A$ and $|c_2| \leq A$, then we get

$$\|u\|_X = \max_{t\in[0,T]} \left\{\|u_1\|_{X_1}, \|u_2\|_{X_2}\right\} \leq \max_{t\in[0,T]} \left\{A t^{1-\beta}, A\right\} = B.$$

Thus the set $\Omega_2$ is bounded.

**Step 3.**

Define the isomorphism $J : \ker L \to \operatorname{Im} Q$ by

$$J\begin{pmatrix} c_1 t^{\beta-1} \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\Delta}\left[\delta_{22}c_1 - \delta_{12}c_2 + (\delta_{11}c_2 - \delta_{21}c_1)\, t^{\beta-1}\right] \end{pmatrix}, \quad \forall t \in [0, T],$$

for $(c_1, c_2) \in \mathbb{R}^2$. Let

$$\Omega_3 = \{u \in \ker L, \rho J u + (1-\rho) QNu = 0, \text{ for some } \rho \in [0, 1]\}.$$

By definition, $u \in \Omega_3$ means that $u = (u_1, u_2)^\top = (c_1 t^{\beta-1}, c_2)^\top$ and $\rho Ju + (1-\rho)QNu = 0$ with $a, b \in \mathbb{R}$. If $\rho = 0$, then $QNu = 0$. By Step 2 we get $\|u\|_X \leq B$. For $\rho = 1$, we obtain $Ju(t) = J(c_1 t^{\beta-1}, c_2)^\top = (0, 0)^\top$, then

$$\begin{cases} \delta_{22}c_1 - \delta_{12}c_2 = 0 \\ \delta_{11}c_2 - \delta_{21}c_1 = 0. \end{cases}$$

Since $\Delta \neq 0$, then $c_1 = c_2 = 0$.

If $0 < \rho < 1$, from $-\rho Ju = (1-\rho)QNu$, we obtain

$$\begin{aligned} &\frac{-\rho}{\Delta}\left[\delta_{22}c_1 - \delta_{12}c_2 + (\delta_{11}c_2 - \delta_{21}c_1)t^{\beta-1}\right] \\ &= \frac{1-\rho}{\Delta}\left[\delta_{22}D_1(N(c_1 t^{\beta-1}, c_2)^\top)_2 - \delta_{12}D_2(N(c_1 t^{\beta-1}, c_2)^\top)_2 + (\delta_{11}D_2(N(c_1 t^{\beta-1}, c_2)^\top)_2\right. \\ &\left. -\delta_{21}D_1(N(c_1 t^{\beta-1}, c_2)^\top)_2)t^{\beta-1}\right], \end{aligned}$$

which implies that

$$-\rho\delta_{22}c_1+\rho\delta_{12}c_2=(1-\rho)\left[\delta_{22}D_1(N(c_1t^{\beta-1},c_2)^\top)_2-\delta_{12}D_2(N(c_1t^{\beta-1},c_2)^\top)_2\right]$$
$$-\rho\delta_{11}c_2+\rho\delta_{21}c_1=(1-\rho)\left[\delta_{11}D_2(N(c_1t^{\beta-1},c_2)^\top)_2-\delta_{21}D_1(N(c_1t^{\beta-1},c_2)^\top)_2\right].$$

Since the determinant $\Delta\neq 0$, then by simple calculations, we obtain

$$\rho c_1=-(1-\rho)D_1(N(c_1t^{\beta-1},c_2)^\top)_2,$$

and

$$\rho c_2=-(1-\rho)D_2(N(c_1t^{\beta-1},c_2)^\top)_2.$$

Then

$$\begin{cases}\rho c_1^2t^{\beta-1}=-(1-\rho)c_1t^{\beta-1}D_1(N(c_1t^{\beta-1},c_2)^\top)_2\\ \rho c_2^2=-(1-\rho)c_2D_2(N(c_1t^{\beta-1},c_2)^\top)_2.\end{cases}$$

By hypothesis $(H_2)$ and from (5.11), we get

$$\rho\left(c_1^2t^{\beta-1}+c_2^2\right)=-(1-\rho)\left[c_1t^{\beta-1}D_1(N(c_1t^{\beta-1},c_2)^\top)_2+c_2D_2(N(c_1t^{\beta-1},c_2)^\top)_2\right]<0$$

And this is a contradiction. So, the set $\Omega_3$ is bounded.

Finally, if we assume that (5.12) holds, then by the same method, we can prove the boundedness of the set $\{u\in\ker L:-\rho Ju+(1-\rho)QNu=0\ ,\text{ for some }\rho\in[0.1]\}$. Next, we prove that all conditions of Theorem 2.3.1 are satisfied: Let $\Omega$ to be an open bounded subset of $X$ such that $\cup_{i=1}^{i=3}\bar{\Omega}_i\subset\Omega$. From Lemma 5.3.2, we known that $L$ is a Fredholm operator of index zero. By Lemma 5.3.3, $N$ is $L$-compact on $\bar{\Omega}$. Since $\Omega_i,(i=1,2,3)$ are bounded sets and $\Omega_i\subset\Omega$, we have

(1) $Lu\neq\rho Nu$ for all $(u,\rho)\in[(dom\ L\backslash\operatorname{Ker}(L))\cap\partial\Omega]\times(0,1)$,

(2) $QNu\neq 0$ for all $u\in\ker L\cap\partial\Omega$.

Finaly we prove that condition (3) of Theorem 2.3.1 is satisfied. Let

$$H(u,\rho)=\pm\rho Ju+(1-\rho)QNu.$$

As $\bar{\Omega}_3\subset\Omega$, for all $u\in\operatorname{Ker}(L)\cap\partial\Omega$ and $\rho\in[0,1]$, we obtain that $H(u,\rho)\neq 0$. So, by the homotopy property of the degree, we conclude that

$$\begin{aligned}\deg\left(QN_{|}\ker L,\ \ker L\cap\Omega,0\right)&=\deg(H(.,0),\ \ker L\cap\Omega,0)\\&=\deg(H(.,1),\ \ker L\cap\Omega,0)\\&=\deg(\pm J,\ \ker L\cap\Omega,0)\neq 0,\end{aligned}$$

which implies that $Lu = Nu$ has at least a solution in $dom\ L \cap \bar{\Omega}$. □

### 5.3.4 An illustrative Example

Consider the following fractional differential equation :

$$(\phi_3(D^{\frac{1}{2}}_{0_+}u(t)))' + \cos t(\tfrac{\sin t}{36\pi}\sin^2 u - \tfrac{5\cos 3t}{36\pi}\phi_3(D^{\frac{1}{2}}_{0_+}u(t)) - \tfrac{\pi+2}{72\pi}) = 0, \quad t \in [0, \tfrac{\pi}{2}], \tag{Ex3}$$

with

$$\begin{cases} \phi_3(D^{\frac{1}{2}}_{0+}u(0)) = \int_0^{\frac{\pi}{2}} \cos t\phi_3(D^{\frac{1}{2}}_{0+}u(t))dt, \\ \phi_3(D^{\frac{1}{2}}_{0+}u(\frac{\pi}{2})) = \int_0^{\frac{\pi}{2}} \cos t\phi_3(D^{\frac{1}{2}}_{0+}u(t))dt, \end{cases} \tag{Cd3}$$

where $\beta = \frac{1}{2}, p = 3, q = \frac{3}{2}, T = \frac{\pi}{2}, g(t) = \cos t, \int_0^{\frac{\pi}{2}} \cos tdt = 1.$
Here $f(t, u, v) = \frac{\sin t}{36\pi}\sin^2 u - \frac{5\cos 3t}{36\pi}v^2 - \frac{\pi+2}{72\pi}$, then

$$\cos t|f(t, u, v)| \leq \frac{\pi+2}{72\pi} + \frac{1}{36\pi}|t^{\frac{1}{2}}u|^2 + \frac{5}{36\pi}|v|^2.$$

So we may take

$$a_1(t) = \frac{\pi+2}{72\pi}, a_2(t) = \frac{1}{36\pi}, a_3(t) = \frac{5}{36\pi},$$

we have $\|a_1\|_{L^1[0,\frac{\pi}{2}]} = \frac{\pi+2}{144}; \|a_2\|_{L^1[0,\frac{\pi}{2}]} = \frac{1}{72}; \|a_3\|_{L^1[0,\frac{\pi}{2}]} = \frac{5}{72}$, and

$$\Delta = \delta_{11}\delta_{22} - \delta_{21}\delta_{12} = 0.81 \neq 0.$$

And we have also

$$\frac{2^{p-2}T^p}{\Gamma(\beta+1)^{p-1}}\|a_2\|_{L^1[0,\frac{\pi}{2}]} + T\|a_3\|_{L^1[0,\frac{\pi}{2}]} = 0.13 < 1.$$

Let $A = \frac{\pi}{2}$, if $|v| > A$, then we get

$$\begin{aligned} & uD_1(N(u,v)^\top)_2 + vD_2(N(u,v)^\top)_2 \\ & = u\left(\frac{\sin^2 u}{36\pi} - \frac{\sin^2 u}{72} + \frac{\pi+2}{72\pi}\right) + v\left(\frac{5}{216\pi}v + (\frac{\pi^2}{8} - 1)(\frac{\pi+2}{72\pi} - \frac{\sin^2 u}{36\pi})\right) \\ & > \frac{1}{72}v\left(\frac{5}{3\pi}v + (\frac{\pi^2}{8} - 1)\right) > 0. \end{aligned}$$

Hence, all conditions of Theorem 5.3.1 hold, which implies that the problem Ex3 - Cd3 has at least one solution in $X$.

# Conclusions

In this thesis we have investigated some questions of existence of solutions of some classes of nonlinear boundary value problems in the resonance case. There are some mathematical difficulties caused by the nonlinearities and the boundary conditions. By utilizing the coincidence topological degree, however we can find some partial solution.

Many related problems have been solved recently, the majority of them have generally focused on second-order differential equations. In this thesis, we have looked at three cases related to higher-order or fractional equations on bounded domains.

Firstly, we have investigated the existence of a solution for a multi-point boundary value problem at resonance for a nonlinear fractional differential equation with a $p$-Laplacian operator by Mawhin's coincidence theorem. Under some conditions on the nonlinear term, we have proved the existence of solution. Next by using the generalized proportional fractional derivative we get new results. Finally, we find the conditions of existence of solutions of a fractional-order $p$-Laplacian boundary value problem at resonance case where the differential operator is nonlinear and has a kernel dimension equal to two, after transforming the nonlinear problem into a semilinear system.

Notice that this method studies only the existence case, in the future, we intend to study some questions related to uniqueness and also to multiplicity of solutions of some boundary value problems in resonance. For this, the application of some other methods may be associated to the coincidence degree, and it can be generalized for different types of operators.

# Bibliography


[1] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo. *Theory and applications of fractional differential equations.* Elsevier B. V. Netherlands, 204, (2006).

[2] A. Yang, C. Miao, and W. Ge. *for second-order nonlocal boundary value problems with a p-Laplacian at resonance on a half-line.* Electronic Journal of Qualitative Theory of Differential Equations, 2009, (2009), pp. 1-15.

[3] B. M. B. Krushna. *Eigenvalues for iterative systems of Riemann–Liouville type p-Laplacian fractional-order boundary-value problems in Banach spaces.* Computational and Applied Mathematics, (2020), 39, pp.1-15.

[4] J. Bai, X. Feng. *Fractional-order anisotropic difusion for image denoising.* Computers and Mathematics with Applications, IEEE Transactions on Image Processing, 16, (2007), pp. 2492- 2502.

[5] J. Mawin. *Topological degree and boundary value problems for nonlinear differential equations in topological methods for ordinary differential equations.* Lecture Notes Math, 1537, (1993), pp. 74-142.

[6] J. Sabatier, O. P. Agrawal, and J. T. Machado. *Advances in fractional calculus.* Springer, 30, (2007), pp. 1-109.

[7] H. Bresis. *Analyse fonctionnelle théorie et applications.* Paris: Masson, (1994).

[8] G. Cupini, C. Marcelli, and F. Papalini. *On the solvability of a boundary value problem on the real line.* Boundary Value Problems, 2011, (2011), pp. 1-17.

[9] K. Allab. *Éléments d'analyse: Fonction d'une variable réelle*. Office des publications universitaires - Alger, édition N 600 Janvier, (1986).

[10] K. Deimling. *Non-linear Functional Analysis*. Springer, Berlin, (1985)

[11] K. R. Prasad, B. M. B Krushna. *Existence of Multiple Positive Solutions for p-Laplacian Fractional Order Boundary Value Problems*. International Journal of Analysis and Applications, 6, (2014), pp. 63-81.

[12] K. R. Prasad, B. L. B. Krushna, and N. Sreedhar. *Even Number of Positive Solutions for the System of $(p, q)$-Laplacian Fractional Order Two-Point Boundary Value Problems*. Differential Equations and Dynamical Systems, 26, (2018), pp. 315-330.

[13] M. Azouzi, L. Guedda. *Existence result for nonlocal boundary value problem of fractional order at resonance with p-laplacian operator*. Azerbaijan Journal of Mathematics, 13, (2023).

[14] M. Azouzi, E. Zeraoulia, and L. Guedda. *Solvability of a resonant fractional-order p-laplacian boundary value problem with two-dimensional kernal*. Journal of Applied and Engineering Mathematics, (2023) (**accepted**).

[15] M. Azouzi, L. Guedda, and Z. Laadjal. *Existence results of generalized proportional fractional differential equations at resonance case by the topological degree theory*. Advances in Mathematics,11, (2022), pp. 613–633.

[16] O. F. Imaga, S. A. Iyase. *Existence of solution for a resonant p-Laplacian second-order m-point boundary value problem on the half-line with two dimensional kernel*. Boundary Value Problems, 2020, (2020), pp. 1-11.

[17] O. F Imaga, and S. A. Iyase, and S. A. Bishop. *Existence results for a resonant third-order problem with two dimensional kernel on the half-line*. Journal of Physics: Conference Series,1734, (2021), pp. 1-7.

[18] O. F. Imaga, S. A. Iyase. *On a fractional-order p-Laplacian boundary value problem at resonance on the half-line with two dimensional kernel*. Advances in Difference Equations, 2021, (2021), pp. 1-14.

[19] O. F. Imaga, and S. O. Edeki, and O. O. Agboola. *On the solvability of a resonant p-Laplacian third-order integral m-point boundary value problem.* IAENG Int. J. Appl. Math, 50,(2020), pp. 256-261.

[20] P. Amster. *Topological methods in the study of boundary value problems.* Springer, (2014).

[21] R. F. Brown. *A topological introduction to nonlinear analysis.* Springer, (1993).

[22] R. L. Magin. *Fractional calculus models of complex dynamics in biological tissues.* Computers & Mathematics with Applications, (2010), 59, pp. 1586-1593.

[23] S. A. Iyase, and O. F. Imaga. *On a singular second order multipoint boundary value problem at resonance.* International Journal of Differential Equations, 2017, (2017), pp. 1-6.

[24] S. Djebali, A. G. Aoun. *Resonant fractional differential equations with multi-point boundary conditions on* $(0, +\infty)$. J. Nonlinear Funct. Anal, 2019, (2019), pp. 1-15.

[25] S. Djebali and L. Guedda. *Nonlocal p-Laplacian boundary value problems of fractional order at resonance.* Panamerican Mathematical Journal, 29, (2019), pp. 45-63.

[26] S. G. Samko, A. A. Kilbas, O. I. Marichev. *Fractional Integrals and Derivatives.* Gordon and breach science publishers, Yverdon Yverdon-les-Bains, Switzerland,1, (1993), pp. 1-973.

[27] W. Ge, and J. Ren. *An extension of Mawhin's continuation theorem and its application to boundary value problems with a p-Laplacian.* Nonlinear Analysis: Theory, Methods & Applications, 58, (2004), pp. 477-488.

[28] X. Lin, and Z. Du, and F. Meng. *A note on a third-order multi-point boundary value problem at resonance.* Mathematische Nachrichten, 284, (2011), pp. 1690-1700.

[29] X. Tang, X. Wang, Z. Wang. *The existence of solutions for mixed fractional resonant boundary value problem with* $p(t)$*-Laplacian operator.* J. Appl. Math. Comput, 61, (2019), pp. 559–572.

[30] Z. Laadjal, T. Abdeljawad, F. Jarad. *On existence–uniqueness results for proportional fractional differential equations and incomplete gamma functions.* Adv Differ Equ, 641, (2020), pp. 1-16.