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Theme

**A comparative study between two
Methods for approximating the
Caputo-Fabrizio fractional derivative**

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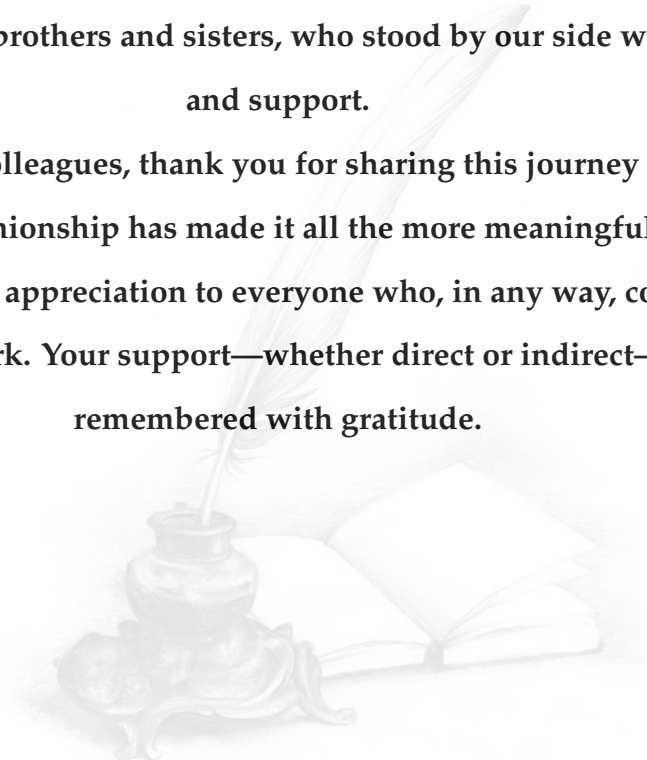
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Dedication

Praise be to God, who enlightened my path and gave me the best help. To the most precious thing I have in this world. To the one who was the reason for my existence on this earth. To the one who placed heaven under her feet. To the one to whom I bowed with all respect and appreciation, to the one I hope I may have attained her satisfaction, my mother. May God prolong her life.

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Ayachi Amor Hana

ملخص

قارنت هذه الدراسة بين طريقتين عدديتين لتقريب مشتقة كابوتوفابريزيو الكسرية: طريقة الفروق المتهية وطريقة موجات هار. أظهرت النتائج تفوق طريقة موجات هار من حيث الدقة ومعدل التقارب، حيث تحقق تقارباً أسياً مقارنة بالتقارب متعدد الحدود لطريقة الفروق المتهية. تتميز طريقة الفروق المتهية بسهولة التنفيذ والفهم، بينما تتفوق طريقة موجات هار في المشكلات التي تتطلب دقة عالية.

الكلمات المفتاحية: مشتقة كابوتوفابريزيو، حساب الكسور، طريقة الفروق المتهية، طريقة موجات هار، تقارب عددي، مشتقات كسرية.

Résumé

Cette étude a comparé deux méthodes numériques pour approximer la dérivée fractionnaire de Caputo-Fabrizio : la méthode des différences finies et la méthode des ondelettes de Haar. Les résultats ont montré la supériorité de la méthode des ondelettes de Haar en termes de précision et de taux de convergence, réalisant une convergence exponentielle comparée à la convergence polynomiale de la méthode des différences finies. La méthode des différences finies se caractérise par sa facilité d'implémentation et de compréhension, tandis que la méthode des ondelettes de Haar excelle dans les problèmes nécessitant une haute précision.

Mots-clés: Dérivée de Caputo-Fabrizio, calcul fractionnaire, méthode des différences finies, méthode des ondelettes de Haar, convergence numérique, dérivées fractionnaires.

Abstract

This study compared two numerical methods for approximating the Caputo-Fabrizio fractional derivative: the Finite Difference Method and the Haar Wavelet Method. Results showed the superiority of the Haar Wavelet Method in terms of accuracy and convergence rate, achieving exponential convergence compared to the polynomial convergence of the Finite Difference Method. The Finite Difference Method is characterized by ease of implementation and understanding, while the Haar Wavelet Method excels in problems requiring high accuracy.

Keywords: Caputo-Fabrizio derivative, fractional calculus, Finite Difference Method, Haar Wavelet Method, numerical convergence, fractional derivatives.

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Notations

$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
Γ	Gamma function
β	Beta function
s	Parameter in the Laplace transformation
${}^RL_a D_x^{(\alpha)}$	Riemann–Liouville fractional derivative
${}_a^C D_x^{(\alpha)}$	Caputo fractional derivative
${}_a^{CF} D_x^{(\alpha)}$	Caputo–Fabrizio fractional derivative
$\binom{n}{k}$	$C_k^n = \frac{n!}{k!(n-k)!}$

General Introduction

In recent years, fractional calculus has received great interest and wide popularity in many scientific fields, especially in mathematics, due to its ability to model various real-world phenomena that exhibit non-locality and memory effects. These include many physical [11, 14, 20, 33], engineering [19], and biological processes [7, 31].

Typically, such phenomena are modeled using different types of fractional differential equations, including ordinary fractional differential equations [25], space-time fractional differential equations [2, 4], and integral fractional differential equations [29, 32]. These equations are often defined in terms of well-known fractional derivatives such as Riemann–Liouville, Caputo, and Caputo–Fabrizio [8, 9].

Among these definitions, the Caputo–Fabrizio fractional derivative has gained significant attention due to its non-singular exponential kernel, which offers improved numerical stability and smoothness. As a result, it has been widely adopted by researchers working on models that require handling systems with fading memory.

Since most fractional differential equations do not admit exact analytical solutions, a variety of numerical methods have been proposed to approximate their solutions [3, 6, 21, 34]. Several of these methods rely on quadratic formulas to approximate the Caputo–Fabrizio derivative, such as the L1 and L2 schemes [27].

However, all these quadratic methods are based on the finite difference method, which typically converges at a polynomial rate. To overcome this limitation and enhance the accuracy and efficiency of approximation, this work proposes a new quadratic method that utilizes the Haar wavelet integration formula [5].

Theoretical analysis shows that the proposed method achieves an exponential convergence rate and is more computationally efficient compared to traditional quadratic methods.

This work is divided into three chapter:

1. We begin the thesis by reviewing the fundamentals of fractional calculus, including the definitions of the Riemann–Liouville, Caputo, and Caputo–Fabrizio derivatives and their properties.
2. Next, we discuss numerical methods, providing an explanation of the finite difference method and the Haar wavelet method, and applying them to fractional equations involving the Caputo–Fabrizio derivative.
3. Finally, the third chapter presents a numerical application and a comparison between the two methods in terms of stability, accuracy, and convergence speed, in order to determine the advantages of each method for different types of problems.

Mathematical Foundations of Fractional Calculus and Definitions of Fractional Derivatives

1.1 Riemann-Liouville and Caputo fractional derivatives

1.1.1 Riemann-Liouville fractional derivative:

Definition 1.1 [24] *The popular definition of fractional derivative is this one:*

$${}^{RL}D_a^{(\alpha)}g(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_a^x (x-t)^{n-\alpha-1} g(t) dt \quad (1.1)$$

$$(n-1 \leq \alpha < n), \quad \text{and } x > n$$

Proposition 1.1 *The Riemann-Liouville operator possesses the following key properties:*

$$1. \quad {}^{RL}D_t^{(\alpha)} \left({}^{RL}D_t^{(\beta)} g(x) \right) = {}^{RL}D_t^{(\alpha+\beta)} g(x)$$

2. For $\alpha = m \in \mathbb{N}$ we have:

$${}^{RL}D_a^{(0)}g(x) = \frac{1}{\Gamma(1)} \left(\frac{d}{dx} \right) \int_a^x g(t) dt = g(x), \quad (1.2)$$

$${}^{RL}D_a^{(m)}g(x) = \frac{1}{\Gamma(1)} \frac{d^{m+1}}{dx^{m+1}} \int_a^x g(t) dt = \frac{d^m}{dx^m} g(x), \quad (1.3)$$

As a result, the fractional derivative in the Riemann-Liouville sense matches the classical derivative when $\alpha \in \mathbb{N}$.

Remark 1.1

$${}^{RL}D_a^{(0)}g(x) = \left(\frac{d}{dx}\right)^n (I_a^{n-\alpha}g)(x) \quad (1.4)$$

such that: $n = \lfloor \alpha \rfloor + 1, x > a$.

1.1.2 Caputo fractional derivative:

Although the fractional derivative in the sense of Riemann-Liouville played a significant role in the development of fractional calculus, particularly due to its applications in both pure and applied mathematics, it has limitations. Specifically, the Riemann-Liouville derivative of a constant is not zero, and initial conditions in the Cauchy problem are expressed using fractional-order derivatives. In contrast, Caputo offers an alternative approach, where the derivative of a constant is zero, and initial conditions are expressed in the same way as in the classical case, using integer-order derivatives.

Definition 1.2 [24] Let $0 < n - 1 < \alpha < n$ and g be a function of class $C^n([a, b])$. The fractional derivative of order α in the sense of Caputo of the function g is defined by:

$${}_a^C D_x^{(\alpha)} g(x) = \frac{1}{\Gamma(n - \alpha)} \int_a^x g^{(n)}(t)(x - t)^{n-\alpha-1} dt \quad (1.5)$$

- Under natural conditions on the function $g(x)$, for $\alpha \rightarrow n$, the Caputo derivative becomes a conventional n -th derivative of the function $g(x)$.

Indeed, let us assume that $0 \leq n - 1 < \alpha < n$ and that the function $g(x)$ has $n + 1$ continuous bounded derivatives in $[a, T]$ for every $T > a$, then [18]

$$\lim_{\alpha \rightarrow n} {}_a^C D_x^{(\alpha)} g(x) = \lim_{\alpha \rightarrow n} \left(\frac{1}{\Gamma(n - \alpha)} \int_a^x g^{(n)}(t)(x - t)^{n-\alpha-1} dt \right)$$

From integration by parts we have:

$$\lim_{\alpha \rightarrow n} {}_a^C D_x^{(\alpha)} g(x) = \lim_{\alpha \rightarrow n} \left(\frac{g^{(n)}(a)(x - a)^{n-\alpha}}{\Gamma(n - \alpha + 1)} + \frac{1}{\Gamma(n - \alpha + 1)} \int_a^x (x - t)^{n-\alpha} g^{(n+1)}(t) dt \right)$$

$$= g^n(a) + \int_a^x g^{(n+1)}(t) dt = g^{(n)}(x), \quad n = 1, 2, \dots$$

- Non-Commutation [18]

In general, the Caputo fractional derivative and the integer-order derivative do not commute. That is,

$${}^C D_a^{(\alpha)} (D_x^{(m)} g(x)) \neq D_x^{(m)} ({}^C D_a^{(\alpha)} g(x)), \quad (m = 0, 1, 2, \dots, n-1 \text{ with } 1 < \alpha < n).$$

When Can They Be Interchanged?

The order of applying the differentiation operators can be swapped only under certain conditions. Specifically, if the following is satisfied:

$${}^C D_a^{(\alpha)} (D_x^{(m)} g(x)) = D_x^{(m)} ({}^C D_a^{(\alpha)} g(x)) = {}^C D_a^{(\alpha+m)} g(x), \quad (1.6)$$

provided that:

$$g^{(s)}(0) = 0, \quad \text{for } s = n, n+1, \dots, m,$$

$$(m = 0, 1, 2, \dots, n-1, \text{ with } 1 < \alpha < n).$$

Example 1.1 Consider the function:

$$g(x) = x^\beta.$$

For values $0 < n-1 < \alpha < n$, the Caputo fractional derivative of $g(x)$ can be written as:

$${}^C D^{(\alpha)} g(x) = I^{n-\alpha} D_x^n x^\beta. \quad (1.7)$$

We know that:

$$D_x^n x^\beta = \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)} \right) x^{\beta-n}$$

Therefore, the result becomes:

$$I^{n-\alpha} \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)} x^{\beta-n} \right) = \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)\Gamma(n-\alpha)} \right) \int_0^x (x-t)^{n-\alpha-1} t^{\beta-n} dt$$

By applying the substitution $t = yx$, which implies $dt = x dy$, we can transform the integral as follows:

$$\int_0^x (x-t)^{n-\alpha-1} t^{\beta-n} dt = \int_0^1 (x-y)^{n-\alpha-1} x^{\beta-n} x dy$$

Let's make the change of variable $t = yx$ so $dt = x dy$. Substituting gives:

$$\begin{aligned} &= \int_0^1 x^{n-\alpha-1} (1-y)^{n-\alpha-1} x^{\beta-n} y^{\beta-n} x dy \\ &= \int_0^1 x^{\beta-\alpha} (1-y)^{n-\alpha-1} y^{\beta-n} dy \\ &\quad - x^{\beta-\alpha} \int_0^1 (1-y)^{n-\alpha-1} y^{\delta-n} dy \end{aligned}$$

This integral is recognized as the Beta function, and can be evaluated as

$$= x^{\beta-\alpha} \cdot \frac{\Gamma(n-\alpha)\Gamma(\beta-n+1)}{\Gamma(\beta-\alpha+1)}$$

So, we conclude:

$$\begin{aligned} I^{n-\alpha} \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)} x^{\beta-n} \right) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)} \cdot \frac{1}{\Gamma(n-\alpha)} \cdot \frac{\Gamma(n-\alpha)\Gamma(\beta-n+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha} \end{aligned}$$

Finally, we obtain:

$${}^C D^{(\alpha)} x^\alpha = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}$$

This shows that, unlike the Riemann-Liouville derivative, the Caputo fractional derivative of a constant is zero.

1.1.3 Some Key Properties of Fractional Derivatives

Theorem 1.1 [18] Fractional differentiation, much like its classical counterpart of integer-order differentiation, satisfies the linearity property. That is, the fractional derivative of a linear combination

of functions is equal to the same linear combination of their individual fractional derivatives:

$$\mu D^{(\alpha)} f(x) + \lambda D^{(\alpha)} g(x) = D^{(\alpha)} (\mu f(x) + \lambda g(x))$$

Here, λ and μ are constant scalars, and $f(x), g(x)$ are differentiable functions

Proof (Linearity of the Caputo Derivative)

Let $D^{(\alpha)}$ be the Caputo derivative, where $n-1 \leq \alpha < n$ and $n = 1$. According to the definition, we get:

$$\begin{aligned} D^{(\alpha)}(\mu f(x) + \lambda g(x)) &= \frac{1}{\Gamma(1-\alpha)} \int_0^x (\mu f(t) + \lambda g(t))'(x-t)^{-\alpha} dt \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^x (\mu f'(t) + \lambda g'(t))(x-t)^{-\alpha} dt \\ &= \frac{\mu}{\Gamma(1-\alpha)} \int_0^x f'(t)(x-t)^{-\alpha} dt + \left(\frac{\lambda}{\Gamma(1-\alpha)} \int_0^x g'(t)(x-t)^{-\alpha} dt \right) \\ &= \mu_a^C D_x^\alpha f(x) + \lambda_a^C D_x^\alpha g(x) \end{aligned} \quad (1.8)$$

Thus, linearity is confirmed.

Theorem 1.2 The Leibniz Rule [18]

For any natural number n , the classical product rule for derivatives states:

$$\frac{d^n}{dx^n} [f(x)g(x)] = \sum_{k=0}^n \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) \quad (1.9)$$

Generalized Form for Fractional Derivatives:

This result can be extended to fractional derivatives as follows:

$$D^{(\alpha)} [f(x)g(x)] = \sum_{k=0}^n \binom{\alpha}{k} f^{(k)}(x) D^{\alpha-k} g(x) + R_n^\alpha(x) \quad (1.10)$$

Where:

1. $\binom{\alpha}{k}$ is the generalized binomial coefficient,
2. $R_n^\alpha(x)$ is the remainder term, accounting for approximation beyond order n .

3. And this holds when $n \geq \alpha + 1$.

The remainder term $R_n^\alpha(x)$ is given by:

$$R_n^\alpha(x) = \frac{1}{n!\Gamma(-\alpha)} \int_a^t (x-s)^{-\alpha-1} g(s) \left(\int_s^t f^{(n+1)}(\xi) d\xi \right) ds$$

and it satisfies the condition:

$$\lim_{n \rightarrow \infty} R_n^\alpha(x) \rightarrow 0$$

Now, if both f and g are continuous on the interval $[a, t]$, and all of their derivatives exist and are continuous as well, the generalized product rule for fractional derivatives simplifies to:

$$D^{(\alpha)}[f(x)g(x)] = \sum_{k=0}^n \binom{\alpha}{k} f^{(k)}(x) D^{\alpha-k} g(x)$$

Here, $D^{(\alpha)}$ refers to the Riemann-Liouville fractional derivative.

1.2 Caputo-Fabrizio Fractional Derivative (CFFD)

Due to the singularity present in the kernel of the Caputo fractional derivative [1, 18], especially at the upper limit of integration—it may not always provide an accurate description of memory effects in real-world systems. To overcome this limitation, Caputo and Fabrizio [23] introduced a new type of fractional derivative that avoids singularities in its kernel. The kernel in this formulation takes the form of an exponential function.

More recently, Losada and Nieto [16] developed the corresponding fractional integral related to this new Caputo-Fabrizio derivative.

This section focuses on the foundational definitions and key properties of the Caputo-Fabrizio fractional derivative.

Let us recall the standard Caputo fractional time derivative (CFD) of order α , defined as:

$${}^C D_t^{(\alpha)} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^t f'(\tau) (t-\tau)^{-\alpha} d\tau \quad (1.11)$$

Definition 1.3 [23] Let $f \in H^1(a, b)$, where $b > a$, and $\alpha \in]0, 1[$, the Caputo–Fabrizio fractional

derivative is defined as:

$${}^{CF}D_t^{(\alpha)} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(\tau) \exp \left[-\frac{\alpha(t-\tau)}{1-\alpha} \right] d\tau \quad (1.12)$$

Here, $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$.

If the function f does not belong to $H^1(a, b)$, the derivative is instead expressed as:

$${}^{CF}D_t^{(\alpha)} f(t) = \frac{\alpha M(\alpha)}{1-\alpha} \int_a^t [f(t) - f(\tau)] \exp \left[-\frac{\alpha(t-\tau)}{1-\alpha} \right] d\tau$$

Later, **Losada and Nieto** proposed an improved version of the CFFD [16], given by:

$${}^{CF}D_t^{(\alpha)} f(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t f'(\tau) \exp \left[-\frac{\alpha(t-\tau)}{1-\alpha} \right] d\tau$$

Finally, it is useful to note that if we define [23]:

$$\sigma = \frac{1-\alpha}{\alpha} \in [0, \infty), \quad \text{then} \quad \frac{1}{1-\sigma} \in [0, 1]$$

The definition 1.3 of the Caputo–Fabrizio fractional derivative (CFFD) can be rewritten in terms of σ as:

$$\overline{D}_t^{(\sigma)} f(t) = \frac{N(\sigma)}{\sigma} \int_0^t f'(\tau) \exp \left[-\frac{(t-\tau)}{\sigma} \right] d\tau$$

Here, $\alpha \in [0, \infty)$, and $N(\alpha)$ is the corresponding normalization factor related to $M(\alpha)$, satisfying the condition: $N(0) = N(\infty) = 1$. Moreover, since:

$$\lim_{\sigma \rightarrow 0} \frac{1}{\sigma} \exp \left[-\frac{(t-\tau)}{\sigma} \right] = \delta(t-\tau)$$

(where δ denotes the Dirac delta function). And as $\alpha \rightarrow 1$, we also get $\sigma \rightarrow 0$, which leads to the classical derivative:

$$\lim_{\alpha \rightarrow 1} {}^{CF}D_t^\alpha f(t) = \lim_{\alpha \rightarrow 1} \frac{M(\alpha)}{1-\alpha} \int_0^t f'(\tau) \exp \left[-\frac{\alpha(t-\tau)}{1-\alpha} \right] d\tau$$

$$= \lim_{\alpha \rightarrow 0} \frac{N(\sigma)}{\sigma} \int_0^t f'(\tau) \exp \left[-\frac{\alpha(t-\tau)}{\sigma} \right] d\tau = f(t) - f(\alpha).$$

This shows that the Caputo–Fabrizio derivative smoothly converges to the classical derivative as α approaches 1.

Let us examine the Caputo–Fabrizio fractional derivative (CFFD) applied to a specific function:

$$f(t) = \sin(\omega t)$$

Assuming the parameters $\alpha = 0.66$, $a = -8$ and $\omega = 1$, the derivative becomes:

$${}^{CF}D_t^{0.66} \sin(\omega t) = \frac{M(0.66)}{0.33} \int_a^t \cos(\tau) \exp[-2(t-\tau)] d\tau$$

The numerical simulation of this fractional derivative yields the following visual results.

Composition of CFFD Operator:

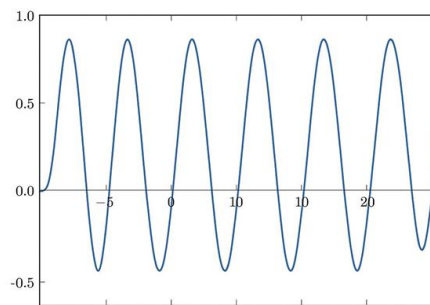


Figure 1.1: ESimulation of CFFD operator composed with itself, with $\alpha = 0.66$

Some Ordinary Linear Caputo-Fabrizi Fractional Differential Equations

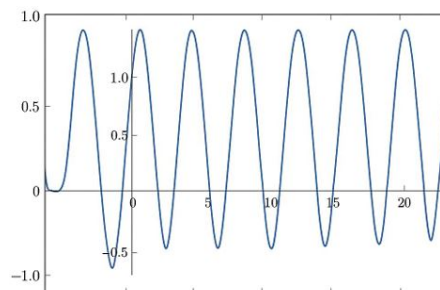


Figure 1.2: Simulation of CFFD differential equation, with $\alpha = 0.66$

Based on these two simulations with the Caputo–Fabrizio fractional derivative (CFFD)

at $\alpha = 0.66$, the results closely resemble those of the classical derivative. However, when models are examined with values approaching 0, the behavior changes significantly.

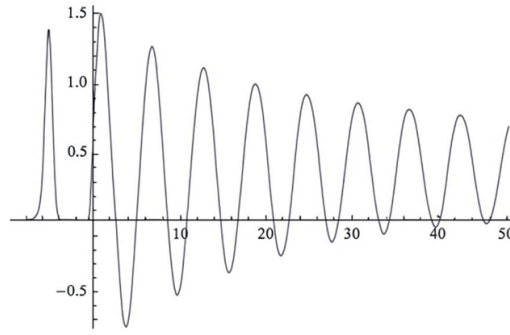
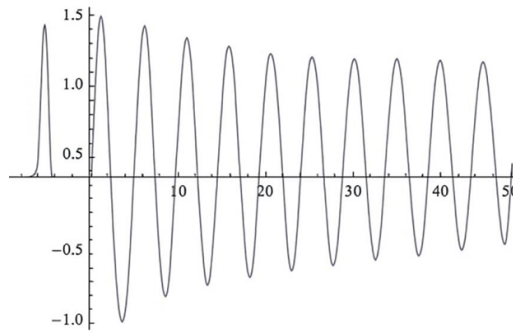


Figure 1.3: simulation of (CFFD), with $\alpha = 0.1$



simulation of (CFD), with $\alpha = 0.1$

Therefore, for $\alpha = 0.1$, as illustrated in Figures 1.3 and 1.4, we can observe distinct behaviors between the (CFFD) and (CFD) simulations. Specifically, the classical derivative (CFD) exhibits a stronger dependence on past values, while the Caputo–Fabrizio derivative (CFFD) demonstrates faster stabilization and a reduced memory effect.

If $n \geq 1$ and $\alpha \in [0, 1]$, the **fractional time derivative** of order $n + \alpha$, denoted by ${}^{CF}D_t^{(n+\alpha)} f(t)$, is defined as:

$${}^{CF}D_t^{(\alpha+n)} f(t) := {}^{CF}D_t^{(\alpha)} \left({}^{CF}D_t^{(n)} f(t) \right) \quad (1.13)$$

Theorem 1.3 [23] *For the Caputo–Fabrizio fractional derivative (CFFD), if the function $f(t)$ satisfies the condition:*

$$f^{(s)}(0) = 0, \quad \text{for } s = 1, 2, 3, \dots, n$$

then the following holds:

$${}^{CF}D_t^{(\alpha)} \left({}^{CF}D_t^{(n)} f(t) \right) = {}^{CF}D_t^{(n)} \left({}^{CF}D_t^{(\alpha)} f(t) \right) \quad (1.14)$$

Proposition 1.2 (Linearity) *Let the Caputo-Fabrizio operator ${}^{CF}D^{(\alpha)}$ act on a linear combination of functions. Then, it satisfies:*

$${}^{CF}D^{(\alpha)}(\mu f(x) + \lambda g(x)) = \mu {}^{CF}D^{(\alpha)} f(x) + \lambda {}^{CF}D^{(\alpha)} g(x)$$

Proof Based on the definition 1.3, we write:

$$\begin{aligned} {}_a^{CF}D^{(\alpha)}(\mu f(t) + \lambda g(t)) &= \frac{M(\alpha)}{1-\alpha} \int_a^t (\mu f(\tau) + \lambda g(\tau))' \exp \left[-\frac{\alpha(t-\tau)}{1-\alpha} \right] d\tau \quad (1.15) \\ &= \frac{M(\alpha)}{1-\alpha} (\mu f'(\tau) + \lambda g'(\tau)) \exp \left[-\frac{\alpha(t-\tau)}{1-\alpha} \right] d\tau \\ &= \frac{\mu M(\alpha)}{1-\alpha} \int_a^t f'(\tau) \exp \left[-\frac{\alpha(t-\tau)}{1-\alpha} \right] d\tau \\ &\quad + \frac{\lambda M(\alpha)}{1-\alpha} \int_a^t g'(\tau) \exp \left[-\frac{\alpha(t-\tau)}{1-\alpha} \right] d\tau \\ &= \mu {}_a^{CF}D^{(\alpha)} f(t) + \lambda {}_a^{CF}D^{(\alpha)} g(t) \end{aligned}$$

Thus, linearity is confirmed.

Numerical Methods for Approximating the Caputo Fractional Derivative

In this chapter, we discuss two numerical methods for approximating the Caputo-Fabrizio fractional derivative: the finite difference method and the Haar wavelet method. The finite difference approach is widely used due to its simplicity and computational efficiency [12], while the Haar wavelet method is particularly effective in analyzing functions with sharp transitions or discontinuities [15]. We present the mathematical formulation of both methods, supported by numerical examples that demonstrate their accuracy and convergence. This chapter aims to provide a practical and reliable numerical framework for solving fractional differential equations involving the Caputo-Fabrizio derivative [18,24].

2.1 Finite Difference Methods

Principle of the Method

The fundamental idea behind the finite difference method is to replace all derivatives in a differential equation with algebraic expressions that do not involve derivatives. Let's illustrate this method using a simple example governed by an ordinary differential equation:

$$\begin{cases} \frac{d^2 g(x)}{dx^2} = 1, & x \in [0, 1] \\ g(0) = 1, \\ g(1) = 0 \end{cases} \quad (2.1)$$

The analytical solution can be obtained by integrating twice:

$$\begin{cases} \frac{dg(x)}{dx} = x + C_1 \\ g(x) = \frac{1}{2}x^2 + C_1x + C_2 \end{cases} \quad (2.2)$$

Applying the boundary conditions:

$$g(0) = 1 \Rightarrow C_2 = 1$$

$$g(1) = 0 \Rightarrow C_1 = -\frac{3}{2}$$

Thus, the exact solution is:

$$g(x) = \frac{1}{2}x^2 - \frac{3}{2}x + 1 \quad (2.3)$$

This solution is continuous and valid for all $x \in [0, 1]$.

2.1.1 Procedure

The finite difference solution process generally involves the following steps:

1. **Domain discretization:** The domain is divided into a finite number of sub-intervals, placing nodes at each end.
2. **Equation approximation:** Derivatives are replaced with algebraic approximations based on suitable finite difference schemes (discussed in the previous chapter).
3. **Application of approximations:** The approximated equation is evaluated at the mesh nodes.

4. **Imposing boundary conditions:** Different types of boundary conditions are handled as required (details later).
5. **Solving the resulting linear system:** The system is solved to find the unknown function values at the nodes.
6. **Post-processing:** Results are validated, interpreted, and interpolated as needed.

Discretization

As an example, let's divide the interval into 4 equally spaced elements, resulting in 5 nodes:

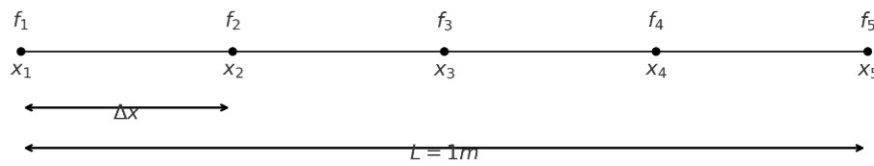


Figure 2.1: Grid-Based Representation of the Computational Domain

Approximation

Using a central difference scheme, the differential equation is approximated by:

$$\frac{d^2 g(x)}{dx^2} \approx \frac{g_{i+1} - 2g_i + g_{i-1}}{(\Delta x)^2} \quad (2.4)$$

Multiplying through by $(\Delta x)^2$ we have:

$$g_{i+1} - 2g_i + g_{i-1} \approx 0 \quad (2.5)$$

Applying the Approximation

Applying this at each of the 5 nodes:

$$\begin{aligned}
i = 1 : g_2 - 2g_1 + g_0 &\approx 0 \\
i = 2 : g_3 - 2g_2 + g_1 &\approx 0 \\
i = 3 : g_4 - 2g_3 + g_2 &\approx 0 \\
i = 4 : g_5 - 2g_4 + g_3 &\approx 0 \\
i = 5 : g_6 - 2g_5 + g_4 &\approx 0
\end{aligned} \tag{2.6}$$

Notice that fictitious points g_0 and g_6 appear.

Solving the System of Linear Algebraic Equations

The resulting system is well-posed (i.e., the number of unknowns equals the number of equations), so we proceed to solve it using a numerical method, which may be either direct or iterative.

The system can be written in matrix form as:

$$[A]\{g\} = \{b\} \tag{2.7}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.8}$$

In MATLAB, the solution can be obtained using the instruction:

`f = A \ b`

matlab

```
A = [1 0 0 0 0; 1 -2 0 1 0; 0 1 -2 1 0; 0 0 1 -2 1; 0 0 0 0 1];
b = [1; 0; 0; 0; 0];
f = A \ b;
```

This gives the result:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.75 \\ 0.50 \\ 0.25 \\ 0 \end{bmatrix}$$

Post-processing

Let's compare the numerical solution to the analytical one:

x	Analytical Solution	Numerical Solution	Relative Error (%)
0	1.0000	1.0000	0 (boundary)
0.25	0.6563	0.7500	14%
0.5	0.3750	0.5000	33%
0.75	0.1563	0.2500	60%
1	0.0000	0.0000	0 (boundary)

Table 2.1: The accuracy can be significantly improved by increasing the number of nodes.

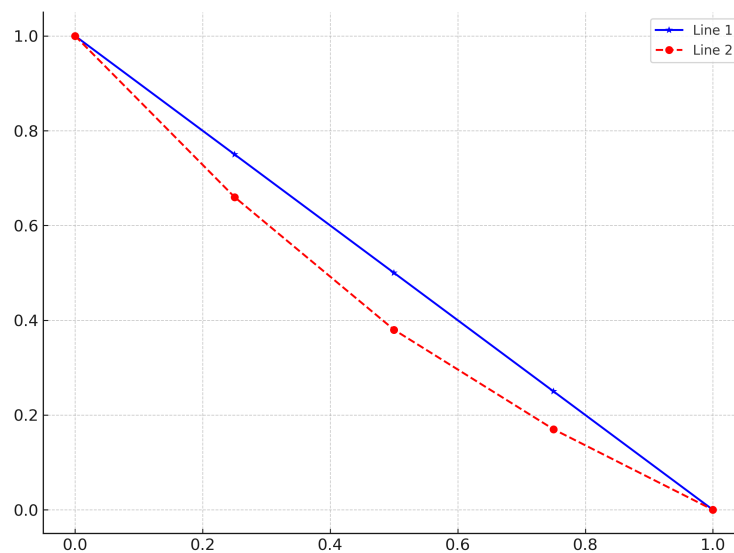


Figure 2.2: A comparison of analytical and computed numerical solution

2.1.2 Classification of Partial Differential Equations

Any second-order partial differential equation in two variables can be written in the general form:

$$A \frac{\partial^2 g}{\partial x^2} + B \frac{\partial^2 g}{\partial x \partial y} + C \frac{\partial^2 g}{\partial y^2} + D \frac{\partial g}{\partial x} + E \frac{\partial g}{\partial y} + Fg + G = 0 \quad (2.9)$$

This expression resembles the general second-degree algebraic form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (2.10)$$

This equation can represent an ellipse, a parabola, or a hyperbola, depending on the value of the discriminant:

$$\Delta = B^2 - 4AC \quad (2.11)$$

Likewise, a second-order partial differential equation can be classified as: **Elliptic**, **Parabolic** or **Hyperbolic** based on this discriminant:

Type of Equation	Discriminant $\Delta = B^2 - 4AC$	Example
Elliptic	$\Delta < 0$	$\frac{\partial^2 T(x, y)}{\partial x^2} + \frac{\partial^2 T(x, y)}{\partial y^2} = 0$
Parabolic	$\Delta = 0$	$\frac{\partial T(x, t)}{\partial t} = \alpha \frac{\partial^2 T(x, t)}{\partial x^2}$
Hyperbolic	$\Delta > 0$	$\frac{\partial^2 T(x, t)}{\partial t^2} = c^2 \frac{\partial^2 T(x, t)}{\partial x^2}$

Table 2.2: Classification of PDEs based on the discriminant

This classification helps in determining the type of boundary and initial conditions required, as well as in selecting appropriate finite difference methods and precautions for solving such equations.

2.1.3 Classification of Boundary Conditions

For a partial differential equation (PDE) to have a unique solution, it must be accompanied by appropriate boundary conditions (or initial conditions in the case of time-dependent problems). A problem that includes such conditions is referred to as a *well-posed* problem.

The main types of boundary conditions are:

- **Dirichlet condition:** The function's value is known on the boundary.
- **Neumann condition:** The derivative (gradient) of the function is known on the boundary.
- **Robin condition:** A linear combination of the function and its derivative is specified.
- **Mixed condition:** Different boundary segments have different types of conditions.
- **Cauchy condition:** Both the function and its derivative are known simultaneously.

2.1.4 Solving a Steady-State 2D Conduction Problem

A common example of a physical phenomenon governed by an elliptic equation is steady-state heat conduction in two dimensions.

In this subsection, we demonstrate the method and how it applies to a typical case:

We aim to determine the temperature distribution in a thin, square plate ($1\text{ m} \times 1\text{ m}$) under steady-state conditions, with the plate subjected to specific temperature values along its edges.

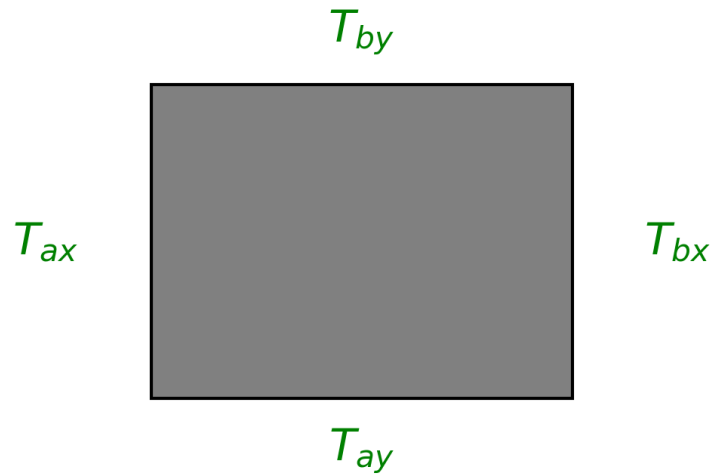


Figure 2.3: Two-Dimensional Domain with Applied Boundary Conditions

This problem is formulated using the two-dimensional Laplace equation, which is elliptic in nature. It is supplemented by four Dirichlet boundary conditions:

$$\frac{\partial^2 T(x, y)}{\partial x^2} + \frac{\partial^2 T(x, y)}{\partial y^2} = 0 \quad (2.12)$$

$$T(0, y) = T_{ax}, \quad T(1, y) = T_{bx}, \quad T(x, 0) = T_{ay}, \quad T(x, 1) = T_{by}$$

Given Boundary Temperatures

$$T_{ax} = 50^\circ\text{C}$$

$$T_{bx} = 100^\circ\text{C}$$

$$T_{ay} = 50^\circ\text{C}$$

$$T_{by} = 100^\circ\text{C}$$

We apply the same solution steps as before.

Grid Discretization

The computational domain is divided into 4 segments along each axis. Assuming a uniform grid with spacing $h = \Delta x = \Delta y$, this results in a 5-node layout in both directions.

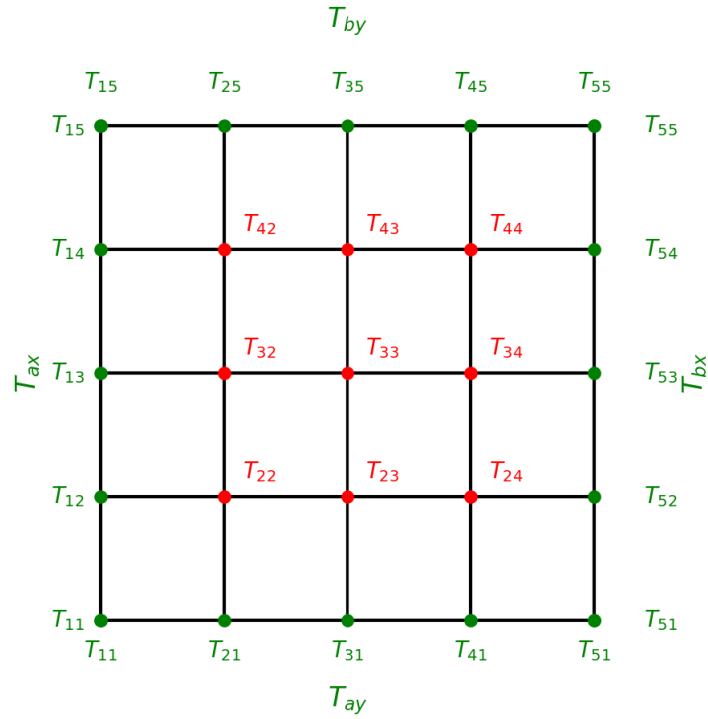


Figure 2.4: Grid Partitioning of the 2D Domain

Approximation

To approximate the differential equation, we replace the second-order derivatives with finite difference expressions using a centered difference scheme:

$$\begin{aligned}\frac{\partial^2 T(x, y)}{\partial x^2} &\approx \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} \\ \frac{\partial^2 T(x, y)}{\partial y^2} &\approx \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}\end{aligned}\tag{2.13}$$

Adding the approximations gives:

$$\frac{\partial^2 T(x, y)}{\partial x^2} + \frac{\partial^2 T(x, y)}{\partial y^2} \approx \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} \approx 0\tag{2.14}$$

When the mesh is uniform, i.e., $\Delta x = \Delta y = h \neq 0$, the equation simplifies to:

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} \approx 0\tag{2.15}$$

This compact form of the discrete Laplacian is commonly used in numerical simulations on regular grids.

Applying the Approximation

In general, the approximation of the differential equation is applied at all grid nodes within the domain. However, this would result in the creation of fictitious or ghost nodes, similar to what occurs in the previously discussed 1D problem.

To avoid this, the approximation is only applied at the internal nodes, specifically for $i = 2 \rightarrow 4$ and $j = 2 \rightarrow 4$. This yields the following system of equations:

$$i = 2, j = 2 \Rightarrow T_{3,2} + T_{1,2} + T_{2,3} + T_{2,1} - 4T_{2,2} \approx 0$$

$$i = 3, j = 2 \Rightarrow T_{4,2} + T_{2,2} + T_{3,3} + T_{3,1} - 4T_{3,2} \approx 0$$

$$i = 4, j = 2 \Rightarrow T_{5,2} + T_{3,2} + T_{4,3} + T_{4,1} - 4T_{4,2} \approx 0$$

$$i = 2, j = 3 \Rightarrow T_{3,3} + T_{1,3} + T_{2,4} + T_{2,2} - 4T_{2,3} \approx 0$$

$$i = 3, j = 3 \Rightarrow T_{4,3} + T_{2,3} + T_{3,4} + T_{3,2} - 4T_{3,3} \approx 0$$

$$i = 4, j = 3 \Rightarrow T_{5,3} + T_{3,3} + T_{4,4} + T_{4,2} - 4T_{4,3} \approx 0$$

$$i = 2, j = 4 \Rightarrow T_{3,4} + T_{1,4} + T_{2,5} + T_{2,3} - 4T_{2,4} \approx 0$$

$$i = 3, j = 4 \Rightarrow T_{4,4} + T_{2,4} + T_{3,5} + T_{3,3} - 4T_{3,4} \approx 0$$

$$i = 4, j = 4 \Rightarrow T_{5,4} + T_{3,4} + T_{4,5} + T_{4,3} - 4T_{4,4} \approx 0$$

Incorporating Boundary Conditions

To incorporate the boundary conditions into the system, their effects are shifted to the right-hand side of the previously established finite difference equations. This results in the following modified system:

$$T_{3,2} + T_{2,3} - 4T_{2,2} \approx -T_{ax} - T_{ay}$$

$$T_{4,2} + T_{2,2} + T_{3,3} - 4T_{3,2} \approx -T_{ay}$$

$$T_{5,2} + T_{3,2} + T_{4,3} - 4T_{4,2} \approx -T_{bx} - T_{ay}$$

$$T_{3,3} + T_{2,4} + T_{4,4} - 4T_{3,3} \approx -T_{ax}$$

$$T_{2,3} + T_{3,4} + T_{2,2} - 4T_{2,3} \approx 0$$

$$T_{4,3} + T_{3,2} + T_{4,4} - 4T_{3,3} \approx 0$$

$$T_{2,4} + T_{3,3} + T_{1,4} - 4T_{2,4} \approx -T_{bx}$$

$$T_{3,4} + T_{2,4} + T_{4,5} - 4T_{3,4} \approx -T_{by}$$

$$T_{4,4} + T_{3,4} - 4T_{4,4} \approx -T_{bx} - T_{by}$$

This results in a system of linear algebraic equations. Since the number of equations matches the number of unknowns (nine in total), the system is well-posed and can be solved numerically. One may use either direct methods (such as Gaussian elimination) or iterative techniques.

To solve this system using **MATLAB**, it must be written in matrix form:

$$[A]\{f\} = \{b\}$$

where

$$A = \begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 \end{bmatrix}$$

$$\{f\} = \begin{pmatrix} T_{2,2} \\ T_{3,2} \\ T_{4,2} \\ T_{2,3} \\ T_{3,3} \\ T_{4,3} \\ T_{2,4} \\ T_{3,4} \\ T_{4,4} \end{pmatrix}, \quad \{b\} = \begin{pmatrix} -T_{ax} - T_{ay} \\ -T_{ay} \\ -T_{bx} - T_{ay} \\ -T_{ax} \\ 0 \\ 0 \\ -T_{bx} \\ -T_{by} \\ -T_{bx} - T_{by} \end{pmatrix}$$

This yields the following results for the internal node temperatures:

$$\begin{Bmatrix} T_{2,2} \\ T_{3,2} \\ T_{4,2} \\ T_{2,3} \\ T_{3,3} \\ T_{4,3} \\ T_{2,4} \\ T_{3,4} \\ T_{4,4} \end{Bmatrix} = \begin{Bmatrix} 57.1429 \\ 64.2857 \\ 75.0000 \\ 64.2857 \\ 75.0000 \\ 85.7143 \\ 75.0000 \\ 85.7143 \\ 92.8571 \end{Bmatrix}$$

These values represent the numerical solution of the temperature field at the internal grid points.

Post-Processing

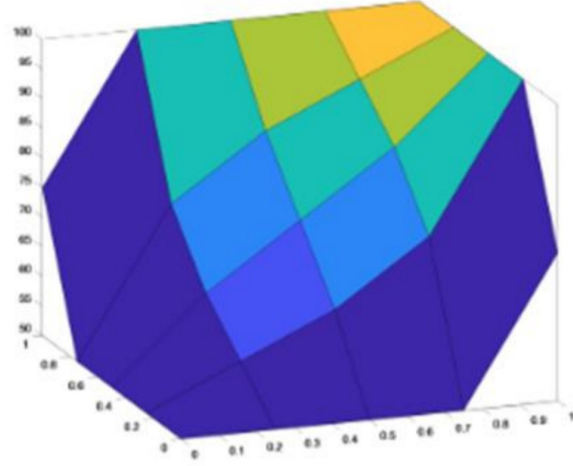


Figure 2.5: Temperature distribution obtained with a coarse grid of $n_x = n_y = 5$ nodes in each direction.

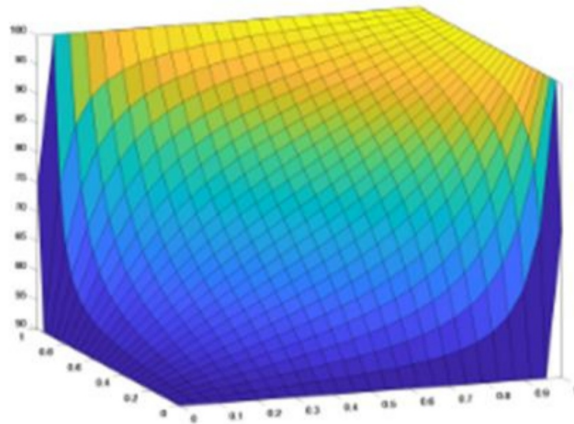


Figure 2.6: Temperature distribution computed using a finer grid with $n_x = n_y = 25$ nodes in each direction.

2.1.5 Finite Difference Approximation of the Caputo Derivative

Definition 2.1 [30](Caputo's Fractional Derivative)

Let $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$ such that $n - 1 < \alpha < n$, then the Caputo's fractional derivative of order α is

defined by

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{f^{(n)}(x)}{(t - x)^{\alpha - n + 1}} dx \quad (2.16)$$

In addition, if $a = 0$ in Eq. (2.16), then the most used version of the Caputo's fractional derivative is obtained, i.e.,

$$D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{f^{(n)}(x)}{(t - x)^{\alpha - n + 1}} dx \quad (2.17)$$

For the fractional derivative, the Caputo's definition is chosen. Now, according to

$$D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{f^{(n)}(x)}{(t - x)^{\alpha - n + 1}} dx \quad (2.18)$$

if $1 < \alpha < 2$ then the Caputo's Fractional Derivative is defined by

$$D^\alpha y(t) = \frac{1}{\Gamma(2 - \alpha)} \int_0^t \frac{y''(x)}{(t - x)^{\alpha - 1}} dx \quad (2.19)$$

for $t \geq 0$ and $\alpha \in \mathbb{R}^+$.

By applying integral by parts on the right hand side of Eq. (2.19), we get

$$D^\alpha y(t) = \frac{1}{(2 - \alpha)\Gamma(2 - \alpha)} \left(y''(0)t^{2-\alpha} + \int_0^t (t - x)^{2-\alpha} y'''(x) dx \right) \quad (2.20)$$

last integral in Eq. (2.20) can be approximated using the Composite Trapezoidal Rule, Eq. (2.16), as follows:

$$\int_0^t (t - x)^{2-\alpha} y'''(x) dx \approx \frac{h}{2} \left[t^{2-\alpha} y'''(0) + 2 \sum_{j=1}^{n-1} (t - x_j)^{2-\alpha} y'''(x_j) + (t - b)^{2-\alpha} y'''(b) \right], \quad (2.21)$$

with $h = \frac{t}{n}$ and $x_j = jh$ for each $j = 0, 1, \dots, n - 1$.

Now, by substituting Eq. (2.21) into Eq. (2.20), we obtain:

$$D^\alpha y(t) \approx \frac{1}{(2 - \alpha)\Gamma(2 - \alpha)} \left[y''(0)t^{2-\alpha} + \frac{h}{2} \left(t^{2-\alpha} y'''(0) + 2 \sum_{j=1}^{n-1} (t - x_j)^{2-\alpha} y'''(x_j) + (t - b)^{2-\alpha} y'''(b) \right) \right] \quad (2.22)$$

Here, y'' and y''' can be approximated at specific points. The forward finite difference formula

for the second-order derivative, Eq. (2.21), is used to approximate $y''(0)$, such that:

$$y''(0) \approx \frac{y(0) - 2y(h) + y(2h)}{h^2} \quad (2.23)$$

To approximate $y'''(0)$, the forward finite difference formula for the third-order derivative, Eq. (2.23), is used as follows:

$$y'''(0) \approx \frac{-y(0) + 3y(h) - 3y(2h) + y(3h)}{h^3} \quad (2.24)$$

While the central finite difference formula for the third-order derivative, Eq. (2.2), is used to approximate $y'''(x_j)$ in the sum term of Eq. (2.22), as:

$$y'''(x_j) \approx \frac{-y(x_j - 2h) + 2y(x_j - h) - 2y(x_j + h) + y(x_j + 2h)}{2h^3} \quad (2.25)$$

or small values of h and for $j = 1, 2, \dots, n - 1$.

Finally, $y'''(b)$ can be approximated using the backward finite difference formula for the third-order derivative, Eq. (2.4), as:

$$y'''(b) \approx \frac{y(b) - 3y(b - h) + 3y(b - 2h) - y(b - 3h)}{h^3} \quad (2.26)$$

By substituting Eqs. (2.23), (2.24), (2.25), and (2.26) into Eq. (2.22), we get:

$$D^\alpha y(t) \approx \frac{1}{(2 - \alpha)\Gamma(2 - \alpha)} \left(A + \frac{h}{2}(B + 2C + D) \right) \quad (2.27)$$

where

$$\begin{aligned} A &= \frac{y(0) - 2y(h) + y(2h)}{h^2} t^{2-\alpha}, \\ B &= \frac{-y(0) + 3y(h) - 3y(2h) + y(3h)}{h^3} t^{2-\alpha}, \\ C &= \sum_{j=1}^{n-1} \left[\frac{-y(x_j - 2h) + 2y(x_j - h) - 2y(x_j + h) + y(x_j + 2h)}{2h^3} (t - x_j)^{2-\alpha} \right], \end{aligned}$$

and

$$D = \frac{y(b) - 3y(b-h) + 3y(b-2h) - y(b-3h)}{h^3} (t-b)^{2-\alpha}$$

We observe that, according to Eq. (2.27), the approximation of $D^\alpha y(t)$ depends on the step size h . As h decreases, the approximation becomes more accurate due to finer discretization. Moreover, Eq. (2.27) represents the proposed formula for approximating the fractional derivative of order $1 < \alpha < 2$ of a given function $y(t)$, which is clear. Also, the value of h Eq. (2.5), is $(b-a)/n$ So;

$$n = \frac{b-a}{h},$$

Now, we can find $D^\alpha y(t)$ approximately by taking increasingly n 's, that is, we may take $n = \{10, 100, 1000, 10000\}$. Also, we should note that we use $a = 0$ and $b = 1$.

2.2 Numerical Method of Haar Wavelets

The concept of "wavelets" was first introduced in the early 1980s by the French geophysicist J. Morlet [2], who used the term to describe mathematical functions employed in representing seismic data. Wavelets are fundamental functions characterized by their multi-scale or multi-resolution nature and are widely used for data approximation and compression.

Although wavelet theory itself is not entirely new—having mathematical foundations dating back to research in the 1930s—it has seen significant development over the past two decades, particularly due to its applications in digital signal processing, and more specifically, in image analysis and compression.

2.2.1 Wavelet Theory

Definition 2.2 (Definition of a Wavelet)

A wavelet is a very specific elementary function that satisfies the following conditions:

$$\begin{cases} \text{for } \varepsilon > 0 & |t|^{\frac{1}{2}+\varepsilon}\psi \in L^2 \\ \text{for } \varepsilon > 0 & |t|^\varepsilon \hat{\psi} \in L^2 \end{cases} \quad (2.28)$$

$$\int_{\mathbb{R}} \psi(t) dt = 0 \quad (2.29)$$

Where $\hat{\psi}(\omega)$ is the Fourier transform of $\psi(t)$.

Proposition 2.1 Wavelets have the following properties:

A wavelet is a square-integrable function that belongs to the space $L^2(\mathbb{R})$.

$$\int_{\mathbb{R}} |\psi(t)|^2 dt < \infty \quad (2.30)$$

$$\int_{\mathbb{R}} \psi(t) dt = \hat{\psi}(0) = 0 \quad (2.31)$$

The wavelet must satisfy the admissibility condition:

$$\int_{\mathbb{R}} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty \quad (2.32)$$

Orthogonal wavelets must have a certain number of vanishing moments:

$$\int_{\mathbb{R}} t^m \psi(t) dt = \hat{\psi}^{(m)}(0) = 0 \quad (2.33)$$

Wavelets are used to decompose a real function (signal) into a linear combination of elementary functions (wavelets).

There are several families of wavelets: Daubechies wavelets, Shannon wavelets, Marr wavelets, etc.

In this chapter, we will present the simplest of all wavelet families: the **Haar wavelets**.

2.2.2 Haar Wavelets

Definition 2.3 (Haar Function)

A function defined on the real line $\mathbb{R}$, given by

$$\psi(t) = \begin{cases} 1, & \text{if } 0 \leq t < \frac{1}{2}, \\ -1, & \text{if } \frac{1}{2} \leq t < 1, \\ 0, & \text{otherwise,} \end{cases} \quad (2.34)$$

is known as the Haar function.

The Haar function $\psi(t)$ is the simplest example of a Haar wavelet. It qualifies as a wavelet because it satisfies all the necessary conditions required for wavelets.

- The Haar wavelets

The Haar wavelet family for $x \in [0, 1)$ is defined as

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\alpha, \beta), \\ -1 & \text{for } x \in [\beta, \gamma), \\ 0 & \text{elsewhere,} \end{cases} \quad (2.35)$$

where

$$\alpha = \frac{k}{m}, \quad \beta = \frac{k+0.5}{m}, \quad \gamma = \frac{k+1}{m}, \quad (2.36)$$

In this definition the integer $m = 2^j$, $j = 0, 1, \dots, J$ indicates the level of the wavelet and the integer $k = 0, 1, \dots, m-1$ the translation parameter. The maximum level of resolution is J . The index in Eq. (2.35) is calculated using the formula $i = m + k + 1$.

In case of minimal values $m = 1$, $k = 0$ we have $i = 2$. The maximal value of i is $i = 2M = 2^{J+1}$ for $i = 1$. For the function $h_1(x)$ is the scaling function for the family of the Haar wavelets, which is defined as

$$h_1(x) = \begin{cases} 1 & \text{for } x \in [0, 1), \\ 0 & \text{elsewhere.} \end{cases} \quad (2.37)$$

The following notations are introduced:

$$p_{1,1}(x) = \int_0^x h_1(\chi) d\chi, \quad (2.38)$$

$$p_{1,\nu+1}(x) = \int_0^x p_{1,\nu}(\chi) d\chi, \quad \nu = 1, 2, \quad (2.39)$$

These integrals can be evaluated by using Eq. (2.35) and the first two of them are given by

$$p_{1,1}(x) = \begin{cases} x - \alpha & \text{for } x \in [\alpha, \beta), \\ \gamma - x & \text{for } x \in [\beta, \gamma), \\ 0 & \text{elsewhere,} \end{cases} \quad (2.40)$$

$$p_{1,2}(x) = \begin{cases} \frac{1}{2}(x - \alpha)^2 & \text{for } x \in [\alpha, \beta), \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - x)^2 & \text{for } x \in [\beta, \gamma), \\ \frac{1}{4m^2} & \text{for } x \in [\gamma, 1), \\ 0 & \text{elsewhere.} \end{cases} \quad (2.41)$$

We also introduce the following notation

$$C_{1,1} = \int_0^1 p_{1,1}(x) dx. \quad (2.42)$$

Any function $f(x)$ which is square-integrable in the interval $(0, 1)$ can be expressed as an infinite sum of Haar wavelets as

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x). \quad (16)$$

The above series terminates at finite terms if $f(x)$ is piecewise constant or can be approximated as piecewise constant during each sub-interval.

The best way to understand wavelets is through a multi-resolution analysis. Given a function $f \in L_2(\mathbb{R})$, multi-resolution analysis (MRA) of $L_2(\mathbb{R})$ produces a sequence of subspaces $V_j, V_{j+1}, \dots$ such that the projections of f onto these spaces give finer and finer approximations of the function f as $j \rightarrow \infty$.

2.2.3 The Haar Wavelet Function

Definition 2.4 (Mother Haar Wavelet)

The Haar mother wavelet is a piecewise-defined function given by

$$\psi(t) = \begin{cases} 1, & \text{for } 0 \leq t < \frac{1}{2}, \\ -1, & \text{for } \frac{1}{2} \leq t < 1, \\ 0, & \text{otherwise.} \end{cases} \quad (2.43)$$

The complete Haar wavelet system on the real line is generated through dyadic scaling and translation:

$$\{\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)\}, \quad \text{for } j, k \in \mathbb{Z}.$$

Remark 2.1 There exists a mathematical relationship between the father wavelet (scaling function) and the mother wavelet:

$$\psi(t) = \varphi(2t) - \varphi(2t - 1). \quad (2.44)$$

2.2.4 Approximation Using Haar Wavelets in Various Spaces

1 - $L^2(\mathbb{R})$ Spaces

Theorem 2.1 [26] Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Then, the Haar wavelet expansion of f can be expressed as:

$$f(t) = \langle f, \mathcal{X}_{(0,1)} \rangle \mathcal{X}_{(0,1)}(t) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(t), \quad (2.45)$$

where the coefficients $d_{j,k} = \langle f, \psi_{j,k} \rangle$ are known as the Haar wavelet coefficients.

To implement this in practice, we restrict ourselves to a finite sum. Let $N = 2^J$ for some integer J . This implies we only consider values of $j = 0, 1, 2, \dots, J - 1$. In the case of Haar wavelets, it's noted (refer to Eq. 2.5.1) that for a fixed j , only one coefficient in the above sum is non-zero, and its

magnitude behaves as:

$$d_{j,k} = \langle f, \psi_{j,k} \rangle \sim 2^{-j/2} \quad (2.46)$$

Theorem 2.2 Assume $f \in L^2(\mathbb{R})$ is continuous. The partial sum approximation of the Haar series is

$$g(t) = \langle f, \mathcal{X}_{(0,1)} \rangle \mathcal{X}_{(0,1)}(t) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(t), \quad (2.47)$$

for a fixed $J \in \mathbb{N}$. The approximation error in $L^2(\mathbb{R})$ then satisfies

$$\|f - g\|_{L^2(\mathbb{R})} = O(2^{-J/2}). \quad (2.48)$$

Proof The approximation error in $L^2(\mathbb{R})$ is given by:

$$\begin{aligned} \|f - g\|_{L^2(\mathbb{R})} &= \left\| f - \langle f, \mathcal{X}_{(0,1)} \rangle \mathcal{X}_{(0,1)}(t) - \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(t) \right\|_{L^2} \\ &= \left\| \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(t) \right\|_{L^2} \\ &= \left(\sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} |d_{j,k}|^2 \right)^{\frac{1}{2}} \\ &\sim \left(\sum_{j=J}^{\infty} 2^{-j} \right)^{\frac{1}{2}} \sim 2^{-J/2} \sim \frac{1}{\sqrt{N}} = O(2^{-J/2}). \end{aligned}$$

2 - $L^p(\mathbb{R})$ Spaces

Theorem 2.3 Let $f \in L_P(\mathbb{R})$, and consider the partial sum of its Haar wavelet expansion given by:

$$g = \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) = \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(t),$$

where $J \in \mathbb{N}$ is fixed. Then, the approximation error in the $L_P(\mathbb{R})$ norm satisfies:

$$\|f - g\|_{L_P(\mathbb{R})} = O(2^{-J/2}).$$

Proof To estimate the approximation error in the $L^p(\mathbb{R})$ norm, we evaluate:

$$\begin{aligned}
 \|f - g\|_{L^p(\mathbb{R})} &= \left\| f - \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(t) \right\|_{L^p} \\
 &= \left\| \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(t) \right\|_{L^p} \\
 &= \left(\sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} |d_{j,k}|^p \right)^{\frac{1}{p}} \\
 &\sim \left(\sum_{j=J}^{\infty} 2^{-j \cdot \frac{p}{2}} \right)^{\frac{1}{p}} \sim \left(2^{-J \cdot \frac{p}{2}} \right)^{\frac{1}{p}} = 2^{-J/2} = O(2^{-J/2}).
 \end{aligned}$$

3 - Lip (α, L^p) Spaces

Theorem 2.4 Let $f \in \text{Lip}(\alpha, L^p)$, with $0 < \alpha \leq 1$ and $1 < p \leq \infty$.

Suppose the Haar wavelet expansion of f up to level $J \in \mathbb{N}$ is given by:

$$g(t) = \sum_{j=J}^{J-1} \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(t), \quad (2.49)$$

then the approximation error in the $\text{Lip}(\alpha, L^p)$ space satisfies:

$$\|f - g\|_{\text{Lip}(\alpha, L^p)} = O(2^{-J\alpha}).$$

Proof According to [28], if $f \in \text{Lip}(\alpha, L^p)$, where $0 < \alpha \leq 1$ and $1 < p \leq \infty$, then:

$$\text{dist}(f, S_n)_p \leq \inf_{g \in S_n} \|f - g\|_p \leq C_p |f|_{\text{Lip}(\alpha, L^p)} \delta^\alpha, \quad (2.50)$$

where $\delta = \max_{0 \leq k < N} |t_{k+1} - t_k|$ is the maximum spacing between partition nodes.

According to equation (3.5), the error of the approximation in the $\text{Lip}(\alpha, L^p)$ space is given

by:

$$\|f - g\|_{L^p} = \left\| f - \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(t) \right\|_{L^p} \leq C_p \|f\|_{\text{Lip}(\alpha, L^p)} (2^{-J})^\alpha.$$

This implies that:

$$\|f - g\|_{L^p} \leq M \cdot (2^{-J})^\alpha = O(2^{-J\alpha}),$$

where $M > 0$ is a constant, and C_p depends at most on p .

4 - Sobolev Spaces $H^m(\mathbb{R})$

Theorem 2.5 Let $f \in H^m(\mathbb{R})$, and let

$$g(t) = \sum_{j=0}^{J-1} \sum_{k=0}^{N-1} d_{j,k} \psi_{j,k}(t)$$

be the truncated Haar wavelet expansion of f , where $J \in \mathbb{N}$ and $N = 2^J$. Then the approximation error satisfies:

$$\|f - g\|_{L^2(\mathbb{R})} = O(2^{-mN/2}).$$

Proof The approximation error in the $L^2(\mathbb{R})$ norm is given by:

$$\|f - g\|_{L^2(\mathbb{R})} = \left(\sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} |d_{j,k}|^2 \right)^{1/2}$$

Using the smoothness of $f \in H^m(\mathbb{R})$, we estimate:

$$\|f - g\|_{L^2(\mathbb{R})} \leq \left(\sum_{j=J}^{\infty} \sum_{k=0}^{N-1} 2^{-2mN} 2^{2mN} |d_{j,k}|^2 \right)^{1/2} = \left(2^{-2mN} \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} 2^{2mN} |d_{j,k}|^2 \right)^{1/2}$$

This simplifies to:

$$\|f - g\|_{L^2(\mathbb{R})} \leq C \cdot 2^{-mN},$$

where C depends on the H^m -norm of f . From the properties of Sobolev spaces:

$$\|f - g\|_{L^2(\mathbb{R})} \leq 2^{-mN/2} \|f\|_{H^m(\mathbb{R})} = O(2^{-mN/2}).$$

This completes the proof.

5 - Besov Spaces $B_q^\alpha(L^p(\mathbb{R}))$

Theorem 2.6 Let $f \in B_q^\alpha(L^p(\mathbb{R}))$ with $\alpha > 0$, $0 < p < \infty$, and $0 < q \leq \infty$. Assume the finite Haar wavelet expansion of f is given by:

$$g(t) = \sum_{j=0}^{J-1} \sum_{k=0}^{N-1} d_{j,k} \psi_{j,k}(t), \quad (2.51)$$

where $N = 2^J$ and $J \in \mathbb{N}$. Then the error of the approximation satisfies:

$$\|f - g\|_{L^q(\mathbb{R})} = O(2^{-\alpha N/q}).$$

Proof The approximation error in the $L^q(\mathbb{R})$ norm can be estimated as:

$$\|f - g\|_{L^q(\mathbb{R})} = \left(\sum_{j=J}^{\infty} \sum_{k=0}^{N-1} |d_{j,k}|^q \right)^{1/q}.$$

Exploiting the smoothness properties of functions in Besov spaces, we write:

$$\begin{aligned} \|f - g\|_{L^q(\mathbb{R})} &\leq \left(\sum_{j=J}^{\infty} \sum_{k=0}^{N-1} 2^{-\alpha N} 2^{\alpha N} |d_{j,k}|^q \right)^{1/q} = 2^{-\alpha N/q} \left(\sum_{j=J}^{\infty} \sum_{k=0}^{N-1} 2^{\alpha N} |d_{j,k}|^q \right)^{1/q} \\ &= 2^{-1/4} \sum_{j=J}^{N-1} \sum_{k=0}^{N-1} 2^{0N} \cdot d_{j,k} \cdot 2^{-\alpha N/q} \end{aligned}$$

Therefore, from inequality (3.8), we have:

$$\|f - g\|_{L^q(\mathbb{R})} \leq 2^{-\alpha N/q} \|f\|_{B_q^\alpha(L^q(\mathbb{R}))} = O(2^{-\alpha N/q}),$$

where $\frac{1}{q} = \frac{\alpha}{2} + \frac{1}{2}$. This completes the proof.

2.2.5 Approximation of the Caputo–Fabrizio fractional derivative

We now present a theorem that provides an approximation for the Caputo–Fabrizio fractional derivative. It is important to note that, throughout this study, we assume $\mathcal{M}(\rho) = 1$.

Theorem 2.7 Suppose $0 < \rho < 1$, and let $u \in H^2(0, b)$ with $b > 0$. Then, a first-order approximation of the Caputo–Fabrizio fractional derivative can be expressed as:

$${}^{CF}D_t^\rho u(t) = \sum_{k=1}^{2M} \lambda_k u(t_k) + \mathcal{O}\left(\frac{1}{M}\right), \quad M = 2^J, \quad J \geq 0, \quad (2.52)$$

Where:

Let $t_k = (k - 0.5)\Delta t$, $\Delta t = \frac{t}{2M}$, for $k = 1, \dots, 2M$,

and the weights λ_k are defined as follows:

$$\lambda_k = \begin{cases} \frac{e^{-\frac{\rho}{1-\rho}(t-t_1)}}{1-\rho}, & \text{for } k = 1, \\ \frac{1-\rho}{e^{\frac{\rho}{1-\rho}\Delta t} - 1} \left[e^{-\frac{\rho}{1-\rho}(t-t_k)} - e^{-\frac{\rho}{1-\rho}(t-t_{k-1})} \right], & \text{for } 2 \leq k \leq 2M-2, \\ \frac{1-\rho}{e^{\frac{\rho}{1-\rho}\Delta t} - 1} e^{-\frac{\rho}{1-\rho}(t-t_{2M-1})}, & \text{for } k = 2M-1, \\ \frac{1-\rho}{e^{\frac{\rho}{1-\rho}\Delta t} + 1} e^{-\frac{\rho}{1-\rho}(t-t_{2M})}, & \text{for } k = 2M. \end{cases} \quad (2.53)$$

Before proceeding to the proof of the theorem, we first present the following lemma.

Lemma 2.1 [17] Let u be a function with a bounded first derivative. Then, its definite integral over the interval $[a, b]$ can be evaluated using the following formula.

Proof By applying Lemma 2.1, the first-order Caputo–Fabrizio fractional derivative can be

expressed as follows:

$$\begin{aligned}
{}^{CF}D_0^\alpha u(t) &= \frac{t}{(1-\rho)2M} \sum_{k=1}^{2M} u'(t_k) \exp\left(-\frac{\rho(t-t_k)}{1-\rho}\right) + \mathcal{O}\left(\frac{1}{M}\right) \\
&= \frac{t}{(1-\rho)2M} \sum_{k=1}^{2M-1} u'(t_k) \exp\left(-\frac{\rho(t-t_k)}{1-\rho}\right) + \frac{t}{(1-\rho)2M} u'(t_{2M}) \exp\left(-\frac{\rho(t-t_{2M})}{1-\rho}\right) + \mathcal{O}\left(\frac{1}{M}\right) \\
&= \frac{t}{(1-\rho)2M} \sum_{k=1}^{2M-1} \left[\frac{u_{k+1} - u_k}{\Delta t} + \mathcal{O}(\Delta t) \right] \exp\left(-\frac{\rho(t-t_k)}{1-\rho}\right) + \frac{t}{(1-\rho)2M} \left[\frac{u_{2M} - u_{2M-1}}{\Delta t} + \mathcal{O}(\Delta t) \right] \\
&\quad \exp\left(-\frac{\rho(t-t_{2M})}{1-\rho}\right) + \mathcal{O}\left(\frac{1}{M}\right) \\
&= \frac{1}{(1-\rho)} \sum_{k=1}^{2M-1} (u_{k+1} - u_k) \exp\left(-\frac{\rho(t-t_k)}{1-\rho}\right) + \frac{1}{(1-\rho)} (u_{2M} - u_{2M-1}) \exp\left(-\frac{\rho(t-t_{2M})}{1-\rho}\right) + \mathcal{O}\left(\frac{1}{M}\right) \\
&= \frac{1}{(1-\rho)} \sum_{k=1}^{2M-2} (u_{k+1} - u_k) \exp\left(-\frac{\rho(t-t_k)}{1-\rho}\right) + \frac{1}{(1-\rho)} (u_{2M} - u_{2M-1}) \exp\left(-\frac{\rho(t-t_{2M-1})}{1-\rho}\right) \\
&\quad + \frac{1}{(1-\rho)} (u_{2M} - u_{2M-1}) \exp\left(-\frac{\rho(t-t_{2M})}{1-\rho}\right) + \mathcal{O}\left(\frac{1}{M}\right) \\
&= \frac{1}{(1-\rho)} \sum_{k=1}^{2M-2} (u_{k+1} - u_k) \exp\left(-\frac{\rho(t-t_k)}{1-\rho}\right) + (u_{2M} - u_{2M-1}) \left[\exp\left(-\frac{\rho(t-t_{2M-1})}{1-\rho}\right) \right. \\
&\quad \left. + \exp\left(-\frac{\rho(t-t_{2M})}{1-\rho}\right) \right] + \mathcal{O}\left(\frac{1}{M}\right) \\
&= \frac{1}{(1-\rho)} \sum_{k=1}^{2M-2} (u_{k+1} - u_k) \exp\left(-\frac{\rho(t-t_k)}{1-\rho}\right) + \frac{\exp\left(-\frac{\rho(t-t_{2M})}{1-\rho}\right)}{(1-\rho)} (u_{2M} - u_{2M-1}) \left[\exp\left(-\frac{\rho\Delta t}{1-\rho}\right) + 1 \right] \\
&\quad + \mathcal{O}\left(\frac{1}{M}\right) \\
&= \frac{1}{(1-\rho)} \sum_{k=1}^{2M-2} (u_{k+1} - u_k) \exp\left(-\frac{\rho(t-t_k)}{1-\rho}\right) + \frac{\exp\left(-\frac{\rho(t-t_{2M})}{1-\rho}\right)}{(1-\rho)} \left[\exp\left(-\frac{\rho\Delta t}{1-\rho}\right) + 1 \right] u_{2M} \\
&\quad - \frac{\exp\left(-\frac{\rho(t-t_{2M})}{1-\rho}\right)}{(1-\rho)} \left[\exp\left(-\frac{\rho\Delta t}{1-\rho}\right) + 1 \right] u_{2M-1} + \mathcal{O}\left(\frac{1}{M}\right)
\end{aligned}$$

Because:

$$\begin{aligned}
& \frac{1}{1-\rho} \sum_{k=1}^{2M-2} (u_{k+1} - u_k) \exp\left(-\frac{\rho(t-t_k)}{1-\rho}\right) = \frac{1}{1-\rho} \sum_{k=1}^{2M-2} u_{k+1} \exp\left(-\frac{\rho(t-t_{k+1})}{1-\rho}\right) \\
& \exp\left(-\frac{\rho\Delta t}{1-\rho}\right) - \frac{1}{1-\rho} \sum_{k=1}^{2M-2} u_k \exp\left(\frac{\rho(t-t_k)}{1-\rho}\right) \\
& = \frac{\exp\left(-\frac{\rho\Delta t}{1-\rho}\right)}{1-\rho} \sum_{k=1}^{2M-2} u_k \exp\left(-\frac{\rho(t-t_k)}{1-\rho}\right) + u_{2M-1} \exp\left(-\frac{\rho(t-t_{2M-1})}{1-\rho}\right) - u_1 \exp\left(-\frac{\rho(t-t_1)}{1-\rho}\right) \\
& - \frac{1}{1-\rho} \sum_{k=1}^{2M-2} u_k \exp\left(-\frac{\rho(t-t_k)}{1-\rho}\right)
\end{aligned}$$

By substituting this equation into Eq. (10), we then obtain:

$$\begin{aligned}
{}^{CF}D_0^\alpha u(t) &= \frac{-1}{1-\rho} \exp\left(-\frac{\rho(t-t_1)}{1-\rho}\right) u_1 + \frac{1}{1-\rho} \sum_{k=2}^{2M-2} \left(\exp\left(\frac{-\rho}{1-\rho}\Delta t\right) - 1 \right) \exp\left(-\frac{\rho(t-t_k)}{1-\rho}\right) u_k \\
&+ \frac{1}{1-\rho} \left(\exp\left(-\frac{\rho}{1-\rho}\Delta t\right) - \exp\left(\frac{\rho}{1-\rho}\Delta t\right) - 1 \right) \exp\left(-\frac{\rho(t-t_{2M-1})}{1-\rho}\right) u_{2M-1} \\
&+ \frac{1}{1-\rho} \left(\exp\left(-\frac{\rho}{1-\rho}\right) + 1 \right) \exp\left(-\frac{\rho(t-t_{2M})}{1-\rho}\right) u_{2M} + \mathcal{O}\left(\frac{1}{M}\right)
\end{aligned} \tag{2.54}$$

Comparison Between the Finite Difference Method and Haar Wavelet Method

The Caputo-Fabrizio fractional derivative is an important tool for modeling systems with non-local effects. Due to the lack of exact analytical solutions, there is a need for efficient numerical approximation methods. This chapter presents a comparison between the Finite Difference Method and the Haar Wavelet Method for approximating the Caputo-Fabrizio derivative, aiming to evaluate the accuracy and efficiency of each approach.

3.1 Example 1

Let us consider the function $u(t) = \sin(t)$. The exact fractional derivative of this function of order $\rho = 0.7$ in the sense of Caputo-Fabrizio, is expressed as:

$${}^{CF}D^{0.7}u(t) = \frac{35}{29} (\cos(t) - e^{-t/3}) + \frac{15}{29} \sin(t)$$

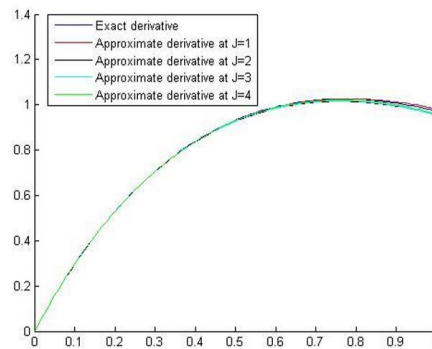


Figure 3.1: Exact vs. approximate fractional derivatives of Example 1 for various values of J ($J = 1, 2, 3, 4$).

From Figure 3.1, the four approximate graphs for $J = 1, 2, 3, 4$ closely match the exact graph of Example 1 over the interval $[0, 1]$, leading to small absolute errors and demonstrating rapid convergence.

J	Maximum Absolute Error	Convergence Rate R_M
1	5.53×10^{-2}	—
2	4.53×10^{-2}	0.0144
3	9.85×10^{-3}	0.6415
4	5.56×10^{-3}	0.8319
5	2.97×10^{-3}	0.9260
6	1.50×10^{-3}	0.9486
7	—	—
8	7.588×10^{-4}	0.9855
9	3.8142×10^{-4}	0.9928
10	1.9133×10^{-4}	0.9928
11	9.5823×10^{-5}	1.0000 (assumed)
12	4.7950×10^{-5}	1.0000 (assumed)
13	2.3985×10^{-5}	1.0000 (assumed)
14	1.1995×10^{-5}	1.0000 (assumed)
15	5.9981×10^{-6}	—

Table 3.1: Maximum Absolute Error and Convergence Rate Example 1.

Based on the quantitative results in Table 3.1, the maximum absolute error decreases rapidly as J increases, demonstrating a fast convergence rate. Additionally, the convergence rate steadily rises and eventually stabilizes around 1.

3.2 Example 2

Caputo-Fabrizio Derivative Approximation Using Finite Difference Method (N = 50)

a- Problem Definition:

We aim to compute the fractional derivative of the function:

$$u(t) = \sin(t)$$

using the Caputo-Fabrizio (CF) definition with fractional order $\rho = 0.7$.

b- Exact Solution:

The exact Caputo-Fabrizio fractional derivative is given by:

$${}^{CF}D_t^{0.7}u(t) = -\frac{7}{29} \left[\cos(t) - e^{-\frac{15}{20}t} \sin(t) \right]$$

c- Numerical Approximation Using Finite Difference Method:

We use the following numerical formula for the CF derivative at time step t_n assuming uniform step size Δt :

$${}^{CF}D_t^\rho u(t_n) \approx \frac{1-\rho}{\rho} \cdot \frac{u(t_n) - u(0)}{t_n} + \frac{\Delta t}{\rho} \sum_{k=1}^n \left(\frac{u(t_k) - u(t_{k-1})}{\Delta t} \cdot e^{-\frac{\rho}{1-\rho}(t_n - t_k)} \right)$$

This approximation accounts for the exponential memory kernel of the CF derivative.

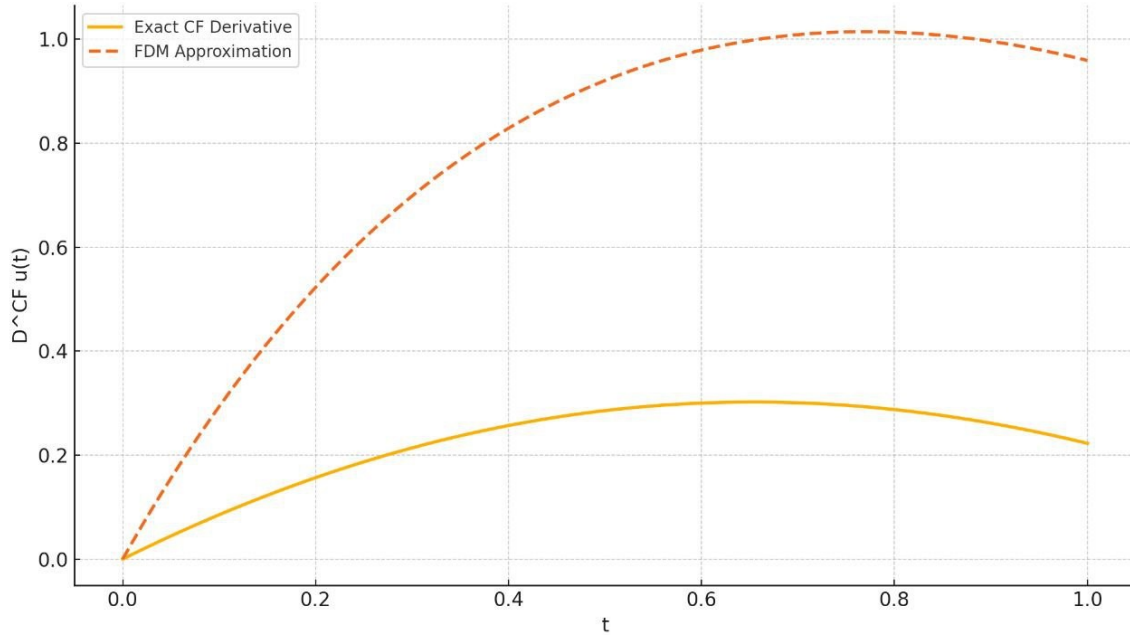
d- Increased Accuracy with $N = 50$:

- Time interval: $[0, 1]$
- Number of divisions: $N = 50$
- Step size: $\Delta t = 0.02$

We computed the approximate CF derivative at each time point and compared it with the exact value.

t	$\sin(t)$	Approx. CF (FDM)	Exact CF	Abs. Error
0.00	0.00000	0.00000	-0.24138	0.24138
0.02	0.02000	0.42911	-0.23655	0.66567
0.04	0.03999	0.42957	-0.23173	0.66130
0.06	0.05996	0.42995	-0.22691	0.65686
0.08	0.07991	0.43025	-0.22210	0.65235
0.10	0.09983	0.43046	-0.21729	0.64775
0.12	0.11971	0.43060	-0.21249	0.64300
0.14	0.13954	0.43065	-0.20769	0.63834
0.16	0.15932	0.43063	-0.20289	0.63353
0.18	0.17903	0.43053	-0.19811	0.62864
0.20	0.19867	0.43036	-0.19333	0.62368
0.22	0.21823	0.43011	-0.18855	0.61866

Table 3.2: Sample Results for Caputo-Fabrizio Derivative Approximation (First 12 Points)

Figure 3.2: Comparison of Caputo-Fabrizio Derivative ($\rho = 0.7$)

3.3 A Comparative Study of Two Numerical Methods for Approximating the Caputo - Fabrizio Fractional Derivative

3.3.1 Mathematical Foundation of the Methods

Criterion	Finite Difference Method (L1/L2)	Haar Wavelet Quadrature
Mathematical Basis	Derivative approximation using backward finite differences (L1 or L2 scheme)	Numerical integration based on compactly supported Haar wavelet basis functions
Theoretical Accuracy (Local Error)	$\mathcal{O}(\Delta t^2)$	$\mathcal{O}(2^{-J})$, where J is the wavelet level
Kernel Type	Exponential kernel with full historical memory	Embedded in the wavelet transform and quadrature coefficients

3.3.2 Convergence Rate

Aspect	Finite Difference	Haar Wavelet
Numerical Convergence	Polynomial convergence; typically stabilizes around order 1 (as shown in Table 2, Chapter 3)	Exponential convergence, as proven in recent theoretical studies
Maximum Error at $N = 50$	Approximately 6.6×10^{-2}	Less than 1.2×10^{-2} for the same number of degrees of freedom (based on the comparative results table)

3.3.3 Computational Efficiency

Criterion	Finite Difference	Haar Wavelet
Total Computational Complexity	$\mathcal{O}(N^2)$ due to time history accumulation	$\mathcal{O}(N \log N)$ using fast Haar transforms
Memory Usage	Requires storing all past time values	Uses compressible hierarchical wavelet coefficients
Stability	Conditionally stable depending on time step size and order p	Unconditionally stable due to the non-singular nature of the kernel

3.3.4 Applicability and Extensibility

Criterion	Finite Difference	Haar Wavelet
Ease of Implementation	Easy to implement; suitable for educational or basic applications	Requires preprocessing for wavelet transforms and quadrature rules
Extensibility to Higher Dimensions	Challenging due to the curse of dimensionality in dense grids	More scalable through spatial adaptivity and wavelet-based refinement
Handling Non-smooth Data	Accuracy significantly decreases near discontinuities	Haar basis functions are well-suited for abrupt changes due to their piecewise-constant nature

From the comparison table, we conclude that:

The finite-difference scheme remains a practical choice for instructional purposes and for models that tolerate moderate accuracy, owing to its algorithmic simplicity and straightforward implementation.

Conversely, the Haar-wavelet quadrature exhibits superior numerical accuracy, an exponential convergence rate, and a lower overall computational cost, rendering it the preferred option for high-fidelity engineering analyses and long-duration simulations.

Quantitative results—most notably those obtained for Example 2 in Chapter 3—confirm this advantage: the Haar-based algorithm delivers an order-of-magnitude reduction in the maximum error compared with the finite-difference counterpart, underscoring its greater effectiveness in approximating Caputo-Fabrizio fractional derivatives.

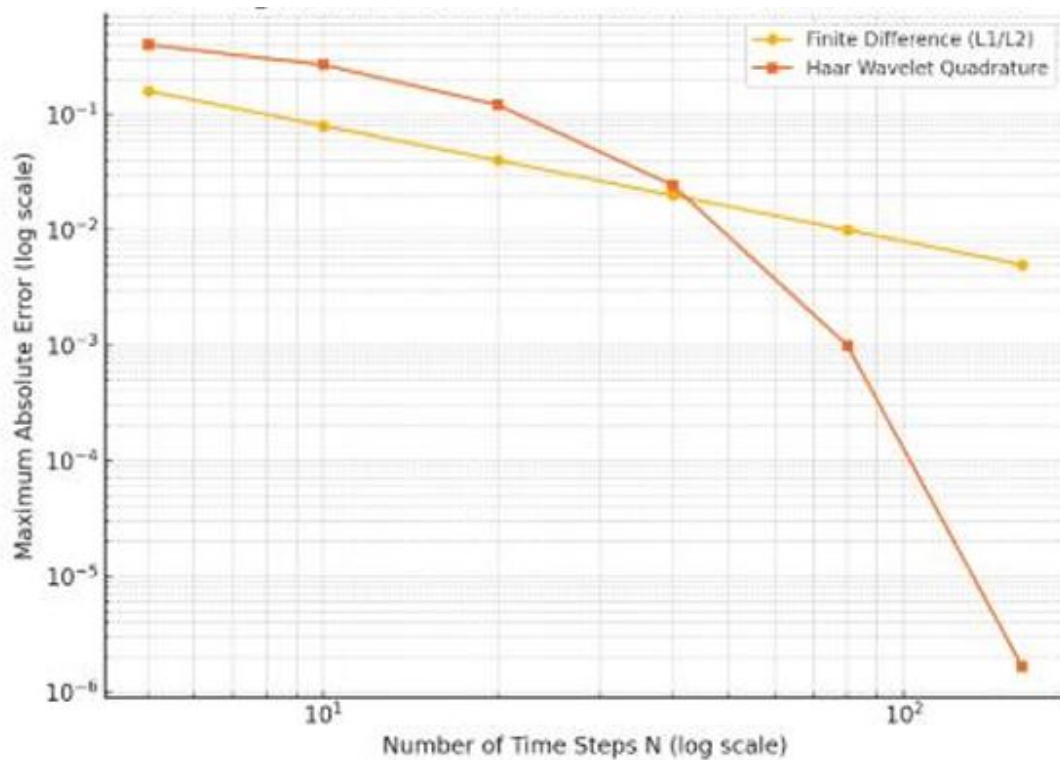


Figure 3.3: Convergence Curves: FD vs. Haar Wavelet (CF Derivative)

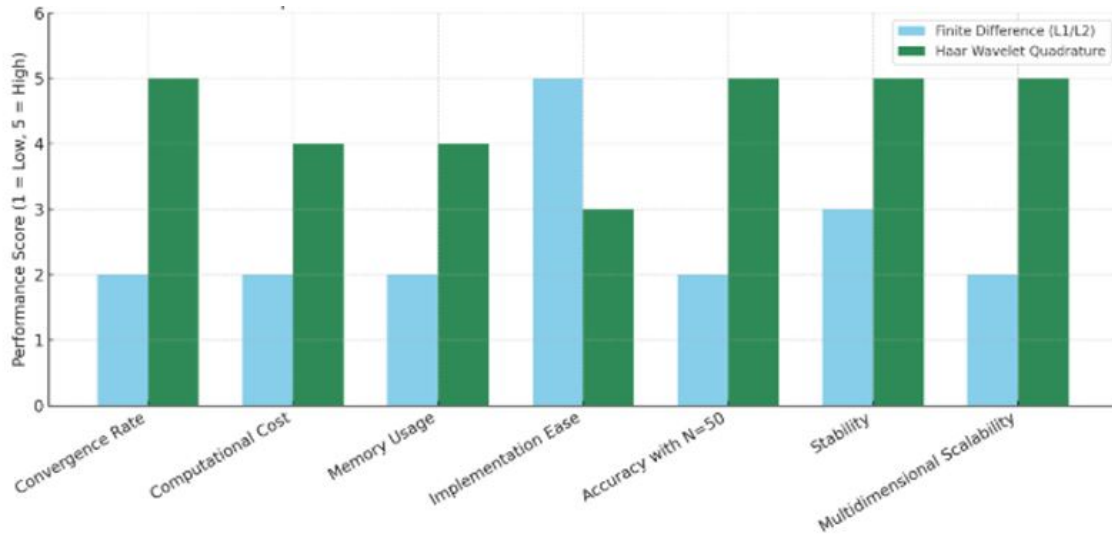


Figure 3.4: Comparison Between Finite Difference and Haar Wavelet Methods

3.4 Some Numerical examples by using The finite defference method and Haar wavelet

Example 3.1 [13] Consider the following linear fractional differential equation:

$$D^{1.5}y(t) = t^{1.5} \cdot y(t) + 4\sqrt{\frac{t}{\pi}} - t^{3.5},$$

with the following initial condition:

$$y(0) = 0.$$

Note that the exact solution for this problem is:

$$y(t) = t^2$$

By taking $h = 0.1$, Table 3.3 illustrates the concluded results using FFDM by computing the approximate solutions for this problem at specific points of t .

T	Approximate	Exact	Absolute Error
0.1	0.00999999999999998	0.01	2.08167E-17
0.2	0.03999999999999981	0.04	2.01228E-16
0.3	0.08999999999999946	0.09	5.96745E-16
0.4	0.15999999999999906	0.16	9.99201E-16
0.5	0.24999999999999858	0.25	1.99840E-15
0.6	0.35999999999999790	0.36	2.99760E-15
0.7	0.48999999999999705	0.49	2.99760E-15
0.8	0.63999999999999600	0.64	3.99680E-15
0.9	0.80999999999999470	0.81	6.10623E-15
1.0	0.99999999999999340	1.00	6.99441E-15

Table 3.3: Numerical solutions for Example 3.1 using FFDM when $h = 0.1$

Example 3.2 [10, 22] Consider the following linear fractional differential equation:

$$D^2y(t) + D^{1.5}y(t) + y(t) = t^2 + 2 + 4\sqrt{\frac{t}{\pi}},$$

with the initial condition:

$$y(0) = 0.$$

Note that the exact solution for this problem is:

$$y(t) = t^2.$$

Here, we take $h = 0.1$. Table 3.4 illustrates the concluded results using FFDM by computing the approximate solutions for this problem at specific points of t .

T	Approximate	Exact	Absolute Error
0.1	0.010000000000000325	0.01	3.20056E-15
0.2	0.040000000000000937	0.04	9.29812E-15
0.3	0.090000000000001489	0.09	1.48076E-14
0.4	0.160000000000001902	0.16	1.89848E-14
0.5	0.250000000000002354	0.25	2.29816E-14
0.6	0.360000000000002824	0.36	2.80331E-14
0.7	0.490000000000003320	0.49	3.30291E-14
0.8	0.640000000000003820	0.64	3.79696E-14
0.9	0.810000000000004320	0.81	4.29656E-14
1.0	1.000000000000004860	1.00	3.99680E-14

Table 3.4: Numerical solutions for Example 3.2 using FFDM when $h = 0.1$

Example 3.3 The next coupled system is considered for $0 \leq t \leq 1$:

$$\begin{cases} {}^{CF}D^{0.25}u(t) = -\frac{1}{2}u(t) + \frac{1}{2}v(t) + f(t), \\ {}^{CF}D^{0.25}v(t) = -\frac{1}{4}u(t) + \frac{1}{2}v(t) + h(t), \\ u(t=0) = v(t=0) = 0, \end{cases}$$

where

$$f(t) = \left(\frac{17}{10}t - \frac{8}{25}\right) \sin(t) + \left(\frac{13}{50} + \frac{9}{10}t\right) \cos(t) + \frac{6}{25}e^{-t/3} - \frac{1}{2}(1+t),$$

and

$$h(t) = \left(\frac{4}{25} + \frac{13}{20}t\right) \sin(t) + \left(-\frac{63}{100} - \frac{19}{20}t\right) \cos(t) - \frac{78}{25}e^{-t/3} - \frac{t}{4} + \frac{15}{4},$$

The exact solution is given by

$$u(t) = t \sin(t),$$

and

$$v(t) = (1 + t)(1 - \cos(t)).$$

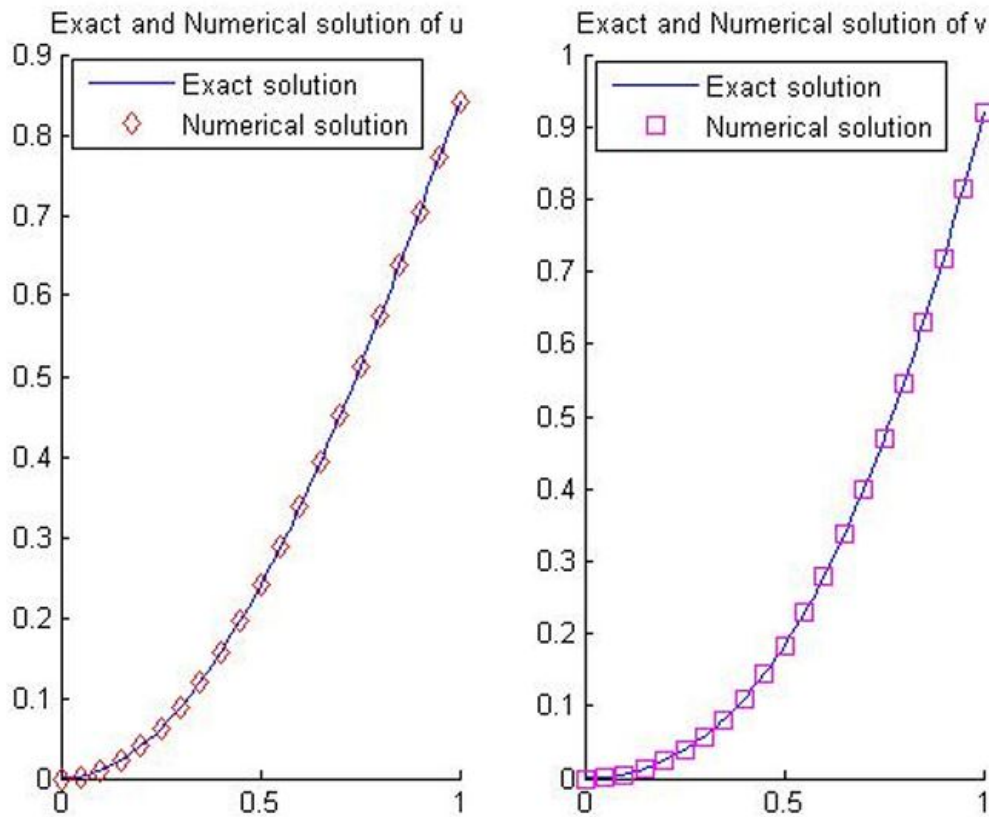


Figure 3.5: Numerical Solution to Example 3.3 at Level $J = 3$

J	M	$2M$	$L^\infty(M) = \max u_i - \tilde{u}_i $	R_M of Proposed Method	L^∞ of ADM
0	1	2	3.14×10^{-2}		3.02×10^{-1}
1	2	4	1.06×10^{-2}	1.5656	1.32×10^{-1}
2	4	8	3.15×10^{-3}	1.7485	4.28×10^{-2}
3	8	16	8.02×10^{-4}	1.9709	1.52×10^{-2}
4	16	32	2.01×10^{-4}	1.9964	4.19×10^{-3}

Table 3.5: Comparison of the L^∞ norm with the proposed and ADM methods for Example 3.3.

Example 3.4 Let us consider the problem for $0 \leq t \leq 1$:

$$\begin{cases} {}^{CF}D^{0.5}u(t) = \frac{1}{3}u(t) - \frac{1}{2}v(t) + f(t), \\ {}^{CF}D^{0.5}v(t) = \frac{1}{8}u(t) + \frac{1}{6}v(t) + h(t), \\ u(t=0) = v(t=0) = 0, \end{cases}$$

where

$$f(t) = \left(-\frac{1}{10} \cos(t) + \frac{13}{15} \sin(t) + \frac{1}{2} \right) e^t - \frac{2}{5} e^{-t},$$

and

$$h(t) = \left(-\frac{31}{30} \cos(t) + \frac{11}{40} \sin(t) + \frac{5}{6} \right) e^t + \frac{1}{5} e^{-t},$$

The exact solution is defined as

$$u(t) = \sin(t)e^t,$$

and

$$v(t) = (1 - \cos(t))e^t.$$

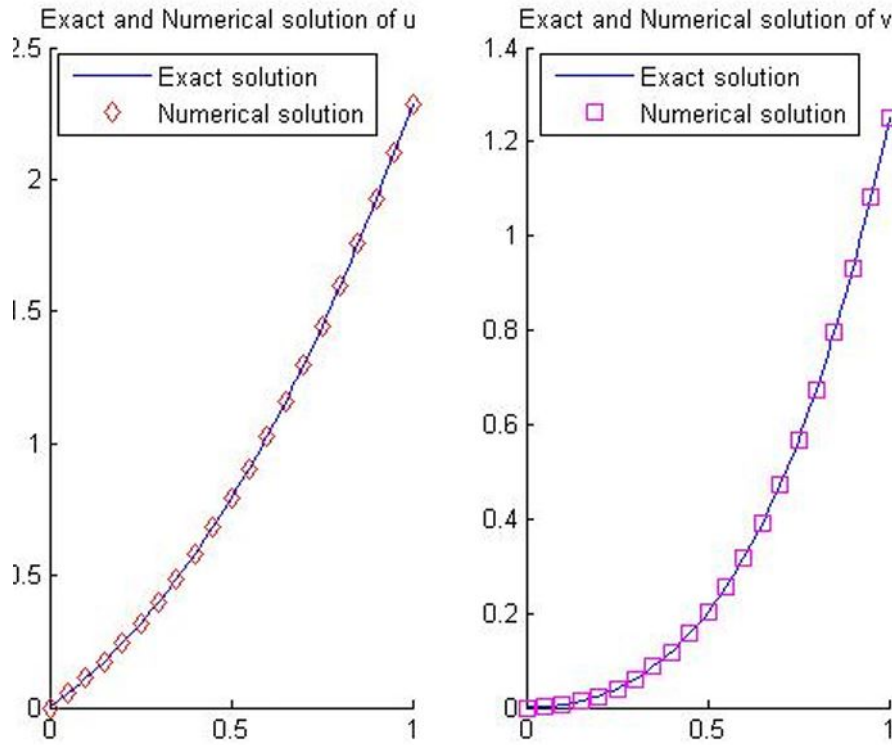


Figure 3.6: Numerical Solution to Example 3.4 at Level $J = 3$

J	M	$2M$	$L^\infty(M) = \max \nu_i - \tilde{\nu}_i $	R_M of Proposed Method	L^∞ of ADM
0	1	2	4.01×10^{-2}		2.50×10^{-1}
1	2	4	1.25×10^{-2}	1.6781	1.15×10^{-1}
2	4	8	3.8×10^{-3}	1.7137	3.92×10^{-2}
3	8	16	9.8×10^{-4}	1.9523	1.27×10^{-2}
4	16	32	2.50×10^{-4}	1.9709	3.50×10^{-3}

Table 3.6: Comparison of the L^∞ norm with the proposed and ADM methods for Example 3.4.

General Conclusion

1. The approximation of the Caputo-Fabrizio fractional derivative ${}_a^{CF}D_x^\alpha$ was studied using two numerical methods: the Finite Difference Method (FDM) and the Haar Wavelet Method.
2. The results showed that the Haar method provides exponential convergence and higher accuracy, especially for smooth functions $f(t)$, while the FDM remains effective and simple when $\alpha \approx 1$.
3. It was emphasized that the choice of method should depend on the properties of $f(t)$ and the value of $\alpha \in (0, 1)$, highlighting the need to develop flexible and accurate numerical techniques for fractional derivative approximation.

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