DEMOCRATIC REPUBLIC ALGERIA

Ministry of Higher Education and Scientific Research

HAMMA LAKHDER UNIVERSITY OF EL OUD

FACULTY OF EXACT SCIENCES

End of study dissertation

ACADEMIC MASTER

Domain: Mathematics and Computer Science
Sector: Mathematics
Specialty: Fundamental and applied mathematics

Theme

Analytical and Numerical study of a Caputo-Fabrizio Fractional differentil System

Presented by: Cheriet.I and Beggar. **A**
Under the supervision of : Beloul Said

Supported before the jury:

M: Abdelaziz Azeb Ahmed.	MAA.	University of El-Oued	President
M: Bachir Dehda.	MC(B).	University of El-Oued	Rapporteur
M Mesai Aoun M.Salah.	MAA.	University of El-Oued	Examiner

Academic year 2023 – 2024

ABSTRACT:

Caputo-Fabrizio's definition of fractional derivation is one of the latest advancements in fractional calculus. This study investigates solutions for a coupled system of linear fractional differential equations with fractional orders using the Picard-Lindelöf approach and fixed-point theory, demonstrating their existence and uniqueness. An efficient computational method for solving the coupled system is presented, utilizing the Caputo-Fabrizio fractional derivative. The approach expands the solution into the Haar wavelet basis, allowing for the determination of Haar wavelet coefficients. Error analysis of the method shows a strong convergence rate. Finally, several numerical examples are provided to demonstrate the precision and efficacy of this approach.

Keywords:

Fractional calculus, Haar wavelet, Caputo- Fabrizio fractional derivative, coupled system with fractional derivative. fractional integral, fractional differential equation,numerical approximation.

RÉSUME:

La définition de la dérivation fractionnelle de Caputo-Fabrizio est l'une des plus récentes pour améliorer les dérivations fractionnelles. C'est mémoire de master examine les solutions d'un système lié d'équations différentielles fractionnelles linéaires avec des ordres fractionnels en utilisant l'approche de Picard-Lindelöf et la théorie du point fixe, en démontrant ainsi leur existence et leur unicité.C'est mémoire master présente une méthode computationnelle efficace pour résoudre un système couplé en utilisant la dérivée fractionnelle de Caputo-Fabrizio. L'approche décompose la solution dans la base des ondelettes de Haar, permettant la découverte des coefficients des ondelettes de Haar. L'analyse d'erreur de la méthode montre un fort taux de convergence. Enfin, quelques exemples numériques sont fournis pour démontrer la précision et l'efficacité de cette approche.

Mots clés et expressions:

dérivé fractionnaire de Caputo-Fabrizio,approximation numérique,Calcul fractionnaire,ondelette de Haar, systéme couple à dérivé fractionnaire. équation différentielle différentielle fractionnaire, approximation numérique.

الملخص:

تعر يف كابوتو - فبريزيو للاشتقاق الكسوري هو واحد من أحدث التطورات في التفاضل والتكامل الكسري .وتبحث هذه الدراسة الحلول لنظام مركب من المعادلات التفاضلية الكسور ية الخطية مع أوامر كسرية باستخدام نهج بيكار – ليندیلوف ونظر ية النقطة الثابتة، مما يدل على وجودها و وحدانيتها .طر يقة حسابية فعالة لحل و يُعرضالنظام المدمج باستخدام مشتقة كابوتو -فبريز يو الكسور ية .وننشر الحل وفق أساس هار مما يسمح بتحديد معاملات حاور الموجي .ويبين تحليل الخطأ في الطر يقة وجود معدل تقارب قوي .وأخيراً تقُدَّم عدة أمثلة رقمية لإثبات دقة هذا النهج وفعاليته.

الكلمات و العبارات الدالة:

المشتق الكسري لكابتو فابريز يو, موجات هار,الحسابات الكسرية,النظام المركب للمشتق الكسري,المعادلات التفاضلية الكسرية,التقريب العددي.

الاهداء

بسم خالقي وميسر اموري وعصمة امري لك كل الحمد والامتنان
يقول الله عز وجل
〈 **واخر دعواهم ان الحمد لله رب العالمين** 〉
من قال انا لها **نالها** ,و انا لها و ان ابت رغما عنها , اتيت بها
نلتها وعانقت اليوم مجدا عظيما رغم الصعوبات وصلت و الحمد لله
ولهذا اهدي هذا النجاح والذي اعتبره انجازا
الى من احمل اسمه بكل فخر , الى من حصد الاشواك من دربي ليمهد لي طريق العلم , الى من غرس في روحي مكارم الاخلاق ,
اهدي نجاحي هذا الى
. . . والدي الحبيب . . .
الى من تحت قدميها الجنة , الى بسمة الحياة وسر الوجود الى من كان دعاؤها سر نجاحي اليك ياسيدتي اهدي تخرجي وكلماتي تنحني اجلالا فشكرا لك
. . . والدتي الطيبة . . .
الى ملاكي في الحياة وملهمي نجاحي من ساندني بكل حب الى تلك الخفية التي زرعت فيا الثقة والاصرار لاكمال هذا البحث
. . . اختاي مريم و ريان . . .
ليس كل ام تلد هناك من تربي إلى من ساندتني وكان لهن الأثر الطيب حتى كبرت
. . . خالاتي الغاليات . . .
الي ضلعي الثابت وامان ايامي الى من شددت عضدي بهم فكانوا لي ينابيع ارتوي بها
. . . اخوالي عبد الرحمن, عمارة, عبد الكريم و توفيق . . .
الى عزي في الحياة واماني
. . . جدي وجدتي . . .
الى من تعلما منه الانضباط والاخلاص في العمل استاذي ومشرف هذا البحث
. . . الدكتور دحده بشير . . .
الى زميلتي في هذا العمل
والى كل من كان عونا وسندا في هذا الطريق والاصدقاء واصحاب الشدائد

الاهداء

بسم خالقي وميسر اموري وعصمة امري لك كل الحمد والامتنان
يقول الله عز وجل
〈 **واخر دعواهم ان الحمد لله رب العالمين** 〉
من قال انا لها نالها ,وانا لهاوان ابت رغما عنها , اتيت بها
نلتها وعانقت اليوم مجدا عظيما رغم الصعوبات وصلت والحمد لله
ولهذا اهدي هذا النجاح والذي اعتبره انجازا
الى من احمل اسمه بكل فخر , الى من حصد الاشواك من دربي ليمهد لي طريق العلم , الى من غرس في روحي مكارم الاخلاق ,
اهدي نجاحي هذا الى
. . . والدي الحبيب. . .
الى من تحت قدميها الجنة , الى بسمة الحياة وسر الوجود الى من كان دعاؤها سر نجاحي اليك ياسيدتي اهدي تخرجي وكلماتي تخني اجلالا فشكرا لك
. . . والدتي الطيبة . . .
ليس كل ام تلد هناك من تربي إلى من ساندتني وكان لها الأثر حتى كبرت
. . . خالتي الغالية . . .
الى ملاكي في الحياة وملهمة نجاحي من ساندتني بكل حب الى تلك الخفية التي زرعت فيا الثقة والاصرار لاكمال هذا البحث
. . ..اختي الغالية . .
الي ضلعي الثابت وامان ايامي الى من شددت عضدي بهم فكانوا لي ينابيع ارتوي بها
. . . اخوتي . . .
الى نبض الفؤاد كما اسميه وجندي المجهول
. . .شاهين . . .
الى من تعلما منه الانضباط والاخلاص في العمل استاذي ومشرف هذا البحث
. . . الدكتور دحده بشير . . .
الى زميلتي في هذا العمل
والى كل من كان عونا وسندا في هذا الطريق والاصدقاء واصحاب الشدائد

CONTENTS

LIST OF FIGURES

Notations :

$\mathbb{N}$	Set of natural numbers.
$\mathbb{N}^*$	Set of natural numbers with nonzero.
$\mathbb{R}$	Set of real numbers.
$\mathbb{R}^+$	Set of positive real numbers.
$\mathbb{C}$	Set of complex numbers.
C	Space of all continuous functions.
C^n	differentiable functions.
$\mathfrak{Re}$	Real part.
Γ	Gamma Function.
β	Beta Function.
E_β	Mittag-Leffler Function.
AC	Space of absolutely continuous functions.
L^1	the space of integrable functions.
L^∞	Space of functions that are essentially bounded.
max	Maximum.
$\parallel . \parallel$	Norm in the spaces.
$W^{m,p}$	Banach space.
I_a^β	Fractional integral of Riemann-Liouville.
D_a^β	The Riemann-Liouville fractional derivative.
${}^C D^{(\alpha)}$	The fractional derivative of Caputo.
${}^{CF} D_x^{(\alpha)}$	The fractional derivative of Caputo-Fabrizio.
(CFFD)	Caputo-Fabrizio fractional derivative.
$erf(x)$	is the error function and is defined as follows:

$$erf(x) = \frac{2}{\sqrt{\Pi}} \int_0^t exp(-x^2)dx.$$

GENERAL INTRODUCTION

Recently, fractional calculus has been used to represent a variety of issues in the physical, biological, engineering sciences,and more (see [1]). The memory effect attribute of this notion has shown tremendous interest in the applied sciences. Numerous definitions, including Riemann-Liouville [2], Caputo[3], and Caputo-Fabrizio fractional integral and derivative ([2],[3]), have been provided in this context for both the integral and the fractional derivative. Despite being employed to model the occurrences initially, the concepts of Riemann-Liouville and Caputo have singularities in their kernels. This has led to the introduction of numerous new definitions of integral and fractional derivatives in the literature. The Caputo-Fabrizio fractional integral and derivative, for example, are more well-liked in the scientific community since they do not cause the singularity problem.

This has led to the introduction of numerous new definitions of integral and fractional derivatives in the literature. In the scientific community,for example, the Caputo-Fabrizio fractional integral and derivative [2] are more well-liked because they do not encounter the singularity problem. The inability to find an analytical solution is the primary obstacle that researchers must overcome in order to solve the Caputo-Fabrizio fractional differential equations and systems [4]. This forces them to resort to numerical methods. As is well known, discontinuities can be seen in the solutions of various numerical techniques. Currently, a popular technique that seeks to increase the convergence rate in accordance with exponential decay is the Haar wavelet approach[5]. As evidence of this, we discover that it has been used to address a wide range of issues, including fractional differential equations (FDEs)[6], ordinary differential equations (ODEs)[7], partial differential equations (PDEs)[8], integral differential equations[9], and so on.

In fact, the compact support, orthogonality, and simplicity of the Haar wavelet are its benefits. In favor of the Haar wavelet technique. In this memory, we apply and investigate it to solve the following system :

$$\begin{cases} {}^{CF}D^{(\alpha)}f(x) = c_1 f(x) + c_2 g(x) + u(x), 0 \leq x \leq 1, \\ {}^{CF}D^{(\alpha)}g(x) = c_3 f(x) + c_4 g(x) + v(x), 0 \leq x \leq 1, \\ f(0) = g(0) = 0. \end{cases}$$

Actually, the existence and uniqueness of the solution to this system have been established by Ikram Mansouri et al.[4]; they also used the Adomian Decomposition Method (ADM) to produce an approximative solution. Nevertheless, the Adomian Decomposition Method's convergence rate decays polynomially, its terms are costly to compute, and a significant number of terms are needed to close the exact answer. In terms of computation costs and convergence rate, the Haar wavelet collocation approach is appropriate to address these flaws.

This work is divided into three chapter:

1. In the first chapter, we will provide some definitions and theorems that we will use in this note.

2. In the second chapter, we will mainly introduce definitions and basic properties of fractional derivatives, Riemann-Liouville fractional derivative, Caputo fractional derivative and Caputo-Fabrizio fractional derivative and some of its properties, etc..

3. In the third chapter, the Picard-Lindelof technique and the Banach fixed point theorem are applied to obtain uniqueness of solutions for system:

$$\begin{cases} {}^{CF}D^{(\alpha)}f(x) = c_1 f(x) + c_2 g(x) + u(x), 0 \leq x \leq 1, \\ {}^{CF}D^{(\alpha)}g(x) = c_3 f(x) + c_4 g(x) + v(x), 0 \leq x \leq 1, \\ f(0) = g(0) = 0. \end{cases}$$

   And we obtain a numerical approximation by using the recently presented derivative of Caputo-Fabrizio fractional order as a basis.

4. In the fourth chapter, We provide the Haar wavelet family, with which our suggested numerical approach to the solution is connected. and we provide examples to demonstrate the precision and potency of our suggested approach.

CHAPTER 1

GENERALITIES IN FUNCTION ANALYSIS

1.1 Some Results from functional analysis

1.1.1 Spaces of Absolutely Continuous and Continuous Functions

Definition 1.1.1.1

Let $\Omega = [a, b](-\infty \leq a < b \leq +\infty)$ be a finite or infinite interval of $\mathbb{R}$.We denote by $L^1(\Omega)$ the space of integrable functions from Ω into $\mathbb{R}$ [10] .

$$L^1(\Omega) = \{f : \Omega \to \mathbb{R}, f \, is \; measurable \; functions\}. \tag{1.1}$$

$$\|f\|_{L^1} = \|f\|_1 = \int_\Omega |f| d\mu = \int |f|. \tag{1.2}$$

Let $p \in \mathbb{R}$ with $1 < p < \infty$, we set

$$L^p = \left\{ f : \Omega \to \mathbb{R}, f \;\; is \; measurable \; functions, |f|^p \in L^1(\Omega) \right\}, \tag{1.3}$$

with

$$\|f\|_{L^p} = \|f\|_p = \left[\int_\Omega |f(x)|^p d\mu \right]^{\frac{1}{p}}. \tag{1.4}$$

Definition 1.1.1.2

We set [10]:

$$L^\infty = \{f : \Omega \to \mathbb{R} \quad | \quad f \quad measurable \; and \; there \; is \; a \; constant \quad C \quad such \; that \quad |f| \leq C \quad a.e.on \quad \Omega \}. \tag{1.5}$$

With

$$\|f\|_{L^\infty} = \|f\|_\infty = inf \, \{C, |f(x)| \leq C \quad a.e.on \quad \Omega\}. \tag{1.6}$$

Definition 1.1.1.3

Let $[a, b]$ be a finite interval. We denote by $AC[a, b]$ the space of primitive functions of integrable functions in the sense of Lebesgue

$$f(x) \in AC[a, b] \Leftrightarrow f(x) = c + \int_a^x \varphi(t) dt, \varphi(t) \in L[a, b], \tag{1.7}$$

and we call $AC[a, b]$ the space of absolutely continuous functions on $[a, b]$ [11] .

1.1.2 Sobolev spaces:

Consider an open subset Ω of $\mathbb{R}^N . D(\Omega)$ is the space of $C^\infty(\mathbb{R}$ or $\mathbb{C})$ functions with compact support in Ω and $D'(\Omega)$ is the space of distributions on Ω.
A distribution $T \in D'(\Omega)$ is said to belong to $L^p(\Omega)(1 \leq p \leq \infty)$ if there exists a function $f \in L^p(\Omega)$ such that

$$\langle T, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx. \tag{1.8}$$

Definition 1.1.2.1

Let $m \in \mathbb{N}$ and let $p \in [1, \infty]$. we define [12] :

$$W^{m,p}(\Omega) = \Big\{ f \in L^p(\Omega) \; D^\alpha f \in L^p(\Omega) \quad \textit{for all} \quad \alpha \in \mathbb{N}^N \quad \textit{such that} \quad |\alpha| \leq m \}. \tag{1.9}$$

$W^{m,p}(\Omega)$ is a Banach space when equipped with the norm :

$$\|f\|_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p}. \tag{1.10}$$

If $p = 2$ and one sets $W^{m,2}(\Omega) = H^m(\Omega)$, then $H^m(\Omega)$ is a Hilbert space with the scalar product

$$\langle u, v \rangle = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u . D^\alpha v dx. \tag{1.11}$$

And it is equipped with the following norm :

$$\|f\|_{H^m} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^2}^2 \right)^{\frac{1}{2}}. \tag{1.12}$$

1.2 Some real analysis properties

- **The continuity:**

Let $f : \mathbb{R} \to \mathbb{R}$ be an application.We say that f is continuous if it is continuous at any point of $\mathbb{R}$. In other words, $f : \mathbb{R} \to \mathbb{R}$ is continuous in an if [13]:

$$\forall a \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}_+^*, \exists \alpha > 0, \forall x \in \mathbb{R} : |x - a| < \alpha \Rightarrow |f(x) - f(a)| < \varepsilon. \tag{1.13}$$

• Uniformly continuous applications:

Let (X, d) and (X', d') be metric spaces. A map $f : X \to X'$
is said to be uniformly continued if, for all $\varepsilon \in \mathbb{R}_+^*$, there exists $\alpha \in \mathbb{R}_+^*$ such that [13]:

$$\forall (x, y) \in X^2, d(x, y) < \alpha \Rightarrow d'(f(x), f(y)) < \varepsilon. \tag{1.14}$$

- **Lipschtzian:**

Let G be a part of $\mathbb{R}^2$, $f : G \to \mathbb{R}$ a application and K a positive real number. We say that f is K-Lipschitzian according to y if [14]:

$$\forall t \in G, \forall (y_1, y_2) \in \mathbb{R} \quad |f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2|. \tag{1.15}$$

- **Bounded Function:**

A function $f : G \to \mathbb{R}$ is bounded if:

$$\exists M > 0, \forall t \in G : |f(t)| \leq M. \tag{1.16}$$

- **Convex function:**

The map f is convex if $\forall x, y, z \in I \subset \mathbb{R}$ with $x \leq y \leq z$ for $y = tx + (1-t)z$, we have [13]:

$$f(y) \leq tf(x) + (1-t)f(z). \tag{1.17}$$

- **Convolution product:**

The convolution product of two real or complex functions (f and g) that are integrable is [15]:

$$(f * g)(x) = \int_0^x f(x-t)g(t)dt = \int_0^x g(x-t)f(t)dt. \tag{1.18}$$

- **The derivation under the symbol of integration:**

Assume that [16]:

1. $f : I \subset \mathbb{R} \times [a, b] \to \mathbb{R}$ is continuous.
2. f admits a partial derivative $\frac{\partial f}{\partial x}$ continue on I .
3. Applications $u : I \to [a, b]$ and $v : I \to [a, b]$ are derivable. then the function

$$\varphi : I \to \mathbb{R}$$

$$x \to \int_{u(x)}^{v(x)} f(x, t)dt,$$

is derivable, or derivative 0

$$\varphi'(x) = \int_{u(x)}^{v(x)} \frac{\partial f(x, t)}{\partial x} dt + v'(x)f(x, v(x)) - u'(x)f(x, u(x)).$$

- **Lebesgue's dominated convergence theorem:**

Let E be a measurable set in $\mathbb{R}$ and let f_n be a sequence of measurable functions such that [12]:

- $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. on E.
- For each $n \in \mathbb{N}, |f_n(x)| \leq g(x)$ a.e. on E, where g is integrable in the sense of Lebesgue on E. So
$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx.$$

• Fubini:

Let $f(x, y)$ be a summable function over the product of measurable spaces (X, μ) and (Y, ν). Then We have the following assertions [17]:

1. For almost all $x \in X$, the function $f(x, y)$ is summable over Y, and its integral over Y is a summable function over X.
2. For ν almost all $y \in Y$, the function $f(x, y)$ is summable over X and its integral over X is a summable function over Y.
3. We have:

$$\begin{aligned}\int_{X\times Y} f(x,y)d(\mu\times\nu)(x,y) &= \int_X \left(\int_Y f(x,y)d\nu(y)\right) d\mu(x)\\ &= \int_Y \left(\int_X f(x,y)d\mu(x)\right) d\nu.\end{aligned}$$

1.3 elements of topology

1.3.1 Definitions of some topological elements

• Norm:

Let E be a vector space on $\mathbb{R}$. We call a norm on E any application $\|.\| : E \to \mathbb{R}$ verify

- $\forall x \in E : \|x\| = 0 \Leftrightarrow x = 0$.
- $\forall \lambda \in \mathbb{R}, \forall x \in E \quad \|\lambda x\| = |\lambda|\|x\|$.
- $\forall x, y \in E : \|x + y\| \leq \|x\| + \|y\|$ " triangular inequality ".

Example 1.3.1.1

Space $C(J, \mathbb{R})$provided with the norm

$$\|y\|_{\infty} := \sup \{|y(t)| : t \in J\}.$$

- **Banach space:**

We call Banach space any space full normalized vector on the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ [18].

Example 1.3.1.2

$C(J, \mathbb{R})$ space of continuous functions on J and with values in $\mathbb{R}$ is Banach.

- **Open parties:**

Let E be a metric space A part A of E is called open if,whenever it contains a point of E, it contains at least one open ball (of radius > 0) having this point as its center:

$$(\forall x \in A)(\exists r > 0) : B_0(x, r) \subset A. \tag{1.19}$$

- **Closed parties:**

We call closed part of E any part of E whose complement is open. Example Any closed ball is a closed part.

- **Compact parts:**

We say that $C \subset \mathbb{R}$ is compact if for any cover of C by open we can extract a finite undercoverage.This translates as follows: if $(U_i)_{i \in I}$ is an open family such that $C \subset (U_i)_{i \in I}$, then there exists a finite subset $J \subset I, C \subset \bigcup_{i \in J} U_i$ [17].

- **Relatively compact parts:**

We say that A is a relatively compact part of a metric space X if its adhesion is a part compact of X [13].

- **Convex parts:**

Let C be a part of E. We say that C is convex in E if, for all $x, y \in C$ and all $t \in [0, 1]$, we have [12]:

$$(1 - t)x + ty \in C. \tag{1.20}$$

- **Operator:**

Let E be a normalized vector space, a linear application A from E in itself is called a linear operator in E. We call domain of A and we denote it by D_A, where [19]:

$$D_A = \{x \in E, Ax \in E\}. \tag{1.21}$$

• Continuous operator:

The operator A is continuous, if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all [19]:

$$(x', x'' \in D_A) : \|x' - x''\| < \delta \Rightarrow \|Ax' - Ax''\| < \varepsilon. \tag{1.22}$$

• Linear Bounded Operators:

Let E be a vector space standard, we call a bounded linear operator any continuous linear application of E in E [19].

- If A is a bounded linear operator, then

  $$(\forall x \in D_A) : VertAx\| \leq \|A\|.\|x\|,$$

  where the norm of A is defined by :

  $$\|A\| = \sup_{\|x\| \leq 1, x \neq 0} \|Ax\| = \sup_{x \in D_A} \frac{\|Ax\|}{\|x\|}. \tag{1.23}$$

• Compact operator:

Operator A is said to be compact if the image of the set $X \subset \mathbb{R}$ by A that is to say the set $A(X)$ is relatively compact [17].

1.4 Special functions related to fractional derivation

1.4.1 The Gamma function

Definition 1.4.1

For $p \in \mathbb{R}$ such that $p > 0$ the Euler Gamma function is defined by the following integral [20]:

$$\Gamma(p) = \int_0^\infty e^{-z} z^{p-1} dz. \tag{1.24}$$

Note

The integral (1.24) converges absolutely on the real half-plane, where x is strictly positive [20]
.

Theorem 1.4.1

The Gamma function checks the following properties [20]:

(1) For all $p \in \mathbb{R}$ with $p > 0$:

$$\Gamma(p+1) = p\Gamma(p), \tag{1.25}$$

In particular, for $n \in \mathbb{N}$

$$\Gamma(n+1) = n!. \tag{1.26}$$

(2) $\Gamma(p)$ is a monotonic and strictly decreasing function for $0 < p \leq 1$ and monotonic and strictly increasing for $p \geq 2$, therefore, it is convex for $p \in [0, +\infty]$.

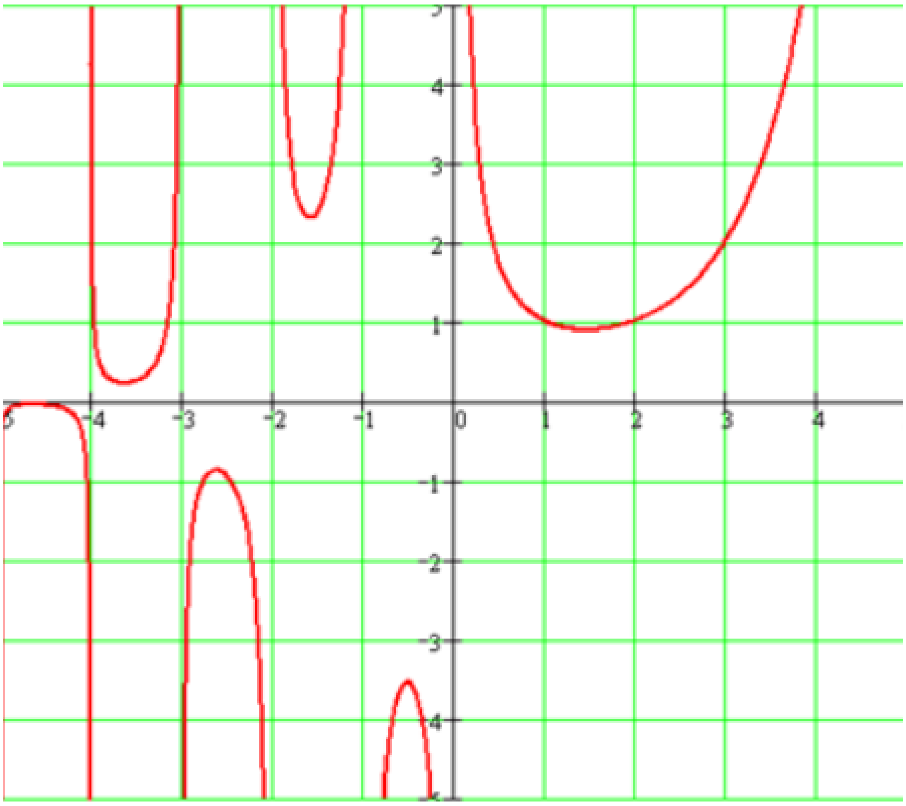


Figure 1.1: the graph of the Gamma function [20] .

Proposition 1.4.1 *we have [20]:*

1. $\Gamma(0^+) = +\infty$.
2. $\Gamma(1) = \Gamma(2) = 1$.
3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
4. $\Gamma(n + \frac{1}{2}) = \frac{\sqrt{\pi}}{2^n}(2n-1)!, \qquad \forall n \in \mathbb{N}$.
5. $\Gamma(-m) = +\infty, \qquad \forall m \in \mathbb{N}$.

1.4.2 The Beta fnction

Definition 1.4.2

For $p, q \in \mathbb{R}$ *such that* $p > 0$ *and* $q > 0$*, the Gamma function is defined by [20]:*

$$A(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx. \tag{1.27}$$

Proposition 1.4.2

The relationship between Euler Beta function and Euler Gamma is given though [15]:

$$A(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

1.4.3 The Mittag-Leffler function

Definition 1.4.3

For $t \in \mathbb{R}$ *and* $\beta > 0$*, the Gamma function is defined by [20]:*

$$E_\beta(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\beta n + 1)}. \tag{1.28}$$

Note *[21]*

$$E_1 = exp(t).$$

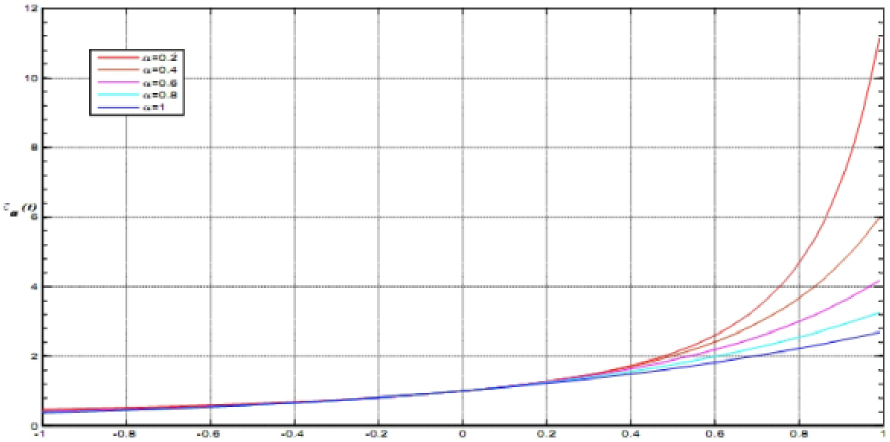


Figure 1.2: The graph of the Mittag-Leffler function [21] .

1.5 Fixed Point Theorems

Definition 1.5.1

Let T be a map of a set S in itself. We call a fixed point of T any point $s \in S$ such that $T(s) = s$.

- ***Banach's principle of contraction***

Theorem 1.5.1

Let S be a completed metric space link and let $T : S \to S$ be a contracting application, that is to say there exists $0 < k < 1$ such that [14]:

$$d(Tx, Ty) \leq k(x, y), \forall x, y \in S. \tag{1.29}$$

Then T admits a unique fixed point $s \in S$. We have

$$\lim_{n \to \infty} T^n(s) = s.$$

With

$$d(T^n(s), s) \leq \frac{k^n}{1-k} d(s, T(s)).$$

Proof: *See ([14])*

- ***Arzela-Ascoli***

Theorem 1.5.2

Let $C(X)$ be the vector normalized space of real functions continuous on a compact metric space X with norm [17]:

$$\|f\| = \sup_{x \in X} |f(x)|.$$

For a family $A \subset C(X)$ is relatively compact, if and only if A is:

- ***Uniformly bounded:***

$$\exists C : |f(x)| \leq C, \forall f \in A, \forall x \in X.$$

- ***Equicontinuous:***

$$\forall \varepsilon > 0, \exists \delta > 0, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon, \forall f \in A.$$

Proof: *See ([17])*

- ***Krasnoselskii's fixed point theorem***

Theorem 1.5.3

Let $X \in E$ *nonempty subset of* E *and* $f, g : X \to E$ *such that [24]:*

- X *: be a closed, convex.*
- E *: Banach space.*
- f *and* g *are continuous,* f *is compact,* g *a contraction and* $f(X) + g(X) \subseteq X$*.*

  Then $f + g$ *admits a fixed point in* X*.*

1.6 Laplace transforms:

Let us recall some basic tools of the Laplace transform.

Definition 1.6.1

The Laplace transform is a practical method for solving differential equations and differential systems, let f *be a function defined for all the variable* $x > 0$ *[15].*

- *Laplace transform of* $f(x)$ *is defined by:*

$$F(s) = \mathcal{L}\left[f(x)\right](s) = \int_0^{+\infty} f(x)e^{-sx}dx, \quad s \in \mathbb{C}. \tag{1.30}$$

- *The original* $f(x)$ *can be restored from the Laplace transform* $F(s)$ *with the help of the inverse Laplace transform,*

$$f(x) = \mathcal{L}^{-1}\left[F(s)\right](x) = \int_{c-i\infty}^{c+i\infty} F(s)e^{sx}ds, \quad c = \mathcal{R}e(s) > c_0. \tag{1.31}$$

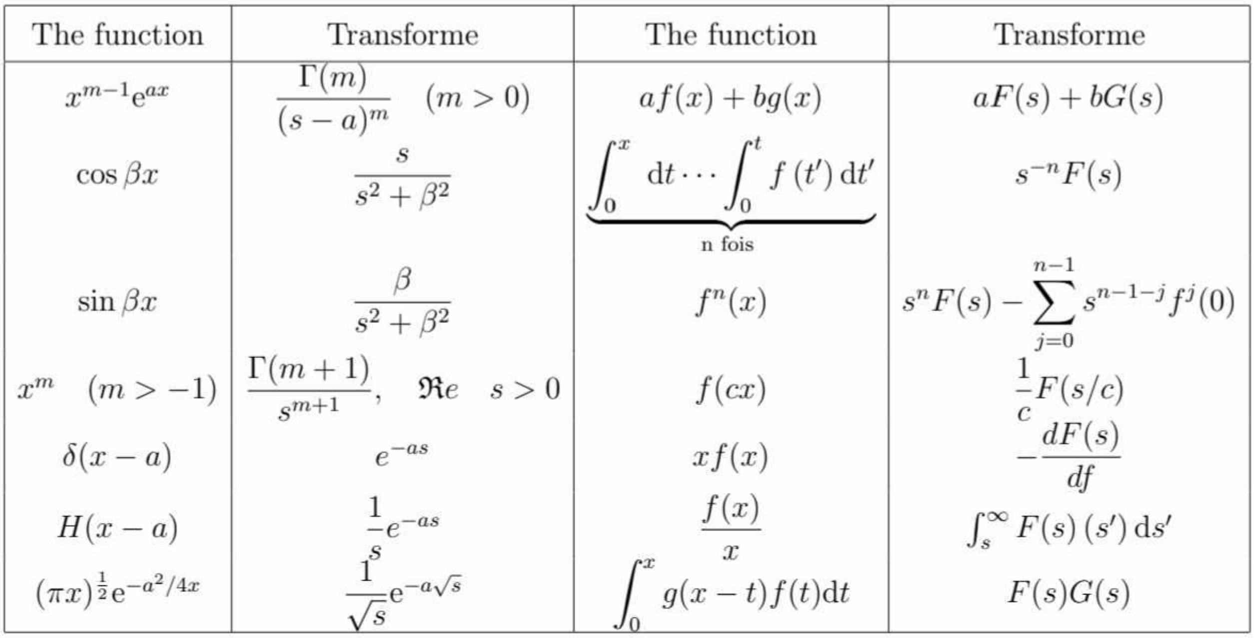
- *Laplace transform of the convolution*

$$\mathcal{L}\left[f(x) * g(x)\right](s) = F(s).G(s).$$

We assume that $F(s)$ *and* $G(s)$ *exist Another useful property which we need is the formula for the Laplace transform of the derivative of an integer order* n *of the function* $f(x)$*:*

$$\mathcal{L}\left[f^n(x)\right](s) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) f^{(n-k-1)}(0).$$

Table summarizes some Laplace transformations of some functions and some properties of Laplace transform ,



The function	Transforme	The function	Transforme
$x^{m-1}e^{ax}$	$\frac{\Gamma(m)}{(s-a)^m} \quad (m>0)$	$af(x)+bg(x)$	$aF(s)+bG(s)$
$\cos\beta x$	$\frac{s}{s^2+\beta^2}$	$\underbrace{\int_0^x \mathrm{d}t \cdots \int_0^t f(t')\,\mathrm{d}t'}_{\text{n fois}}$	$s^{-n}F(s)$
$\sin\beta x$	$\frac{\beta}{s^2+\beta^2}$	$f^n(x)$	$s^nF(s)-\sum_{j=0}^{n-1} s^{n-1-j}f^j(0)$
$x^m \quad (m>-1)$	$\frac{\Gamma(m+1)}{s^{m+1}}, \quad \Re e \quad s>0$	$f(cx)$	$\frac{1}{c}F(s/c)$
$\delta(x-a)$	e^{-as}	$xf(x)$	$-\frac{dF(s)}{df}$
$H(x-a)$	$\frac{1}{s}e^{-as}$	$\frac{f(x)}{x}$	$\int_s^\infty F(s)\,(s')\,\mathrm{d}s'$
$(\pi x)^{\frac{1}{2}}e^{-a^2/4x}$	$\frac{1}{\sqrt{s}}e^{-a\sqrt{s}}$	$\int_0^x g(x-t)f(t)\mathrm{d}t$	$F(s)G(s)$

CHAPTER 2

GENERALITIES IN FRACTIONAL CALCULATIONS

2.1 Riemann-Liouville fractional integrals

Definition 2.1.1

Let $\alpha \in \mathbb{R}_+^*$*,the fractional integral of Riemann-Liouville of ordre* α *(a left) the fonction* f *is defined by [21],[24],[25]:*

$$I_{a^+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^a (x-y)^{\alpha-1} f(y)dy, \qquad (-\infty \leq a < y < \infty), \tag{2.1}$$

for the view that the right side in (2.1) exist almost everywhere.

Definition 2.1.2

Let $\beta \in \mathbb{R}_+^*$*, the fractional integral of Riemann-Liouville of ordre* β *(a right) the fonction* f *is defined by [25]:*

$$I_{b^-}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_b^x (y-x)^{\beta-1} f(y)dy, \qquad b \in \left]-\infty, +\infty\right[, \tag{2.2}$$

for the view that the right side in (2.2) exist almost everywhere.

Note

We can write $I_{b^-}^{\alpha}$ *and* $I_{a^+}^{\beta}$ *in the following form [22]:*

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x-a}^{0} y^{\alpha-1} f(x-y)dy,$$

$$I_{b^-}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_{0}^{x-b} y^{\beta-1} f(y-x)dy.$$

When $\beta, \alpha = n \in \mathbb{N}$ *the definitions 2.1.1 and 2.1.2 coincide with the integrals of the form*

$$I_{a^+}^{\alpha} f(x) = \int_a^x dy_1 \int_a^{y_1} dy_2 ... \int_a^{y_{n-1}} f(y_n)dy_n = \frac{1}{(n-1)!} \int_x^a (x-y)^{n-1} f(y)dy \quad (n \in \mathbb{N}), \tag{2.3}$$

and

$$I_{b^-}^{\beta} f(x) = \int_x^b dy_1 \int_{y_1}^b dy_2 ... \int_{y_{n-1}}^b f(y_n)dy_n = \frac{1}{(n-1)!} \int_b^x (y-x)^{n-1} f(y)dy \quad (n \in \mathbb{N}). \tag{2.4}$$

Example 2.1.1

Let $f(x) = (x-a)^{\alpha}$ *for a fixed* $x > a$ *we have*

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^a (x-y)^{\alpha-1} f(y)dy = \frac{1}{\Gamma(\alpha)} \int_x^a (x-y)^{\alpha-1} (y-a)^{\alpha} dy.$$

Let's put

$$y = a + (x-a)u.$$

So

$$dy = (x-a)du.$$

So

$$\begin{aligned} I^{\alpha}_{a^+}f(x) &= \frac{1}{\Gamma(\alpha)}\int_0^1 (x-a)^{\alpha-1}(1-u)^{\alpha-1}(x-a)^{\alpha}u^{\alpha}(x-a)du \\ &= \frac{(x-a)^{2\alpha}}{\Gamma(\alpha)}\int_0^1 (1-u)^{\alpha-1}u^{\alpha}du \\ &= \frac{(x-a)^{2\alpha}}{\Gamma(\alpha)}\beta(\alpha,\alpha+1), \end{aligned}$$

where is the Euler beta function defined by

$$\beta(x_1,x_2) = \frac{\Gamma(x_1)\Gamma(x_2)}{\Gamma(x_1+x_2)}.$$

That's to say

$$I^{\alpha}_{a^+}f(x) = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}(x-a)^{2\alpha}.$$

Proposition 2.1.1

Let f and g be two continuous functions, λ and $\mu \in \mathbb{R}$ where $\mathbb{C}$.
The fractional integral operator has the following properties The integral is a linear operator[22]:

(a) $I^{\alpha}_{a^+}\left(\lambda f(y) + \mu g(y)\right) = \lambda I^{\alpha}_{a^+}f(y) + \mu I^{\alpha}_{a^+}g(y).$

(b) $I^{\alpha}_{a^+}I^{\gamma}_{a^+}f = I^{\alpha+\gamma}_{a^+}$ *and* $I^{\alpha}_{b^-}I^{\gamma}_{b^-} = I^{\alpha+\gamma}_{b^-}.$

2.2 Riemann-Liouville fractional derivative

Definition 2.2.1

The Riemann-Liouville fractional derivative $D^{\alpha}_{a^+}$ and $D^{\alpha}_{b^-}$ order $\alpha \in \mathbb{C}$ $(Re(\alpha) > 0)$ are defined by[25]:

$$D^{\alpha}_{a^+}f(x) := \left(\frac{d}{dx}\right)^n I^{n-\alpha}_{a^+}f(x),$$

$$D^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{f(y)dy}{(x-y)^{\alpha-n+1}} \quad (n = [Re(\alpha)]+1, x > a), \tag{2.5}$$

and

$$D^{\alpha}_{b^-}f(x) := \left(-\frac{d}{dx}\right)^n I^{n-\alpha}_{b^-}f(x),$$

$$D^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b \frac{f(y)dy}{(y-x)^{\alpha-n+1}} \quad (n = [Re(\alpha)]+1; x < b). \tag{2.6}$$

Note

In particular, when $\alpha = n \in \mathbb{N}_0$ *so*

$$D^0_{a^+}f(x) = D^0_{b^-}f(x), D^0_{a^+}f(x) = f^{(n)}(x), D^0_{b^-}f(x) = (-1)^n f^{(n)}(x). \tag{2.7}$$

Where $f^{(n)}(x)$ *is the usual derivative of* $f(x)$ *order* n *[25] .*

If $0 < Re(\alpha) < 1$ *then*

$$D^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(y)dy}{(x-y)^{\alpha-[Re(\alpha)]}} \quad (0 < Re(\alpha) < 1, x > a), \tag{2.8}$$

$$D^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{f(y)dy}{(y-x)^{\alpha-[Re(\alpha)]}} \quad (0 < Re(\alpha) < 1, x < b). \tag{2.9}$$

When an $\alpha \in \mathbb{R}^+$*, then (2.5) and (2.6) take the following forms:*

$$D^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{f(y)dy}{(x-y)^{\alpha-[n+1]}} \quad (n = [\alpha]+1, x > a), \tag{2.10}$$

$$D^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b \frac{f(y)dy}{(y-x)^{\alpha-[n+1]}} \quad (n = [\alpha]+1, x < b), \tag{2.11}$$

while (2.8) and (2.9) are given by

$$D^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(y)dy}{(x-y)^{\alpha}} \quad (0 < \alpha < 1, x > a), \tag{2.12}$$

and

$$D^{\alpha}_{b^-}f(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\frac{1}{\Gamma(1-\alpha)}\int_x^b \frac{f(y)dy}{(y-x)^{\alpha}} \quad (0 < \alpha < 1, x < b). \tag{2.13}$$

Example 2.2.1

Calculate the fractional derivative of the power function

$$f : [a,b] \to \mathbb{R}$$

$$x \mapsto f(x) = (x-a)^{\alpha}, \alpha > 0.$$

Indeed: according to the definition of $D^{\alpha}_{a^+}$ *we have*

$$D^{\alpha}_{a^+}f(x) = \left(\frac{d}{dx}\right)^n I^{n-\alpha}_{a^+}f(x) = \frac{1}{n-\alpha}\left(\frac{d}{dx}\right)^n \int_x^a \frac{f(y)dy}{(x-y)^{\alpha-n+1}},$$

$$I_{a^+}^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (y-a)^{\alpha}(x-y)^{-\alpha+n-1} dy,$$

by changing the variable $y = x - x_1(x-a)$,

we have $dt = -(x-a)dx_1$ *and then*

$$\begin{aligned}
I_{a^+}^{n-\alpha} f(x) &= -(x-a)^{-\alpha+n-1}(1-x_1)^{\alpha}(x-a)^{\alpha}(x-a)dx_1\\
&= \frac{(x-a)^{n+\gamma-\alpha}}{\Gamma(n-\alpha)} \int_0^1 (1-x_1)^{\alpha} x_1^{-\alpha+n-1} dx_1\\
&= \frac{(x-a)^{n+\gamma-\alpha} B(\gamma+1, n-\alpha)}{\Gamma(n-\alpha)}\\
&= \frac{(x-a)^{n+\gamma-\alpha}\Gamma(\gamma+1)\Gamma(n-\alpha)}{\Gamma(\gamma+1-\alpha+n)\Gamma(n-\alpha)}\\
&= \frac{(x-a)^{n+\gamma-\alpha}\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha+n)},
\end{aligned}$$

$$\left(\frac{d}{dx}\right)^n I_{a^+}^{n-\alpha} f(x) = \frac{\Gamma(\gamma+1)(n+\gamma-\alpha-1)...(\gamma-\alpha+1)}{\Gamma(\gamma+1-\alpha+n)}(x-a)^{\gamma-\alpha},$$

because

$$\Gamma(\gamma-\alpha+n) = (\gamma-\alpha)(\gamma-\alpha+1)...(\gamma-\alpha+n-1)\Gamma(\gamma-\alpha),$$

we obtain

$$\left(\frac{d}{dx}\right)^n I_{a^+}^{n-\beta} f(x) = \frac{\Gamma(\gamma+1)\Gamma(\gamma-\alpha)+n(n+\gamma-\alpha)}{(\gamma-\alpha)\Gamma(\gamma-\alpha)\Gamma(n+\gamma-\alpha+1}(x-a)^{\gamma-\alpha}.$$

From where

$$D_{a^+}^{\alpha} f(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}(x-a)^{\gamma-\alpha}.$$

Note

if we take $\gamma = 0$ *we obtain the following result*

$$D_{a^+}^{\alpha} I(x) = \frac{1}{\Gamma(1-\alpha)}(x-a)^{-\alpha}.$$

that is to say that the Riemann Liouville derivative of a constant is no longer zero [25].

2.2.1 Some Properties of Riemann-Liouville fractional derivative

Proposition 2.2.1.1

Differentiation is a linear operator [25]:

$$D_{a^+}^{\alpha}\left(\lambda f(y) + \mu g(y)\right) = \lambda D_{+a}^{\alpha} f(y) + \mu D_{+a}^{\alpha} g(y).$$

Proposition 2.2.1.2

Let f and g be two continuous functions,λ and $\mu \in \mathbb{R}$ where $\mathbb{C}$. The fractional derivative operator has the following properties [25]:

(b) *Either $\alpha > 0$ and $f \in L_p([a,b])$, $1 \leq p \leq +\infty$ then*
$D_{\alpha}^{a^+} I_{\alpha}^{a^+} f(x) = f(x)$ *almost everything in* $[a,b]$.
$D_{\alpha}^{b^-} I_{\alpha}^{b^-} f(x) = f(x)$ *almost everything in*$[a,b]$.

(b) *Let $0 < \alpha < 1$,$f \in L_1([a,b])$ and $I_{a^+}^{1-\alpha} \in C([a,b])$ then on a*
$I_{\alpha}^{a^+} D_{a^+}^{\alpha} f(x) = f(x) - \dfrac{I^{1-\alpha} f(a)}{\Gamma(\alpha)} (x-a)^{\alpha-1}.$
$I_{b^-}^{\alpha} D_{b^-}^{\alpha} f(x) = f(x) - \dfrac{I^{1-\alpha} f(a)}{\Gamma(\alpha)} (b-x)^{\alpha-1}.$

2.3 Caputo fractional derivative

Riemann-Liouville fractional derivation significantly influenced fractional calculus development due to its applications in pure and applied mathematics. Caputo proposes a different approach, where the derivative of a constant is zero and initial conditions are expressed using whole order derivatives.

Definition 2.3.1

Let $0 < n-1 < \alpha < n, n \in \mathbb{N}, \alpha \in \mathbb{R}$ and f be a function of class $C^n([a,b])$. The fractional derivative of order α in the sense of Caputo of the function f is defined by [26]:

$$^{C}D^{(\alpha)} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x f^n(t)(x-t)^{n-\alpha-1} dt. \tag{2.14}$$

- *Under natural conditions on the function $f(x)$,for $\alpha \to n$ the Caputo derivative becomes a conventional $n-1$the derivative of the function $f(x)$.*
  Indeed let us assume that $0 \leq n-1 < \alpha < n$ and that the function $f(x)$ has $n+1$ continuous bounded derivatives in$[a,T]$ for every $T > a$ then [15]

$$\lim_{\alpha \to n} {}_a^C D_x^{(\alpha)} f(x) = \lim_{\alpha \to n} \left(\frac{1}{\Gamma(n-\alpha)} \int_a^x f^n(t)(x-t)^{n-\alpha-1} dt \right),$$

from integration by parts we have

$$\begin{aligned}\lim_{\alpha\to n}{}^{C}_{a}D^{(\alpha)}_{x}f(x) =& \lim_{\alpha\to n}\left(\frac{f^{(n)}(a)(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}+\frac{1}{\Gamma(n-\alpha+1)}\int_a^x (x-t)^{n-\alpha}f^{(n+1)}(t)dt\right)\\ =&f^{(n)}(a)+\int_a^x f^{(n+1)}(t)dt\\ =&f^{(n)}(x), \quad n=1,2,... \quad .\end{aligned}$$

- *Non-commutation*

$${}^{C}_{a}D^{(\alpha)}_{x}\left({}^{C}_{a}D^{(m)}_{x}f(x)\right) ={}^{C}_{a}D^{(\alpha+m)}_{x}f(x) \neq {}^{C}_{a}D^{(m)}_{x}\left({}^{C}_{a}D^{(\alpha)}_{x}f(x)\right), \quad (m=1,2,..., \quad n-1<\alpha<n).$$

The interchange of the differentiation operators in formulas is allowed under different conditions [15]:

$${}^{C}_{a}D^{(\alpha)}_{x}\left({}^{C}_{a}D^{(m)}_{x}f(x)\right) = {}^{C}_{a}D^{(m)}_{x}\left({}^{C}_{a}D^{(\alpha)}_{x}f(x)\right) = {}^{C}_{a}D^{(\alpha+m)}_{x}f(x),$$

$$f^{(s)}(0)=0, \quad s=n,n+1,...,m, \quad (m=0,1,2,...,n-1<\alpha<n).$$

Definition 2.3.2

If $\alpha \in \mathbb{R}_+$ and $n=[\alpha]+1$, for all $f \in C^n[a,b]$ then The fractional derivative on the left and the fractional derivative on the right in the sense of Caputo of order $\alpha>0$ of the function f in $x\in[a,b]$, are are defined respectively by [27]:

- ***fractional derivative of caputo on the left***

$$\forall x>a, \quad {}^{C}_{a}D^{\alpha}_{x}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}f^{(n)}(t)dt. \tag{2.15}$$

- ***fractional derivative of caputo on the right***

$$\forall x<b, \quad {}^{C}_{x}D^{\beta}_{b}f(x)=\frac{1}{\Gamma(n-\beta)}(-1)^n\int_x^b (t-x)^{n-\beta-1}f^{(n)}(t)dt. \tag{2.16}$$

Note that f is a function such that ${}^{C}_{a}D^{\alpha}_{x}f(x)$ and ${}^{C}_{x}D^{\beta}_{b}f(x)$ are defined.
Operators whose integral relates to $[a,x]$ (respectively $[x,b]$) will be qualified as past operators (repectively future operators).

- *In particular, if $0<\alpha<1$,one have*

$${}^{C}_{a}D^{\alpha}_{x}f(x)=\frac{1}{\Gamma(1-\alpha)}\int_a^x (x-t)^{-\alpha}f'(t)dt,$$

and

$${}^{C}_{x}D^{\beta}_{b}f(x)=\frac{-1}{\Gamma(1-\beta)}\int_x^b (t-x)^{-\beta}f'(t)dt.$$

Example 2.3.1

Consider the function:

$$f(x) = x^{\beta},$$

for $0 < n-1 < \alpha < n$,*we have:*

$${}^{C}D^{(\alpha)} f(x) = I^{n-\alpha}(D^{n} x^{\beta}),$$

or

$$D^{n} x^{\beta} = \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)} x^{\beta-n} \right).$$

As a result:

$$I^{n-\alpha} \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)} x^{\beta-n} \right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)\Gamma(n-\alpha)} \int_{0}^{x} (x-t)^{n-\alpha-1} t^{\beta-n} dt,$$

by performing the change of variable$t = yx$ *so* $dt = xdy$, *we obtain:*

$$\begin{aligned} \int_{0}^{x} (x-t)^{n-\alpha-1} t^{\beta-n} dt &= \int_{0}^{1} (x-xy)^{n-\alpha-1} (xy)^{\beta-n} x dy \\ &= \int_{0}^{1} x^{n-\alpha-1} (1-y)^{n-\alpha-1} y^{\beta-n} x^{\beta-n+1} dy \\ &= \int_{0}^{1} x^{\beta-\alpha} (1-y)^{n-\alpha-1} y^{\beta-n} dy \\ &= x^{\beta-\alpha} \int_{0}^{1} (1-y)^{n-\alpha-1} y^{\beta-n} dy \\ &= x^{\beta-\alpha} B(n-\alpha, \beta-n+1) \\ &= x^{\beta-\alpha} \frac{\Gamma(n-\alpha)\Gamma(\beta-n+1)}{\Gamma(\beta-n+1)}. \end{aligned}$$

So

$$I^{n-\alpha} \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)} x^{\beta-n} \right) = \frac{(\beta+1)}{\Gamma(\beta+1-n)} \quad \frac{1}{\Gamma(n-\alpha)} \quad \frac{\Gamma(n-\alpha)\Gamma(\beta-n+1)}{\Gamma(\beta-n+1)} x^{\beta-\alpha},$$

finally,we obtain

$${}^{C}D^{(\alpha)} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}.$$

In particular, for $\beta = 0$, *we have:*

$${}^{C}D^{(\alpha)} x^{0} = D^{(\alpha)} 1 = 0,$$

unlike the Riemann-Liouville derivative, the fractional order derivative in the Caputo sense of a constant is zero.

Remark 2.3.1

- *As in the case of the Riemann-Liouville operators, we see that the Caputo derivatives are not local either.*
- *The main advantage of Caputo's approach is that the initial conditions of the fractional derivative in Caputo's sense of the differential equations take the same form as in the case of differential equtions of integer order.*
- *Thefractional derivative in the sense of Riemann-Liouville of order* $\alpha \in]n-1, n[$ *is obtained by an application of the fractional integration operator of order a followed by a classical derivation of order* $n-\alpha$*,while the fractional derivative in the sense of Caputo is the result of the premutation of these two operations [11].*

Remark 2.3.2

taking into account the definition of $(R-L)$*,we have:*

$$^{C}D^{\alpha}_{a^{+}}f(x) := (I^{n-\alpha}_{a^{+}}D^{n}f)(x), \tag{2.17}$$

and

$$^{C}D^{\alpha}_{b^{-}}f(x) := (-1)^{n}(I^{n-\alpha}_{b^{-}}D^{n}f)(x),$$

in particular,if $0 < \alpha < 1$ *we have:*

$$^{C}D^{\alpha}_{a^{+}}f(x) := (I^{1-\alpha}_{a^{+}}D^{1}f)(x),$$

and

$$^{C}D^{\alpha}_{b^{-}}f(x) := (-1)(I^{1-\alpha}_{b^{-}}D^{1}f)(x),$$

or

$$D^{n} = \frac{d^{n}}{dx^{n}}.$$

2.3.1 Some properties of Caputo fractional derivative

Proposition 2.3.1.1

Let $\alpha > 0$ *and* $n = [\alpha] + 1$ *such as* $n \in \mathbb{N}^{*}$ *then, the followng equalities [15]:*

1.

$$^{C}D^{\alpha}I^{\alpha}_{a}f = f. \tag{2.18}$$

2.

$$I_a^\alpha({}^C D^\alpha f(x)) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(x-a)^k}{k!}, \tag{2.19}$$

are true for all $x \in [a,b]$.

Proof.

by Theorem.2.4.3.1 and the use of the semi-group property , we find:

1. $({}^C D^\alpha f))(x) := (I_a^{\alpha-n} D^n I_a^\alpha f)(x) = I_a^0 f.$

2. $(I_a^\alpha({}^C D^\alpha f))(x) := (I_a^\alpha I_a^{n-\alpha} D^\alpha) f(x),$

   according to the semi-group property,we have:

$$\begin{aligned}(I_a^\alpha I_a^{n-\alpha} D^\alpha f)(x) &= I_a^\alpha I_a^n I_a^{-\alpha} D^n f(x)\\ &= I_a^n D^n f(t),\end{aligned}$$

and like

$$(I_a^n D^n f)(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

we find

$$I_a^\alpha({}^C D^\alpha f(x)) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

therefore the caputo differentiation operator is a left inverse of the fractional integration oper tor but it is not a right inverse.

Lemma 2.3.1.1

Let f be a continue function over $[a,b]$ and $\alpha > 0$ [28].

$$\lim_{x\to a^+} (I_a^\alpha f)(x) = 0.$$

Proof.

$$\begin{aligned}|(I_a^\alpha f)(x)| &\le \frac{1}{\Gamma(\alpha)} \int_a^x (x-a)^{\alpha-1} |f(t)| dt\\ &\le \frac{\|f\|_\infty}{\Gamma(\alpha)} int_a^x (x-a)^{\alpha-1} dt\\ &\le \frac{\|f\|_\infty}{\Gamma(\alpha+1)} (x-a)^\alpha.\end{aligned}$$

Corollary 2.3.1.1

If $0 < \alpha < 1$ and f of class C^1 then[29] :

$$(I_a^\alpha \circ^R D_a^\alpha) f = f \quad ({}^c D_a^\alpha \circ I_a^\alpha) f = f.$$

Corollary 2.3.1.2

If $\alpha \leq 0$, $\beta \leq 1$ *with*$\alpha + \beta \leq 1, f$ *of class* C^1 *then*

$$({}^C D_a^a \circ^C D_a^\beta)f = ({}^C D_a^{a+\beta})f = ({}^C D_a^\beta \circ {}^C D_a^a)f.$$

Proof.

It is easy to see that

$$\begin{aligned}
({}^C D_a^a \circ^C D_a^\beta)f &= (I_a^{1-\alpha} \circ \frac{d}{dx} \circ I_a^{1-\beta} \circ \frac{d}{dx})f \\
&= (I_a^{1-\alpha-\beta} \circ I_a^\beta \circ \frac{d}{dx} \circ I_a^{1-\beta} \circ \frac{d}{dx})f \\
&= (I_a^{1-\alpha-\beta} \circ {}^C D_b^{1-\beta} \circ I_a^{1-\beta} \circ \frac{d}{dx})f \\
&= (I_a^{1-\alpha-\beta} \circ \frac{d}{dx})f \\
&=^C D_a^{\alpha+\beta} f.
\end{aligned}$$

Lemma 2.3.1.2

Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ *and* $n = [\alpha] + 1$, *if* $f \in C^n([a,b])$ *so almost everywhere*

$$\begin{aligned}
\lim_{\alpha \to n^-} {}_a^C D_x^{(\alpha)} f(x) &= f^{(n)}(x), \\
\lim_{\alpha \to n^-} {}_x^C D_b^{(\alpha)} f(x) &= (-1)^n f^{(n)}(x).
\end{aligned}$$

Proof.

as $f^{(n)} \in L^1([a,b])$, *from [22], passing* $\beta = n - \alpha$, $\lim\limits_{\beta \to 0^+} {}_a I_x^\beta f^{(n)} = f^{(n)}$. *almost everywhere.* *The same reasoning applies for* ${}_x^C D_b^{(\alpha)}$.

Proposition 2.3.1.2

for $n - 1 \leq \alpha \leq n$,*if* $f(x) \in C^n([a,b])$ *the derivative* $({}^C D_{a^+}^{(\alpha)} f)(x)$ *exists almost everywhere on* $[a,b]$ *[31].*

- *if* $\alpha \notin \mathbb{N}$ *The derivative* $({}^C D_{a^+}^\alpha f)(x)$ *can be represented by:*

$$({}^C D_{a^+}^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^\alpha + 1 - n} dt = (I_{a^+}^{n-\alpha} D^n f)(t), \tag{2.20}$$

in particular, if $0 < \alpha < 1$ *and* $f(x) \in C[a,b]$

$$({}^C D_{a^+}^\alpha f)(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f'(t)}{(x-t)^\alpha} dt = (I_{a^+}^{1-\alpha} D f)(t),$$

- *if* $\alpha = n \in \mathbb{N}$:

$$({}^{C}D^{\alpha}_{a^+}f)(x) = f^n(x). \tag{2.21}$$

Proof. *(See [31])*

Proposition 2.3.1.3

if $\alpha \in \mathbb{N}_0$*then the fractional derivatives of caputo* ${}^{C}_{a}D^{\alpha}_{x}f(x)$*and* ${}^{C}_{x}D^{\alpha}_{b}f(x)$ *of ordre* $\alpha \in \mathbb{R}_+$ *of the function* f *exist at the same time with the fractional derivatives of Riemann-Liouville* ${}_{a}D^{\alpha}_{x}f(x)$ *and* ${}_{x}D^{\alpha}_{b}f(x)$ *as follows [12]:*

$${}^{C}_{a}D^{\alpha}_{x}f(x) = {}_{a}D^{\alpha}_{x}f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha},$$

and

$${}^{C}_{x}D^{\alpha}_{b}f(x) = {}_{x}D^{\alpha}_{b}f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-x)^{k-\alpha},$$

where $n = [\alpha]+1$*, in particular,when* f*, we have*

$${}^{C}_{a}D^{\alpha}_{x}f(x) = {}_{a}D^{\alpha}_{x}f(x) - \frac{f(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha},$$

and

$${}^{C}_{x}D^{\alpha}_{b}f(x) = {}_{x}D^{\alpha}_{b}f(x) - \frac{f(a)}{\Gamma(1-\alpha)}(b-x)^{-\alpha}.$$

Proposition 2.3.1.4

Let $\alpha \in \mathbb{R}_+$ *and* $n = [\alpha]+1$ *for* $\alpha \notin \mathbb{N}_0$*,*$n = \alpha$ *for* $\alpha \in \mathbb{N}_0$ *if* $f \in AC^n([a,b],\mathbb{R}^n)$*, then the fractional derivatives of caputo* ${}^{C}_{a}D^{\alpha}_{x}f(x)$ *and* ${}^{C}_{x}D^{\alpha}_{b}f(x)$ *exist almost everywhere on* $[a,b]$*.*
If $\alpha \notin \mathbb{N}_0$*,* ${}^{C}_{a}D^{\alpha}_{x}f(x)$ *and* ${}^{C}_{x}D^{\alpha}_{b}f(x)$ *are represented by [12]:*

$${}^{C}_{a}D^{\alpha}_{x}f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\int_a^x (x-t)^{n-\alpha-1}f^{(n)}(t)dt\right),$$

and

$${}^{C}_{x}D^{\alpha}_{b}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\left(\int_x^b (t-x)^{n-\alpha-1}f^{(n)}(t)dt\right),$$

respectively, such that $n = [\alpha]+1$*. In particular, when* $0 < \alpha < 1$ *and* $f \in C^1([a,b],\mathbb{R}^n)$

$${}^{C}_{a}D^{\alpha}_{x}f(x) = \frac{1}{\Gamma(1-\alpha)}\left(\int_a^x (x-t)^{-\alpha}f'(t)dt\right),$$

and

$${}^{C}_{x}D^{\alpha}_{b}f(x) = -\frac{(1)}{\Gamma(1-\alpha)}\left(\int_x^b (t-x)^{-\alpha}f'(t)dt\right).$$

Proposition 2.3.1.5

- *Let* $\alpha > 0$ *and* $f \in L^{\infty}([a,b],\mathbb{R}^n)$ *or* $f \in C([a,b],\mathbb{R}^n)$*,then* [12]*:*

$$ {}^C_aD^{\alpha}_x({}_aD^{-\alpha}_x f(x)) = f(x) \quad \textit{and} \quad {}^C_xD^{\alpha}_b({}_xD^{-\alpha}_b f(x)) = f(x). $$

- *Let* $\alpha > 0$ *and* $n = [\alpha] + 1$ *for* $\alpha \notin \mathbb{N}_0$, $n = \alpha$ *for* $\alpha \in \mathbb{N}_0$*, if* $f \in AC^n([a,b],\mathbb{R}^n)$ *or* $f \in C^n([a,b],\mathbb{R}^n)$*,then* [12]*:*

$$ {}_aD^{-\alpha}_x({}^C_aD^{\alpha}_x f(x)) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k, $$

and

$$ {}_xD^{-\alpha}_b({}^C_xD^{\alpha}_b f(x)) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(a)}{k!}(b-x)^k, $$

In particular, if $0 < \alpha \leq 1$ *and* $f \in AC^n([a,b],\mathbb{R}^n)$ *or* $f \in C^n([a,b],\mathbb{R}^n)$*,then* [12]*:*

$$ {}_aD^{-\alpha}_x({}^C_aD^{\alpha}_x f(x)) = f(x) - f(a) \quad \textit{and} \quad {}_xD^{-\alpha}_b({}^C_xD^{\alpha}_b f(x)) = f(x) - f(b). $$

Properties

- ***Composition with whole order derivatives***

If f *is a fairly regular function over the interval* $[a,b]$*, and* $n-1 \leq \alpha \leq n, n \in \mathbb{R}^*$*.*

So

- $\frac{d^n}{dx^n}({}^C_aD^{\alpha}_x f(x)) = ({}^C_aD^{n+\alpha}_x f(x))$.
- ${}^C_aD^{\alpha}_x\left(\frac{d^n}{dx^n}f(x)\right) = {}^C_aD^{n+\alpha}_x f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(x-a)^{k-\alpha-n}}{\Gamma(1+k-\alpha-n)}$.

- ***Composition with fractional derivatives***

If f *is a fairly regular function over the interval* $[a,b]$*, for all* $n-1 \leq \alpha < n, m-1 \leq \beta < m$*.*

We have

$$ {}^C_aD^{\alpha}_x({}^C_aD^{\beta}_x f(x)) = {}^C_aD^{\alpha+\beta} f(x) - \sum_{k=1}^{n} \left[{}^C_aD^{\alpha-k}_x f(x)\right]_{x=a} \frac{(x-a)^{-\alpha-k}}{\Gamma(1-\alpha-k)}. $$

2.3.2 Relationship between fractional derivatives with in the sense of Caputo and those of Riemann-Liouville

Theorem 2.3.2.1

Let $a > 0$ *with* $n-1 < \alpha < n, (n \in \mathbb{N}^*)$*, suppose* f *is a function such that* ${}^{C}D_a^{\alpha} f(x)$ *and* ${}^{R}D_a^{\alpha} f(x)$ *exist,then [11]:*

$$ {}^{C}D_a^{\alpha} f(x) = {}^{R}D_a^{\alpha} f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a). $$

Proof.*(See [11])*

Proposition 2.3.2.1

If $\alpha \in \mathbb{R}_+$ *and* f *is a fairly regular function over the interval* $[a,b]$*, then the fractional derivative to the left (resp- to the right) of Caputo is linked to Riemann- Liouville's the fractional derivative to the left (resp- to the right) by*

$$ {}_a^C D_x^{\alpha} f(x) = {}_a^R D_x^{\alpha} f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(1-\alpha+k)} (x-a)^{k-\alpha}, \quad n = [\alpha]+1, $$

resp

$$ {}_a^C D_b^{\alpha} f(x) = {}_x^R D_b^{\alpha} f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(1-\alpha+k)} (b-x)^{k-\alpha}, \quad n = [\alpha]+1. $$

In particular, when $0 < \alpha < 1$ *then*

$$ {}_a^C D_x^{\alpha} f(x) = {}_a^R D_x^{\alpha} f(x) - \frac{f(a)}{\Gamma(1-\alpha)} (x-a)^{-\alpha}, \quad n = [\alpha]+1, $$

and

$$ {}_x^C D_b^{\alpha} f(x) = {}_x^R D_b^{\alpha} f(x) - \frac{f(a)}{\Gamma(1-\alpha)} (b-x)^{-\alpha}, \quad n = [\alpha]+1. $$

Proof *(See[15])*

2.3.3 Some general properties of fractional derivatives

- ***Linearity***

Similarly to integer-order differentiation, fractional differentiation is a linear operator [15]:

$$ D^{(\alpha)}(\lambda f(x) + \mu g(x)) = \lambda D^{(\alpha)} f(x) + \mu D^{(\alpha)} g(x). $$

Proof

For example, if $D^{(\alpha)}$ *is the Caputo operator (where* $n-1\leq\alpha<n$ *and* $n=1$*), by definition we have:*

$$\begin{aligned}{}_a^C D_x^{(\alpha)}(\lambda f(x)+\mu g(x)) &= \frac{1}{\Gamma(1-\alpha)}\int_a^x(\lambda f(t)+\mu g(t))'(x-t)^{-\alpha}dt\\ &= \frac{1}{\Gamma(1-\alpha)}\int_a^x(\lambda f'(t)+\mu g(t)')(x-t)^{-\alpha}dt\\ &= \frac{\lambda}{\Gamma(1-\alpha)}\int_a^x f'(t)(x-t)^{-\alpha}dt+\frac{\mu}{\Gamma(1-\alpha)}\int_a^x g(t)'(x-t)^{-\alpha}dt\\ &= \lambda_a^C D_x^{(\alpha)}f(x)+\mu_a^C D_x^{(\alpha)}g(x).\end{aligned}$$

- ***The Leibniz Rule***

For all $n\in\mathbb{N}$. *we have [15]*

$$\frac{d^n}{dt^n}(f(x)g(x))=\sum_{k=0}^{n}\begin{pmatrix}n\\ k\end{pmatrix}f^{(k)}(x)g^{(n-k)}(x).$$

The generalization of this formula gives us

$$D^{(\alpha)}(f(x)g(x))=\sum_{k=0}^{n}\begin{pmatrix}\alpha\\ k\end{pmatrix}f^{(k)}(x)D^{(\alpha)}g^{(\alpha-k)}(x)+R_n^{\alpha}(x).$$

Or $n\geq\alpha+1$ *and*

$$R_n^{\alpha}(x)=\frac{1}{n!\Gamma(-\alpha)}\int_a^t(x-s)^{-\alpha-1}g(s)ds\int_s^t f^{(n+1)}(\xi)d\xi,$$

with

$$\lim_{n\to\infty}R_n^{\alpha}(x)=0.$$

If f *and* g *are continuous in* $[a,t]$ *as well as all their derivatives, the formula becomes:*

$$D^{\alpha}(f(x)g(x))=\sum_{k=0}^{n}\begin{pmatrix}\alpha\\ k\end{pmatrix}f^{(k)}(x)D^{(\alpha)}g^{(\alpha-k)}(x).$$

where$D^{(\alpha)}$*is the fractional derivative in the sense of Riemann-Liouville.*

- ***Integration by parts***

In this paragraph, we are interested in the integration formulz by parties for fractional derivatives of Riemann-Liouville and those of Caputo.

Let us first give the integration formula by parties for the fractional derivatives of Riemann-Liouville:

Theorem 2.3.3.1

let $0 < \alpha < 1$ *and* $a < x < b$*,then:*

$$\int_a^x \left[{}_a^R D_x^\alpha f(t)\right] g(t)dt = \int_a^x f(t) \left[{}_t^R D_x^\alpha g(t)\right] dt,$$

$$\int_x^b \left[{}_t^R D_b^\alpha f(t)\right] g(t)dt = \int_x^b f(t) \left[{}_x^R D_t^\alpha g(t)\right] dt.$$

In particular,

$$\int_a^b \left[{}_a^R D_b^\alpha f(t)\right] g(t)dt = \int_a^b f(t) \left[{}_t^R D_b^\alpha g(t)\right] dt.$$

Proof.

$$\begin{aligned}
\int_a^x \left[{}_a^R D_x^\alpha f(t)\right] g(t)dt &= \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{d}{dt} \left(\int_a^t (t-u)^{-\alpha} f(u)du \right) g(t)dt \\
&= -\frac{1}{\Gamma(1-\alpha)} \int_a^x \left(\int_a^t (t-u)^{-\alpha} f(u)du \right) g'(t)dt \\
&+ \left[\frac{1}{\Gamma(1-\alpha)} g(t) \int_a^t (t-u)^{-\alpha} f(u)du \right]_{t=a}^{t=x} \\
&= -\frac{1}{\Gamma(1-\alpha)} \int_a^x \left(\int_u^x (t-u)^{-\alpha} g'(t)dt \right) f(u)du \\
&+ g(t) \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-u)^{-\alpha} f(u)du \\
&= \int_a^x f(u) \left[{}_u^C D_x^\alpha g(u)\right] du \\
&+ g(x) \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-u)^{-\alpha} f(u)du \\
&= \int_a^x f(u) \left[{}_u^R D_x^\alpha g(u) - g(x) \frac{(x-u)^{-\alpha}}{\Gamma(1-\alpha)} \right] du \\
&+ g(x) \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-u)^{-\alpha} f(u)du \\
&= \int_a^x f(u) \left[{}_u^R D_x^\alpha g(u)\right] du \\
&= \int_a^x f(t) \left[{}_t^R D_x^\alpha g(t)\right] dt.
\end{aligned}$$

The second formula shows itself in the same way.

Unlike the classic case, no term appears in this formula, it is not the same for the derivative of Caputo.

Theorem 2.3.3.2

Let $0 < \alpha < 1$ *and* $a < t < b$,*then:*

$$\int_a^x \left[{}_a^C D_x^\alpha f(t)\right] g(t)dt = \int_a^x f(t)\left[{}_t^C D_x^\alpha g(t)\right] dt + g(x)_a D_x^{-(1-\alpha)} f(x) - f(a)_a D_x^{-(1-\alpha)} g(a),$$

$$\int_x^b \left[{}_t^C D_b^\alpha f(t)\right] g(t)dt = \int_x^b f(t)\left[{}_x^C D_t^\alpha g(t)\right] dt + g(x)_x D_b^{-(1-\alpha)} f(x) - f(b)_x D_b^{-(1-\alpha)} g(b),$$

In particular,

$$\int_a^b \left[{}_a^C D_x^\alpha f(x)\right] g(x)dx = \int_a^b f(x)\left[{}_b^C D_x^\alpha g(x)\right] dx + g(b)_a D_b^{-(1-\alpha)} f(b) - f(a)_a D_b^{-(1-\alpha)} g(a).$$

Proof.

$$\begin{aligned}
\int_a^x \left[{}_a^C D_x^\alpha f(t)\right] g(t)dt &= \int_a^x \left[{}_a^R D_x^\alpha f(t)\right] g(t)dt - \frac{f(a)}{\Gamma(1-\alpha)}\int_a^x (t-a)^{-\alpha} g(t)dt \\
&= \int_a^x \left[{}_t^R D_x^\alpha g(t)\right] dt - f(a)_a D_x^{-(1-\alpha)} g(a) \\
&= \int_a^x f(t)\left[{}_t^C D_x^\alpha g(t)\right] dt + \frac{g(x)}{\Gamma(1-\alpha)}\int_a^x (x-t)^{-\alpha} f(t)dt - f(a)_a D_x^{-(1-\alpha)} g(a) \\
&= \int_a^x f(t)\left[{}_t^C D_x^\alpha g(t)\right] dt + g(x)_a D_x^{-(1-\alpha)} f(x) - f(a)_a D_x^{-(1-\alpha)} g(a).
\end{aligned}$$

The same goes for the other formula.

2.4 Caputo-Fabrizio fractional derivative (CFFD):

Recently, a new derivative was launched by Caputo and Fabrizio [32] and it was followed by some related theoretical and applied results (see for example Refs. [33] and the references therein). We recall that the existing fractional derivatives have been used in many real world problems with great success (see for example Refs.[32] and the references therein) but still there are many thinks to be done in this direction. Because of the singularity in the kernel of the Caputo fractional derivative[11] at the end point of the interval of integration, the Caputo fractional derivative is not always a suitable kernel to accurately describe the memory effect in a real system. Caputo and Fabrizio[32] has recently proposed a new fractional derivative without any singularity in its kernel. The kernel of the new fractional derivative has the form of an exponential function. More recently, Losada and Nieto[33] derived the fractional integral associated with the new fractional Caputo-Fabrizio fractional derivative.

This section is devoted to studying the basic definitions and results about the Caputo-Fabrizio fractional derivative. Let us recall the usual Caputo fractional time derivative (CFD) of order α*, is given by :*

$$^{C}D_x^{(\alpha)}f(x) = \frac{1}{\Gamma(1-\alpha)}\int_a^t f'(t)(x-t)^{-\alpha}dt.$$

Definition 2.4.1 *Let* $f \in H^1(a,b), b > a, \alpha \in [0,1]$*the Caputo-Fabrizio fractional derivative is defined as*

$$^{CF}D_x^{(\alpha)}f(x) = \frac{M(\alpha)}{1-\alpha}\int_a^t f'(t)exp\left[-\alpha\frac{(x-t)}{1-\alpha}\right]dt,$$

where $M(\alpha)$ *is a normalization function such that* $M(0) = M(1) = 1$*. If the function does not belong to* $H^1(a,b)$ *then, the derivative can be reformulated as*

$$^{CF}D_x^{(\alpha)}f(x) = \frac{\alpha M(\alpha)}{1-\alpha}\int_a^x (f(x)-f(t))exp\left[-\alpha\frac{(x-t)}{1-\alpha}\right]dt.$$

The definition of the CFFD was improved by Losada and Nieto to become [28].

$$^{CF}D_x^{(\alpha)}f(x) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_a^x f'(t)exp\left[-\alpha\frac{(x-t)}{1-\alpha}\right]dt.$$

Now, it is worth to observe that if we put [32].

$$\sigma = \frac{1-\alpha}{\alpha} \in [0,\infty], \alpha = \frac{1}{1+\sigma} \in [0,1],$$

the definition 2.4.1 of CFFD assumes the form

$$\tilde{D}_x^{(\sigma)}f(x) = \frac{N(\sigma)}{\sigma}\int_a^x f'(t)\exp\left[-\frac{(x-t)}{\sigma}\right]dt,$$

where $\sigma \in [0,\infty]$ *and* $N(\sigma)$ *is the corresponding normalization term of* $M(\alpha)$*, such that* $N(0) = N(\infty) = 1$*. Moreover because*

$$\lim_{\sigma\to 0}\frac{1}{\sigma}exp\left[-\frac{(x-t)}{\sigma}\right] = \delta(x-t),$$

and for $\alpha \to 1$*, we have* $\sigma \to 0$*.*

$$\begin{aligned}\lim_{\alpha\to 1} {}^{CF}D_x^{(\alpha)}f(x) &= \lim_{\alpha\to 1}\frac{M(\alpha)}{1-\alpha}\int_a^x f'(t)\exp\left[-\frac{\alpha(x-t)}{1-\alpha}\right]dt \\ &= \lim_{\sigma\to 0}\frac{N(\sigma)}{\sigma}\int_a^x f'(t)\exp\left[-\frac{(x-t)}{\sigma}\right]dt = f'(x).\end{aligned}$$

Otherwise, when $\alpha \to 0$*, then* $\sigma \to +\infty$ *Hence,*

$$\begin{aligned}\lim_{\alpha\to 0} {}^{CF}D_x^{(\alpha)}f(x) &= \lim_{\alpha\to 0}\frac{M(\alpha)}{1-\alpha}\int_a^x f'(t)\exp\left[-\frac{\alpha(x-t)}{1-\alpha}\right]dt \\ &= \lim_{\sigma\to +\infty}\frac{N(\sigma)}{\sigma}\int_a^x f'\exp\left[-\frac{(x-t)}{\sigma}\right]dt = f(x)-f(a).\end{aligned}$$

Let us consider the (CFFD) of a particular function,as $f(x) = \sin\omega x$, *for* $\alpha = 0.66, a = -8$ *and* $\omega = 1$

$$^{CF}D_x^{0.66} \sin\omega x = \frac{M(0.66)}{0.33}\int_a^x \cos t \exp -2(x-t)dt.$$

The simulation of this derivative produces the following pictures:

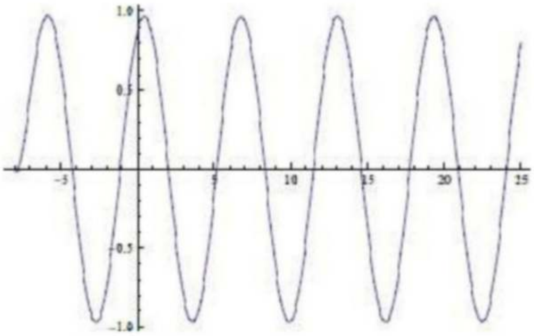


Figure 2.1: Simulation of (CFFD) [32] .

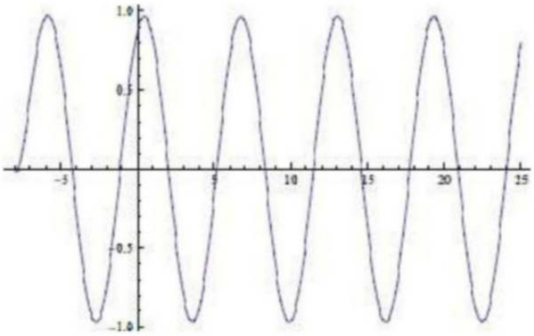


Figure 2.2: Simulation of (CFD) [32] .

From these two simulations with (CFFD) $\alpha = 0.66$, *it appears as the classical is very similar to the (CFD). Otherwise, when we study models with* α *close to* 0, *we see a different behavior.*

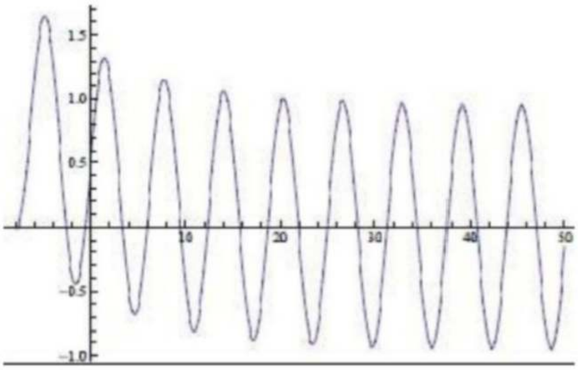


Figure 2.3: Simulation of (CFFD) [32] .

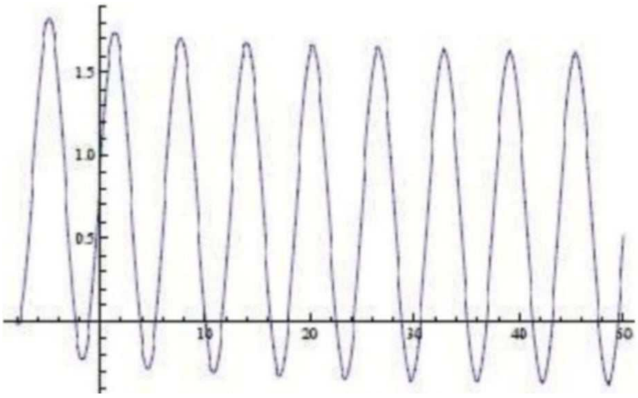


Figure 2.4: Simulation of (CFD) [32] .

So that, for $\alpha = 0 : 1$ *in figuore* 2.3 *and figuore* 2.4 *we observe different actions between (CFFD) and (CFD) simulations. In particular the classical (CFD) is more affected by past,compared with the (CFFD) which show a rapid stabilization.*

If $n \geq 1$*, and* $\alpha \in [0,1]$ *the fractional time derivative CFD* ${}^{CF}D_x^{(\alpha+n)}f(x)$ *of order* $(n+\alpha)$ *is defined by:*

$$ {}^{CF}D_x^{(\alpha+n)}f(x) := {}^{CF}D_x^{(\alpha)}\left({}^{CF}D_x^{(n)}f(x)\right). \tag{2.22}$$

Theorem 2.4.1

For (CFFD), if the function $f(t)$ *is such that*

$$f^{(s)}(a) = 0, \quad s = 1, 2, ..., n \quad ,$$

then, we have [32]

$$ {}^{CF}D_x^{(\alpha)}\left({}^{CF}D_x^{(n)}f(x)\right) = {}^{CF}D_x^{(n)}\left({}^{CF}D_x^{(\alpha)}f(x)\right).$$

Proof.

We begin considering $n = 1$*, then from definition 2.22 of* $\;{}^{CF}D_x^{(\alpha+1)}f(x)$*, we obtain*

$$ {}^{CF}D_x^{(\alpha)}\left({}^{CF}D_x^{(1)}f(x)\right) = \frac{M(\alpha)}{1-\alpha}\int_a^x f''(t)\exp\left[-\frac{\alpha(x-t)}{1-\alpha}\right]dt. $$

Hence, after an integration by parts and assuming $f'(a) = 0$*, we have*

$$ \begin{aligned} {}^{CF}D_x^{(\alpha)}\left({}^{CF}D_x^{(1)}f(x)\right) =& \frac{M((\alpha)}{1-\alpha}\int_a^x \left(\frac{d}{dt}f'(t)\right)\exp\left[-\frac{\alpha(x-t)}{1-\alpha}\right]dt \\ =& \frac{M(\alpha)}{1-\alpha}\left[\int_a^x \frac{d}{dt}f'(t)\exp -\frac{\alpha(x-t)}{1-\alpha}dt - \frac{\alpha}{1-\alpha}\int_a^x f'(t)\exp\frac{\alpha(x-t)}{1-\alpha}dt\right] \\ =& \frac{M(\alpha)}{1-\alpha}\left[f'(x) - \frac{\alpha}{1-\alpha}\int_a^x f'(t)\exp\left(-\frac{\alpha(x-t)}{1-\alpha}\right)dt\right], \end{aligned} $$

otherwise

$$ \begin{aligned} {}^{CF}D_x^{(1)}\left({}^{CF}D_x^{(\alpha)}f(x)\right) =& \frac{d}{dx}\left(\frac{M(\alpha)}{1-\alpha}\int_a^x f'(t)\exp\left(-\frac{\alpha(x-t)}{1-\alpha}\right)dt\right) \\ =& \frac{M(\alpha)}{1-\alpha}\left[f'(x) - \frac{\alpha}{1-\alpha}\int_a^x f'(t)\exp\left(-\frac{\alpha(x-t)}{1-\alpha}\right)dt\right]. \end{aligned} $$

So, can be generalized for any $n > 0$*,*

$$ {}^{CF}D_x^{(\alpha)}\left({}^{CF}D_x^{(n)}f(x)\right) = {}^{CF}D_x^{(n)}\left({}^{CF}D_x^{(\alpha)}f(x)\right). $$

Proposition 2.4.1

- ***Linearity***

Let ${}^{CF}D_x^{(\alpha)}$ *Caputo-Fabrizio operator satisfy :*

$$ {}^{CF}D_x^{(\alpha)}\left(\lambda f(x) + \mu g(x)\right) = \lambda{}^{CF}D_x^{(\alpha)}f(x) + \mu{}^{CF}D_x^{(\alpha)}g(x). $$

2.4.1 Laplace transform of the CFFD

Definition 2.4.1.1

It is well known that Laplace Transform plays an important role in the study of ordinary differential equations. In the case of this new fractional definition (CFFD) with $a = 0$ *the Laplace Transform becomes like this, for* $0 < \alpha < 1$ *[32].*

$$ \mathcal{L}\left[{}^{CF}D_x^{(\alpha)}f(x)\right](s) = \frac{s\mathcal{L}\left[f(x)\right](s) - f(0)}{2(s+\alpha(1-s))} \quad, s > 0. \tag{2.23} $$

- ***Lemma***

The Laplace transform of the Caputo-Fabrizio fractional of order $\sigma = \alpha + n$ for $\alpha \in (0,1)$ and $n \in \mathbb{N}$ is given by :

$$\mathcal{L}\left[{}^{CF}D_x^{(\sigma)} f(x)\right](s) = \frac{s^{n+1}\mathcal{L}\left[f(x)\right](s) - s^n f(0) - s^{n-1}f'(0) - ... - f^{(n)}(0)}{s+\alpha(1-s)}. \tag{2.24}$$

Proof.

We apply the Laplace Transform to definition 2.4.1, we suppose the function $M(\alpha) = 1$,

$$\mathcal{L}\left[{}^{CF}D_x^{(\alpha)} f(x)\right](s) = \frac{1}{1-\alpha}\int_a^\infty \exp(-sx)\int_a^x f'(t)\exp -\frac{\alpha(x-t)}{1-\alpha}dtdx.$$

Hence, from the property of Laplace transform of a convolution, we have :

$$\begin{aligned}
\mathcal{L}\left[{}^{CF}D_x^{(\alpha)} f(x)\right](s) &= \frac{1}{1-\alpha}\mathcal{L}\left[f'(x)\right]\mathcal{L}\left[\exp -\frac{\alpha x}{1-\alpha}\right] \\
&= \frac{1}{1-\alpha}\left(\int_a^\infty f'(x)\exp(-sx)dx\right)\left(\int_a^\infty \exp(-\frac{\alpha x}{1-\alpha}\exp(-sx)dx\right) \\
&= \frac{1}{1-\alpha}\left(-f(0) + s\int_a^\infty f(x)\exp(-sx)dx\right)\left(\int_a^\infty \exp(-x(s+\frac{\alpha}{1-\alpha}))dx\right) \\
&= \left(s\mathcal{L}\left[f(x)\right] - f(0)\right)\left(\frac{1}{s+\alpha(1-s)}\right) \\
&= \frac{s\mathcal{L}\left[f(x)\right] - f(0)}{s+\alpha(1-s)},
\end{aligned}$$

in the same way

$$\begin{aligned}
\mathcal{L}\left[{}^{CF}D_x^{(\alpha)} f(x)\right](s) &= \frac{1}{1-\alpha}\mathcal{L}\left[f''(x)\right]\mathcal{L}\left[\exp -\frac{\alpha x}{1-\alpha}\right] \\
&= \frac{1}{1-\alpha}\left(\int_a^\infty f''\exp(-sx)dx\right) \\
&\left(\int_a^\infty \exp(-\frac{\alpha x}{1-\alpha}\exp(-sx)dx\right) \\
&= \frac{1}{1-\alpha}\left(-f'(0) + s\int_a^\infty f'(x)\exp(-sx)dx\right) \\
&\left(\int_a^\infty \exp(-x(s+\frac{\alpha}{1-\alpha}))dx\right) \\
&= \left(-f'(0) - sf(0) + s^2\int_a^\infty f(x)\exp(-sx)dx\right) \\
&\left(\frac{1}{s+\alpha(1-s)}\right) \\
&= \frac{s^2\mathcal{L}\left[f(x)\right] - sf(0) - f'(0)}{s+\alpha(1-s)}.
\end{aligned}$$

Finally

$$\begin{aligned}
\mathcal{L}\left[{}^{CF}D_x^{(\alpha+n)} f(x)\right](s) &= \frac{1}{1-\alpha}\mathcal{L}\left[f^{(n+1)}(x)\right]\mathcal{L}\left[\exp -\frac{\alpha x}{1-\alpha}\right] \\
&= \frac{s^{n+1}\mathcal{L}\left[f(x)\right] - s^n f(0) - s^{n-1}f'(0) - ... - f^{(n)}(0)}{s+\alpha(1-s)}.
\end{aligned}$$

- **The Inverse Laplace Transform**

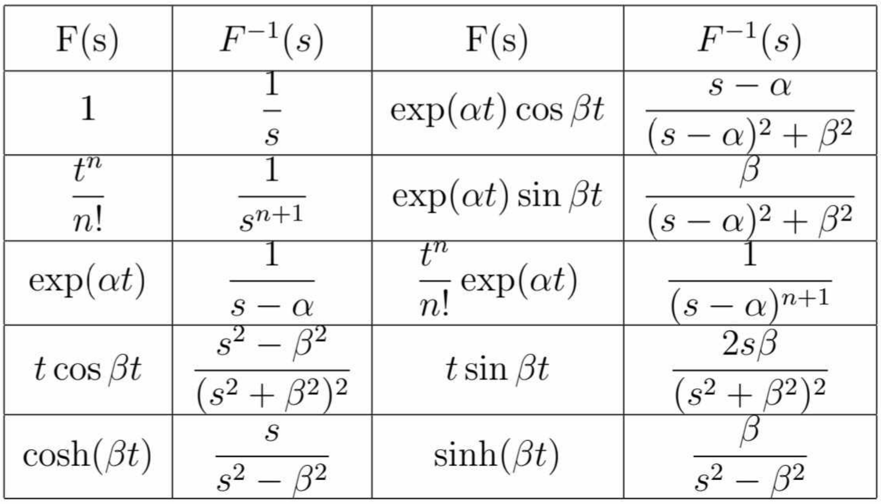
If $G(s) = \mathcal{L}[g(x)](s)$, then the inverse transform of $G(s)$, is defined as [34]:

$$\mathcal{L}^{-1}G(s) = g(x).$$

properties of the inverse Laplace transform

- $\mathcal{L}^{-1}\left[aG_1(s) + bG_2(s)\right] = ag_1(x) + bg_2(x).$
- $\mathcal{L}^{-1}G(s-a) = e^{ax}g(x).$
- $\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right] = \int_0^x g(x)dx.$

- ***Images of basic elementary functions***



F(s)	$F^{-1}(s)$	F(s)	$F^{-1}(s)$
1	$\frac{1}{s}$	$\exp(\alpha t)\cos\beta t$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$	$\exp(\alpha t)\sin\beta t$	$\frac{\beta}{(s-\alpha)^2+\beta^2}$
$\exp(\alpha t)$	$\frac{1}{s-\alpha}$	$\frac{t^n}{n!}\exp(\alpha t)$	$\frac{1}{(s-\alpha)^{n+1}}$
$t\cos\beta t$	$\frac{s^2-\beta^2}{(s^2+\beta^2)^2}$	$t\sin\beta t$	$\frac{2s\beta}{(s^2+\beta^2)^2}$
$\cosh(\beta t)$	$\frac{s}{s^2-\beta^2}$	$\sinh(\beta t)$	$\frac{\beta}{s^2-\beta^2}$

Table: Comparaison between models [34].

2.4.2 The fractional integral associated to the CFFD

After the notion of fractional derivative of order $0 < \alpha < 1$, that of fractional integral of order $0 < \alpha < 1$ becomes a natural requirement. In this section we obtain the fractional integral associated to the Caputo-Fabrizio fractional derivative previously introduced.

Definition 2.4.2.1

the associated fractional integral is defined as:

$$ {}^{CF}I_a^{\alpha} f(x) = \frac{1}{M(\alpha)} \left[(1-\alpha)(f(x) - f(a)) + \alpha \int_a^x f(t)dt \right]. \tag{2.25} $$

Proposition 2.4.2.1

Let $0 < \alpha < 1$, The fractional integral of order α of a function f is defined by [33]:

$$ {}^{CF}I^{(\alpha)} f(x) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f(x) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^x f(s)ds, \quad x \geq 0. \tag{2.26} $$

Proof.

Let $0 < \alpha < 1$, Consider now the following fractional differential equation:

$$ {}^{CF}D^{(\alpha)} f(x) = u(x), \quad x \geq 0, \tag{2.27} $$

using Laplace transform, we obtain:

$$ \mathcal{L}\left[{}^{CF}D^{(\alpha)} f(x) \right](s) = U(s), \quad s > 0. $$

That is, using (2.24), we have that

$$ \frac{(2-\alpha)M(\alpha)}{2(s+\alpha(1-s))}(sF(s) - f(0)) = U(s), \quad s > 0, $$

or equivalently,

$$ sF(s) - f(0) = \frac{2(s+\alpha(1-s))}{(2-\alpha)M(\alpha)} U(s), \quad s > 0, $$

$$ sF(s) = f(0) + \frac{2(s+\alpha(1-s))}{(2-\alpha)M(\alpha)} U(s), \quad s > 0, $$

$$ F(s) = \frac{1}{s} f(0) + \frac{2\alpha}{(2-\alpha)M(\alpha)} U(s) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} U(s), \quad s > 0. $$

Hence, using now well known properties of inverse Laplace transform, we that

$$ \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{1}{s}(f(0))\right] + \frac{2\alpha}{s(2-\alpha)M(\alpha)} \mathcal{L}^{-1}\left[\frac{U(s)}{s}\right] + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \mathcal{L}^{-1}[U(s)]. $$

From the properties of the inverse Laplace transform, we get the following expressions

$$ f(0)\mathcal{L}^{-1}\left[\frac{1}{s}\right] = f(0), \quad \mathcal{L}^{-1}\left[\frac{U(s)}{s}\right] = \int_0^x u(x)dx, \quad \mathcal{L}^{-1}[U(s)] = u(x), $$

$$ f(x) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} u(s) + \frac{2\alpha}{s(2-\alpha)M(\alpha)} \int_0^x u(x)dx + f(0), \quad x \geq 0. \tag{2.28} $$

In other words, the function defined as

$$f(x) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(x) + \frac{2\alpha}{s(2-\alpha)M(\alpha)}\int_0^x u(s)dx + c, \quad x \geq 0, \tag{2.29}$$

where $c \in R$ *is a constant, is also a solution of (2.27).*

we can also rewrite fractional differential equation (2.27) as

$$\frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_0^x \exp(-\frac{\alpha}{1-\alpha}(x-s))f'(s)ds = u(x), \quad x \geq 0. \tag{2.30}$$

Or equivalently

$$\int_0^x \exp(-\frac{\alpha}{1-\alpha}(x-s))f'(s)ds = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(x), \quad x \geq 0,$$

$$\int_0^x \exp(-\frac{\alpha x}{1-\alpha})\exp(\frac{\alpha s}{1-\alpha})f'(s)ds = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(x), \quad x \geq 0,$$

$$\exp(-\frac{\alpha x}{1-\alpha})\int_0^x \exp(\frac{\alpha s}{1-\alpha})f'(s)ds = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(x), \quad x \geq 0,$$

$$\int_0^x \exp(\frac{\alpha s}{1-\alpha})f'(s)ds = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\exp(\frac{\alpha x}{1-\alpha})u(x), \quad x \geq 0.$$

Differentiating both sides of the latter equation, we obtain that,

$$\exp(\frac{\alpha x}{1-\alpha})f'(x) = \exp(\frac{\alpha x}{1-\alpha})\left(\frac{2(1-\alpha)}{1-\alpha}\right)\left[\frac{\alpha}{1-\alpha}u(x) + u'(x)\right],$$

$$f'(x) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left(\frac{\alpha}{1-\alpha}u(x) + u'(x)\right), \quad x \geq 0.$$

Hence, integrating now from 0 *to* x*, we deduce as in (2.28), that*

$$\int_0^x f'(x)dx = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\int_0^x \left[u'(s) + \frac{\alpha}{1-\alpha}u(s)\right]ds,$$

$$[f(x) - f(0)] = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}[u(x) - u(0)] + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\int_0^x u(s)ds,$$

$$f(x) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}[u(x) - u(0)] + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\int_0^x u(s)ds + f(0), \quad x \geq 0. \tag{2.31}$$

2.4.3 Composition of CFFD Operators

Here we present a theoretical property related to the Caputo-Fabrizio fractional derivative.

Theorem 2.4.3.1

Let be $n \in \mathbb{N} - \{0\}, a, b \in \mathbb{R}(a < b)$ *and* $u \in C^n([a,b])$*. Then the equality*

$$\frac{d^n}{dx^n}({}^{CF}D_{ax}^{(\alpha)}u(x)) = \sum_{i=1}^{n}(-1)^{n-i}\frac{a^{n-i}}{(1-a)^{n+1-i}}u^{(i)}(x) + (-1)^n(\frac{\alpha}{1-\alpha})^{n\,CF}D_{ax}^{(\alpha)}u(x),$$

is true [35].

Proof: *See ([35]).*

Corollary 2.4.3.1

Let be $a, b \in \mathbb{R}(a < b)$ *and* $C^1([a,b])$*, Then the equality*

$$\int_a^b ({}^{CF}D_{ax}^{(\alpha)}u(x))dx = \frac{1}{\alpha}(u(b) - u(a)) - \frac{1-\alpha}{\alpha}({}^{CF}D_{ab}^{(\alpha)}u(b)),$$

is true.

In this section, we give some theoretical properties concerning the composition of Caputo- Fabrizio fractional operators.

Theorem 2.4.3.2

Let be $a, \alpha, \beta \in \mathbb{R}$ *such that* $0 < \alpha, \beta < 1(\alpha \neq \beta)$*, Then the equality*

$$ {}^{CF}D_{ax}^{(\alpha)}({}^{CF}D_{ax}^{(\beta)}u(x)) = \frac{1}{\beta - \alpha}(\beta \quad {}^{CF}D_{ax}^{(\beta)}u(x) - \alpha \quad {}^{CF}D_{ax}^{(\alpha)}u(x)), \tag{2.32}$$

is true [35].

Proof.

On the first side, from the definition of Caputo-Fabrizio derivative, we deduce that

$$\begin{aligned}
{}^{CF}D_{ax}^{(\alpha)}({}^{CF}D_{ax}^{(\beta)}u(x)) &= \frac{1}{1-\alpha}\int_a^x \exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right)\left({}^{CF}D_{at}^{(\beta)}u(t)\right)' dt \\
&= \frac{1}{1-\alpha}\frac{1}{1-\beta}\int_a^x \exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right) \\
&\left[u'(t) - \int_a^t \frac{\beta}{1-\beta}\exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right)u'(s)dx\right]dt \\
&= \frac{1}{1-\alpha}\frac{-\beta}{1-\beta}\int_a^x \exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right){}^{CF}D_{at}^{(\beta)}u(t)dt + \frac{1}{1-\beta}{}^{CF}D_{ax}^{(\beta)}u(x),
\end{aligned}$$

which is equivalent to

$$\int_a^x \exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right){}^{CF}D_{at}^{(\beta)}u(t)dt = -\frac{(1-\alpha)(1-\beta)}{\beta}{}^{CF}D_{ax}^{(\alpha)}\left({}^{CF}D_{ax}^{(\beta)}u(x)\right) + \frac{1-\alpha}{\beta}{}^{CF}D_{ax}^{(\alpha)}u(x). \tag{2.33}$$

On the other side, integrating by parts and considering that

$${}^{CF}D_{aa}^{(\alpha)}u(a) = 0,$$

we obtain

$$\begin{aligned}
{}^{CF}D_{ax}^{(\alpha)}({}^{CF}D_{ax}^{(\beta)}u(x)) &= \frac{1}{1-\alpha}\int_a^x \exp-\frac{\alpha}{1-\alpha}(x-t)\left({}^{CF}D_{at}^{(\beta)}u(t)\right)' dt \\
&= \frac{1}{1-\alpha}{}^{CF}D_{ax}^{(\beta)}u(x) \\
&- \frac{\alpha}{(1-\alpha)^2}\int_a^x \exp-\frac{\alpha}{1-\alpha}(x-t)\ {}^{CF}D_{at}^{(\beta)}u(t)dt,
\end{aligned}$$

which is equivalent to

$$\int_a^x \exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right) {}^{CF}D_{at}^{(\beta)}u(t)dt = \frac{1-\alpha}{\alpha}\ {}^{CF}D_{ax}^{(\beta)}u(x) - \frac{(1-\alpha)^2}{\alpha}\ {}^{CF}D_{ax}^{(\alpha)}\left({}^{CF}D_{ax}^{(\beta)}u(x)\right). \tag{2.34}$$

Combining (2.33) with (2.34), we obtain

$$\left(\frac{(1-\alpha)(1-\beta)}{\beta} - \frac{(1-\alpha)^2}{\alpha}\right) {}^{CF}D_{ax}^{(\alpha)}\left({}^{CF}D_{ax}^{(\beta)}u(x)\right) = \left(-\frac{1-\alpha}{\beta} + \frac{1-\alpha}{\alpha}\right) {}^{CF}D_{ax}^{(\beta)}u(x).$$

$$\left(\frac{(1-\beta)}{\beta} - \frac{(1-\alpha)}{\alpha}\right) {}^{CF}D_{ax}^{(\alpha)}\left({}^{CF}D_{ax}^{(\beta)}u(x)\right) = -\frac{1}{\beta}{}^{CF}D_{ax}^{(\alpha)} + \frac{1}{\alpha}{}^{CF}D_{ax}^{(\beta)}u(x).$$

$$\left(-\frac{\beta-\alpha}{\alpha\beta}\right) {}^{CF}D_{ax}^{(\alpha)}\left({}^{CF}D_{ax}^{-(\beta)}u(x)\right) = -\frac{1}{\beta}\ {}^{CF}D_{ax}^{(\alpha)} + \frac{1}{\alpha}\ {}^{CF}D_{ax}^{(\beta)}u(x).$$

$${}^{CF}D_{ax}^{(\alpha)}\left({}^{CF}D_{ax}^{(\beta)}u(x)\right) = \frac{-\alpha}{\beta-\alpha}\ {}^{CF}D_{ax}^{(\beta)}u(x).$$

$${}^{CF}D_{ax}^{(\alpha)}\left({}^{CF}D_{ax}^{(\beta)}u(x)\right) = \frac{1}{\beta-\alpha}\left(\beta\ {}^{CF}D_{ax}^{(\beta)}u(x) - \alpha\ {}^{CF}D_{ax}^{(\alpha)}u(x)\right).$$

Theorem 2.4.3.3

Let be $a, \alpha \in R$ *such that* $0 < \alpha < 1$, *Then the equality*

$${}^{CF}D_{ax}^{(\alpha)}\left({}^{CF}I_{ax}^{(\alpha)}u(x)\right) = u(x) - \exp\left(-\frac{\alpha}{1-\alpha}(x-a)\right)u(a), \tag{2.35}$$

is true [35].

Proof.

$$\begin{aligned}
{}^{CF}D_{ax}^{(\alpha)}\left({}^{CF}I_{a\alpha}^{(\alpha)}u(x)\right) &= \frac{1}{1-\alpha}\int_a^x \exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right)\left[I_{at}^{(\alpha)}u(t)\right]' dt, \\
&= \frac{1}{1-\alpha}\int_a^x \exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right)\left[(1-\alpha)u(t) + \alpha\int_a^t u(s)ds\right]' dt, \\
&\int_a^x \exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right)u'(t)dt + \frac{\alpha}{1-\alpha}\int_a^x \exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right)u(t)dt, \\
&= \int_a^x \exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right)u'(t)dt + u(x) - \exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right)u(a) \\
&- \int_a^x \exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right)u'(t)dt \\
&= u(x) - \exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right)u(a).
\end{aligned}$$

Theorem 2.4.3.4

Let $a, \alpha \in \mathbb{R}$ *such that* $0 < \alpha < 1$. *Then the equality*

$${}^{CF}I_{ax}^{(\alpha)}\left({}^{CF}D_{ax}^{(\alpha)}u(x)\right) = u(x) - u(a),$$

is true [35].

Proof.

On the one side, using definition of Caputo-Fabrizio integral, we obtain

$$ {}^{CF}I^{(\alpha)}_{ax}\left({}^{CF}D^{(\alpha)}_{ax}u(x)\right) = (1-\alpha)\ {}^{CF}D^{(\alpha)}_{ax}u(x) + \alpha\int_a^x\ {}^{CF}D^{(\alpha)}_{as}u(s)ds. \tag{2.36} $$

On the other side, applying Theorem (2.4.3.1), we obtain

$$ \frac{d}{ds}\left[{}^{CF}D^{(\alpha)}_{as}u(s)\right] = -\frac{\alpha}{1-\alpha}{}^{CF}D^{(\alpha)}_{as}u(s) + \frac{1}{1-\alpha}u'(s). \tag{2.37} $$

Integrating (2.37) respect to s over (a,x) to t and considering that ${}^{CF}D^{(\alpha)}_{aa}u(a) = 0$, we obtain

$$ \begin{aligned} {}^{CF}D^{(\alpha)}_{ax}u(x) &= -\frac{\alpha}{1-\alpha}\int_a^x\ {}^{CF}D^{(\alpha)}_{ax}u(s)ds + \frac{1}{1-\alpha}\int_a^x u'(s)ds, \\ {}^{CF}D^{(\alpha)}_{ax}u(x) - \frac{1}{1-\alpha}\left[u(x)-u(a)\right] &= -\frac{\alpha}{1-\alpha}\int_a^x\ {}^{CF}D^{(\alpha)}_{ax}u(s)ds, \\ \int_a^x\ {}^{CF}D^{(\alpha)}_{as}u(s)ds &= \frac{1}{\alpha}\left[u(x)-u(a)\right] - \frac{1-\alpha}{\alpha}\ {}^{CF}D^{(\alpha)}_{ax}u(x). \end{aligned} \tag{2.38} $$

Inserting the right hand side of (2.38) into (2.36), we obtain

$$ {}^{CF}I^{(\alpha)}\left({}^{CF}D^{(\alpha)}_{ax}u(x)\right) = (1-\alpha)\ {}^{CF}D^{(\alpha)}_{ax}u(x) + \left[u(x)-u(a)\right] - (1-\alpha)\ {}^{CF}D^{(\alpha)}_{ax}u(x) = u(x) - u(a). $$

Theorem 2.4.3.5

Let $0 < \alpha < 1, a \in \mathbb{R}$, Then the equality

$$ {}^{CF}I^{(\alpha)}_{ax}\left(I^{(\alpha)}_{ax}u(x)\right) = (1-\alpha)I^{(\alpha)}_{ax}u(x) + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}I^{(\alpha+1)}_{ax}u(x), $$

is true [35].

Proof.

$$ \begin{aligned} {}^{CF}I^{(\alpha)}_{ax}\left(I^{(\alpha)}_{ax}u(x)\right) &= (1+\alpha) + I^{(\alpha)}_{ax}u(x) + \alpha\int_a^x I^{(\alpha)}_{at}u(t)dt, \\ &= (1-\alpha)I^{(\alpha)}_{ax}u(x) + \alpha\int_a^x\left[\frac{1}{\Gamma(\alpha)}\int_a^t (t-\xi)^{\alpha-1}u(\xi)d\xi\right]dt, \\ &= (1-\alpha)I^{(\alpha)}_{ax}u(x) + \frac{\alpha}{\Gamma(\alpha)}\int_a^x u(\xi)d\xi\int_\xi^x (t-\xi)^{\alpha-1}dt, \\ &= (1-\alpha)I^{(\alpha)}_{ax}u(x) + \frac{\alpha}{\Gamma(\alpha)}\int_a^x \frac{1}{\alpha}(x-\xi)^{\alpha}u(\xi)d\xi, \\ &= (1-\alpha)I^{(\alpha)}_{ax}u(x) + \frac{1}{\Gamma(\alpha)}\int_a^x (x-\xi)^{(\alpha+1)-1}u(\xi)d\xi, \\ &= (1-\alpha)I^{(\alpha)}_{ax}u(x) + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)(\alpha+1)}\int_a^x (x-\xi)^{(\alpha+1)-1}u(\xi)d\xi, \\ &= (1-\alpha)I^{(\alpha)}_{ax}u(x) + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}I^{(\alpha+1)}_{ax}u(x). \end{aligned} $$

CHAPTER 3

AN ANALYTICAL AND THEORETICAL STUDY

In this chapter, we mention the definitions of Caputo-Fabrizio fractional and integral derivatives as well as their main properties.

And, we apply and investigate the ***Picard-Lindelof technique*** *and the* ***Banach fixed point theorem****,the* ***numerical approximation*** *to study analysis and theoretically the following system [4] :*

$$\begin{cases} {}^{CF}D^{(\alpha)}f(x) = c_1 f(x) + c_2 g(x) + u(x), 0 \leq x \leq 1, \\ {}^{CF}D^{(\alpha)}g(x) = c_3 f(x) + c_4 g(x) + v(x), 0 \leq x \leq 1, \\ f(0) = g(0) = 0. \end{cases} \tag{3.1}$$

3.1 Definitions and main characteristics of Caputo-Fabrizio derivatives

Definition 3.1.1 *The fractional derivative* ***Caputo-Fabrizio*** *of the function $f \in H^1[a,b]$, $b > a$ of order $\alpha \in]0,1[$ is given as follows [4]:*

$${}^{CF}D_x^{(\alpha)}f(x) = \frac{M(\alpha)}{1-\alpha}\int_a^t f'(\mu) exp\left[-\frac{\alpha(x-\mu)}{1-\alpha}\right] d\mu. \tag{3.2}$$

Definition 3.1.2 *for $\alpha \in [0,1]$,for all $n \geqslant 1$ The fractional derivative ${}^{CF}D^{(n+\alpha)}$ defined by [36]:*

$${}^{CF}D^{(\alpha+n)}f(x) = \frac{M(\alpha)}{1-\alpha}\int_t^a f^{(n+1)}(x) exp\left[-\frac{\alpha(x-\mu)}{1-\alpha}\right] d\mu.$$

Note

For $n \geq 1$, $\alpha \in [0,1]$we get [36]:

$${}^{CF}D^{(\alpha+n)}f(x) = ^{CF} D^{(\alpha)}\left({}^{CF}D^{(n)}f(x)\right).$$

Definition 3.1.3

The fractional integral ***Caputo-Fabrizio*** *is defined as follows [4]:*

$$I_a^{\alpha}f(x) = \frac{1}{M(\alpha)}\left[(1-\alpha)(f(x) - f(a)) + \alpha\int_a^t f(\mu)d\mu\right]. \tag{3.3}$$

Where $M(\alpha)$ is a normalization function that has the following main properties $M(0) = M(1) = 1$.

Theorem 3.1.1

For all $n \geq 1$,and $f \in \mathscr{C}^1[a,b]$,for all $\alpha \in [0,1]$, we get [36]:

$$I^{n+\alpha}f(x) = \frac{1}{M(\alpha).n!}\int_a^t (x-\mu)^{n-1}[\alpha(x-\mu) + n(1-\alpha)]f(\mu)d\mu.$$

$M(\alpha)$ is a normalization function.

Proof.

By definition 3.1.1 and 3.1.2 we get [36]:

$$^{CF}D^{(\alpha+n)}f(x) = \frac{M(\alpha)}{1-\alpha}\int_t^a f^{(n+1)}(x)exp\left[-\frac{\alpha(x-\mu)}{1-\alpha}\right]d\mu,$$

and the Leibniz integral rule gives the formula

$$\frac{d}{ds}\left(^{CF}D^{(\alpha+n)}f(x)\right) = \frac{M(\alpha)}{1-\alpha}f^{(n+1)}(x) - \frac{\alpha}{1-\alpha}\frac{M(\alpha)}{1-\alpha}\int_t^a f^{(n+1)}(x)exp\left[-\frac{\alpha(x-\mu}{1-\alpha}\right]d\mu,$$

$$\begin{aligned}\Rightarrow&\frac{d}{ds}\left(^{CF}D^{(\alpha+n)}f(x)\right) = \frac{M(\alpha)}{1-\alpha}f^{(n+1)}(x) - \frac{\alpha}{1-\alpha}{}^{CF}D^{(\alpha+n)}f(x),\\ \Rightarrow&f^{(n+1)}(x) = \frac{1}{M(\alpha)}\left[(1-\alpha)\frac{d}{ds}\left(^{CF}D^{(\alpha+n)}f(x)\right) + \alpha^{CF}D^{(\alpha+n)}f(x)\right],\end{aligned}$$

we now use the Cauchy formula for evaluating the $(n+1)^{th}$ integration of the function $f(n+1)(x)$

$$f(x) = \frac{1}{n!M(\alpha)}\int_a^t (x-\mu)^n\left[(1-\alpha)\frac{d}{ds}\left(^{CF}D^{(\alpha+n)}f(x)\right) + \alpha^{CF}D^{(\alpha+n)}f(x)\right]d\mu.$$

$$\begin{aligned}\Rightarrow I^{n+\alpha}f(x) =&\frac{1}{M(\alpha).n!}\int_a^t (x-\mu)^n[\alpha f(\mu) + (1-\alpha)f'(\mu)]d\mu,\\ =&\frac{1}{M(\alpha).n!}\int_a^t (x-\mu)^{(n-1)}[\alpha(x-\mu) + n(1-\alpha)]f(\mu)d\mu.\end{aligned}$$

Lemma 3.1.1

Let $\lambda \in (n, n+1), n = [\lambda]$. And $f \in \mathscr{C}^n[a,b]$,then those statements hold :

1. *if $f(a) = 0$, then $^{CF}D_x^{(\lambda)}(I_a^\lambda f(x)) = f(x)$, [4].*
2. $I_a^\lambda\left(^{CF}D_x^{(\lambda)}f(x)\right) = f(x) + \sum_{i=0}^n a_i x^i, a_i \in \mathbb{R} \quad i = 0, 1, ..., n$, *[4] .*
3. *if $\alpha \in (0,1)$,then $^{CF}I_a^\alpha\left(^{CF}D_a^{(\alpha)}f(x)\right) = f(x) - f(a)$, [35] .*

   Proof

   It can be written in the form $\lambda = n + \alpha$, [36].

(a)

$$
\begin{aligned}
{}^{CF}D^{(\lambda)}(I_a^{\lambda}f(x)) =& \frac{\alpha}{(1-\alpha)}\int_a^t \frac{d^{(n+1)}}{dx^{(n+1)}}\left[\frac{1}{n!}\int_a^x (x-t)^n f(x)dt\right] exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right] dx \\
&+\int_a^t \frac{d^{(n+1)}}{dx^{(n+1)}}\left[\frac{1}{n!}\int_a^x (x-t)^n f'(x)dt\right] exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right] dx \\
=& \frac{\alpha}{(1-\alpha)}\int_a^t f(x)exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right] dx + \int_a^t f'(x)exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right] dx \\
=& \frac{\alpha}{(1-\alpha)}\int_a^t f(x)exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right] dx \\
&+ f(x)exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right] \Big|_a^x - \frac{\alpha}{(1-\alpha)}\int_a^t f(x)exp\left[-\frac{\alpha(t-x)}{1-\alpha}\right] dx \\
=& f(x) - f(a)exp\left[-\frac{\alpha(t-a)}{1-\alpha}\right] = f(x).
\end{aligned}
$$

(b)

$$
\begin{aligned}
I_a^{\lambda}\left({}^{CF}D_x^{(\lambda)}f(x)\right) =& \frac{1}{M(\alpha).n!}\int_a^t (t-x)^{n-1}[\alpha(t-x)+n(1-\alpha)]^{CF}D^{(\lambda)}f(x)dx, \\
=& \frac{\alpha}{M(\alpha).n!}\int_a^t (t-x)^{n\,CF}D^{(\lambda)}f(x)dx \\
&+ \frac{(1-\alpha)}{M(\alpha).n!}\int_a^t (t-x)^n \frac{d}{dx}\left({}^{CF}D^{(\lambda)}f(x)\right) dx \\
=& \frac{\alpha}{M(\alpha).n!}\int_a^t (t-x)^{n\,CF}D^{(\lambda)}f(x)dx \\
&+ \frac{(1-\alpha)}{M(\alpha).n!}\int_a^t (t-x)^n \left(\frac{M(\alpha)}{1-\alpha}f^{(n+1)}(x) - \frac{\alpha}{1-\alpha}{}^{CF}D^{(\lambda)}f(x)\right) dx \\
=& \frac{1}{n!}\int_a^t (t-x)^n f^{(n+1)}(x)dx = f(x) + \sum_{i=0}^{n} a_i t^i. \quad a_i \in \mathbb{R} i = 0, 1, ..., n.
\end{aligned}
$$

Proposition 3.1.1

(a) *And let there be* $0 < \alpha < 1$ *and* f *be a solution of the*

$$ {}^{CF}D^{(\alpha)}f(x) = 0, \qquad x \geqslant 0. $$

Then, f *is a constant function [34].*

(b) *For* $\gamma \in [n, n+1]$ *and* $f^{(n)} \in H^1(a, b)$*, that suppose* $\alpha = \gamma - n$*,and* $\alpha \in [0, 1]$ *we get [35].*

$$ {}^{CF}I_x^{\gamma}f(x) = I^{(n)CF}I_x^{\alpha}f^{(n)}(x). $$

(c) *Let* $\gamma \in [n, n+1]$*, and* $f(x)$ *defined in* (a, b) *we have [35]*

$$ {}^{CF}I_x^{\gamma CF}D^{\gamma}{}_x f(x) = f(x) - \sum_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k. $$

3.2 The associated linear system

Lemma 3.2.1

Let $0 < \alpha > 1$,$r, w \in C^1(\Omega), f, g : \Omega \longrightarrow \mathbb{R}$ *be given continuous functions, and* c_i *real constants and* $i = 1, 2, 3, 4$.

Then the solution of linear coupled system (3.1) is given by [4]:

$$U(x) = H(x) + (1-\alpha)(c_1U(x) + c_2V(x)) + \alpha\int_0^t (c_1U(\xi) + c_2V(\xi))d\xi, \tag{3.4}$$

$$V(x) = P(x) + 1(1-\alpha)(c_3(x)U(x) + c_4V(x)) + \alpha\int_0^t (c_3U(\xi) + c_4V(\xi))d\xi. \tag{3.5}$$

So

$H(x) = (1-\alpha)(f(x) - f(0)) + \alpha\int_0^t f(\xi)d\xi$,*and* $P(x) = (1-\alpha)(g(x) - g(0)) + \alpha\displaystyle\int_0^t g(\xi)d\xi$.

Proof

Through Lemma 3.2.1, we can write Eq. (3.1) to the following equivalent integral equations [4] :

$$U(x) + \epsilon = (1-\alpha)(c_1U(x) + c_2V(x) + f(x) - f(0)) + \alpha\int_0^t (c_1U(\xi) + c_2V(\xi) + f(\xi))d\xi, \tag{3.6}$$

$$V(x) + \varepsilon = (1 - \alpha(c_3U(x) + c_4V(x) + g(x)g(0)) + \alpha\int_0^t (c_3U(\xi) + c_4V(\xi) + g(\xi))d\xi.$$

So $\epsilon, \varepsilon \in \mathbb{R}$. *Using initial conditions* $U(0) = V(0) = 0$, *We find* $\epsilon = \varepsilon = 0$.

Hence(3.6) writes

$$U(x) = (1-\alpha)(c_1U(x) + c_2V(x) + f(x) - f(0)) + \alpha\int_0^t (c_1U(\xi) + c_2V(\xi) + f(\xi))d\xi, \tag{3.7}$$

$$V(x) = (1-\alpha)(c_3U(x) + c_4V(x) + g(x) - g(0)) + \alpha\int_0^t (c_3U(\xi) + c_4V(\eta) + g(\xi))d\xi.$$

Hence equation (3.7) is written

$$U(x) = (1-\alpha)(f(x) - f(0)) + \alpha\int_0^t f(\xi)d\xi + (1-\alpha)(c_1U(x) + c_2V(x)) + \alpha\int_0^t c_1U(\xi) + c_2V(\xi))d\xi, \tag{3.8}$$

$$V(x) = (1-\alpha) + g(x) - g(0)) + \alpha\int_0^t g(\xi)d\xi + (1-\alpha)(c_3U(x) + c_4V(x)) + \alpha\int_0^t (c_3U(\xi) + c_4V(\xi))d\xi.$$

In the end, the unique solution of problem (3.1) is

$$U(x) = H(x) + (1-\alpha)(c_1U(x) + c_2V(x)) + \alpha\int_0^t c_1U(\xi) + c_2V(\xi))d\xi,$$

$$V(x) = P(x) + (1-\alpha)(c_3U(x) + c_4V(x)) + \alpha\int_0^t (c_3U(\xi) + c_4V(\xi))d\xi.$$

3.3 Existence and uniqueness of the solution

Let $(U_0(x), V_0(x)) = (H(x), P(x))$*, Picard iteration is defined as follows* [4]

$$U_{i+1}(x) = (1-\alpha)(c_1U_i(x) + c_2V_i(x)) + \alpha\int_0^t (c_1U_i(\xi) + c_2V_i(\xi))d\xi, \tag{3.9}$$
$$V_{i+1}(x) = (1-\alpha)(c_3U_i(x) + c_4V_i(x)) + \alpha\int_0^t (c_3U_i(\xi) + c_4V_i(\xi))d\xi.$$

In order to clarify the unity of the solution, we know

$$Z_1(x, U, V) = c_1U + c_2V, \tag{3.10}$$
$$Z_2(x, U, V) = c_3U + c_4V,$$

and $\varphi(x) = (U(x), V(x))$.

Lemma 3.3.1

Let $Z_1, Z_2 : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be a continuous function. Then* $Z_1(x, U, V)$ *and* $Z_2(x, U, V)$ *are contraction with respect to* U *and* V *if* [4]

$$\theta_1 < 1 \qquad and \qquad \theta_2 < 1, \tag{3.11}$$

where $\theta_1 = max(\mid c_1 \mid, \mid c_2 \mid), \quad \theta_2 = max(\mid c_3 \mid, \mid c_4 \mid)$.

Proof

Let $U_i, V_i \in \mathbb{R}, \quad i = 1, 2$

$\forall x \in \Omega$*, we have*

$$\begin{aligned} \mid Z_1(x, U_1, V_1) - L_1(x, U_2, V_2) \mid \leqslant & \mid c_1 \mid\parallel U_1(x) - V_2(x) \parallel + \mid c_2 \mid\parallel V_1(x) - V_2(x) \parallel \\ \leqslant & max(\mid c_1 \mid, \mid c_2 \mid)(\parallel U_1(x) - U_2(x) \parallel + \parallel V_1(x) - V_2(x) \parallel) \\ \leqslant & \theta_1 \parallel \varphi_1(x) - \varphi_2(x) \parallel . \end{aligned}$$

In the same manner we find :

$$\mid Z_2(x, U_1, V_1) - L_2(x, U_2, V_2) \mid \leqslant \theta_2 \parallel \varphi_1(x) - \varphi_2(x) \parallel .$$

Hence,

$$\parallel Z_1(x, U_1, V_1) - Z_1(x, U_2, V_2) \parallel \leqslant \theta_1 \parallel \varphi_1(x) - \varphi_2(x) \parallel, \tag{3.12}$$
$$\parallel Z_2(x, U_1, V_1) - Z_2(x, U_2, V_2) \parallel \leqslant \theta_2 \parallel \varphi_1(x) - \varphi_2(x) \parallel . \tag{3.13}$$

which, in view of (3.11), implies that $Z_1(x, U, V)$ *and* $Z_2(x, U, V)$ *are contraction with respect to* U *and* V.

Threorem 3.3.1

Let $Z_1, Z_2 : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be a continuous function.Therefore, the system has a unique solution in* Ω*, which is known as such*

$$\theta = \theta_1 + \theta_2 < 1.$$

So $\theta_1 = max(\mid c_1 \mid, \mid c_2 \mid), \quad \theta_2 = max(\mid c_3 \mid, \mid c_4 \mid)$.*[4]*

Proof.

In order to show the existence of a unique solution, we have $Z_1(x, U, V)$ *and* $Z_2(x, U, V)$ *are contraction with respect to* U *and* V. *Then the Picard operator can there fore be defined as follows:*

$$\omega(\varphi(x)) = \varphi_0 + (1 - \alpha)\delta(x, \varphi(x)) + \alpha \int_0^t \delta(\xi, \varphi(\xi))d\xi. \tag{3.14}$$

where $\delta(x\varphi(x)) = (Z_1(x, U(x), V(x)), Z_2(x, U(x), V(x)) and \varphi_0 = (H(x), P(x))$. *We know from the above that the solution to the fractional problem is bounded. In addition,*Z_1 *and* Z_2 *are contractions,*

so they count as

$$\parallel \delta(x, \varphi_1(x)) - \delta(x, \varphi_2(x)) \parallel \leqslant \theta \parallel \varphi_1(x) - \varphi_2(x) \parallel . \tag{3.15}$$

So $\theta = \theta_1 + \theta_2 < 1$. *Also, by using Eq 3.5, we obtain*

$$\begin{aligned}\parallel \varphi(x) - \varphi_0 \parallel =& \parallel (1 - \alpha)\delta(x, \varphi(x)) + \alpha \int_0^t \delta(\xi, \varphi(\xi))d\xi \parallel \\ \leqslant&(1 - \alpha) \parallel \delta(x, \varphi(x)) \parallel + \alpha \int_0^t \parallel \delta(\xi, \varphi(\xi)) \parallel d\xi \\ \leqslant&(1 - \alpha + \alpha x)\theta \\ \leqslant&\theta.\end{aligned} \tag{3.16}$$

where $\theta < 1$.

Now, by using the definition of the Picard operator ((3.15),we prove the contraction property of ω*. We have*

$$\begin{aligned}\| \omega(\delta_1(x)) - \omega(\delta_2(x)) \| = & \| (1-\alpha)(\delta(x,\varphi_1(x)) - \delta(x,\varphi_2(x))) + \alpha \int_0^t (\delta(\xi,\varphi_1(\xi)) - \delta(\xi,\varphi_2(\xi)))d\xi \| \\ & \leqslant (1-\alpha) \| \delta(x,\varphi_1(x)) - \delta(x,\varphi_2(x)) \| + \alpha \int_0^t \| \delta(\xi,\varphi_1(\xi)) - \delta(\xi,\varphi_2(\xi)) \| d\xi \\ & \leqslant (1-\alpha)\theta \| \varphi_1(x) - \varphi_2(x) \| + \alpha\theta \int_0^t \| \varphi_1(\xi) - \varphi_2(\xi) \| d\xi \qquad (3.17) \\ & \leqslant (1-\alpha+\alpha x)\theta \| \varphi_1(x) - \varphi_2(x) \| \\ & \leqslant \theta \| \varphi_1(x) - \varphi_2(x) \|,\end{aligned}$$

since $\theta < 1$ *by Eq.(3.17) . Therefore, the defined operator* ω *is a contraction.*
Thus, by Banach's fixed point theorem [30], the operator ω *has a unique fixed point, which is the unique solution of (3.1).*

3.4 Caputo-Fabrizio approximations

$$^{Cf}D_a^{\alpha} f(x) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(\mu) exp\left[-\frac{\alpha(x-\mu)}{1-\alpha}\right] d\mu. \qquad (3.18)$$

For some positive integer N, the grid size in time for finite difference technique is defined by $z = \dfrac{1}{N}$.
The grid points in the time interval [0,X] are labeled $x_n = nz$,$n = 0, 1, 2, ..., XN$*. The value of the function* f *at the grid point is* $f_i = f(x_i)$ *[37] .*

A discrete approximation to the Caputo-Fabrizio derivative of fractional order can be obtained by simple quadrature formula as follows:

$$^{Cf}D_a^{\alpha} f(x_n) = \frac{M(\alpha)}{1-\alpha} \int_0^{x_n} f'(t) exp\left[-\frac{\alpha(x_n-\mu)}{1-\alpha}\right] d\mu. \qquad (3.19)$$

When using the first-order approximation, the equation becomes

$$^{Cf}D_a^{\alpha} f(x_i) = \frac{M(\alpha)}{1-\alpha} \sum_{i=1}^{n} \int_{(i-1)z}^{iz} \left(\frac{f^{z+1} - f^z}{\Delta x} + O(\Delta x)\right) exp\left[-\alpha \frac{x_i - \mu}{1-\alpha}\right] d\mu. \qquad (3.20)$$

Before integration we get

$$\frac{M(\alpha)}{1-\alpha} \sum_{i=1}^{n} \left(\frac{f^{i+1} - f^i}{\Delta x} + O(\Delta x)\right) \int_{(i-1)z}^{iz} exp\left[-\alpha \frac{x_n - \mu}{1-\alpha}\right] d\mu, \qquad (3.21)$$

$$^{Cf}D_a^\alpha f(x_i)\frac{M(\alpha)}{\alpha}\sum_{i=1}^{n}\left(\frac{f^{i+1}-f^i}{\Delta x}+O(\Delta x)\right)d_{i,z}.$$

So

$$d_{i,z}=exp\left[-\alpha\frac{z}{1-\alpha}(n-i+1)\right]-exp\left[-\alpha\frac{z}{1-\alpha}(n-i)\right]. \tag{3.22}$$

In the end we get

$$^{Cf}D_a^\alpha f(x_i)=\frac{M(\alpha)}{\alpha}\sum_{n}^{i=1}\left(\frac{f^{i+1}-f^i}{\Delta x}\right)d_{j,z}+\frac{M(\alpha)}{\alpha}\sum_{i=1}^{n}d_{j,z}O(\Delta x). \tag{3.23}$$

Theorem 3.4.1

Let $f(x)\in C^2[a,b]$,and let the order of the fractional derivative be $0<\alpha\leq 1$,So the first-order approximation of the Caputo-Fabrizio derivative at a point x_n is [37]

$$^{Cf}D_a^\alpha f(x_i)=\frac{M(\alpha)}{\alpha}\sum_{n}^{i=1}\left(\frac{f^{i+1}-f^i}{\Delta x}\right)d_{i,z}+O((\Delta x)^2). \tag{3.24}$$

Proof.

From equation (3.22) we have [37] :

$$\begin{aligned} ^{Cf}D_a^\alpha f(x_i)=&\frac{M(\alpha)}{\alpha}\sum_{n}^{i=1}\left(\frac{f^{i+1}-f^i}{\Delta x}\right)d_{j,k}\\ &+\frac{M(\alpha)}{\alpha}\sum_{n}^{i=1}\left(exp\left[-\alpha\frac{z}{1-\alpha}(n-i+1)\right]-exp\left[-\alpha\frac{z}{1-\alpha}(n-i)\right]\right)O(\Delta x).\end{aligned}$$

However

$$\sum_{n}^{i=1}\left(exp\left[-\alpha\frac{z}{1-\alpha}(n-i+1)\right]-exp\left[-\alpha\frac{z}{1-\alpha}(n-i)\right]\right)=exp\left[-\alpha\frac{z}{1-\alpha}(n)\right]-1. \tag{3.25}$$

Through this we obtain the approximation of the exponential function

$$exp\left[-\alpha\frac{z}{1-\alpha}(n)\right]\approx 1-\alpha\frac{z}{1-\alpha}(n). \tag{3.26}$$

By substituting into equation (3.25), we get:

$$\sum_{n}^{i=1}\left(exp\left[-\alpha\frac{z}{1-\alpha}(n-i+1)\right]-exp\left[-\alpha\frac{z}{1-\alpha}(n-i)\right]\right)\approx 1-\alpha\frac{z}{1-\alpha}(n). \tag{3.27}$$

Then equation (3.25) becomes

$$^{Cf}D_a^\alpha f(x_i)=\frac{M(\alpha)}{\alpha}\sum_{n}^{i=1}\left(\frac{f^{i+1}-f^i}{\Delta x}\right)d_{i,k}+\frac{M(\alpha)z}{1-\alpha}(n)O(\Delta x). \tag{3.28}$$

$$^{Cf}D_a^\alpha f(x_i)=\frac{M(\alpha)}{\alpha}\sum_{n}^{i=1}\left(\frac{f^{i+1}-f^i}{\Delta x}\right)d_{i,k}+O((\Delta x)^2). \tag{3.29}$$

For some positive integer N, the grid sizes in time for finite difference technique is defined by $j = \dfrac{1}{p}$.
The grid points in the time interval [0, X] are labeled $x_j = c_j, c = 0, 1, 2, ..., XC$. *The value of the function* f *at the grid point is* $f_j^m = f(x_j, t_m)$.
We have

$$^{Cf}D_a^{\alpha}\left(f(x_c, t_j)\right) = \frac{M(\alpha)}{\sqrt{\Pi}(1-\alpha)} \int_0^{x_c} \frac{\partial}{\partial s} f'(s, t_j) exp\left[-\alpha^2 \frac{(x_c - s)^2}{(1-\alpha)}\right] ds. \tag{3.30}$$

Now employing the Crank-Nicolson approximation for the first-order derivative, the above equation is converted to

$$\begin{aligned} ^{Cf}D_a^{\alpha}\left(f(x_c, t_m)\right) =& \frac{M(\alpha)}{\sqrt{\Pi}}(1-\alpha) \int_0^{x_c} \left(\frac{(f_{j+1}^{m+1} - f_{j-1}^{m+1}) - (f_{j+1}^{m} - f_{j-1}^{m})}{4\Delta x} + O(\Delta x) \right) \\ & \times exp\left[-\alpha^2 \frac{(x_c - s)^2}{(1-\alpha)^2}\right] ds. \end{aligned} \tag{3.31}$$

Convert the last equation

$$\begin{aligned} ^{Cf}D_a^{\alpha}\left(f(x_c, t_j)\right) =& \frac{M(\alpha)}{\sqrt{\Pi}}(1-\alpha) \sum_{q=1}^{c} \left\{ \frac{f_{q+1}^{m+1} - f_{q-1}^{m+1}) - (f_{q+1}^{m} - f_{q-1}^{m})}{4\Delta x} + O(j) \right\} \\ & \times \int_{(q-1)j}^{qj} exp\left[-\alpha^2 \frac{(jc - s)^2}{(1-\alpha)^2}\right] ds, \end{aligned} \tag{3.32}$$

where the integral part is given as

$$\int_{(q-1)j}^{qj} exp\left[-\alpha^2 \frac{(jc - s)^2}{(1-\alpha)^2}\right] ds = \frac{(1-\alpha)\sqrt{\Pi}}{2\alpha} \left\{ erf\left[(cj - qj)\frac{\alpha}{1-\alpha}\right] - erf\left[(cj - qj + j)\frac{\alpha}{1-\alpha}\right] \right\}. \tag{3.33}$$

So

$$\begin{aligned} ^{Cf}D_a^{\alpha}\left(f(x_c, t_j)\right) =& \frac{M(\alpha)}{1-\alpha} \sum_{q=1}^{c} \left\{ \frac{(f_{q+1}^{m+1} - f_{q-1}^{m+1}) - (f_{q+1}^{m} - f_{q-1}^{m})}{4\Delta x} + O(j) \right\} \\ & \times \frac{(1-\alpha)}{2\alpha} \left\{ erf\left[(c - q)\frac{\alpha j}{1-\alpha}\right] - erf\left[(c - q + 1)\frac{\alpha j}{1-\alpha}\right] \right\}. \end{aligned} \tag{3.34}$$

From the above we conclude

$$\begin{aligned}{}^{Cf}D_a^{\alpha}\left(f(x_c,t_j)\right) =& \frac{M(\alpha)}{1-\alpha}\sum_{q=1}^{c}\left\{\frac{(f_{q+1}^{m+1}-f_{q-1}^{m+1})-(f_{q+1}^{m}-f_{q-1}^{m})}{4\Delta x}\frac{(1-\alpha)\sqrt{\Pi}}{2\alpha}\right\}\\ &\times\left\{erf\left[(c-q)\frac{\alpha j}{1-\alpha}\right]-erf\left[(c-q+1)\frac{\alpha j}{1-\alpha}\right]\right\}\\ &+O(j)\frac{(1-\alpha)}{2\alpha}\sum_{q=1}^{c}\left\{erf\left[(c-q)\frac{\alpha j}{1-\alpha}\right]-erf\left[(c-q+1)\frac{\alpha j}{1-\alpha}\right]\right\}. \end{aligned} \tag{3.35}$$

Theorem 3.4.2

Let $f(x,t)\in C^2([a,b]\times[0,T])$*,and* $0<\alpha\leq 1$*is the order of the fractional derivative.Then the first-order approximation of the Caputo-Fabrizio derivative at a point* (x_c,t_n) *is [37]*

$${}^{Cf}D_a^{\alpha}\left(f(x_c,t_j)\right) = \frac{M(\alpha)}{2\alpha}\sum_{q=1}^{c}\left\{\frac{(f_{q+1}^{m+1}-f_{q-1}^{m+1})-(f_{q+1}^{m}-f_{q-1}^{m})}{4\Delta x}\right\}d_{j,q}+L(\alpha,j,q). \tag{3.36}$$

Where

$$d_{j,q}=\left\{erf\left[(c-q)\frac{\alpha j}{1-\alpha}\right]-erf\left[(c-q+1)\frac{\alpha j}{1-\alpha}\right]\right\},\qquad \parallel L(\alpha,j,q)\parallel< M.$$

Proof

From equation (3.36) we have that

$$\begin{aligned}{}^{Cf}D_a^{\alpha}\left(f(x_c,t_j)\right) =& \frac{M(\alpha)}{2\alpha}\sum_{q=1}^{c}\Big(\frac{(f_{q+1}^{m+1}-f_{q-1}^{m+1})-(f_{q+1}^{m}-f_{q-1}^{m})}{4\Delta x}\Big(erf\left[(c-q)\frac{\alpha j}{1-\alpha}\right]\\ &-erf\left[(c-q+1)\frac{\alpha j}{1-\alpha}\right]\Big)\Big)\\ &+O(j)\frac{M(\alpha)}{2\alpha}\sum_{q=1}^{c}\left\{erf\left[(c-q)\frac{\alpha j}{1-\alpha}\right]-erf\left[(c-q+1)\frac{\alpha j}{1-\alpha}\right]\right\}. \end{aligned}$$

In place

$$L(\alpha,j,q)=O(j)\frac{M(\alpha)}{2\alpha}\sum_{q=1}^{c}\left\{erf\left[(c-q)\frac{\alpha j}{1-\alpha}\right]-erf\left[(c-q+1)\frac{\alpha j}{1-\alpha}\right]\right\}. \tag{3.37}$$

Then taking the norm to both sides, we have

$$\| L(\alpha, j, q) \| = \left\| O(j) \frac{M(\alpha)}{2\alpha} \sum_{q=1}^{c} \left\{ erf \left[(c-q) \frac{-\alpha j}{1-\alpha} \right] - erf \left[(c-q+1) \frac{\alpha j}{1-\alpha} \right] \right\} \right\|, \tag{3.38}$$

$$\| L(\alpha, j, q) \| = \left\|, O(j) \frac{M(\alpha)}{2\alpha} \left(erf \left[c \frac{-\alpha j}{1-\alpha} \right] \right) \right\|.$$

CHAPTER 4

A NUMERICAL AND APPLIED STUDY

*we apply and investigate the **Haar wavelet basis** to solve the following system [4] :*

$$\begin{cases} {}^{CF}D^{(\alpha)}f(x) = c_1 f(x) + c_2 g(x) + u(x), 0 \leq x \leq 1, \\ {}^{CF}D^{(\alpha)}g(x) = c_3 f(x) + c_4 g(x) + v(x), 0 \leq x \leq 1, \\ f(0) = g(0) = 0. \end{cases} \tag{4.1}$$

4.1 Haar wavelet basis

The scaling function on interval $[0,1)$ *is* $w_1(x) = 1$ *for* $x \in [0,1)$ *[38],[39].*

The wavelets functions: $\forall i \geq 2$,

$$w_i(x) = \begin{cases} 1, & \text{if } x \in [\theta_1, \Phi_2), \\ -1, & \text{if } x \in [\theta_2, \Phi_3), \\ 0, & \text{otherwise}, \end{cases} \tag{4.2}$$

For $\theta_1 = \dfrac{k}{z}$, $\theta_2 = \dfrac{k+0.5}{z}$, *and* $\theta_3 = \dfrac{k+1}{z}$, $z = 2^j$, $j = 0, 1, ..., J$ J *is the resolution level of wavelet approximation,* $k = 0, ..., z-1$ *represents the translation parameter.*

The relation between i, z *and* p *is given by* $i = z + k + 1$.

In case of the minimal values of $z = 1$, $k = 0$, *then* $i = 2$. *The maximum value of* i *is* $i = 2N, N = 2^k$.

Any function $u \in L^2([0,1))$ *can be expanded as:*

$$u(x) = \sum_{i=1}^{+\infty} a_i w_i(x). \tag{4.3}$$

Where $a_i = \int_0^1 u(x) w_i(x) dx$.

In practice, only the first $2N$ *terms of Eq.(4.3) are considered, where* N *is a power of* 2 $\left(N = 2^J\right)$.

That is:

$$u(x) \simeq u_{2N}(x) = \sum_{i=1}^{2N} a_i w_i(x). \tag{4.4}$$

By integrating σ *times* $w_1(x)$ *and Eq. (4.2) as:*

$$F_{i,\sigma}(x) = \int_0^x \int_0^x ... \int_0^x w_i(x) dx^\sigma, \tag{4.5}$$

we obtain the following formula : $F_{1,\sigma}(x) = \dfrac{x^{\sigma}}{\sigma!}$ *and* $\forall i \geq 2$,

$$F_{i,\sigma}(x) = \frac{1}{\sigma!}\begin{cases} 0, & \text{if } x \in [0,\theta_1), \\ (x-\theta_1)^{\sigma}, & \text{if } x \in [\theta_1,\theta_2), \\ (x-\theta_1)^{\sigma} - 2(x-\theta_2)^{\sigma}, & \text{if } x \in [\theta_2,\theta_3), \\ (x-\theta_1)^{\sigma} - 2(x-\theta_2)^{\sigma} + (x-\theta_3)^{\sigma}, & \text{if } x \in [\theta_3,1). \end{cases} \tag{4.6}$$

4.2 Method of solution

Consider the system (4.1)

$$\begin{cases} {}^{CF}D^{(\alpha)}f(x) = c_1 f(x) + c_2 g(x) + u(x), \quad 0 \leq x \leq 1, \\ {}^{CF}D^{(\alpha)}g(x) = c_3 f(x) + c_4 g(x) + v(x), \quad 0 \leq x \leq 1, \\ f(0) = g(0) = 0. \end{cases} \tag{4.7}$$

Suppose that

$$\begin{cases} {}^{CF}D^{(\alpha)}f(x) \simeq \widetilde{{}^{CF}D^{(\alpha)}}f(x) = \sum_{i=1}^{2N} a_i w_i(x), \\ {}^{CF}D^{(\alpha)}g(x) \simeq \widetilde{{}^{CF}D^{(\alpha)}}g(x) = \sum_{i=1}^{2N} b_i w_i(x), \end{cases} \tag{4.8}$$

where $a_i, b_i, i = 1, ..., 2N$ *are the Haar wavelet [39]. coeffcients to be determined. By integrating Eq (4.8) in the Caputo-Fabrizio sense and taking into account that* $f(0) = g(0) = 0$, *we get*

$$\begin{cases} f(x) \simeq f_{2N}(x) = \sum_{i=1}^{2N} a_i I^{(\alpha)} w_i, \\ g(x) \simeq g_{2N}(x) = \sum_{i=1}^{2N} b_i I^{(\alpha)} w_i. \end{cases} \tag{4.9}$$

Using Eq. (2.25), we have

$$I^{(\alpha)} w_i(x) = (1-\alpha)(w_i(x) - w_i(0)) + \alpha F_{i,1}(x), \forall i = 1, ..., 2N, \tag{4.10}$$

by substituting Eq. (4.9) in Eq. (4.10), we get

$$\begin{cases} f(x) \simeq f_{2N}(x) = \sum_{i=1}^{2N} a_i \left[(1-\alpha)(w_i(x) - w_i(0)) + \alpha F_{i,1}(x)\right], \\ g(x) \simeq g_{2N}(x) = \sum_{i=1}^{2N} b_i \left[(1-\alpha)(w_i(x) - w_i(0)) + \alpha F_{i,1}(x)\right]. \end{cases} \tag{4.11}$$

Now, by substituting Eq.(4.8) and Eq.(4.11) in Eq.(4.7), we obtain the following system of equations:

$$\begin{cases} \sum_{i=1}^{2N} a_i L(i,x) + \sum_{i=1}^{2N} b_i G(i,x) = -u(x), \\ \sum_{i=1}^{2N} a_i \tilde{L}(i,x) + \sum_{i=1}^{2N} b_i \tilde{G}(i,x) = -v(x), \end{cases} \tag{4.12}$$

where

$$\begin{cases} L(i,x) = (c_1(1-\alpha)-1)w_i(x) + \alpha c_1 F_{i,1}(x) - c_1(1-\alpha)w_x(0), \\ G(i,x) = c_2(1-\alpha)w_i(x) + \alpha c_2 F_{i,1}(x) - c_2(1-\alpha)w_x(0), \\ \tilde{L}(i,x) = c_3(1-\alpha)w_i(x) + \alpha c_3 F_{i,1}(x) - c_3(1-\alpha)w_i(0), \\ \tilde{G}(i,x) = (c_4(1-\alpha)-1)w_i(x) + \alpha c_4 F_{i,1}(x) - c_4(1-\alpha)w_i(0). \end{cases} \tag{4.13}$$

By defining the collocation points $x_l = \frac{l-0.5}{2N}, l = 1,...,2N$, and replace them into the system. (4.12), we have the following $4N \times 4N$ linear system of equations:

$$\begin{cases} \sum_{i=1}^{2N} a_i L(i,x_l) + \sum_{i=1}^{2N} b_i G(i,x_l) = -u(x_l), \quad l = 1,...,2N, \\ \sum_{i=1}^{2N} a_i \tilde{L}(i,x_l) + \sum_{i=1}^{2N} b_i \tilde{G}(i,x_l) = -v(x_l), \quad l = 1,...,2N. \end{cases} \tag{4.14}$$

By solving this system, we obtain the unknown coeffcients $a_i, b_i, i = 1,...,2N$, and by substituting them into Eq.(4.11), we get the numerical solution of the system(4.1).

4.3 Error Analysis

Here, we study the error of approximation using our proposed method of the solution [40].

Lemma 4.3.1 *Let u be a differentiable function of $L^2([0,1))$ with bounded first derivative on $(0,1)$ and $\tilde{u}$ is its Haar wavelet approximation defined by Eq. (4.4), then*

$$\|u - \tilde{u}\|_{L^2([0,1))} \leq C2^{-J}, \tag{4.15}$$

where C is a constant.

Lemma 4.3.2 *The Haar wavelet coefficients that are given in Eq. (4.8) can be estimated as:*

$$a_i = \mathrm{O}\left(\frac{1}{2^{j+1}}\right), i = 2^j + K + 1. \tag{4.16}$$

Proof.

We have first,

$$\begin{aligned} a_i &= \int_0^1 {}^{CF}D^{(\alpha)}f(x)w_i(x)dx \\ &= \int_{\theta_1}^{\theta_2} {}^{CF}D^{(\alpha)}f(x)dx - \int_{\theta_2}^{\theta_3} {}^{CF}D^{(\alpha)}f(x)dx \\ &= (\theta_2-\theta_1)^{CF}D^{(\alpha)}f(\eta_1) - (\theta_3-\theta_2)^{CF}D^{(\alpha)}f(\eta_2), \end{aligned}$$

where $\eta_1 \in [\theta_1, \theta_2]$ *and* $\eta_2 \in [\theta_2, \theta_3]$.

Since $\theta_2 - \theta_1 = \theta_3 - \theta_2 = \frac{1}{2m} = \frac{1}{2^{j+1}}$, *then*

$$a_i = \frac{\eta_1-\eta_2}{2^{j+1}} \frac{d \quad {}^{CF}D^{(\alpha)}f}{dx}(\vartheta), \quad \vartheta \in \,]\eta_1, \eta_2[\,,$$

this implies that

$$|a_i| \leq \frac{1}{2^{j+1}} \left\| \frac{d \quad {}^{CF}D^{(\alpha)}f}{dx} \right\|_\infty . \tag{4.17}$$

On the other hand,

$$\begin{aligned} \frac{d \quad {}^{CF}D^{(\alpha)}f}{dx}(x) &= \frac{1}{1-\alpha}\left[f'(x) + \int_0^x f'(t)(\frac{-\alpha}{1-\alpha})\exp(-\alpha\frac{x-t}{1-\alpha})dt\right] \\ &= \frac{1}{1-\alpha}\left[f'(x) - \alpha^{CF}D^{(\alpha)}f(x)\right]. \end{aligned}$$

This expression leads to

$$\left\| \frac{d \quad {}^{CF}D^{(\alpha)}f}{dx} \right\|_\infty \leq \frac{1}{1-\alpha}\|f'\|_\infty + \frac{\alpha}{1-\alpha}\left\| {}^{CF}D^{(\alpha)}f \right\|_\infty . \tag{4.18}$$

Now, about $\left\| {}^{CF}D^{(\alpha)}f \right\|_\infty$ *we have*

$$\begin{aligned} \left| {}^{CF}D^{(\alpha)}f(x) \right| &\leq \frac{1}{1-\alpha}\|f'\|_\infty \int_0^x \exp\left(-\alpha\frac{x-t}{1-\alpha}\right)dt \\ &\leq \frac{\|f'\|_\infty}{\alpha}\left[1-\exp\left(-\alpha\frac{x}{1-\alpha}\right)\right] \\ &\leq \frac{\|f'\|_\infty}{\alpha}\left[1-\exp\left(\frac{-\alpha}{1-\alpha}\right)\right], \end{aligned}$$

thus

$$\left\| {}^{CF}D^{(\alpha)}f(x) \right\|_\infty \leq \frac{\|f'\|_\infty}{\alpha}\left[1-\exp\left(\frac{-\alpha}{1-\alpha}\right)\right], \tag{4.19}$$

inserting Eq. (4.19) in Eq.(4.18) yields

$$\left\| \frac{d \quad {}^{CF}D^{(\alpha)}f}{dx} \right\|_\infty \leq \frac{2-\exp(\frac{-\alpha}{1-\alpha})}{1-\alpha}|f'|_\infty .$$

Therefore,

$$|a_i| \leq \left(\frac{2-\exp(\frac{-\alpha}{1-\alpha})}{1-\alpha}\|f'\|_\infty \right)\frac{1}{2^{j+1}}.$$

Lemma 4.3.3 *The function $F_{i,1}$ for $i \geq 2$ verifies the following inequality:*

$$\|F_{i,1}\| \leq \frac{1}{2^{j+1}}, \quad i = 2^j + k + 1. \tag{4.20}$$

Proof.

From Eq. 4.6 we have, $i \geq 2$

$$F_{i,1}(x) \begin{cases} 0, & \text{if } \ x \in [0, \theta_1), \\ x - \theta_1, & \text{if } \ x \in [\theta_1, \theta_2), \\ -x + 2\theta_2 - \theta_1, & \text{if } \ x \in [\theta_2, \theta_3), \\ 2\theta_2 - \theta_3 - \theta_1, & \text{if } \ x \in [\theta_3, 1). \end{cases} \tag{4.21}$$

Note that in the interval $[\theta_1, \theta_2)$, the function $F_{i,1}$ is positive and increasing, then $|F_{i,1}| \leq \theta_2 - \theta_1 = \frac{1}{2^{j+1}}$.
In the interval $[\theta_2, \theta_3)$, the function $F_{i,1}$ is positive and decreasing, then $|F_{i,1}| \leq \theta_2 - \theta_1 = \frac{1}{2^{j+1}}$.
Otherwise, the function $F_{i,1}$ is null.
Thus, $\|F_{i,1}\|_\infty \leq \frac{1}{2^{j+1}}$.

Lemma 4.3.4 *For $i \geq 2$ we define the function p_i as:*

$$p_i(x) = (1 - \alpha)(w_i(x) - w_i(0)) + \alpha F_{i,1}(x), \quad i = 2^j + k + 1, 0 \leq x < 1. \tag{4.22}$$

Then, we have

1. $\forall i \geq 2, |p_i(x)| \leq 2 - \alpha.$
2. *If $k \neq 0$, then* $\int_0^1 |p_i(x)| dx \leq (1 - \frac{\alpha}{2}) \frac{1}{2^j}.$

Proof.

1. $\forall i \geq 2$, *we have*

$$\begin{aligned} |p_i(x)| &\leq (1 - \alpha)|w_i(x) - w_i(0) + \alpha \|F_{i,1}(x)\|_\infty \\ &\leq 2(1 - \alpha) + \alpha \frac{1}{2^{j+1}} \\ &\leq 2(1 - \alpha) + \alpha = 2 - \alpha. \end{aligned}$$

2. *If* $k \neq 0$, *then* $w_i(0) = 0$ *and we have*

$$\int_0^1 |p_i(x)|dx \leq (1-\alpha)\int_0^1 |w_i(x)|dx + \alpha \int_0^1 \|F_{i,1}(x)\|_\infty dx.$$

Note that

$$\int_0^1 |w_i(x)|dx = \int_{\theta_1}^{\theta_2} dx + \int_{\zeta_2}^{\theta_3} dx = \frac{1}{2^j}.$$

Hence,

$$\begin{aligned}\int_0^1 |p_i(x)|dx &\leq (1-\alpha)\frac{1}{2^j} + \alpha\frac{1}{2^{j+1}} \\ &\leq (1-\frac{\alpha}{2})\frac{1}{2^j}.\end{aligned}$$

Theorem 4.3.1

Let (f, g) *be the exact solution of the problem (4.1), and* (f_{2N}, g_{2N}) *be its approximation formula that is expressed in Eq. (4.11), then the convergence rate is estimated by*

$$\|f - f_{2N}\|_{L^2([0,1))} = \mathrm{O}\left(\frac{1}{\sqrt{N}}\right), \tag{4.23}$$

and

$$\|g - g_{2N}\|_{L^2([0,1))} = \mathrm{O}\left(\frac{1}{\sqrt{N}}\right). \tag{4.24}$$

Proof.

From Eq. (4.11), the error square at the Jth level resolution for the function f *can be written as:*

$$\begin{aligned}E_N^2 = \|f - f_{2N}\|^2_{L^2([0,1))} &= \| \sum_{i=2N+1}^{+\infty} a_i \left[(1-\alpha)(w_i(x) - w_0(0) + \alpha F_{i,1}(x)\right] \|^2_{L^2([0,1))} \\ &= \sum_{l=2N+1}^{+\infty} \sum_{i=2N+1}^{+\infty} a_i a_l \int_0^1 p_i(x) p_l(x) dx.\end{aligned}$$

By putting $i = 2^j + k + 1$ *and* $l = 2^r + s + 1$, *we get*

$$\begin{aligned}E_N^2 &= \sum_{r=J+1}^{+\infty} \sum_{s=0}^{2^r-1} \sum_{j=J+1}^{+\infty} \sum_{k=0}^{2^j-1} a_{2^j+k+1} a_{2^r+s+1} \int_0^1 p_{2^j+k+1}(x) p_{2^r+s+1}(x) dx \\ &= \sum_{r=J+1}^{+\infty} \sum_{j=J+1}^{+\infty} a_{2^j+1} a_{2^r+1} \int_0^1 p_{2^j+1}(x) p_{2^r+1}(x) dx \\ &+ \sum_{r=J+1}^{+\infty} \sum_{s=1}^{2^r-1} \sum_{j=J+1}^{+\infty} a_{2^j+1} a_{2^r+s+1} \int_0^1 p_{2^j+1}(x) p_{2^r+s+1}(x) dx \\ &+ \sum_{r=J+1}^{+\infty} \sum_{j=J+1}^{+\infty} \sum_{k=1}^{2^j-1} a_{2^j+k+1} a_{2^r+1} \int_0^1 p_{2^j+k+1}(x) p_{2^r+1}(x) dx \\ &+ \sum_{r=J+1}^{+\infty} \sum_{s=1}^{2^r-1} \sum_{j=J+1}^{+\infty} \sum_{k=1}^{2^j-1} a_{2^j+k+1} a_{2^r+s+1} \int_0^1 p_{2^j+k+1}(x) p_{2^r+s+1}(x) dx.\end{aligned}$$

This implies that,

$$\begin{aligned}
E_N^2 \leq & \sum_{r=J+1}^{+\infty} \sum_{j=J+1}^{+\infty} |a_{2^j+1}||a_{2^r+1}| \int_0^1 |p_{2^j+1}(x)||p_{2^r+1}(x)|dx \\
& + \sum_{r=J+1}^{+\infty} \sum_{s=1}^{2^r-1} \sum_{j=J+1}^{+\infty} |a_{2^j+1}||a_{2^r+s+1}| \int_0^1 |p_{2^j+1}(x)||p_{2^r+s+1}(x)|dx \\
& + \sum_{r=J+1}^{+\infty} \sum_{j=J+1}^{+\infty} \sum_{k=1}^{2^j-1} |a_{2^j+k+1}||a_{2^r+1}| \int_0^1 |p_{2^j+k+1}(x)||p_{2^r+1}(x)|dx \\
& + \sum_{r=J+1}^{+\infty} \sum_{s=1}^{2^r-1} \sum_{j=J+1}^{+\infty} \sum_{k=1}^{2^j-1} |a_{2^j+k+1}||a_{2^r+s+1}| \int_0^1 |p_{2^j+k+1}(x)||p_{2^r+s+1}(x)|dx.
\end{aligned}$$

By using Lemma (4.3.2), Lemma (4.3.3) and Lemma (4.3.4), we obtain

$$\begin{aligned}
E_N^2 \leq & \sum_{r=J+1}^{+\infty} \sum_{j=J+1}^{+\infty} C \frac{1}{2^{j+1}} \frac{1}{2^{r+1}} \int_0^1 (2-\alpha)^2 dx \\
& + \sum_{r=J+1}^{+\infty} \sum_{s=1}^{2^r-1} \sum_{j=J+1}^{+\infty} C \frac{1}{2^{j+1}} \frac{1}{2^{r+1}} \int_0^1 (2-\alpha)|p_{2^r+s+1}(x)|dx \\
& + \sum_{r=J+1}^{+\infty} \sum_{j=J+1}^{+\infty} \sum_{k=1}^{2^j-1} C \frac{1}{2^{j+1}} \frac{1}{2^{r+1}} \int_0^1 (2-\alpha)|p_{2^j+k+1}(x)|dx \\
& + \sum_{r=J+1}^{+\infty} \sum_{s=1}^{2^r-1} \sum_{j=J+1}^{+\infty} \sum_{k=1}^{2^j-1} C \frac{1}{2^{j+1}} \frac{1}{2^{r+1}} \sqrt{\int_0^1 |p_{2^j+k+1}(x)|^2 dx \int_0^1 |p_{2^r+s+1}(x)|^2 dx.}
\end{aligned}$$

Note that

$$\begin{aligned}
\sqrt{\int_0^1 |p_{2^j+k+1}(x)|^2 dx \int_0^1 |p_{2^r+s+1}(x)|^2 dx} &= \sqrt{\int_0^1 |p_{2^j+k+1}(x)||p_{2^j+k+1}(x)|dt \int_0^1 |p_{2^r+s+1}(x)||p_{2^r+s+1}(x)|dx} \\
&\leq (2-\alpha)\sqrt{\int_0^1 |p_{2^j+k+1}(x)|dx \int_0^1 |p_{2^r+s+1}(x)|dx} \\
&\leq (2-\alpha)(1-\frac{\alpha}{2}) \frac{1}{2^{\frac{j}{2}}} \frac{1}{2^{\frac{r}{2}}}.
\end{aligned}$$

Then

$$
\begin{aligned}
E_N^2 \le & \sum_{r=J+1}^{+\infty}\sum_{j=J+1}^{+\infty} C\frac{1}{2^{j+1}}\frac{1}{2^{r+1}}(2-\alpha)^2 \\
& + \sum_{r=J+1}^{+\infty}\sum_{s=1}^{2^r-1}\sum_{j=J+1}^{+\infty} C\frac{1}{2^{j+1}}\frac{1}{2^{r+1}}(2-\alpha)(1-\frac{\alpha}{2})\frac{1}{2^r} \\
& + \sum_{r=J+1}^{+\infty}\sum_{j=J+1}^{+\infty}\sum_{k=1}^{2^j-1} C\frac{1}{2^{j+1}}\frac{1}{2^{r+1}}(2-\alpha)(1-\frac{\alpha}{2})\frac{1}{2^j} \\
& + \sum_{r=J+1}^{+\infty}\sum_{j=J+1}^{+\infty} C\frac{1}{2^{\frac{j}{2}+1}}\frac{1}{2^{\frac{r}{2}+1}}(2-\alpha)(1-\frac{\alpha}{2}) \\
\le & \frac{C(2-\alpha)^2}{4}(\frac{1}{2^J})^2 + \frac{C(2-\alpha)(1-\frac{\alpha}{2})}{4}\left(\frac{1}{2^J}\right)^2 \\
& + \frac{C(2-\alpha)(1-\frac{\alpha}{2})}{4}(\frac{1}{2^J})^2 + \frac{C(2-\alpha)(1-\frac{\alpha}{2})}{4(\sqrt{2}-1)^2}\left(\frac{1}{2^{\frac{J}{2}}}\right)^2 \\
\le & \left[\frac{C(2-\alpha)^2}{4} + \frac{C(2-\alpha)(1-\frac{\alpha}{2})}{2} + \frac{C(2-\alpha)(1-\frac{\alpha}{2})}{4(\sqrt{2}-1)^2}\right]\left(\frac{1}{2^J}\right)^2.
\end{aligned}
$$

Therefore,

$$E_N(f) = O\left(\frac{1}{\sqrt{N}}\right).$$

In a similar way, we prove that $E_N(g) = O\left(\frac{1}{\sqrt{N}}\right)$.

Now, to verify the convergence analysis, we must prove that the following system converges to the system Eq. (4.1).

$$
\begin{cases}
\widetilde{{}^{CF}D^{(\alpha)}}f(x) = c_1 f_{2N}(x) + c_2 g_{2N}(x) + u(x) + R_N^1(x), \quad 0 \le x \le 1, \\
\widetilde{{}^{CF}D^{(\alpha)}}g(x) = c_3 f_{2N}(x) + c_4 g_{2N}(x) + v(x) + R_N^2(x), \quad 0 \le x \le 1,
\end{cases}
\tag{4.25}
$$

where $R_N^1(x)$ *and* $R_N^2(x)$ *represent the remainders.*

By subtracting Eq. (4.25) from Eq. (4.1), we get

$$
\begin{cases}
R_N^1(x) = c_1(f(x) - f_{2N}(x)) + c_2(g(x) - g_{2N}(x)) - \left({}^{CF}D^{(\alpha)}f(x) - \widetilde{{}^{CF}D^{(\alpha)}}f(x)\right), 0 \le x \le 1, \\
R_N^2(x) = c_3(f(x) - f_{2N}(x)) + c_4(g(x) - g_{2N}(x)) - \left({}^{CF}D^{(\alpha)}g(x) - \widetilde{{}^{CF}D^{(\alpha)}}g(x)\right), 0 \le x \le 1.
\end{cases}
\tag{4.26}
$$

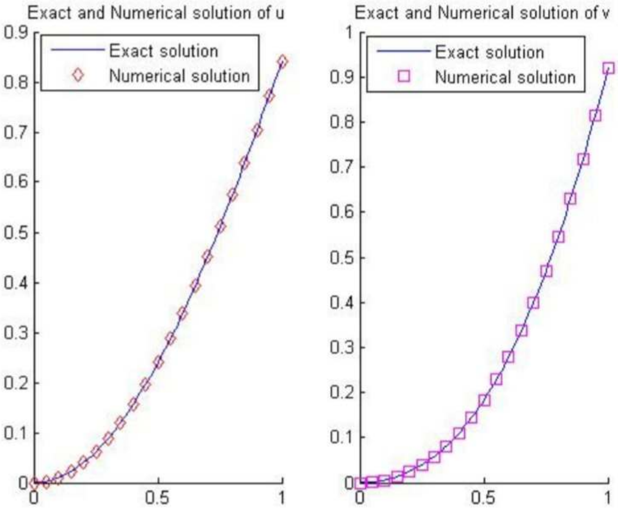


Figure 4.1: Numerical solution of Example (4.4.1) at level $J = 3$. [40] .

Then

$$\begin{cases} \|R_N^1\|_{L^2([0,1])} \leq c_1\|f - f_{2N}\|_{L^2([0,1]} + c_2\|g - g_{2N}\|_{L^2([0,1]} + \|{}^{CF}D^{(\alpha)}f - \widetilde{{}^{CF}D^{(\alpha)}}f\|_{L^2([0,1]}, \\ \|R_N^2\|_{L^2([0,1])} \leq c_3\|f - f_{2N}\|_{L^2([0,1]} + c_4\|g - g_{2N}\|_{L^2([0,1]} + \|{}^{CF}D^{(\alpha)}g - \widetilde{{}^{CF}D^{(\alpha)}}g\|_{L^2([0,1])}, \end{cases}$$

when $N \to +\infty$, *we obtain* $\|f - f_{2N}\|_{L^2([0,1[)} \to 0, \|f - f_{2N}\|_{L^2([0,1)} \to 0$, $\|{}^{CF}D^{(\alpha)}f - \widetilde{{}^{CF}D^{(\alpha)}}f\|_{L^2([0,1]} \to 0, \|{}^{CF}D^{(\alpha)}g - \widetilde{{}^{CF}D^{(\alpha)}}g\|_{L^2([0,1]} \to 0$. *Therefore,* $\|R_N^1\|_{L^2([0,1])} \to 0$ *and* $\|R_N^2\|_{L^2([0,1])} \to 0$.

4.4 Some Numerical examples by using Haar wavelet basis:

- ***Numerical examples:***

Here, we look at three instances of problem (4.1) to demonstrate the effectiveness of our suggested approach. Fig.4.1 Fig.4.2 and Fig.4.3 show the numerical results of all computations, which are carried out using Matlab.

Example 4.4.1

Consider the following coupled system:

$$\begin{cases} {}^{CF}D^{(0.25)}f(x) = -\frac{1}{2}f(x) + \frac{1}{2}g(x) + u(x), \quad 0 \le x \le 1, \\ {}^{CF}D^{(0.25)}g(x) = -\frac{1}{4}g(x) + \frac{1}{4}g(x) + v(x), \quad 0 \le x \le 1, \\ f(0) = g(0) = 0, \end{cases}$$

where $u(x) = (\frac{17}{10}x - \frac{8}{25})\sin(x) + (\frac{13}{50} + \frac{9}{10}x)\cos(x) + \frac{6}{25}e^{-\frac{x}{3}} - \frac{1}{2}(x+1)$ *and*
$v(x) = (\frac{4}{25} + \frac{13}{20}x)\sin(x) + (\frac{-63}{100} - \frac{19}{20}x)\cos(x) - \frac{78}{25}e^{-\frac{x}{3}} - \frac{x}{4} + \frac{15}{4}$.
The exact solution is given by $f(x) = x\sin(x)$ *and* $g(x) = (x+1)(1-\cos(x))$.

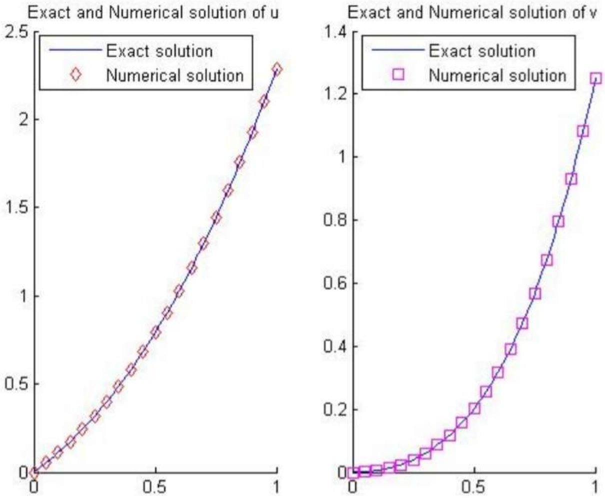


Figure 4.2: Numerical solution of Example (4.4.2) at level $J = 3$. [40] .

Example 4.4.2

Consider the following problem:

$$\begin{cases} {}^{CF}D^{(0.5)}f(x) = -\frac{1}{3}f(x) + \frac{1}{2}g(x) + u(x), \quad 0 \le x \le 1, \\ {}^{CF}D^{(0.5)}g(x) = -\frac{1}{8}f(x) + \frac{1}{6}g(x) + v(x), \quad 0 \le x \le 1, \\ f(0) = g(0) = 0, \end{cases}$$

where $u(x) = (-\frac{1}{10}\cos(x) + \frac{13}{15}\sin(x) + \frac{1}{2})e^{x} - \frac{2}{5}e^{-x}$ *and*
$v(x) = (-\frac{31}{35}\cos(x) + \frac{11}{40}\sin(x) + \frac{5}{6})e^{x} + \frac{1}{5}e^{-x}$.
The exact solution is given by $f(x) = \sin(x)e^{x}$ *and* $g(x) = (1-\cos(x))e^{x}$.

Example 4.4.3

Consider the following problem:

$$\begin{cases} {}^{CF}D^{(0.75)}f(x) = -\frac{1}{5}f(x) + \frac{2}{3}g(x) + u(x), \quad 0 \leq x \leq 1, \\ {}^{CF}D^{(0.75)}g(x) = -\frac{1}{6}f(x) + \frac{1}{12}g(x) + v(x), \quad 0 \leq x \leq 1, \\ f(0) = g(0) = 0, \end{cases}$$

where $u(x) = (\frac{60}{169} + \frac{93}{65}x)\sin(2x) + (\frac{48}{13}x - \frac{288}{169})\cos^2(x) + \frac{144}{169}e^{-3x} - \frac{24}{13}x + \frac{144}{169} - \frac{2}{3}(x-1)(1\cos(x))$ *and* $v(x) = (-\frac{48}{25} + \frac{6}{5}x)\sin(x) + (-\frac{14}{25} - \frac{2}{5}x)\cos(x) - \frac{1}{6}x\sin(2x) - \frac{58}{75}e^{-3x} - \frac{1}{12}(x-1)(1-\cos(x)) + \frac{4}{3}$. *The exact solution is given by* $f(x) = x\sin(2x)$ *and* $g(x) = (x-1)(1-cos(x))$.

Conicts of Interest:

The writers say they have no competing interests.

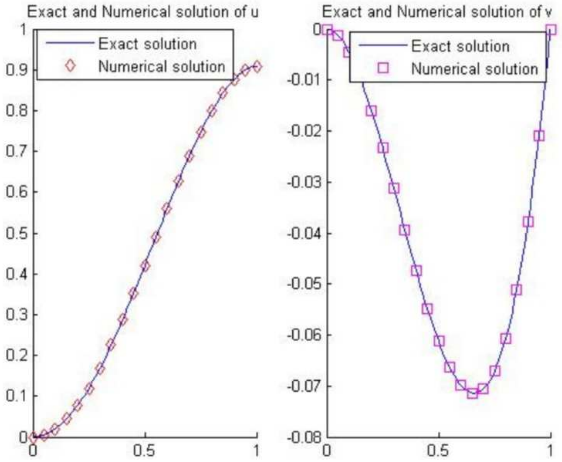


Figure 4.3: Numerical solution of Example (4.4.3) at level $J = 3$. [40] .

CONCLUSION

- *We have studied a new coupled system of linear fractional differential equations with non-singular exponential kernels that involve Caputo-Fabrizio fractional derivatives. The Picard-Lindelof technique and fixed point theory were utilized to examine the existence and uniqueness of the solution.*
- *Also, in this study this coupled system with the Caputo-Fabrizio fractional derivative has been solved using the Haar wavelet collocation approach. Our suggested strategy has an exponential convergence rate, according to error analysis. there is a strong agreement between the numerical and exact solutions.*

  Furthermore, this approach is advised and successful.Hence, we can state that the suggested approach offers a very high degree of accuracy in the solution estimation.

  Additionally, the proposed method is easy to implement for the computation of solutions to various problems of fractional order differential equations.
- *As a future work, we will solve various kinds of coupled systems and implicit fractional differential equations Caputo-Fabrizio by using the Haar wavelet collocation approach.*

BIBLIOGRAPHY

[1] Hilfer, R. (2000). Fractional diffusion based on Riemann-Liouville fractional derivatives.The Journal of Physical Chemistry B, 104(16), 3914–3917.

[2] Bachir, Dehda., Abdelaziz, Azeb.Ahmed., Fares, Yazid., Fatima, Siham, and Djeradi.: Numerical solution of a class of Caputo-Fabrizio derivative problem using Haar wavelet collocation method. Journal of Applied Mathematics and Computing, volume 69, pages 2761–2774 (2023).

[3] Sania, Qureshi., Norodin, A.Rangaig., Dumitru, Baleanu.: New Numerical Aspects of Caputo-Fabrizio Fractional Derivative Operator. Mathematics 2019, 7(4), 374;

[4] Ikram,Mansouri.Mohammed,Moumen.Bekkouche.,Abdelaziz, Azeb.Ahmed.:NUMERICAL SOLUTION OF A FRACTIONAL COUPLED SYSTEM WITH THE CAPUTO-FABRIZIO FRACTIONAL DERIVATIVE.Journal of Applied Mathematics and Computational Mechanics 2023, 22(1), 46–56.

[5] S.C. Shiralashetti, E. Harishkumar.: Haar wavelet matrices for the numerical solution of system of ordinary differential equations. Malaya Journal of Matematik, Vol. S, No. 1, 144–147, 2020.

[6] Mujeeb, Urrehman., and Rahmat, Ali.Khan.: A numerical method for solving boundary value problems for fractional differential equations. Applied Mathematical Modeling 36 (2012), 894–907.

[7] *Mohammad, Heydari., Zakieh, Avazzadeh., and Narges, Hosseinzadeh.: Haar Wavelet Method for Solving High-Order Differential Equations with Multi-Point Boundary Conditions. J. Appl. Comput. Mech., 8(2) (2022), 528–544.*

[8] *Siraj-ul, Islam; Imran, Aziz; and Masood, Ahmad.: Numerical solution of two-dimensional elliptic PDEs with nonlocal boundary conditions. Computers and Mathematics with Applications 69 (2015), 180–205.*

[9] *E. Babolian, A. Shahsavaran.: Numerical solution of nonlinear Fredholm integral equations of the second kind using Haar wavelets. Journal of Computational and Applied Mathematics, 225 (2009), 87–95.*

[10] *H. Brezis, Functional Analysis, Sobolev Spaces, and Partial Differential Equations, Masson,Paris, 2010.*

[11] *A.A.Kilbas,H.M.Srivastava,J.J.Trujillo. Theory And Applications Of Fractional Differential Equations. North-Holland Mathematical studies 204, Ed van Mill, Amsterdam , 2006.*

[12] *Y.Zhou,Basic Theory Of Fractional Differential Equations.Xiangtan University, China,2014.*

[13] *G. Skandalis, Topologie Et Analyse 3e Année, 2004.*

[14] *B. Bollobas, W. Fulton, A. Katok, F. Kirwan, P. Sarnak. Fixed Point Theory and Applications,2004.*

[15] *I. Podlubny. Fractional differential equations. Academic Press , 1999.*

[16] *L.Pujo-Menjouet.Cours d'Analyse IV Suites et Séries de fonctions .Université Claude Bernard,Lyon I 43,boulevard 11 novembre 1918.*

[17] *A. Kirillov, A. Gvichiani, Théorèmes Et Problèmes D'analyse Fonctionnelle, Éditions Mir Moscou , 1982.*

[18] *L. Schwartz, Analyse Topologie générale et analyse fonctionnelle, Édition Corrigée , Paris,2008.*

[19] *N. Boccara, Analyse Fonctionnelle une introduction pour physiciens, 1984.*

[20] *M. Henry, La théorie de la fonction gamma, Limbourg, 1858.*

[21] Kai.Diethelm, The Analysis of Fractional Differential Equation, An ApplicationOriented Exposition Using Differential Operators of Caputo Type, Ed Springer, 2004 .

[22] A.A.Kilbas,H.M.Srivastavaand J.J.Trujllo.Theory and application of fractional differencial equation . Elsvier,2006.

[23] Eiman,K. Shah,M. Sarwar and D. Baleanu,Study on Krasnoselskii's fixed point theoremfor Caputo–Fabrizio fractional differential equations,University of Malakand, Khyber Pakhtulkhwa, Pakistan,2020:178.

[24] A. A. Kilbass, S. G. Samko, O. I. Marichev, Fractional integrals and derivatives theory and applications, Gordon and Breach Science publisher, Amsterdam, 1993.

[25] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations, North-Holland Mathematical Studies 204, Ed Jan Van Mill Amsterdam, (2006).

[26] M.Rahimy.Applications of Fractional Differential Equations.Department of Mathematics,Faculty of Sciences Islamic Azad University of Aligudarz, Lorestan, Iran.Vol.4,2010,no.50,2453-2461.

[27] I. Podlubny, Geometric and physical interpretation of fractional integration and fractional differentiation. Fractional Calculus and Applied Analysis. 5(2002), 367-386, .

[28] lgor Podlubny, Mathematics In Science And Engineering, Fractional equations,Technical University of Kosice, Slovak Republic, Vol 198.

[29] Constantin Milici, Gheorghe Draganescu,J.Tenreiro Machado, Introduction to Fractional Differential equations, Vol 25, Springer.

[30] S.G. Samko, A.A. Kilbas, and O.I. Marichev. Fractional integrals and derivatives :theory and applications. Gordon and Breach (1993).

[31] M.Caputo and F.Mainardi ,Linear models of dissipation in anelastic solids, Rivistadel Nuovo Cimento (Ser. II), Vol. 1, pp. 161 - 198(1971).

[32] M. Caputo M. Fabrizio, A new definition of fractional derivative without singular kernel,Progr. Fract. Differ.Appl, 1(2015), 73-85.

[33] J. Losada, J.J. Nieto, Properties of a new fractional derivative without singular kernel,Progr. Fract. Differ. Appl., 1 (2016), 87-92

[34] J.Tenreiro Machado , Introduction to Fractional Differential Equations ,Nonlinear Systems and Complexity ,volume 25, series Editor: Abert C.J.Luo,Southern Illinos University Edwardsville,IL,USA.

[35] G.A.M. Boro Nchama, Properties of Caputo-Fabrizio fractional operators,Universidad Nacional de Guinea Ecuatorial, Malabo, Guinea Ecuatorial,NTMSCI 8, No. 1,1-25 (2020).

[36] M.Moumen Bekkouche ,H.Guebbai :ANALYTICAL AND NUMERICAL STUDY FOR A FRACTIONAL BOUNDARY VALUE PROBLEM WITH A CONFORMABLE FRACTIONAL DERIVATIVE OF CAPUT AND ITS FRACTIONAL INTEGRAL .Journal of Applied and Computational Mechanics 2020,19(2),31-42.

[37] Abdon Atangana ,Rubayyi T Alqahtani :Numerical approximation of the space-time Caputo-Fabrizio fractional derivative and application to groundwater pollution equation.Atangana and Alqahtani Advances in Difference Equations (2016) 2016:156.

[38] Rohul, Amin., Kamal, Shah., Muhammad, Asif., Imran, Khan., Faheem Ullah.: An efficient algorithm for numerical solution of fractional integro-differential equations via Haar wavelet. Journal of Computational and Applied Mathematics (2020),

[39] Siddu, Channabasappa.Shiralashetti., Aravind, B.Deshi.: An efficient Haar wavelet collocation method for the numerical solution of multi-term fractional differential equations. August 2015. Nonlinear Dynamics 83(1-2).

[40] Hussam, Alrabaiah., Israr, Ahmad., Rohul, Amin., Kamal, Shah.: A numerical method for fractional variable order pantograph diferential equations based on Haar wavelet. June 2022. Engineering with Computers 38(3):1-14.

ABSTRACT:

Caputo-Fabrizio’s definition of fractional derivation is one of the latest advancements in fractional calculus. This study investigates solutions for a coupled system of linear fractional differential equations with fractional orders using the Picard-Lindelöf approach and fixed-point theory, demonstrating their existence and uniqueness. An efficient computational method for solving the coupled system is presented, utilizing the Caputo-Fabrizio fractional derivative. The approach expands the solution into the Haar wavelet basis, allowing for the determination of Haar wavelet coefficients. Error analysis of the method shows a strong convergence rate. Finally, several numerical examples are provided to demonstrate the precision and efficacy of this approach.

Keywords:

Fractional calculus, Haar wavelet, Caputo- Fabrizio fractional derivative, coupled system with fractional derivative. fractional integral, fractional differential equation,numerical approximation.

RÉSUME:

La définition de la dérivation fractionnelle de Caputo-Fabrizio est l’une des plus récentes pour améliorer les dérivations fractionnelles. C'est mémoire de master examine les solutions d’un système lié d’équations différentielles fractionnelles linéaires avec des ordres fractionnels en utilisant l’approche de Picard-Lindelöf et la théorie du point fixe, en démontrant ainsi leur existence et leur unicité.C'est mémoire master présente une méthode computationnelle efficace pour résoudre un système couplé en utilisant la dérivée fractionnelle de Caputo-Fabrizio. L’approche décompose la solution dans la base des ondelettes de Haar, permettant la découverte des coefficients des ondelettes de Haar. L’analyse d’erreur de la méthode montre un fort taux de convergence. Enfin, quelques exemples numériques sont fournis pour démontrer la précision et l’efficacité de cette approche.

Mots clés et expressions:

dérivé fractionnaire de Caputo-Fabrizio,approximation numérique,Calcul fractionnaire,ondelette de Haar, systéme couple à dérivé fractionnaire. équation différentielle différentielle fractionnaire, approximation numérique.

الملخص:

تعريف كابوتو - فبريزيو للاشتقاق الكسوري هو واحد من أحدث التطورات في التفاضل والتكامل الكسري .وتبحث هذه الدراسة الحلول لنظام مركب من المعادلات التفاضلية الكسورية الخطية مع أوامر كسرية باستخدام نهج بيكار – ليندیلوف ونظرية النقطة الثابتة، مما يدل على وجودها و وحدانيتها .طريقة حسابية فعالة لحل و يُعرضالنظام المدمج باستخدام مشتقة كابوتو -فبريزيو الكسورية .وننشر الحل وفق أساس هار مما يسمح بتحديد معاملات حاور الموجي .ويبين تحليل الخطأ في الطريقة وجود معدل تقارب قوي .وأخيراً، تقُدَّم عدة أمثلة رقمية لإثبات دقة هذا النهج وفعاليته.

الكلمات و العبارات الدالة:

المشتق الكسري لكابتو فابريزيو, موجات هار,الحسابات الكسرية,النظام المركب للمشتق الكسري,المعادلات التفاضلية الكسرية,التقريب العددي.

Thanks

First of all, we thank Allah, who gave us the strength and the will to carry out this work. We extend our sincere thanks to our supervisor

Dr. Bechir dehda who was responsible for providing advice and guidance for each small and each large in this memory.

We extend our sincere thanks to all the teachers of the Mathematics department of Echahid Hamma Lakhdar University, El-Oued.

Also to our colleagues from the 2023–2024 promotion. We also thank all those who participated closely or far to develop this work.