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Theme

Improvment of some inequalities between the spectral values of operators in B(H)

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شكر وتقدير

(وَقُلِ اعْمَلُوا فَسَيَرَى اللَّهُ عَمَلَكُمْ وَرَسُولُهُ وَالْمُؤْمِنُونَ)
(سورة التوبة، الآية 105)

أولًا، أشكر الله الذي وفقني ويسّر لي سُبل العلم،
وأعانني على إتمام هذا العمل، فله الحمد والشكر دائمًا وأبدًا.

كما أتوجه بخالص الشكر والتقدير لأستاذي المشرف
البروفيسور عبد الوهاب منصور.
على ما قدمه لي من دعم علمي وتوجيهات قيّمة، وعلى حرصه الدائم
في متابعة كل تفاصيل هذا العمل، فله مني كل الامتنان والعرفان.

ولا يفوتني أن أشكر أساتذة تخصص رياضيات على ما قدموه
من علم وجهد خلال سنوات الدراسة.

كما أخص بالشكر عائلتي الكريمة التي وقفت إلى جانبي بالدعم والدعاء.

وأصدقائي وزملائي الذين كانوا خير رفيق في درب العلم والعمل.

ولا ننسى فضل الأستاذ يونس بن عمر له خالص الشكر والتقدير

إهداء

الحمد لله حمدًا وشكرًا وامتنانًا على البدء والختام
(وآخر دعواهم أن الحمد لله رب العالمين)

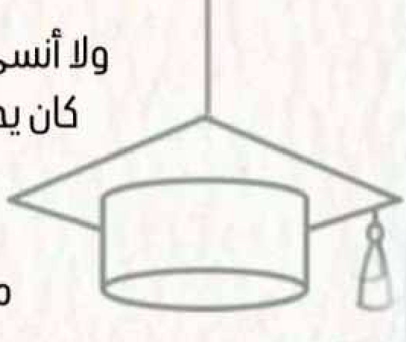
لم تكن الرحلة قصيرة ولا الطريق مفروشًا بالورود، لكنها كانت مليئة بالتحديات، فالحمد لله الذي يسّر البدايات وبلغنا النهايات بفضله وكرمه.

أُهدي هذا النجاح أولًا لنفسي الطموحة التي آمنت دومًا أن الجهد لا يضيع، ثم لكل من ساندني في مسيرتي الجامعية، ولإخوتي وأخواتي الذين كانوا سندًا وعونًا في كل خطوة.

بكل حب أُهدي ثمرة تعبي ونجاحي وتخرجي إلى والدي العزيز نعرورة مبروك، النور الذي أنار دربي، والسراج الذي لا ينطفئ نوره، بذل من أجلي الكثير، وكان سندي الذي أفتخر به دائمًا.

وإلى من علمتني الحرف قبل أن أكتب، وغرست في قلبي الصبر والقيم، إلى من تحملت آلام تعبي وسهرت على راحتي، أهديكِ يا حابي سالمة، يا أمي الحبيبة، نجاحي هذا عربون وفاء ومحبة وامتنان.

ولا أنسى أن أرسل سلامًا يلامس السماء إلى روح أخي الغالي عبد كريم، الذي كان يحلم بأن يراني في هذا اليوم. غيابك مؤلم، لكن طيفك حاضر في كل لحظة. هذا التخرج لك، كما هو لي، بل وأكثر.



ما كنت لأصل إلى هذا اليوم لولا توفيق الله، ثم صبري وجهدي.
فالحمد لله الذي بنعمته تتم الصالحات، وبكرمه ننعم بالفرح
وننسى مشقة الطريق.

Naroura Heddi

إهداء

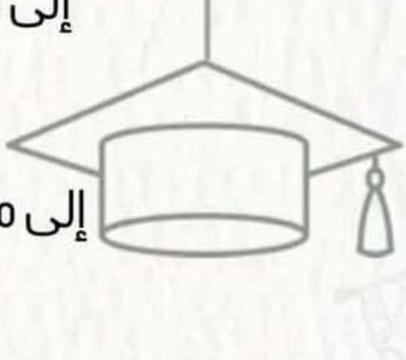
لم تكن الرحلة قصيرة ولا الطريق محفوفاً بالتسهيلات، لكنني فعلتها، فالحمد لله الذي يسر البدايات وبلغنا النهايات بفضله وكرمه.اهدي هذا النجاح لنفسي الطموحة.

إلى من غرس بداخلي القوة لأبحر في هذه الحياة، ها أنا اليوم أهدي تخرجي لأمير ابنته ومُدللها، أبي الحبيب مصباح ، كنت لنا السند الداعم دائمًا ، حفظك اللّٰه

إلى من كانت الداعم الأول لتحقيق طموحي ،
إلى من كانت ملجأي و مصدر قوتي أمي الغالية حياة.
إلى القلب الحنون من كانت دعواتها تحيطني خالتي العزيزة امال.

إلى من تسعد العين برؤيتهم . ويطرب القلب بـ نجواهم، وتهدأ نفسي بلقياهم ويبتسم الثغر بمحياهم، إخوتي و أخواتي

إلى من تسعد العين برؤيته، و ضلعي الثابت و بحر الأمان,و رفيق الدرب شكرا لتشجيعك، زوجي حبيبي محسن.



إلى من حبهم يعلو فوق كل حب، لكل من كان سندا و عونا في هذا الطريق، الى جميع افراد عائلتي

Laouini Maria

- ملخص:

تهدف هذه المذكرة إلى دراسة وتطوير بعض المتباينات المهمة بين القيم الطيفية (نصف القطر الطيفي ، نصف القطر العددي ، القيم الشاذة ، النظيم الجديد ... إلخ) للمؤثرات . و لتحقيق ذلك ؛ تم تعريف النظيم الجديد $\|.\|_{a,b}$ في $\mathcal{B}(\mathcal{H})$ حيث ان $\mathcal{B}(\mathcal{H})$ هو فضاء هيلبرتي مركب ، و a، b ينتميان إلى مجموعة الأعداد الطبيعية غير المعدومة . تمت دراسة بعض خصائص هذا النظيم وتطبيقاته على المتباينات بين القيم الطيفية للمؤثرات .

- الكلمات المفتاحية:

مؤثر خطي محدود ، نصف القطر الطيفي ، القيمة الشاذة ، نظيم مؤثرات الخطية الاعتيادية ، نصف القطر العددي ، المتباينات .

- **Abstract:**

This memory aims to study and develope some important inequalities between the spectral values (spectral radius , numerical radius , singular value , new norm ...) of operators . To achieve this ; the new norm $\|\cdot\|_{a,b}$ is defined on $\mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the complex Hilbert space , and a, b in the set of natural numbers without zero . Some properties of this norm and their applications to inequalities between the spectral values of operators are investigated .

- **Keywords:**

Bounded linear operators , spectral radius , singular values , usual operator norm , numerical radius , inequalities .

- **Résumé :**

Ce mémoire a pour objectif d'étudier et de développer certaines inégalités importantes entre les valeurs spectrales (rayon spectral , rayon numérique, valeur singulière, nouvelle norme...) des opérateurs . Pour ce faire; une nouvelle norme $\|.\|_{a,b}$ est définie sur $\mathcal{B}(\mathcal{H})$, où $\mathcal{B}(\mathcal{H})$ est un espace de Hilbert complexe , et a, b appartiennent à l'ensemble des nombres naturels non nuls . Certaines propriétés de cette norme ainsi que ses applications aux inégalités entre les valeurs spectrales des opérateurs sont étudiées .

- **Mots clés :**

Opérateur linéaire borné , rayon spectral , valeur singulière , norme usuelle des opérateurs linéaires , rayon numérique , inégalités .

CONTENTS

NOTATIONS

- $\mathbb{N}^*$: The set of natural numbers without zero $\{1, 2, 3, \cdots\}$.
- $\mathcal{H}$: Complex Hilbert space.
- $\mathcal{B}(\mathcal{H})$: Banach algebra of all bounded linear operators on Hilbert space $\mathcal{H}$.
- $||x||$: The norm of x
- $\langle \cdot, \cdot \rangle$: The inner product .
- $\overline{M}$: The closure of M.
- $M^{\perp}$: The orthogonal complement of M.
- $||T||$: The norm of T.
- T^{-1} : The inverse operator of T.
- T^* : The adjoint operator of T.
- $Re(T)$: The real part of T.
- $Im(T)$: The imaginary part of T.
- $|T|$: The absolute value of T.
- $R(T)$: The range of T.
- $N(T)$: The kernel of T.
- $\rho(T)$: The resolvent of T.
- $r(T)$: Spectral radius of T.
- $W(T)$: The numerical range of T.
- $\omega(T)$: The numerical radius of T.

INTRODUCTION

In the domain of functional analysis and linear algebra, normed spaces and inner product spaces serve as fundamental structures. These concepts find vital applications across various scientific fields, including quantum mechanics, signal theory, and control theory. Among the most important of these spaces stands the Hilbert space, characterized by its rich geometric and analytical properties that provide powerful tools for understanding infinite-dimensional spaces.

Over recent decades, operator theory has witnessed remarkable development and increasing interest among researchers due to its theoretical depth and wide-ranging applications. In this thesis, we provide an overview of how this theory has evolved and highlight key contributions that have shaped its development.

Historically, quadratic forms played an essential role in the emergence of the concept known as the numerical range of an operator. Let $\mathcal{H}$ be a non-trivial complex Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, the numerical range was introduced by Toeplitz in [26] as:

$$W(T) = \{\langle Tx, x \rangle \mid x \in \mathcal{H}, \|x\| = 1\}.$$

The Toeplitz-Hausdorff Theorem establishes that $W(T)$ is a convex subset of the complex plane(see [14]) . This concept has been extensively studied in the last few decades due to its significance and utility in analyzing matrices and operators.

Closely related to the numerical range is the concept of the *numerical radius*, which is defined for $T \in \mathcal{B}(\mathcal{H})$ as:

$$\omega(T) = \sup_{\|x\|=1} |\langle Tx, x \rangle|.$$

The numerical radius is equivalent to the usual norm on $\mathcal{B}(\mathcal{H})$ and has been the subject of extensive research, leading to many important results and inequalities. Prominent researchers such as F. Kittaneh and S.S. Dragomir have contributed significantly to this field . Numerous refinements and new inequalities continue to emerge, enhancing our understanding of operator behavior.

Another major notion is the *spectrum* of a bounded linear operator, which generalizes the set of eigenvalues of a matrix. For $T \in \mathcal{B}(\mathcal{H})$, the spectrum is defined by:

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not invertible}\}.$$

The spectrum possesses a rich structure and various classifications, which we shall explore in due course. The study of spectral properties, known as *spectral theory*, is fundamental in many fields—most notably in the mathematical formulation of quantum mechanics.

Another significant value is the *spectral radius*, given by:

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|.$$

Together, the numerical radius, the spectral radius, the singular value and the usual norm in $\mathcal{B}(\mathcal{H})$ constitute essential tools in the analysis of bounded operators and their properties.

This thesis aims to explore several advanced concepts related to normed spaces, focusing on the inner product, projections, and the structure of norm-related quantities, particularly in connection with special elements . Special attention is given to key inequalities that are indispensable tools for estimation and proof construction in mathematical analysis.

Among the topics that have attracted researchers' interest for decades is the relationship between the ordinary norm of a linear operator and its numerical radius. Starting from the classical works of Toeplitz and Hausdorff in the 1930s, the importance of the numerical range and numerical radius in understanding the behavior of operators began to emerge.Then came the famous work by Gustafson and Rao in 1974 (see [14]), which became a fundamental reference in this field, as they deeply explored the relationship between numerical ranges and systems, laying the groundwork for advanced analysis of these quantities. And there is another reference that studies this (see [16] and [19]). One of the most famous inequalities [14]

$$\frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|.$$

The work of researcher F. Kittaneh also stood out, as he presented improved inequalities (see [19] [20] and [21] , that appeared several reseach articles , such as . Interest in these relationships has continued in recent decades, with significant contributions from researchers such as S.S. Dragomir, who worked on precise subtle improvement to these inequalities using analytical techniques (see [9]).

These works did not only present the inequalities, but also demonstrated how these relationships can be exploited to prove spectral results and analyze stability in control systems and signal theory.

The memoire is organized into four main chapters:

- **Chapter 1**: We provide an overview of *norms* in vector spaces, with a special focus on the *Hilbert space*. We study the *inner product* and its role in defining norms, delve into *orthogonal projections*, and explore the concept of *some cless operators* and their properties.

- **Chapter 2**: This chapter is dedicated to exploring classical *inequalities* such as the *Cauchy-Schwarz inequality*, *Hölder's inequality*, and *Minkowski's inequality*, presenting their proofs and immediate applications in mathematical analysis.

- **Chapter 3**: We focus on the quantities the spectral radius, the numerical radius, and the singular value, discussing the fundamental inequalities that govern their behavior and providing rigorous proofs that illustrate the intricate relationships between these notions.

- **Chapter 4**: We introduce a *new norm* closely associated with the quantity the numerical radius, establish the necessary conditions for it to be a valid norm, and propose a set of *new inequalities* connected to it, discussing their implications and potential applications.

Through this study, we aim to deepen the theoretical understanding of norms, vector spaces, and numerical ranges, while laying a solid foundation for further research in this area.

Main results:

New inequalities were obtained, including:

- Let $T \in \mathcal{B}(\mathcal{H})$ and $a, b \in \mathbb{N}^*$.

$$\|T\|_{a,b} = \|a|T| + b|T^*|\|$$

- Let $T \in \mathcal{B}(\mathcal{H})$ and $a, b \in \mathbb{N}^*$. Then:

$$\|T\| \leq \|T\|_{a,b}$$

- Let $T \in \mathcal{B}(\mathcal{H})$ and $a, b \in \mathbb{N}^*$

$$\frac{1}{2(a+b)}\|T\|_{a,b} \leq \omega(T) \leq \frac{1}{2}\max\{\frac{1}{a}, \frac{1}{b}\}\|T\|_{a,b}$$

- Let $T, S \in \mathcal{B}(\mathcal{H})$ and $a, b \in \mathbb{N}^*$. Then :

$$\|TS\|_{a,b} \leq 4(a+b)\omega(T)\omega(S)$$

CHAPTER

1

PRELIMINARIES

In this chapter, we present the fundamental definitions and properties related to Hilbert spaces and operator theory, which are essential for the remainder of this thesis . We denote by $\mathbb{H}$ a complex Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$, and by $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators on $\mathcal{H}$. This chapter in intended to be as self-contained as possible, and therefore we have collected the most important results needed later in the thesis, while omitting the proofs of many theorems for the sake of brevity.

1.1 Generalities

Definition 1.1 *Let $\mathcal{H}$ be a complex vector space.*

1. *A **norm** on $\mathcal{H}$ is a map $\|\cdot\| : \mathcal{H} \longrightarrow \mathbb{R}_+$ such that for all $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{C}$:*
   - *(a)* $\|x\| \geq 0$ *(strictly positive).*
   - *(b)* $\|x\| = 0$ *if and only if* $x = 0$.
   - *(c)* $\|\alpha x\| = |\alpha|\|x\|$ *(strictly homogeneous).*
   - *(d)* $\|x + y\| \leq \|x\| + \|y\|$ *(triangle inequality).*

Example 1.1 *Common instances of norms include*

- *Euclidian norm in* $\mathbb{R}^2$: $\|x\| = \sqrt{x_1^2 + x_2^2}$
- $\|x\|_1 = \sum_{i=1}^{n} |x_i|, x \in \boldsymbol{\ell}^1$
- $\|x\|_\infty = \max_{1\leq i\leq n} |x_i|$

*A vector space $\mathcal{H}$ on which there is a norm is called a **normed vector space**, or just a **normed space**.*

*A **Complete space** if every cauchy sequence in X converges to a limit that lies in X.*

*A **Banach space** is a complete complex normed vector space.*

Definition 1.2 *Let $\mathcal{H}$ be a complex vector space.*

1. *An **inner product** on $\mathcal{H}$ is a map $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H} \longrightarrow C$ such that for all $x,y,z \in \mathcal{H}$, and $\alpha,\beta \in \mathbb{C}$*
    - *(a)* $\langle x,x\rangle \geq 0$.
    - *(b)* $\langle x,x\rangle = 0$ *if and only if* $x = 0$.
    - *(c)* $\langle \alpha x + \beta y, z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle$.
    - *(d)* $\langle x,y\rangle = \overline{\langle y,x\rangle}$.

Example 1.2 *Let $x,y \in \mathcal{H}$*

- *Inner product in $\mathbb{R}^n$*

$$\langle x,y\rangle = \sum_{i=1}^{n} x_i y_i$$

- *Inner product in $\mathbb{C}^n$*

$$\langle x,y\rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

- *Inner product in $L^2([a,b])$:*

$$\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}\,dx$$

*A complex vector space $\mathcal{H}$ with an inner product $\langle\cdot,\cdot\rangle$ is called an **inner product space**.*

*An inner product space which is complete with respect to the metric associated with the norm induced by the inner product is called a **Hilbert space**.*

Remark 1 *It comes directly from the definition of inner product that*

(1) $\langle x, y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$ *for all $x,y,z \in X$.*

(2) $\langle x, \lambda y\rangle = \overline{\lambda}\langle x,y\rangle$ *for all $x,y \in X$ and $\lambda \in \mathbb{K}$.*

Proof

(2) for all $x,y \in X$ and $\lambda \in \mathbb{K}$

$$\begin{aligned}\langle x, \lambda y\rangle &= \overline{\langle \lambda y, x\rangle} = \overline{\lambda\langle y,x\rangle} \\ &= \overline{\lambda}\,\overline{\langle y,x\rangle} = \overline{\lambda}\langle x,y\rangle\end{aligned}$$

Definition 1.3 *Let $\mathcal{H}$ be a vector space over $\mathbb{C}$. It is said to be a Hilbert space if it is an inner product space and $\mathcal{H}$ with associated norm is a Banach space, and we denote $(\mathcal{H},\langle\cdot,\cdot\rangle)$ a complex Hilbert space.*

Example 1.3

(1) The Euclidian inner product in $\mathbb{C}^n$ ($n \in \mathbb{N}^*$) is defined as follows

$$\forall x, y \in \mathbb{C}^n : \langle x, y\rangle = \sum_{k=1}^{n} x_k \overline{y_k}$$

$\mathbb{C}^n$ with its Euclidian inner product $(\mathbb{C}^n, \langle\cdot,\cdot\rangle)$ is a Hilbert space.

(2) Consider the following vector space over $\mathbb{C}$

$$\ell^2(\mathbb{C}) = \{x = (x_n)_{n\in\mathbb{N}} \subset \mathbb{C} : \textstyle\sum_{n=1}^{\infty} |x_n|^2 < \infty\}$$

(3) The fuction space $L^2([a,b])$: the set of square-integrable function on the interval [a, b]:

$$L^2([a,b]) = \left\{ f : \int_a^b |f(x)|^2 dx < \infty \right\}$$

This space is a Hilbert space when it is endowed with the following inner product

$$\forall x, y \in \ell^2 : \langle x, y\rangle = \textstyle\sum_{n=1}^{\infty} x_n \overline{y_n}$$

Corollary 1.1 *Relation between Norm and Inner product is*

$$\|x\| = \sqrt{\langle x, x\rangle}$$

1.2 Orthogonal Decomposition of Hilbert space

From now we consider $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ a complex Hilbert space, and is the associated norm of the inner product.

Definition 1.4 *M is said to be a closed linear subspace of $\mathcal{H}$ if it is a linear subspace of $\mathcal{H}$ and $\overline{M} = M$.*

Corollary 1.2 *Let M be a linear subspace of $\mathcal{H}$. Then $\overline{M}$ is a linear subspace of $\mathcal{H}$ as well.*

Definition 1.5 *Let M and F be two linear subspaces of $\mathcal{H}$. Then $\mathcal{H}$ is said to be the direct sum of M and F, and we write $\mathcal{H} = M \oplus F$, if $\mathcal{H} = M + F$ and $M \cap F = \{0\}$.*

Lemme 1.1 *Let M and F be two linear subspaces of $\mathcal{H}$. Then $\mathcal{H} = M \oplus F$ if and only if for every $x \in \mathcal{H}$ there exist unique vectors $y \in M$ and $z \in F$ such that $x = y + z$.*

Definition 1.6 *The vectors $x, y \in \mathcal{H}$ are said to be orthogonal if $\langle x, y\rangle = 0$, and we write $x \perp y$.*

Corollary 1.3 *If $x, y \in \mathcal{H} \setminus \{0\}$ are orthogonal. Then they are linearly independent.*

Theorem 1.1 *(Pythagoras's theorem)*
If $x, y \in \mathcal{H}$ are orthogonal, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Definition 1.7 *Let M be subset of $\mathcal{H}$. The orthogonal complement of M is the set*

$$M^{\perp} = \{x \in \mathcal{H} : \langle x, y \rangle = 0 \quad \forall y \in M\}$$

Proposition 1.1 *Let M and F be two subsets of $\mathcal{H}$ then:*

(1) $\mathcal{H}^{\perp} = \{0\}$ *and* $\{0\}^{\perp} = \mathcal{H}$.

(2) $M^{\perp}$ *is a closed linear subspace of* $\mathcal{H}$.

(3) If $0 \in M$ *then* $M \cap M^{\perp} = \{0\}$, *otherwise* $M \cap M^{\perp} = \emptyset$.

(4) If $M \subset F$ *then* $F^{\perp} \subset M^{\perp}$.

(5) $M \subset (M^{\perp})^{\perp}$.

Theorem 1.2 *(Orthogonal Decomposition)*
Let M be a closed linear subspace of $\mathcal{H}$. Then $\mathcal{H} = M \oplus M^{\perp}$ is the direct sum of M and $M^{\perp}$, i.e. for every $x \in \mathcal{H}$ there exist a unique $y \in M$ and a unique $z \in M^{\perp}$ such that $x = y + z$.

Corollary 1.4 *Let M be a linear subspace of $\mathcal{H}$. Then*

(1) $\mathcal{H} = \overline{M} \oplus M^{\perp}$

(2) $(M^{\perp})^{\perp} = \overline{M}$

Definition 1.8 *Let f be a mapping from $\mathcal{H}$ to $\mathbb{C}$. Then f is said to be a linear functional if it satisfies*

$$\forall x, y \in \mathcal{H}, \forall \lambda \in \mathbb{C} : f(\lambda x + y) = \lambda f(x) + f(y)$$

Definition 1.9 *Let f be a linear functional from $\mathcal{H}$ to $\mathbb{C}$. Then f is said to be a bounded linear functional if it satisfies*

$$\exists c > 0, \quad \forall x \in \mathcal{H} : |f(x)| \leq c\|x\|$$

1.3 Types of operators in Hilbert space

Definition 1.10 *Let $\mathcal{H}$ be a Hilbert space. An operator $T : \mathcal{H} \to \mathcal{H}$ is*

- ***Linear operator*** *if*

  $$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y),$$

  for all $x, y \in \mathcal{H}$, and scalars α, β.

- ***Bounded Linear operator*** *if T is linear and there exists a positive number k such that*

$$||Tx|| \leq k||x|| \quad \textit{for all } x \in \mathcal{H}.$$

Corollary 1.5 *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$. Then*

(1) $\forall x \in \mathcal{H}, ||Tx|| \leq ||T||||x||$

(2) $||TS|| \leq ||T||||S||$

(3) $||T^n|| \leq ||T||^n$

Definition 1.11 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

(1) *The range of T is the set*

$$R(T) = \{Tx : x \in \mathcal{H}\}$$

(2) *The kernel of T is the set*

$$N(T) = \{x \in \mathcal{H} : Tx = 0\}$$

(3) *The identity I is the operator given by $I(x) = x, \quad \forall x \in \mathcal{H}$*

Proposition 1.2 *Let $T \in \mathcal{B}(\mathcal{H}))$. Then*

(1) *$R(T)$ is a linear subspace of $\mathcal{H}$.*

(2) *$N(T)$ is a closed linear subspace of $\mathcal{H}$.*

Lemme 1.2 *Let $\mathcal{H}, \mathcal{K}$ be two Hilbert spaces over $\mathbb{C}$ and $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, then*

1. $N(T) = R(T^*)^{\perp}$.
2. $N(T)^* = R(T)^{\perp}$.
3. *$N(T)^* = \{0\}$ if and only if $R(T)$ is dense in $\mathcal{K}$.*

Proof

(1) *Let $x \in N(T)$ and $z \in R(T^*)^{\perp}$, then $\exists y \in \mathcal{K}$ such that $z = T^*y$, we have*

$$\langle x, z \rangle = \langle x, T^*y \rangle = \langle Tx, y \rangle = \langle 0, y \rangle = 0$$

this shows that $x \in R(T^*)^{\perp}$ and hence $N(T) \subseteq R(T^*)^{\perp}$. On the other hand, suppose that $x \in R(T^*)^{\perp}$. Since $T^*Tx \in R(T)^*$, then

$$\langle x, T^*Tx \rangle = 0$$

$$\langle x, T^*Tx \rangle = \langle Tx, Tx \rangle = ||Tx||^2 = 0$$

therefore $Tx = 0$, hence $x \in N(T)$ and $R(T^*)^{\perp} \subseteq N(T)$, consequently

$$N(A) = R(T^*)^{\perp}$$

(2) From (1) we deduce, $N(T)^* = (R(T^*)^*)^{\perp} = R(T)^{\perp}$

(3) Recall that $(F^{\perp})^{\perp} = \overline{F}$ if F is closed, $(F^{\perp})^{\perp} = F$. Suppose that $R(T^*) = \{0\}$, then from (2) we have $R(T)^{\perp} = \{0\}$, then $(R(T)^{\perp})^{\perp} = \{0\}^{\perp}$ which gives $\overline{R} = \{0\}^{\perp} = \mathcal{K}$.

Conversely, suppose that $\overline{R(T)} = \mathcal{K}$, that is $(R(T)^{\perp})^{\perp} = \mathcal{K}$. Therefore

$$\left(\left(R(T^*)^{\perp}\right)^{\perp}\right)^{\perp} = \mathcal{K}^{\perp} = \{0\}.$$

Since $R(T)^{\perp}$ is closed we have $\left((R(T^*)^{\perp})^{\perp}\right)^{\perp} = R(T)^{\perp} = N(T^*)$. Consequently

$$N(T)^* = \{0\}$$

Corollary 1.6 *Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. The following statements are equivalent*

1. *(1) T is invertible .*
2. *(2) $N(T)^* = \{0\}$ and there exists $\alpha > 0$ such that $\|Tx\| \geq \alpha\|x\|, \forall x \in \mathcal{H}$.*

Proof
(1) $\Rightarrow$ (2) Suppose that T is invertible, then $R(T) = \mathcal{H}$ and from number 3 of the previous lemma, $N(T)^* = \{0\}$. On the other hand, since T is invertible T^{-1} is bounded, then

$$\exists c > 0 : \|T^{-1}y\| \leqslant c\|y\|$$

But $R(T) = \mathcal{H}$, then

$$\|x\| \in \mathcal{H}, \exists y \in \mathcal{H} : y = Tx, x = T^{-1}y.$$

Thus, for $\alpha = \frac{1}{c}$ we have

$$\|T^{-1}y\| \leqslant c\|y\| \Longleftrightarrow \alpha\|x\| \leqslant \|Tx\|.$$

(2) $\Rightarrow$ (1) If $N(T^*) = \{0\}$, then $R(T)$ is dense in $\mathcal{H}$. Let $y \in \mathcal{H}$ and $\{y_n\} \subset R(T)$ be a sequence that converges to y. Then, $\{y_n\}$ is a Cauchy sequence and

$$\|y_n - y_m\| = \|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \geqslant \alpha\|x_n - x_m\|$$

therefore $\{x_n\}$ is a Cauchy too. Since $\mathcal{H}$ is complete, $\{x_n\}$ converges to an $x \in \mathcal{H}$. Moreover, since T is continuous

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = Tx$$

Consequently, $y \in R(T)$ and $R(T)$ is closed, which shows that $R(T) = \mathcal{H}$. On the other hand, if $x \in N(T)$, one has $Tx = 0$, then $0 = \|Tx\| \geqslant \alpha\|x\| \geqslant 0$, which shows that T is injective, hence bijective. Since T is also continuous, we deduce by Banach theorem that T is invertible.

Definition 1.12 *Let T be a bounded linear operator. The norm of T is defined by*

$$\|T\| = \sup_{\|x\|=1} \|Tx\|$$

An equivalent definition of the operator norm is

$$\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in \mathcal{H} \setminus \{0\}\right\}$$

Theorem 1.3 *: Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\} = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in \mathcal{H} \setminus \{0\}\right\} = \sup\{|\langle Tx, y\rangle| : \|x\| = \|y\| = 1\}$$

Definition 1.13 *To each operator $T \in \mathcal{B}(\mathcal{H})$ corresponds a unique operator $T^* \in \mathcal{B}(\mathcal{H})$ that satisfies*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \text{for all } x, y \in \mathcal{H}.$$

The operator T^* is called the ***adjoint*** *of the operator T. If $T = T^*$, then T is self-adjoint .*

Example 1.4 *Let the linear operator $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by*
$T(x) = Ax$, where $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$
we have $\langle u, v\rangle = u^t v$

$$A^t = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

So

$$T^*(x) = A^t x = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} x$$

Proposition 1.3 *Let $T \in \mathcal{B}(\mathcal{H})$, then*

- $\|T\| = \|T^*\|$ *and* $\|T\|^2 = \|T^*T\| = \|TT^*\|$.
- $(\lambda T)^* = \overline{\lambda} T^*$, $\forall \lambda \in \mathbb{C}$.
- $(T^*)^* = T$.
- $(T^*)^{-1} = (T^{-1})^*$, *if T is invertible .*

Definition 1.14 *The set of all bounded linear operators on $\mathcal{H}$ is a unitary algebra over $\mathbb{C}$ denoted by $\mathcal{B}(\mathcal{H})$, where the addition , the external product , and the product are defined as follows . Let $T, S \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$*

(1) $\forall x \in \mathcal{H}$: $(T + S)x = Tx + Sx$

(2) $\forall x \in \mathcal{H}$: $(\lambda T)x = \lambda Tx$

(3) $\forall x \in \mathcal{H}$: $(TS)x = T(Sx)$

And the unitary element is the identity $I(x) = x, \quad \forall x \in \mathcal{H}$. Moreover, $\mathcal{B}(\mathcal{H})$ is a Banach algebra when we equip it with the following norm

$$\|T\| = \inf\{c > 0 : \|Tx\| \leq c\|x\| \quad \text{for all } x \in \mathcal{H}\}$$

Theorem 1.4 *(Polarisation identity)*

Let $\mathcal{H}$ be an inner product space, and let $x, y \in \mathcal{H}$. Then

$$\langle x,y\rangle = \frac{1}{4}[\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x+iy\|^2]$$

Proof

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex inner-product space, and set

$$\|x\|^2 = \langle x,x\rangle$$

We claim that for all $x, y \in \mathcal{H}$

$$\langle x,y\rangle = \frac{1}{4}\Big[\|x+y\|^2 - \|x-y\|^2 \ + \ i\,\|x+iy\|^2 - i\,\|x-iy\|^2\Big]$$

let's first prove the real part

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2$$
$$\|x-y\|^2 = \langle x-y, x-y\rangle = \|x\|^2 - \langle x,y\rangle - \langle y,x\rangle + \|y\|^2$$

Subtracting gives

$$\|x+y\|^2 - \|x-y\|^2 = 2\Big(\langle x,y\rangle + \langle y,x\rangle\Big) = 4\,Re\langle x,y\rangle$$

And now we prove the imaginary part

$$\|x+iy\|^2 = \langle x+iy,\, x+iy\rangle = \|x\|^2 + \langle x,iy\rangle + \langle iy,x\rangle + \|y\|^2$$
$$\langle x,iy\rangle = i\,\langle x,y\rangle, \quad \langle iy,x\rangle = \overline{\langle x,iy\rangle} = -i\,\overline{\langle x,y\rangle}$$

So

$$\|x+iy\|^2 = \|x\|^2 + \|y\|^2 + i\,\langle x,y\rangle - i\,\overline{\langle x,y\rangle} = \|x\|^2 + \|y\|^2 - 2\,Im\langle x,y\rangle$$

Similarly

$$\|x-iy\|^2 = \|x\|^2 + \|y\|^2 + 2\,Im\langle x,y\rangle$$

Hence

$$\|x+iy\|^2 - \|x-iy\|^2 = -4\,Im\langle x,y\rangle$$

3. Combine

$$\Big(\|x+y\|^2 - \|x-y\|^2\Big) + i\Big(\|x+iy\|^2 - \|x-iy\|^2\Big) = 4\,Re\langle x,y\rangle - 4\,i\,Im\langle x,y\rangle = 4\,\langle x,y\rangle$$

Dividing by 4 *, we get what is required .*

Theorem 1.5 *(Generalized Polarization Identity): For each $T \in \mathcal{B}(\mathcal{H}) and x, y \in \mathcal{H}$, we have*

$$\langle Tx,y\rangle = \frac{1}{4}[\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle] + \frac{i}{4}[\langle T(x+iy), x+iy\rangle - \langle T(x-iy), x-iy\rangle]$$

1.3.1 Some classes of linear operators in $\mathcal{B}(\mathcal{H})$

Definition 1.15 *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is called:*

- ***Self-adjoint operator*** *if $T^* = T$.*

  Example 1.5 *:*
  $$T(f)(x) = xf(x), \quad \textit{for } f \in L^2([0,1])$$
  This operator is self-adjoint because:
  $$\langle Tf, g\rangle = \int_0^1 xf(x)\overline{g(x)}\,dx = \langle f, Tg\rangle$$

  Normal operator *if $TT^* = T^*T$.*

  Positive operator *$\forall x \in \mathcal{H} : \langle Tx, x\rangle \geq 0$, and we denote $T \geq 0$. (T is strictly positive operator if $\forall x \in \mathcal{H} \setminus \{0\} : \langle Tx, x\rangle > 0$, and we denote $T > 0$)*

  Example 1.6 *:*
  $$T(f)(x) = x^2 f(x), \quad \textit{for } f \in L^2([0,1])$$
  Since $x^2 \geq 0$, then $\langle Tf, f\rangle \geq 0$, and T is self-adjoint, hence positive.

  Unitary operator *if $T^*T = TT^* = I$.*

  Example 1.7 *:*
  $$T(f)(x) = f(x+a), \quad \textit{for } f \in L^2(\mathbb{R})$$
  Translation operator is unitary since it preserves inner product and is invertible.

  Isometry operator *if $T^*T = I$.*

  Example 1.8 *:*
  $$T(f)(x) = \sqrt{2}f(2x), \quad \textit{from } L^2([0,\tfrac{1}{2}]) \to L^2([0,1])$$
  We have $\|Tf\| = \|f\|$, so it's an isometry, but not necessarily surjective.

  Projection operator *if $T^2 = T$.*

  Example 1.9 *:*
  $$T(f)(x) = \int_0^1 f(t)\,dt$$
  Constant projection operator. Clearly, $T^2 = T$, so it's a projection.

  Orthogonal projection operator *if $T^2 = T = T^*$. Same as above. Since the range is orthogonal to the kernel, it is an orthogonal projection.*

  Quasinormal operator*: $T(T^*T) = (T^*T)T$.*

  Example 1.10 *: In ℓ^2:*
  $$T((x_1, x_2, x_3, \ldots)) = (0, x_1, x_2, \ldots)$$
  This is the forward shift operator. It satisfies $T(T^*T) = (T^*T)T$.

  Hyponormal operator *if $T^*T \geq TT^*$. ($T \geq S$ if and only if $T - S \geq 0$)*

Example 1.11 *:*

$$T(f)(x) = \phi(x)f(x), \quad \phi \in L^{\infty}, \phi \geq 0$$

Multiplication by a non-negative function is hyponormal.

Corollary 1.7 *If $T \in \mathcal{B}(\mathcal{H})$ is positive , so is self-adjoint.*

Theorem 1.6 *Let $T \in \mathcal{B}(\mathcal{H})$. Then there exists self-adjoint operators $Re(T), Im(T) \in \mathcal{B}(\mathcal{H})$ such that*

$$T = Re(T) + iIm(T)$$

Necessarily $Re(T) = \dfrac{T+T^*}{2}$, and $Im(T) = \dfrac{T-T^*}{2i}$. The decomposition is called the Cartesian decomposition of T. The operators $Re(T)$ and $Im(T)$ are called real part, and imaginary part of T respectively.

Proposition 1.4 *Let $T \in \mathcal{B}(\mathcal{H})$ be a positive operator . Then*

(1) $\|T^{\frac{1}{2}}\| = \|T\|^{\frac{1}{2}}$

(2) $N(T^{\frac{1}{2}}) = N(T)$

Proof

(1) Let $x \in \mathcal{H}$:

$$\|T^{\frac{1}{2}}x\|^2 = \langle T^{\frac{1}{2}}x, T^{\frac{1}{2}}x\rangle = \langle Tx, x\rangle \Longrightarrow \sup_{\|x\|=1} \|T^{\frac{1}{2}}x\|^2 = \sup_{\|x\|=1} \langle Tx, x\rangle$$

Since $T \geq 0$, then T is self-adjoint :

$$\|T\| = \sup_{\|x\|=1} \langle Tx, x\rangle \quad (\langle Tx, x\rangle \geq 0)$$

Then $\|T^{\frac{1}{2}}\|^2 = \|T\|$
hence $\|T^{\frac{1}{2}}\| = \|T\|^{\frac{1}{2}}$
That is to say

$$\|T^{\frac{1}{2}}\| = \|T\|^{\frac{1}{2}}$$

(2) $N(T) \subset N(T^{\frac{1}{2}})$, let $x \in N(T) \Longleftrightarrow Tx = 0$
and we have $\|T^{\frac{1}{2}}x\|^2 = \langle Tx, x\rangle = 0$

Then $\|T^{\frac{1}{2}}x\| = 0 \Longleftrightarrow T^{\frac{1}{2}}x = 0 \Longleftrightarrow x \in N(T^{\frac{1}{2}})$,
therefore $N(T) \subset N(T^{\frac{1}{2}})$.

$N(T^{\frac{1}{2}}) \subset N(T)$, let $x \in N(T^{\frac{1}{2}}) \Longleftrightarrow T^{\frac{1}{2}}x = 0 \Longrightarrow T^{\frac{1}{2}}T^{\frac{1}{2}}x = 0 \Longrightarrow Tx = 0.$

Hence

$$N(T^{\frac{1}{2}}) = N(T)$$

Definition 1.16 *An operator $U \in \mathcal{B}(\mathcal{H})$ is called a partial isometry if*

$$\|Ux\| = \|x\| \quad \textit{for all } x \in N(U)^{\perp}.$$

In that case $N(U)^{\perp}$ is called the initial space of U and $R(U)$ is called the final space .
The range of U is always closed .

1.3.2 Closed operator

Definition 1.17 *The operator $T : D(T) \subset X \longrightarrow Y$, is said to be closed if $D(A) \times R(T)$ is closed in the space $X \times Y$, that is*

$$\forall (x_n) \subset D(T) : \lim x_n = x, \;\; x \in D(T), \;\; \lim Tx_n = Tx$$

$$\|x\|_{D(T)} = \|x\|_X + \|Tx\|_Y$$

Remark 2 *Let $T : D(T) \subset X \to Y$ be an operator. We sometimes endow the domain $D(T)$ by the norm*

Theorem 1.7 *(closed Graph Theorem) Let X, Y be Banach spaces and $T : D(T) \subset X \to Y$ be a linear operator. If the graph $G(T)$ is closed in the topology of $D(T)$, then the operator T is bounded.*

Proof

Since $X \times Y$ is a Banach space and $G(T)$ is closed, then $G(T)$ is a Banach subspace of $X \times Y$.
Define the linear transformation $R : G(T) \to D(T)$ by $R(x, Tx) = x$. Then, R is a bijection between G and $D(T)$. Moreover

$$\|R(x, Tx)\| = \|x\| \leq \|x\| + \|Tx\| = \|(x, Tx)\|_{G(T)}$$

Therefore, R is bounded and $\|R\| \leq 1$. Consequently, from the open mapping theorem, there exists $S : D(T) \to G(T)$ such that $SR = I_{G(T)}$ and $RS = I_{D(T)}$. In particular

$$Sx = (x, Tx), \textit{ for all } x \in D(T)$$

Thus

$$\|Tx\| \leq \|x\| + \|Tx\| = \|(x, Tx)\| = \|Sx\| \leq \|S\|\|x\|$$

which shows that T is bounded.

Remark 3 *If T is a closed operator, then $\ker T$ is closed in X.*

1.3.3 Invertible operators

Definition 1.18 *An operator $T \in \mathcal{L}(X)$ is said to be invertible if there exists an operator $S : R(T) \subset Y \to X$, such that $S \in \mathcal{L}(X)$ and $ST = I_{D(T)}$ and $TS = I_{R(T)}$. In this case S is denoted T^{-1}.*

Example 1.12 *For $f \in \mathcal{C}([0,1])$ and defined $T_f \in \mathcal{L}(L^2([0,1]))$ by*

$$(T_f u)(x) = f(x)u(x), \quad \forall u \in L^2([0,1]).$$

Clearly, $T_f \in \mathcal{L}(L^2([0,1]))$. Let f be the function defined by $f(x) = 1+x$. Then , T_f is invertible. Indeed, for $g(x) = \frac{1}{1+x}$, we have $T_g \in \mathcal{L}(L^2([0,1]))$.
Moreover

$$(T_f T_g u)(x) = f(x)g(x)u(x) = u(x)$$

and

$$(T_g T_f u)(x) = g(x)f(x)u(x) = u(x)$$

which shows that T_f is invertible and $T_f^{-1} = T_g$.

Theorem 1.8 *Let X be a Banach space and $T \in \mathcal{L}(X)$ with $\|I - T\| < 1$, then T is invertible with*

$$T^{-1} = \sum_{n \geq 0} (I - T)^n$$

Proof
Since $\|I - T\| < 1$, the series $\sum_{n\geq 0} \|I - T\|^n$ converges. On the other hand

$$\|(I - T)^k\| \leq \|I - T\|^k$$

then the series $\sum_{n\geq 0}(I - T)^n$ converges and $\sum_{n\geq 0}(I - T)^n$ is absolutely convergent series. Let S be its limit and $S_k = \sum_{n=0}^{k}(I - T)^n$, then we have

$$\|TS_k - I\| = \|(I - (I - T))S_k - I\| = \|(I - T)^{k+1}\| \leq \|I - T\|^{k+1}$$

Thus

$$0 \leq \lim_{k\to\infty} \|TS_k - I\| \leq \lim_{k\to\infty} \|(I - T)\|^{k+1} = 0$$

Therefore

$$TS - I = \lim_{k\to\infty} (TS_k - I) = 0$$

Similarly

$$ST - I = \lim_{k\to\infty} (S_k T - I) = 0$$

which completes the proof.

Theorem 1.9 *Let T be a linear operator from normed linear space X into normed linear space Y. Then , T^{-1} exists and is continuous, if and only if there $m > 0$, such that*

$$\|Tx\| \geq m\|x\|, \quad \forall x \in X.$$

Definition 1.19 *Let X, Y be normed linear spaces. If an invertible operator $T \in \mathcal{L}(X, Y)$ exists, then X, Y are isomorphic, and T is an isomorphism (between X and Y).*

Lemme 1.3 *If the normed linear spaces X, Y are isomorphic, then*

(1) $\dim X < \infty$ *if and only if* $\dim Y < \infty$, *in which case* $\dim X = \dim Y$

(2) X is separable if and only if Y is separable

(3) X is complete (i.e., Banach) if and only if Y is complete (i.e., Banach)

Theorem 1.10 *Let X and Y two are Banach space , so if $T \in \mathcal{L}(X, Y)$ is bijective, it is invertible.*

1.3.4 Compact operators

Definition 1.20 *Let X, Y be normed spaces. A linear operator $T \in \mathcal{L}(X)$ is said to be compact if the image by T of every bounded set B of X is relatively compact in Y. The set of all compact operators from X into Y is denoted by $\mathcal{L}(X)$.*

Proposition 1.5 *Let X, Y be normed spaces and $T \in \mathcal{L}(X)$. The following statements are equivalent*

1. *T is compact.*

2. *The image of the unit ball $B_X(0,1)$ of X is relatively compact in Y.*
3. *Every bounded sequence $\{x_n\}$ in X has a subsequence x_{n_k} such that $\{Tx_{n_k}\}$ converges in Y.*

Lemme 1.4 *Every compact operator $T \in \mathcal{L}(X)$ is continuous.*

Theorem 1.11 *The set $\mathcal{L}(X)$ is a closed subspace of $\mathcal{L}(X)$ for the operator norm.*

Proposition 1.6 *Let $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(X)$. If at least one of the operators S,T is compact then ST is compact.*

Proof

Let $\{x_n\}$ be a bounded sequence in X. If T is compact then, there exists a subsequence $\{x_{n_k}\}$ such that $\{Tx_{n_k}\}$ converges in Y, and since S is continuous, the sequence $\{STx_{n_k}\}$ is still convergent. If T is not compact , then $\{Tx_n\}$ is still bounded and, since S is compact there exists a subsequence $\{Tx_{n_k}\}$ such that $\{STx_{n_k}\}$ converges, therefore ST is compact.

Definition 1.21 *An operator T is said to be of finite rank, if its range (image) $R(T)$ is of finite dimension. In this case we note* $\dim R(T) = r(T)$.

Theorem 1.12 *Let $\{T_n\} \subset \mathcal{L}(X)$ be a sequence of bounded operators of finite range and let $T \in \mathcal{L}(X)$ be its limit, then T is compact.*

Proof

Since every operator of finite range is compact, and the set $\mathcal{L}(X)$ is closed, T is compact.

1.4 Model of T

Definition 1.22 *Let $T \in \mathcal{B}(\mathcal{H})$, the absolute value of T is the unique positive square root of the positive operator T^*T, and we denote it by $|T|$, that is $|T| = \sqrt{T^*T}$.*

Proposition 1.7 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

1. $\|T\| = \||T|\|$
2. $|T| = T$ *if and only if* $T \geq 0$
3. $|T| = |T^*|$ *if and only if T is normal*

Proof

1. $\|T\|^2 = \|T^*T\| = \||T|^2\| = \||T|\|^2 = \|T\|^2 \Rightarrow \|T\| = \||T|\|$
2. *First assume that $|T| = T$, since $\sqrt{T^*T} = T$ and $\sqrt{T^*T} \geq 0$, then $T \geq 0$.*

   Now suppose that $T \geq 0$, then

$$T^* = T \Rightarrow T^*T = T^2 \Rightarrow \sqrt{T^*T} = T \Rightarrow |T| = T$$

3. *Assume that* $|T| = |T^*|$, *then*

$$\sqrt{T^*T} = \sqrt{TT^*} \Rightarrow \sqrt{T^*T}^2 = \sqrt{TT^*}^2 \Rightarrow T^*T = TT^* \Leftrightarrow T \text{ is normal}$$

   Suppose that T *is normal, then* $T^*T = TT^*$, *so*

$$\sqrt{T^*T} = \sqrt{TT^*} \Rightarrow |T| = |T^*|$$

Remark 4 *If* T *is a self-adjoint operator, this doesn't imply that* $|T| = T$.
Considering the following self-adjoint matrix

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, T^*T = T^2 = I \Rightarrow |T| = \sqrt{T^*T} = I$$

.
So $|T|$ *equals* I *not* T.

Proposition 1.8 *Let* $T \in \mathcal{B}(\mathcal{H})$. *Then* $\||T||T^*|\| = \|T^2\|$.

Proof *Since* $T|T|^2 = TT^*T = |T^*|^2T$, *then* $T|T|^2 = |T^*|^2T$, *also* $\||T|^2\| = \||T^*|^2\| = \|T\|^2$, *we obtain* $|T^*|T = T|T|$
(becouse when taking T, S *be two positive operators and* $X \in \mathcal{B}(\mathcal{H})$ *such that* $TX = XS$ *and* $\|T\| = \|S\|$ *we get* $\sqrt{T}X = \sqrt{X}T$*)* .

- *Let* $x \in \mathcal{H}$ *then*

$$\begin{aligned}
\||T||T^*|Tx\|^2 =& \langle |T||T^*|Tx, |T||T^*|Tx\rangle \\
=& \langle |T|^2|T^*|Tx, |T^*|Tx\rangle \\
=& \langle T^*T|T^*|Tx, |T^*|Tx\rangle \\
=& \langle T|T^*|Tx, T|T^*|Tx\rangle \\
=& \langle TT|T|x, TT|T|x\rangle \\
=& \|T^2|T|x\|^2 \\
\leq& \|T^2\|^2\||T|x\|^2
\end{aligned}$$

  Since $\||T|x\| = \|Tx\|$, *then* $\||T||T^*|Tx\| \leq \|T^2\|\|Tx\|$ *for all* $x \in \mathcal{H}$.
  Therefore

$$\||T||T^*|x\| \leq \|T^2\|\|x\| \quad \forall x \in R(T)$$

  And it can be extended as usual to $\overline{R(T)}$, *so* $\||T||T^*|x\| \leq \|T^2\|\|x\| \quad \forall x \in \overline{R(T)}$.
  Since $\mathcal{H} = \overline{R(T)} \oplus N(T^*)$, *let* $x \in N(T^*) \Leftrightarrow T^*x = 0 \Rightarrow TT^*x = 0 \Rightarrow \sqrt{TT^*}x = 0 \Rightarrow |T^*|x = 0$
  Then $|T||T^*|x = 0$ *for all* $x \in N(T^*)$.
  $\forall x \in \mathcal{H}$, *there exist* $x_1 \in R(T)$ *and* $x_2 \in N(T^*)$
  such that $x = x_1 + x_2$ *and* $\|x\| = \sqrt{\|x_1\|^2 + \|x_2\|^2}$, *then*

$$\||T||T^*|x\| = \||T||T^*|(x_1 + x_2)\| = \||T||T^*|x_1\| \leq \|T^2\| \cdot \|x_1\| \leq \|T^2\|\sqrt{\|x_1\|^2 + \|x_2\|^2} = \|T^2\| \cdot \|x\|$$

  Thus $\||T||T^*|\| \leq \|T^2\|\|x\|$ *for all* $x \in \mathcal{H}$, *hence* $\||T||T^*|\| \leq \|T^2\|$.

- *Let $x \in \mathcal{H}$ then*

$$\begin{aligned} T^2T^*x\|^2 &= \langle T^2T^*x, T^2T^*x\rangle \\ &= \langle T^*TTT^*x, TT^*x\rangle \\ &= \langle |T|^2|T^*|^2x, |T^*|^2x\rangle \\ &= \langle |T||T^*||T^*|x, |T||T^*||T^*|x\rangle \\ &= \||T||T^*||T^*|x\|^2 \\ &\le \||T||T^*|\|\||T^*x\|^2 \end{aligned}$$

Since $\||T^*|x\| = \|T^*x\|$, *then* $\|T^2T^*x\| \le \||T||T^*|\|\|T^*x\|$ *for all $x \in \mathcal{H}$. Therefore:*

$$\|T^2x\| \le \||T||T^*|\|\|x\| \quad \text{for all } x \in R(T^*)$$

And it can be extended to $\overline{R(T^*)}$, *then* $\|T^2x\| \le \||T||T^*|\|\|x\|$ *for all* $x \in \overline{R(T^*)}$.

Let $x \in N(T) \iff Tx = 0 \Rightarrow T^2x = 0$.

We have that $\mathcal{H} = \overline{R(T^*)} \oplus N(T)$, *then* $\forall x \in \mathcal{H}, \exists x_1 \in \overline{R(T^*)}$ *and* $\exists x_2 \in N(T)$ *such that* $x = x_1 + x_2$, *then*

$$\|T^2x\| = \|T^2(x_1+x_2)\| = \|T^2x_1\| \le \||T||T^*|\|\|x_1\| \le \||T||T^*|\|\sqrt{\|x_1\|^2 + \|x_2\|^2} = \||T||T^*|\|\|x\|$$

Then $\|T^2x\| \le \||T||T^*|\|\|x\|$ *for all $x \in \mathcal{H}$, therefore* $\|T^2\| \le \||T||T^*|\|$.

Hence $\||T||T^*|\| = \|T^2\|$.

Remark 5 *The inequality in*

$$\forall y \in \mathcal{H} \text{ and } \forall x \in \mathcal{H}: \quad |\langle Tx, y\rangle|^2 \le \langle |T|x, x\rangle \cdot \langle |T^*|y, y\rangle$$

the Generalized Schwarz inequality, because if T Positive operator, then $|T| = |T^*| = T$.

Theorem 1.13 *Let $T \in \mathcal{B}(\mathcal{H})$. Then there exists a partial isometry $U \in B(\mathcal{H})$ such that*

$$T = U|T| \quad \text{where} \quad |T| = (T^*T)^{\frac{1}{2}}. \tag{1.1}$$

Furthermore, U may be chosen such that $R(|T|) = R(T) = \ker(T)^{\perp}$ is the initial space of U and in that case the decomposition (1.1) is unique and is called the Polar decomposition of T.

For every $T \in \mathcal{B}(\mathcal{H})$, the Aluthge transform of T denoted by $\tilde{T} \in \mathcal{B}(\mathcal{H})$ was first defined by Aluthge as

$$\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$$

The generalised Aluthge transform of T denoted by $\tilde{T}_t$ is defined by

$$\tilde{T}_t = |T|^tU|T|^{1-t} \quad \text{for } t \in [0,1].$$

Lemme 1.5 *Let $\{0\} \neq K$ be a nonempty compact subset of $\mathbb{C}$. Then there exists $v \in K$, $v \neq 0$ such that*

$$|K + v| = |K| + |v|$$

Proof

By a compactness argument, there exists $v \in K$, $v \neq 0$ such that $|K| = |v|$.
So, $|K+v| \le |K| + |v| = 2|v|$. But $2v \in K + v$, then $2|v| \le |K+v|$ and therefore $|K+v| = |K| + |v|$.

Theorem 1.14 *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent statements*

(1) T is normaloid .

(2) $\|T+\lambda\| = \|T\| + |\lambda|$ for some $\lambda \in \mathbb{C}^*$.

Proof

(1) ⇒ (2). *Assume that T is normaloid, that is $|W(T)| = \omega(T) = \|T\|$. Since $W(T)$ is a compact subset of $\mathbb{C}$, there exists $\lambda \in W(T)$ ($|\lambda| = \omega(T)$) satisfying*

$$|\overline{W(T)} + \lambda| = |\overline{W(T)}| + |\lambda| = \|T\| + |\lambda|$$

But $\overline{W(T)} + \lambda = \overline{W(T+\lambda)}$, then $|\overline{W(T)} + \lambda| = |\overline{W(T+\lambda)}| = w(T+\lambda) \leq \|T+\lambda\|$.
It results that $\|T\| + |\lambda| \leq \|T+\lambda\|$ and so, $\|T+\lambda\| = \|T\| + |\lambda|$.

(2) ⇒ (1). *Let λ be a nonzero scalar such that $\|T+\lambda\| = \|T\| + |\lambda|$. By Barraa-Boumazgour , we have*

$$|\lambda|\|T\| \in \overline{W(\overline{\lambda}T)} = \overline{\lambda}\,\overline{W(T)}$$

It results that $\|T\| \leq \omega(T)$ and hence $\|T\| = \omega(T)$. This is exactly to say that T is normaloid.

Corollary 1.8 *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is normaloid if and only if*

$$\sup_{\lambda \in \overline{W(T)}} \|T+\lambda\| = 2\|T\|$$

Proof

Note first that, since $|\lambda| \leq \|T\|$ for any $\lambda \in \overline{W(T)}$, we always have

$$\sup_{\lambda \in \overline{W(T)}} \|T+\lambda\| \leq \sup_{\lambda \in \overline{W(T)}} (\|T\| + |\lambda|) \leq 2\|T\|$$

If T is normaloid, then states that there exists $\lambda \in \overline{W(T)}$, $|\lambda| = \omega(T) = \|T\|$ and $\|T+\lambda\| = \|T\| + |\lambda| = 2\|T\|$. Hence,

$$\sup_{\lambda \in \overline{W(T)}} \|T+\lambda\| = 2\|T\|$$

Conversely, since $\overline{W(T)}$ is compact, there exists $\mu \in \overline{W(T)}$ such that

$$\sup_{\lambda \in \overline{W(T)}} (\|T\| + |\lambda|) = \|T\| + |\mu|$$

If T is not normaloid, then $|\mu| < \|T\|$ and we obtain

$$\sup_{\lambda \in \overline{W(T)}} \|T+\lambda\| = \|T\| + |\mu| < 2\|T\|$$

CHAPTER

2

FUNCTIONAL INEQUALITIES

In this chapter we pressent some famous inequalities (Cauchy-Schwarz, Hölder, Minkowski,...) with their proofs and applications. These are vital tools for advanced mathematics and real-world problems .

2.1 Cauchy-Schwarz inequaliy

Let X is a vector space over $\mathbb{R}$ or $\mathbb{C}$, and $\langle .,. \rangle$ an inner product on X ; then $(X, \langle .,. \rangle)$ be an inner product space. Then $\forall x, y \in X$, the following inequality holds

$$| \langle x,y \rangle | \leq \sqrt{\langle x,x \rangle} \sqrt{\langle y,y \rangle} \tag{2.1}$$

Proof

- *if $y = 0$ then*

  $$| \langle x,y \rangle | = | \langle x,0 \rangle | = 0$$

  and

  $$\sqrt{\langle x,x \rangle} \sqrt{\langle 0,0 \rangle} = 0$$

  and therfore

  $$| \langle x,y \rangle | = 0 \leq \sqrt{\langle x,x \rangle} = 0$$

  the inequality is satisfied.

- *The general case when $y \neq 0$, We define the function*

  $$g(t) = \|x + ty\|^2 = \langle x + ty, x + ty \rangle \geq 0 \quad \forall t \in \mathbb{C}$$

  We calculate $g(t)$

  $$g(t) = \langle x,x \rangle + t\langle x,y \rangle + \bar{t}\langle y,x \rangle + |t|^2 \langle y,y \rangle$$

  We choose a specific complex number

  $$t = -\langle y,x \rangle / \langle y,y \rangle$$

By substituting into $g(t)$, we obtain

$$g(t) = \langle x,x\rangle - |\langle y,x\rangle|^2/\langle y,y\rangle \geq 0$$

We move the second term to the other side

$$|\langle y,x\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle$$

Therefor

$$|\langle x,y\rangle| \leq \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle}$$

Further if $|\langle x,y\rangle| = \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle}$, then x and y are linearly dependent.

Theorem 2.1 *Let $(X, \langle .,.\rangle)$ be an inner product space . So $\|x\| = \sqrt{\langle x,x\rangle}$ defines a norm.*

Remark 6 *We are able to write Cauchy-Schwarz inequality that way*

$$|\langle x,y\rangle| \leq \|x\|\|y\| \quad \forall x,y \in X$$

Theorem 2.2 *(Cauchy-Schwarz inequaliy by sum)*
For any real numbers $a_1, a_2, ..., a_k$ and $b_1, b_2, ..., b_k$ the following inequality holds.

$$(a_1b_1 + a_2b_2 + \cdots + a_kb_k)^2 \leq (a_1^2 + a_2^2 + \cdots + a_k^2)(b_1^2 + b_2^2 + \cdots + b_k^2)$$

$$\left(\sum_{i=1}^{k} a_ib_i\right)^2 \leq \left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} b_i^2\right)$$

Example 2.1 *(Fourier Transform)*
Let $f,g \in L^2$

$$\left|\int f(t)g(t)\,dt\right|^2 \leq \left(\int |f(t)|^2\,dt\right)\left(\int |g(t)|^2\,dt\right)$$

Example 2.2 *Suppose that $a_1, a_2, a_3 \geq 1$ and $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} = 2$*
We have

$$\sqrt{a_1 - 1} + \sqrt{a_2 - 1} + \sqrt{a_3 - 1} \leq \sqrt{a_1 + a_2 + a_3}$$

We estabish it as follows
By Cauchy-Schwartz, we have

$$\begin{aligned}
\sqrt{a_1 - 1} + \sqrt{a_2 - 1} + \sqrt{a_3 - 1} &= \left(\sum_{i=1}^{3} \frac{\sqrt{a_i - 1}}{\sqrt{a_i}}\sqrt{a_i}\right) \\
&\leq \left(\sum_{i=1}^{3} \frac{a_i - 1}{a_i}\right)^{1/2} (a_1 + a_2 + a_3)^{1/2} \\
&= \left(3 - \frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3}\right)^{1/2} \sqrt{(a_1 + a_2 + a_3} \\
&= \sqrt{a_1 + a_2 + a_3}.
\end{aligned}$$

That is to say

$$\sqrt{a_1 - 1} + \sqrt{a_2 - 1} + \sqrt{a_3 - 1} \leq \sqrt{a_1 + a_2 + a_3}$$

2.2 Young's inequality

Let $a,b \geq 0$ *and* $1 < p,q < \infty$ *betwo conjugate exponents . Then*

$$ab \leqslant \frac{1}{p}a^p + \frac{1}{q}b^q \tag{2.2}$$

Proof

- *if* $a = 0$ *or* $b = 0$ *it's easy*
- $a,b > 0$ *the function e is convex , which means that* $\forall x,y \in X$ *and* $\forall t \in [0,1]$ *. Then*

  $$e(tx + (1-t)y) \leq te(x) + (1-t)e(y)$$

  we have

$$\begin{aligned} a.b =& e^{\ln(a.b)} \\ \leq& e^{\left(\frac{\ln a^p}{p} + \frac{\ln b^q}{q}\right)} \\ \leq& \frac{1}{p}e^{(\ln a^p)} + \frac{1}{q}e^{(\ln b^q)} \\ =& \frac{1}{p}a^p + \frac{1}{q}b^q \end{aligned}$$

Hence

$$a.b \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

2.3 Hölder's inequality

Let $f \in \mathbf{L}^p$ *and* $g \in \mathbf{L}^q$ *with* $1 \leq p,q \leq \infty$, *such that* $\frac{1}{q} + \frac{1}{p} = 1$ *. Then*

$$f,g \in L^1(\Omega), \qquad \int_\Omega |f(x)g(x)|dx \leq \|f\|_{L^p}\|g\|_{L^q}$$

Proof

The conclusion is obvious if $p = \infty$, $q = 1$ *. Let's assume that* $1 < p,q < \infty$, *we apply Young's inequality (2.2) .*
The proof of Young's inequality is evident: the log function being concave on $\mathbb{R}_+^*$ *,*
we have

$$\log\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \geq \frac{1}{p}\log a^p + \frac{1}{q}\log b^q = \log ab$$

So

$$|f(x)g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q$$

It resulted that $f,g \in \mathbf{L}^1$ *and that*

$$\int_\Omega |f(x)g(x)|dx \leq \frac{1}{p}\|f\|_{L^p}^p + \frac{1}{q}\|g\|_{L^q}^q$$

To find the desired inequality, we replace the pair (f,g) *with* $(tf,\frac{1}{t}g)$, *where* $t>0$. *We choose* $t=\frac{\|g\|_q^{1/p}}{\|f\|_p^{1/q}}$. *So*

$$\int_\Omega |f(x)g(x)|dx \le \left(\frac{1}{p}+\frac{1}{q}\right)\|f\|_p\|g\|_q$$

which is indeed the announced inequality. If $p=1$ *or* $p=\infty$, *we have* $q=\infty$ *or* $q=1$ *and the inequality is immediate by choosing a representative of g such that sup* $|g(x)|=\|g\|_\infty$
then we have

$$\int_\Omega |f(x)g(x)|dx \le \sup|g(x)|\int_\Omega |f(x)|dx = \|g\|_\infty\|f\|_{L^1}$$

Theorem 2.3 *(Hölder's Inequality by sum)*
Let $p,q>0$ *are real numbers such that* $\frac{1}{q}+\frac{1}{p}=1$ *for all* $a_1,..,a_k\in\mathbb{R}^+$ *and* $b_1,..,b_k\in\mathbb{R}^+$ *we have* $a_1b_1+\cdots+a_kb_k\le\left(a_1^p+\cdots+a_k^p\right)^{\frac{1}{p}}\left(b_1^q+\cdots+b_k^q\right)^{\frac{1}{q}}$ *what is written using the sum signs.*

$$\sum_{i=1}^k |a_ib_i| \le \left(\sum_{i=1}^k |a_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^k |b_i|^q\right)^{\frac{1}{q}}$$

for $p=2=q$ *(which are suitable since* $\frac{1}{2}+\frac{1}{2}=1$*) . We find exactly the Cauchy-Schwarz inequaliy*

$$\left(\sum_{i=1}^k a_ib_i\right)^2 \le \left(\sum_{i=1}^k a_i^2\right)\left(\sum_{i=1}^k b_i^2\right)$$

2.4 Minkowski inequality

Let $p\in\mathbb{R}_+^*$ *and let* $a_1,\dots,a_k$ *and* $b_1,\dots,b_k\in\mathbb{R}^+$

Proof

- *if* $p>1$.

$$((a_1+b_1)^p+\cdots+(a_k+b_k)^p)^{\frac{1}{p}} \le (a_1^p+\cdots+a_k^p)^{\frac{1}{p}}+(b_1^p+\cdots+b_k^p)^{\frac{1}{p}}$$

this can also be written,using thesummation notation

$$\left(\sum_{i=1}^k |a_i+b_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^k |a_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^k |b_i|^p\right)^{\frac{1}{p}}$$

- *if* $0<p<1$, *on the contrary*

$$\left(\sum_{i=1}^k |a_i+b_i|^p\right)^{\frac{1}{p}} \ge \left(\sum_{i=1}^k |a_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^k |b_i|^p\right)^{\frac{1}{p}}$$

Example 2.3 *Let* $x,y\in\mathbb{R}^3$ $x=(1,2,3)$, $y=(4,5,6)$. *By simple calculation, we get*

$$\|x+y\|_2 \approx 12,45 \le 12,51 \approx \|x\|_2+\|y\|_2$$

Remark 7 *The Minkowski inequality allows us to assert that* L^p *spaces are normed, and that* $\mathbb{R}^n$ *can be normed in infinitely many ways.*

2.5 Jensen inequality

Let f be a convex function on a real interval I and x a random variable taking values in I such that the expectation $E(f(x))$ exists, then

$$f(E(x)) \leq E(f(x))$$

Proof
Let's assume that x is a discrete random variable , that takes values in $x \in D$. $D = x_1, x_2, ..., x_k$ with probabilities $P(x = x_i) = k_i$ or $t_i > 0$ with $\sum_{i=1}^{k} t_i = 1$ then

$$E(x) = \sum_{i=1}^{k} x_i \lambda_i$$

the expectation of a function $f : D \longrightarrow \mathbb{R}$, such that

$$\begin{aligned} E(f(x)) =& \sum_{i=1}^{k} f(x_i) t_i \\ \geq& f\left(\sum_{i=1}^{k} x_i t_i\right) \\ =& f(E(x)) \end{aligned}$$

2.6 AM-GM Inequality (Arithmetic-Geometric Mean Inequality)

Definition 2.1 *Let $a_1, ..., a_k$ be k positive real numbers,*
Arithmetic Mean *The arithmetic Mean of k real numbers $a_1, ..., a_k$ is defined as*

$$AM = \frac{a_1 + ... + a_k}{k}$$

Geometric Mean *The geometric Mean of k real numbers $a_1, ..., a_k$ is defined as*

$$GM = (a_1 ... a_k)^{\frac{1}{k}}$$

Theorem 2.4 *$(AM \geq GM)$ For non negative real numbers $a_1, ..., a_k$ we have*

$$\frac{a_1 + ... + a_k}{k} \geq (a_1 ... a_k)^{\frac{1}{k}}$$

Equality holds , if and only if $a_1 = a_2 = ... = a_k$.
Proof
Let $a_1, ..., a_k$ be k positive real numbers.
If

$$a_1 a_2 ... a_k = 1$$

Then

$$a_1 + a_2 + ... + a_k \geq k$$

- $k = 1$
  is trivial .

- $k \geq 2$
  Let $a_1, a_2, ..., a_k$ *be positive numbers*
  with

  $$a_1 a_2 ... a_k = 1$$

  Without loss of generality, we can assume that

  $$a_1 = \max_{1 \leq i \leq k} a_i \geq 1 \text{ and } a_2 = \min_{1 \leq i \leq k} a_i \leqslant 1$$

  Thus we have

  $$a_1 + a_2 - a_1 a_2 - 1 = (a_1 - 1)(1 - a_2) \geq 0$$

  So

  $$a_1 + a_2 - a_1 a_2 \geq 1 \tag{2.3}$$

  Since
  $a_1 a_2, ..., a_k$ *are* $(k-1)$ *positive numbers with product equal to* 1 *, by the induction hypothesis we have*

  $$a_1 a_2 + a_3 + ... + a_k \geq k - 1 \tag{2.4}$$

  By adding the two equation 2.3 *and* 2.4 *, we get*

  $$a_1 + a_2 + a_3 + ... + a_k \geq k \tag{2.5}$$

  If equality holds in 2.5 *then we must have equality in* 2.4*, that is* $a_1 = 1$ *or* $a_2 = 1$
  hence

  $$a_1 = a_2 = ... = a_k = 1$$

  On the other hand .
  For positive real numbers $a_1, a_2, .., a_k$ *,*
  we set $x_1 = \frac{a_1}{(a_1 a_2 \cdots a_k)^{\frac{1}{k}}}, \quad \ldots, \quad x_k = \frac{a_k}{(a_1 a_2 \cdots a_k)^{\frac{1}{k}}}$
  the numbers $x_1, x_2, ..., x_k$ *satisfy the condition* $x_1 x_2 ... x_k = 1$
  So we have

  $$\frac{a_1}{(a_1 ... a_k)^{\frac{1}{k}}} + \cdots + \frac{a_k}{(a_1 ... a_k)^{\frac{1}{k}}} \geq k$$

  hence we obtain

  $$\frac{a_1 + \cdots + a_k}{k} \geq (a_1 ... a_k)^{\frac{1}{k}}$$

  with the equality holding only for $a_1 = a_2 = ... = a_k$.

Example 2.4 *Let* $a_1, a_2, ..., a_k$ *be a positive real numbers , such that*

$$a_1 a_2 ... a_k = 1$$

We have

$$(1 + a_1)(1 + a_2)...(1 + a_k) \geq 2^k$$

We estabish it as follows

$$\begin{aligned} 1 + a_1 \geq& 2\sqrt{a_1} \\ 1 + a_2 \geq& 2\sqrt{a_2} \\ &. \\ &. \\ &. \\ 1 + a_n \geq& 2\sqrt{a_n} \end{aligned}$$

We get

$$\begin{aligned} (1 + a_1)(1 + a_2)...(1 + a_k) \geq& 2^k \sqrt{a_1 a_2 ... a_k} \\ \geq& 2^k \sqrt{1} \end{aligned}$$

Hence

$$(1 + a_1)(1 + a_2)...(1 + a_k) \geq 2^k$$

2.7 Other inequalities

Theorem 2.5 *(Generalized Schwarz inequality)*
Let $T \in \mathcal{B}(\mathcal{H})$ be a positive operator.
Then

$$|\langle Tx, y\rangle|^2 \leq \langle Tx, x\rangle\langle Ty, y\rangle \quad \forall x, y \in \mathcal{H}$$

Remark 8 *To prove this theory , we use the previous proof 2.1 , but by taking $g(t) = \|T(x + ty)\|^2$*

Theorem 2.6 *[15] Let $T, S \in B(\mathcal{H})$ be two positive operators . Then*

$$\|T + S\| \leq \frac{1}{2}\left(\|T\| + \|S\| + \sqrt{(\|T\| - \|S\|)^2 + 4\|\sqrt{T}\sqrt{S}\|^2}\right) \tag{2.6}$$

Corollary 2.1 *Let $T, S \in \mathcal{B}(\mathcal{H})$ be two positive operators . Then*

$$\|T + S\| \leq \max\{\|T\|, \|S\|\} + \|\sqrt{T}\sqrt{S}\| \quad (2.12)$$

Proof
By 2.6, we have that

$$\begin{aligned} \|T + S\| \leq& \frac{1}{2}\left(\|T\| + \|S\| + \sqrt{(\|T\| - \|S\|)^2 + 4\|\sqrt{T}\sqrt{S}\|^2}\right) \\ \leq& \frac{1}{2}\left(\|T\| + \|S\| + |\|T\| - \|S\|| + 2\|\sqrt{T}\sqrt{S}\|^2\right) \end{aligned}$$

Since

$$\max\{\|T\|, \|S\|\} = \frac{1}{2}(\|T\| + \|S\| + |\|T\| - \|S\||)$$

then

$$\|T+S\| \leq \frac{1}{2}\left(2\max\{\|T\|,\|S\|\}+2\|\sqrt{T}\sqrt{S}\|\right)$$

Therefore

$$\|T+S\| \leq \max\{\|T\|,\|S\|\}+\|\sqrt{T}\sqrt{S}\|$$

Theorem 2.7 *(Mixed Schwartz inequality)*
Let $T \in \mathcal{B}(\mathcal{H})$ *, then*

$$|\langle Tx,y\rangle| \leq \sqrt{\langle |T|x,x\rangle}\sqrt{\langle |T^*|y,y\rangle} \forall x,y \in \mathcal{H}$$

Proof

we have $|T^*| \geq 0$*, then*
$\forall x,y \in \mathcal{H} \quad |\langle |T^*|x,y\rangle| \leq \sqrt{\langle |T^*|x,x\rangle}\sqrt{\langle |T^*|y,y\rangle} \quad (Generalized\ \ Schwarz\ \ inequality)$
Since

$$T|T|^2 = TT^*T = |T^*|^2T \textit{ and } \||T|^2\| = \||T^*|^2\| = \|T\|^2$$

we get

$$T\sqrt{|T|^2} = \sqrt{|T^*|^2}T \quad is\ \ equivalent \quad T|T| = |T^*|T$$

$x = Tx, \forall x,y \in \mathcal{H}$ *we obtain*

$$\begin{aligned}|\langle |T^*|Tx,y\rangle|^2 \leq& \langle |T^*|Tx,Tx\rangle\langle |T^*|y,y\rangle \\ \langle T|T|x,y\rangle|^2 \leq& \langle T^*T|T|x,x\rangle\langle |T^*|y,y\rangle \\ =&\langle |T|^2|T|x,x\rangle\langle |T^*|y,y\rangle \\ =&\langle |T||T|x,|T|x\rangle\langle |T^*|y,y\rangle\end{aligned}$$

Then, $\forall x,y \in \mathcal{H}$*:*

$$\langle T|T|x,y\rangle|^2 \leq \langle |T||T|x,|T|x\rangle\langle |T^*|y,y\rangle$$

Thus $\forall x \in R(T)$ *and* $\forall y \in \mathcal{H}$*:*

$$|\langle Tx,y\rangle|^2 \leq \langle |T|x,x\rangle\langle |T^*|y,y\rangle$$

We can extend it to $\overline{R(T)}$ *as follows.*
Let $x \in \overline{R(T)}$*,* $\exists (x_n) \subset R(T)$ *such that* $\lim_{n\to\infty} x_n = x$*, then* $\forall n \in \mathbb{N}$

$$\begin{aligned}&|\langle Tx_n,y\rangle|^2 \leq \langle |T|x_n,x_n\rangle\langle |T^*|y,y\rangle \\ \Longrightarrow &\lim_{n\to\infty}|\langle Tx_n,y\rangle|^2 \leq \lim_{n\to\infty}\langle |T|x_n,x_n\rangle\langle |T^*|y,y\rangle\end{aligned}$$

Then, $\forall x \in \overline{R(T)}$ *and* $\forall y \in \mathcal{H}$*,*
we have

$$|\langle Tx,y\rangle|^2 \leq \langle |T|x,x\rangle\langle |T^*|y,y\rangle$$

Let $x \in N(|T|) itisequivalentto |T|x = 0$*, then*

$$\begin{aligned} \||T|x\|^2 &= \langle |T|x, |T|x\rangle \\ &= \langle |T|^2 x, x\rangle \\ &= \langle T^*Tx, x\rangle \\ &= \langle Tx, Tx\rangle \\ &= \|Tx\|^2 \\ \Longrightarrow\ \||T|x\| &= \|Tx\| \end{aligned}$$

Then $\|Tx\| = 0$ *it is equivalent to* $Tx = 0$

Therefore for all $x \in N(|T|)$: $Tx = 0$.
By orthogonal decomposition, $\mathcal{H} = \overline{R(|T|)} \oplus R(|T|)^{\perp}$, *but*

$$R(|T|)^{\perp} = N(|T|^*) = N(|T|)$$

Then

$$\mathcal{H} = \overline{R(|T|)} \oplus N(|T|)$$

so that $\forall x \in \mathcal{H}$
there exist $x_1 \in R(|T|)$ *and* $x_2 \in N(|T|)$ *where* $x = x_1 + x_2$.
Then

$$\begin{aligned} |\langle Tx, y\rangle|^2 &= |\langle T(x_1 + x_2), y\rangle|^2 \\ &= |\langle Tx_1, y\rangle + \langle Tx_2, y\rangle|^2 \\ &= |\langle Tx_1, y\rangle|^2 \\ &\leq \langle Tx_1, y\rangle|^2 \\ &\leq \langle |T|x_1, x_1\rangle\langle |T^*|y, y\rangle \end{aligned}$$

But

$$\begin{aligned} \langle |T|x_2, x_2\rangle &= \langle |T|x_2, x_1\rangle \\ &= \langle |T|x_1, x_2\rangle \\ &= 0 \end{aligned}$$

Thus

$$(\langle |T|x_2, x_2\rangle + \langle |T|x_2, x_1\rangle + \langle |T|x_1, x_2\rangle)\langle |T^*|y, y\rangle = 0$$

And we have that

$$\begin{aligned} |\langle |T|x_1, x_1\rangle|\langle |T^*|y, y\rangle &= \langle |T|x_1, x_1\rangle\langle |T^*|y, y\rangle + (\langle |T|x_2, x_2\rangle + \langle |T|x_2, x_1\rangle| + \langle |T|x_1, x_2\rangle|)\langle |T^*|y, y\rangle| \\ &= \langle |T^*|x, x\rangle\langle |T^*|y, y\rangle| \end{aligned}$$

Therefor $\forall x, y \in \mathcal{H}$;

$$|\,\langle Tx, y\rangle\,|^2 \leq \langle |T|x, x\rangle\langle |T^*|y, y\rangle$$

Hence

$$|\,\langle Tx, y\rangle\,| \leq \sqrt{\langle |T|x, x\rangle}\sqrt{\langle |T^*|y, y\rangle}$$

Lemme 2.1 *[18] Let* $T \in \mathcal{B}(\mathcal{H})$ *a positive operator and let* $x \in \mathcal{H}$ *a unit vector .*
Then

(1) $\langle Tx,x\rangle^r \le \langle T^r x,x\rangle$ *for* $r \ge 1$.

(2) $\langle T^r x,x\rangle \le \langle Tx,x\rangle^r$ *for* $0 < r \le 1$.

Lemme 2.2 *[18] Let* $T \in \mathcal{B}(\mathcal{H})$ *self-adjoint. Then*

$$|\langle Tx,x\rangle| \le \langle |T|x,x\rangle \quad \forall x \in \mathcal{H}$$

Lemme 2.3 *[18] Let* $T \in B(\mathcal{H})$ *and* $0 \le \alpha \le 1$. *Then*

$$|\langle Tx,y\rangle|^2 \le \langle |T|^{2\alpha}x,x\rangle\langle |T^*|^{2(1-\alpha)}y,y\rangle \quad \forall x,y \in \mathcal{H}$$

Lemme 2.4 *[7] Let* $T,S \in \mathcal{B}(\mathcal{H})$ *be two positive operators. Then*

$$\|(T+S)^r\| \le \|T^r + S^r\| \quad for \quad 0 < r \le 1. \tag{2.7}$$

Singular Values Inequalities

Definition 2.2 *The singular values of A, denoted by*

$$s_1(A), s_2(A), \ldots, s_n(A)$$

are the eigenvalues of $|A|$
(Where eigenvalues of A is a complex nember λ *, such that* $\quad Ax = \lambda x \quad \forall x \in \mathcal{H}\backslash\{0\}$ *)*

The singular values of $A \in M_n$ *are arranged as*

$$s_1(A) \ge s_2(A) \ge \cdots \ge s_n(A).$$

Note that $s_j(A) = s_j(A^*) = s_j(|A|)$, *for* $j = 1,\ldots,n$. *It follows from Weyl's monotonicity principle that if* $A,B \in M_n$ *are positive semidefinite and* $A \le B$, *then*

$$s_j(A) \le s_j(B) \quad for\ j = 1,\ldots,n.$$

It is well known that

$$s_j(A) \le s_j(B) \quad if\ and\ only\ if \quad s_j(A \oplus A) \le s_j(B \oplus B), \quad j = 1,\ldots,2n$$

$$s_j(A \oplus B) = s_j\left(\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right), \quad j = 1,\ldots,2n$$

for any $A,B \in M_n$.

Moreover, if $A_1,A_2,B_1,B_2 \in M_n$ *are such that* $s_j(A_1) \le s_j(B_1)$ *and* $s_j(A_2) \le s_j(B_2)$, *then*

$$s_j(A_1 \oplus A_2) \le s_j(B_1 \oplus B_2), \quad j = 1,\ldots,2n$$

The following results, arithmetic geometric mean inequality for singular values.

Theorem 2.8 *[8] Let* $A,B \in M_n$. *Then*

$$2s_j(AB^*) \le s_j(A^*A + B^*B), \quad j = 1,\ldots,n \tag{2.8}$$

On the other hand, Zhan has proved the following theorem.

Theorem 2.9 *Let $A, B \in M_n$ be positive semidefinite. Then*

$$s_j(A-B) \leq s_j(A \oplus B), \quad j = 1, \ldots, n \tag{2.9}$$

In addition, Zhan (2002) proved that the inequalities 2.8 and 2.9 are equivalent. Tao gave a lower bound for singular value of a 2×2 positive semidefinite block matrices and another equivalent form of these inequalities.

Theorem 2.10 *Let $A, B, C \in M_n$ be such that*

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$$

is positive semidefinite. Then

$$2s_j(B) \leq s_j\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right), \quad j = 1, \ldots, n$$

Theorem 2.11 *The following statements are equivalent*

1. *Let $A, B \in M_n$ be positive semidefinite matrices. Then*

$$s_j(A-B) \leq s_j(A \oplus B), \quad j = 1, \ldots, n.$$

2. *For any $X, Y \in M_n$*

$$2s_j(XY^*) \leq s_j(X^*X + Y^*Y), \quad j = 1, \ldots, n.$$

3. *Let $M, N, K \in M_n$ be such that*

$$\begin{bmatrix} M & K \\ K^* & N \end{bmatrix} \geq 0$$

Then

$$2s_j(K) \leq s_j\left(\begin{bmatrix} M & K \\ K^* & N \end{bmatrix}\right), \quad j = 1, \ldots, n.$$

Theorem 2.12 *Let $A, B, C, D \in M_n$. Then*

$$2s_j(AB^* + CD^*) \leq s_j^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right), \quad j = 1, 2, \ldots, n.$$

Proof

Suppose that

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Then

$$\begin{aligned} s_j(T) = s_j(T^*) &= \lambda_j^{\frac{1}{2}}(TT^*) \\ &= \frac{1}{2}\lambda_j^{\frac{1}{2}}\left(\begin{bmatrix} AA^* + CC^* & AB^* + CD^* \\ BA^* + DC^* & BB^* + DD^* \end{bmatrix}\right) \\ &= \frac{1}{2}s_j^{\frac{1}{2}}\left(\begin{bmatrix} AA^* + CC^* & AB^* + CD^* \\ BA^* + DC^* & BB^* + DD^* \end{bmatrix}\right) \end{aligned}$$

So, we get

$$2s_j(AB^* + CD^*) \leq s_j^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$$

This completes the proof.

CHAPTER

3

SOME ADVANCED SPECTRAL VALUES NOTION

This chapter aims to present the basic concepts related to the spectrum, spectral radius, numerical range, and numerical radius of bounded linear operators on a Hilbert space. These concepts are essential tools in spectral analysis and operator theory. Additionally, we review some important inequalities that link the numerical radius and the operator norm, such as the classical inequalities, along with references to some recent improvements on them.

3.1 The spectrum

Definition 3.1 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

(a) The spectrum of T denoted by $\sigma(T)$ is the non-empty compact set of all complex numbers λ defined by

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$$

(b) the resolvent of T is denoted by $\rho(T)$, and $\rho(T) = \mathbb{C} \setminus \sigma(T)$.

Lemme 3.1 *Let $T \in \mathcal{B}(\mathcal{H})$, then $\sigma(T^*) = \{\overline{\lambda} : \lambda \in \sigma(T)\}$.*

Proof

$$\begin{aligned} \lambda \in \rho(T) &\Longleftrightarrow T - \lambda I \text{ is invertible} \\ &\Longleftrightarrow (T - \lambda I)^* \text{ is invertible} \\ &\Longleftrightarrow T^* - \overline{\lambda I} \text{ is invertible} \\ &\Longleftrightarrow \overline{\lambda} \in \rho(T^*) \end{aligned}$$

Therefore : $\lambda \notin \rho(T) \Longleftrightarrow \overline{\lambda} \notin \rho(T^*)$
Thus $\sigma(T^*) = \{\overline{\lambda} : \lambda \in \sigma(T)\}$

Theorem 3.1 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

1. *If ρ is a polynomial then $\sigma(\rho(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$.*
2. *If T is invertible then $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$.*

Corollary 3.1 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

(1) *$\sigma(T)$ is compact.*

(2) *$\rho(T)$ is an open non-empty set.*

Proof

(1) *The spectrum $\sigma(T)$ is bounded (since $r(T) \leq \|T\|$) and closed (as the complement of the resolvent set $\rho(T)$, which is open). In finite dimensions, it is also finite. In infinite dimensions, it is closed and bounded, hence compact.*

(2) *The resolvent set $\rho(T)$ is open because for any $\lambda \in \rho(T)$, there exists $\epsilon > 0$ such that $B_\epsilon(\lambda) \subset \rho(T)$. It is non-empty because $\sigma(T)$ is bounded, so $\rho(T)$ contains all λ with $|\lambda| > r(T)$.*

3.2 Spectral radius

The spectral radius of T is the number given by

$$r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}.$$

$r(T)$ is the radius of the smallest closed disk centred at the origin of the complex plane and containing $\sigma(T)$. The most important property of the spectral radius is the Gelfand formula

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

It is well known that for all $T \in \mathcal{B}(\mathcal{H})$,

$$r(T) \leq \|T\|.$$

Proposition 3.1 *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

- *$\sigma(TS) \setminus \{0\} = \sigma(ST) \setminus \{0\}$.*
- *$r(TS) = r(ST)$.*

Example 3.1 *: Real Eigenvalues*
Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Step 1: Find the eigenvalues of A
Solve the characteristic equation:

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = 0$$

$$(2-\lambda)^2 = 1 \Rightarrow 2-\lambda = \pm 1 \Rightarrow \lambda = 1, 3$$

Step 2: Spectral Radius
The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$

$$\rho(A) = \max(|\lambda_1|, |\lambda_2|) = \max(1,3) = 3$$

Example 3.2 *: Complex Eigenvalues*
Let

$$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Step 1: Find the eigenvalues of B .
Solve the characteristic equation

$$\det(B - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

$$\lambda^2 = -1 \Rightarrow \lambda = i, -i$$

Step 2: Spectral Radius

$$|\lambda_1| = |i| = 1, \quad |\lambda_2| = |-i| = 1$$

$$\rho(B) = \max(|i|, |-i|) = 1$$

Theorem 3.2 *Let* $T \in \mathcal{B}(\mathcal{H})$. *Then*

(1) *If* $|\lambda| > \|T\|$, *then* $\lambda \in \rho(T)$.

(2) $\sigma(T)$ *is closed set.*

Proof

(1) If $|\lambda| > \|T\|$ *, then* $\|\lambda^{-1}T\| < 1$ *,* $I - \lambda^{-1}T$ *is invertible , then* $T - \lambda I$*is invertible. Hence* $\lambda \in r(T)$.

Proposition 3.2 *Let* $\mathcal{H}$ *be a Hilbert spaces and let* $T \in \mathcal{B}(\mathcal{H})$ *,* $S \in \mathcal{B}(\mathcal{H})$. *Then*

(a) $r\begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix} = \sqrt{r(TS)}$

(b) $\left\|\begin{matrix} 0 & T \\ S & 0 \end{matrix}\right\| = \max\{\|T\|, \|S\|\}$

Proof

(a) *Consider the block operator* $A = \begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix}$. *Then*

$$A^2 = \begin{pmatrix} TS & 0 \\ 0 & ST \end{pmatrix}.$$

The spectral radius satisfies $r(A)^2 = r(A^2)$. *Since* A^2 *is block-diagonal ,* $r(A^2) = \max\{r(TS), r(ST)\}$ *.*
But $r(TS) = r(ST)$, *so*

$$r(A) = \sqrt{r(TS)}.$$

(b) For any $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$ *, we have*

$$\left\| A \begin{bmatrix} x \\ y \end{bmatrix} \right\| = \left\| \begin{bmatrix} Ty \\ Sx \end{bmatrix} \right\| = \sqrt{\|Ty\|^2 + \|Sx\|^2}.$$

The operator norm is

$$\|A\| = \sup_{\|x\|^2+\|y\|^2=1} \sqrt{\|Ty\|^2 + \|Sx\|^2} = \max\{\|T\|, \|S\|\}$$

Theorem 3.3 *Let* $T \in \mathcal{B}(\mathcal{H})$ *be a diagonal operator matrix (or* $T = \bigoplus_{i=1}^{n} T_{ii}$*) . Then*

$$\|T\| = \max(\|T_{11}\|, \|T_{22}\|, \ldots, \|T_{nn}\|)$$

$$r(T) = \max(r(T_{11}), r(T_{22}), \ldots, r(T_{nn}))$$

Theorem 3.4 *[11] Let* $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ *be Hilbert spaces and let* $T = [T_{ij}]_{n\times n}$ *be an operator matrix, with* $T_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$ *for all* $i, j = 1, 2, \ldots, n$ *and let* $A = [\|T_{ij}\|]_{n\times n}$ *. Then*

$$\|T\| \le \|A\| = \|[\|T_{ij}\|]_{n\times n}\|$$

$$r(T) \le r(A) = r([\|T_{ij}\|]_{n\times n})$$

Definition 3.2 *Let* $T \in \mathcal{B}(\mathcal{H})$ *. Then*

(1) The ponctual spectrum of T *is the set*

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : N(T - \lambda I) \neq \{0\}\}$$

$\lambda \in \sigma_p(T)$ *is called eigenvalue of* T *, and the* $x \in \mathcal{H} \setminus \{0\}$ *that verifies* $Tx = \lambda x$ *is called eigenvector of* λ *.*

(2) The continuous spectrum of T *is the set*

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : N(T - \lambda I) = \{0\}, \overline{R(T - \lambda I)} = \mathcal{H}\}$$

(3) The residual spectrum of T *is the set*

$$\sigma_r(T) = \{\lambda \in \mathbb{C} : N(T - \lambda I) = \{0\}, \overline{R(T - \lambda I)} \subsetneq \mathcal{H}\}$$

(4) The approximate spectrum of T *is the set*

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \exists (x_n) \subset \mathcal{H} : \|x_n\| = 1, \lim_{n\to\infty} (T - \lambda I)x_n = 0\}$$

Corollary 3.2 *If* $\mathcal{H}$ *is a finite-dimensional Hilbert space, then* $\sigma(T) = \sigma_p(T)$ *for all* $T \in \mathcal{B}(\mathcal{H})$*.*

3.3 Numerical Range

Definition 3.3 *The* Numerical Range *(also known as field of values) of an operator* $T \in \mathcal{B}(\mathcal{H})$ *is the non empty subset of the complex numbers* $\mathbb{C}$ *, given by*

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$$

Proposition 3.3 *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *,* F *subspace of* $\mathcal{H}$ *and* $\alpha, \beta \in \mathbb{C}$ *. Then*

1. $W(\alpha I + \beta T) = \alpha + \beta W(T)$
2. $W(T^*) = \{\overline{\lambda} : \lambda \in W(T)\}$
3. $W(U^*TU) = W(T)$ *for any unitary* $U \in B(\mathcal{H})$
4. $W(T|_F) \subset W(T)$
5. $W(T+S) \subset W(T) + W(S)$
6. $W(\mathrm{Re}(T)) = \mathrm{Re}(W(T))$ *and* $W(\mathrm{Im}(T)) = \mathrm{Im}(W(T))$

Proof

1. *For any unit vector* $x \in \mathcal{H}$ *(i.e.,* $\|x\| = 1$*), we have*

   $$(\alpha I + \beta T)x = \alpha x + \beta Tx$$

   Therefore

   $$\langle(\alpha I + \beta T)x, x\rangle = \alpha\langle x, x\rangle + \beta\langle Tx, x\rangle = \alpha + \beta\langle Tx, x\rangle$$

   Since $\langle Tx, x\rangle \in W(T)$*, it follows that*

   $$W(\alpha I + \beta T) = \{\alpha + \beta\lambda : \lambda \in W(T)\} = \alpha + \beta W(T)$$

2. *For any unit vector* $x \in \mathcal{H}$

   $$\langle T^*x, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$$

   If $\lambda = \langle Tx, x\rangle \in W(T)$ *, then* $\overline{\lambda} \in W(T^*)$ *.*
   Hence,

   $$W(T^*) = \{\overline{\lambda} : \lambda \in W(T)\}$$

3. *For any unit vector* $x \in \mathcal{H}$ *, define* $y = Ux$*.*
   Since U *is unitary,* $\|y\| = \|x\| = 1$*. Thus,*

   $$\langle U^*TUx, x\rangle = \langle TUx, Ux\rangle = \langle Ty, y\rangle.$$

   Since y *is a unit vector,* $\langle Ty, y\rangle \in W(T)$*.*
   Therefore, $W(U^*TU) \subseteq W(T)$*.*
   For the reverse inclusion, note that U *is invertible and* U^* *is also unitary.*
   For any $\lambda \in W(T)$ *, there exists* y *such that* $\lambda = \langle Ty, y\rangle$*. Let* $x = U^*y$*. Then,*

   $$\langle U^*TUx, x\rangle = \langle Ty, y\rangle = \lambda$$

   Hence, $W(T) \subseteq W(U^*TU)$ *, and thus* $W(U^*TU) = W(T)$

4. *For any unit vector $x \in F$ (where $\|x\| = 1$):*

$$\langle T|_F x, x\rangle = \langle Tx, x\rangle \in W(T)$$

Therefore, $W(T|_F) \subseteq W(T)$.

5. *For any unit vector $x \in \mathcal{H}$*

$$\langle (T+S)x, x\rangle = \langle Tx, x\rangle + \langle Sx, x\rangle$$

Since $\langle Tx, x\rangle \in W(T)$ and $\langle Sx, x\rangle \in W(S)$, it follows that

$$W(T+S) \subseteq W(T) + W(S)$$

6. *Recall that*

$$Re(T) = \frac{T+T^*}{2}, \quad Im(T) = \frac{T-T^*}{2i}$$

Using properties 1 and 5:

$$W(Re(T)) = \frac{1}{2}W(T+T^*) \subseteq \frac{1}{2}(W(T) + W(T^*)) = Re(W(T))$$

where $W(T^*) = \{\overline{\lambda} : \lambda \in W(T)\}$.
To show equality, note that:

$$Re(\langle Tx, x\rangle) = \langle Re(T)x, x\rangle$$

and thus $Re(W(T)) \subseteq W(Re(T))$.

Therefore $W(Re(T)) = Re(W(T))$

Similarly $W(Im(T)) = Im(W(T))$

Example 3.3 *Let $T \in \mathcal{B}(\mathcal{H})$ be the unilateral shift on ℓ_2, the Hilbert space of square summable sequences. For any $x = (x_1, x_2, x_3, \ldots) \in \mathcal{H}$, $\|x\| = 1$, we have $Tx = (x_2, x_3, x_4, \ldots)$ and hence consider*

$$\langle Tx, x\rangle = x_1\overline{x_2} + x_2\overline{x_3} + x_3\overline{x_4} + \cdots$$

with

$$|x_1|^2 + |x_2|^2 + |x_3|^2 + \cdots = 1$$

Notice that

$$\begin{aligned}|\langle Tx, x\rangle| &\leq |x_1||x_2| + |x_2||x_3| + |x_3||x_4| + \cdots \\ &\leq \frac{1}{2}\left[|x_1|^2 + 2|x_2|^2 + 2|x_3|^2 + \cdots\right] \\ &\leq \frac{1}{2}\left[2 - |x_1|^2\right]\end{aligned}$$

Hence $|\langle Tx, x\rangle| < 1$ if $|x_1| \neq 0$. For $|x_1| = 0$ and x containing a finite number of nonzero entries, we can show in the same way that $|\langle Ax, x\rangle| < 1$ by considering the minimum natural number n for which $x_n \neq 0$.

Thus $W(T)$ is contained in the open disk $\{z : |z| < 1\}$. We now show that it is in fact the open unit disk. Let $z = re^{i\theta}, 0 \leq r < 1$, be any point of this disk.

Consider

$$x = (\sqrt{1-r^2}, r\sqrt{1-r^2}e^{i\theta}, r^2\sqrt{1-r^2}e^{2i\theta}, \ldots)$$

Observe that

$$\|x\|^2 = 1 - r^2 + r^2(1-r^2) + r^4(1-r^2) + \cdots = 1$$

Furthermore

$$\langle Tx, x\rangle = r(1-r^2)e^{i\theta} + r^3(1-r^2)e^{i\theta} + \cdots = re^{i\theta}$$

Thus $z \in W(T)$*, so that*

$$W(T) = \{z : |z| < 1\}$$

Theorem 3.5 *[26] (Toeplitz-Hausdorff) : For any operator* $T \in \mathcal{B}(\mathcal{H})$ *on a Hilbert space* $\mathcal{H}$*:*

1. *The numerical range* $W(T)$ *is convex*
2. *The spectrum satisfies* $\sigma(T) \subseteq \overline{W(T)}$

Proof *(of Convexity)*
Let $\lambda_1, \lambda_2 \in W(T)$ *and* $t \in [0,1]$*. We construct* $\lambda_t = t\lambda_1 + (1-t)\lambda_2 \in W(T)$*:*

1. *Choose unit vectors* x_1, x_2 *with* $\langle Tx_i, x_i\rangle = \lambda_i$
2. *For* $\theta \in [0, 2\pi]$*, define*
$$x_\theta = \sqrt{t}e^{i\theta}x_1 + \sqrt{1-t}x_2$$
3. *Normalize* $y_\theta = x_\theta / \|x_\theta\|$
4. *The function*
$$f(\theta) = \langle Ty_\theta, y_\theta\rangle$$
is continuous and satisfies $f(0) = \lambda_2$*,* $f(\pi) = \lambda_1$
5. *By Intermediate Value Theorem,* f *attains all values between* λ_1 *and* λ_2

Proof *(of Spectral Inclusion):*
For any $\lambda \in \sigma(T)$*:*
Case 1: λ *is an eigenvalue*

- $\exists x \in \mathcal{H}$ *with* $\|x\| = 1$ *and* $Tx = \lambda x$
- *Then* $\langle Tx, x\rangle = \lambda \in W(T)$

Case 2: λ *is not an eigenvalue*

- *There exists a sequence* $\{x_n\}$ *with* $\|x_n\| = 1$ *and* $\|(T-\lambda)x_n\| \to 0$
- *We have*
$$|\langle Tx_n, x_n\rangle - \lambda| = |\langle (T-\lambda)x_n, x_n\rangle| \le \|(T-\lambda)x_n\| \to 0$$
- *Thus* $\lambda = \lim\langle Tx_n, x_n\rangle \in \overline{W(T)}$

Corollary 3.3 *(Spectrum Bounding): For* $W(T) = \{z \in \mathbb{C} : |z| < 1\}$

- $\sigma(T) \subseteq \overline{\mathbb{D}}$ *(closed unit disk)*
- *More generally,* $\sigma(T)$ *is contained in the smallest closed convex set containing* $W(T)$

Applications

- *For self-adjoint operators:* $W(T) \subseteq \mathbb{R} \Rightarrow \sigma(T) \subseteq \mathbb{R}$

- *For contractions:* $\|T\| \leq 1 \Rightarrow W(T) \subseteq \mathbb{D} \Rightarrow \sigma(T) \subseteq \mathbb{D}$

Theorem 3.6 *[14] Let T be an operator on a two-dimensional space. Then $W(T)$ is an ellipse whose foci are the eigenvalues of T.*

Proof

Let $T = \begin{bmatrix} \lambda_1 & a \\ 0 & \lambda_2 \end{bmatrix}$*, where λ_1 and λ_2 are the eigenvalues of T.*

First, if $\lambda_1 = \lambda_2 = \lambda$, we have

$$T - \lambda I = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$$

Then $W(T - \lambda I) \subseteq \{z : |z| \leq \frac{|a|}{2}\}$

We now show that $W(T - \lambda I) = \{z : |z| \leq \frac{|a|}{2}\}$.

Let $z = re^{i\theta}$, $0 \leq r \leq \frac{|a|}{2}$, *and let* $x = (\overline{a}\cos\alpha, \frac{1}{|a|}\sin\alpha e^{i\theta})$, *where* $\sin 2\alpha = \frac{2}{|a|}r \leq 1$ *and* $0 \leq \alpha \leq \frac{\pi}{4}$, *then*

$$\langle (T - \lambda I)x, x\rangle = |a|e^{i\theta}\cos\alpha\sin\alpha = |a|e^{i\theta}\frac{\sin 2\alpha}{2} = re^{i\theta}$$

so that

$$W(T - \lambda I) = \{z : |z| \leq \frac{|a|}{2}\}.$$

Corollary 3.4 *Let $T \in \mathcal{B}(\mathcal{H})$. If $W(T)$ is a line segment, then T is normal.*

Proof

There exist $\alpha, \beta \in \mathbb{C}$ and self-adjoint operator S such that $T = \alpha S + \beta I$.
Then $T^* = \overline{\alpha}S + \overline{\beta}I$, therefore

$$TT^* = (\alpha S + \beta I)(\overline{\alpha}S + \overline{\beta}I) = |\alpha|^2 S + |\beta|^2 I + \alpha\overline{\beta}S + \beta\overline{\alpha}S$$
$$T^*T = (\overline{\alpha}S + \overline{\beta}I)(\alpha S + \beta I) = |\alpha|^2 S + |\beta|^2 I + \alpha\overline{\beta}S + \beta\overline{\alpha}S$$

Hence $TT^* = T^*T$

3.4 Numerical Radius

As the spectrum, the numerical range also links a set with each operator, this set called the set of valued-function of operators and this link generates a numerical function called Numerical Radius.

Definition 3.4 *The* numerical radius *of an operator $T \in B(\mathcal{H})$ denoted by $\omega(\cdot)$ and given by*

$$\omega(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

Obviously, $\forall x \in \mathcal{H}$, we have

$$|\langle Tx, x\rangle| \leq \omega(T)\|x\|^2$$

Proposition 3.4 *For all $T, S \in \mathcal{B}(\mathcal{H})$, and $\lambda \in \mathbb{C}$*

- $\omega(T) \geq 0$ *and* $\omega(T) = 0$ *if and only if* $T = 0$

- $\omega(\lambda T) = |\lambda|\omega(T)$
- $\omega(T + S) \leq \omega(T) + \omega(S)$

This norm is unitary invariant, means that

$$\omega(A) = \omega(U^* AU)$$

for all unitary operator U and it is equivalent to the usual operator norm, we present this equivalence in this Theorem.

Example 3.4 *Consider the operator* $T \in \mathcal{B}(\mathbb{C}^{\in})$ *represented by the matrix*

$$T = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

Computing the Numerical Radius :
The numerical radius *of T is defined as*

$$\omega(T) = \sup\{|\langle Tx, x\rangle| : x \in \mathbb{C}^2, \|x\| = 1\}$$

1. *Compute the numerical range: For a unit vector* $x = (x_1, x_2) \in \mathbb{C}^2$ *with* $|x_1|^2 + |x_2|^2 = 1$, *we have:*
$$\langle Tx, x\rangle = 2x_2\overline{x}_1$$
2. *Parameterize using angles:*
Let $x_1 = e^{i\theta}\cos\phi$, $x_2 = e^{i\psi}\sin\phi$ *where* $\phi \in [0, \pi/2]$, $\theta, \psi \in [0, 2\pi]$. *Then*
$$|\langle Tx, x\rangle| = 2|\sin\phi\cos\phi| = |\sin(2\phi)| \leq 1$$
3. *The maximum value 1 is achieved when* $\phi = \pi/4$.

Comparison with Operator Norm :
The operator norm of T is

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sqrt{\lambda_{max}(T^*T)} = 2$$

Thus for this operator we have

$$\omega(T) = 1 < 2 = \|T\|$$

This illustrates the general inequality

$$\frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|$$

Geometric Interpretation :
The numerical range $W(T)$ *is the closed unit disk in* $\mathbb{C}$, *while the numerical radius* $\omega(T) = 1$ *is its radius. This shows that for non-normal operators, the numerical radius can be strictly less than the operator norm.*

Theorem 3.7 *[14] Let $T \in B(\mathcal{H})$. Then*

$$\frac{1}{2}\|T\| \le \omega(T) \le \|T\|$$

Proof
Let $\lambda = \langle Tx, x\rangle$, with $\|x\| = 1$ and let $y \in \mathcal{H}$, we have by the Schwarz inequality

$$|\lambda| = |\langle Tx, x\rangle| \le \|Tx\| \le \|T\|$$

To prove the second inequality, we use the polarization identity, which may be verified by direct computation

$$\begin{aligned} 4\langle Tx, y\rangle &= \langle T(x+y),(x+y)\rangle - \langle T(x-y),(x-y)\rangle \\ &\quad + i\langle T(x+iy),(x+iy)\rangle - i\langle T(x-iy),(x-iy)\rangle \end{aligned}$$

Hence

$$\begin{aligned} 4\langle Tx, y\rangle &\le \omega(T)\left[\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2\right] \\ &= 4\omega(T)\left(\|x\|^2 + \|y\|^2\right) \end{aligned}$$

Choosing $\|x\| = \|y\| = 1$, we get

$$4\langle Tx, y\rangle \le 8\omega(T)$$

Thus

$$\|T\| \le 2\omega(T)$$

The next Theorem gives a useful characterization of the numerical radius.

Theorem 3.8 *Let [14] $T \in \mathcal{B}(\mathcal{H})$. If $R(T) \perp R(T^*)$, then*

$$\omega(T) = \frac{\|T\|}{2}$$

Proof
Let $x = x_1 + x_2$ be a unit vector in $\mathcal{H} = N(T) \oplus \overline{R(T^*)}$, where $x_1 \in N(T)$ and $x_2 \in \overline{R(T^*)}$. Thus we have

$$|\langle Tx, x\rangle| = |\langle T(x_1 + x_2), x_1 + x_2\rangle| = |\langle Tx_2, x_1\rangle|$$

Since $Tx_1 = 0$ and $\langle Tx_2, x_2\rangle = \langle x_2, T^*x_2\rangle = 0$, we get

$$|\langle Tx, x\rangle| \le \|T\|\|x_1\|\|x_2\| \le \frac{\|T\|}{2}(\|x_1\| + \|x_2\|) = \frac{\|T\|}{2}$$

then

$$\frac{\|T\|}{2} \le \omega(T) \le \frac{\|T\|}{2}$$

Example 3.5 *Let*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We want to compute the ***numerical radius*** *$\omega(A)$, which is defined as*

$$\omega(A) = \sup\{|x^*Ax| : x \in \mathbb{C}^n, \|x\| = 1\}$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^2$*, with* $\|x\| = 1 \Rightarrow |x_1|^2 + |x_2|^2 = 1$

$$x^* A x = \begin{bmatrix} \overline{x_1} & \overline{x_2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overline{x_1} x_2$$

So

$$|x^* A x| = |\overline{x_1} x_2| = |x_1||x_2|$$

To maximize this subject to $|x_1|^2 + |x_2|^2 = 1$*, let* $|x_1| = \cos\theta, |x_2| = \sin\theta$*, then*

$$|x_1||x_2| = \cos\theta \sin\theta = \frac{1}{2}\sin(2\theta)$$

The maximum value of $\sin(2\theta)$ *is 1, so:*

$$\omega(A) = \max |x^* A x| = \frac{1}{2}$$

$$\omega(A) = \frac{1}{2}$$

Theorem 3.9 *Let* $T \in \mathcal{B}(\mathcal{H})$*. If* T *is idempotent and* $\omega(T) \leq 1$*, then* T *is an orthogonal projection.*

Proof

To prove this theorem it is sufficient to prove that T *is null on* $R(T)^{\perp}$*.*
Let $x \in R(T)^{\perp}$ *and* $y = Tx$*. Then for* $t \geq 0$*, we have*

$$T(x + ty) = y + tT^2 y = (1 + t)y$$

As $x \perp y$*, we have*

$$\begin{aligned} \langle T(x + ty), x + ty \rangle =& \langle (1 + t)y, x + ty \rangle \\ =& \langle (1 + t)y, ty \rangle = (1 + t)t\|y\|^2 \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (1 + t)t\|y\|^2 =& |\langle T(x + ty), x + ty \rangle| \\ \leq& \omega(T)\|x + ty\|^2 = \|x\|^2 + t\|y\|^2 \end{aligned}$$

because $\omega(T) \leq 1$*. Thus*

$$t\|y\|^2 \leq \|x\|^2$$

Since t *is arbitrary, we conclude that* $\|y\| = 0$ *and* $T = 0$ *on* $R(T)^{\perp}$*.*

Corollary 3.5 *we have*

a) If $\omega(T) = \|T\|$*, then* $r(T) = \|T\|$*.*

b) If $\lambda \in W(T) and |\lambda| = \|T\|$ *, then* $\lambda \in \sigma_p(T)$*.*

Theorem 3.10 *[24] For any operator* $T \in \mathcal{B}(\mathcal{H}), n \in \mathbb{N}$*, we have*

$$\omega(T^n) \leq \omega^n(T)$$

Proof

We proceed by induction on n.

Base case (n = 1): Trivially true since $w(T^1) = w(T)$*.*

Inductive step: Assume $w(T^k) \leq w^k(T)$ holds for some $k \geq 1$. Then for any unit vector $x \in \mathcal{H}$:

$$\begin{aligned}
|\langle T^{k+1}x, x\rangle| =& |\langle T(T^k x), x\rangle| \\
\leq& w(T)\|T^k x\| \\
\leq& w(T) \cdot w(T^k) \\
\leq& w(T) \cdot w^k(T) \\
=& w^{k+1}(T)
\end{aligned}$$

Taking supremum over all unit vectors x gives

$$w(T^{k+1}) \leq w^{k+1}(T)$$

The result follows by mathematical induction.

Theorem 3.11 *[16] For all operator $T \in \mathcal{B}(\mathcal{H})$, we have*

$$\omega(T) \leq \frac{1}{2}(\|T\| + \|T^2\|^{\frac{1}{2}})$$

Proof

For any unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned}
|\langle Tx, x\rangle|^2 \leq& \langle |T|x, x\rangle\langle |T^*|x, x\rangle \quad \text{(by (2.1))} \\
\leq& \frac{1}{4}(\langle |T|x, x\rangle + \langle |T^*|x, x\rangle)^2 \\
\leq& \frac{1}{4}(\|T\| + \|T^2\|^{1/2})^2
\end{aligned}$$

The last inequality follows because

- $\langle |T|x, x\rangle \leq \||T|\| = \|T\|$
- $\langle |T^*|x, x\rangle \leq \||T^*|\| = \|T\|$
  But more precisely
  $$\langle |T^*|x, x\rangle^2 \leq \langle |T^*|^2 x, x\rangle = \langle TT^* x, x\rangle \leq \|T^2\|$$
  since $|T^*|^2 = TT^*$ and $\langle TT^* x, x\rangle = \|T^* x\|^2 \leq \|T^2\|$ when $\|x\| = 1$.

Taking square roots and supremum over all unit vectors x gives the result.

Theorem 3.12 *[10] If for an operator $T \in \mathcal{B}(\mathcal{H})$ we denote $|T| = \sqrt{T^*T}$, then*

$$\omega^r(T) \leq \frac{1}{2}\left\||T|^{2\alpha r} + |T^*|^{2(1-\alpha)r}\right\| \tag{3.1}$$

$$\omega^{2r}(T) \leq \left\|\alpha|T|^{2r} + (1-\alpha)|T^*|^{2r}\right\| \tag{3.2}$$

where $\alpha \in (0,1)$ and $r \geq 1$.

If we take $\alpha = \frac{1}{2}$ and $r = 1$ we get

$$\omega(T) \leq \frac{1}{2}\||T| + |T^*|\|$$

$$\omega^2(T) \leq \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|$$

Theorem 3.13 *[19] Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega(T)^2 \leq \frac{1}{2}\|T^*T + TT^*\|$$

Proof

Let $x \in \mathcal{H}$ where $\|x\| = 1$, we get

$$\begin{aligned}
|\langle Tx, x\rangle|^2 &\leq \langle |T|x, x\rangle\langle |T^*|x, x\rangle \\
&\leq \frac{1}{2}\left(\langle |T|x, x\rangle^2 + \langle |T^*|x, x\rangle^2\right) \\
&\leq \frac{1}{2}\left(\||T|x\|^2 + \||T^*|x\|^2\right) \\
&= \frac{1}{2}\left(\langle T^*Tx, x\rangle + \langle TT^*x, x\rangle\right) \\
&= \frac{1}{2}\langle (T^*T + TT^*)x, x\rangle \\
&\leq \frac{1}{2}\|T^*T + TT^*\|
\end{aligned}$$

Hence

$$\omega(T)^2 \leq \frac{1}{2}\|T^*T + TT^*\| \tag{3.3}$$

- *$T = Re(T) + i\ Im(T)$ such that $Re(T) = \frac{1}{2}(T + T^*)$ and $Im(T) = \frac{1}{2i}(T - T^*)$, and by simple calculation, we get that $T^*T + TT^* = 2(Re(T)^2 + Im(T)^2)$.*

Let $x \in \mathcal{H}$ where $\|x\| = 1$, since $Re(T)$ and $Im(T)$ are self-adjoint, and using the convexity of the square function, we obtain

$$\begin{aligned}
|\langle Tx, x\rangle|^2 &= (Re(T)x, x)^2 + (Im(T)x, x)^2 \\
&\leq \frac{1}{2}\left((Re(T)x, x)^2 + (Im(T)x, x)^2\right) \\
&= \frac{1}{2}\left((Re(T)x, x)^2 + (Im(T)x, x)^2\right)
\end{aligned}$$

$$\Rightarrow \omega(T)^2 = \sup_{\|x\|=1} |\langle Tx, x\rangle|^2 = \frac{1}{2}\sup_{\|x\|=1}\left(\langle Re(T)x, x\rangle^2 + \langle Im(T)x, x\rangle^2\right) \leq \frac{1}{2}\|Re(T)^2 + Im(T)^2\|$$

Therefore

$$\begin{aligned}
2(\omega(T)^2) &\geq \|Re(T)^2 + Im(T)^2\| \\
&= \frac{1}{2}\|T^*T + TT^*\|
\end{aligned}$$

Thus

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega(T)^2$$

Theorem 3.14 *Let $A, B \in M_n$. Then*

$$w(AB) \leq w^2\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$$

Proof

We have

$$w\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}^2 \le w^2\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}$$

Thus

$$w\begin{bmatrix}AB & 0\\ 0 & BA\end{bmatrix} \le w^2\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}$$

So

$$\max(w(AB), w(BA)) \le w^2\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}$$

$$w(AB) \le w^2\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}$$

This completes the proof.

The following theorem gives a new upper bound for the numerical radius of product matrices.

Theorem 3.15 *Let $A, B \in M_n$. Then*

$$w(AB) \le \frac{1}{2}(\|BA\| + \|A\|\|B\|)$$

Proof

For $\theta \in \mathbb{R}$, we have

$$\left\|\mathrm{Re}(e^{i\theta}AB)\right\| = r\left(\mathrm{Re}(e^{i\theta}AB)\right) = \frac{1}{2}r\left(e^{i\theta}AB + e^{-i\theta}B^*A^*\right)$$

$$= \frac{1}{2}r\left(\begin{bmatrix}e^{i\theta}AB + e^{-i\theta}B^*A^* & 0\\ 0 & 0\end{bmatrix}\right)$$

$$= \frac{1}{2}r\left(\begin{bmatrix}e^{i\theta}A & B^*\\ 0 & 0\end{bmatrix}\begin{bmatrix}e^{-i\theta}B & 0\\ 0 & A^*\end{bmatrix}\right)$$

Using a commutativity property of the spectral radius, we have

$$\left\|\mathrm{Re}(e^{i\theta}AB)\right\| = \frac{1}{2}r\left(\begin{bmatrix}e^{-i\theta}A^* & 0\\ 0 & B\end{bmatrix}\begin{bmatrix}e^{i\theta}A & B^*\\ 0 & 0\end{bmatrix}\right)$$

$$= \frac{1}{2}r\left(\begin{bmatrix}e^{i\theta}BA & BB^*\\ A^*A & e^{-i\theta}A^*B\end{bmatrix}\right)$$

$$\le \frac{1}{2}r\left(\begin{bmatrix}\|BA\| & \|BB^*\|\\ \|A^*A\| & \|A^*B\|\end{bmatrix}\right)$$

$$= \frac{1}{2}(\|BA\| + \|A\|\|B\|)$$

Taking the maximum over $\theta \in \mathbb{R}$ in both sides, we get

$$w(AB) \le \frac{1}{2}(\|BA\| + \|A\|\|B\|)$$

This completes the proof.

At the end of this section, we estimate the numerical raduis of produducts and sums of matrices.

Theorem 3.16 *[10] If $T = A + iB$ is the Cartesian decomposition of T , then*

$$\omega^r(T) \le \||A|^r + |B|^r\|$$

for $r \in [0,2]$.

If $r \ge 2$, then

$$\omega^r(T) \le 2^{\frac{r}{2}-1} \||A|^r + |B|^r\|$$

and

$$2^{-\frac{r}{2}-1} \||A+B|^r + |A-B|^r\| \le \omega^r(T) \le \frac{1}{2} \||A+B|^r + |A-B|^r\|$$

We observe that for $r = 1$, we get

$$\omega(T) \le \||A| + |B|\|$$

while for $r = 2$, we get

$$\omega^2(T) \le \left\||A|^2 + |B|^2\right\|$$

$$\frac{1}{4}\left\||A+B|^2 + |A-B|^2\right\| \le \omega^2(T) \le \frac{1}{2}\left\||A+B|^2 + |A-B|^2\right\|.$$

Theorem 3.17 *[10] For any $A, B, C, D, S, T \in \mathcal{B}(\mathcal{H})$, we have*

$$w(ATB + CSD) \le \frac{1}{2}\left\|A|T^*|^{2(1-\alpha)}A^* + B^*|T|^{2\alpha}B + C|S^*|^{2(1-\alpha)}C^* + D^*|S|^{2\alpha}D\right\|,$$

where $\alpha \in [0,1]$.

Following we list here some particular inequalities of interest. If we take $T = I$ and $S = 0$ we get

$$w(AB) \le \frac{1}{2}\|AA^* + B^*B\|$$

In addition to this we have the related inequality

$$w(AB) \le \frac{1}{2}\|A^*A + BB^*\|$$

If we choose $T = S = I$, $C = B$ and $D = \pm A$, we get

$$w(AB \pm BA) \le \frac{1}{2}\|A^*A + AA^* + BB^* + B^*B\|$$

which provides an upper bound for the numerical radius of the commutator $AB - BA$.

If we take $\alpha = \frac{1}{2}$, we also can derive the inequality

$$w(AB \pm B^*A) \le \frac{1}{2}\||A| + |A^*| + B^*(|A| + |A^*|)B\|.$$

CHAPTER

4

SOME ADVANCED INEQUALITIES

This chapter presents some refined inequalities linking the operator norm and the numerical radius on Hilbert spaces. By introducing the(a,b)-norm $\|.\|_{a,b}$, sharper and more effective bounds are established. Applications include various operator forms and structured operator matrices.

4.1 The(a,b)-norm and its properties

Theorem 4.1 *Let $T \in \mathcal{B}(\mathcal{H})$ and $a, b \in \mathbb{N}^*$. Then the mapping given by*

$$\|T\|_{a,b} = \|a|T| + b|T^*|\|$$

defines a norm on $\mathcal{B}(\mathcal{H})$. Where $|T| = \sqrt{T^*T}$

Proof

- *If $T = 0$, it is obvious that $\||T\|_{a,b} = 0$.*
  Conversely, assume that $\|T\|_{a,b} = 0$, then
  for all $x \in \mathcal{H}$ where $\|x\| = 1$:

  $$\|(a|T| + b|T^*|)x\| = 0 \Rightarrow |Tx| = |T^*x| = 0$$

  Thus $|T| = 0, \quad$ hence $T = 0$.
  Therefore

  $$T = 0 \iff \|T\|_{a,b} = 0$$

- *Let $\lambda \in \mathbb{C}$ and $T \in \mathcal{B}(\mathcal{H})$, then*

  $$\begin{aligned}\|\lambda T\|_{a,b} =& \|a|(\lambda T)x| + b|(\lambda T)^*x|\| \\ =& \|a|\lambda T| + b|\bar{\lambda}T^*|\| \\ =& \|a|\lambda||Tx| + b|\bar{\lambda}||T^*x|\| \\ =& |\lambda|\|a|Tx| + b|T^*x|\| \\ =& |\lambda|\|T\|_{a,b}\end{aligned}$$

Then $\|\lambda T\|_{a,b} = |\lambda|\|T\|_{a,b}$.

- *Let* $T, S \in \mathcal{B}(\mathcal{H})$, *then*

$$\|T + S\|_{a,b} = \|a|(T + S)| + b|(T + S)^*|\|$$

We have

$$\begin{aligned} |T + S| &\leq |T| + |S| \\ |T + S|^2 &= (\sqrt{(T + S)^*(T + S)}\ )^2 \\ &= T^*T + T^*S + S^*T + S^*S \\ &= |T|^2 + T^*S + S^*T + |S|^2 \end{aligned}$$

and

$$\begin{aligned} (|T| + |S|)^2 &= |T|^2 + |T||S| + |S||T| + |S|^2 \\ &= T^*T + |T||S| + |S||T| + S^*S \end{aligned}$$

Prove that

$$T^*S + S^*T \leq 2|T||S|$$

we notice that $T^*S + S^*T = T^*S + (T^*S)^*$
It is well known

$$\|Tx\| = \||T|x\|$$

$$\|Sx\| = \||S|x\|$$

So

$$|\langle Tx, Sx\rangle| \leq \||T|x\|\||S|x\|$$

.

$$\begin{aligned} \langle (T^*S + S^*T)x, x\rangle &= \langle (T^*S + (T^*S)^*)x, x\rangle \\ &= \langle T^*Sx, x\rangle + \langle x, T^*Sx\rangle \\ &= \langle Sx, Tx\rangle + \langle Tx, Sx\rangle = 2\ Im(\langle Tx, Sx\rangle) \end{aligned}$$

.

$$\begin{aligned} |\langle (T^*S + S^*T)x, x\rangle| &= 2|Im(\langle Tx, Sx\rangle| \\ &\leq 2|\langle Tx, Sx\rangle| \\ &\leq 2\||T|x\|\ \||S|x\| \end{aligned}$$

That is

$$\langle (T^*S + S^*T)x, x\rangle \leq 2\||T|x\|\ \||S|x\|$$

$\forall x \in \mathcal{H}$

$$T^*S + S^*T \leq 2|T||S|$$

$$\begin{aligned} |T + S|^2 &= |T|^2 + T^*S + S^*T + |S|^2 \\ &\leq |T|^2 + 2|T||S| + |S|^2 \\ &= (|T| + |S|)^2 \end{aligned}$$

Since

$$\begin{cases}|T+S| \leq |T|+|S| \\ |(T+S)^*| \leq |T^*|+|S^*|\end{cases}$$

$$\begin{cases}a|T+S| \leq a|T|+a|S| \\ b|(T+S)^*| \leq b|T^*|+b|S^*|\end{cases}$$

$$\begin{aligned} a|T+S|+b|(T+S)^*| &\leq a|T|+a|S|+b|T^*|+b|S^*| \\ &\leq a|T|+b|T^*|+a|S|+b|S^*| \\ Since \qquad \||a|T+S|+b|T+S|^*\| &\leq \||a|T|+b|T^*|+a|S|+b|S^*|\| \\ &\leq \||a|T|+b|T^*|\|| + \|+a|S|+b|S^*|\| \\ &= \|T\|_{a,b}+\|S\|_{a,b} \end{aligned}$$

Hence $\|\cdot\|_{a,b}$ *defines a norm on* $\mathcal{B}(\mathcal{H})$.

Remark 9 *If* $a=0$ *and* $b \in \mathbb{N}^*$, *then* $\|T\|_{a,b} = b\||T^*|\| = b\|T\|$,
and the same if $b=0$ *and* $a \in \mathbb{N}^*$, *then* $\|T\|_{a,b} = a\||T|\|$, *that is why we take* $a, b \in \mathbb{N}^*$.

Lemme 4.1 *Let* $T \in \mathcal{B}(\mathcal{H})$ *and* $a, b \in \mathbb{N}^*$. *Then we have*

$$\max\{a,b\}\|T\| \leq \|T\|_{a,b} \leq (a+b)\|T\| \tag{4.1}$$

Proof
Let $x \in \mathcal{H}$ *such that* $\|x\|=1$, *then*

$$\|T\|_{a,b} = \|a|T|+b|T^*|\,\| \leq (a+b)\||T|\| = (a+b)\|T\|$$

Therefore

$$\|T\|_{a,b} \leq (a+b)\|T\|$$

we have that

$$a\|T\| \leq \||a|T|+b|T^*|\|| = \|T\|_{a,b}$$

and the same , we have

$$b\|T\| \leq \||a|T|+b|T^*|\|| = \|T\|_{a,b}$$

Hence

$$\max\{a,b\}\|T\| \leq \|T\|_{a,b} \leq (a+b)\|T\|$$

Corollary 4.1 *Let* $T \in \mathcal{B}(\mathcal{H})$ *and* $a, b \in \mathbb{N}^*$. *Then*

$$\|T\| \leq \|T\|_{a,b}$$

Proof

$$\begin{aligned}\|T\| \leq & a\|T\| \\ = & \|a|T|\| \\ \leq & \|a|T| + b|T^*|\,\|\end{aligned}$$

because : $1 \leq a, b$

Hence

$$\|T\| \leq \|T\|_{a,b}$$

Remark 10 *By 4.1 mean that* $\|.\|_{a,b}$ *and* $\|.\|$ *are equivalent norms .*

Proposition 4.1 *Let* $T \in \mathcal{B}(\mathcal{H})$ *and* $a, b \in \mathbb{N}^*$*. Then*

- $\|T^*\|_{a,b} = \|T\|_{b,a}$
- *if T is normal :* $\|T^n\|_{a,b} = (a+b)\|T\|^n$
- *if T is normal :* $\|T^n\|_{a,b} = (a+b)^{1-n}\|T\|^n_{a,b}$

Proof
Let $T \in \mathcal{B}(\mathcal{H})$ *and* $a, b \in \mathbb{N}^*$*. Then*

- $$\begin{aligned}\|T^*\|_{a,b} = & \|a|T^*| + b|(T^*)^*|\,\| \\ = & \|a|T^*| + b|T|\,\| \\ = & \|T\|_{b,a}\end{aligned}$$

- $$\begin{aligned}\|T\|_{a,b} = & \|a|T| + b|T^*|\,\| \\ = & \|(a+b)|T|\,\| \\ = & (a+b)\|T\|\end{aligned}$$

  By placing T^n *in T*

  $$\|T^n\|_{a,b} = (a+b)\|T^n\|$$

  by T normal

  $$\|T^n\| = \|T\|^n$$

  Hence

  $$\|T^n\|_{a,b} = (a+b)\|T\|^n \tag{4.2}$$

- *we have*

$$\begin{aligned}(a+b)\|T\| = & \|T\|_{a,b} \\ \|T\| = & (a+b)^{-1}\|T\|_{a,b} \\ \|T\|^n = & (a+b)^{-n}\|T\|^n_{a,b} \\ \frac{1}{(a+b)}\|T\|^n_{a,b} = & (a+b)^{-n}\|T\|^n_{a,b} \qquad (By 4.2) \\ \|T^n\|_{a,b} = & (a+b)(a+b)^{-n}\|T\|^n_{a,b} \\ \|T^n\|_{a,b} = & (a+b)^{1-n}\|T\|^n_{a,b}\end{aligned}$$

Example 4.1

- *Let ℓ^2 be the Hilbert space*

$$\ell^2 = \left\{ (x_1, x_2, x_3, \dots) \,\middle|\, \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}$$

Define the operator $T : \ell^2 \to \ell^2$ as

$$T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$$

*This is called the **unilateral shift operator**.*
It shifts all coordinates one position to the right and inserts 0 at the beginning.

and

The adjoint of T is

$$T^*(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$$

This operator shifts components one step to the left, dropping the first entry.
Clearly , $T \neq T^*$, so T is not self-adjoint.

***Compute** $|T|$ **and** $|T^*|$*
We use the polar decomposition definitions

$$|T| = \sqrt{T^*T}, \qquad |T^*| = \sqrt{TT^*}$$

We compute

$$T^*T = I \quad \Rightarrow \quad |T| = \sqrt{I} = I$$

$$TT^* = P \quad \text{where } P \text{ is a projection operator (i.e., } P^2 = P) \Rightarrow |T^*| = \sqrt{P} = P$$

***Compute the** $\|\cdot\|_{a,b} Norms$*

First

$$\|T\|_{b,a} = \|b|T| + a|T^*|\| = \|bI + aP\|$$

Since P is a projection, its spectrum is contained in {0, 1}, so

$$\|bI + aP\| = \max(b, b + a)$$

Similarly

$$\|T^*\|_{a,b} = \|a|T^*| + b|T|\| = \|aP + bI\| = \max(a, b + a)$$

Hence

$$\|T^*\|_{a,b} = \|T\|_{b,a}$$

This confirms the identity for the unilateral shift operator, which is not self-adjoint.

- $\|T^n\|_{a,b} = (a+b)\|T^n\|$ *for Normal Operators*

  Definition of the operator T

  We consider the operator

  $$T = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^{\in})$$

  This is a diagonal operator and thus ***normal****, since*

  $$T^*T = TT^*$$

  Computing T^n

  Since T is diagonal, its powers are computed directly

  $$T^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}$$

  Computing $\|T^n\|$ *:*

  As T^n *is diagonal:*

  $$\|T^n\| = \max(|2^n|, |3^n|) = 3^n \quad \textit{for all } n \in \mathbb{N}^*$$

  Definition of the New Norm $\|\cdot\|_{a,b}$

  For any bounded operator $T \in \mathcal{B}(\mathcal{H})$ *and integers* $a, b \in \mathbb{N}^*$

  $$\|T\|_{a,b} = \|a|T| + b|T^*|\|$$

  Computing $\|T^n\|_{a,b}$

  Since T^n *is normal*

  $$|T^n| = |T^{n*}| = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}$$

  Therefore

  $$\|T^n\|_{a,b} = \|a|T^n| + b|T^{n*}|\| = \|(a+b)|T^n|\| = (a+b)\cdot\|T^n\|$$

  Since

  $$\|T^n\| = 3^n$$

  We conclude

  $$\|T^n\|_{a,b} = (a+b)\cdot 3^n$$

- $\|T^n\|_{a,b} = (a+b)^{1-n}\cdot\|T\|_{a,b}^n$ *for a Normal Operator*

  Recall from the previous example , we have

  $$\|T\| = 3, \quad \|T\|_{a,b} = (a+b)\cdot 3, \quad \|T^n\| = 3^n$$

  Since T is diagonal, it is normal, and so is T^n *for all* $n \in \mathbb{N}$.

  Compute $\|T^n\|_{a,b}$ *:*

  Since T^n *is normal, we have*

  $$|T^n| = |T^{n*}| = T^n \Rightarrow \|T^n\|_{a,b} = \|a|T^n| + b|T^{n*}|\| = \|(a+b)\cdot T^n\| = (a+b)\cdot\|T^n\|$$

$$\Rightarrow \|T^n\|_{a,b} = (a+b)\cdot 3^n$$

Compute the Right-Hand Side *:*

From the previous example

$$\|T\|_{a,b} = (a+b)\cdot 3 \Rightarrow \|T\|_{a,b}^n = ((a+b)\cdot 3)^n = (a+b)^n \cdot 3^n$$

Then

$$(a+b)^{1-n}\cdot \|T\|_{a,b}^n = (a+b)^{1-n}\cdot (a+b)^n \cdot 3^n = (a+b)\cdot 3^n$$

Conclusion

We have:

$$\|T^n\|_{a,b} = (a+b)\cdot 3^n = (a+b)^{1-n}\cdot \|T\|_{a,b}^n$$

Hence

$$\|T^n\|_{a,b} = (a+b)^{1-n}\cdot \|T\|_{a,b}^n$$

4.2 Some developed inequalities

Theorem 4.2 *Let $T \in \mathcal{B}(\mathcal{H})$ and $a, b \in \mathbb{N}^*$.*
Then

$$\frac{1}{2(a+b)}\|T\|_{a,b} \leq \omega(T) \leq \frac{1}{2}\max\{\frac{1}{a},\frac{1}{b}\}\|T\|_{a,b} \tag{4.3}$$

Proof

$T \in \mathcal{B}(\mathcal{H})$ with $\|x\| = 1$

- *Since*

$$\begin{aligned}
|\langle Tx, x\rangle|^2 \leq& \langle |T|x, x\rangle \langle |T^*|x, x\rangle \\
|\langle Tx, x\rangle| \leq& \sqrt{\langle |T|x, x\rangle}\sqrt{\langle |T^*|x, x\rangle} \\
\leq& \frac{1}{2}(\langle |T|x, x\rangle + \langle |T^*|x, x\rangle) \\
\leq& \frac{1}{2}(\langle \frac{a}{a}|T|x, x\rangle + \langle \frac{b}{b}|T^*|x, x\rangle) \\
\leq& \frac{1}{2}\max\{\frac{1}{a},\frac{1}{b}\}(\langle (a|T| + b|T^*|)x, x\rangle) \\
\leq& \frac{1}{2}\max\{\frac{1}{a},\frac{1}{b}\}\|T\|_{a,b}
\end{aligned}$$

Therefore

$$\omega(T) \leq \frac{1}{2}\max\{\frac{1}{a},\frac{1}{b}\}\|T\|_{a,b}$$

- *On the other hand.*
  We have

$$\|T\|_{a,b} \leq (a+b)\|T\|$$

So

$$\begin{aligned}
\|T\|_{a,b} &\le (a+b)\|T\| \\
&\le (a+b) \sup_{\|x\|=\|y\|=1} |\langle Tx, y\rangle| \\
&\le (a+b) \sup_{\|x\|=\|y\|=1} \frac{1}{4}(|\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle| + |\langle T(x+iy), x+iy\rangle| \\
&\qquad + |\langle T(x-iy), x-iy\rangle|) \\
&\le (a+b) \sup_{\|x\|=\|y\|=1} \frac{\omega(T)}{4}(\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2) \\
&\le \frac{(a+b)}{4}\omega(T) \sup_{\|x\|=\|y\|=1} 4(\|x\|^2 + \|y\|^2) \\
&\le 2(a+b)\omega(T)
\end{aligned}$$

Therefore

$$\frac{1}{2(a+b)}\|T\|_{a,b} \le \omega(T)$$

Hence

$$\frac{1}{2(a+b)}\|T\|_{a,b} \le \omega(T) \le \frac{1}{2}\max\{\frac{1}{a}, \frac{1}{b}\}\|T\|_{a,b}$$

Remark 11 *By 4.3 , we can say that $\|\|_{a,b}$ and ω are equivalent norms.*

Lemme 4.2 *Let $T \in \mathcal{B}(\mathcal{H})$ $\quad a, b \in \mathbb{N}^*$. Then*

- *$|T|$ is normaloid , So*

$$\|T\| = \||T|\| = \omega(|T|)$$

- *$|T^*|$ is normaloid , So*

$$\|T^*\| = \||T^*|\| = \omega(|T^*|)$$

  We obtain

$$\|T\| = \omega(|T^*|) = \omega(|T|) \tag{4.4}$$

- *$a|T| + b|T^*|$ is normaloid (Because it is the sum of two positive operators). So*

$$\begin{aligned}
\|T\|_{a,b} &= \|a|T| + b|T^*|\| \\
&= \omega(a|T| + b|T^*|) \\
&\le \omega(a|T|) + \omega(b|T^*|) \\
&\le a\omega(|T|) + b\omega(|T^*|) \\
&= (a+b)\omega(|T|) \qquad (4.5)
\end{aligned}$$

Remark 12 *From the recent inequality 4.5 and 4.4 , we obtain*

$$\|T\|_{a,b} \leq (a+b)\omega(|T|)$$

or

$$\|T\|_{a,b} \leq (a+b)\omega(|T^*|)$$

or

$$\|T\|_{a,b} \leq (a+b)\|T\|$$

(we received it earlier)

Theorem 4.3 *Let $T \in \mathcal{B}(\mathcal{H})$ and $a, b \in \mathbb{N}^*$. Then*

$$\omega(T) \leq \frac{1}{2}\|T\|_{a,b} + \frac{1}{\sqrt{\max\{a,b\}}}\sqrt{\|T^2\|_{a,b}}$$

Proof

- *Since*

  $$\begin{aligned} a\|T^2\| &\leq \|T^2\|_{a,b} \\ b\|T^2\| &\leq \|T^2\|_{a,b} \end{aligned}$$

  Then

  $$\begin{aligned} \max\{a,b\}\|T^2\| \leq& \|T^2\|_{a,b} \\ \|T^2\| \leq& \frac{1}{\max\{a,b\}}\|T^2\|_{a,b} \\ \sqrt{\|T^2\|} \leq& \sqrt{\frac{1}{\max\{a,b\}}\|T^2\|_{a,b}} \\ \sqrt{\|T^2\|} \leq& \frac{1}{\sqrt{\max\{a,b\}}}\sqrt{\|T^2\|_{a,b}} \end{aligned}$$

Therefore

$$\begin{aligned} \omega(T) \leq& \frac{1}{2}\left(\|T\| + \sqrt{\|T^2\|}\right) \\ \leq& \frac{1}{2}\left(\|T\|_{a,b} + \frac{1}{\sqrt{\max\{a,b\}}}\sqrt{\|T^2\|_{a,b}}\right) \end{aligned}$$

Hence

$$\omega(T) \leq \frac{1}{2}\left(\|T\|_{a,b} + \frac{1}{\sqrt{\max\{a,b\}}}\sqrt{\|T^2\|_{a,b}}\right)$$

Theorem 4.4 *Let $T \in \mathcal{B}(\mathcal{H})$ and $a, b \in \mathbb{N}^*$, then*

$$\frac{1}{(2(a+b))^2}\|T\|_{a,b}^2 \leq \omega(T)^2 \leq \frac{1}{2}\|T\|_{a,b}^2$$

Proof

- *Using 2.4 , we have*

$$\|(T+S)^r\| \le \|T^r + S^r\|$$

for $0 < r \le 1$.
Setting $r = \frac{1}{2}$ *and* $T = T^*T, S = TT^*$, *we get*

$$\begin{aligned}
\|(T^*T + TT^*)^{\frac{1}{2}}\| \le& \|(T^*T)^{\frac{1}{2}} + (TT^*)^{\frac{1}{2}}\| \\
\|(T^*T + TT^*)\|^{\frac{1}{2}} \le& \|(T^*T)^{\frac{1}{2}} + (TT^*)^{\frac{1}{2}}\| \\
=& \||T| + |T^*|\| \\
\le& \|a|T| + b|T^*|\| \qquad (Because 1 \le a, b) \\
\le& \|T\|_{a,b} \\
\|(T^*T + TT^*)\| \le& \|T\|_{a,b}^2
\end{aligned}$$

Therefore

$$\begin{aligned}
\omega(T)^2 \le& \frac{1}{2}\|(T^*T + TT^*)\| \qquad (By 3.3) \\
\le& \frac{1}{2}\|T\|_{a,b}^2
\end{aligned}$$

Hence

$$\omega(T)^2 \le \frac{1}{2}\|T\|_{a,b}^2$$

- *another side*

$$\begin{aligned}
\frac{1}{(2(a+b))^2}\|T\|_{a,b}^2 =& \frac{1}{4}\frac{1}{a+b}\|T\|_{a,b}\frac{1}{a+b}\|T\|_{a,b} \\
\le& \frac{1}{4}\|T\|\|T\| \\
\le& \frac{1}{4}\|T\|^2 \\
\le& \frac{1}{4}\|TT^* + T^*T\| \\
\le& \omega(T)^2
\end{aligned}$$

Hence

$$\frac{1}{(2(a+b))^2}\|T\|_{a,b}^2 \le \omega(T)^2$$

Theorem 4.5 *Let* $T \in \mathcal{B}(\mathcal{H})$ *and* $a, b \in \mathbb{N}^*$.
Then

$$\omega(T^n) \le \frac{1}{\max\{a,b\}^n}\|T\|_{a,b}^n$$

Proof

$$\begin{aligned}
\omega(T^n) \le& \omega^n(T) \\
\le& \|T\|^n \\
\le& \frac{1}{\max\{a,b\}^n}\|T\|_{a,b}^n
\end{aligned}$$

4.3 inequalities for two operators

Theorem 4.6 *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $a, b \in \mathbb{N}^*$. Then*

$$\|TS\|_{a,b} \leq 4(a+b)\omega(T)\omega(S)$$

Proof

$$\begin{aligned}\|TS\|_{a,b} =& \|a|TS| + b|(TS)^*|\| \\ \leq& a\|TS\| + b\|(TS)^*\| \\ \leq& (a+b)\|T\|\|S\| \\ \leq& 4(a+b)\omega(T)\omega(S)\end{aligned}$$

4.4 inequalities between operator matrices

Theorem 4.7 *Let $B, C \in \mathcal{B}(\mathcal{H})$ and $a, b \in \mathbb{N}^*$. Then*

$$\omega\left(\begin{bmatrix} 0 & 0 & B \\ 0 & C & 0 \\ B & 0 & 0 \end{bmatrix}\right) \leq \max\left(\frac{1}{2\max\{a,b\}}\|B\|_{a,b}, \omega(C)\right)$$

Proof

Let $T = \begin{bmatrix} 0 & 0 & B \\ 0 & C & 0 \\ B & 0 & 0 \end{bmatrix}$

and

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ *be a unit vector in* $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$ *(with* $\|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2 = 1$*).*

Then

$$\begin{aligned}|\langle Tx, x\rangle| &= |\langle Bx_3, x_1\rangle + \langle Cx_2, x_2\rangle + \langle Bx_1, x_3\rangle| \\ &\leq |\langle Bx_3, x_1\rangle| + |\langle Cx_2, x_2\rangle| + |\langle Bx_1, x_3\rangle| \\ &\leq \langle |B|x_3, x_3\rangle^{1/2}\langle |B^*|x_1, x_1\rangle^{1/2} + \langle |B|x_1, x_1\rangle^{1/2}\langle |B^*|x_3, x_3\rangle^{1/2} + \omega(C)\|x_2\|^2 \qquad (\textit{by theorem}2.7) \\ &\leq (\langle |B^*|x_1, x_1\rangle + \langle |B|x_1, x_1\rangle)^{1/2}(\langle |B|x_3, x_3\rangle + \langle |B^*|x_3, x_3\rangle)^{1/2} + \omega(C)\|x_2\|^2 \qquad (\textit{by theorem}2.2)(n=2) \\ &\leq \langle (|B^*| + |B|)x_1, x_1\rangle^{1/2}\langle (|B| + |B^*|)x_3, x_3\rangle^{1/2} + \omega(C)\|x_2\|^2 \\ &\leq \||B^*| + |B|\|\|x_1\|\|x_3\| + \omega(C)\|x_2\|^2 \\ &\leq \||B| + |B^*|\|\frac{(\|x_1\|^2 + \|x_3\|^2)}{2} + \omega(C)\|x_2\|^2 \qquad (\textit{by theorem}2.4) \\ &\leq \frac{1}{\max\{a,b\}}\|a|B| + b|B^*|\|\frac{(\|x_1\|^2 + \|x_3\|^2)}{2} + \omega(C)\|x_2\|^2 \\ &\leq \max\left(\frac{\|B\|_{a,b}}{2\max\{a,b\}}, \omega(C)\right)\end{aligned}$$

Hence

$$\omega(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle| \leq \max\left(\frac{1}{2\max\{a,b\}}\|B\|_{a,b}, \omega(C)\right)$$

Corollary 4.2 *Let $B, C \in B(\mathcal{H})$ and $a, b \in \mathbb{N}^*$.*
Then

$$\omega\left(\begin{bmatrix} 0 & B & 0 \\ B & 0 & 0 \\ 0 & 0 & C \end{bmatrix}\right) \leq \max\left(\frac{1}{2\max\{a,b\}}||B||_{a,b}, \omega(C)\right)$$

$$\omega\left(\begin{bmatrix} C & 0 & 0 \\ 0 & 0 & B \\ 0 & B & 0 \end{bmatrix}\right) \leq \max\left(\frac{1}{2\max\{a,b\}}||B||_{a,b}, \omega(C)\right)$$

To ***proof*** *this, we follow the same steps as the previous proof.*

Theorem 4.8 *Let $B, C \in \mathcal{B}(\mathcal{H})$ and $a, b \in \mathbb{N}^*$. Then*

$$\omega\left(\begin{bmatrix} B & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & B \end{bmatrix}\right) \leq \max\left(\frac{1}{2\max\{a,b\}}||B||_{a,b}, \omega(C)\right)$$

we obtained the same inequality

Proof

Let $T = \begin{bmatrix} B & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & B \end{bmatrix}$
and
$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ *be a unit vector in $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$ (with $||x_1||^2 + ||x_2||^2 + ||x_3||^2 = 1$).*
Then

$$\begin{aligned}
|\langle Tx, x\rangle| &= |\langle Bx_1, x_1\rangle + \langle Cx_2, x_2\rangle + \langle Bx_3, x_3\rangle| \\
&\leq |\langle Bx_1, x_1\rangle| + |\langle Cx_2, x_2\rangle| + |\langle Bx_3, x_3\rangle| \\
&\leq \langle |B|x_1, x_1\rangle^{1/2}\langle |B^*|x_1, x_1\rangle^{1/2} + \langle |B|x_3, x_3\rangle^{1/2}\langle |B^*|x_3, x_3\rangle^{1/2} + \omega(C)||x_2||^2 \qquad (by \ \ theorem 2.7) \\
&\leq \frac{1}{2}(\langle |B|x_1, x_1\rangle + \langle |B^*|x_1, x_1\rangle) + \frac{1}{2}(\langle |B|x_3, x_3\rangle + \langle |B^*|x_3, x_3\rangle) + \omega(C)||x_2||^2 \\
&\leq \frac{1}{2}(\langle (|B^*| + |B|)x_1, x_1\rangle + \langle (|B| + |B^*|)x_3, x_3\rangle) + \omega(C)||x_2||^2 \\
&\leq \frac{1}{2}|||B^*| + |B|||(||x_1||^2 + ||x_3||^2) + \omega(C)||x_2||^2 \\
&\leq \frac{1}{\max\{a,b\}}||a|B| + b|B^*|||\frac{(||x_1||^2 + ||x_3||^2)}{2} + \omega(C)||x_2||^2 \qquad (by \ \ theorem 2.4) \\
&\leq \max\left(\frac{||B||_{a,b}}{2\max\{a,b\}}, \omega(C)\right)
\end{aligned}$$

Corollary 4.3 *Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega\left(\begin{bmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}\right) \leq \max\left(\frac{1}{2\max\{a,b\}}||B||_{a,b}, \omega(C)\right)$$

$$\omega\left(\begin{bmatrix} C & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{bmatrix}\right) \leq \max\left(\frac{1}{2\max\{a,b\}}||B||_{a,b}, \omega(C)\right)$$

we obtained the same inequality .
To prove this, we follow the same steps as the previous proof.

CONCLUSION

At the conclusion of this thesis, we have surveyed a range of advanced concepts and analytical tools within the framework of normed spaces, focusing particularly on Hilbert spaces and the associated numerical ranges.
Our study began with a detailed exploration of norms and inner products, including the pivotal role of orthogonal projections and the characterization of normaloid elements.

In the second chapter, we examined classical inequalities — namely Cauchy-Schwarz, Hölder, and Minkowski — emphasizing their importance in establishing key estimates within mathematical analysis. Subsequently, the third chapter addressed the definitions and interrelationships between the spectral radius, the numerical radius, and the singular values , and provided precise proofs for the major inequalities connecting them.

Finally, in the fourth chapter, we introduced a new norm derived from the numerical radius , identified the sufficient conditions for it to qualify as a norm, and presented a set of new inequalities that extend the analytical framework surrounding the numerical radius.

The findings of this thesis open up new avenues for refining spectral analysis and numerical analysis, particularly in infinite-dimensional settings. They suggest further research into the connections between the numerical ranges and newly constructed norms, with promising applications in differential equations, stability analysis, and signal processing.

We hope that this thesis contributes meaningfully to the academic study of normed spaces and encourages deeper investigation into the rich interactions between geometry, analysis, and algebra.

BIBLIOGRAPHY

[1] *N. Akhiezer and I. M. Glazman, The Theory of Linear Operators in Hilbert Space, Vols. I, II, Ungar, New York, 1961.*

[2] *F. A. Berezin, A. D. Gvishiani, E. A. Gorin, and A. A. Kirillov, Collection of Problems in Functional Analysis, Moscow State Univ., Moscow, 1977 (in Russian).*

[3] *R. Bhatia, Matrix Analysis, Springer-Verlag, New York, 1997.*

[4] *R. Bhatia, Positive Definite Matrices, Princeton University Press, 2007.*

[5] *R. Bhatia and C. Davis, More matrix forms of the arithmetic-geometric mean inequality, SIAM J. Matrix Anal. Appl.,* ***14*** *(1993), 132–136.*

[6] *R. Bhatia and F. Kittaneh, The matrix arithmetic-geometric mean inequality revisited, Linear Algebra Appl.,* ***428*** *(2008), 2177–2191.*

[7] *R. Bhatia, and F . Kittaneh, Norm inequalities for positive operators, Lett. Math. Phys. 43 (1998), 225-231.*

[8] *R. Bhatia and F. Kittaneh, On the singular values of a product of operators, SIAM J. Matrix Anal. Appl.,* ***11*** *(1990), 272–277.*

[9] *S. S. Dragomir, Some inequalities for the numerical radius of linear operators in Hilbert spaces, Linear Algebra Appl.,* ***387*** *(2004), 271–282.*

[10] *M. El-Haddad and F. Kittaneh, Numerical radius inequalities for Hilbert space operators. II, Studia Math.,* ***182****(2) (2007), 133–140.*

[11] *J. C. Hou and H. K, Du. Norm inequalities of positive operator matrices, Integr. Equ.Oper. Theory 22 (1995), 281-294.*

[12] *M. Goldberg and E. Tadmor, On the numerical radius and its applications, Linear Algebra Appl.,* ***42*** *(1982), 263–284.*

[13] *H. Guelfen, Sur le rayon numérique et ses applications, Rapport de stage, 2018/2019.*

[14] *K. E. Gustafson and D. K. M. Rao, Numerical Range, Springer, New York, 1997.*

[15] *J. A. R. Holbrook, Multiplicative properties of the numerical radius in operator theory, J. Reine Angew. Math.,* ***237*** *(1969), 166–174.*

[16] *F. Kittaneh, A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, Studia Math.,* ***158*** *(2003), 11–17.*

[17] *F. Kittaneh, Norm inequalities for certain operator sums, J. Funct. Anal.,* ***143*** *(1997), 337–348.*

[18] *F. Kittaneh, Norm inequalities for sums of positive operators, J. Operator Theory,* ***48*** *(2002), 95–103.*

[19] *F. Kittaneh, Numerical radius inequalities for Hilbert space operators, Studia Math.,* ***168****(1) (2005), 73–80.*

[20] *F. Kittaneh, Spectral radius inequalities for Hilbert space operators, Proc. Amer. Math. Soc.,* ***134*** *(2006), 385–390.*

[21] *F. Kittaneh, M. S. Moslehian, and T. Yamazaki, Cartesian decomposition and numerical radius inequalities, Linear Algebra Appl.,* ***471*** *(2015), 46–53.*

[22] *A. Mansour, Résolution de deux types d'équations opératorielles et interactions, Thèse de doctorat, Université de Lyon, 2016.*

[23] *S. J. Miller, The arithmetic and geometric mean inequality, The Ohio State University, Columbus, 2003.*

[24] *C. Pearcy, An elementary proof of the power inequality for the numerical radius, Michigan Math. J.,* ***13*** *(1966), 289–291.*

[25] *V. Oxman, M. Stupel, and J. M. Jahangiri, Finding extrema without resorting to calculus, Journal of Mathematics Education, 2018.*

[26] *O. Toeplitz, Das algebraische Analogon zu einem Satze von Fejér, Math. Z.,* ***2*** *(1918), 187–197.*

[27] *A. Treibergs, Inequalities of analysis, University of Utah, Fall 2014.*

[28] *H. L. Vasudeva, Elements of Hilbert Spaces and Operator Theory, Springer, 2017.*

[29] *A. W. Wafula, On invariance of the numerical range and some classes of operators in Hilbert space, Thèse de doctorat, Université de Nairobi, 2013.*

[30] *H. Wu and S. Wu, Various proofs of the Cauchy-Schwarz inequality, Octogon Math. Mag.,* ***17****(1) (2009), 221–229.*

[31] *K. Zaiz and A. Mansour, On numerical range and numerical radius of convex function operators, Korean J. Math.,* ***27****(4) (2019), 879–898.*

[32] *X. Zhan, Matrix Inequalities, Lecture Notes in Mathematics, vol. 1790, Springer-Verlag, Berlin, 2002.*

[33] *X. Zhan, Singular values of differences of positive semidefinite matrices, SIAM J. Matrix Anal. Appl.,* ***22*** *(2000), 819–823.*

[34] *F. Zhang, Matrix Theory, Springer-Verlag, New York, 1999.*

[35] *Y. Zhang and D. Chen, Singular value inequalities for real and imaginary parts of matrices, 2016.*