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Ministry Of Higher Education and Scientific Research
ECHAHID HAMMA LAKHDAR UNIVERSITY – EL-OUED
Faculty of Exact Sciences
Department of mathematics

# THESIS

**For obtaining the degree of the 3rd cycle LMD in mathematics.**

**Specialty: Mathematical analysis and applications.**

# On the behavior of some porous thermoelastic systems with type III thermoelasticity and with delay term

**Presented by: NID Zineb**

**Front of the jury:**

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RÉPUBLIQUE ALGÉRIENNE DÉMOCRATIQUE ET POPULAIRE

Ministère de L'Enseignement Supérieur et de la

Recherche Scientifique

UNIVERSITÉ ECHAHID HAMMA LAKHDAR - EL-OUED

Faculté des Sciences Exactes

Department de mathématiques

# THÈSE

**Pour l'obtention du diplôme de 3ème cycle LMD en mathématiques.**

**Spécialité: Analyse mathématiques et applications.**

# Sur le comportement de certains systèmes thermoélastiques poreux à thermoélasticité de type III et à term de retard

**Présenté par: NID Zineb**

**Devant le jury:**

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Année: 2024/2025

# Dedication

This work is dedicated to the individual who fostered within my heart a spirit of perseverance and imparted lessons in resilience during times of adversity - my esteemed father who tirelessly labored to elevate our standing to the highest levels. May Allah grace him with mercy and grant him eternal peace.

To the epitome of warmth revitalized and the constant wellspring of affection; to the individual whose supplications served as the keystone to my achievements - my cherished mother. May Allah bestow upon her longevity, good health, and well-being.

Acknowledgment is extended to the individual fate bestowed upon me, who graciously accommodated my imperfections, consistently provided unwavering support - my life partner, my husband. May Allah safeguard him and grant him prolonged life.

To the blossoms embellishing the garden of my existence, my daughters Jana and Yasmine; and to all the esteemed members of my gracious family who have bolstered and continue to bolster me...

# Acknowledgements

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# Abstract

In this thesis, we investigate the longtime behavior of certain porous thermoelastic systems characterized by type-III thermoelasticity. Specifically, we study three problems, two of which involve delay terms.

The first problem addresses a porous thermoelastic system in which the heat conduction is governed by the type III law of Green and Naghedi, and incorporating a constant delay term. We examined both equal and non-equal wave speed scenarios. In both cases, we established a well-posedness result using a semigroup approach and the Hille-Yosida theorem. Additionally, for the case of equal speeds, we demonstrated that an exponential rate of decay occurs provides an inequality between the weights of the delay and thermal dissipation. Conversely, in the case of unequal speeds, we derived a polynomial rate of decay. In both scenarios, we employed multiplier and energy methods.

In the second problem, we replaced the constant delay from the first problem with a distributed delay term. The results obtained are similar to those of the first problem.

The third problem addressed a porous thermoelastic system characterized by Type-III thermoelasticity and a nonlinear damping term acting in the porous equation. We established a well-posedness result and demonstrated an explicit general decay result in the case of equal wave speeds. This was accomplished by employing properties of convex functions and the multiplier method.

**Keywords:** Porous thermoelasticity, Type-III thermoelasticity, Distributed delay, Nonlinear damping, Well-posedness, Exponential decay, Polynomial decay, General decay.

# Résumé


Dans cette thèse, nous étudions le comportement asymptotique à long terme de certains systèmes thermoélastiques poreux, dont la conduction thermique est décrite par la loi de type III de Green et Nqghedi. Plus précisément, nous avons analysé trois problèmes, dont les deux premiers incluent des termes de retard.

Le premier problème concerne un système thermoélastique poreux de type III, avec un retard constant agissant sur l'équation de la chaleur. Nous avons démontré l'existence et l'unicité d'une solution en utilisant une approche par semi-groupes et le théorème de Hille-Yosida. Concernant le comportement asymptotique, nous avons examiné les cas des vitesses égales et des vitesses inégales. Dans le premier cas, nous avons établi un taux de décroissance exponentielle en fonction d'une relation entre les poids du retard et de la dissipation thermique. Dans le second cas, nous avons obtenu un taux de décroissance polynomial. Dans les deux situations, nous avons utilisé les méthodes des multiplicateurs et de l'énergie.

Pour le second problème, nous avons remplacé le terme retard dans le premier problème par un retard distribué et nous avons démontré des résultats similaires à ceux obtenus pour le premier problème.

Le troisième problème concerne un système thermoélastique poreux de type III avec un amortissement non linéaire. Nous avons démontré l'existence et l'unicité d'une solution, et établi un résultat de décroissance explicite et général dans le cas de vitesses égales. Cela a été réalisé en utilisant les propriétés des fonctions convexes et la méthode des multiplicateurs.



**Mots clés:** Thermoélasticité poreuse, Thermoélasticité de type-III, Retard distribué, Ammortissement nonlinéaire, Existence et unicité, Décroissance exponentielle, décroissance polynomial, Décroissance général.

# ملخص


موضوع دراستنا في هذه الاطروحة، هو السلوك التقاربي عند الزمن اللانهائي لبعض جمل المعادلات التفاضلية الجزئية التي تنمذج لأنظمة حرارية لأوساط مسامية مرنة، حيث معادلة الحرارة هي من النوع الثالث لـ Green-Naghdi.

المسألة الأولى هي جملة ثلاث معادلات حرارية في وسط مرن بوجود حرارة من النوع الثالث وبمعامل تأخير يؤثر في معادلة الحرارة، حيث قمنا بدراسة حالتين: الأولى عند تساوي السرعات وأثبتنا التقارب الأسي، والثانية عند اختلاف السرعات وأثبتنا التقارب الجبري.

أما المسألة الثانية فتخص وسط مسامي مرن حراري من النوع الثالث يحتوي على تأخير موزع في معادلة الحرارة، حيث تم الحصول على نفس النتائج للمسألة السابقة.

المسألة الثالثة، فتتسم بكونها تحتوي على تخميد غير خطي، حيث استثمرنا شكل الدالة المحدب بالقرب من المبدأ وتوصلنا إلى معدل اضمحلال عام في حالة تساوي السرعات.

في المسائل الثلاث طبقنا طريقة الجداءات في بناء تابع ليابونوف لإثبات التقارب الأسي أو الجبري أو العام، ونظرية أنصاف الزمر من أجل أثبات وجود الحلول.



**كلمات مفتاحية:** المرونة الحرارية المسامية - المرونة الحرارية من النوع الثالث – تأخير موزع – تخميد غير خطي - وجود ووحدانية - استقرار أسي - استقرار جبري – اضمحلال عام.

# Published papers

- On the decay of a porous thermoelasticity type III with constant delay, Nid Zineb, Fareh Abdelfeteh, Apalara Tijani Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat, 117(2), 67 (2023) https://doi.org/10.1007/s13398-023-01396-9.

## Submitted paper

- The decay of a porous thermoelastic system of type III with distributed delay.

- General decay of the solution of a porous thermoelastic system of type III with non-linear damping term.

# Notation

| | |
|---|---|
| $\Omega$ | a bounded domain in $\mathbb{R}^n$, |
| $C_0^\infty(\Omega)$ | the test functions space, |
| $L^p(\Omega),\ L^\infty(\Omega)$ | the Lebesgue space, |
| $L^{p'}(\Omega) = (L^p(\Omega))'$ | the dual space of $L^p(\Omega)$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1.$ |
| $H^1(\Omega), H^2(\Omega)$ | Sobolev space, |
| $H_0^1(\Omega)$ | the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$ |
| $L^2(0,T;H^m(0,L))$ | $= \left\{ f : (0,T) \times (0,L) \to \mathbb{R};\ \int_0^L \|f(t)\|_{H^m}^m dt < \infty \right\}$, |
| $L^\infty(0,T;H^m(0,L))$ | $= \left\{ f : (0,T) \times (0,L) \to \mathbb{R};\ \sup\limits_{0 \le t \le T} \|f(t)\|_{H^m} dt < \infty \right\}$, |
| $H^{-1}(\Omega)$ | $= \mathscr{L}\left(H_0^1(\Omega), \mathbb{R}\right)$ the dual space of $H_0^1(\Omega)$, |
| $\langle \cdot, \cdot \rangle$ | the inner Product in a Hilbert space, |
| $D\left(\mathcal{B}\right)$ | the domain of the operator $\mathcal{B}$, |
| $\|\cdot\|_Y$ | the norm on a normed space $Y$, |
| $\mathscr{L}(X)$ | the space of all bounded linear operators over $X$, |
| $Re\langle ., . \rangle$ | the real part of the inner product, |
| $a.e.$ | almost everywhere (except on a negligible set). |

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# Introduction

In recent years, the theory of continuous models of porous elastic bodies has been intensively studied. The material's response to external forces is significantly influenced by its internal structure. However, classical elasticity theory overlooks the impact of this inner structure by considering only the translational degrees of freedom of material points within the body.

The theory of elastic materials with voids is a simple extension of classical elasticity theory employed to analyze porous solids. In this framework, the matrix material is elastic, while the interstices are devoid of material. This theory serves as a valuable tool for describing the behavior of porous materials, such as rocks, gases, soils, and manufactured porous bodies. The theory of elastic materials with voids was developed by Nunziato and Cowin [13] and has recently garnered significant attention.

To the best of our knowledge, the study of the long-time behavior of porous systems was investigated first, by Quintanilla. In his pioneering paper [49], he studied the system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x, & \text{in } (0,L)\times(0,\infty), \\ \rho_0\kappa\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi - \tau\varphi_t, & \text{in } (0,L)\times(0,\infty). \end{cases} \tag{1}$$

He applied the Routh-Hurwitz theorem to demonstrate the slow decay of the solution. Subsequently, various dissipation mechanisms were employed to achieve exponential stabilization of the system (1). Magaña and Quintanilla [31] substituted the porous damping term $\tau\varphi_t$ with the viscoelastic dissipation term $\gamma u_{txx}$ in the first equation, and they showed that the solution continued to exhibit slow decay.

The coupling with thermal effects was considered by Magaña and Quintanilla [31]. They studied the system

$$
\begin{cases}
\rho u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x + \gamma u_{txx}, & \text{in } (0,L)\times(0,\infty)\,, \\
\rho_0\kappa\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi + m\theta, & \text{in } (0,L)\times(0,\infty)\,, \\
c\theta_t = \kappa\theta_{xx} - \beta u_{tx} - m\varphi_t, & \text{in } (0,L)\times(0,\infty)\,,
\end{cases}
\tag{2}
$$

where thermal and viscoelastic dissipation mechanisms are acting simultaneously, the system continues to decay slowly. Casas and Quintanilla [11] studied the following system

$$
\begin{cases}
\rho u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x, & \text{in } (0,L)\times(0,\infty)\,, \\
\rho_0\kappa\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi + m\theta - \tau\varphi_t, & \text{in } (0,L)\times(0,\infty)\,, \\
c\theta_t = \kappa\theta_{xx} - \beta u_{tx} - m\varphi_t, & \text{in } (0,L)\times(0,\infty)
\end{cases}
\tag{3}
$$

They established a result demonstrating exponential stability in the presence of thermal and porous dissipation.

Later, Apalara [4] and Santos *et al.* [52] established separately, an exponential rate of decay for the solution of (1), if and only if the equality of the wave speeds

$$
\frac{\mu}{\rho} = \frac{\alpha}{\rho_0\kappa}
\tag{4}
$$

holds. The same result was obtained by Santos *et al.* [52] for an elastic damping $\gamma u_t$ in the elastic equation of (1). Apalara [5] extended the results of [4, 52] to the following porous system with memory term

$$
\begin{cases}
\rho u_{tt} = \mu u_{xx} + b\varphi_x, & \text{in } (0,L)\times(0,\infty)\,, \\
\rho_0\kappa\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi - \displaystyle\int_0^t g(t-s)\varphi_{xx}(s)ds, & \text{in } (0,L)\times(0,\infty)\,.
\end{cases}
\tag{5}
$$

Recently, Ahmima and co-authors [1, 2] obtained a similar result for porous systems with microtemperature, and temperature with delay term.

## Thermal effects

The effects of heat on the deformation of an elastic solid, as well as the reciprocal effects of deformation on the solid thermal state, are the primary focus of classical thermoelasticity. Fourier's law of heat conduction, which states that the heat flux is proportional to the temperature gradient, governs the heat flux in the classical model of heat propagation, and is given by

$$q = -\delta\theta_x,$$

where $q$ is the heat flux vector, $\theta$ is the temperature and $\delta > 0$ is a constant.

The substitution of the heat flux from Fourier's law into the energy conservation law

$$\kappa\theta_t = -q_x,$$

leads to the linear parabolic heat equation

$$\kappa\theta_t = \delta\theta_{xx},$$

where $\kappa$ is the specific heat capacity and $\delta$ is the thermal conductivity of the material.

It is well known that Fourier's law leads to the paradox of infinite heat speed, where the effect of a heat pulse is felt instantaneously throughout its surrounding medium. This nonphysical consequence of the heat equation is caused by the over simplification in Fourier's assumption.

Over the last fifty years, an intense interest has been accorded to propose an alternative theory that addresses this paradox. In the 1990s, Green and Naghedi [18, 20, 21] introduced three new theories based on an entropy equality rather than the entropy inequality. They labeled them thermoelasticity of type I, type II and type III respectively. In each of these theories, the heat flux is given by a different constitutive assumption. The linearized Type I coincides with the classical theory of heat conduction. The Type II and Type III constitutive equations for heat flux in the one-dimensional case are, respectively, given by:

$$q = -f_1\zeta_x x \;\; and \;\; q = -f_1\zeta_x - f_2\theta_x,$$

where

$$\zeta = \int_0^t \theta(x,\tau)d\tau + \zeta_0(x)$$

is the thermal displacement, $\zeta_0$ is the initial value at $t = 0$ and $f_1, f_2$ are positive constants. Though the type II theory does not permit energy dissipation, it is a limiting case of the type III thermoelasticity. The type III thermoelasticity states that the heat transport equation takes on a hyperbolic form, resolving the paradox of infinite speed.

To the best of our knowledge, the first contribution in the framework of thermoelasticity of type III was made by Zhang and Zuazua [56]. They studied the system

$$\begin{cases} y_{tt} - \mu\Delta y - (\lambda+\mu)\nabla\nabla\cdot y + \nabla\theta = 0 & \text{in } (0,\infty)\times\Omega, \\ \theta_{tt} - \Delta\theta_t - \Delta\theta + \nabla\cdot y_{tt} = 0 & \text{in } (0,\infty)\times\Omega, \end{cases}$$

where $\Omega$ is a bounded domain or entire space $\mathbb{R}^n$. For the first case, they demonstrated that, for most domains, the energy of the system does not decay uniformly. However, under suitable conditions on the domain, the energy of the system decays exponentially. For the problem in the entire space $\mathbb{R}^n$, they employed Fourier analysis and Lyapunov's second method to demonstrate that the energy of longitudinal and thermal waves in the system decays similarly to that of the classical heat equation.

In the same context, Quintanilla and Racke [50] examined the system

$$\begin{cases} u_{tt} - \alpha u_{xx} + \beta\theta_x = 0 & \text{in } (0,L)\times(0,\infty), \\ \theta_{tt} - \delta\theta_{xx} + \gamma u_{ttx} - \kappa\theta_{txx} = 0 & \text{in } (0,L)\times(0,\infty), \end{cases} \tag{6}$$

and proved the exponential stability of the associated energy.

## Porous elasticity with delay term

It is worth noting that in many phenomena, the current state depends not only on the present moment but also on certain past characteristics. This is because the response of some materials to an applied force can be delayed. Time delays are observed in various problems across physics, chemistry, biology, and other fields. The first contribution in the study of the effect of time delay on the stabilization of solutions was made by Datko [16]. He considered the equation

$$u_{tt} - u_{xx} + 2u_t(x,t-\tau) = 0,\ x\in(0,1)\times(0,+\infty),$$

and He demonstrated that the presence of a time delay, $\tau>0$, in the damping associated with the velocity term can destabilize the system. Also, Datko *et al.* [17] studied the problem

$$\begin{cases} u_{tt} - u_{xx} + 2au_t + a^2u = 0 & (0,1)\times(0,+\infty), \\ u(0,t) = 0,\ u_x(1,t) = -ku_t(1,t-\tau) & (0,+\infty), \end{cases} \tag{7}$$

where the delay is acting on the boundary. They proved that for $a > 0$ and $k \geq \dfrac{1-e^{-2a}}{1+e^{-2a}}$, there exists a sequence of delay terms for which the solutions of (7) can not be exponentially stable. Nicaise and Pignotti [42] considered a damped wave equation with a delay in the boundary conditions and proved an exponential decay result under the assumption that the weight of delay is less than the weight of the damping. They also investigated the case of frictional damping in the internal feedback delay and discovered a sequence of small delays for which the corresponding solution is unstable. Additionally, they extended the findings of [42] to a scenario involving distributed delay at the boundary, as presented in [43].

Time delay is also considered in thermoelastic problems. Racke [51] studied the system

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t-\tau_1) + b\theta_x(x,t) = 0 & \text{in } (0,L)\times(0,\infty), \\ \theta_t - d\theta_{xx}(x,t-\tau_2) + bu_{tx}(x,t) = 0 & \text{in } (0,L)\times(0,\infty), \end{cases} \tag{8}$$

with two delay terms in the displacement and temperature functions. They proved the ill-posedness and the unstability of the system (8), for $\tau_1 > 0$ or $\tau_2 > 0$. However, for $\tau_1 = \tau_2 = 0$, the system is exponentially stable. Later, Mustafa and Kafini [41] considered the following thermoelastic system with internal distributed delay in the temperature function

$$\begin{cases} au_{tt} - du_{xx} + \beta\theta_x = 0 & \text{in } (0,L)\times(0,\infty), \\ b\theta_t - k_1\theta_{xx} - \displaystyle\int_{\tau_1}^{\tau_2} k_2(s)\theta_{xx}(t-s)ds + \beta u_{tx} = 0 & \text{in } (0,L)\times(0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x) & x \in (0,L), \\ \theta_x(x,-t) = f_0(x,t), \quad x \in (0,L), & t \in (0,\tau_2), \end{cases}$$

and established the exponential stability of the solution provided that $\displaystyle\int_{\tau_1}^{\tau_2} |k_2(s)|ds < k_2$.

The time delay in the temperature equation is also considered by Kafini *et al.* [25] for Timoshenko-type systems. They investigated the system

$$\begin{cases} \rho_1\phi_{tt} - K(\phi_x + \psi)_x = 0 & \text{in } (0,L)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\phi_x + \psi) + \beta\theta_{tx} = 0 & \text{in } (0,L)\times(0,\infty), \\ \rho_3\theta_{tt} - \delta\theta_{xx} - k\theta_{txx} - \displaystyle\int_{\tau_1}^{\tau_2} g(s)\theta_{txx}(x,t-s)ds + \gamma\psi_{tx} = 0 & \text{in } (0,L)\times(0,\infty), \end{cases}$$

and proved the stability of the solution in the case of equal and non-equal wave speeds.

Problems with delay term for porous thermoelastic problems was considered by Liu and Chen [55]. They studied the following porous thermoelastic system with second sound and time varying delay

$$\begin{cases} \rho_0 u_{tt} - a u_{xx} - b\phi_x + \gamma_1 u_t + \gamma_2 u_t\left(\cdot, t - \tau\left(t\right)\right) = 0 & \text{in } (0,1) \times (0,\infty), \\ J\phi_{tt} - \alpha\phi_{xx} + b u_x + \xi\phi - \beta\theta_x = 0 & \text{in } (0,1) \times (0,\infty), \\ c\theta_t + q_x - \beta\phi_{tx} + \delta\theta = 0 & \text{in } (0,1) \times (0,\infty), \\ \tau_0 q_t + q + k\theta_x = 0 & \text{in } (0,1) \times (0,\infty), \end{cases}$$

and established an exponential decay result under specific assumptions on the weights of damping and delay. Borges and Santos [9] investigated the porous elastic system with time varying delay term

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0 & \text{in } (0,1) \times (0,\infty), \\ J\phi_{tt} - \delta\phi_{xx} + b u_x + \xi\phi + \mu_1\phi_t + \mu_2\phi_t\left(t - \tau\left(t\right)\right) = 0 & \text{in } (0,1) \times (0,\infty), \end{cases}$$

with Dirichlet boundary conditions and established an exponential rate of decay.

Recently, Ouchenane *et al.* [47] studied the following system

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b\phi_x, \\ \rho_2\phi_{tt} = \delta\phi_{xx} - b u_x - \xi\phi - \beta\theta_x - \mu_1\phi_t - \displaystyle\int_{\tau_1}^{\tau_2} \left|\mu_2\left(s\right)\right| \phi_t\left(x, t - s\right) ds, \\ \rho_3\theta_{tt} = l\theta_{xx} - \gamma\phi_{ttx} + k\theta_{txx}, \end{cases}$$

for $(x, s, t) \in (0,1) \times (\tau_1, \tau_2) \times (0, \infty)$ and with the assumption

$$\int_{\tau_1}^{\tau_2} \left|\mu_2\left(s\right)\right| ds < \mu_1. \tag{9}$$

They demonstrated exponential stability when the wave speeds are equal, specifically when $\dfrac{\mu}{\rho_1} = \dfrac{\delta}{\rho_2}$, and polynomial decay in the case of unequal wave speeds.

The remainder of this thesis is structured as follows: Chapter I presents some preliminary remarks and the basic tools that we use for the rest of our thesis. In chapter II, we consider a porous thermoelastic system of type III with a constant delay. Chapter III is devoted to the study of a porous thermoelastic system of type III with a distributed delay. In Chapter IV, we study a porous thermoelastic system of type III with a nonlinear damping term.

# Chapter 1

# Preliminaries

In this chapter, we present several functional spaces, along with essential definitions, theorems, and fundamental inequalities that are utilized throughout this thesis.

## 1.1 Functional spaces

Throughout this chapter, we denote by $\Omega$ a bounded domain in $\mathbb{R}^n$, where $n \in \mathbb{N}$.

### 1.1.1 Lebesgue and Sobolev spaces

**Definition 1.1.1.** *Let $p \in [1, +\infty[$. The Lebesgue space $L^p(\Omega)$ is defined by*

$$L^p(\Omega) := \left\{ \widetilde{f} : \Omega \to \mathbb{R},\ f \text{ is measurable and } \int_\Omega |f(x)|^p dx < \infty \right\},$$

*and for $p = \infty$, by*

$$L^\infty(\Omega) := \left\{ \widetilde{f} : \Omega \to \mathbb{R},\ f \text{ is measurable } ess \sup(f) < \infty \right\}.$$

*Note that $\widetilde{f}$ is the class of equivalent of $f$ with respect to the relation*

$$f \sim g \Leftrightarrow f = g \text{ a.e in } \Omega.$$

**Definition 1.1.2.** *Let $m \in \mathbb{N}$. The Sobolev space $H^m(\Omega)$ is defined by*

$$H^m(\Omega) := \left\{ f \in L^2(\Omega);\ \text{for } |\alpha| \leq m\ D^\alpha f \in L^2(\Omega) \right\},$$

*where* $|\alpha| = \alpha_1 + ... + \alpha_n$ *and* $D^\alpha f$ *is the* $\alpha$*-weak derivative of* $f$ *defined as the function* $g_\alpha$ *satisfied*

$$\int_\Omega f(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_\Omega g_\alpha(x) \varphi(x) dx, \qquad \forall \varphi \in C_c^\infty(\Omega).$$

$H$ *is a Hilbert space with respect to the inner product*

$$\langle f, g \rangle_{H^m} = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha f(x) D^\alpha g(x) dx, \quad \forall f, g \in H^m(\Omega).$$

*The associated norm is*

$$\|f\|_{H^m} = \left( \sum_{|\alpha| \leq m} \|D^\alpha f\|_2^2 \right)^{\frac{1}{2}}.$$

In the case when $m = 1$, $H^m(\Omega)$, coincides with $L^2(\Omega)$.

**Definition 1.1.3.** *The space* $H_0^1(\Omega)$ *is the closure of* $C_0^\infty(\Omega)$ *in* $H^m(\Omega)$.

$$H_0^1(\Omega) = \overline{C_c^\infty(\Omega)^{H^1(\Omega)}}.$$

**Definition 1.1.4.** *The denote by* $H^{-1}(\Omega)$ *dual space of* $H_0^1(\Omega)$,

$$H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)' = \mathscr{L}\left(H_0^1(\Omega), \mathbb{R}\right).$$

**Definition 1.1.5.** *Let* $X$ *be a Banach space and* $T > 0$, *we define the space*

$$L^2(0, T; X) := \left\{ \widetilde{f} : [0, T] \to X; \quad \int_0^T \|f(t)\|_X^2 dt < \infty \right\}$$

*and*

$$L^\infty(0, T; X) := \left\{ \widetilde{f} : [0, T] \to X; \ ess \sup \|f(t)\|_X < \infty \right\}.$$

**Proposition 1.** *[10] Let* $F \in H^{-1}(\Omega)$, *then there exists,* $f_0, f_i \in L^2(\Omega)$. $1 \leq i \leq n$ *such that*

$$\langle F, u \rangle = \int_\Omega f_0 u dx + \sum_{1 \leq i \leq n} \int_\Omega f_i \frac{\partial u}{\partial x_i}.$$

*If* $\Omega$ *is bounded, we can take* $f_0 = 0$.

**Proposition 2.** *Let* $\Omega$ *is bounded. The subspace* $H_*^1(\Omega)$ *of* $H^1(\Omega)$, *given by*

$$H_*^1(\Omega) = \left\{ u \in H^1(\Omega); \int_\Omega u(x) dx = 0 \right\},$$

*is a Hilbert space.*

*Proof.* Let $F : H^1(\Omega) \to \mathbb{R}$ be defined by $F(v) = \displaystyle\int_\Omega v(x)dx$, then

$$\begin{aligned} |F(u) - F(v)| &= \left| \int_\Omega (u(x) - v(x))\, dx \right| \\ &\leq \int_\Omega |u(x) - v(x)| dx \end{aligned}$$

By using the Cauchy-Schwarz inequality, we get

$$|F(u) - F(v)| \leq mes(\Omega) \left[ \int_\Omega |u(x) - v(x)|^2 \right]^{1/2} \leq mes(\Omega) \|u - v\|_{H^1}$$

which implies that $F$ is continous.

Since $\{0\}$ is closed, $H^1_*(\Omega) = F^{-1}(\{0\})$ is closed in $H^1(\Omega)$, consequentely $H^1_*(\Omega)$ is a Hilbert space. □

**Remark 3.** *The same result holds for*

$$L^2_*(\Omega) = \left\{ u \in L^2(\Omega);\ \int_\Omega u(x)dx = 0 \right\}.$$

**Lemma 4.** *Let $[a, b]$ be a bounded interval in $\mathbb{R}$ and $c \in [a, b]$. The subspace $V_c$ of $H^1([a, b])$ defined by*

$$V_c = \left\{ u \in H^1([a, b])\,;\ u(c) = 0 \right\}$$

*is a Hilbert spaces.*

*Proof.* Let $(\psi_n) \subset V_c$ be a convergent sequence to $\psi$ in $H^1(\Omega)$, then for any $n \in \mathbb{N}$, we have

$$|\psi(c)| = |\psi_n(c) - \psi(c)| \leq \|\psi_n - \psi\|_\infty,$$

using the continuous embedding of $H^1([a, b]) \hookrightarrow L^\infty([a, b])$, we deduce that

$$|\psi(c)| \leq M \|\psi_n - \psi\|_{H^1}.$$

As $n \to \infty$, $|\psi(c)| \leq 0$. Consequently, $\psi(c) = 0$ which completes the proof. □

**Theorem 5 (Lax-Milgram).** *[10] Let $a(.,.) : H \times H \to \mathbb{R}$ be a bilinear, continuous and coercive form. Then, given any $L \in H'$, there exists a unique element $u \in H$ such that*

$$a(u, v) = \langle L, v \rangle \quad \forall v \in H.$$

*Moreover, if $a$ is symmetric, then $u$ is characterized by the property*

$$\frac{1}{2} a(u, u) - \langle L, u \rangle = \min_{v \in H} \left\{ \frac{1}{2} a(v, v) - \langle L, v \rangle \right\}.$$

## 1.2 Some useful inequalities

**Theorem 1.2.1 (Young's inequality).** *[10] Let $p$ and $p'$ be two real conjugate $\left(\frac{1}{p}+\frac{1}{p'}=1\right)$, then*

$$\forall a, b \in \mathbb{R}, \quad |ab| \leq \frac{|a|^p}{p} + \frac{|b|^{p'}}{p'}.$$

*In particular if $f, g \in L^2(\Omega)$ we have, for any $\varepsilon > 0$,*

$$\int_\Omega |fg| dx \leq \varepsilon \int_\Omega |f|^2 dx + \frac{1}{4\varepsilon} \int_\Omega |g|^2 dx \quad \forall \varepsilon > 0.$$

**Theorem 1.2.2 (Hölder's inequality).** *[10]*
*Let $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \leq p \leq \infty$, then $fg \in L^1(\Omega)$ and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

**Theorem 1.2.3 (Cauchy-Schwarz's inequality).** *[10]*
*Let $H$ be a Hilbert space endowed with the inner product $\langle ., . \rangle$, we have*

$$|\langle f, g \rangle| \leq \langle f, f \rangle^{\frac{1}{2}} \langle g, g \rangle^{\frac{1}{2}} \quad \forall f, g \in H.$$

**Theorem 1.2.4 (Poincaré's inequality).** *[10]*
*Let $\Omega$ be bounded (at least in one direction). Then for any $u \in H_0^1(\Omega)$, there exists a constant $C_P > 0$, depending only on $mes(\Omega)$ such that*

$$\|u\|_{L^2} \leq C_P \|\nabla u\|_{L^2}.$$

**Remark 6.** *Poincaré's inequality remains valid for functions in $H_*^1(\Omega)$ and, in the one-dimensional case, for $V_c$ defined in Lemma 4.*

**Lemma 7 (Jensen's inequality).** *[3] Let $G : [a, b] \to \mathbb{R}$ be a convex function. Assume that the functions $f : (0, L) \to [a, b]$ and $r : (0, L) \to \mathbb{R}$ are integrable such that $r(x) \geq 0$, for any $x \in (0, L)$ and $\int_0^L r(x)dx = k > 0$. Then,*

$$G\left(\frac{1}{k}\int_0^L f(x)r(x)dx\right) \leq \frac{1}{k}\int_0^L G(f(x))r(x)dx$$

## 1.3 Semigroup of operators

**Definition 1.3.1.** *[48] Let $Y$ be a Banach space. A one parameter family of bounded linear operators $\{T(t), t \geq 0\} \subset \mathscr{L}(Y)$ is a semigroup of bounded linear operators, if it satisfies:*

*i) $T(0) = I$, ($I$ is the identity operator on $Y$).*

*ii) $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$ (the semigroup property).*

*In addition, the semigroup $T(t)$ is said to be strongly continuous, or $C_0-$semigroup, if*

$$\lim_{t\to 0^+} T(t)y = y \quad \text{for every} \quad y \in Y.$$

*The infinitesimal generator of a semigroup $T(t)$, is the operator $\mathcal{A}$ defined on*

$$D(\mathcal{A}) = \left\{ y \in Y : \lim_{t\to 0^+} \frac{T(t)y - y}{t} \qquad \text{exists} \right\},$$

*by*

$$\mathcal{A}y = \lim_{t\to 0^+} \frac{T(t)y - y}{t} \quad \text{for} \quad y \in D(\mathcal{A}).$$

**Definition 1.3.2.** *[48] The semigroup $\{T(t), t \geq 0\}$, is said to be uniformly continuous if it satisfies*

$$\lim_{t\to 0^+} \|T(t) - I\| = 0.$$

**Theorem 8.** *[48] Let $\{T(t), t \geq 0\}$ be a $C_0-$semigroup. There exist two constants $w \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{wt} \qquad \text{for} \qquad 0 \leq t < \infty.$$

*If $w = 0$, $T(t)$ is called uniformly bounded and if moreover $M = 1$, then the semigroup is called a $C_0-$ semigroup of contractions.*

**Definition 1.3.3. (Maximal Monotone Operators):** *[10]*
*Let $A : D(A) \subset H \to H$ be an unbounded linear operator. We say that $A$ is monotone ($-A$ is dissipative) if it satisfies*

$$\langle Av, v \rangle \geq 0 \quad \forall v \in D(A).$$

*The operator $A$ is said to be maximal if $R(I + A) = H$, i.e.*

$$\forall f \in H,\ \exists u \in D(A) \text{ such that } u + Au = f.$$

**Proposition 1.3.1.** *[10]*
*Let $A$ be a maximal monotone operator on a Hilbert space $H$. Then*

*a) $D(A)$ is dense in $H$.*

*b) $A$ is a closed operator.*

*c) For every $\lambda > 0$, $(I + \lambda A)$ is bijective from $D(A)$ into $H$, $(I + \lambda A)^{-1}$ is a bounded operator and $\|(I + \lambda A)^{-1}\|_{\mathscr{L}(H)} \leq 1$.*

**Theorem 9 (Hille-Yosida).** *[10, 54] Let $\mathcal{A}$ be a maximal monotone operator on Hilbert space $H$. Then, for any $\Psi_0 \in H$, the problem*

$$\begin{cases} \dfrac{d\Psi}{dt} + \mathcal{A}\Psi = 0 \text{ on } \; [0, +\infty[\,, \\ \Psi(0) = \Psi_0 \end{cases} \tag{1.1}$$

*has a unique solution satisfies*

$$\Psi \in C([0, +\infty[\,; H).$$

*Moreover, if $\Psi_0 \in D(\mathcal{A})$, then*

$$\Psi \in C^1([0, +\infty[\,; H) \cap C([0, +\infty[\,; D(\mathcal{A})).$$

**Theorem 10 (Lumer-Phillips).** *[48, 54] Let $H$ be a Hilbert space and $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be a densely defined operator on $H$. Then $\mathcal{A}$ generates a $C_0-$semigroup of contractions on $H$ if and only if*

**i)** *$\mathcal{A}$ is dissipative,($-\mathcal{A}$ monotone).*

**ii)** *there exists $\lambda_0 > 0$ such that $\lambda_0 I - \mathcal{A}$ is surjective.*

**Definition 1.3.4. (Globally Lipschitz continuous)***[24]*
*Let $\Omega \subset \mathbb{R}$ be open, and let $g : \Omega \to \mathbb{R}$. We say that $g$ is globally Lipschitz continuous if there exists $L > 0$ such that*

$$\forall (u, v) \in \Omega \times \Omega : \; |g(u) - g(v)| \leq L|u - v|.$$

**Definition 11.** *( [8], p.61-64) Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable and convex function. We define the Young conjugate of $f$ as the function $g:\mathbb{R}\to\mathbb{R}$, given by $g(s)=s\widetilde{x}-f(\widetilde{x})$, where $\widetilde{x}>0$ is the real for which $g(s)$ is maximal.*

**Remark 12.** *From the definition of $g$ we deduce that $f'(\widetilde{x})=s$, for all $s\geq 0$,*

$$g(s)=s(f')^{-1}(s)-f\left((f')^{-1}(s)\right)$$

*and*

$$sx\leq g(s)+f(x),\ \textit{for all}\ 0\leq x\leq x_0, \tag{1.2}$$

*where $x_0$ is the real that satisfies $sx_0=f(x_0)$.*

**Theorem 13.** *[27],[48] Let $H$ be a Hilbert space, $\mathcal{A}:D(\mathcal{A})\subset H\to H$ be a maximal monotone operator and $F$ is globally Lipschitz. Then for any $u_0\in D(\mathcal{A})$, the problem*

$$\begin{cases}\dfrac{du}{dt}+\mathcal{A}u=F(u)\ \ on\ \ [0,+\infty[\,,\\ u(0)=u_0\end{cases} \tag{1.3}$$

*has a unique solution satisfying*

$$u\in C\left([0,+\infty[\,;H\right).$$

# Chapter 2

# Decay rate for a porous system with type III thermoelasticity and constant delay

The results of this chapter is the object of our paper [45]

In this chapter, we investigate a porous elastic system with thermoelasticity of type III and in the presence of a constant delay term acting on the thermal equation. We prove the existence of a unique solution by the use of a semigroup theory, and we employ the multiplier method to prove the exponential stability of the solution in the case of equal wave speeds. In the opposite case, we establish a polynomial decay rate.

## 2.1 Setting of the problem

In one dimension, the evolution equations for the theory of elastic solids with voids in the presence of type-III thermal effects are given by

$$\rho u_{tt} = T_x, \tag{2.1}$$

$$J\phi_{tt} = h_x + g, \tag{2.2}$$

$$\rho\eta_t = q_x, \tag{2.3}$$

where $T$ is the stress, $h$ is the equilibrated stress, $g$ is the equilibrated body force, $\eta$ is the entropy and $q$ is the heat flux. The coefficients $\rho, J$ are, respectively, the mass density and the product of the mass density by the equilibrated inertia.

The constitutive equations are

$$T = \mu u_x + b\phi, \quad h = \delta\phi_x, \quad g = -bu_x - \xi\phi - \beta\theta_x, \tag{2.4}$$

$$q = \kappa\zeta_x + k_1\theta_x + k_2\theta_x\,(t-\tau) \tag{2.5}$$

and

$$\rho\eta = \alpha\theta + \beta\phi_x, \tag{2.6}$$

where $\zeta$ is the thermal displacement given by

$$\zeta\,(x,t) = \zeta_0\,(x) + \int_0^t \theta\,(x,s)\,ds,$$

$\zeta_0$ is the initial value of $\zeta$ at time $t = 0$, $\tau > 0$ is the time delay, $\mu$ is the bulk modulus, $\delta$ and $\xi$ are porosity diffusion coefficients, $\alpha$ is the heat capacity, $\kappa$ is a thermal diffusion coefficient, $b$ is a porosity coefficient, $\beta$ is a thermal expansion coefficient and $k_1$, $k_2$ are constants that represent the weights of the heat damping and the heat delay, respectively. We assume that the above constitutive coefficients satisfy

$$\mu > 0,\ \delta > 0,\ \xi > 0,\ \kappa > 0, k_1 > 0,\ \alpha > 0\ \text{ and }\ \mu\xi\ > b^2. \tag{2.7}$$

Since the coupling and the time delay are considered, the coefficients $b, \beta$ and $k_2$ must be different from 0 but their signs does not matter in the analysis.
Differentiating the evolution equation (2.3) with respect to $t$, then substituting the constitutive equations (2.4)–(2.5) into the evolution equations, recalling that $\zeta_t = \theta$, we obtain

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, & \text{in } (0,\pi)\times(0,\infty)\,, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \beta\theta_x = 0, & \text{in } (0,\pi)\times(0,\infty)\,, \\ \alpha\theta_{tt} - \kappa\theta_{xx} + \beta\phi_{ttx} - k_1\theta_{txx} - k_2\theta_{txx}(x,t-\tau) = 0, & \text{in } (0,\pi)\times(0,\infty)\,. \end{cases} \tag{2.8}$$

We consider the system (2.8) with the following initial and boundary conditions:

$$u\,(0,t) = u\,(\pi,t) = \phi_x\,(0,t) = \phi_x\,(\pi,t) = \theta\,(0,t) = \theta\,(\pi,t) = 0, \quad \text{in } (0,\infty)\,, \tag{2.9}$$

$$\begin{array}{l} u(x,0)=u_0(x),\ \phi(x,0)=\phi_0(x),\ \theta(x,0)=\theta_0(x), \\ u_t(x,0)=u_1(x),\ \phi_t(x,0)=\phi_1(x),\ \theta_t(x,0)=\theta_1(x), \end{array} \quad \text{in } (0,\pi). \tag{2.10}$$

Notice that since Neumann boundary conditions for $\phi$ are considered, we are not able to utilize Poincaré's inequality, we make the following transformation. From boundary conditions (2.9) and the second equation of (2.8), it follows that

$$\frac{d^2}{dt^2}\int_0^\pi \phi(x,t)\,dx = -\frac{\xi}{J}\int_0^\pi \phi(x,t)\,dx.$$

When we solve this differential equation, we get

$$\int_0^\pi \phi(x,t)\,dx = \left(\int_0^\pi \phi_0(x)\,dx\right)\cos\left(\sqrt{\frac{\xi}{J}}t\right) + \sqrt{\frac{J}{\xi}}\left(\int_0^\pi \phi_1(x)\,dx\right)\sin\left(\sqrt{\frac{\xi}{J}}t\right),$$

consequently, if we set

$$\overline{\phi}(x,t) = \phi(x,t) - \frac{1}{\pi}\left(\int_0^\pi \phi_0(x)\,dx\right)\cos\left(\sqrt{\frac{\xi}{J}}t\right) - \frac{1}{\pi}\sqrt{\frac{J}{\xi}}\left(\int_0^\pi \phi_1(x)\,dx\right)\sin\left(\sqrt{\frac{\xi}{J}}t\right),$$

then

$$\int_0^\pi \overline{\phi}(x,t)\,dx = 0,$$

which allows the application of Poincaré's inequality on $\overline{\phi}$. Moreover, $\left(u,\overline{\phi},\theta\right)$ satisfies (2.8)–(2.9) with the same initial conditions (2.10) for $u$ and $\theta$, and

$$\overline{\phi}(x,0) = \phi_0(x) - \frac{1}{\pi}\left(\int_0^\pi \phi_0(x)\,dx\right),\ \overline{\phi}_t(x,0) = \phi_1(x) - \frac{1}{\pi}\left(\int_0^\pi \phi_1(x)\,dx\right).$$

In the sequel, we work with $\overline{\phi}$ but we write $\phi$ for simplicity.

Next, to deal with the delay term, we introduce; as in [42], the variable

$$z(x,\omega,t) = \theta_t(x,t-\omega\tau),\ \text{ for } 0\le\omega\le 1,$$

which satisfies

$$\tau z_t(x,\omega,t) + z_\omega(x,\omega,t) = 0 \ \text{ for all } (x,\omega,t)\in(0,\pi)\times(0,1)\times(0,+\infty). \tag{2.11}$$

By differentiating the first two equations of (2.8) with respect to $t$ and setting $\varphi = u_t$ and

$\psi = \phi_t$, (2.8)-(2.11) become

$$\begin{cases} \rho\varphi_{tt} - \mu\varphi_{xx} - b\psi_x = 0 & \text{in } (0,\pi)\times(0,\infty), \\ J\psi_{tt} - \delta\psi_{xx} + b\varphi_x + \xi\psi + \beta\theta_{tx} = 0 & \text{in } (0,\pi)\times(0,\infty), \\ \alpha\theta_{tt} - \kappa\theta_{xx} + \beta\psi_{tx} - k_1\theta_{txx} - k_2 z_{xx}(x,1,t) = 0 & \text{in } (0,\pi)\times(0,\infty), \\ \tau z_t(x,\omega,t) + z_\omega(x,\omega,t) = 0 & \text{in } (0,\pi)\times(0,1)\times(0,\infty), \\ z(x,0,t) = \theta_t(x,t) & \text{in } (0,\pi)\times(0,\infty), \\ \varphi(0,t) = \varphi(\pi,t) = \psi_x(0,t) = \psi_x(\pi,t) = \theta(0,t) = \theta(\pi,t) = 0 & \text{in } (0,\infty), \\ (\varphi(x,0),\psi(x,0),\theta(x,0)) = (\varphi_0(x),\psi_0(x),\theta_0(x)) & \text{in } (0,\pi), \\ (\varphi_t(x,0),\psi_t(x,0),\theta_t(x,0)) = (\varphi_1(x),\psi_1(x),\theta_1(x)) & \text{in } (0,\pi), \\ z(x,\omega,0) = f_0(x,\omega) & \text{in } (0,\pi)\times(0,1). \end{cases} \tag{2.12}$$

we define the energy functional associated with the solution of the system (2.12) by

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^\pi \left(\rho\varphi_t^2 + J\psi_t^2 + \alpha\theta_t^2 + \mu\varphi_x^2 + \xi\psi^2 + 2b\varphi_x\psi + \delta\psi_x^2 + \kappa\theta_x^2\right)dx \\ + \frac{k_1\tau}{2}\int_0^\pi\int_0^1 z_x^2(x,\omega,t)\,d\omega dx. \end{aligned} \tag{2.13}$$

**Remark 14.** *We notice that*

$$\begin{aligned} E(t) = \; & \frac{1}{2}\int_0^\pi \left(\rho\varphi_t^2 + J\psi_t^2 + \alpha\theta_t^2 + \delta\psi_x^2 + \kappa\theta_x^2\right)dx + \frac{k_1\tau}{2}\int_0^\pi\int_0^1 z_x^2(x,w,t)\,d\omega dx \\ & + \frac{\mu}{4}\int_0^\pi \left(\varphi_x + \frac{b}{\mu}\psi\right)^2 dx + \frac{\xi}{4}\int_0^\pi \left(\frac{b}{\xi}\varphi_x + \psi\right)^2 dx \\ & + \frac{1}{4}\left(\mu - \frac{b^2}{\xi}\right)\int_0^\pi \varphi_x^2 dx + \frac{1}{4}\left(\xi - \frac{b^2}{\mu}\right)\int_0^\pi \psi^2 dx, \end{aligned}$$

*which is a positive definite form by virtue of $\mu\xi > b^2$.*

## 2.2 Well posedness

In this subsection, we establish the existence, uniqueness, and regularity of the solution of the problem (2.12). First, we introduce the vector function $U = (\varphi, w, \psi, v, \theta, q, z)^T$, where $w = \varphi_t$, $v = \psi_t$, and $q = \theta_t$.

To rewrite the system (2.12) in the semigroup setting, we define the Hilbert space,

$\mathcal{H} = H_0^1(0,\pi)\times L^2(0,\pi)\times H_*^1(0,\pi)\times L_*^2(0,\pi)\times H_0^1(0,\pi)\times L^2(0,\pi)\times L^2((0,1);H_0^1(0,\pi)),$

where

$$L_*^2(0,\pi) = \left\{ w \in L^2(0,\pi) : \int_0^\pi w dx = 0 \right\},$$
$$H_*^1(0,\pi) = H^1(0,\pi) \cap L_*^2(0,\pi).$$

The space $\mathcal{H}$ is equipped with the following inner product

$$\langle U, \tilde{U} \rangle_{\mathcal{H}} = \int_0^\pi \left( \rho w \widetilde{w} + \mu \varphi_x \widetilde{\varphi}_x + J v \widetilde{v} + \alpha q \widetilde{q} + \delta \psi_x \widetilde{\psi}_x + \xi \psi \widetilde{\psi} + \kappa \theta_x \widetilde{\theta}_x \right) dx$$
$$+ b \int_0^\pi (\varphi_x \widetilde{\psi} dx + \psi \widetilde{\varphi}_x) dx + k_1 \tau \int_0^\pi \int_0^1 z_x \widetilde{z}_x d\omega dx,$$

for $U = (\varphi, w, \psi, v, \theta, q, z)$ and $\tilde{U} = \left( \widetilde{\varphi}, \widetilde{w}, \widetilde{\psi}, \widetilde{v}, \widetilde{\theta}, \widetilde{q}, \widetilde{z} \right) \in \mathcal{H}$. Thus, the system (2.12) becomes

$$\begin{cases} U_t(t) + \mathcal{A} U(t) = 0, \text{for all } t > 0, \\ U(0) = U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \theta_1, f_0), \end{cases} \tag{2.14}$$

where $\mathcal{A}$ is the operator defined by

$$\mathcal{A} U = \begin{pmatrix} -w \\ -\frac{\mu}{\rho} \varphi_{xx} - \frac{b}{\rho} \psi_x \\ -v \\ -\frac{\delta}{J} \psi_{xx} + \frac{b}{J} \varphi_x + \frac{\xi}{J} \psi + \frac{\beta}{J} q_x \\ -q \\ \frac{-\kappa}{\alpha} \theta_{xx} + \frac{\beta}{\alpha} v_x - \frac{k_1}{\alpha} q_{xx} - \frac{k_2}{\alpha} z_{xx}(1) \\ \frac{1}{\tau} z_\omega \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \left\{ U \in \mathcal{H} \left| \begin{array}{ll} \varphi \in H^2(0,\pi) \cap H_0^1(0,\pi), & \psi \in H_*^2(0,\pi) \cap H_*^1(0,\pi), \\ w, q \in H_0^1(0,\pi), v \in H_*^1(0,\pi), & z \in H^1((0,1); H_0^1(0,\pi)), \\ (\kappa\theta + k_1 q + k_2 z(1)) \in H^2(0,\pi) & z(0) = q. \end{array} \right. \right\},$$

where

$$H_*^2(0,\pi) = \left\{ u \in H_*^1(0,\pi) : u_x \in H_0^1(0,\pi) \right\}.$$

The well-posedness result reads as follows.

**Theorem 15.** *Suppose that*

$$|k_2| < k_1. \tag{2.15}$$

*Then for any $U_0 \in \mathcal{H}$, the problem (2.14) has a unique solution $U \in C([0,+\infty[\,,\mathcal{H})$. Moreover, if $U_0 \in D(\mathcal{A})$, then $U \in C([0,+\infty[\,, D(\mathcal{A})) \cap C^1([0,+\infty[\,,\mathcal{H})$.*

*Proof.* By virtue of Theorem 9, It suffices to establish the monotonicity and the maximality of $\mathcal{A}$ .

First, the monotony of $\mathcal{A}$, for any $U \in D(\mathcal{A})$, we have

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} =& -\mu \int_0^\pi \varphi_{xx} w dx - b\int_0^\pi \psi_x w dx - \mu \int_0^\pi w_x \varphi_x dx - \delta \int_0^\pi \psi_{xx} v dx + b \int_0^\pi \varphi_x v dx \\
&+ \xi \int_0^\pi \psi v dx + \beta \int_0^\pi q_x v dx - \kappa \int_0^\pi \theta_{xx} q dx + \beta \int_0^\pi v_x q dx - k_1 \int_0^\pi q_{xx} q dx \\
&- k_2 \int_0^\pi z_{xx}(1) q dx - \delta \int_0^\pi v_x \psi_x dx - \xi \int_0^\pi v\psi dx - \kappa \int_0^\pi q_x \theta_x dx \\
&+ b \int_0^\pi (-w_x \psi - v\varphi_x) dx + k_1 \int_0^\pi \int_0^1 z_{\omega x} z_x d\omega dx \\
=& k_1 \int_0^\pi q_x^2 dx + k_2 \int_0^\pi z_x(1) q_x dx + \frac{k_1}{2} \int_0^\pi \int_0^1 \frac{d}{d\omega} z_x^2 dw dx \\
=& k_1 \int_0^\pi q_x^2 dx + k_2 \int_0^\pi z_x(1) q_x dx + \frac{k_1}{2} \int_0^\pi z_x^2(1) dx - \frac{k_1}{2} \int_0^\pi q_x^2 dx \\
=& \frac{k_1}{2} \int_0^\pi q_x^2 dx + k_2 \int_0^\pi z_x(1) q_x dx + \frac{k_1}{2} \int_0^\pi z_x^2(1) dx.
\end{aligned} \tag{2.16}$$

Young's inequality allows us to deduce that

$$-k_2 \int_0^\pi z_x(x,1) q_x dx \leq \frac{|k_2|}{2} \int_0^\pi q_x^2 dx + \frac{|k_2|}{2} \int_0^\pi z_x^2(x,1) dx. \tag{2.17}$$

From (2.15), we deduce that

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \geq& \frac{k_1}{2} \int_0^\pi q_x^2 dx - \frac{|k_2|}{2} \int_0^\pi q_x^2 dx - \frac{|k_2|}{2} \int_0^\pi z_x^2(x,1) dx \\
&+ \frac{k_1}{2} \int_0^\pi z_x^2(x,1) dx = \left(\frac{k_1}{2} - \frac{|k_2|}{2}\right) \int_0^\pi q_x^2 dx + \left(\frac{k_1}{2} - \frac{|k_2|}{2}\right) \int_0^\pi z_x^2(x,1) dx \geq 0.
\end{aligned}$$

which implies that $\mathcal{A}$ is monotone.

Secondly, let $G = (h_1, h_2, h_3, h_4, h_5, h_6, h_7) \in \mathcal{H}$, we want to prove that there exists $U \in D(\mathcal{A})$ satisfying

$$(I + \mathcal{A})U = G. \tag{2.18}$$

That is,

$$\left\{\begin{aligned} \varphi - w = &\ h_1 \\ \rho w - \mu\varphi_{xx} - b\psi_x = &\ \rho h_2 \\ \psi - v = &\ h_3 \\ Jv - \delta\psi_{xx} + b\varphi_x + \xi\psi + \beta q_x = &\ Jh_4 \\ \theta - q = &\ h_5 \\ \alpha q - \kappa\theta_{xx} + \beta v_x - k_1 q_{xx} - k_2 z_{xx}(1) = &\ \alpha h_6 \\ \tau z + z_\omega = &\ \tau h_7 \end{aligned}\right. \tag{2.19}$$

From the first, the third and the fifth equation of (2.19) we have

$$w = \varphi - h_1,\ v = \psi - h_3,\ q = \theta - h_5. \tag{2.20}$$

By solving the seventh equation of (2.19), we get

$$z\left(\omega\right) = \tilde{C}e^{-\tau\omega} + \tau\int_0^\omega e^{\tau(\sigma-\omega)}h_7(\sigma)d\sigma \tag{2.21}$$

Using the fact that $z(0) = q$, we infer

$$\tilde{C} = q$$

and

$$z(\omega) = qe^{-\tau\omega} + \tau\int_0^\omega e^{\tau(\sigma-\omega)}h_7(\sigma)d\sigma,$$

Thus, (2.20) yields

$$z(\omega) = (\theta - h_5)e^{-\tau\omega} + \tau\int_0^\omega e^{\tau(\sigma-\omega)}h_7(\sigma,.)d\sigma. \tag{2.22}$$

Next, plugging (2.20) and (2.22) into the second, the fourth and the sixth equations of (2.19), we arrive at

$$\left\{\begin{aligned} -\mu\varphi_{xx} + \rho\varphi - b\psi_x &= f_1, \\ -\delta\psi_{xx} + b\varphi_x + (\xi + J)\psi + \beta\theta_x &= f_2, \\ -(\kappa + k_1 + k_2 e^{-\tau})\theta_{xx} + \alpha\theta + \beta\psi_x &= f_3, \end{aligned}\right. \tag{2.23}$$

where

$$f_1 = \rho(h_2 + h_1) \in L^2(0,\pi), \tag{2.24}$$

$$f_2 = J(h_4 + h_3) + \beta(h_5)_x \in L^2_*(0,\pi) \tag{2.25}$$

and

$$f_3 = \alpha(h_6 + h_5) + \beta(h_3)_x - \left(k_1 + k_2 e^{-\tau}\right)(h_5)_{xx} + \tau k_2 e^{-\tau} \int_0^1 e^{\sigma\tau}(h_7)_{xx}(\sigma) d\sigma \in H^{-1}(0, \pi).$$

We consider on $\mathcal{V} = H_0^1(0,\pi) \times H_*^1(0,\pi) \times H_0^1(0,\pi)$, the variational formulation

$$B((\varphi, \psi, \theta), (\widetilde{\varphi}, \widetilde{\psi}, \widetilde{\theta})) = L(\widetilde{\varphi}, \widetilde{\psi}, \widetilde{\theta}), \tag{2.26}$$

where $B$ is a bilinear form given by

$$\begin{aligned} B((\varphi, \psi, \theta), (\tilde{\varphi}, \tilde{\psi}, \tilde{\theta})) =& \mu \int_0^\pi \varphi_x \widetilde{\varphi}_x dx + \rho \int_0^\pi \varphi \widetilde{\varphi} dx + b \int_0^\pi \left(\psi \widetilde{\varphi}_x + \varphi_x \widetilde{\psi}\right) dx \\ &+ \delta \int_0^\pi \psi_x \widetilde{\psi}_x dx + (\xi + J) \int_0^\pi \psi \widetilde{\psi} dx + \beta \int_0^\pi (\theta_x \widetilde{\psi} + \psi_x \widetilde{\theta}) dx \\ &+ (\kappa + k_1 + k_2 e^{-\tau}) \int_0^\pi \theta_x \widetilde{\theta}_x dx + \alpha \int_0^\pi \theta \widetilde{\theta} dx \end{aligned}$$

and $L$ is the linear form

$$L(\widetilde{\varphi}, \widetilde{\psi}, \widetilde{\theta}) = \int_0^\pi f_1 \widetilde{\varphi} dx + \int_0^\pi f_2 \widetilde{\psi} dx + \int_0^\pi \langle f_3, \widetilde{\theta} \rangle_{H^{-1}, H_0^1} dx.$$

over $\mathcal{V} = H_0^1(0,\pi) \times H_*^1(0,\pi) \times H_0^1(0,\pi)$, equipped with the norm

$$\|(\varphi, \psi, \theta)\|_{\mathcal{V}}^2 = \|\varphi_x\|^2 + \|\psi\|^2 + \|\psi_x\|^2 + \|\theta_x\|^2.$$

It is evident that $\mathcal{V}$ is a Hilbert space and it is easy to verify that $B$ and $L$ are bounded. Additionally, we have

$$\begin{aligned} B((\varphi, \psi, \theta), (\varphi, \psi, \theta)) =& \mu \int_0^\pi \varphi_x^2 dx + \rho \int_0^\pi \varphi^2 dx + 2b \int_0^\pi \varphi_x \psi dx + (\xi + J) \int_0^\pi \psi^2 dx \\ &+ \delta \int_0^\pi \psi_x^2 dx + \alpha \int_0^\pi \theta^2 dx + (\kappa + k_1 + k_2 e^{-\tau}) \int_0^\pi \theta_x^2 dx. \end{aligned}$$

Young's inequality yields

$$\begin{aligned} B((\varphi, \psi, \theta), (\varphi, \psi, \theta)) \geq & \rho \|\varphi\|^2 + (\mu - \frac{b^2}{\xi}) \|\varphi_x\|^2 + J \|\psi\|^2 + \delta \|\psi_x\|^2 \\ &+ \alpha \|\theta\|^2 + (\kappa + k_1 + k_2 e^{-\tau}) \|\theta_x\|^2. \end{aligned}$$

Thus,

$$B((\varphi, \psi, \theta), (\varphi, \psi, \theta)) \geq N \|(\varphi, \psi, \theta)\|_{\mathcal{V}}^2,$$

for $N > 0$, which shows the coercivity of $B$. Thus, the Lax-Milgram theorem assures that there is a unique solution to the system (2.23), that satisfies

$$\varphi \in H_0^1(0,\pi), \psi \in H_*^1(0,\pi), \theta \in H_0^1(0,\pi).$$

Substituting $\varphi, \psi, \theta$ into (2.20), we get

$$w \in H_0^1(0,\pi), v \in H_*^1(0,\pi), q \in H_0^1(0,\pi). \tag{2.27}$$

Recalling (2.22) and using the fact that $\theta \in H_0^1(0,\pi)$ , we get

$$z \in L^2((0,1); H_0^1(0,\pi))$$

and

$$z_\omega = \tau(h_7 - z) \in L^2((0,1); H_0^1(0,\pi)), z(0) = q.$$

Consequently, we have,

$$z \in H^1((0,1); H_0^1(0,\pi)). \tag{2.28}$$

Additionally, taking $(\widetilde{\psi}, \widetilde{\theta}) \equiv (0,0) \in H_*^1(0,\pi) \times H_0^1(0,\pi)$ in (2.26), then using integration by parts, we get

$$\mu \int_0^\pi \varphi_x \widetilde{\varphi}_x dx = -\int_0^\pi (\rho\varphi - b\psi_x - f_1)\, \widetilde{\varphi} dx \;\forall \widetilde{\varphi} \in H_0^1(0,\pi), \tag{2.29}$$

The elliptic regularity theory entails that

$$\varphi \in H^2(0,\pi). \tag{2.30}$$

Integrating the left-hand side of (2.29) by parts, we arrive at

$$-\mu \int_0^\pi \varphi_{xx} \widetilde{\varphi} dx = -\int_0^\pi (\rho\varphi - b\psi_x - f_1)\, \widetilde{\varphi} dx \quad \forall \widetilde{\varphi} \in H_0^1(0,\pi)$$

$$\int_0^\pi (\mu\varphi_{xx} - \rho\varphi + b\psi_x + f_1)\, \widetilde{\varphi} dx = 0 \quad \forall \widetilde{\varphi} \in H_0^1(0,\pi)$$

thus,

$$\mu\varphi_{xx} - \rho\varphi + b\psi_x + f_1 = 0.$$

Also, from (2.20) and (2.24), we obtain

$$-\mu\varphi_{xx} + \rho w - b\psi_x = \rho h_2,$$

which solves the second equation of (2.19). Similarly, taking $(\widetilde{\varphi}, \widetilde{\theta}) \equiv 0 \in H_0^1(0,\pi) \times H_0^1(0,\pi)$, then using integration by parts and (2.20), we obtain

$$\delta \int_0^\pi \psi_x \widetilde{\psi}_x dx = \int_0^\pi (f_2 - (J+\xi)\,\psi - b\varphi_x - \beta\theta_x)\widetilde{\psi} dx,\ \forall \widetilde{\psi} \in H_*^1(0,\pi). \tag{2.31}$$

Here we are not able to apply the elliptic regularity. To address this difficulty, let $\Psi \in H_0^1(0,\pi)$ and set

$$\widetilde{\Psi}(x) = \Psi(x) - \frac{1}{\pi}\int_0^\pi \Psi(x)\,dx.$$

Clearly $\widetilde{\Psi} \in H_*^1(0,\pi)$. Plugging $\widetilde{\Psi}$ into (2.31) and recalling that

$$f_2 - (J+\xi)\,\psi - b\varphi_x - \beta\theta_x \in L_*^2(0,\pi),$$

we arrive at

$$\delta \int_0^\pi \psi_x \Psi_x dx = \int_0^\pi (f_2 - (J+\xi)\,\psi - b\varphi_x - \beta\theta_x)\Psi dx, \qquad \forall \Psi \in H_0^1(0,\pi). \tag{2.32}$$

which means that $\psi \in H^2(0,\pi)$, with

$$\psi_{xx} = \eta,$$

where $\eta = \dfrac{-1}{\delta}(f_2 - (J+\xi)\,\psi - b\varphi_x + \beta\theta_x)$. From (2.20) and (2.25), we get

$$\delta\psi_{xx} - b\varphi_x - Jv - \xi\psi - \beta q_x = -Jh_4.$$

So, $\psi$ solves the fourth equation in (2.19). Moreover, we have, from (2.31)

$$\int_0^\pi \psi_{xx}\widetilde{\psi} dx = \int_0^\pi \eta\widetilde{\psi} dx,\ \forall \widetilde{\psi} \in H_*^1(0,\pi).$$

Integration by parts yields

$$\psi_x(\pi)\,\widetilde{\psi}(\pi) - \psi_x(0)\,\widetilde{\psi}(0) - \int_0^\pi \psi_x\widetilde{\psi}_x dx = \int_0^\pi \eta\widetilde{\psi} dx,\ \forall \widetilde{\psi} \in H_*^1(0,\pi).$$

Using (2.32), we arrive at

$$\psi_x(\pi)\,\widetilde{\psi}(\pi) - \psi_x(0)\,\widetilde{\psi}(0) = 0,\ \forall \widetilde{\psi} \in H_*^1(0,\pi),$$

which yields that

$$\psi_x(\pi) = \psi_x(0) = 0,$$

and then $\psi \in H_*^2(0,\pi)$. Therefore,

$$\psi \in H_*^2(0,\pi) \cap H_*^1(0,\pi). \tag{2.33}$$

Finally, taking $\left(\widetilde{\varphi}, \widetilde{\psi}\right) \equiv (0,0) \in H_0^1(0,\pi) \times H_*^1(0,\pi)$ in (2.26), substituting $h_5$ by $\theta - q$ and recalling that

$$z(1) = qe^{-\tau} + \tau \int_0^1 e^{\tau(\sigma-1)} h_7(\sigma) d\sigma,$$

we obtain

$$\int_0^\pi [\kappa\theta_x + k_1 q_x + k_2 z_x(1)] \widetilde{\theta}_x dx = \int_0^\pi (\alpha h_6 - \alpha q - \beta v_x) \widetilde{\theta} dx \quad \forall \widetilde{\theta} \in H_0^1(0,\pi) \tag{2.34}$$

The regularity theory of elliptic equation entails that

$$\kappa\theta + k_1 q + k_2 z(1) \in H^2(0,\pi) \tag{2.35}$$

Integrating the left-hand side of (2.34) by parts, we arrive at

$$\int_0^\pi [-\kappa\theta_{xx} - k_1 q_{xx} - k_2 z_{xx}(1) - \alpha h_6 + \alpha q + \beta v_x] \widetilde{\theta} dx = 0 \quad \forall \widetilde{\theta} \in H_0^1(0,\pi)$$

Thus,

$$-\kappa\theta_{xx} - k_1 q_{xx} - k_2 z_{xx}(1) + \alpha q + \beta v_x = \alpha h_6$$

which solves the sixth equation in (2.19). The combination of (2.28), (2.27), (2.30), (2.33) and (2.35) shows that the solution $U$ belongs to $D(\mathcal{A})$. As a result, $\mathcal{A}$ is a maximal operator. This completes the proof of Theorem 15. $\square$

## 2.3 Exponential stability

In this section, we establish the exponential stability of the solution to the system (2.12) under the equal wave-speed propagation (2.36), using the energy method.
Additionally, we employ c as a positive generic constant.

**Theorem 16.** *Suppose that* (2.15) *hold and that*

$$\chi = \frac{\mu}{\rho} - \frac{\delta}{J} = 0 \tag{2.36}$$

*Then, there exist two positive constants $A, \eta$, for which the solution of problem* (2.12) *satisfies*

$$E(t) \leq Ae^{-\eta t}, \ \forall t \geq 0. \tag{2.37}$$

Several lemmas will be used to establish the proof of this theorem.

**Lemma 17.** *Suppose that* (2.15) *holds. Then, the energy $E(t)$ along the solution of the system* (2.12), *satisfies the estimate*

$$\frac{dE(t)}{dt} \leq -k_0 \left( \int_0^\pi \theta_{tx}^2 dx + \int_0^\pi z_x^2(x,1,t)\, dx \right) \leq 0. \tag{2.38}$$

*where* $k_0 := \dfrac{k_1 - |k_2|}{2} > 0.$

*Proof.* Multiplying the first four equations of system (2.12) by $\varphi_t, \psi_t, \theta_t$, and $k_1 z_{xx}$, respectively, then integrating the first three equations with respect to $x$ over $(0,\pi)$ and the last equation with respect to $(x,\omega)$ over $(0,\pi)\times(0,1)$ respectively, we get

$$\frac{\rho}{2}\frac{d}{dt}\int_0^\pi \varphi_t^2 dx + \frac{\mu}{2}\frac{d}{dt}\int_0^\pi \varphi_x^2 dx + b\int_0^\pi \psi\varphi_{xt} dx = 0,$$

$$\frac{J}{2}\frac{d}{dt}\int_0^\pi \psi_t^2 dx + \frac{\delta}{2}\frac{d}{dt}\int_0^\pi \psi_x^2 dx + b\int_0^\pi \varphi_x\psi_t dx + \frac{\xi}{2}\frac{d}{dt}\int_0^\pi \psi^2 dx + \beta\int_0^\pi \theta_{tx}\psi_t dx = 0,$$

$$\frac{\alpha}{2}\frac{d}{dt}\int_0^\pi \theta_t^2 dx + \frac{\kappa}{2}\frac{d}{dt}\int_0^\pi \theta_x^2 dx - \beta\int_0^\pi \psi_t\theta_{tx} dx + k_1\int_0^\pi \theta_{tx}^2 dx + k_2\int_0^\pi z_x(1)\,\theta_{tx} dx = 0,$$

and

$$\frac{\tau k_1}{2}\frac{d}{dt}\int_0^\pi\int_0^1 z_x^2(\omega)\, d\omega dx + \frac{k_1}{2}\int_0^\pi\int_0^1 \frac{d}{d\omega} z_x^2(\omega)\, d\omega dx = 0$$

Summing up the above identities, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^\pi \left(\rho\varphi_t^2 + J\psi_t^2 + \alpha\theta_t^2 + \mu\varphi_x^2 + \xi\psi^2 + 2b\varphi_x\psi + \delta\psi_x^2 + \kappa\theta_x^2\right) dx \\ &\qquad + \frac{\tau k_1}{2}\frac{d}{dt}\int_0^\pi\int_0^1 z_x^2(\omega)\, d\omega dx \\ &= -\frac{k_1}{2}\int_0^\pi \theta_{tx}^2 dx - k_2\int_0^\pi z_x(1)\theta_{tx} dx - \frac{k_1}{2}\int_0^\pi z_x^2(1)\, dx. \end{aligned}$$

Using Young's inequality and assumption (2.15), we arrive at (2.38) □

**Lemma 18.** *The functional*

$$\mathcal{I}_1(t) := J\int_0^\pi \psi\psi_t dx - \frac{\rho b}{\mu}\int_0^\pi \varphi_t \int_0^x \psi(y)\, dydx$$

*along the solution of problem (2.12) satisfies, for any* $\varepsilon_1 > 0$, *the estimate*

$$\mathcal{I}_1'(t) \le -\frac{\delta}{2}\int_0^\pi \psi_x^2 dx - b_0\int_0^\pi \psi^2 dx + \varepsilon_1\int_0^\pi \varphi_t^2 dx + c\int_0^\pi \theta_{tx}^2 dx + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^\pi \psi_t^2 dx, \tag{2.39}$$

*where* $b_0 = \left(\xi - \frac{b^2}{\mu}\right) > 0.$

*Proof.* Direct differentiation, using the first and the second equations of (2.12) and integration by parts, yields

$$\begin{aligned}
\mathcal{I}_1'(t) =& J\int_0^\pi \psi_t^2 dx + J\int_0^\pi \psi\psi_{tt} dx - \frac{\rho b}{\mu}\int_0^\pi \varphi_{tt}\int_0^x \psi(y)\, dydx - \frac{\rho b}{\mu}\int_0^\pi \varphi_t\int_0^x \psi_t(y)\, dydx \\
=& J\int_0^\pi \psi_t^2 dx - \delta\int_0^\pi \psi_x^2 dx - b\int_0^\pi \varphi_x\psi dx - \xi\int_0^\pi \psi^2 dx + \beta\int_0^\pi \theta_t\psi_x dx \\
& - b\int_0^\pi \varphi_{xx}\int_0^x \psi(y)\, dydx - \frac{b^2}{\mu}\int_0^\pi \psi_x\int_0^x \psi(y)\, dydx - \frac{\rho b}{\mu}\int_0^\pi \varphi_t\int_0^x \psi_t(y)\, dydx
\end{aligned}$$

We apply integration by parts and $\int_0^\pi \psi dx = 0$ to obtain

$$\int_0^\pi \varphi_{xx}\int_0^x \psi(y)\, dydx = -\int_0^\pi \varphi_x\psi dx$$

and

$$\int_0^\pi \psi_x\int_0^x \psi(y)\, dydx = -\int_0^\pi \psi^2 dx$$

So,

$$\begin{aligned}
\mathcal{I}_1'(t) = & -\delta\int_0^\pi \psi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^\pi \psi^2 dx + J\int_0^\pi \psi_t^2 dx + \beta\int_0^\pi \theta_t\psi_x dx \\
& - \frac{\rho b}{\mu}\int_0^\pi \varphi_t\int_0^x \psi_t(y)\, dydx.
\end{aligned} \tag{2.40}$$

Using the Cauchy-Schwarz, Poincaré and Young's inequalities, we get

$$\beta\int_0^\pi \theta_t\psi_x dx \le \frac{\delta}{2}\int_0^\pi \psi_x^2 dx + \frac{\beta^2 c_p^2}{2\delta}\int_0^\pi \theta_{tx}^2 dx$$

and, for any $\varepsilon_1 > 0$, we obtain

$$
\begin{aligned}
-\frac{\rho b}{\mu}\int_0^\pi \varphi_t \int_0^x \psi_t(y)\,dydx \leq & \varepsilon_1 \int_0^\pi \varphi_t^2 dx + \frac{c}{\pi\varepsilon_1}\int_0^\pi \left(\int_0^x \psi_t(y)dy\right)^2 dx \\
\leq & \varepsilon_1 \int_0^\pi \varphi_t^2 dx + \frac{c}{\pi\varepsilon_1}\int_0^\pi \left(\int_0^\pi \psi_t(y)dy\right)^2 dx \\
\leq & \varepsilon_1 \int_0^\pi \varphi_t^2 dx + \frac{c}{\pi\varepsilon_1}\left(\int_0^\pi \psi_t(y)dy\right)^2 \int_0^\pi dx \\
\leq & \varepsilon_1 \int_0^\pi \varphi_t^2 dx + \frac{c}{\varepsilon_1}\int_0^\pi \psi_t^2 dx.
\end{aligned}
\tag{2.41}
$$

Then, by substituting the above inequalities into (2.40), we arrive at (2.39). □

**Lemma 19.** *The functional*

$$
\mathcal{I}_2(t) := -\alpha \int_0^\pi \theta_t \int_0^x \psi_t(y)\,dydx
$$

*satisfies along the solution of problem (2.12) and for any* $\varepsilon_2 > 0$, *the estimate*

$$
\begin{aligned}
\mathcal{I}_2'(t) \leq & -\frac{\beta}{2}\int_0^\pi \psi_t^2 dx + c\left(1+\frac{1}{\varepsilon_2}\right)\int_0^\pi \theta_{tx}^2 dx + c\varepsilon_2 \int_0^\pi \left(\varphi_x^2 + \psi_x^2\right) dx + c\int_0^\pi \theta_x^2 dx \\
& + c\int_0^\pi z_x^2(1)\,dx.
\end{aligned}
\tag{2.42}
$$

*Proof.* The differentiation of $\mathcal{I}_2(t)$ and the use the third equation of (2.12) lead to

$$
\begin{aligned}
\mathcal{I}_2'(t) = & -\alpha \int_0^\pi \theta_{tt} \int_0^x \psi_t(y)\,dydx - \alpha \int_0^\pi \theta_t \int_0^x \psi_{tt}(y)\,dydx \\
= & -\kappa \int_0^\pi \theta_{xx}\left(\int_0^x \psi_t(y)\,dy\right)dx + \beta \int_0^\pi \psi_{tx}\left(\int_0^x \psi_t(y)\,dy\right)dx \\
& - k_1 \int_0^\pi \theta_{txx}\left(\int_0^x \psi_t(y)\,dy\right)dx - k_2 \int_0^\pi z_{xx}(1)\left(\int_0^x \psi_t(y)\,dy\right)dx \\
& - \alpha \int_0^\pi \theta_t \int_0^x \psi_{tt}(y)\,dydx \\
= & \kappa \int_0^\pi \theta_x \psi_t dx - \beta \int_0^\pi \psi_t^2 dx + k_1 \int_0^\pi \theta_{tx}\psi_t dx + k_2 \int_0^\pi z_x(1)\psi_t dx \\
& - \alpha \int_0^\pi \theta_t \int_0^x \psi_{tt}(y)\,dydx.
\end{aligned}
$$

We now estimate the terms in the right-hand side of the above identity. With the use of (2.12), the Cauchy-Schwarz and Young's inequalities, and manipulations as in (2.41), we find

$$
\kappa \int_0^\pi \theta_x \psi_t dx \leq \frac{\beta}{6}\int_0^\pi \psi_t^2 dx + \frac{3\kappa^2}{2\beta}\int_0^\pi \theta_x^2 dx,
$$

$$k_1 \int_0^\pi \theta_{tx}\psi_t dx \leq \frac{\beta}{6}\int_0^\pi \psi_t^2 dx + \frac{3k_1^2}{2\beta}\int_0^\pi \theta_{tx}^2 dx,$$

$$k_2 \int_0^\pi z_x(x,1,t)\psi_t dx \leq \frac{\beta}{6}\int_0^\pi \psi_t^2 dx + \frac{3k_2^2}{2\beta}\int_0^\pi z_x^2(x,1,t) dx,$$

and, for any $\varepsilon_2 > 0$, we infer that

$$\begin{aligned}
-\alpha \int_0^\pi \theta_t \int_0^x \psi_{tt}(y)\, dy dx =& \frac{-\alpha}{J}\int_0^\pi \theta_t \left(\int_0^x (\delta\psi_{yy} - b\varphi_y - \xi\psi - \beta\theta_{ty})\, dy\right) dx \\
=& -\frac{\alpha\delta}{J}\int_0^\pi \theta_t\psi_x dx + \frac{\alpha b}{J}\int_0^\pi \theta_t\varphi dx + \frac{\alpha\xi}{J}\int_0^\pi \theta_t \int_0^x \psi(y) dy dx \\
&+ \frac{\alpha\beta}{J}\int_0^\pi \theta_t^2 dx \\
\leq& \varepsilon_2 \int_0^\pi \psi_x^2 dx + \frac{c}{\varepsilon_2}\int_0^\pi \theta_t^2 dx + \varepsilon_2 \int_0^\pi \varphi^2 dx + \frac{c}{\varepsilon_2}\int_0^\pi \theta_t^2 dx \\
&+ \varepsilon_2 \int_0^\pi \psi^2 dx + \frac{c}{\varepsilon_2}\int_0^\pi \theta_t^2 dx + \frac{\alpha\beta}{J}\int_0^\pi \theta_t^2 dx
\end{aligned}$$

Thus, using the above estimate and Poincaré's inequality, we obtain

$$\begin{aligned}
\mathcal{I}_2'(t) \leq& -\frac{\beta}{2}\int_0^\pi \psi_t^2 dx + \left(\frac{\alpha\beta}{J} + \frac{c}{\varepsilon_2}\right)\int_0^\pi \theta_t^2 dx + \varepsilon_2 \int_0^\pi \psi^2 dx + \varepsilon_2 \int_0^\pi \psi_x^2 dx + \varepsilon_2 \int_0^\pi \varphi^2 dx \\
&+ \frac{3k_1^2}{2\beta}\int_0^\pi \theta_{tx}^2 dx + \frac{3\kappa^2}{2\beta}\int_0^\pi \theta_x^2 dx + c\int_0^\pi z_x^2(1) dx \\
\leq& -\frac{\beta}{2}\int_0^\pi \psi_t^2 dx + c\left(1 + \frac{1}{\varepsilon_2}\right)\int_0^\pi \theta_{tx}^2 dx + c\varepsilon_2 \int_0^\pi \left(\varphi_x^2 + \psi_x^2\right) dx + c\int_0^\pi \theta_x^2 dx \\
&+ c\int_0^\pi z_x^2(x,1,t)\, dx.
\end{aligned}$$

□

**Lemma 20.** *Suppose that* (2.36) *holds, then the functional*

$$\mathcal{I}_3(t) := \frac{\mu}{\rho}\int_0^\pi \psi_t\varphi_x dx + \frac{\delta}{J}\int_0^\pi \psi_x\varphi_t dx$$

*satisfies along the solution of problem (2.12), the estimate*

$$\mathcal{I}_3'(t) \leq -\frac{b\mu}{2\rho J}\int_0^\pi \varphi_x^2 dx + c\int_0^\pi \psi^2 dx + c\int_0^\pi \theta_{tx}^2 dx + c\int_0^\pi \psi_x^2 dx. \tag{2.43}$$

*Proof.* Direct differentiation, using the first and the second equation of (2.12) and integration by parts, yields

$$\begin{aligned}
\mathcal{I}_3'(t) =&\frac{\mu}{\rho}\int_0^\pi \psi_{tt}\varphi_x dx + \frac{\mu}{\rho}\int_0^\pi \psi_t\varphi_{tx}dx + \frac{\delta}{J}\int_0^\pi \psi_{tx}\varphi_t dx + \frac{\delta}{J}\int_0^\pi \psi_x\varphi_{tt}dx \\
=&\frac{\mu\delta}{\rho J}\int_0^\pi \psi_{xx}\varphi_x dx - \frac{\mu b}{\rho J}\int_0^\pi \varphi_x^2 dx - \frac{\mu\xi}{\rho J}\int_0^\pi \psi\varphi_x dx - \frac{\beta\mu}{\rho J}\int_0^\pi \theta_{tx}\varphi_x dx \\
&+ \frac{\mu}{\rho}\int_0^\pi \psi_t\varphi_{tx}dx + \frac{\delta}{J}\int_0^\pi \psi_{tx}\varphi_t dx - \frac{\delta\mu}{\rho J}\int_0^\pi \varphi_x\psi_{xx}dx + \frac{\delta b}{\rho J}\int_0^\pi \psi_x^2 dx \\
=&-\frac{b\mu}{\rho J}\int_0^\pi \varphi_x^2 dx - \frac{\mu\xi}{\rho J}\int_0^\pi \varphi_x\psi dx - \frac{\mu\beta}{\rho J}\int_0^\pi \varphi_x\theta_{tx}dx + \frac{\delta b}{\rho J}\int_0^\pi \psi_x^2 dx - \chi\int_0^\pi \varphi_t\psi_{tx}dx \\
=&-\frac{b\mu}{\rho J}\int_0^\pi \varphi_x^2 dx - \frac{\mu\xi}{\rho J}\int_0^\pi \varphi_x\psi dx - \frac{\mu\beta}{\rho J}\int_0^\pi \varphi_x\theta_{tx}dx + \frac{\delta b}{\rho J}\int_0^\pi \psi_x^2 dx. \qquad (2.44)
\end{aligned}$$

By using Young's inequality, we get,

$$\begin{aligned}
\mathcal{I}_3'(t) \leq& -\frac{b\mu}{\rho J}\int_0^\pi \varphi_x^2 dx + \frac{b\mu}{4\rho J}\int_0^\pi \varphi_x^2 dx \\
&+ c\int_0^\pi \psi^2 dx + \frac{b\mu}{4\rho J}\int_0^\pi \varphi_x^2 dx + c\int_0^\pi \theta_{tx}^2 dx + c\int_0^\pi \psi_x^2 dx. \\
\leq& -\frac{b\mu}{2\rho J}\int_0^\pi \varphi_x^2 dx + c\int_0^\pi \psi^2 dx + c\int_0^\pi \theta_{tx}^2 dx + c\int_0^\pi \psi_x^2 dx.
\end{aligned}$$

□

**Lemma 21.** *The functional*

$$\mathcal{I}_4(t) := -\rho\int_0^\pi \varphi_t\varphi dx$$

*satisfies along the solution of problem* (2.12), *the estimate*

$$\mathcal{I}_4'(t) \leq -\rho\int_0^\pi \varphi_t^2 dx + \frac{3\mu}{2}\int_0^\pi \varphi_x^2 dx + c\int_0^\pi \psi_x^2 dx. \qquad (2.45)$$

*Proof.* Direct computations, using the first equation of (2.12), give

$$\begin{aligned}
\mathcal{I}_4'(t) =& -\rho\int_0^\pi \varphi_{tt}\varphi dx - \rho\int_0^\pi \varphi_t^2 dx \\
=& -\mu\int_0^\pi \varphi_{xx}\varphi dx - b\int_0^\pi \psi_x\varphi dx - \rho\int_0^\pi \varphi_t^2 dx \\
=&\mu\int_0^\pi \varphi_x^2 dx + b\int_0^\pi \psi\varphi_x dx - \rho\int_0^\pi \varphi_t^2 dx
\end{aligned}$$

Using Young's and Poincaré's inequalities, we infer that

$$\begin{aligned}\mathcal{I}_4'(t) \leq & -\rho\int_0^\pi \varphi_t^2 dx + \mu\int_0^\pi \varphi_x^2 dx + \frac{\mu}{2}\int_0^\pi \varphi_x^2 dx + \frac{b^2}{2\mu}\int_0^\pi \psi^2 dx \\ \leq & -\rho\int_0^\pi \varphi_t^2 dx + \frac{3\mu}{2}\int_0^\pi \varphi_x^2 dx + c\int_0^\pi \psi_x^2 dx.\end{aligned}$$

□

**Lemma 22.** *The functional*

$$\mathcal{I}_5(t) := \alpha\int_0^\pi \theta\theta_t dx + \frac{k_1}{2}\int_0^\pi \theta_x^2 dx + \beta\int_0^\pi \psi_x\theta dx$$

*satisfies along the solution of problem* (2.12) *and for any* $\varepsilon_2 > 0$, *the estimate*

$$\mathcal{I}_5'(t) \leq -\frac{\kappa}{2}\int_0^\pi \theta_x^2 dx + \varepsilon_2\int_0^\pi \psi_x^2 dx + c\left(1+\frac{1}{\varepsilon_2}\right)\int_0^\pi \theta_{tx}^2 dx + c\int_0^\pi z_x^2(1)\,dx. \tag{2.46}$$

*Proof.* Direct differentiation of $\mathcal{I}_5(t)$ and the use of the third equation of (2.12) lead to

$$\begin{aligned}\mathcal{I}_5'(t) =& \alpha\int_0^\pi \theta_t^2 dx + \alpha\int_0^\pi \theta\theta_{tt} dx + k_1\int_0^\pi \theta_x\theta_{tx} dx + \beta\int_0^\pi \psi_{xt}\theta dx + \beta\int_0^\pi \psi_x\theta_t dx \\ =& \alpha\int_0^\pi \theta_t^2 dx + \kappa\int_0^\pi \theta_{xx}\theta dx - \beta\int_0^\pi \psi_{tx}\theta dx + k_1\int_0^\pi \theta_{txx}\theta dx + k_2\int_0^\pi z_{xx}(1)\,\theta dx \\ & + k_1\int_0^\pi \theta_x\theta_{tx} dx + \beta\int_0^\pi \psi_{xt}\theta dx + \beta\int_0^\pi \psi_x\theta_t dx \\ =& -\kappa\int_0^\pi \theta_x^2 dx + \alpha\int_0^\pi \theta_t^2 dx + \beta\int_0^\pi \psi_x\theta_t dx - k_2\int_0^\pi z_x(1)\,\theta_x dx.\end{aligned}$$

Next, by applying Young's and Poincaré's inequalities, we arrive at

$$\begin{aligned}\mathcal{I}_5'(t) \leq & -\kappa\int_0^\pi \theta_x^2 dx + \alpha\int_0^\pi \theta_t^2 dx + \varepsilon_2\int_0^\pi \psi_x^2 dx + \frac{\beta^2}{4\varepsilon_2}\int_0^\pi \theta_t^2 dx + \frac{\kappa}{2}\int_0^\pi \theta_x^2 dx \\ & + \frac{k_2^2}{2\kappa}\int_0^\pi z_x^2(x,1,t)\,dx \\ \leq & -\frac{\kappa}{2}\int_0^\pi \theta_x^2 dx + \varepsilon_2\int_0^\pi \psi_x^2 dx + c\left(1+\frac{1}{\varepsilon_2}\right)\int_0^\pi \theta_{tx}^2 dx + c\int_0^\pi z_x^2(1)\,dx.\end{aligned}$$

□

**Lemma 23.** *The functional*

$$\mathcal{I}_6(t) := \tau\int_0^\pi\int_0^1 e^{-\tau\omega} z_x^2(x,\omega,t)\,d\omega dx.$$

*satisfies along the solution of problem* (2.12) *the estimate*

$$\mathcal{I}_6'(t) \leq -c_0\tau\int_0^\pi\int_0^1 z_x^2(\omega)\,d\omega dx - c_0\int_0^\pi z_x^2(1)\,dx + \int_0^\pi \theta_{xt}^2 dx, \tag{2.47}$$

*where* $c_0 := e^{-\tau}$.

*Proof.* A simple differentiation of $\mathcal{I}_6(t)$, using (2.11), yields

$$\begin{aligned}
\mathcal{I}_6'(t) &= 2\tau \int_0^\pi \int_0^1 e^{-\tau\omega} z_x(\omega) z_{xt}(\omega) d\omega dx \\
&= -2 \int_0^\pi \int_0^1 e^{-\tau\omega} z_x(\omega) z_{x\omega}(\omega) d\omega dx \\
&= -\int_0^\pi \int_0^1 e^{-\tau\omega} \frac{\partial}{\partial\omega} z_x^2(\omega) d\omega dx \qquad (2.48)
\end{aligned}$$

We differentiate the term $e^{-\tau\omega} z_x^2(\omega)$ with respect to $\omega$, we get

$$\frac{\partial}{\partial\omega}\left(e^{-\tau\omega} z_x^2(\omega)\right) = -\tau e^{-\tau\omega} z_x^2(\omega) + e^{-\tau\omega} \frac{\partial}{\partial\omega} z_x^2(\omega).$$

Then, by substituting into (2.48) and using the fact that $z(0) = \theta_t$, we obtain

$$\begin{aligned}
\mathcal{I}_6'(t) &= -\tau \int_0^\pi \int_0^1 e^{-\tau\omega} z_x^2(\omega) d\omega dx - \int_0^\pi \int_0^1 \frac{\partial}{\partial\omega}\left(e^{-\tau\omega} z_x^2(\omega)\right) d\omega dx \\
&= -\tau \int_0^\pi \int_0^1 e^{-\tau\omega} z_x^2(\omega) d\omega dx - e^{-\tau} \int_0^\pi z_x^2(1) dx + \int_0^\pi \theta_{tx}^2 dx.
\end{aligned}$$

Exploiting Young's inequality with the fact that $e^{-\tau} \le e^{-\tau\omega}$, for $\omega \in [0,1]$, we end up with (2.47). □

For positive constants $N, N_1, N_2$, and $N_3$ sufficiently large, we defined the Lyapunov functional by

$$\mathcal{L}(t) := NE(t) + N_1\mathcal{I}_1(t) + N_2\mathcal{I}_2(t) + \frac{5\rho J}{b}\mathcal{I}_3(t) + \mathcal{I}_4(t) + N_3\mathcal{I}_5(t) + \mathcal{I}_6(t) \qquad (2.49)$$

**Lemma 24.** *The Lyapunov functional $\mathcal{L}(t)$ satisfies the equivalent relation*

$$m_1 E(t) \le \mathcal{L}(t) \le m_2 E(t), \ \forall t \ge 0, \qquad (2.50)$$

*where $m_1$ and $m_2$ are positive constants .*

*Proof.* By using the functionals defined in the preceding lemmas along with Cauchy-Schwarz, Young's and Poincaré's inequalities, we get

$$\begin{aligned}
|\mathcal{L}(t) - NE(t)| \le & cN_1 \left(\int_0^\pi \psi^2 dx + \int_0^\pi \psi_t^2 dx + \int_0^\pi \varphi_t^2 dx\right) + cN_2 \left(\int_0^\pi \psi_t^2 dx + \int_0^\pi \theta_t^2 dx\right) \\
& + c\frac{5\rho J}{b}\left(\int_0^\pi \varphi_x^2 dx + \int_0^\pi \psi_x^2 dx + \int_0^\pi \psi_t^2 dx + \int_0^\pi \varphi_t^2 dx\right) \\
& + c\left(\int_0^\pi \varphi_x^2 dx + \int_0^\pi \varphi_t^2 dx\right) + cN_3\left(\int_0^\pi \theta_t^2 dx + \int_0^\pi \theta_x^2 dx + \int_0^\pi \psi_x^2 dx\right) \\
& + c\int_0^\pi \int_0^1 z^2(x,\omega,t) d\omega dx.
\end{aligned}$$

Therefore, there is a sufficiently large $C > 0$ such that

$$\begin{aligned}
|\mathcal{L}(t) - NE(t)| &\le C \int_0^\pi \left( \varphi_t^2 + \psi_t^2 + \theta_t^2 + \varphi_x^2 + \psi_x^2 + \theta_x^2 + \psi^2 + \int_0^1 z^2(\omega)\, d\omega \right) dx \\
&\le CE(t).
\end{aligned}$$

Consequently,

$$(N - C)\, E(t) \le \mathcal{L}(t) \le (N + C)\, E(t)$$

By choosing $N$ sufficiently large, (2.50) follows. □

Substituting (2.38), (2.39), (2.42), (2.43), (2.45), (2.46) and (2.47) in the expression of $\mathcal{L}'(t)$, we get

$$\begin{aligned}
\mathcal{L}'(t) \le & -Nk_0 \int_0^\pi \theta_{tx}^2 dx - Nk_0 \int_0^\pi z_x^2(1)\, dx - N_1 \frac{\delta}{2} \int_0^\pi \psi_x^2 dx - N_1 b_0 \int_0^\pi \psi^2 dx \\
& + N_1 \varepsilon_1 \int_0^\pi \varphi_t^2 dx + N_1 c \int_0^\pi \theta_{tx}^2 dx + N_1 c \left(1 + \frac{1}{\varepsilon_1}\right) \int_0^\pi \psi_t^2 dx - N_2 \frac{\beta}{2} \int_0^\pi \psi_t^2 dx \\
& + N_2 c \left(1 + \frac{1}{\varepsilon_2}\right) \int_0^\pi \theta_{tx}^2 dx + N_2 c \varepsilon_2 \int_0^\pi \left(\varphi_x^2 + \psi_x^2\right) dx + N_2 c \int_0^\pi \theta_x^2 dx \\
& + N_2 c \int_0^\pi z_x^2(1)\, dx - \frac{5}{2}\mu \int_0^\pi \varphi_x^2 dx + \frac{5Jc\rho}{b} \int_0^\pi \psi^2 dx + \frac{5Jc\rho}{b} \int_0^\pi \theta_{tx}^2 dx \\
& + \frac{5Jc\rho}{b} \int_0^\pi \psi_x^2 dx - \rho \int_0^\pi \varphi_t^2 dx + \frac{3\mu}{2} \int_0^\pi \varphi_x^2 dx + c \int_0^\pi \psi_x^2 dx - N_3 \frac{\kappa}{2} \int_0^\pi \theta_x^2 dx \\
& + N_3 \varepsilon_2 \int_0^\pi \psi_x^2 dx + N_3 c \left(1 + \frac{1}{\varepsilon_2}\right) \int_0^\pi \theta_{tx}^2 dx + N_3 c \int_0^\pi z_x^2(1)\, dx \\
& - c_0 \tau \int_0^\pi \int_0^1 z_x^2(\omega)\, d\omega dx - c_0 \int_0^\pi z_x^2(1)\, dx + \int_0^\pi \theta_{xt}^2 dx.
\end{aligned}$$

That is,

$$\begin{aligned}
\mathcal{L}'(t) \le & - \left( k_0 N - cN_1 - cN_2 \left(1 + \frac{1}{\varepsilon_2}\right) - c\left(1 + \frac{1}{\varepsilon_2}\right) N_3 - \frac{5Jc\rho}{b} - 1 \right) \int_0^\pi \theta_{xt}^2 dx \\
& - (\mu - c\varepsilon_2 N_2) \int_0^\pi \varphi_x^2 dx - (\rho - \varepsilon_1 N_1) \int_0^\pi \varphi_t^2 dx - \left( b_0 N_1 - \frac{5Jc\rho}{b} \right) \int_0^\pi \psi^2 dx \\
& - \left( \frac{\delta N_1}{2} - \varepsilon_2 (cN_2 + N_3) - \frac{5Jc\rho}{b} \right) \int_0^\pi \psi_x^2 dx - \left( \frac{\kappa}{2} N_3 - cN_2 \right) \int_0^\pi \theta_x^2 dx \\
& - \left( \frac{\beta}{2} N_2 - c\left(1 + \frac{1}{\varepsilon_1}\right) N_1 \right) \int_0^\pi \psi_t^2 dx - c_0 \tau \int_0^\pi \int_0^1 z_x^2(\omega)\, d\omega dx \\
& - (k_0 N + c_0 - cN_2 - cN_3) \int_0^\pi z_x^2(1)\, dx.
\end{aligned}$$

At this stage, we have to choose the constants $N, N_1, N_2, N_3, \varepsilon_1, \varepsilon_2$ carefully. First we take $N_1$ large enough such that

$$b_0 N_1 - \frac{5Jc\rho}{b} > 0 \text{ and } d_0 := \frac{\delta}{2} N_1 - \frac{5Jc\rho}{b} > 0.$$

Next, we select a sufficiently small value for $\varepsilon_1$ so that

$$\rho - \varepsilon_1 N_1 > 0.$$

Then, we pick a large $N_2$ such that

$$\frac{\beta}{2} N_2 - c\left(1 + \frac{1}{\varepsilon_1}\right) N_1 > 0$$

and choose $N_3$ such that

$$\frac{\kappa}{2} N_3 > cN_2,$$

After that, we take a sufficiently small $\varepsilon_2$ so that

$$\mu - c\varepsilon_2 N_2 > 0 \text{ and } d_0 - \varepsilon_2 \left(cN_2 + N_3\right) > 0.$$

Bearing in mind (2.50) , we choose $N$ so large such that

$$k_0 N - cN_1 - cN_2\left(1 + \frac{1}{\varepsilon_2}\right) - c\left(1 + \frac{1}{\varepsilon_2}\right) N_3 - \frac{5Jc\rho}{b} - 1 > 0$$

and

$$k_0 N - cN_2 - cN_3 > 0$$

and $\mathcal{L} \sim E$. Consequently, we get for a positive constant $\lambda_0$,

$$\mathcal{L}'(t) \leq -\lambda_0 E(t), \ \forall t > 0, \tag{2.51}$$

A combination of (2.50) and (2.51) gives

$$\mathcal{L}'(t) \leq -\eta \mathcal{L}(t), \ \forall t > 0, \tag{2.52}$$

for $\eta > 0$. The integration of (2.52) over $(0, t)$ yields

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-\eta t}, \ \forall t > 0.$$

Using (2.50) once againth, we get our etimate (2.37); which completes the proof of Theorem 16.

## 2.4 Polynomial stability

In this section, we examine the case in which (2.36) does not hold, and demonstrate the polynomial decay of the strong solution for (2.12). First we define the second-order energy by

$$E_*(t) = \frac{1}{2}\int_0^\pi \left(\rho\varphi_{tt}^2 + J\psi_{tt}^2 + \alpha\theta_{tt}^2 + \delta\psi_{tx}^2 + \mu\varphi_{tx}^2 + \xi\psi_t^2 + 2b\varphi_{tx}\psi_t + \kappa\theta_{tx}^2\right) dx + \frac{k_1\tau}{2}\int_0^\pi\int_0^1 z_{xt}^2(1)\, d\omega dx.$$

Similar calculations, as in (2.38), lead to

$$E_*'(t) \le -k_0\left(\int_0^\pi \theta_{ttx}^2 dx + \int_0^\pi z_{tx}^2(1)\, dx\right) \le 0. \tag{2.53}$$

We consider the following lemma in order to handle the term $\chi$ that appears in the proof of $\mathcal{I}_3$.

**Lemma 25.** *Let $(\varphi, \psi, \theta, z)$ be the strong solution of (2.12). Then the functional*

$$\widetilde{\mathcal{I}}_3(t) := \beta\mathcal{I}_3(t) - \chi\int_0^\pi \varphi_x\left(k_1\theta_{tx} + \kappa\theta_x + k_2 z_x(1)\right) dx. \tag{2.54}$$

*satisfies, for any $\varepsilon_3 > 0$, the estimate*

$$\widetilde{\mathcal{I}}_3'(t) \le -\frac{b\beta\mu}{2\rho J}\int_0^\pi \varphi_x^2 dx + \varepsilon_3\int_0^\pi \varphi_t^2 dx + c\left(\int_0^\pi \psi^2 dx + \int_0^\pi \theta_{tx}^2 dx + \int_0^\pi z_{tx}^2(1)\right) dx + c\int_0^\pi \psi_x^2 dx + c\left(1 + \frac{1}{\varepsilon_3}\right)\int_0^\pi \theta_{ttx}^2 dx. \tag{2.55}$$

*Proof.* A simple differentiation of (2.54) leads

$$\widetilde{\mathcal{I}}_3'(t) = \beta\mathcal{I}_3'(t) - \chi k_1\int_0^\pi \varphi_{xt}\theta_{tx} dx - \chi k_1\int_0^\pi \varphi_x\theta_{ttx} dx - \chi\kappa\int_0^\pi \varphi_{xt}\theta_x dx - \chi\kappa\int_0^\pi \varphi_x\theta_{xt} dx - \chi k_2\int_0^\pi \varphi_{xt} z_x(1) dx - \chi k_2\int_0^\pi \varphi_x z_{tx}(1) dx. \tag{2.56}$$

By utilizing the third equation of (2.12) and integration by parts for the second term on the right side of (2.56), we obtain

$$-\chi k_1\int_0^\pi \varphi_{xt}\theta_{tx} dx = \chi k_1\int_0^\pi \varphi_t\theta_{txx} dx$$

$$= \chi\alpha\int_0^\pi \varphi_t\theta_{tt} dx + \chi\kappa\int_0^\pi \varphi_{xt}\theta_x dx + \chi\beta\int_0^\pi \varphi_t\psi_{xt} dx + \chi k_2\int_0^\pi \varphi_{xt} z_x(1)\, dx. \tag{2.57}$$

Substituting (2.57) and (2.44) into (2.56), we obtain

$$\widetilde{\mathcal{I}}_3'(t) = -\frac{\beta b\mu}{\rho J}\int_0^\pi \varphi_x^2 dx - \frac{\mu\xi\beta}{\rho J}\int_0^\pi \varphi_x\psi dx - \left(\frac{\beta^2\mu}{\rho J} + \chi\kappa\right)\int_0^\pi \varphi_x\theta_{tx}dx$$
$$+ \frac{\delta\beta b}{\rho J}\int_0^\pi \psi_x^2 dx + \chi\alpha\int_0^\pi \varphi_t\theta_{tt}dx - \chi k_2\int_0^\pi \varphi_x z_{tx}(1)\, dx - \chi k_1\int_0^\pi \varphi_x\theta_{ttx}dx.$$

By using Poincaré's and Young's inequalities, we get (2.55). □

The following theorem provides our polynomial decay result.

**Theorem 26.** *Assume that*

$$\chi = \frac{\mu}{\rho} - \frac{\delta}{J} \neq 0,$$

*and let $(\varphi, \psi, \theta, z)$ be a strong solution of (2.12). Then, there is $\eta > 0$ such that $E(t)$ satisfies,*

$$E(t) \leq \frac{\eta}{t}, \quad \forall t > 0. \tag{2.58}$$

*Proof.* We define the Lyapunov functional by

$$\mathcal{L}_*(t) := N^*\left(E(t) + E_*(t)\right) + N_1^*\mathcal{I}_1(t) + N_2^*\mathcal{I}_2(t) + N_3^*\widetilde{\mathcal{I}}_3(t) + \mathcal{I}_4(t) + N_5^*\mathcal{I}_5(t) + \mathcal{I}_6(t).$$

Differentiating $\mathcal{L}_*(t)$ and substituting (2.38), (2.53), (2.39), (2.42), (2.45), (2.46), (2.47), (2.55), and setting $\varepsilon_1 = \frac{\rho}{4N_1^*}$ and $\varepsilon_3 = \frac{\rho}{4N_3^*}$, we arrive at

$$\mathcal{L}_*'(t) \leq -\left[k_0N^* - cN_1^* - cN_2^*\left(1 + \frac{1}{\varepsilon_2}\right) - cN_3^* - cN_5^*\left(1 + \frac{1}{\varepsilon_2}\right) - 1\right]\int_0^\pi \theta_{tx}^2 dx$$
$$- \frac{\rho}{2}\int_0^\pi \varphi_t^2 dx - \left[\frac{\beta}{2}N_2^* - c\left(1 + \frac{4N_1^*}{\rho}\right)N_1^*\right]\int_0^\pi \psi_t^2 dx - \left[b_0N_1^* - cN_3^*\right]\int_0^\pi \psi^2 dx$$
$$- \left[\frac{\delta N_1^*}{2} - cN_3^* - c - c\varepsilon_2\left(N_2^* + N_5^*\right)\right]\int_0^\pi \psi_x^2 dx - \tau c_0\int_0^\pi\int_0^1 z_x^2(\omega)\, d\omega dx$$
$$- \left[\frac{\kappa}{2}N_5^* - cN_2^*\right]\int_0^\pi \theta_x^2 dx - \left[\frac{b\beta\mu}{2\rho J}N_3^* - \frac{3\mu}{2} - c\varepsilon_2 N_2\right]\int_0^\pi \varphi_x^2 dx$$
$$- \left[k_0N^* - cN_2^* - cN_5^* + c_0\right]\int_0^\pi z_x^2(1)\, dx$$
$$- \left[k_0N^* - c\left(1 + \frac{4N_3^*}{\rho}\right)N_3^*\right]\int_0^\pi \theta_{ttx}^2 dx - \left[k_0N^* - cN_3^*\right]\int_0^\pi z_{tx}^2(1)\, dx.$$

First, we choose a large $N_3^*$ such that

$$C_1 := \frac{b\beta\mu}{2\rho J}N_3^* - \frac{3\mu}{2} > 0,$$

then, we let $N_1^*$ large so that

$$c_1 := b_0 N_1^* - cN_3^* > 0 \qquad \text{and} \qquad C_2 = \frac{\delta}{2} N_1^* - cN_3^* - c > 0.$$

Next, we select a sufficiently large $N_2^*$ such that

$$c_2 := \frac{\beta}{2} N_2^* - c\left(1 + \frac{4N_1^*}{\rho}\right) N_1^* > 0,$$

subsequently selecting $N_5^*$ large enough guarantees

$$c_3 := \frac{\kappa}{2} N_5^* - cN_2^* > 0.$$

Then, we choose a sufficiently small $\varepsilon_2$ so that

$$c_4 := C_2 - c\varepsilon_2 \left(N_2^* + N_5^*\right) > 0$$

and

$$c_5 := C_1 - c\varepsilon_2 N_2^* > 0.$$

In the end, we select a very large $N^*$ such that

$$c_6 = k_0 N^* - cN_1^* - cN_2^* \left(1 + \frac{4N_3^*}{\rho}\right) - cN_3^* - N_5^* c \left(1 + \frac{4N_3^*}{\rho}\right) - 1 > 0,$$

$$c_7 = k_0 N^* - cN_2^* - cN_5^* - 1 > 0,$$

$$c_8 = k_0 N^* - c\left(1 + \frac{1}{\varepsilon_3}\right) N_3^* > 0$$

and

$$c_9 = k_0 N^* - cN_3^* > 0.$$

Therefore,

$$\begin{aligned}
\mathcal{L}_*'(t) \leq &- \left(\frac{\rho}{2}\int_0^\pi \varphi_t^2 dx + c_2 \int_0^\pi \psi_t^2 dx + c_6 \int_0^\pi \theta_{tx}^2 dx + c_5 \int_0^\pi \varphi_x^2 dx + c_1 \int_0^\pi \psi^2 dx\right) \\
&- \left(c_4 \int_0^\pi \psi_x^2 dx + c_3 \int_0^\pi \theta_x^2 dx + \tau e^{-\tau} \int_0^\pi \int_0^1 z_x^2(\omega)\, d\omega dx\right) \\
&- \left(c_7 \int_0^\pi z_x^2(1)\, dx + c_8 \int_0^\pi \theta_{ttx}^2 dx + c_9 \int_0^\pi z_{tx}^2(1)\, dx\right).
\end{aligned}$$

By denoting $\eta_0 = \min\left\{\frac{\rho}{2}, c_1, c_2, c_3, c_4, c_5, c_6, \tau e^{-\tau}\right\}$ we conclude

$$\mathcal{L}_*'(t) \leq -\eta_0 E(t). \tag{2.59}$$

Integrating (2.59) over $(0,t)$, taking into account that $E(t)$ is non-increasing, we obtain

$$tE(t) \leq \int_0^t E(s)\, ds \leq \frac{1}{\eta_0}\left(\mathcal{L}_*(0) - \mathcal{L}_*(t)\right) \leq \frac{\mathcal{L}_*(0)}{\eta_0},\ \forall t \geq 0 \tag{2.60}$$

which implies that

$$E(t) \leq \frac{\eta}{t},\ \forall t > 0,$$

for $\eta > 0$. As a result, Theorem 26 is proved. □

# Chapter 3

# Exponential and polynomial decay for a porous system of thermoelasticity of type III with distributed delay.

In this section, we are concerned with the following porous thermoelastic type III system with distributed delay acting on the heat equation

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, & \text{in } (0,\pi) \times (0,\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \beta\theta_x = 0, & \text{in } (0,\pi) \times (0,\infty), \\ \alpha\theta_{tt} - \kappa\theta_{xx} + \beta\phi_{ttx} - k_1\theta_{txx} - \displaystyle\int_{\tau_1}^{\tau_2} k_2(s)\,\theta_{txx}(x, t-s)\,ds = 0, & \text{in } (0,\pi) \times (0,\infty), \end{cases} \tag{3.1}$$

where $k_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function, $\tau_1$ and $\tau_2$ are two real numbers satisfying $0 \leq \tau_1 < \tau_2$.
subjected to the following initial and boundary conditions

$$u(0,t) = u(\pi,t) = \phi_x(0,t) = \phi_x(\pi,t) = \theta(0,t) = \theta(\pi,t) = 0, \quad \text{in } (0,\infty), \tag{3.2}$$

$$\begin{aligned} &(u(x,0), \phi(x,0), \theta(x,0)) = (u_0(x), \phi_0(x), \theta_0(x)), && \text{in } (0,\pi), \\ &(u_t(x,0), \phi_t(x,0), \theta_t(x,0)) = (u_1(x), \phi_1(x), \theta_0(x)). && \text{in } (0,\pi). \end{aligned} \tag{3.3}$$

We study the well-posedness and the asymptotic stability of (3.1)-(3.3). By using the semigroup theory, we prove the existence and uniqueness of the solution. We then exploit the energy method to obtain the exponential decay result for the case of equal wave speeds and a polynomial decay result in the opposite case.

## 3.1 Setting of the problem

Note first, since Neumann boundary conditions are considered for $\phi$, which prevents the application of Poincaré's inequality for $\phi$. To avoid these inconveniences, we proced, as follows. From boundary conditions (3.2) and equation $(3.1)_2$, we infer that

$$\frac{d^2}{dt^2}\int_0^\pi \phi(x,t)\,dx = -\frac{\xi}{J}\int_0^\pi \phi(x,t)\,dx.$$

Thus,

$$\int_0^\pi \phi(x,t)\,dx = \left(\int_0^\pi \phi_0(x)\,dx\right)\cos\left(\sqrt{\frac{\xi}{J}}t\right) + \sqrt{\frac{J}{\xi}}\left(\int_0^\pi \phi_1(x)\,dx\right)\sin\left(\sqrt{\frac{\xi}{J}}t\right),$$

therefore, if we set

$$\begin{aligned}\overline{\phi}(x,t) = \phi(x,t) &- \frac{1}{\pi}\left(\int_0^\pi \phi_0(x)\,dx\right)\cos\left(\sqrt{\frac{\xi}{J}}t\right)\\ &-\frac{1}{\pi}\sqrt{\frac{J}{\xi}}\left(\int_0^\pi \phi_1(x)\,dx\right)\sin\left(\sqrt{\frac{\xi}{J}}t\right),\end{aligned}$$

then

$$\int_0^\pi \overline{\phi}(x,t)\,dx = 0,$$

which allows the application of Poincaré's inequality. Moreover, $\left(u,\overline{\phi},\theta\right)$ satisfies (3.1)–(3.3) with the same initial conditions (3.2) for $u$ and $\theta$, and

$$\overline{\phi}(x,0) = \phi_0(x) - \frac{1}{\pi}\left(\int_0^\pi \phi_0(x)\,dx\right),\ \overline{\phi}_t(x,0) = \phi_1(x) - \frac{1}{\pi}\left(\int_0^\pi \phi_1(x)\,dx\right).$$

In the sequel we work with $\overline{\phi}$ but for simplicity we write $\phi$. Next; as in [42], we introduce the new variable to deal with the delay term.

$$z(x,\omega,s,t) = \theta_t(x,t-\omega s),\ \text{ for }\ 0\le\omega\le 1\ \textit{and}\ \tau_1\le s\le\tau_2.$$

Clearly, we have

$$sz_t(x,\omega,s,t) + z_\omega(x,\omega,s,t) = 0,\ \forall\ (x,\omega,s,t)\in(0,\pi)\times(0,1)\times(\tau_1,\tau_2)\times(0,+\infty). \quad (3.4)$$

Differentiating the first two equations of (3.1) with respect to $t$, setting $\varphi = u_t$ and $\psi = \phi_t$, the system (3.1), the equation (3.4) and the conditions (3.3) and (3.2) become

$$\begin{cases} \rho\varphi_{tt} - \mu\varphi_{xx} - b\psi_x = 0 & \text{in } (0,\pi)\times(0,+\infty), \\ J\psi_{tt} - \delta\psi_{xx} + b\varphi_x + \xi\psi + \beta\theta_{tx} = 0 & \text{in } (0,\pi)\times(0,+\infty), \\ \alpha\theta_{tt} - \kappa\theta_{xx} + \beta\psi_{tx} - k_1\theta_{txx} & \\ \qquad\qquad - \displaystyle\int_{\tau_1}^{\tau_2} k_2(s) z_{xx}(1,s)\,ds = 0 & \text{in } (0,\pi)\times(0,+\infty), \\ s z_t(\omega,s) + z_\omega(\omega,s) = 0 & \text{in } (0,\pi)\times(0,1)\times(\tau_1,\tau_2)\times(0,+\infty), \end{cases} \tag{3.5}$$

$$\varphi(0,t) = \varphi(\pi,t) = \psi_x(0,t) = \psi_x(\pi,t) = \theta(0,t) = \theta(\pi,t) = 0 \quad \text{in } (0,\infty), \tag{3.6}$$

$$\begin{aligned} &(\varphi(x,0),\psi(x,0),\theta(x,0)) = (\varphi_0(x),\psi_0(x),\theta_0(x)) && \text{in } (0,\pi), \\ &(\varphi_t(x,0),\psi_t(x,0),\theta_t(x,0)) = (\varphi_1(x),\psi_1(x)q_0(x)) && \text{in } (0,\pi), \\ &z(x,\omega,s,0) = f_0(x,\omega,s) && \text{in } (0,\pi)\times(0,1)\times(\tau_1,\tau_2). \end{aligned} \tag{3.7}$$

We introduce the energy associated with the system (3.5)–(3.7) by

$$\begin{aligned} E(t) = &\ \frac{1}{2}\int_0^\pi \left(\rho\varphi_t^2 + J\psi_t^2 + \alpha\theta_t^2 + \mu\varphi_x^2 + \xi\psi^2 + 2b\varphi_x\psi + \delta\psi_x^2 + \kappa\theta_x^2\right)dx \\ &+\frac{1}{2}\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} s\,|k_2(s)|\, z_x^2(\omega,s)\,ds\,d\omega\,dx. \end{aligned} \tag{3.8}$$

**Remark 27.** *We can rewrite the energy $E(t)$ as*

$$\begin{aligned} E(t) = &\ \frac{1}{2}\int_0^\pi \left(\rho\varphi_t^2 + J\psi_t^2 + \alpha\theta_t^2 + \delta\psi_x^2 + \kappa\theta_x^2\right)dx \\ &+\frac{1}{2}\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} s\,|k_2(s)|\, z_x^2(\omega,s)\,ds\,d\omega\,dx + \frac{\mu}{4}\int_0^\pi \left(\varphi_x + \frac{b}{\mu}\psi\right)^2 dx \\ &+\frac{\xi}{4}\int_0^\pi \left(\frac{b}{\xi}\varphi_x + \psi\right)^2 dx + \frac{1}{4}\left(\mu - \frac{b^2}{\xi}\right)\int_0^\pi \varphi_x^2 dx + \frac{1}{4}\left(\xi - \frac{b^2}{\mu}\right)\int_0^\pi \psi^2 dx, \end{aligned}$$

*which is a positive definite form by virtue of $\mu\xi > b^2$.*

## 3.2 Well posedness

In this section, we demonstrate that the problem given by (3.5)–(3.7), has a unique solution. To rewrite the problem (3.5)–(3.7) in the semigroup setting, we define the new variables. $w = \varphi_t$, $v = \psi_t$ and $q = \theta_t$ and introduce the Hilbert space

$$\begin{aligned} \mathcal{H} =&H_0^1(0,\pi)\times L^2(0,\pi)\times H_*^1(0,\pi)\times L_*^2(0,\pi) \\ &\times H_0^1(0,\pi)\times L^2(0,\pi)\times L^2((0,1);H_0^1(0,\pi)), \end{aligned}$$

where

$$L_*^2(0,\pi) = \left\{ w \in L^2(0,\pi) : \int_0^\pi w dx = 0 \right\},$$
$$H_*^1(0,\pi) = \left\{ w \in H^1(0,\pi) : \int_0^\pi w dx = 0 \right\} = H^1(0,\pi) \cap L_*^2(0,\pi).$$

The space $\mathcal{H}$ is equipped with the inner product

$$\langle U, \tilde{U} \rangle_{\mathcal{H}} = \int_0^\pi \left( \rho w \widetilde{w} + \mu \varphi_x \widetilde{\varphi}_x + J v \widetilde{v} + \alpha q \widetilde{q} + \delta \psi_x \widetilde{\psi}_x + \xi \psi \widetilde{\psi} + \kappa \theta_x \widetilde{\theta}_x \right) dx$$
$$+ b \int_0^\pi (\varphi_x \widetilde{\psi} dx + \psi \widetilde{\varphi}_x) dx + \int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} |k_2(s)| \, z_x(\omega, s) \widetilde{z}_x(\omega, s) ds d\omega dx,$$

where $U = (\varphi, w, \psi, v, \theta, q, z)$ and$\tilde{U} = \left( \widetilde{\varphi}, \widetilde{w}, \widetilde{\psi}, \widetilde{v}, \widetilde{\theta}, \widetilde{q}, \widetilde{z} \right) \in \mathcal{H}$. According to the new notations the problem (3.5),(3.7) can be written

$$\begin{cases} U_t(t) + \mathcal{A} U(t) = 0, \text{for all } \ t > 0, \\ U(0) = U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0, z_0), \end{cases} \tag{3.9}$$

where $\mathcal{A}$ is the operator defined by

$$\mathcal{A} U = \begin{pmatrix} -w \\ -\frac{\mu}{\rho} \varphi_{xx} - \frac{b}{\rho} \psi_x \\ -v \\ -\frac{\delta}{J} \psi_{xx} + \frac{b}{J} \varphi_x + \frac{\xi}{J} \psi + \frac{\beta}{J} q_x \\ -q \\ \frac{-\kappa}{\alpha} \theta_{xx} + \frac{\beta}{\alpha} v_x - \frac{k_1}{\alpha} q_{xx} - \frac{1}{\alpha} \int_{\tau_1}^{\tau_2} k_2(s) z_{xx}(1, s) ds \\ z_\omega \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \left\{ U \in \mathcal{H} \left| \begin{array}{l} \varphi \in H^2(0,\pi) \cap H_0^1(0,\pi), \quad \psi \in H_*^2(0,\pi) \cap H_*^1(0,\pi), \\ w, q \in H_0^1(0,\pi), \ v \in H_*^1(0,\pi), \ z \in H^1((0,1); H_0^1(0,\pi)), \\ \kappa\theta + k_1 q + \displaystyle\int_{\tau_1}^{\tau_2} k_2(s) z(1,s) \, ds \in H^2(0,\pi), \ \ z(0,s) = q \end{array} \right. \right\},$$

where

$$H_*^2(0,\pi) = \left\{ u \in H_*^1(0,\pi); u_x \in H_0^1(0,\pi) \right\}$$

The main result of this section is given by the following theorem.

**Theorem 28.** *Suppose that*

$$\int_{\tau_1}^{\tau_2} |k_2(s)|\, ds \leq k_1. \tag{3.10}$$

*Then, for any $U_0 \in \mathcal{H}$, the problem (3.9) has a unique solution $U \in C\left([0, +\infty[\,, \mathcal{H}\right)$. Moreover, if $U_0 \in D\left(\mathcal{A}\right)$, then $U \in C\left(\mathbb{R}^+, D\left(\mathcal{A}\right)\right) \cap C^1\left(\mathbb{R}^+, \mathcal{H}\right)$.*

To prove Theorem 28, we use the semigroup approach. According to the Lumer Phillips theorem, it suffices to prove that $\mathcal{A}$ is monotone and maximal.
First, for any $U \in D\left(\mathcal{A}\right)$, we have

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} =& -\mu \int_0^\pi \varphi_{xx} w dx - b\int_0^\pi \psi_x w dx - \mu \int_0^\pi w_x \varphi_x dx - \delta \int_0^\pi \psi_{xx} v dx + b \int_0^\pi \varphi_x v dx \\
&+ \xi \int_0^\pi \psi v dx + \beta \int_0^\pi q_x v dx - \kappa \int_0^\pi \theta_{xx} q dx + \beta \int_0^\pi v_x q dx - k_1 \int_0^\pi q_{xx} q dx \\
&- \int_0^\pi q \int_{\tau_1}^{\tau_2} k_2(s) z_{xx}(1,s) ds dx - \delta \int_0^\pi v_x \psi_x dx - \xi \int_0^\pi v\psi dx - \kappa \int_0^\pi q_x \theta_x dx \\
&+ b \int_0^\pi (-w_x \psi - v\varphi_x) dx + \int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} |k_2(s)| z_{\omega x}(w,s) z_x(w,s) ds d\omega dx \\
=& \, k_1 \int_0^\pi q_x^2 dx + \int_0^\pi q_x \int_{\tau_1}^{\tau_2} k_2(s) z_x(1,s) ds dx \\
&+ \frac{1}{2} \int_0^\pi \int_{\tau_1}^{\tau_2} |k_2(s)|\, z_x^2(1,s) ds dx - \frac{1}{2} \int_0^\pi \int_{\tau_1}^{\tau_2} |k_2(s)|\, q_x^2 ds dx,
\end{aligned} \tag{3.11}$$

Using Young's inequality for the second term of (3.11), we reach

$$-\int_0^\pi q_x \int_{\tau_1}^{\tau_2} k_2(s) z_x(1,s) ds dx \leq \frac{1}{2} \left( \int_{\tau_1}^{\tau_2} |k_2(s)|\, ds \right) \int_0^\pi q_x^2 dx + \frac{1}{2} \int_0^\pi \int_{\tau_1}^{\tau_2} |k_2(s)|\, z_x^2(1) ds dx \tag{3.12}$$

So,

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \geq& \, k_1 \int_0^\pi q_x^2 dx - \frac{1}{2} \left( \int_{\tau_1}^{\tau_2} |k_2(s)|\, ds \right) \int_0^\pi q_x^2 dx - \frac{1}{2} \int_0^\pi \int_{\tau_1}^{\tau_2} |k_2(s)|\, z_x^2(1) ds dx \\
&+ \frac{1}{2} \int_0^\pi \int_{\tau_1}^{\tau_2} |k_2(s)|\, z_x^2(1) ds dx - \frac{1}{2} \int_0^\pi \int_{\tau_1}^{\tau_2} |k_2(s)|\, q_x^2 ds dx
\end{aligned}$$

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \geq (k_1 - \int_{\tau_1}^{\tau_2} |k_2(s)|\, ds) \int_0^\pi q_x^2 dx \geq 0,$$

which shows that $\mathcal{A}$ is monotone.
Secondly, let $G = (g^1, g^2, g^3, g^4, g^5, g^6, g^7) \in \mathcal{H}$ and find $U \in D\left(\mathcal{A}\right)$ such that

$$U + \mathcal{A}U {=} G. \tag{3.13}$$

That is,

$$\begin{cases} \varphi - w = & g^1 \\ \rho w - \mu\varphi_{xx} - b\psi_x = & \rho g^2 \\ \psi - v = & g^3 \\ Jv - \delta\psi_{xx} + b\varphi_x + \xi\psi + \beta q_x = & Jg^4 \\ \theta - q = & g^5 \\ \alpha q - \kappa\theta_{xx} + \beta v_x - k_1 q_{xx} - \int_{\tau_1}^{\tau_2} k_2(s) z_{xx}(1,s)ds = & \alpha g^6 \\ z + z_\omega = & g^7 \end{cases} \tag{3.14}$$

From the first, the third and the fifth equations of (3.14), we have

$$w = \varphi - g^1,\ v = \psi - g^3,\ q = \theta - g^5. \tag{3.15}$$

By solving the seventh equation of (3.14), we get

$$z\left(x, \omega, s, t\right) = \tilde{C}e^{-\omega} + \int_0^\omega e^{(\sigma-\omega)} g^7(\sigma)d\sigma \tag{3.16}$$

Using the fact that $z(0, s) = q$, we infer

$$\tilde{C} = q$$

and

$$z(\omega, s) = qe^{-\omega} + \int_0^\omega e^{(\sigma-\omega)} g^7(\sigma)d\sigma,$$

Thus, (3.15) yields

$$z(\omega, s) = (\theta - g^5)e^{-\omega} + \int_0^\omega e^{(\sigma-\omega)} g^7(\sigma)d\sigma. \tag{3.17}$$

Next, plugging (3.15) and (3.17) into $(3.14)_2$, $(3.14)_4$ and $(3.14)_6$, we obtain

$$\begin{cases} -\mu\varphi_{xx} + \rho\varphi - b\psi_x = & h_1, \\ -\delta\psi_{xx} + b\varphi_x + (\xi + J)\psi + \beta\theta_x = & h_2, \\ -\left(\kappa + k_1 + \displaystyle\int_{\tau_1}^{\tau_2} e^{-1} k_2(s)ds\right)\theta_{xx} + \alpha\theta + \beta\psi_x = & h_3, \end{cases} \tag{3.18}$$

where

$$h_1 = \rho(g^2 + g^1) \in L^2(0, \pi), \tag{3.19}$$

$$h_2 = J(g^4 + g^3) + \beta g_x^5 \in L_*^2(0, \pi) \tag{3.20}$$

and

$$\begin{aligned} h_3 =& \alpha(g^6+g^5)+\beta g_x^3-\left(-k_1+\int_{\tau_1}^{\tau_2} e^{-1}k_2(s)ds\right) g_{xx}^5 \\ &+\int_0^1\int_{\tau_1}^{\tau_2} k_2(s)e^{(\sigma-1)}g_{xx}^7(\sigma,s)dsd\sigma \in H^{-1}(0,\pi), \end{aligned}$$

where $H^{-1}(0,\pi)$ is the dual space of $H_0^1(0,\pi)$.

We consider on $\mathcal{V}=H_0^1(0,\pi)\times H_*^1(0,\pi)\times H_0^1(0,\pi)$, the variational formulation

$$B((\varphi,\psi,\theta),(\widetilde{\varphi},\widetilde{\psi},\widetilde{\theta}))=F(\widetilde{\varphi},\widetilde{\psi},\widetilde{\theta}), \tag{3.21}$$

where $B$ is the bilinear form over $\mathcal{V}=H_0^1(0,\pi)\times H_*^1(0,\pi)\times H_0^1(0,\pi)$, defined by

$$\begin{aligned} B((\varphi,\psi,\theta),(\tilde{\varphi},\tilde{\psi},\tilde{\theta})) =& \mu\int_0^\pi \varphi_x\widetilde{\varphi}_x dx+\rho\int_0^\pi \varphi\widetilde{\varphi}dx+b\int_0^\pi\left(\psi\widetilde{\varphi}_x+\varphi_x\widetilde{\psi}\right)dx \\ &+\delta\int_0^\pi \psi_x\widetilde{\psi}_x dx+(\xi+J)\int_0^\pi \psi\widetilde{\psi}dx+\beta\int_0^\pi(\theta_x\widetilde{\psi}+\psi_x\widetilde{\theta})dx \\ &+\left(\kappa+k_1+\int_{\tau_1}^{\tau_2} e^{-1}k_2(s)ds\right)\int_0^\pi \theta_x\widetilde{\theta}_x dx+\alpha\int_0^\pi \theta\widetilde{\theta}dx \end{aligned}$$

and $F$ is the linear form over $\mathcal{V}$,given by

$$F(\widetilde{\varphi},\widetilde{\psi},\widetilde{\theta})=\int_0^\pi h_1\widetilde{\varphi}dx+\int_0^\pi h_2\widetilde{\psi}dx+\langle h_3,\widetilde{\theta}\rangle_{H^{-1}\times H_0^1},$$

where $\langle\cdot,\cdot\rangle_{H^{-1}\times H_0^1}$ is the $H^{-1}\times H_0^1$ pairing.

The space $\mathcal{V}$ is equipped with an equivalent norm to

$$\|(\varphi,\psi,\theta)\|_{\mathcal{V}}^2=\|\varphi_x\|_2^2+\|\psi\|_2^2+\|\psi_x\|_2^2+\|\theta_x\|_2^2.$$

Easily one can check that $B$ and $F$ are bounded.

Further, we have

$$\begin{aligned} B((\varphi,\psi,\theta),(\varphi,\psi,\theta)) =& \mu\int_0^\pi \varphi_x^2dx+\rho\int_0^\pi \varphi^2dx+2b\int_0^\pi \varphi_x\psi dx \\ &+(\xi+J)\int_0^\pi \psi^2dx+\delta\int_0^\pi \psi_x^2dx+\alpha\int_0^\pi \theta^2dx \\ &+\left(\kappa+k_1+\int_{\tau_1}^{\tau_2} e^{-1}k_2(s)ds\right)\int_0^\pi \theta_x^2dx, \end{aligned}$$

Young's inequality yields

$$\begin{aligned} B((\varphi,\psi,\theta),(\varphi,\psi,\theta)) \geq& \rho\|\varphi\|^2+(\mu-\frac{b^2}{\xi})\|\varphi_x\|^2+J\|\psi\|^2+\delta\|\psi_x\|^2 \\ &+\alpha\|\theta\|^2+\left(\kappa+k_1+\int_{\tau_1}^{\tau_2} e^{-1}k_2(s)ds\right)\|\theta_x\|^2. \end{aligned}$$

Thus, there exists $M > 0$, such that

$$B((\varphi, \psi, \theta), (\varphi, \psi, \theta)) \geq M \left\| (\varphi, \psi, \theta) \right\|_{\mathcal{V}}^2,$$

which shows that $B$ is coercive. Thus, the Lax-Milgram theorem ensures that there is a unique solution $(\varphi, \psi, \theta)$ to the system (3.18), such that

$$\varphi \in H_0^1(0,\pi),\ \psi \in H_*^1(0,\pi),\ \theta \in H_0^1(0,\pi).$$

Substituting $\varphi, \psi$ and $\theta$ into (3.15), we get

$$w \in H_0^1(0,\pi), v \in H_*^1(0,\pi), q \in H_0^1(0,\pi). \tag{3.22}$$

Moreover, from (3.17) we entail that

$$z \in L^2((0,1); H_0^1(0,\pi)) \tag{3.23}$$

and

$$z_\omega = g^7 - z \in L^2((0,1); H_0^1(0,\pi)). \tag{3.24}$$

Consequently,

$$z \in H^1((0,1); H_0^1(0,\pi)).$$

Next, replacing $(\widetilde{\psi}, \widetilde{\theta}) \equiv 0 \in H_*^1(0,\pi) \times H_0^1(0,\pi)$ into (3.21), and using integration by parts, we obtain

$$\mu \int_0^\pi \varphi_x \tilde{\varphi}_x dx = -\int_0^\pi (\rho\varphi - b\psi_x - h_1)\, \widetilde{\varphi} dx,\ \forall \widetilde{\varphi} \in H_0^1(0,\pi). \tag{3.25}$$

Thus,

$$\varphi \in H^2(0,\pi) \tag{3.26}$$

Integrating the left-hand side of (3.25) by parts, we arrive at

$$-\mu \int_0^\pi \varphi_{xx} \widetilde{\varphi} dx = -\int_0^\pi (\rho\varphi - b\psi_x - h_1)\, \widetilde{\varphi} dx,\ \forall \widetilde{\varphi} \in H_0^1(0,\pi)$$
$$\int_0^\pi (\mu\varphi_{xx} - \rho\varphi + b\psi_x + h_1)\, \widetilde{\varphi} dx = 0,\ \forall \widetilde{\varphi} \in H_0^1(0,\pi)$$

the elliptic regularity theory implies that

$$\mu\varphi_{xx} - \rho\varphi + b\psi_x + h_1 = 0.$$

Moreover, from (3.15) and (3.19), we obtain

$$-\mu\varphi_{xx} + \rho w - b\psi_x = \rho g^2,$$

which solves $(3.14)_2$.

Similarly, taking $(\widetilde{\varphi}, \widetilde{\theta}) \equiv 0 \in H_0^1(0,\pi) \times H_0^1(0,\pi)$, and using integration by parts and (3.15), we obtain

$$\delta \int_0^\pi \psi_x \widetilde{\psi}_x dx = \int_0^\pi \left(h_2 - (J+\xi)\,\psi - b\varphi_x - \beta\theta_x\right) \widetilde{\psi} dx, \ \forall \widetilde{\psi} \in H_*^1(0,\pi). \tag{3.27}$$

Here, we can't use the regularity theorem directly, because $\widetilde{\psi} \in H_*^1(0,\pi)$. Let $\Psi \in H_0^1\,(0,\pi)$ and set

$$\widetilde{\Psi}\,(x,t) = \Psi\,(x,t) - \frac{1}{\pi} \int_0^\pi \Psi\,(x,t)\,dx.$$

Clearly $\widetilde{\Psi} \in H_*^1\,(0,\pi)$. Plugging $\widetilde{\Psi}$ into (3.27) and recalling that

$$h_2 - (J+\xi)\,\psi - b\varphi_x - \beta\theta_x \in L_*^2\,(0,\pi)\,,$$

we get

$$\delta \int_0^\pi \psi_x \Psi_x dx = \int_0^\pi \left(h_2 - (J+\xi)\,\psi - b\varphi_x - \beta\theta_x\right) \Psi dx, \ \forall \Psi \in H_0^1(0,\pi), \tag{3.28}$$

which means that $\psi \in H^2\,(0,\pi)$, with

$$\psi_{xx} = \frac{-1}{\delta}\left(h_2 - (J+\xi)\,\psi - b\varphi_x - \beta\theta_x\right) = \eta.$$

From (3.15) and (3.20) we get

$$\delta\psi_{xx} - b\varphi_x - Jv - \xi\psi - \beta q_x = Jg^4$$

and then, $\psi$ solves $(3.14)_4$. Moreover, we have, from (3.27)

$$\int_0^\pi \psi_{xx} \widetilde{\psi} dx = \int_0^\pi \eta \widetilde{\psi}, \, \forall \widetilde{\psi} \in H_*^1(0,\pi).$$

Integration by parts yields

$$\psi_x\,(\pi)\,\widetilde{\psi}\,(\pi) - \psi_x\,(0)\,\widetilde{\psi}\,(0) - \int_0^\pi \psi_x \widetilde{\psi}_x dx = \int_0^\pi \eta \widetilde{\psi}, \, \forall \widetilde{\psi} \in H_*^1(0,\pi).$$

Using (3.28) we arrive at

$$\psi_x(\pi)\,\widetilde{\psi}(\pi) - \psi_x(0)\,\widetilde{\psi}(0) = 0,\ \forall\widetilde{\psi} \in H_*^1(0,\pi),$$

Since $\widetilde{\psi} \in H_*^1(0,\pi)$ is arbitrary. So,

$$\psi_x(\pi) = \psi_x(0) = 0,$$

and then $\psi \in H_*^2(0,\pi)$. Therefore,

$$\psi \in H_*^2(0,\pi) \cap H_*^1(0,\pi). \tag{3.29}$$

Finally, taking $\left(\widetilde{\varphi}, \widetilde{\psi}\right) \equiv 0 \in H_0^1(0,\pi) \times H_*^1(0,\pi)$ in (3.21), replacing $g^5$ by $\theta - q$ and recalling that

$$z(1,s) = qe^{-1} + \int_0^1 e^{(\sigma-1)} g^7(\sigma,.)d\sigma,$$

we obtain

$$\int_0^\pi \left[\kappa\theta_x + k_1 q_x + \int_{\tau_1}^{\tau_2} k_2(s) z_x(1,s)\,ds\right]\widetilde{\theta}_x dx = \int_0^\pi \left(\alpha g^6 - \alpha q - \beta v_x\right)\widetilde{\theta} dx, \quad \forall\tilde{\theta} \in H_0^1(0,\pi) \tag{3.30}$$

the elliptic regularity theory implies that

$$\kappa\theta + k_1 q + \int_{\tau_1}^{\tau_2} k_2(s) z(1,s)\,ds \in H^2(0,\pi) \tag{3.31}$$

Integrating the left-hand side of (3.30) by parts, we arrive at

$$\int_0^\pi \left(\kappa\theta_{xx} + k_1 q_{xx} + \int_{\tau_1}^{\tau_2} k_2(s) z_{xx}(1,s)\,ds + \alpha g^6 - \alpha q - \beta v_x\right)\tilde{\theta} dx = 0 \quad \forall\tilde{\theta} \in H_0^1(0,\pi)$$

Thus,

$$-\kappa\theta_{xx} - k_1 q_{xx} - \int_{\tau_1}^{\tau_2} k_2(s) z_{xx}(1,s)\,ds + \alpha q + \beta v_x = \alpha g^6$$

and $(3.14)_6$ is satisfied. Combining (3.22), (3.23), (3.24), (3.26), (3.29) and (3.31), we conclude that the solution $U$ belongs to $D(\mathcal{A})$. As a result, the operator $\mathcal{A}$ is maximal. Then, the result of Theorem 28, follows immediately.

## 3.3 Exponential stability

In this section, we prove the exponential stability of the solution $(\varphi, \psi, \theta, z)$ to the system (3.5), provided that

$$\chi = \frac{\mu}{\rho} - \frac{\delta}{J} = 0. \tag{3.32}$$

The main tools used is the proof are the multiplier method and Lyapunov functionals. First, we have this result.

**Lemma 29.** *Assume that*

$$\int_{\tau_1}^{\tau_2} |k_2(s)|\, ds \leq k_1, \tag{3.33}$$

*then, the energy $E(t)$ satisfies along the solution $(\varphi, \psi, \theta, z)$ of the system (3.5) the estimate*

$$\frac{dE(t)}{dt} \leq -\left(k_1 - \int_{\tau_1}^{\tau_2} |k_2(s)|\, ds\right) \int_0^\pi \theta_{tx}^2 dx \leq 0. \tag{3.34}$$

*Proof.* Multiplying the first three equations of system (3.5) by $\varphi_t, \psi_t$ and $\theta_t$ respectively, then integrating with respect to $x$ over $(0, \pi)$, we obtain

$$\frac{\rho}{2}\frac{d}{dt}\int_0^\pi \varphi_t^2 dx + \frac{\mu}{2}\frac{d}{dt}\int_0^\pi \varphi_x^2 dx - b\int_0^\pi \psi_x \varphi_t dx = 0, \tag{3.35}$$

$$\frac{J}{2}\frac{d}{dt}\int_0^\pi \psi_t^2 dx + \frac{\delta}{2}\frac{d}{dt}\int_0^\pi \psi_x^2 dx + b\int_0^\pi \varphi_x \psi_t dx + \frac{\xi}{2}\frac{d}{dt}\int_0^\pi \psi^2 dx + \beta\int_0^\pi \theta_{tx}\psi_t dx = 0, \tag{3.36}$$

and

$$\begin{aligned}\frac{\alpha}{2}\frac{d}{dt}\int_0^\pi \theta_t^2 dx + \frac{\kappa}{2}\frac{d}{dt}\int_0^\pi \theta_x^2 dx - \beta\int_0^\pi \psi_t \theta_{tx} dx + k_1\int_0^\pi \theta_{tx}^2 dx \\ + \int_0^\pi \theta_{tx}\int_{\tau_1}^{\tau_2} k_2(s)\, z_x(1, s)\, ds dx = 0.\end{aligned}$$

On the other hand, multiplying the last equation of system (3.5) by $|k_2(s)|\, z_{xx}$ and integrating with respect to $(x, \omega, s)$ over $(0, \pi) \times (0, 1) \times (\tau_1, \tau_2)$ respectively, we get

$$\begin{aligned}&\int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} s\, |k_2(s)|\, z_t(w, s) z_{xx}(w, s)\, (\omega, s)\, ds d\omega dx \\ &+ \int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} z_\omega(w, s)\, |k_2(s)|\, z_{xx}(\omega, s)\, ds d\omega dx = 0,\end{aligned}$$

which gives

$$\frac{1}{2}\frac{d}{dt}\int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} s\, |k_2(s)|\, z_x^2(\omega, s)\, ds d\omega dx = -\frac{1}{2}\int_0^\pi \int_0^1 \frac{d}{d\omega}\int_{\tau_1}^{\tau_2} |k_2(s)|\, z_x^2(\omega, s)\, ds d\omega dx = 0$$

Thus, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} s\left|k_2(s)\right| z_x^2\left(\omega,s\right)dsd\omega dx \\
&= -\frac{1}{2}\int_0^\pi\int_{\tau_1}^{\tau_2}\left|k_2(s)\right| z_x^2\left(1,s\right)dsdx+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|k_2(s)\right|ds\right)\int_0^\pi\theta_{tx}^2dx.
\end{aligned}
\tag{3.37}
$$

Adding (3.35)-(3.37) , we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^\pi\left(\rho\varphi_t^2+J\psi_t^2+\alpha\theta_t^2+\mu\varphi_x^2+\xi\psi^2+2b\varphi_x\psi+\delta\psi_x^2+\kappa\theta_x^2\right)dx \\
&+\frac{1}{2}\frac{d}{dt}\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} s\left|k_2(s)\right| z_x^2\left(\omega,s\right)dsd\omega dx=-k_1\int_0^\pi\theta_{tx}^2dx \\
&-\int_0^\pi\theta_{tx}\int_{\tau_1}^{\tau_2}k_2\left(s\right)z_x\left(1,s\right)dsdx-\frac{1}{2}\int_0^\pi\int_{\tau_1}^{\tau_2}\left|k_2(s)\right|z_x^2\left(1,s\right)dsdx \\
&+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|k_2(s)\right|ds\right)\int_0^\pi\theta_{tx}^2dx.
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
\frac{dE\left(t\right)}{dt}=&-\left(k_1-\frac{1}{2}\int_{\tau_1}^{\tau_2}\left|k_2(s)\right|ds\right)\int_0^\pi\theta_{tx}^2dx-\int_0^\pi\theta_{tx}\int_{\tau_1}^{\tau_2}k_2(s)z_x\left(1,s\right)dsdx \\
&-\frac{1}{2}\int_0^\pi\int_{\tau_1}^{\tau_2}\left|k_2(s)\right|z_x^2(1,s)dsdx.
\end{aligned}
$$

Using Cauchy-Schwarz's and Young's inequalities to the second term on the right-hand side yield

$$
\begin{aligned}
-\int_0^\pi\theta_{tx}\int_{\tau_1}^{\tau_2}k_2(s)z_x\left(1,s\right)dsdx\leq&\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|k_2(s)\right|ds\right)\int_0^\pi\theta_{tx}^2dx \\
&+\frac{1}{2}\int_0^\pi\int_{\tau_1}^{\tau_2}\left|k_2(s)\right|z_x^2\left(1,s\right)dsdx
\end{aligned}
$$

and (3.34) follows immediately. □

The main result of this Section reads as follows:

**Theorem 30.** Suppose that (3.32) is true and let $(\varphi,\psi,\theta,z)$ be the solution of problem (3.5)–(3.7*). Then there exist* $A>0$ *and* $\eta>0$*, for which the energy functional (*3.8*)* satisfies the estimate

$$
E\left(t\right)\leq Ae^{-\eta t},\ \forall t\geq 0. \tag{3.38}
$$

To prove Theorem 30, we need to establish some essential lemmas.

**Lemma 31.** *The functional*

$$\mathcal{G}_1(t) := J\int_0^\pi \psi\psi_t dx - \frac{\rho b}{\mu}\int_0^\pi \varphi_t \int_0^x \psi(y)\,dydx,$$

*satisfies, along the solution of the problem (3.5)–(3.7) and for any $\varepsilon_1 > 0$, the estimate*

$$\begin{aligned}\mathcal{G}_1'(t) \le & -\frac{\delta}{2}\int_0^\pi \psi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^\pi \psi^2 dx + \varepsilon_1 \int_0^\pi \varphi_t^2 dx + \frac{\beta^2 c_p^2}{2\delta}\int_0^\pi \theta_{tx}^2 dx\\ & + \left(J + \frac{c}{\varepsilon_1}\right)\int_0^\pi \psi_t^2 dx,\end{aligned} \tag{3.39}$$

*where henceforth, $c$ is a generic positive constant.*

*Proof.* Direct differentiation of $\mathcal{G}_1(t)$, using $(3.5)_2$, $(3.5)_1$ and integration by parts, yields

$$\begin{aligned}J\frac{d}{dt}\int_0^\pi \psi\psi_t dx =& J\int_0^\pi \psi_t^2 dx + J\int_0^\pi \psi\psi_{tt} dx\\ =& J\int_0^\pi \psi_t^2 dx - \delta\int_0^\pi \psi_x^2 dx - b\int_0^\pi \varphi_x\psi dx - \xi\int_0^\pi \psi^2 dx + \beta\int_0^\pi \theta_t\psi_x dx.\end{aligned} \tag{3.40}$$

Similarly,

$$\begin{aligned}-\frac{\rho b}{\mu}\frac{d}{dt}\int_0^\pi \varphi_t\int_0^x \psi(y)\,dydx =& -\frac{\rho b}{\mu}\int_0^\pi \varphi_{tt}\int_0^x \psi(y)\,dydx - \frac{\rho b}{\mu}\int_0^\pi \varphi_t\int_0^x \psi_t(y)\,dydx\\ =& b\int_0^\pi \varphi_x\psi dx + \frac{b^2}{\mu}\int_0^\pi \psi^2 dx - \frac{\rho b}{\mu}\int_0^\pi \varphi_t\int_0^x \psi_t(y)\,dydx.\end{aligned} \tag{3.41}$$

The addition of (3.40) and (3.41) yields

$$\begin{aligned}\mathcal{G}_1'(t) =& -\delta\int_0^\pi \psi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^\pi \psi^2 dx + J\int_0^\pi \psi_t^2 dx\\ & + \beta\int_0^\pi \theta_t\psi_x dx - \frac{\rho b}{\mu}\int_0^\pi \varphi_t\int_0^x \psi_t(y)\,dydx.\end{aligned}$$

Consequently, (3.39) follows from Poincaré's, Cauchy-Schwarz and Young's inequalities. □

**Lemma 32.** *The functional*

$$\mathcal{G}_2(t) := -\alpha\int_0^\pi \theta_t\int_0^x \psi_t(y)\,dydx$$

*satisfies, along the solution of the problem (3.5)–(3.7), and for any $\varepsilon_2 > 0$, the estimate*

$$\begin{aligned}\mathcal{G}_2'(t) \le & -\frac{\beta}{2}\int_0^\pi \psi_t^2 dx + c\left(1 + \frac{1}{\varepsilon_2}\right)\int_0^\pi \theta_{tx}^2 dx + c\varepsilon_2\int_0^\pi \left(\varphi_x^2 + \psi_x^2\right)dx\\ & + \frac{3\kappa^2}{2\beta}\int_0^\pi \theta_x^2 dx + \frac{3k_1}{2\beta}\int_0^\pi\int_{\tau_1}^{\tau_2} |k_2(s)| z_x^2(1,s)\,dsdx.\end{aligned} \tag{3.42}$$

*Proof.* The differentiation of $\mathcal{G}_2(t)$ and the use of $(3.5)_2$, $(3.5)_3$ and (3.33) lead to

$$\begin{aligned}
\mathcal{G}_2'(t) =& -\alpha\int_0^\pi \theta_{tt}\int_0^x \psi_t(y)\,dydx - \alpha\int_0^\pi \theta_t\int_0^x \psi_{tt}(y)\,dydx\\
=& -\kappa\int_0^\pi \theta_{xx}\left(\int_0^x \psi_t(y)\,dy\right)dx + \beta\int_0^\pi \psi_{tx}\left(\int_0^x \psi_t(y)\,dy\right)dx\\
& - k_1\int_0^\pi \theta_{txx}\left(\int_0^x \psi_t(y)\,dy\right)dx - \int_0^\pi\int_{\tau_1}^{\tau_2} k_2(s)z_{xx}(1,s)ds\left(\int_0^x \psi_t(y)\,dy\right)dx\\
& - \alpha\int_0^\pi \theta_t\int_0^x \psi_{tt}(y)\,dydx\\
=&\kappa\int_0^\pi \theta_x\psi_t dx - \beta\int_0^\pi \psi_t^2 dx + k_1\int_0^\pi \theta_{tx}\psi_t dx + \int_0^\pi \psi_t\int_{\tau_1}^{\tau_2} k_2(s)z_x(1,s)dsdx\\
& - \alpha\int_0^\pi \theta_t\int_0^x \psi_{tt}(y)\,dydx.
\end{aligned}$$

We now estimate the terms in the right-hand side of the above identity. Using and Cauchy-Schwarz's and Young's inequalities, (3.33) and calculations as in (2.41), we find

$$\kappa\int_0^\pi \theta_x\psi_t dx \le \frac{\beta}{6}\int_0^\pi \psi_t^2 dx + \frac{3\kappa^2}{2\beta}\int_0^\pi \theta_x^2 dx,$$

$$k_1\int_0^\pi \theta_{tx}\psi_t dx \le \frac{\beta}{6}\int_0^\pi \psi_t^2 dx + \frac{3k_1^2}{2\beta}\int_0^\pi \theta_{tx}^2 dx,$$

$$\begin{aligned}
\int_0^\pi \psi_t\int_{\tau_1}^{\tau_2} k_2(s)z_x(1,s)dsdx \le& \frac{\beta}{6}\int_0^\pi \psi_t^2 dx + \frac{3}{2\beta}\int_0^\pi\left(\int_{\tau_1}^{\tau_2} k_2(s)z_x(1,s)ds\right)^2 dx\\
\le& \frac{\beta}{6}\int_0^\pi \psi_t^2 dx + \frac{3}{2\beta}\int_{\tau_1}^{\tau_2}|k_2(s)|ds\int_0^\pi\int_{\tau_1}^{\tau_2}|k_2(s)|z_x^2(1,s)dsdx\\
\le& \frac{\beta}{6}\int_0^\pi \psi_t^2 dx + \frac{3k_1}{2\beta}\int_0^\pi\int_{\tau_1}^{\tau_2}|k_2(s)|z_x^2(1,s)dsdx,
\end{aligned}$$

and, for any $\varepsilon_2 > 0$, we infer

$$\begin{aligned}
& -\alpha\int_0^\pi \theta_t\int_0^x \psi_{tt}(y)\,dydx = \frac{-\alpha}{J}\int_0^\pi \theta_t\left(\int_0^x (\delta\psi_{yy} - b\varphi_y - \xi\psi - \beta\theta_{ty})\,dy\right)dx\\
=& -\frac{\alpha\delta}{J}\int_0^\pi \theta_t\psi_x dx + \frac{\alpha b}{J}\int_0^\pi \theta_t\varphi dx + \frac{\alpha\xi}{J}\int_0^\pi \theta_t\int_0^x \psi(y)dydx + \frac{\alpha\beta}{J}\int_0^\pi \theta_t^2 dx\\
\le& \varepsilon_2\int_0^\pi \psi_x^2 dx + \frac{c}{\varepsilon_2}\int_0^\pi \theta_t^2 dx + \varepsilon_2\int_0^\pi \varphi^2 dx + \frac{c}{\varepsilon_2}\int_0^\pi \theta_t^2 dx + \varepsilon_2\int_0^\pi \psi^2 dx\\
& + \frac{c}{\varepsilon_2}\int_0^\pi \theta_t^2 dx + \frac{\alpha\beta}{J}\int_0^\pi \theta_t^2 dx
\end{aligned}$$

Then, by the above estimate and Poincaré's inequality , we obtain

$$\begin{aligned}
\mathcal{G}_2'(t) \leq & -\frac{\beta}{2}\int_0^{\pi}\psi_t^2dx + \left(\frac{\alpha\beta}{J}+\frac{c}{\varepsilon_2}\right)\int_0^{\pi}\theta_t^2dx + \varepsilon_2\int_0^{\pi}\psi^2dx + \varepsilon_2\int_0^{\pi}\psi_x^2dx + \varepsilon_2\int_0^{\pi}\varphi^2dx \\
& + \frac{3k_1^2}{2\beta}\int_0^{\pi}\theta_{tx}^2dx + \frac{3\kappa^2}{2\beta}\int_0^{\pi}\theta_x^2dx + \frac{3k_1}{2\beta}\int_0^{\pi}\int_{\tau_1}^{\tau_2}|k_2(s)|z_x^2(1,s)dsdx \\
\leq & -\frac{\beta}{2}\int_0^{\pi}\psi_t^2dx + c\left(1+\frac{1}{\varepsilon_2}\right)\int_0^{\pi}\theta_{tx}^2dx + c\varepsilon_2\int_0^{\pi}\left(\varphi_x^2+\psi_x^2\right)dx + \frac{3\kappa^2}{2\beta}\int_0^{\pi}\theta_x^2dx \\
& + \frac{3k_1}{2\beta}\int_0^{\pi}\int_{\tau_1}^{\tau_2}|k_2(s)|z_x^2(1,s)dsdx.
\end{aligned}$$

□

**Lemma 33.** *Assume that (3.32) is true, then the functional*

$$\mathcal{G}_3(t) := \frac{\mu}{\rho}\int_0^{\pi}\psi_t\varphi_xdx + \frac{\delta}{J}\int_0^{\pi}\psi_x\varphi_tdx,$$

*satisfies, along the solution of the problem (3.5)–(3.7), and for any $\varepsilon_3 > 0$, the estimate*

$$\mathcal{G}_3'(t) \leq -\frac{b\mu}{2\rho J}\int_0^{\pi}\varphi_x^2dx + \frac{\mu\xi^2}{\rho Jb}\int_0^{\pi}\psi^2dx + \frac{\beta^2\mu}{\rho Jb}\int_0^{\pi}\theta_{tx}^2dx + \frac{\delta b}{\rho J}\int_0^{\pi}\psi_x^2dx. \tag{3.43}$$

*Proof.* Direct differentiation of $\mathcal{G}_3(t)$, using $(3.5)_2$ and the integration by parts, yields

$$\begin{aligned}
\frac{\mu}{\rho}\frac{d}{dt}\int_0^{\pi}\psi_t\varphi_xdx =& \frac{\mu}{\rho}\int_0^{\pi}\psi_{tt}\varphi_xdx + \frac{\mu}{\rho}\int_0^{\pi}\psi_t\varphi_{tx}dx \\
=& \frac{\mu\delta}{\rho J}\int_0^{\pi}\varphi_x\psi_{xx}dx - \frac{b\mu}{\rho J}\int_0^{\pi}\varphi_x^2dx - \frac{\mu\xi}{\rho J}\int_0^{\pi}\varphi_x\psi dx \\
& - \frac{\beta\mu}{\rho J}\int_0^{\pi}\varphi_x\theta_{tx}dx + \frac{\mu}{\rho}\int_0^{\pi}\psi_t\varphi_{xt}dx.
\end{aligned} \tag{3.44}$$

Similarly, using $(3.5)_1$, we obtain

$$\begin{aligned}
\frac{\delta}{J}\frac{d}{dt}\int_0^{\pi}\psi_x\varphi_tdx =& \frac{\delta}{J}\int_0^{\pi}\psi_{tx}\varphi_tdx + \frac{\delta}{J}\int_0^{\pi}\psi_x\varphi_{tt}dx \\
=& \frac{\delta}{J}\int_0^{\pi}\psi_{xt}\varphi_tdx - \frac{\delta\mu}{\rho J}\int_0^{\pi}\varphi_x\psi_{xx}dx + \frac{\delta b}{J\rho}\int_0^{\pi}\psi_x^2dx.
\end{aligned} \tag{3.45}$$

Summing up (3.44) - (3.45), yields

$$\begin{aligned}
\mathcal{G}_3'(t) =& -\frac{b\mu}{\rho J}\int_0^{\pi}\varphi_x^2dx - \frac{\mu\xi}{\rho J}\int_0^{\pi}\varphi_x\psi dx - \frac{\beta\mu}{\rho J}\int_0^{\pi}\theta_{tx}\varphi_xdx \\
& + \frac{\delta b}{\rho J}\int_0^{\pi}\psi_x^2dx - \chi\int_0^{\pi}\psi_{xt}\varphi_tdx.
\end{aligned} \tag{3.46}$$

Considering (3.32) and by using Young's inequality, the estimate (3.43) follows. □

**Lemma 34.** The functional

$$\mathcal{G}_4(t) := -\rho \int_0^\pi \varphi_t \varphi dx$$

satisfies, along the solution of problem (3.5)–(3.7), the estimate

$$\mathcal{G}_4'(t) \leq -\rho \int_0^\pi \varphi_t^2 dx + \frac{3\mu}{2} \int_0^\pi \varphi_x^2 dx + c \int_0^\pi \psi_x^2 dx. \tag{3.47}$$

*Proof.* Direct computations, using $(3.5)_1$, gives

$$\begin{aligned}
\mathcal{G}_4'(t) =& -\rho \int_0^\pi \varphi_{tt}\varphi dx - \rho \int_0^\pi \varphi_t^2 dx \\
=& -\mu \int_0^\pi \varphi_{xx}\varphi dx - b \int_0^\pi \psi_x \varphi dx - \rho \int_0^\pi \varphi_t^2 dx \\
=& \mu \int_0^\pi \varphi_x^2 dx + b \int_0^\pi \psi \varphi_x dx - \rho \int_0^\pi \varphi_t^2 dx
\end{aligned}$$

Then, the use of Poincaré's and Young's inequalities leads to

$$\begin{aligned}
\mathcal{G}_4'(t) \leq& -\rho \int_0^\pi \varphi_t^2 dx + \mu \int_0^\pi \varphi_x^2 dx + \frac{\mu}{2} \int_0^\pi \varphi_x^2 dx + \frac{b^2}{2\mu} \int_0^\pi \psi^2 dx \\
\leq& -\rho \int_0^\pi \varphi_t^2 dx + \frac{3\mu}{2} \int_0^\pi \varphi_x^2 dx + c \int_0^\pi \psi_x^2 dx.
\end{aligned}$$

□

**Lemma 35.** The functional

$$\mathcal{G}_5(t) := \alpha \int_0^\pi \theta\theta_t dx + \frac{k_1}{2} \int_0^\pi \theta_x^2 dx + \beta \int_0^\pi \psi_x \theta dx$$

satisfies, along the solution of problem (3.5)–(3.7), and for any $\varepsilon_2 > 0$, the estimate

$$\begin{aligned}
\mathcal{G}_5'(t) \leq& -\frac{\kappa}{2} \int_0^\pi \theta_x^2 dx + \varepsilon_2 \int_0^\pi \psi_x^2 dx + \left(\alpha + \frac{\beta^2}{4\varepsilon_2}\right) c_p^2 \int_0^\pi \theta_{tx}^2 dx \\
&+ \frac{k_1}{2\kappa} \int_0^\pi \int_{\tau 1}^{\tau_2} |k_2(s)| z_x^2(1,s)\, ds dx.
\end{aligned} \tag{3.48}$$

*Proof.* Direct differentiation of $\mathcal{G}_5(t)$ and the use of $(3.5)_3$ lead to

$$\begin{aligned}
\mathcal{G}_5'(t) =& \alpha \int_0^\pi \theta_t^2 dx + \alpha \int_0^\pi \theta\theta_{tt} dx + k_1 \int_0^\pi \theta_x \theta_{tx} dx + \beta \int_0^\pi \psi_{xt}\theta dx + \beta \int_0^\pi \psi_x \theta_t dx \\
=& \alpha \int_0^\pi \theta_t^2 dx + \kappa \int_0^\pi \theta_{xx}\theta dx - \beta \int_0^\pi \psi_{tx}\theta dx + k_1 \int_0^\pi \theta_{txx}\theta dx \\
&+ \int_0^\pi \int_{\tau_1}^{\tau_2} k_2(s) z_{xx}(1,s)\, \theta ds dx + k_1 \int_0^\pi \theta_x \theta_{tx} dx + \beta \int_0^\pi \psi_{xt}\theta dx + \beta \int_0^\pi \psi_x \theta_t dx \\
=& -\kappa \int_0^\pi \theta_x^2 dx + \alpha \int_0^\pi \theta_t^2 dx + \beta \int_0^\pi \psi_x \theta_t dx - \int_0^\pi \theta_x \int_{\tau_1}^{\tau_2} k_2(s) z_x(1,s)\, ds dx.
\end{aligned}$$

Using Cauchy-Schwarz's and Young's inequalities to the last term on the right-hand side yield

$$\begin{aligned}-\int_0^\pi \theta_x \int_{\tau_1}^{\tau_2} k_2(s) z_x(1,s)\, dsdx \leq & \frac{\kappa}{2}\int_0^\pi \theta_x^2 dx + \frac{1}{2\kappa}\left(\int_{\tau_1}^{\tau_2} |k_2(s)| ds\right)\int_0^\pi \int_{\tau_1}^{\tau_2} |k_2(s)| z_x^2(1,s) dsdx \\ \leq & \frac{\kappa}{2}\int_0^\pi \theta_x^2 dx + \frac{k_1}{2\kappa}\int_0^\pi \int_{\tau_1}^{\tau_2} |k_2(s)| z_x^2(1,s) dsdx\end{aligned}$$

and (3.48) follows. □

**Lemma 36.** The functional

$$\mathcal{G}_6(t) := \int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\omega} |k_2(s)|\, z_x^2(\omega, s)\, dsd\omega dx.$$

satisfies

$$\begin{aligned}\mathcal{G}_6'(t) \leq & -e^{-\tau_2}\int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} s\, |k_2(s)|\, z_x^2(\omega, s)\, dsd\omega dx \\ & - e^{-\tau_2}\int_0^\pi |k_2(s)|\, z_x^2(1,s)\, dsdx + k_1\int_0^\pi \theta_{xt}^2 dx.\end{aligned} \tag{3.49}$$

*Proof.* Direct differentiation of $\mathcal{G}_6(t)$ infer to

$$\mathcal{G}_6'(t) = 2\int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\omega} |k_2(s)|\, z_x(w,s) z_{xt}(w,s) dsd\omega dx.$$

Using (3.4) we get

$$\begin{aligned}\mathcal{G}_6'(t) &= -2\int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\omega} |k_2(s)|\, z_x(w,s) z_{x\omega}(\omega,s)\, dsd\omega dx \\ &= -\int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\omega} |k_2(s)|\, \frac{dz_x^2}{d\omega}(\omega,s)\, dsd\omega dx\end{aligned} \tag{3.50}$$

By differentiating the term $e^{-s\omega} z_x^2(\omega,s)$ with respect to $\omega$, we get

$$\frac{\partial}{\partial\omega}\left(e^{-s\omega} z_x^2(\omega,s)\right) = -se^{-s\omega} z_x^2(\omega,s) + e^{-s\omega}\frac{\partial}{\partial\omega} z_x^2(\omega,s)$$

Then, by substituting into (3.50) and using the fact that $z(0,s) = \theta_t$, we obtain

$$\begin{aligned}\mathcal{G}_6'(t) =& -\int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} s\, |k_2(s)|\, e^{-s\omega} z_x^2(\omega,s)\, dsd\omega dx \\ & -\int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} |k_2(s)|\, \frac{d}{d\omega}\left(e^{-s\omega} z_x^2(\omega,s)\right) dsd\omega dx. \\ =& -\int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} s\, |k_2(s)|\, e^{-s\omega} z_x^2(\omega,s)\, dsd\omega dx - \int_0^\pi \int_{\tau_1}^{\tau_2} |k_2(s)|\, e^{-s} z_x^2(1,s)\, dsdx \\ & + \int_0^\pi \int_{\tau_1}^{\tau_2} |k_2(s)|\, \theta_{tx}^2 dsdx.\end{aligned}$$

Thus, (3.49) is satisfied. □

Now, we defined the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) := NE(t) + N_1\mathcal{G}_1(t) + N_2\mathcal{G}_2(t) + \frac{5\rho J}{b}\mathcal{G}_3(t) + \mathcal{G}_4(t) + N_3\mathcal{G}_5(t) + N_4\mathcal{G}_6(t), \quad (3.51)$$

where $N, N_1, N_2, N_3, N_4$ are positive constants to be determined appropriately.

**Lemma 37.** For $N$ sufficiently large, the functional $\mathcal{L}$ satisfies

$$m_1 E(t) \leq \mathcal{L}(t) \leq m_2 E(t), \forall t \geq 0, \quad (3.52)$$

for $m_1 > 0$ and $m_2 > 0$.

*Proof.* Let

$$\mathcal{K}(t) := \mathcal{L}(t) - NE(t) = N_1\mathcal{G}_1(t) + N_2\mathcal{G}_2(t) + \frac{5\rho J}{b}\mathcal{G}_3(t) + \mathcal{G}_4(t) + N_3\mathcal{G}_5(t) + N_4\mathcal{G}_6(t).$$

By using Poincaré's, Cauchy-Schwarz's and Young's inequalities and (3.8), we get

$$\begin{aligned}|\mathcal{K}(t)| \leq\, & cN_1\left(\int_0^\pi \psi^2 dx + \int_0^\pi \psi_t^2 dx + \int_0^\pi \varphi_t^2 dx\right) + cN_2\left(\int_0^\pi \psi_t^2 dx + \int_0^\pi \theta_t^2 dx\right)\\ & + c\frac{5\rho J}{b}\left(\int_0^\pi \varphi_x^2 dx + \int_0^\pi \psi_x^2 dx + \int_0^\pi \psi_t^2 dx + \int_0^\pi \varphi_t^2 dx\right)\\ & + c\left(\int_0^\pi \varphi_x^2 dx + \int_0^\pi \varphi_t^2 dx\right) + cN_3\left(\int_0^\pi \theta_t^2 dx + \int_0^\pi \theta_x^2 dx + \beta\int_0^\pi \psi_x^2 dx\right)\\ & + cN_4\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} s\,|k_2(s)|\, z_x^2(x,\omega,s,t)\,dsd\omega dx.\end{aligned}$$

Therefore, there is a sufficiently large $M > 0$ such that

$$\begin{aligned}|\mathcal{K}(t)| \leq & M\int_0^\pi \left(\varphi_t^2 + \psi_t^2 + \theta_t^2 + \varphi_x^2 + \psi_x^2 + \theta_x^2 + \psi^2\right.\\ & \left. + \int_0^1\int_{\tau_1}^{\tau_2} s\,|k_2(s)|\, z_x^2(x,\omega,s,t)\,dsd\omega\right) dx\\ \leq & ME(t).\end{aligned}$$

Consequently,

$$(N - M)E(t) \leq \mathcal{L}(t) \leq (N + M)E(t)$$

By selecting $N$ sufficiently large, (3.52) follows. □

**Proof of Theorem 30.** Differentiating $\mathcal{L}(t)$ and using estimates (3.34), (3.39), (3.42), (3.43), (3.47), (3.48) and (3.49), we arrive at

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left(N\left(k_1 - \int_{\tau_1}^{\tau_2} |k_2(s)|\, ds\right) - \frac{\beta^2 c_p^2 N_1}{2\delta} - cN_2\left(1 + \frac{1}{\varepsilon_2}\right)\right. \\
& -\left.\left(\alpha + \frac{\beta^2}{4\varepsilon_2}\right) c_p^2 N_3 - \frac{5\beta^2\mu}{b^2} - k_1 N_4\right) \int_0^\pi \theta_{xt}^2 dx \\
& -(\mu - c\varepsilon_2 N_2) \int_0^\pi \varphi_x^2 dx - (\rho - \varepsilon_1 N_1) \int_0^\pi \varphi_t^2 dx \\
& -\left(\left(\xi - \frac{b^2}{\mu}\right) N_1 - \frac{5\mu\xi^2}{b^2}\right) \int_0^\pi \psi^2 dx \\
& -\left(\frac{\delta}{2} N_1 - \varepsilon_2 (cN_2 + N_3) - 5\delta - c\right) \int_0^\pi \psi_x^2 dx \\
& -\left(\frac{\beta}{2} N_2 - \left(J + \frac{c}{\varepsilon_1}\right) N_1\right) \int_0^\pi \psi_t^2 dx - \left(\frac{\kappa}{2} N_3 - \frac{3\kappa^2}{2\beta} N_2\right) \int_0^\pi \theta_x^2 dx \\
& -\left(e^{-\tau_2} N_4 - \frac{3k_1}{2\beta} N_2 - \frac{k_1}{2\kappa} N_3\right) \int_0^\pi \int_{\tau_1}^{\tau_2} |k_2(s)|\, z_x^2(1, s)\, ds dx \\
& - e^{-\tau_2} N_4 \int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} s\, |k_2(s)|\, z_x^2(\omega, s)\, ds d\omega dx.
\end{aligned}$$

First, we select a sufficiently large $N_1$ so that

$$\left(\xi - \frac{b^2}{\mu}\right) N_1 - \frac{5\mu\xi^2}{b^2} > 0 \text{ and } \frac{\delta}{2} N_1 - 5\delta - c > 0.$$

Secondly, we choose a sufficiently small $\varepsilon_1$ so that

$$\rho - \varepsilon_1 N_1 > 0.$$

Next, we pick $N_2$ large such that

$$\frac{\beta}{2} N_2 - \left(J + \frac{c}{\varepsilon_1}\right) N_1 > 0.$$

Then, choosing $N_3$ such that

$$\frac{\kappa}{2} N_3 > \frac{3\kappa^2}{2\beta} N_2.$$

After that, we choose $\varepsilon_2$ small enough such that

$$\mu - c\varepsilon_2 N_2 > 0 \text{ and } \frac{\delta}{2} N_1 - 5\delta - c - \varepsilon_2 (cN_2 + N_3) > 0.$$

Then we select $N_4$ such that

$$e^{-\tau_2} N_4 - \frac{3k_1}{2\beta} N_2 - \frac{k_1}{2\kappa} N_3 > 0.$$

Finally, we choose $N$ so large such that (3.52) holds and

$$N\left(k_1-\int_{\tau_1}^{\tau_2}|k_2(s)|\,ds\right)-\frac{\beta^2c_p^2N_1}{2\delta}$$
$$-cN_2\left(1+\frac{1}{\varepsilon_2}\right)-\left(\alpha+\frac{\beta^2}{4\varepsilon_2}\right)c_p^2N_3-\frac{5\beta^2\mu}{b^2}-k_1N_4>0$$

Therefore, there exists $\lambda_0>0$, so that

$$\mathcal{L}'(t)\leq-\lambda_0E(t)\,,\ \forall t>0. \tag{3.53}$$

A combination of (3.52) and (3.53) leads

$$\mathcal{L}'(t)\leq-\eta\mathcal{L}(t)\,,\ \forall t>0, \tag{3.54}$$

for $\eta>0$. The integration of (3.54) over $(0,t)$ yields

$$\mathcal{L}(t)\leq\mathcal{L}(0)\,e^{-\eta t},\ \forall t>0,$$

By using (3.52) again, we get (3.38), This completes the proof of theorem 30.

## 3.4 Polynomial stability

In this section, we assume that (3.32) does not true and prove that the solution of (3.5)–(3.7) decays polynomially. We define the second-order energy by

$$E_*(t)=\frac{1}{2}\int_0^\pi\left(\rho\varphi_{tt}^2+J\psi_{tt}^2+\alpha\theta_{tt}^2+\delta\psi_{tx}^2+\mu\varphi_{tx}^2+\xi\psi_t^2+2b\varphi_{tx}\psi_t+\kappa\theta_{tx}^2\right)dx$$
$$+\frac{1}{2}\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2}s\,|k_2(s)|\,z_{xt}^2(\omega,s)\,dsd\omega dx.$$

Similarly to (3.34), $E_*(t)$ satisfies

$$\frac{dE_*(t)}{dt}\leq-(k_1-\int_{\tau_1}^{\tau_2}|k_2(s)|\,ds)\int_0^\pi\theta_{ttx}^2dx\leq0. \tag{3.55}$$

Our decay result reads as follows.

**Theorem 38.** *Suppose that (3.32) does not hold, that is*

$$\chi=\frac{\mu}{\rho}-\frac{\delta}{J}\neq0,$$

*and let $(\varphi,\psi,\theta,z)$ be the strong solution of (3.5)–(3.7). Then, there exists $\eta>0$ so that the energy $E(t)$ satisfies,*

$$E(t)\leq\frac{\eta}{t},\forall t>0.$$

The proof of Theorem 38 will be done through the following Lemmas.

**Lemma 39.** *Let $(\varphi, \psi, \theta, z)$ be the strong solution of (3.5)–(3.7). Then the functional*

$$\widetilde{\mathcal{G}}_3(t) := \beta \mathcal{G}_3(t) - \chi \int_0^\pi \varphi_x \left( k_1 \theta_{tx} + \kappa \theta_x + \int_{\tau_1}^{\tau_2} k_2(s) z_x(1,s)\, ds \right) dx, \tag{3.56}$$

*satisfies, for any $\varepsilon_3 > 0$, the estimate*

$$\begin{aligned}\widetilde{\mathcal{G}}_3'(t) \leq & -\frac{b\beta\mu}{2\rho J} \int_0^\pi \varphi_x^2 dx + \varepsilon_3 \int_0^\pi \varphi_t^2 dx + \frac{\delta\beta b}{\rho J} \int_0^\pi \psi_x^2 dx + c\left(1 + \frac{1}{\varepsilon_3}\right) \int_0^\pi \theta_{ttx}^2 dx \\ & + c\left( \int_0^\pi \psi^2 dx + \int_0^\pi \theta_{tx}^2 dx + \int_0^\pi \int_{\tau_1}^{\tau_2} |k_2(s)| z_{tx}^2(1,s)\, ds \right) dx. \end{aligned} \tag{3.57}$$

*Proof.* A simple differentiation of (3.56) yields

$$\begin{aligned}\widetilde{\mathcal{G}}_3'(t) = & \beta \mathcal{G}_3'(t) - \chi k_1 \int_0^\pi \varphi_{xt}\theta_{tx} dx - \chi k_1 \int_0^\pi \varphi_x \theta_{ttx} dx - \chi\kappa \int_0^\pi \varphi_{xt}\theta_x dx \\ & - \chi\kappa \int_0^\pi \varphi_x \theta_{xt} dx - \chi \int_0^\pi \varphi_{xt} \int_{\tau_1}^{\tau_2} k_2(s) z_x(1,s)\, ds dx \\ & - \chi \int_0^\pi \varphi_x \int_{\tau_1}^{\tau_2} k_2(s) z_{tx}(1,s)\, ds dx. \end{aligned} \tag{3.58}$$

Integrating by parts the second term in the right-hand side of (3.58) and exploiting $(3.5)_3$, we obtain

$$\begin{aligned}-\chi k_1 \int_0^\pi \varphi_{xt}\theta_{tx} dx = \chi k_1 \int_0^\pi \varphi_t \theta_{txx} dx = & \chi\alpha \int_0^\pi \varphi_t \theta_{tt} dx + \chi\kappa \int_0^\pi \varphi_{xt}\theta_x dx \\ & + \chi\beta \int_0^\pi \varphi_t \psi_{xt} dx + \chi \int_0^\pi \varphi_{xt} \int_{\tau_1}^{\tau_2} k_2(s) z_x(1,s)\, ds dx. \end{aligned} \tag{3.59}$$

Substituting (3.59) and (3.46) into (3.58), we get

$$\begin{aligned}\widetilde{\mathcal{G}}_3'(t) = & -\frac{\beta b\mu}{\rho J} \int_0^\pi \varphi_x^2 dx - \frac{\mu\xi\beta}{\rho J} \int_0^\pi \varphi_x \psi dx - \left( \frac{\beta^2\mu}{\rho J} + \chi\kappa \right) \int_0^\pi \varphi_x \theta_{tx} dx \\ & + \frac{\delta\beta b}{\rho J} \int_0^\pi \psi_x^2 dx + \chi\alpha \int_0^\pi \varphi_t \theta_{tt} dx \\ & - \chi \int_0^\pi \varphi_x \int_{\tau_1}^{\tau_2} k_2(s) z_{tx}(x,1,s,t)\, ds dx - \chi k_1 \int_0^\pi \varphi_x \theta_{ttx} dx. \end{aligned}$$

Consequently, (3.57) follows from Poincaré's and Young's inequalities. □

**Lemma 40.** *Let $(\varphi, \psi, \theta, z)$ be the strong solution of (3.5)–(3.7). Then the functional*

$$\mathcal{G}_7(t) := \int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\omega} |k_2(s)|\, z_{tx}^2(x,\omega,s,t)\, ds d\omega dx.$$

*satisfies*

$$\mathcal{G}_7'(t) \leq -e^{-\tau_2} \int_0^\pi \int_0^1 |k_2(s)| z_{tx}^2(1,s)\, dsdx - e^{-\tau_2} \int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} s\,|k_2(s)| z_{tx}^2(\omega,s)\, dsd\omega dx + k_1 \int_0^\pi \theta_{ttx}^2 dx. \tag{3.60}$$

*Proof.* Direct differentiation of $\mathcal{G}_7(t)$ infer to

$$\mathcal{G}_7'(t) = 2\int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\omega} |k_2(s)| z_{ttx}(\omega,s)\, z_{xt}(\omega,s)\, dsd\omega dx.$$

Using (3.4), we get

$$\begin{aligned}
\mathcal{G}_7'(t) &= -2\int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\omega} |k_2(s)| z_{tx}(\omega,s)\, z_{tx\omega}(\omega,s)\, dsd\omega dx \\
&= -\int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\omega} |k_2(s)| \frac{d}{d\omega} z_{tx}^2(\omega,s)\, dsd\omega dx \\
&= -\int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} s\,|k_2(s)|\, e^{-s\omega} z_{tx}^2(\omega,s)\, d\omega dx \\
&\quad - \int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} |k_2(s)| \frac{d}{d\omega}\left(e^{-s\omega} z_{tx}^2(\omega,s)\right) dsd\omega dx,
\end{aligned}$$

$$\mathcal{G}_7'(t) = \left(\int_{\tau_1}^{\tau_2} |k_2(s)ds|\right) \int_0^\pi \theta_{ttx}^2(x,t)\, dx - \int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} s\,|k_2(s)|\, e^{-s\omega} z_{tx}^2(\omega,s)\, d\omega dx - \int_0^\pi \int_{\tau_1}^{\tau_2} |k_2(s)|\, e^{-s} z_{tx}^2(1,s)\, dsdx.$$

Thus, (3.60) follows from (3.33). □

Now, we define the Lyapunov functional

$$\mathcal{L}_*(t) := N^*(E(t) + E_*(t)) + N_1^*\mathcal{G}_1(t) + N_2^*\mathcal{G}_2(t) + N_3^*\widetilde{\mathcal{G}}_3(t) + \mathcal{G}_4(t) + N_5^*\mathcal{G}_5(t) + N_6^*\mathcal{G}_6(t) + N_7^*\mathcal{G}_7(t).$$

Differentiating $\mathcal{L}_*(t)$ and substituting (3.34), (3.55), (3.39), (3.42), (3.47), (3.48), (3.49), (3.57) and (3.60) we reach

$$\begin{aligned}
\mathcal{L}_*'(t) \leq &-\left[N^*\left(k_1 - \int_{\tau_1}^{\tau_2} |k_2(s)|\, ds\right) - c\left(1 + \frac{1}{\varepsilon_3}\right) N_3^* - k_1 N_7^*\right] \int_0^\pi \theta_{ttx}^2 dx \\
&- [\rho - \varepsilon_1 N_1^* - \varepsilon_3 N_3^*] \int_0^\pi \varphi_t^2 dx - \left[\frac{\beta}{2} N_2^* - \left(J + \frac{c}{\varepsilon_1}\right) N_1^*\right] \int_0^\pi \psi_t^2 dx \\
&- \left[\frac{\kappa}{2} N_5^* - \frac{3\kappa^2}{2\beta} N_2^*\right] \int_0^\pi \theta_x^2 dx - \left[\frac{b\beta\mu}{2\rho J} N_3^* - \frac{3\mu}{2} - c\varepsilon_2 N_2^*\right] \int_0^\pi \varphi_x^2 dx
\end{aligned}$$

$$
\begin{aligned}
&-\left[\frac{\delta}{2}N_1^* - \frac{\delta\beta b}{\rho J}N_3^* - c - \varepsilon_2\left(cN_2^* + N_5^*\right)\right]\int_0^\pi \psi_x^2 dx \\
&-\left[e^{-\tau_2}N_6^* - \frac{3k_1}{2\beta}N_2^* - \frac{k_1}{2\kappa}N_5^*\right]\int_0^\pi\int_{\tau_1}^{\tau_2} |k_2(s)|\, z_x^2\left(1,s\right) ds dx \\
&-e^{-\tau_2}N_6^*\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} s\,|k_2(s)|\, z_x^2\left(\omega,s\right) ds d\omega dx \\
&-e^{-\tau_2}N_7^*\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} s\,|k_2(s)|\, z_{tx}^2\left(\omega,s\right) ds d\omega dx \\
&-\left[e^{-\tau_2}N_7^* - cN_3^*\right]\int_0^\pi\int_{\tau_1}^{\tau_2} |k_2(s)| z_{tx}^2\left(1,s\right) ds dx \\
&-\left[\left(\xi - \frac{b^2}{\mu}\right)N_1^* - cN_3^*\right]\int_0^\pi \psi^2 dx \\
&-\left[N^*\left(k_1 - \int_{\tau_1}^{\tau_2}|k_2(s)|\,ds\right) - \frac{\beta^2 c_p^2}{2\delta}N_1^* - cN_2^*\left(1+\frac{1}{\varepsilon_2}\right)\right. \\
&\left. -cN_3^* - N_5^*\left(\alpha + \frac{\beta^2}{4\varepsilon_2}\right)c_p^2 - k_1 N_6^*\right]\int_0^\pi \theta_{tx}^2 dx.
\end{aligned}
$$

First, we take $\varepsilon_1 = \frac{\rho}{4N_1^*}$ and $\varepsilon_3 = \frac{\rho}{4N_3^*}$ to get

$$\rho - \varepsilon_1 N_1^* - \varepsilon_3 N_3^* = \frac{\rho}{2}.$$

Next, we select $N_3^*$ large enough such that

$$\frac{b\beta\mu}{2\rho J}N_3^* - \frac{3\mu}{2} = C_1 > 0.$$

After that, recalling that $\mu\xi > b^2$, we pickt $N_1^*$ large so that

$$c_1 = \left(\xi - \frac{b^2}{\mu}\right)N_1^* - cN_3^* > 0$$

and

$$\frac{\delta}{2}N_1^* - \frac{\delta\beta b}{\rho J}N_3^* - c > 0.$$

Then, we pick $N_2^*$ so large that

$$c_2 = \frac{\beta}{2}N_2^* - \left(J + \frac{c}{\varepsilon_1}\right)N_1^* > 0,$$

Next, the selection of sufficiently large $N_5^*$ guarantees

$$c_3 = \frac{\kappa}{2}N_5^* - \frac{3\kappa^2}{2\beta}N_2^* > 0.$$

Now, we pick $N_6^*$ large to get

$$e^{-\tau_2}N_6^* - \frac{3k_1}{2\beta}N_2^* - \frac{k_1}{2\kappa}N_5^* > 0.$$

We choose $\varepsilon_2$ small enough so that

$$c_4 = \frac{\delta}{2}N_1^* - \frac{\delta\beta b}{\rho J}N_3^* - c - \varepsilon_2\left(cN_2^* + N_5^*\right) > 0$$

and

$$c_5 = C_1 - c\varepsilon_2 N_2^* > 0.$$

Then we select $N_7^*$ so large that

$$c_6 = e^{-\tau_2}N_7^* - cN_3^* > 0$$

Finally, we choose $N^*$ so large so that $\mathcal{L}_*(t) \sim E(t) + E_*(t)$ and

$$\begin{aligned} c_7 =& N^*\left(k_1 - \int_{\tau_1}^{\tau_2}|k_2(s)|\,ds\right) - \frac{\beta^2 c_p^2}{2\delta}N_1^* - cN_2^*\left(1+\frac{1}{\varepsilon_2}\right) \\ &- cN_3^* - N_5^*\left(\alpha + \frac{\beta^2}{4\varepsilon_2}\right)c_p^2 - k_1 N_6^* > 0, \end{aligned}$$

$$c_8 = N^*\left(k_1 - \int_{\tau_1}^{\tau_2}|k_2(s)|\,ds\right) - c\left(1+\frac{1}{\varepsilon_3}\right)N_3^* - k_1 N_7^* > 0,$$

Therefore,

$$\begin{aligned} \mathcal{L}_*'(t) \le& -\left(\frac{\rho}{2}\int_0^\pi \varphi_t^2dx + c_2\int_0^\pi \psi_t^2dx + c_7\int_0^\pi \theta_{tx}^2dx + c_5\int_0^\pi \varphi_x^2dx\right) \\ &-\left(c_1\int_0^\pi \psi^2dx + c_4\int_0^\pi \psi_x^2dx + N_6^*e^{-\tau_2}\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} s\,|k_2(s)|\,z_x^2(\omega,s)\,dsd\omega dx\right) \\ &-\left(c_8\int_0^\pi \theta_{ttx}^2dx + c_3\int_0^\pi \theta_x^2dx + c_6\int_0^\pi\int_{\tau_1}^{\tau_2}|k_2(s)|z_{tx}^2(1,s)\,dsdx\right). \end{aligned}$$

Hence, there exists $\lambda_0^* > 0$

$$\lambda_0^* = \min\left\{\frac{\rho}{2}, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, N_6^*e^{-\tau_2}\right\}$$

so that

$$\mathcal{L}_*'(t) \le -\lambda_0^* E(t). \tag{3.61}$$

Integrating (3.61) over $(0,t)$, taking into account that $E(t)$ is non-increasing, we infer that

$$tE(t) \leq \int_0^t E(s)\,ds \leq \frac{1}{\lambda_0^*}\left(\mathcal{L}_*(0) - \mathcal{L}_*(t)\right), \ \forall t \geq 0. \tag{3.62}$$

Since $\mathcal{L}_*(t) \geq 0$ and $\mathcal{L}_*(0) \sim E(0) + E_*(0)$, inequality (3.62) leads to

$$E(t) \leq \frac{\eta\left(E(0) + E_*(0)\right)}{t}, \ \forall t > 0,$$

for $\eta > 0$. Consequently, Theorem 38 has been fully proved.

# Chapter 4

# General decay of the solution of a porous thermoelastic system of type III with a nonlinear damping term.

In this chapter, we consider the following porous thermoelasticity type III with nonlinear term

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, & \text{in } (0,\pi) \times (0,\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \beta\theta_x + h(\phi_t(x,t)) = 0, & \text{in } (0,\pi) \times (0,\infty), \\ \alpha\theta_{tt} - \kappa\theta_{xx} + \beta\phi_{ttx} - \gamma\theta_{txx} = 0, & \text{in } (0,\pi) \times (0,\infty). \end{cases} \tag{4.1}$$

where $u$ is the longitudinal displacement, $\phi$ is the volume fraction, $\theta$ is the difference of temperature, and $\rho, J, \beta, \mu, \delta, \xi, \kappa, \alpha, \gamma$ are constitutive constants which are positive, with $\mu, \xi, b$ satisfying $\mu\xi > b^2$. System (4.1) is supplemented by the following initial and boundary conditions:

$$u(0,t) = \phi_x(0,t) = \theta(0,t) = 0, \quad u_x(\pi,t) = \phi(\pi,t) = \theta(\pi,t) = 0, \text{ in } (0,\infty), \tag{4.2}$$

$$\begin{matrix} u(x,0) = u_0(x),\ \phi(x,0) = \phi_0(x),\ \theta(x,0) = \theta_0(x), \\ u_t(x,0) = u_1(x),\ \phi_t(x,0) = \phi_1(x),\ \theta_t(x,0) = \theta_1(x), \end{matrix} \quad \text{in } (0,\pi). \tag{4.3}$$

We will show the well-posedness, using the semigroup theory, and establish an explicit and general decay result, using some properties of convex functions and the multiplier method. provided the wave speeds are equal.

## 4.1 Statement of the problem

In order to exhibit the dissipative nature of system (4.1), we follow Zheng and Zuazua [46] and introduce the new variable

$$\vartheta(x,t)=\int_0^t \theta(x,s)ds+\chi(x), \tag{4.4}$$

where $\chi(x)$ is the solution of

$$\begin{cases} \kappa\chi_{xx}=\alpha\theta_1-\gamma\theta_{0xx}+\beta\phi_{1x}, \quad \text{in } (0,\pi), \\ \chi(0)=\chi(\pi)=0. \end{cases} \tag{4.5}$$

Then, integrating the third equation in (4.1), with respect to $t$, and using (4.5), we have

$$\alpha\theta_t-\kappa\int_0^t \theta_{xx}(x,s)ds+\beta\phi_{tx}-\gamma\theta_{xx}-\kappa\chi_{xx}=0 \tag{4.6}$$

by using (4.4), we get

$$\theta_t=\vartheta_{tt}, \quad \theta_{xx}=\vartheta_{txx}, \quad \kappa\int_0^t \theta_{xx}(x,s)ds=\kappa\vartheta_{xx}-\kappa\chi_{xx}$$

then, by substituting into (4.6) lead to

$$\alpha\vartheta_{tt}-\kappa\vartheta_{xx}+\beta\phi_{tx}-\gamma\vartheta_{txx}=0.$$

Therefore, the problem (4.1)-(4.3) takes the form

$$\begin{cases} \rho u_{tt}-\mu u_{xx}-b\phi_x=0 & \text{in } (0,\pi)\times(0,\infty), \\ J\phi_{tt}-\delta\phi_{xx}+bu_x+\xi\phi+\beta\vartheta_{tx}+h\left(\phi_t\left(x,t\right)\right)=0 & \text{in } (0,\pi)\times(0,\infty), \\ \alpha\vartheta_{tt}-\kappa\vartheta_{xx}+\beta\phi_{tx}-\gamma\vartheta_{txx}=0 & \text{in } (0,\pi)\times(0,\infty), \\ u\left(0,t\right)=\phi_x\left(0,t\right)=\vartheta\left(0,t\right)=0, \quad u_x\left(\pi,t\right)=\phi\left(\pi,t\right)=\vartheta\left(\pi,t\right)=0 & \text{in } (0,\infty), \\ \left(u\left(x,0\right),\phi\left(x,0\right),\vartheta\left(x,0\right)\right)=\left(u_0(x),\phi_0(x),\chi(x)\right) & \text{in } (0,\pi), \\ \left(u_t\left(x,0\right),\phi_t\left(x,0\right),\vartheta_t(x,0)\right)=\left(u_1(x),\phi_1(x),\theta_0(x)\right) & \text{in } (0,\pi). \end{cases} \tag{4.7}$$

The energy functional associated with problem (4.7) is given by

$$E\left(t\right)=\frac{1}{2}\int_0^\pi \left(\rho u_t^2+J\phi_t^2+\alpha\vartheta_t^2+\mu u_x^2+\xi\phi^2+2bu_x\phi+\delta\phi_x^2+\kappa\vartheta_x^2\right)dx. \tag{4.8}$$

**Remark 41.** *By virtue of* $\mu\xi > b^2$, *we can rewrite the energy* $E(t)$ *as*

$$\begin{aligned} E\left(t\right) = \ & \frac{1}{2}\int_0^\pi \left(\rho u_t^2 + J\phi_t^2 + \alpha\vartheta_t^2 + \delta\phi_x^2 + \kappa\vartheta_x^2\right) dx \\ & + \frac{\mu}{4}\int_0^\pi \left(u_x + \frac{b}{\mu}\phi\right)^2 dx + \frac{\xi}{4}\int_0^\pi \left(\frac{b}{\xi}u_x + \phi\right)^2 dx \\ & + \frac{1}{4}\left(\mu - \frac{b^2}{\xi}\right)\int_0^\pi u_x^2 dx + \frac{1}{4}\left(\xi - \frac{b^2}{\mu}\right)\int_0^\pi \phi^2 dx, \end{aligned}$$

*which is a positive definite form.*

## 4.2 Preliminaries and well-posedness

In this section, we present some materials needed in the proof of the well-posedness and the stability of the solution of the problem (4.1)–(4.3). Throughout this chapter, $c$ is used to denote a generic positive constant. For the nonlinearity term, we assume the following hypothesis.

(B) $h : \mathbb{R} \to \mathbb{R}$ is a nondecreasing $C^1-$ function such that there exist positive constants $c_1, c_2, \varepsilon$, and a strictly increasing function $h_0 \in C^1([0, +\infty))$, with $h_0(0) = 0$ and $H(s) = \sqrt{\frac{s}{2}}h_0(\sqrt{\frac{s}{2}})$ is strictly convex $C^2$ function on $(0, \varepsilon^2]$ when $h_0$ is nonlinear, such that

$$\begin{cases} h_0(|s|) \le |h(s)| \le h_0^{-1}(|s|), & \text{for all } |s| \le \varepsilon, \\ c_1|s| \le |h(s)| \le c_2|s|, & \text{for all } |s| \ge \varepsilon. \end{cases}$$

**Remark 42.**

- *From the first inequality in (B), we get:*

$$H(2s^2) = |s|h_0(|s|) \le sh(s), \ \textit{for } |s| \le \varepsilon,$$

*and*

$$H(2h^2(s)) = |h(s)|h_0(|h(s)|) \le |h(s)|h_0(h_0^{-1}|s|) = sh(s), \ \textit{for } |s| \le \varepsilon.$$

*Consequently,*

$$s^2 + h^2(s) \le H^{-1}(sh(s)), \ \textit{for } |s| \le \varepsilon. \tag{4.9}$$

- *For* $|s| \ge \varepsilon$, *we have*

$$h^2(s) \le c_2 sh(s) \ \textit{and } c_1 s^2 \le sh(s).$$

*Thus,*

$$s^2 + h^2(s) \leq csh(s), \text{ for all } |s| \geq \varepsilon. \tag{4.10}$$

- *Hypothesis (B) implies that* $sh(s) > 0$*, for all* $s \neq 0$.

  *Indeed, for* $|s| \leq \varepsilon$*, we have*

  $$s^2 + h^2(s) \leq H^{-1}(sh(s)), \text{ for } |s| \leq \varepsilon.$$

  *since* $H$ *is strictly increasing, we get*

  $$0 = H(0) < H(s^2 + h^2(s)) \leq sh(s), \text{ for } |s| \leq \varepsilon \text{ and } s \neq 0$$

  *for* $|s| \geq \varepsilon$*, we have*(4.10)

- *Hypothesis (B) implies that* $h$ *is globally Lipschitz.*

  *Indeed, in the bounded interval* $[-\varepsilon, \varepsilon]$ $, h$ *is countinously differentiable, then* $h'$ *is bounded and for any* $(s,t) \in [-\varepsilon, \varepsilon]^2$ *there exists,* $\xi \in (s,t)$ *such that*

  $$|h(s) - h(t)| = |h'(\xi)(s-t)| \leq \eta|s-t|.$$

  *For* $|s| \geq \varepsilon$*, since* $sh(s) > 0$*, we have*

  $$c_1|s| \leq |h(s)| \leq c_2|s|.$$

  *This fact implies that* $h$ *grows linearly, as* $|s| \to \infty$*, with its magnitude bounded between* $c_1|s|$ *and* $c_2|s|$*. Then, its graph confined between the lines* $c_1 s$ *and* $c_2 s$*. Since* $h$ *is nondecreasing,* $h'$ *must be bounded; otherwise, the graph of* $h$ *would exceed the bounds* $c_1|s| \leq h(s) \leq c_2|s|$*. Consequently, there exists a positive constant* $\widetilde{\eta} > 0$ *such that* $|h'(s)| \leq \widetilde{\eta}$ *for* $|s| \geq \varepsilon$*. Thus,* $h$ *is globally Lipschitz.*

- *In [29], Lasiecka and Tataru established the existence of* $H$ *as defined in (B) by using the monotonicity and continuity of* $h$*.*

To rewrite the system (4.7) in the semigroups setting, we present the new variables $w = u_t$, $v = \phi_t, \psi = \vartheta_t$, and define the state space

$$\mathcal{H} = V_1 \times L^2(0,\pi) \times V_2 \times L^2(0,\pi) \times H_0^1(0,\pi) \times L^2(0,\pi),$$

where

$$V_1 = \{f \in H^1(0,\pi) : f(0) = 0\}$$

and

$$V_2 = \{f \in H^1(0,\pi) : f(\pi) = 0\}.$$

The space $\mathcal{H}$ is a Hilbert space with respect to the inner product

$$\begin{aligned}\langle U, \tilde{U}\rangle_{\mathcal{H}} = &\int_0^\pi \left(\rho w\widetilde{w} + \mu u_x\widetilde{u}_x + Jv\widetilde{v} + \alpha\psi\widetilde{\psi} + \delta\phi_x\widetilde{\phi}_x + \xi\phi\widetilde{\phi} + \kappa\vartheta_x\widetilde{\vartheta}_x\right)dx \\ &+ b\int_0^\pi (u_x\widetilde{\phi}dx + \phi\widetilde{u}_x)dx\end{aligned}$$

for $U = (u, w, \phi, v, \vartheta, \psi)$, $\tilde{U} = \left(\widetilde{u}, \widetilde{w}, \widetilde{\phi}, \widetilde{v}, \widetilde{\vartheta}, \widetilde{\psi}\right) \in \mathcal{H}$.

The system (4.7) can be written as

$$\begin{cases} U_t(t) + \mathcal{A}U(t) = F(U), \text{for all } \ t > 0, \\ U(0) = U_0 = (u_0, u_1, \phi_0, \phi_1, \chi, \theta_0), \end{cases} \tag{4.11}$$

where $F(U) = \left(0, 0, 0, -\dfrac{1}{J}h(v), 0, 0\right)^T$ and $\mathcal{A}$ is the operator defined by

$$\mathcal{A}U = \begin{pmatrix} -w \\ -\dfrac{\mu}{\rho}u_{xx} - \dfrac{b}{\rho}\phi_x \\ -v \\ -\dfrac{\delta}{J}\phi_{xx} + \dfrac{b}{J}u_x + \dfrac{\xi}{J}\phi + \dfrac{\beta}{J}\psi_x \\ -\psi \\ \dfrac{-\kappa}{\alpha}\vartheta_{xx} + \dfrac{\beta}{\alpha}v_x - \dfrac{\gamma}{\alpha}\psi_{xx} \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \left\{ U \in \mathcal{H} \mid \begin{array}{r} u \in W_1 \cap V_1, \quad \phi \in W_2 \cap V_2, \quad w \in V_1, \\ v \in V_2, \quad \psi \in H_0^1(0,\pi), \quad (\kappa\vartheta + \gamma\psi) \in H^2(0,\pi) \end{array} \right\},$$

where

$$W_1 = \left\{g \in H^2(0,\pi); g_x(\pi) = 0\right\}$$

and

$$W_2 = \left\{g \in H^2(0,\pi); g_x(0) = 0\right\}.$$

The following theorem provides the primary result of this section.

**Theorem 43.** *Let $(u_0, u_1, \phi_0, \phi_1, \chi, \theta_0) \in \mathcal{H}$ be given, where $\chi$ satisfies (4.5). Then the problem (4.11) has a unique mild solution $(u, w, \phi, v, \vartheta, \psi)$ such that*

$$(u, w, \phi, v, \vartheta, \psi)^T \in C\left([0, +\infty[\,; \mathcal{H}\right).$$

*Proof.* It is sufficient to demonstrate that $\mathcal{A}$ is monotone maximal and $F$ is globally Lipschitz, by virtue of Theorem 13.

First, for any $U \in D(\mathcal{A})$ we have

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} =& -\mu \int_0^\pi u_{xx} w dx - b\int_0^\pi \phi_x w dx - \mu \int_0^\pi w_x u_x dx - \delta \int_0^\pi \phi_{xx} v dx + b\int_0^\pi u_x v dx \\
&+ \xi \int_0^\pi \phi v dx + \beta \int_0^\pi \psi_x v dx - \kappa \int_0^\pi \vartheta_{xx}\psi dx + \beta \int_0^\pi v_x \psi dx - \gamma \int_0^\pi \psi_{xx}\psi dx \\
&- \delta \int_0^\pi v_x \phi_x dx - \xi \int_0^\pi v\phi dx - \kappa \int_0^\pi \psi_x \vartheta_x dx \\
&+ b\int_0^\pi (-w_x\phi - v u_x) dx \\
=& \gamma \int_0^\pi \psi_x^2 dx \geq 0;
\end{aligned}$$

which shows that $\mathcal{A}$ is monotone.

Next, let $F = (f^1, f^2, f^3, f^4, f^5, f^6) \in \mathcal{H}$. We seek a solution $U \in D(\mathcal{A})$ that satisfies

$$(I + \mathcal{A})U = F, . \tag{4.12}$$

This can be expressed in components as

$$\left\{\begin{aligned}
u - w =&\ \ f^1, \\
\rho w - \mu u_{xx} - b\phi_x =&\ \ \rho f^2, \\
\phi - v =&\ \ f^3, \\
Jv - \delta\phi_{xx} + bu_x + \xi\phi + \beta\psi_x =&\ \ Jf^4, \\
\vartheta - \psi =&\ \ f^5, \\
\alpha\psi - \kappa\vartheta_{xx} + \beta v_x - \gamma\psi_{xx} =&\ \ \alpha f^6.
\end{aligned}\right. \tag{4.13}$$

The first, the third and the fifth equation of (4.13) give

$$w = u - f^1,\ v = \phi - f^3,\ \psi = \vartheta - f^5. \tag{4.14}$$

Next, plugging (4.14) into the second, fourth and sixth equations of (4.13) we reach

$$\begin{cases} \rho(u-f^1)-\mu u_{xx}-b\phi_x = & \rho f^2 \\ J(\phi-f^3)-\delta\phi_{xx}+bu_x+\xi\phi+\beta(\vartheta_x-f^5_x) = & Jf^4 \\ \alpha(\vartheta-f^5)-\kappa\vartheta_{xx}+\beta(\phi_x-f^3_x)-\gamma(\vartheta_{xx}-f^5_{xx}) = & \alpha f^6 \end{cases} \tag{4.15}$$

So, we have

$$\begin{cases} -\mu u_{xx}+\rho u-b\phi_x & = h_1, \\ -\delta\phi_{xx}+bu_x+(\xi+J)\phi+\beta\vartheta_x & = h_2, \\ -(\kappa+\gamma)\vartheta_{xx}+\alpha\vartheta+\beta\phi_x & = h_3, \end{cases} \tag{4.16}$$

where

$$h_1=\rho(f^2+f^1)\in L^2(0,\pi), \tag{4.17}$$

$$h_2=J(f^4+f^3)+\beta f^5_x\in L^2(0,\pi) \tag{4.18}$$

and

$$h_3=\alpha(f^6+f^5)+\beta f^3_x-\gamma f^5_{xx}\in H^{-1}(0,\pi).$$

Multiplying the equations of (4.16) "formally" by $\widetilde{u}\in V_1, \widetilde{\phi}\in V_2$ and $\widetilde{\vartheta}\in H^1_0(0,\pi)$, respectively, then integrating with respect to $x$ over $(0,\pi)$ and summing up, we arrive at We consider on $\mathcal{V}=V_1\times V_2\times H^1_0(0,\pi)$, the Variational formulation

$$B((u,\phi,\vartheta),(\widetilde{u},\widetilde{\phi},\widetilde{\vartheta}))=L(\widetilde{u},\widetilde{\phi},\widetilde{\vartheta}), \tag{4.19}$$

where,

$$\begin{aligned} B((u,\phi,\vartheta),(\tilde{u},\tilde{\phi},\tilde{\vartheta})) =& \mu\int_0^\pi u_x\widetilde{u}_x dx+\rho\int_0^\pi u\widetilde{u}dx+b\int_0^\pi\left(\phi\widetilde{u}_x+u_x\widetilde{\phi}\right)dx \\ &+\delta\int_0^\pi \phi_x\widetilde{\phi}_x dx+(\xi+J)\int_0^\pi\phi\widetilde{\phi}dx+\beta\int_0^\pi(\vartheta_x\widetilde{\phi}+\phi_x\widetilde{\vartheta})dx \\ &+(\kappa+\gamma)\int_0^\pi\vartheta_x\widetilde{\vartheta}_x dx+\alpha\int_0^\pi\vartheta\widetilde{\vartheta}dx \end{aligned}$$

and

$$L(\widetilde{u},\widetilde{\phi},\widetilde{\vartheta})=\int_0^\pi h_1\widetilde{u}dx+\int_0^\pi h_2\widetilde{\phi}dx+\langle h_3,\widetilde{\vartheta}\rangle_{H^{-1}\times H^1_0},$$

where $\langle\cdot,\cdot\rangle_{H^{-1}\times H^1_0}$ is the $H^{-1}\times H^1_0$ pairing.

Let $\mathcal{V}=V_1\times V_2\times H^1_0(0,\pi)$ be the Hilbert space equipped with the norm

$$\|(u,\phi,\vartheta)\|^2_{\mathcal{V}}=\|u\|^2+\|u_x\|^2+\|\phi\|^2+\|\phi_x\|^2+\|\vartheta_x\|^2.$$

Clearly, $B$ and $L$ are bilinear and linear forms over $\mathcal{V}$. On the other hand, we easily check that $B$ and $L$ are bounded. Furthermore, we have

$$B((u,\phi,\vartheta),(u,\phi,\vartheta)) = \mu\int_0^\pi u_x^2dx + \rho\int_0^\pi u^2dx + 2b\int_0^\pi u_x\phi dx + (\xi+J)\int_0^\pi \phi^2dx + \delta\int_0^\pi \phi_x^2dx + \alpha\int_0^\pi \vartheta^2dx + (\kappa+\gamma)\int_0^\pi \vartheta_x^2dx,$$

Young's inequality yields

$$B((u,\phi,\vartheta),(u,\phi,\vartheta)) \geq \rho\left\|u\right\|^2 + (\mu - \frac{b^2}{\xi})\left\|u_x\right\|^2 + J\left\|\phi\right\|^2 + \delta\left\|\phi_x\right\|^2 + \alpha\left\|\vartheta\right\|^2 + (\kappa+\gamma)\left\|\vartheta_x\right\|^2.$$

Thus,

$$B((u,\phi,\vartheta),(u,\phi,\vartheta)) \geq N\left\|(u,\phi,\vartheta)\right\|_{\mathcal{V}}^2,$$

for $N > 0$, which implies that $B$ is coercive. As a result, the Lax-Milgram theorem guarantees that the system (4.16) has a unique weak solution

$$u \in V_1, \phi \in V_2, \vartheta \in H_0^1(0,\pi).$$

Substituting $u,\phi,\vartheta$ into (4.14), we get

$$w \in V_1,\ v \in V_2,\ \psi \in H_0^1(0,\pi). \tag{4.20}$$

Moreover, taking $(\widetilde{\phi},\widetilde{\vartheta}) \equiv 0 \in V_2 \times H_0^1(0,\pi)$ in (4.19), we get

$$\mu\int_0^\pi u_x\tilde{u}_xdx = \int_0^\pi (h_1 - \rho u + b\phi_x)\,\widetilde{u}dx \quad \forall\widetilde{u} \in V_1, \tag{4.21}$$

The regularity theory of elliptic equations entails that $u \in H^2(0,\pi)$.
Moreover, considering (4.21) with $\widetilde{u} \in C_0^1(0,\pi) \subset V_1$ and integrating the left-hand side by parts, we arrive at

$$-\mu\int_0^\pi u_{xx}\widetilde{u}dx = \int_0^\pi (h_1 - \rho u + b\phi_x)\,\widetilde{u}dx \ \forall\widetilde{u} \in C_0^1(0,\pi), \tag{4.22}$$

which leads to

$$-\mu u_{xx} + \rho u - b\phi_x = h_1. \tag{4.23}$$

Recalling (4.14) and (4.17), we get

$$\rho w - \mu u_{xx} - b\phi_x = \rho f^2.$$

Thus, $u$ satisfies the second equation of (4.13).

In addition, integration by parts of the left hand side of (4.21) yields

$$\mu u_x(\pi)\widetilde{u}_x(\pi) - \mu \int_0^\pi u_{xx}\widetilde{u}dx = \int_0^\pi (h_1 - \rho u + b\phi_x)\,\widetilde{u}dx, \forall \widetilde{u} \in V_1.$$

Keeping in mind (4.23) and the arbitrariness of $\widetilde{u}$, we get $u_x(\pi) = 0$, which shows that $u \in W_1$ and consequently,

$$u \in W_1 \cap V_1. \tag{4.24}$$

Similarly, by inserting $(\widetilde{u}, \widetilde{\vartheta}) = (0, 0)$ in (4.19), we get

$$\delta \int_0^\pi \phi_x \widetilde{\phi}_x dx = \int_0^\pi (h_2 - bu_x - (J+\xi)\,\phi - \beta\vartheta_x)\widetilde{\phi}dx,\ \forall \widetilde{\phi} \in V_2. \tag{4.25}$$

The elliptic regularity theory yields $\phi \in H^2(0, \pi)$.

Next, integrating by parts the left-hand side of (4.25) we arrive at

$$-\delta\phi_x(0)\widetilde{\phi}(0) - \delta \int_0^\pi \phi_{xx}\widetilde{\phi}dx = \int_0^\pi (h_2 - bu_x - (J+\xi)\,\phi - \beta\vartheta_x)\widetilde{\phi}dx,\ \forall \widetilde{\phi} \in V_2. \tag{4.26}$$

So, for $\widetilde{\phi} \in C_0^1(0, \pi) \subset V_2$, (4.26) gives

$$-\delta\phi_{xx} + bu_x + (J+\xi)\,\phi + \beta\vartheta_x = h_2,$$

then using (4.14) and (4.18), we get

$$Jv - \delta\phi_{xx} + bu_x + \xi\phi + \beta\psi_x = Jf^4.$$

Again, (4.26) and the fact that $\widetilde{\phi}$ is arbitrary implies that $\phi_x(0) = 0$. which shows that $\phi \in W_2$, hence

$$\phi \in W_2 \cap V_2. \tag{4.27}$$

Finally, taking $\left(\widetilde{u}, \widetilde{\phi}\right) = (0, 0)$ in (4.19) and substituting $f^3$ by $\phi - v$ and $f^5$ by $\vartheta - \psi$ we obtain

$$\int_0^\pi [\kappa\vartheta_x + \gamma\psi_x]\,\widetilde{\vartheta}_x dx = \int_0^\pi \left(\alpha f^6 - \alpha\psi - \beta v_x\right)\widetilde{\vartheta}dx,\ \forall \widetilde{\vartheta} \in H_0^1(0, \pi), \tag{4.28}$$

The elliptic regularity theory yields

$$\kappa\vartheta + \gamma\psi \in H^2(0,\pi). \tag{4.29}$$

Moreover, integrating the left-hand side of (4.28) by parts entails

$$\kappa\vartheta_{xx} + \gamma\psi_{xx} = \alpha f^6 - \alpha\psi - \beta v_x, \text{ in } L^2(0,\pi),$$

which shows that $\psi$ solves the sixth equation in (4.13).

The combination of (4.20), (4.24), (4.27) and (4.29) shows that the solution $U$ belongs to $D(\mathcal{A})$. Consequently, the operator $\mathcal{A}$ is maximal. Thus, the proof of Theorem 43 is completes. □

## 4.3 General decay

In this section, we prove a general decay result for the solution of system (4.7) provided that

$$\zeta = \frac{\mu}{\rho} - \frac{\delta}{J}. \tag{4.30}$$

Before addressing the decay result, we need to explain some technical lemmas concerning the derivatives of certain auxiliary functionals.

**Lemma 44.** *The energy $E(t)$ satisfies, along the solution of the system (4.7), the estimate*

$$\frac{dE(t)}{dt} = -\gamma\int_0^\pi \vartheta_{tx}^2 dx - \int_0^\pi \phi_t h(\phi_t)\, dx \le 0. \tag{4.31}$$

*Proof.* Multiplying the equations of the system (4.7) by $u_t$, $\phi_t$, and $\vartheta_t$ respectively, then integrating the equations with respect to $x$ over $(0,\pi)$, we get
The first equation

$$\frac{\rho}{2}\frac{d}{dt}\int_0^\pi u_t^2 dx + \frac{\mu}{2}\frac{d}{dt}\int_0^\pi u_x^2 dx - b\int_0^\pi \phi_x u_t dx = 0. \tag{4.32}$$

The second equation

$$\frac{J}{2}\frac{d}{dt}\int_0^\pi \phi_t^2 dx + \frac{\delta}{2}\frac{d}{dt}\int_0^\pi \phi_x^2 dx + b\int_0^\pi u_x\phi_t dx + \frac{\xi}{2}\frac{d}{dt}\int_0^\pi \phi^2 dx + \beta\int_0^\pi \vartheta_{tx}\phi_t dx + \int_0^\pi \phi_t h(\phi_t)\, dx = 0. \tag{4.33}$$

The third equation

$$\frac{\alpha}{2}\frac{d}{dt}\int_0^\pi \vartheta_t^2 dx + \frac{\kappa}{2}\frac{d}{dt}\int_0^\pi \vartheta_x^2 dx - \beta\int_0^\pi \phi_t\vartheta_{tx}dx + \gamma\int_0^\pi \vartheta_{tx}^2 dx = 0. \tag{4.34}$$

Adding (4.32)-(4.34), we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^\pi \left(\rho u_t^2 + J\phi_t^2 + \alpha\vartheta_t^2 + \mu u_x^2 + \xi\phi^2 + 2bu_x\phi + \delta\phi_x^2 + \kappa\vartheta_x^2\right)dx &= -\gamma\int_0^\pi \vartheta_{tx}^2 dx \\ &\quad - \int_0^\pi \phi_t h\left(\phi_t\right)dx.\end{aligned}$$

Thus, (4.31) is achieved. □

**Lemma 45.** *The functional*

$$\mathcal{F}_1(t) := J\int_0^\pi \phi\phi_t dx - \frac{\rho b}{\mu}\int_0^\pi u_t\int_0^x \phi(y)\,dydx$$

*satisfies, along the solution of the problem (4.7), the estimate*

$$\begin{aligned}\mathcal{F}_1'(t) \le &-\frac{\delta}{2}\int_0^\pi \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^\pi \phi^2 dx + \varepsilon_1\int_0^\pi u_t^2 dx \\ &+ c\int_0^\pi \vartheta_{tx}^2 dx + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^\pi \phi_t^2 dx + c\int_0^\pi h^2\left(\phi_t\right)dx,\end{aligned} \tag{4.35}$$

*for any $\varepsilon_1 > 0$.*

*Proof.* Direct differentiation, using the second equation of (4.7) and integration by parts, yields

$$\begin{aligned}J\frac{d}{dt}\int_0^\pi \phi\phi_t dx &= J\int_0^\pi \phi_t^2 dx + J\int_0^\pi \phi\phi_{tt}dx \\ &= J\int_0^\pi \phi_t^2 dx - \delta\int_0^\pi \phi_x^2 dx - b\int_0^\pi u_x\phi dx - \xi\int_0^\pi \phi^2 dx + \beta\int_0^\pi \vartheta_t\phi_x dx \\ &\quad - \int_0^\pi \phi h\left(\left(\phi_t\right)dx.\right.\end{aligned} \tag{4.36}$$

Similarly,

$$\begin{aligned}-\frac{\rho b}{\mu}\frac{d}{dt}\int_0^\pi u_t\int_0^x \phi(y)\,dydx &= \frac{-\rho b}{\mu}\int_0^\pi u_{tt}\int_0^x \phi(y)\,dydx - \frac{\rho b}{\mu}\int_0^\pi u_t\int_0^x \phi_t(y)\,dydx \\ &= b\int_0^\pi u_x\phi dx + \frac{b^2}{\mu}\int_0^\pi \phi^2 dx - \frac{\rho b}{\mu}\int_0^\pi u_t\int_0^x \phi_t(y)\,dydx.\end{aligned} \tag{4.37}$$

The addition of (4.36) and (4.37) yields

$$\begin{aligned}\mathcal{F}_1'(t) = &-\delta\int_0^\pi \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^\pi \phi^2 dx + J\int_0^\pi \phi_t^2 dx \\ &+\beta\int_0^\pi \vartheta_t\phi_x dx - \int_0^\pi \phi h(\phi_t)\,dx - \frac{\rho b}{\mu}\int_0^\pi u_t\int_0^x \phi_t(y)\,dydx. \qquad (4.38)\end{aligned}$$

By using Cauchy-Schwarz's, Poincaré's and Young's inequalities, we have

$$\beta\int_0^\pi \vartheta_t\phi_x dx \le \frac{\delta}{4}\int_0^\pi \phi_x^2 dx + \frac{\beta^2}{\delta}\int_0^\pi \vartheta_{tx}^2 dx,$$

$$\int_0^\pi \phi h(\phi_t)\,dx \le \frac{\delta}{4}\int_0^\pi \phi_x^2 dx + \frac{c_p^2}{\delta}\int_0^\pi h^2(\phi_t)\,dx$$

and, for any $\varepsilon_1 > 0$, we obtain

$$\begin{aligned}-\frac{\rho b}{\mu}\int_0^\pi u_t\int_0^x \phi_t(y)\,dydx \le&\varepsilon_1\int_0^\pi u_t^2 dx + \frac{c}{\pi\varepsilon_1}\int_0^\pi\left(\int_0^x \phi_t(y)dy\right)^2 dx \\ \le&\varepsilon_1\int_0^\pi u_t^2 dx + \frac{c}{\pi\varepsilon_1}\int_0^\pi\left(\int_0^\pi \phi_t(y)dy\right)^2 dx \\ \le&\varepsilon_1\int_0^\pi u_t^2 dx + \frac{c}{\pi\varepsilon_1}\left(\int_0^\pi \phi_t(y)dy\right)^2\int_0^\pi dx \\ \le&\varepsilon_1\int_0^\pi u_t^2 dx + \frac{c}{\varepsilon_1}\int_0^\pi \phi_t^2 dx.\end{aligned}$$

Then, by substituting the above inequalities into (4.38), we end up with (4.35). □

**Lemma 46.** *Suppose that* (4.30) *holds, then the functional*

$$\mathcal{F}_2(t) := \frac{\mu}{\rho}\int_0^\pi \phi_t u_x dx + \frac{\delta}{J}\int_0^\pi \phi_x u_t dx$$

*along the solution of problem (4.7) satisfies the estimate*

$$\mathcal{F}_2'(t) \le -\frac{b\mu}{2\rho J}\int_0^\pi u_x^2 dx + c\int_0^\pi \phi^2 dx + c\int_0^\pi \vartheta_{tx}^2 dx + c\int_0^\pi \phi_x^2 dx + c\int_0^\pi h^2(\phi_t)\,dx. \qquad (4.39)$$

*Proof.* Direct differentiation, using the second equation of (4.7) and integration by parts, yields

$$\begin{aligned}\frac{\mu}{\rho}\frac{d}{dt}\int_0^\pi \phi_t u_x dx =& \frac{\mu}{\rho}\int_0^\pi \phi_{tt}u_x dx + \frac{\mu}{\rho}\int_0^\pi \phi_t u_{tx} dx \\ =& \frac{\mu\delta}{\rho J}\int_0^\pi u_x\phi_{xx} dx - \frac{b\mu}{\rho J}\int_0^\pi u_x^2 dx - \frac{\mu\xi}{\rho J}\int_0^\pi u_x\phi dx - \frac{\beta\mu}{\rho J}\int_0^\pi u_x\vartheta_{tx} dx \\ &-\frac{\mu}{\rho J}\int_0^\pi u_x h(\phi_t)\,dx + \frac{\mu}{\rho}\int_0^\pi \phi_t u_{xt} dx. \qquad (4.40)\end{aligned}$$

Similarly, using $(4.7)_1$, we get

$$\begin{aligned}\frac{\delta}{J}\frac{d}{dt}\int_0^\pi \phi_x u_t dx &= \frac{\delta}{J}\int_0^\pi \phi_{tx} u_t dx + \frac{\delta}{J}\int_0^\pi \phi_x u_{tt} dx \\ &= \frac{\delta}{J}\int_0^\pi \phi_{xt} u_t dx - \frac{\delta\mu}{\rho J}\int_0^\pi u_x \phi_{xx} dx + \frac{\delta b}{J\rho}\int_0^\pi \phi_x^2 dx. \qquad (4.41)\end{aligned}$$

The addition of (4.40) and (4.41), infer to

$$\begin{aligned}\mathcal{F}_2'(t) =& -\frac{b\mu}{\rho J}\int_0^\pi u_x^2 dx - \frac{\mu\xi}{\rho J}\int_0^\pi u_x \phi dx - \frac{\beta\mu}{\rho J}\int_0^\pi \vartheta_{tx} u_x dx + \frac{\delta b}{\rho J}\int_0^\pi \phi_x^2 dx \\ & - \frac{\mu}{\rho J}\int_0^\pi u_x h(\phi_t) dx - \zeta\int_0^\pi \phi_{xt} u_t dx. \qquad (4.42)\end{aligned}$$

Taking into account (4.30) and using Young's inequality, we get

$$\begin{aligned}\mathcal{F}_2'(t) \le& -\frac{b\mu}{\rho J}\int_0^\pi u_x^2 dx + \frac{b\mu}{6\rho J}\int_0^\pi u_x^2 dx + c\int_0^\pi \phi^2 dx + \frac{b\mu}{6\rho J}\int_0^\pi u_x^2 dx + c\int_0^\pi \vartheta_{tx}^2 dx \\ & + \frac{\delta b}{\rho J}\int_0^\pi \phi_x^2 dx + \frac{b\mu}{6\rho J}\int_0^\pi u_x^2 dx + c\int_0^\pi h^2(\phi_t) dx \\ \le& -\frac{b\mu}{2\rho J}\int_0^\pi u_x^2 dx + c\int_0^\pi \phi^2 dx + c\int_0^\pi \vartheta_{tx}^2 dx + c\int_0^\pi \phi_x^2 dx + c\int_0^\pi h^2(\phi_t) dx.\end{aligned}$$

Thus, (4.39) is established. □

**Lemma 47.** *The functional*

$$\mathcal{F}_3(t) := -\rho\int_0^\pi u_t u dx$$

*satisfies, along the solution of (4.7), the estimate*

$$\mathcal{F}_3'(t) \le -\rho\int_0^\pi u_t^2 dx + \frac{3\mu}{2}\int_0^\pi u_x^2 dx + \frac{b^2 C_p}{2\mu}\int_0^\pi \phi_x^2 dx, \qquad (4.43)$$

*where $C_p$ is Poincaré's constant.*

*Proof.* Direct computations using the first equation of (4.7) give

$$\begin{aligned}\mathcal{F}_3'(t) =& -\rho\int_0^\pi u_{tt} u dx - \rho\int_0^\pi u_t^2 dx \\ =& -\mu\int_0^\pi u_{xx} u dx - b\int_0^\pi \phi_x u dx - \rho\int_0^\pi u_t^2 dx \\ =& \mu\int_0^\pi u_x^2 dx + b\int_0^\pi u_x \phi dx - \rho\int_0^\pi u_t^2 dx.\end{aligned}$$

By using Poincaré's and Young's inequalities, we obtain

$$\begin{aligned}\mathcal{F}_3'(t) \leq & -\rho\int_0^\pi u_t^2dx+\mu\int_0^\pi u_x^2dx+\frac{\mu}{2}\int_0^\pi u_x^2dx+\frac{b^2}{2\mu}\int_0^\pi \phi^2dx\\ \leq & -\rho\int_0^\pi u_t^2dx+\frac{3\mu}{2}\int_0^\pi u_x^2dx+c\int_0^\pi \phi_x^2dx.\end{aligned}$$

□

**Lemma 48.** *The functional*

$$\mathcal{F}_4(t):=\alpha\int_0^\pi \vartheta\vartheta_t dx+\frac{\gamma}{2}\int_0^\pi \vartheta_x^2dx$$

*satisfies, along the solution of (4.7), the estimate*

$$\mathcal{F}_4'(t)\leq -\frac{\kappa}{2}\int_0^\pi \vartheta_x^2dx+c\int_0^\pi \phi_t^2dx+c\int_0^\pi \vartheta_{tx}^2dx. \tag{4.44}$$

*Proof.* Direct differentiation of $\mathcal{F}_4(t)$ and the use of the third equation of (4.7) lead to

$$\begin{aligned}\mathcal{F}_4'(t)=&\alpha\int_0^\pi \vartheta_t^2dx+\alpha\int_0^\pi \vartheta\vartheta_{tt}dx+\gamma\int_0^\pi \vartheta_{tx}\vartheta_x dx\\ =&\alpha\int_0^\pi \vartheta_t^2dx+\kappa\int_0^\pi \vartheta_{xx}\vartheta dx-\beta\int_0^\pi \phi_{tx}\vartheta dx+\gamma\int_0^\pi \vartheta_{txx}\vartheta dx+\gamma\int_0^\pi \vartheta_{tx}\vartheta_x dx\\ =&-\kappa\int_0^\pi \vartheta_x^2dx+\alpha\int_0^\pi \vartheta_t^2dx+\beta\int_0^\pi \phi_t\vartheta_x dx.\end{aligned}$$

Consequently, (4.44) follows by using Young's and Poincaré's inequalities. □

At this point, we define the Lyapunov functional

$$\mathcal{L}(t):=NE(t)+N_1\mathcal{F}_1(t)+\frac{5\rho J}{b}\mathcal{F}_2(t)+\mathcal{F}_3(t)+\mathcal{F}_4(t), \tag{4.45}$$

where $N$ and $N_1$ are positive constants that will be determined later.

**Lemma 49.** *For $N$ sufficiently large, there exists $m_1, m_2>0$, such that*

$$m_1E(t)\leq\mathcal{L}(t)\leq m_2E(t). \tag{4.46}$$

*Proof.* A simple computation leads to

$$\begin{aligned}|\mathcal{L}(t)-NE(t)|\leq &JN_1\int_0^\pi|\phi\phi_t|dx+\frac{\rho bN_1}{\mu}\int_0^\pi\left|u_t\int_0^x\phi(y)\,dy\right|dx\\ &+\frac{5\mu J}{b}\int_0^\pi|\phi_t u_x|dx+\frac{5\rho\delta}{b}\int_0^\pi|\phi_x u_t|dx+\rho\int_0^\pi|uu_t|dx\\ &+\alpha\int_0^\pi|\vartheta\vartheta_t|dx+\frac{\gamma}{2}\int_0^\pi|\vartheta_x^2|dx.\end{aligned}$$

Young's, Cauchy-Schwarz's and Poincaré's inequalities lead to

$$\begin{aligned}|\mathcal{L}(t)-NE(t)| \leq & c\left(\int_0^\pi u_t^2dx+\int_0^\pi u_x^2dx+\int_0^\pi \phi_t^2dx+\int_0^\pi \phi^2dx+\int_0^\pi \phi_x^2dx\right.\\ & \left.+\int_0^\pi \vartheta_t^2dx+\int_0^\pi \vartheta_x^2dx\right) \leq \lambda E(t),\end{aligned}$$

for some positive constant $\lambda$. Consequently,

$$(N-\lambda)\, E(t) \leq \mathcal{L} \leq (N+\lambda)\, E(t).$$

It suffices to choose $N$ large such that $N-\lambda>0$ and take $m_1=N-\lambda$ and $m_2=N+\lambda$. $\square$

**Lemma 50.** *Let $(u,\phi,\vartheta)$ be the solution of system (4.7). Then, for $N, N_1>0$ sufficiently large, the Lyapunov functional $\mathcal{L}(t)$ satisfies, for some positive constants $d$ and $C$, the estimate*

$$\mathcal{L}'(t) \leq -dE(t)+C\int_0^\pi \left(\phi_t^2+h^2(\phi_t)\right)dx,\ \forall t>0. \tag{4.47}$$

*Proof.* Differentiating $\mathcal{L}(t)$ then using estimates (4.31), (4.35), (4.39), (4.43) and (4.44), we arrive at

$$\begin{aligned}\mathcal{L}'(t) \leq & -\left(\gamma N-c(N_1+1)-\frac{5\rho Jc}{b}\right)\int_0^\pi \vartheta_{xt}^2dx\\ & -\mu\int_0^\pi u_x^2dx-(\rho-\varepsilon_1N_1)\int_0^\pi u_t^2dx-\left(\left(\xi-\frac{b^2}{\mu}\right)N_1-\frac{5Jc\rho}{b}\right)\int_0^\pi \phi^2dx\\ & -\left(\frac{\delta N_1}{2}-\frac{5\rho Jc}{b}-\frac{b^2C_p^2}{2\mu}\right)\int_0^\pi \phi_x^2dx-\frac{\kappa}{2}\int_0^\pi \vartheta_x^2dx\\ & +\left(c\left(1+\frac{1}{\varepsilon_1}\right)N_1+c\right)\int_0^\pi \phi_t^2dx\\ & +\left(cN_1+\frac{c5\rho J}{b}\right)\int_0^\pi h^2(\phi_t)\,dx.\end{aligned}$$

At this point, we carefully select the constants $\varepsilon_1, N$ and $N_1$. Initially, we select a sufficiently large $N_1$ such that

$$\left(\xi-\frac{b^2}{\mu}\right)N_1-\frac{5Jc\rho}{b}>0 \text{ and } \frac{\delta N_1}{2}-\frac{5\rho Jc}{b}-\frac{b^2C_p}{2\mu}>0.$$

Next, we select $\varepsilon_1$ small enough so that

$$\rho-\varepsilon_1N_1>0.$$

Finally, we choose a sufficiently large value of $N$ that satisfies (4.46) and

$$\gamma N - c(N_1 + 1) - \frac{5\rho J c}{b} > 0.$$

Thus, we end with the inequality

$$\mathcal{L}'(t) \leq -\lambda_0 \int_0^\pi \left(u_t^2 + \phi_t^2 + \vartheta_t^2 + u_x^2 + \phi^2 + \phi_x^2 + \vartheta_x^2\right) dx + C \int_0^\pi \left(\phi_t^2 + h^2(\phi_t)\right) dx$$

for some $\lambda_0, C > 0$. Consequently, (4.47) follows for some $d > 0$. $\square$

We now report and demonstrate our primary stability finding, which is as follows.

**Theorem 51.** *Let $(u, \phi, \vartheta)$ be the solution of problem (4.7). Assume that the hypothesis (B) holds. Then there exist positive constants $k_2, k_3, k_4, \varepsilon_0$ such that*

$$E(t) \leq k_4 G_1^{-1}(k_2 t + k_3), \text{ for all } t \in \mathbb{R}_+, \tag{4.48}$$

*where*

$$G_1(\tau) = \int_\tau^1 \frac{1}{sH'(\varepsilon_0 s)} ds.$$

*Proof.* We examine two cases:

**Case 1**: $h_0$ is linear on $[0, \varepsilon]$.

In this case, $H(s) = \sqrt{\frac{s}{2}} h_0\left(\sqrt{\frac{s}{2}}\right) = cs$ is also linear. According to hypothesis (B), we deduce that there exist positive constants $c_1' < c_2'$ such that

$$c_1'|s| \leq |h(s)| \leq c_2'|s|, \text{ for all } s \neq 0$$

and, consequently,

$$h^2(s) \leq c_2' s h(s), \text{ for all } s. \tag{4.49}$$

Therefore, using (4.9), (4.10), (4.31) and (4.47), we arrive at

$$\mathcal{L}'(t) \leq -dE(t) + C \int_0^\pi \phi_t h(\phi_t)\, dx \leq -dE(t) - CE'(t) \quad \forall t > 0.$$

Recalling (4.46), we have $\mathcal{F} := \mathcal{L}(t) + CE(t) \sim E(t)$ and satisfies $\mathcal{F}'(t) \leq -d\mathcal{F}(t)$.
A simple integration yields, for some $\eta > 0$,

$$E(t) \leq C\mathcal{F}(0)e^{-\eta t} = G_1^{-1}(\eta t + C(\mathcal{F}(0))), \ \forall t \geq 0. \tag{4.50}$$

**Case 2:** $h_0$ is nonlinear on $[0, \varepsilon]$.

In this case, as in [29], we choose $0 < \varepsilon_1 \leq \varepsilon$ such that

$$sh(s) \leq min\left\{\varepsilon, h_0(\varepsilon)\right\}, \text{ for all } |s| \leq \varepsilon_1.$$

First, for $\varepsilon_1 < |s| \leq \varepsilon$, we have

$$|h(s)| \leq \frac{h_0^{-1}(|s|)}{|s|}|s| \leq \frac{h_0^{-1}(|s|)}{\varepsilon_1}|s| \leq \frac{h_0^{-1}(\varepsilon)}{\varepsilon_1}|s|$$

and

$$|h(s)| \geq \frac{h_0(|s|)}{|s|}|s| \geq \frac{h_0(|s|)}{\varepsilon}|s| \geq \frac{h_0(\varepsilon_1)}{\varepsilon}|s|,$$

which, with (B) and (4.9), gives

$$\begin{cases} c_1'|s| \leq |h(s)| \leq c_2'|s|, \text{ for all } \varepsilon_1 < |s| \leq \varepsilon, \\ s^2 + h^2(s) \leq H^{-1}(sh(s)), \text{ for all } |s| \leq \varepsilon_1. \end{cases} \tag{4.51}$$

Next, for $\varepsilon_1 < |s| \leq \varepsilon$, we have

$$h^2(s) \leq c_2' sh(s) \text{ and } c_1' s^2 \leq sh(s),$$

therefore,

$$s^2 + h^2(s) \leq csh(s), \text{ for all } \varepsilon_1 < |s| \leq \varepsilon, \tag{4.52}$$

At this stage, we will address the integral in (4.47). For this purpose, we consider, as in [27], the following partition:

$$S_1 = \left\{y \in (0, \pi) : |\phi_t(y)| \leq \varepsilon_1\right\}, \quad S_2 = \left\{y \in (0, \pi) : |\phi_t(y)| > \varepsilon_1\right\}$$

and define

$$J(t) := \int_{S_1} \phi_t h\left(\phi_t\right) dx.$$

Jensen's inequality and the fact that $H^{-1}$ is concave lead to

$$H^{-1}(J(t)) \geq c \int_{S_1} H^{-1}\left(\phi_t h\left(\phi_t\right)\right) dx. \tag{4.53}$$

Thus, combining (4.51), (4.52) and (4.53), we arrive at

$$\begin{aligned}\int_0^{\pi}\left(\phi_t^2+h^2\left(\phi_t\right)\right)dx &= \int_{S_1}\left(\phi_t^2+h^2\left(\phi_t\right)\right)dx+\int_{S_2}\left(\phi_t^2+h^2\left(\phi_t\right)\right)dx\\ &\leq \int_{S_1}H^{-1}\left(\phi_t h\left(\phi_t\right)\right)dx+c\int_{S_2}\phi_t h\left(\phi_t\right)dx\\ &\leq cH^{-1}(J(t))-CE'(t).\end{aligned}$$

Hence, (4.47) becomes

$$\mathscr{L}_0'(t)\leq -dE(t)+cH^{-1}(J(t)),\quad \forall t\geq 0, \tag{4.54}$$

where $\mathscr{L}_0=\mathcal{L}+CE\sim E$.

Now, for $\delta_0>0$ and $\varepsilon_0<\varepsilon$ , we introduce the functional

$$\mathscr{L}_1(t):=H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathscr{L}_0(t)+\delta_0 E(t).$$

Recalling that $E'\leq 0$, $H'>0$, $H''>0$ on $(0,\varepsilon]$ and using (4.54), we deduce that the functional $\mathscr{L}_1$ satisfies, for some $\alpha_1,\alpha_2>0$, the estimate

$$\alpha_1\mathscr{L}_1(t)\leq E(t)\leq \alpha_2\mathscr{L}_1(t),\ \text{for all } t>0 \tag{4.55}$$

and for *a.e.* $t\geq t_0$,

$$\begin{aligned}\mathscr{L}_1'(t)&=\varepsilon_0\frac{E'(t)}{E(0)}H''\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathscr{L}_0(t)+H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathscr{L}_0'(t)+\delta_0E'(t)\\ &\leq -dE(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)+cH'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)H^{-1}(J(t))+\delta_0E'(t).\end{aligned} \tag{4.56}$$

Now, let $H^*$ be the convex Young conjugate of $H$, then for $s\in(0,H'(\varepsilon)]$,

$$H^*(s)=s(H')^{-1}(s)-H[(H')^{-1}(s)]\leq s(H')^{-1}(s). \tag{4.57}$$

By using the general Young inequality (1.2), for $s=H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)$ and $x=H^{-1}(J(t))$, we get

$$H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)H^{-1}(J(t))\leq H^*\left(H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right)+J(t).$$

Recalling (4.31) and using (4.57), we arrive at

$$cH'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)H^{-1}(J(t))\leq c\varepsilon_0\frac{E(t)}{E(0)}H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)-cE'(t). \tag{4.58}$$

Combining (4.56) and (4.58), we obtain

$$\begin{aligned}\mathscr{L}_1'(t) &\leq -dE(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)+c\varepsilon_0\frac{E(t)}{E(0)}H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)-CE'(t)+\delta_0E'(t)\\ &=-\left(dE(0)-C\varepsilon_0\right)\frac{E(t)}{E(0)}H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)-\left(C-\delta_0\right)E'(t).\end{aligned}$$

Thus, by selecting $\delta_0$ high enough such that $C-\delta_0<0$ and $\varepsilon_0$ small enough such that $dE(0)-C\varepsilon_0>0$, we obtain, for all $t\in\mathbb{R}^+$,

$$\mathscr{L}_1'(t)\leq -k\frac{E(t)}{E(0)}H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)=-kG_2\left(\frac{E(t)}{E(0)}\right), \tag{4.59}$$

where $G_2(s)=sH'(\varepsilon_0 s)$.

From hypothesis (B), we infer that $G_2\in C^1\left([0,+\infty)\right)$ and $G_2(0)=0$. In addition, the convexity of $H$ entails that

$$G_2'(t)=H'(\varepsilon_0 t)+\varepsilon_0 tH''(\varepsilon_0 t)>0,\ \text{for all } t>0.$$

Thus, with $R(t)=\dfrac{\alpha_1}{E(0)}\mathscr{L}_1(t)$, and by the use of (4.55) and (4.59), we have

$$R(t)\sim E(t) \tag{4.60}$$

and

$$R'(t)\leq -k_2G_2\left(R(t)\right), \tag{4.61}$$

for some $k_2>0$, which leads to

$$\frac{dG_1(R(t))}{dt}=-\frac{R'(t)}{G_2(R(t))}\geq k_2,\ \text{for all } t>0.$$

Thus, by integrating over $[0,t]$, bearing in mind the properties of $G_2$, and the fact that $G_1$ is strictly decreasing on $(0,1]$ we obtain, for some $k_3>0$,

$$R(t)\leq G_1^{-1}\left(k_2t+k_3\right)\qquad \forall t\in\mathbb{R}^+. \tag{4.62}$$

Finally, (4.48) follows from (4.50), (4.60) and (4.62). The proof of Theorem 51 is completes.

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