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ASYMPTOTIC ANALYSIS OF A NON-STATIONARY OF THE BINGHAM FLUID

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Abstract: The aim of this paper is to study the asymptotic behavior of an incompressible Bingham fuid in a dynamic regime occupying a bounded domain of R^3 with nonlinear friction of Tresca type. Firstly, the existence and uniqueness of weak solution is proved. Then we show the estimates for the velocity field and the pressure independently of the parameter ε. Finally, we give a specic Reynolds equation associated with variational inequalities and prove the uniqueness. The proof uses the asymptotic behavior when the dimension of the domain tends to zero.
Keywords: A priori inequalities; Asymptotic approach; Bingham fluid; Reynolds equation; Tresca law; Weak solution...
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1 Introduction

In this work we analyses the asymptotic behavior of an incompressible Bingham fluid in a dynamic regime occupying a bounded homogeneous domain $\Omega^\varepsilon \subset \mathbb{R}^3$ with boundary $\Gamma^\varepsilon = \bar{\Gamma}_1^\varepsilon \cup \bar{\Gamma}_L^\varepsilon \cup \bar{\omega}$,
The model of Bingham fluid is a non-Newtonian fluid, whose flow properties differ in any way from those of any Newtonian fluids. There are many phenomena in nature and industry exhibiting the behavior of the Bingham fluid medium. For instance, the flow of metals, plastic solids and some polymers. Let us mention the work which is realized by the authors in [1], in which they mainly consider a problem describing the motion of an incompressible, isothermal and non-Newtonian fluid in a three-dimensional thin domain with Tresca law . The existence and uniqueness solutions a two dimensional Navier-Stokes shear ow with time dependent boundary driving is prove in [2]. In [3], the author has given in the last chapter of his doctoral thesis the asymptotic behavior of a Bingham fluid in a thin domain. Unfortunately this work is not done due to the difficulty encountered in this study which resides on the choice of test functions because of the boundary conditions imposed. Then in [4], the authors studied the same problem, in which, only the Dirichlet conditions on the boundary have been considered. The asymptotic analysis of a Bingham fluid in a three dimensional bounded domain with Fourier and Tresca boundary conditions are also studied in [5, 6]. The authors in [7] studied the existence and the results of regularity for the d - dimensional Bingham fluid flow problem with Dirichlet boundary conditions. We note here, these papers were restricted to the stationary case only.
This work is a companion of the result in [6]. The novelty here consist in the fact that we study the asymptotic behavior of the same problem with Tresca free boundary friction conditions but this time in a dynamic regime occupying a bounded homogeneous domain $\Omega^\varepsilon \subset \mathbb{R}^3$. A special attention is devoted to the appearance of the punctual derivation $\frac{du}{dt} = \frac{\partial u}{\partial t} + u\nabla u$. For this, our initial domain will transpose to a new one that is independent of the parameter ε. This desired goal is achieved by the change of scale of the third component by $z = x_3/\varepsilon$ and we use the same change of variables as in [1, 6]

2 Problem statement and variational formula- tion

Let ω be fixed region in plan $x = (x_1, x_2) \in \mathbb{R}^2$. We suppose that ω has a Lipschitz boundary and is the bottom of the fluid domain. The upper surface Γ_1^ε is defined by $x_3 = \varepsilon h(x)$ where $(0 < \varepsilon < 1)$ is a small parameter that will tend to zero and h a smooth bounded function such that $0 < \underline{h} < h(x) < \bar{h}$ for all $(x, 0) \in \omega$. We denote by Ω^ε the domain of the flow:

$$\Omega^{\varepsilon}=\left\{(x,x_3)\in\mathbb{R}^3:(x,0)\in\omega,0<x_3<\varepsilon h(x)\right\}$$

and Γ^{ε} its boundary : $\Gamma^{\varepsilon}=\bar{\Gamma}_1^{\varepsilon}\cup\bar{\Gamma}_L^{\varepsilon}\cup\bar{\omega}$, where $\bar{\Gamma}_L^{\varepsilon}$ is the lateral boundary.
- The flow during the time $[0,T]$ is given by the equation

$$\frac{du^{\varepsilon}}{dt}-\operatorname{div}\sigma^{\varepsilon}=f^{\varepsilon}\text{ in }\Omega^{\varepsilon}\times(0,T)\tag{2.1}$$

where $\frac{du}{dt}=\frac{\partial u}{\partial t}+u\nabla u$ is the punctual derivation, and $\operatorname{div}(\sigma^{\varepsilon})=\left(\sigma_{ij,j}^{\varepsilon}\right)$.
- The fluid is supposed to be viscoplastic, and the relation between σ^{ε} and $D(u^{\varepsilon})$ is given by

$$\begin{cases}\sigma_{ij}^{\varepsilon}=\tilde{\sigma}_{ij}^{\varepsilon}-p^{\varepsilon}\delta_{ij}\\ \tilde{\sigma}^{\varepsilon}=\alpha^{\varepsilon}\frac{D(u^{\varepsilon})}{|D_{II}(u^{\varepsilon})|}+2\mu D(u^{\varepsilon})\ \text{ if } D(u^{\varepsilon})\neq 0,\\ |\tilde{\sigma}^{\varepsilon}|\leq\alpha^{\varepsilon}\text{ if } D(u^{\varepsilon})=0\end{cases}\tag{2.2}$$

where σ^{ε} represents the constitutive law of a Bingham fluid, $\alpha^{\varepsilon}\geq 0$ is the yield stress, μ is the constant viscosity, u^{ε} is the velocity field, p^{ε} the pressure, δ_{ij} is the Kronecker symbol, and $D(u^{\varepsilon})=\frac{1}{2}\left(\nabla u^{\varepsilon}+(\nabla u^{\varepsilon})^T\right)$. For any tensor $D=d_{ij}$, the notation $|D|$ represents the matrix norm: $|D_{II}|=\left(\sum_{i,j}^{3}\frac{1}{2}d_{ij}d_{ij}\right)^{\frac{1}{2}}$
- The incompressibility equation

$$\operatorname{div}(u^{\varepsilon}(t))=0\text{ in }\Omega^{\varepsilon}\times(0,T)\tag{2.3}$$

Our boundary conditions is describe as
- At the surface $\Gamma_1\cup\Gamma_L$ we assume

$$u^{\varepsilon}(t)=0\text{ in }\ (\Gamma_1\cup\Gamma_L)\times(0,T)\tag{2.4}$$

- On ω, there is a no-flux condition across ω so that

$$u^{\varepsilon}(t)\times n=0\tag{2.5}$$

the tangential velocity on ω is unknown and satisfies Tresca boundary conditions:

$$\begin{cases}|\sigma_{\tau}^{\varepsilon}|<k^{\varepsilon}\Longrightarrow u_{\tau}^{\varepsilon}(t)=0\\ |\sigma_{\tau}^{\varepsilon}|=k^{\varepsilon}\Longrightarrow\exists\lambda\geq 0,u_{\tau}^{\varepsilon}(t)=-\lambda\sigma_{\tau}^{\varepsilon}\end{cases}\quad\text{in }\omega\tag{2.6}$$

where k^{ε} is the friction yield coefficient.
- The conditions initials

$$u^{\varepsilon}(0)=\frac{du^{\varepsilon}}{dt}(0)=0\tag{2.7}$$

Here $|.|$ is the euclidean norm in $\mathbb{R}^2$; $n=(n_1,n_2,n_3)$ is the unit outward normal vector on the boundary Γ^{ε}. The normal and the tangential components on the boundary ω are given by

$$u_n^{\varepsilon}=u^{\varepsilon}\cdot n=u_i^{\varepsilon}\cdot n_i,\quad u_{\tau_i}^{\varepsilon}=u_i^{\varepsilon}-u_n^{\varepsilon}n_i,$$
$$\sigma_n^{\varepsilon}=(\sigma\cdot n)n=\sigma_{ij}^{\varepsilon}n_in_j,\quad\sigma_{\tau_i}^{\varepsilon}=\sigma_{ij}^{\varepsilon}n_j-\sigma_n^{\varepsilon}n_i,$$

The complete problem therefore consists in finding a velocity field u^{ε} and a pressure p^{ε} satisfying the following equations and boundary conditions:

$$(\text{ pb})\begin{cases}\frac{du^{\varepsilon}}{dt}-\operatorname{div}\sigma^{\varepsilon}=f^{\varepsilon}\text{ in }\Omega^{\varepsilon}\times(0,T).\\ \frac{du}{dt}=\frac{\partial u}{\partial t}+u\nabla u\\ \operatorname{div}(u^{\varepsilon}(t))=0\text{ in }\Omega^{\varepsilon}\times(0,T)\\ u^{\varepsilon}(t)=0\text{ in }\bar{\Gamma}_1^{\varepsilon}\cup\bar{\Gamma}_L^{\varepsilon}\times(0,T)\\ u^{\varepsilon}(t)\times n=0\\ u^{\varepsilon}(0)=\frac{du^{\varepsilon}}{dt}(0)=0\\ \left.\begin{array}{c}|\sigma_{\tau}^{\varepsilon}|<k^{\varepsilon}\Longrightarrow u_{\tau}^{\varepsilon}(t)=0\\ |\sigma_{\tau}^{\varepsilon}|=k^{\varepsilon}\Longrightarrow\exists\lambda\geq 0,u_{\tau}^{\varepsilon}(t)=-\lambda\sigma_{\tau}^{\varepsilon}\end{array}\right\}\quad\text{in }\omega.\end{cases}\tag{2.8}$$

To get a weak formulation, we consider the functional framework on Ω^ε

$$K^\varepsilon=\left\{\varphi\in H^1\left(\Omega^\varepsilon\right)^3:\varphi=0 \text{ on } \Gamma_1\cup\Gamma_L,\varphi\cdot n=0 \text{ on } \omega\right\},$$
$$K^\varepsilon_{div}=\left\{\varphi\in K^\varepsilon:\operatorname{div}(\varphi)=0\right\},$$
$$L^2_0\left(\Omega^\varepsilon\right)=\left\{q\in L^2\left(\Omega^\varepsilon\right):\int_{\Omega^\varepsilon} q dx dx_3=0\right\}.$$

Also, we use the following notions

$$a\left(u^\varepsilon(t),v^\varepsilon\right)=2\mu\int_{\Omega^\varepsilon} D\left(u^\varepsilon(t)\right)D\left(v^\varepsilon\right)dxdx_3,$$
$$B\left(u^\varepsilon(t),u^\varepsilon(t),v^\varepsilon\right)=\int_{\Omega^\varepsilon} u^\varepsilon(t)\nabla u^\varepsilon(t)v^\varepsilon dxdx_3,$$
$$\left(p^\varepsilon,div(\varphi)\right)=\int_{\Omega^\varepsilon} p^\varepsilon div\varphi dxdx_3,$$
$$\phi\left(v^\varepsilon\right)=2\alpha^\varepsilon\int_{\Omega^\varepsilon}\left|D\left(v^\varepsilon\right)\right|dxdx_3+\int_\omega k^\varepsilon\left|v^\varepsilon\right|dx$$
$$\left(f^\varepsilon(t),v^\varepsilon\right)=\int_{\Omega^\varepsilon} f^\varepsilon(t)v^\varepsilon dxdx_3.$$

A formal application of Green's formula, using (1) - (7) leads to
Problem 1. Find $(u^\varepsilon,p^\varepsilon)$ where $u^\varepsilon(t)\in K^\varepsilon_{div}$, $\frac{\partial u^\varepsilon}{\partial t}(t)\in K^\varepsilon$ and $p^\varepsilon(t)\in L^2_0\left(\Omega^\varepsilon\right)$, such that

$$\left.\begin{array}{c}\left(\frac{\partial u^\varepsilon}{\partial t}(t),\varphi-u^\varepsilon(t)\right)+a\left(u^\varepsilon(t),\varphi-u^\varepsilon(t)\right)+B\left(u^\varepsilon(t),u^\varepsilon(t),\varphi\right)\\ -\left(p^\varepsilon,\operatorname{div}(\varphi)\right)+\phi(\varphi)-\phi\left(u^\varepsilon(t)\right)\geq\left(f^\varepsilon(t)-\varphi-u^\varepsilon(t)\right),\forall\varphi\in K^\varepsilon\\ u^\varepsilon(0)=\frac{du^\varepsilon}{dt}(0)=0.\end{array}\right\}\tag{2.9}$$

3 Change of the domain and some estimates

According to the change of variables $z=\frac{x_3}{\varepsilon}$, we define the fixed domain Ω which is independent of ε

$$\Omega=\left\{(x,z)\in\mathbb{R}^3:(x,0)\in\omega,0<z<h(x)\right\}$$

We denote by $\Gamma=\bar\Gamma_1\cup\bar\Gamma_L\cup\bar\omega$ its boundary, then we define the following functions in Ω

$$\begin{aligned}&\hat u^\varepsilon_i(x,z,t)=u^\varepsilon_i\left(x,x_3,t\right);i=1,2,\hat u^\varepsilon_3(x,z,t)=\varepsilon^{-1}u^\varepsilon_3\left(x,x_3,t\right) \text{ and}\\ &\hat p^\varepsilon(x,z)=\varepsilon^2p^\varepsilon\left(x,x_3\right)\end{aligned}$$

Let us assume that

$$\hat f^\varepsilon(x,z,t)=f^\varepsilon\left(x,x_3,t\right),\hat\alpha=\varepsilon\alpha^\varepsilon,\hat k=\varepsilon k^\varepsilon.\tag{3.1}$$

Now we introduce the functional framework on Ω. For this, we write

$$\begin{aligned}K&=\left\{\hat\varphi\in\left(H^1(\Omega)\right)^3:\hat\varphi=0 \text{ on } \Gamma_1\cup\Gamma_L,\hat\varphi\cdot n=0 \text{ on } \omega\right\},\\ K_{\text{div}}&=\{\hat\varphi\in K:\operatorname{div}(\hat\varphi)=0\},\\ V_z&=\left\{\hat\varphi\in\left(L^2(\Omega)\right)^2;\frac{\partial\hat\varphi_i}{\partial z}\in L^2(\Omega):\hat\varphi=0 \text{ on } \Gamma_1\cup\Gamma_L\right\},\\ \tilde V_z&=\{\hat\varphi\in V_z:\hat\varphi \text{ satisfy (3.2) },.\end{aligned}$$

where the condition (3.2) is given by

$$\int_\Omega\left(\hat\varphi_1\frac{\partial\theta}{\partial x_1}+\hat\varphi_2\frac{\partial\theta}{\partial x_2}\right)dxdz=0, \text{ for all } \hat\varphi\in\left(L^2(\Omega)\right)^2 \text{ and } \theta\in C^\infty_0(\Omega)\tag{3.2}$$

and V_z is the Banach space with the norm

$$\|v\|_{V_z}=\left(\sum_{i=1}^{2}\left(\|v_i\|_{0,\Omega}^2+\left\|\frac{\partial v_i}{\partial z}\right\|_{0,\Omega}^2\right)\right)^{\frac{1}{2}}$$

By injecting the new data and unknown factors in (8) then after multiplication by ε, we deduce

$$\left.\begin{array}{c}\left(\frac{\partial \hat{u}^{\varepsilon}}{\partial t}(t),\hat{\varphi}-\hat{u}^{\varepsilon}(t)\right)+a_0\left(\hat{u}^{\varepsilon}(t),\hat{\varphi}-\hat{u}^{\varepsilon}(t)\right)+B_0\left(\hat{u}^{\varepsilon}(t),\hat{u}^{\varepsilon}(t),\hat{\varphi}\right)-\\ \left(\hat{p}^{\varepsilon},\operatorname{div}\left(\hat{\varphi}-\hat{u}^{\varepsilon}(t)\right)\right)+\phi_0(\hat{\varphi})-\phi_0\left(\hat{u}^{\varepsilon}(t)\right)\geq\left(\hat{f},\hat{\varphi}-\hat{u}^{\varepsilon}(t)\right),\forall\hat{\varphi}\in K\\ \hat{u}^{\varepsilon}(0)=\frac{d\hat{u}^{\varepsilon}}{dt}(0)=0\end{array}\right\}\tag{3.3}$$

In the next, we establish the estimates and convergences for the velocity field $\hat{u}^{\varepsilon}$ and the pressure $\hat{p}^{\varepsilon}$ in the domain Ω. These estimates will be useful in order to prove the convergence of $(\hat{u}^{\varepsilon},\hat{p}^{\varepsilon})$ toward the expected functions. For this we need to establish the following result.

Theorem 1. If $f^{\varepsilon},\frac{\partial f^{\varepsilon}}{\partial t}\in L^2\left(0,T;L^2\left(\Omega^{\varepsilon}\right)^3\right),k^{\varepsilon}\in C_0^{\infty}(\omega),k^{\varepsilon}>0$ does not depend on t, then there exists a constant C and C' independent of ε such that

$$\begin{array}{c}\sum_{i;j=1}^{2}\left\|\varepsilon\frac{\partial\hat{u}_i^{\varepsilon}}{\partial x_j}\right\|_{L^2(0,T,L^2(\Omega))}^2+\left\|\varepsilon\frac{\partial\hat{u}_3^{\varepsilon}}{\partial z}\right\|_{L^2(0,T,L^2(\Omega))}^2+\left\|\varepsilon^2\frac{\partial\hat{u}_3^{\varepsilon}}{\partial t}\right\|_{L^2(0,T,L^2(\Omega))}^2+\\ \sum_{i=1}^{2}\left\|\frac{\partial\hat{u}_i^{\varepsilon}}{\partial z}\right\|_{L^2(0,T,L^2(\Omega))}^2+\sum_{i=1}^{2}\left\|\varepsilon^2\frac{\partial\hat{u}_3^{\varepsilon}}{\partial x_i}\right\|_{L^2(0,T,L^2(\Omega))}^2+\sum_{i=1}^{2}\left\|\frac{\partial\hat{u}_i^{\varepsilon}}{\partial t}\right\|_{L^2(0,T,L^2(\Omega))}^2\leq C\end{array}\tag{3.4}$$

$$\begin{array}{c}\sum_{i;j=1}^{2}\left\|\varepsilon\frac{\partial^2\hat{u}_i^{\varepsilon}}{\partial x_j\partial t}\right\|_{L^2(0,T,L^2(\Omega))}^2+\left\|\varepsilon\frac{\partial^2\hat{u}_3^{\varepsilon}}{\partial z\partial t}\right\|_{L^2(0,T,L^2(\Omega))}^2+\left\|\varepsilon^2\frac{\partial^2\hat{u}_3^{\varepsilon}}{\partial t^2}\right\|_{L^2(0,T,L^2(\Omega))}^2+\\ \sum_{i=1}^{2}\left\|\frac{\partial^2\hat{u}_i^{\varepsilon}}{\partial z\partial t}\right\|_{L^2(0,T,L^2(\Omega))}^2+\sum_{i=1}^{2}\left\|\varepsilon^2\frac{\partial^2\hat{u}_3^{\varepsilon}}{\partial x_i\partial t}\right\|_{L^2(0,T,L^2(\Omega))}^2+\sum_{i=1}^{2}\left\|\varepsilon\frac{\partial^2\hat{u}_i^{\varepsilon}}{\partial t^2}\right\|_{L^2(0,T,L^2(\Omega))}^2\leq C\end{array}\tag{3.5}$$

$$\left\|\frac{\partial\hat{p}^{\varepsilon}}{\partial x_i}\right\|_{H^{-1}(\Omega)}\leq C'\quad\text{for } i=1,2\ ,\ \left\|\frac{\partial\hat{p}^{\varepsilon}}{\partial z}\right\|_{H^{-1}(\Omega)}\leq\varepsilon C'\tag{3.6}$$

4 Convergence results and limit problem

The question which naturally arises is to know what will be the asymptotic behavior of the fluid when the thickness of the thin film is very small.

Mathematically, it's about knowing: Do the speed field and the pressure admit a limit when ε tends towards zero and what is the limit problem who should check this limit?

The answer to the first question is given in Theorem 2. However, the answer to the second question will be dealt with in Theorems $(3,4,5)$.

Theorem 2. Under the same assumptions as in Theorem 1, there exist $u^*=(u_1^*,u_2^*)\in L^2\left(0,T,\tilde{V}_z\right)$ and $p^{\star}\in L_0^2(\Omega)$ such that

$$\left\{\begin{array}{c}\hat{u}_i^{\varepsilon}(t)\rightarrow u_i^*(t)\\ \frac{\partial\hat{u}_i^{\varepsilon}}{\partial t}(t)\rightarrow\frac{\partial u_i^*}{\partial t}(t)\end{array}\right.,\quad i=1,2\left\{\begin{array}{c}\text{weakly in } L^2\left(0,T,\tilde{V}_z\right)\\ \text{weakly in } L^{\infty}\left(0,T,\tilde{V}_z\right)\end{array}\right.\quad\left\{\begin{array}{c}\varepsilon\frac{\partial\hat{u}_i^{\varepsilon}}{\partial x_j}(t)\rightarrow 0\\ \varepsilon\frac{\partial^2\hat{u}_i^{\varepsilon}}{\partial x_j\partial t}(t)\rightarrow 0\end{array}\right.,\quad i,j=1,2\left\{\begin{array}{c}\text{weakly in } L^2\left(0,T,L^2(\Omega)\right)\\ \text{weakly in } L^{\infty}\left(0,T,\tilde{V}_z\right)\end{array}\right.$$

$$\left\{\begin{array}{l} \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}(t) \rightarrow 0 \\ \varepsilon^2 \frac{\partial^2 \hat{u}_3^\varepsilon}{\partial x_i \partial t}(t) \rightarrow 0 \end{array}\right., \quad i=1,2\left\{\begin{array}{l} \text{weakly in } L^2\left(0,T,L^2(\Omega)\right) \\ \text{weakly in } L^\infty\left(0,T,\tilde{V}_z\right) \end{array}\right. \quad \left\{\begin{array}{l} \varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial z}(t) \rightarrow 0 \\ \varepsilon \frac{\partial^2 \hat{u}_3^\varepsilon}{\partial z \partial t}(t) \rightarrow 0 \end{array}\right., \left\{\begin{array}{l} \text{weakly in } L^2\left(0,T,L^2(\Omega)\right) \\ \text{weakly in } L^\infty\left(0,T,\tilde{V}_z\right) \end{array}\right.$$

$$\varepsilon \hat{u}_3^\varepsilon(t) \rightarrow 0, \left\{\begin{array}{l} \text{weakly in } L^2\left(0,T,\tilde{V}_z\right), \\ \text{weakly in } L^\infty\left(0,T,\tilde{V}_z\right) \end{array}\right.$$

$$\left\{\ \hat{p}^\varepsilon \rightarrow p^*, \text{ weakly in } L_0^2(\Omega), p^* \text{ depend only of } x.\right.$$

In the following theorem, we prove that the limit solution satisfies a variational inequality.
Theorem 3. With the same assumptions of Theorem 2, the solution (u^*, p^*) satisfying the following relations

$$\begin{aligned} &\sum_{i=1}^{2} \int_{\Omega} \mu \frac{\partial u_i^*}{\partial z}(t) \frac{\partial\left(\hat{\varphi}_i - u_i^*(t)\right)}{\partial z} dxdz - \int_{\Omega} p^* \left(\frac{\partial \hat{\varphi}_1}{\partial x_1} + \frac{\partial \hat{\varphi}_2}{\partial x_2}\right) dxdz \\ &+\hat{\alpha} \int_{\Omega} \left(\left|\frac{\partial \hat{\varphi}}{\partial z}\right| - \left|\frac{\partial u^*(t)}{\partial z}\right|\right) dxdz + \int_{\omega} \hat{k}\left(|\hat{\varphi}| - |u^*(t)|\right) dx \\ &\geq \sum_{i=1}^{2} \int_{\Omega} \hat{f}_i(t)\left(\hat{\varphi} - u_i^*(t)\right) dxdz, \hat{\varphi} \in \Pi(K) \end{aligned} \tag{4.1}$$

Moreover if $\hat{\varphi}$ satisfy (3.2), then

$$\begin{aligned} &\sum_{i=1}^{2} \int_{\Omega} \mu \frac{\partial u_i^*}{\partial z}(t) \frac{\partial\left(\hat{\varphi}_i - u_i^*(t)\right)}{\partial z} dxdz + \hat{\alpha} \int_{\Omega} \left(\left|\frac{\partial \hat{\varphi}}{\partial z}\right| - \left|\frac{\partial u^*(t)}{\partial z}\right|\right) dxdz \\ &+ \int_{\omega} \hat{k}\left(|\hat{\varphi}| - |u^*(t)|\right) dx \geq \sum_{i=1}^{2} \int_{\Omega} \hat{f}_i(t)\left(\hat{\varphi} - u_i^*(t)\right) dxdz \end{aligned} \tag{4.2}$$

where

$$\Pi(K) = \left\{\hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2) \in H^1(\Omega) : \exists \hat{\varphi}_3 \text{ such that } \varphi = (\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3) \in K\right\}$$

Theorem 4. The variational inequality (4.2) is equivalent to the following system

$$\begin{gathered} \mu \int_{\Omega} \left|\frac{\partial u^*(t)}{\partial z}\right|^2 dxdz + \hat{\alpha} \int_{\Omega} \left|\frac{\partial u^*(t)}{\partial z}\right| dxdz + \\ \int_{\omega} \hat{k}\left|u^*\right| dx - \int_{\Omega} \hat{f}(t) u^*(t) dxdz = 0 \end{gathered}$$

and

$$\begin{gathered} \mu \int_{\Omega} \frac{\partial u^*(t)}{\partial z} \frac{\partial \hat{\psi}}{\partial z} dxdz + \hat{\alpha} \int_{\Omega} \left|\frac{\partial \hat{\psi}}{\partial z}\right| dxdz + \\ \int_{\omega} \hat{k}|\hat{\psi}| dx \geq \int_{\Omega} \hat{f}(t) \hat{\psi} dxdz, \forall \hat{\psi} \in \Sigma(K) \end{gathered}$$

where

$$\Sigma(K) = \left\{\hat{\psi} = \left(\hat{\psi}_1, \hat{\psi}_2\right) \in H^1(\Omega)^2 : \hat{\psi} \text{ satisfies } (11)\right\}$$

Theorem 5. Let us set

$$\sigma^* = \tilde{\sigma}^* - \nabla p^* \quad \text{and } \tilde{\sigma}^* = \mu \frac{\partial u^*}{\partial z}(t) + \hat{\alpha}\pi,$$

then

$$-\frac{\partial}{\partial z}\left[\mu \frac{\partial u^*(t)}{\partial z} + \hat{\alpha}\frac{\partial u^*(t)/\partial z}{|\partial u^*(t)/\partial z|}\right] = \hat{f}(t) - \nabla p^* \text{ in } L^2(\Omega)^2$$

where $\pi \in L^\infty(\Omega)^2$ and $\|\pi\|_{\infty,\Omega} \leq 1$
In the following theorem, we give the convergence of our problem towards the Reynolds equation.
Theorem 6. Under the assumptions of preceding theorems, u^* and p^* satisfy the following equality where

$$\tilde{F} = \int_0^h F(x,y,t)dy - \frac{h}{2}F(x,h,t), \quad F(x,y,t) = \int_0^y \int_0^\zeta \hat{f}(x,\xi,t)d\xi d\zeta$$

The uniqueness of the limit velocity and pressure are given in the following theorem:

Theorem 7. The solution (u^*, p^*) of inequality (4.1) is unique in $L^2(0,T,V_z) \times \left(L_0^2(\omega) \cap H^1(\omega)\right)$.

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LIOUVILLE THEOREMS FOR NONLINEAR FRACTIONAL WAVE EQUATION ON THE HEISENBERG GROUP

MENECEUR BEKKAR

Abstract. This paper deals with the non-existence result for solutions to the problem:

$$(NLFWE):\begin{cases} \mathbf{D}^{\alpha}_{0/t}u-(\Delta_{\mathbb{H}})^{\delta/2}(|u|^m)=|\ u\ |^p, \quad (\eta,t)\in\mathbb{H}^N\times]0,+\infty[\\ u(\eta,0)=u_0(\eta)\geq 0\ ,\ \frac{\partial u}{\partial t}(\eta,0)=u_1(\eta)\geq 0, \text{ forall } \ \eta\in\mathbb{H}^N \end{cases}$$

where $\mathbf{D}^{\alpha}_{0/t}$ is the time-fractional derivative of order $\alpha\in(1,2)$ in the sense of Caputo, $(\Delta_{\mathbb{H}})^{\delta/2}$ is the fractional Laplacian of order $\delta/2$ with $1<\delta\leq 2$ in the $(2N+1)$-dimensional Heisenberg group $\mathbb{H}^N$. These non existence result Related to p less than the critical exponents that depends on α, δ, $Q=2N+2$, and $m\in\mathbb{N}$. For $\alpha=2$ we retrieve the result obtained by B. Ahmed et all [1] from the wave equation.

1. Introduction

Pohozaev and Véron [25] have established the question of nonexistence results for solutions of semilinear hyperbolic inequalities of the type:

$$(WI):\frac{\partial^2 u}{\partial t^2}-\Delta_{\mathbb{H}}(au)\geq |u|^p \tag{1.1}$$

it is shown that no weak solution u exists provided that:

$$\int_{\mathbb{R}^{2N+1}} u_1(\eta)d\eta\geq 0, \qquad \text{and} \quad 1<p\leq\frac{Q+1}{Q-1} \tag{1.2}$$

Their result have been generalized by B.Ahmed et all [1] to nonlinear non-local wave equation of the form:

$$(NLWE):\frac{\partial^2 u}{\partial t^2}-(\Delta_{\mathbb{H}})^{\delta/2}(|u|^m)=|u|^p \tag{1.3}$$

where they proved that the equation (NLWE) admits no solution defined in $\mathbb{H}^N$ whenever

$$1<p<\frac{2mQ+\delta}{2Q-\delta} \quad \text{and} \quad \int_{\mathbb{R}^{2N+1}} u_1(\eta)d\eta\geq 0. \tag{1.4}$$

In this paper we generalize this result to non-local wave equation with temporal fractional derivative of the type:

$$(NLFWE):\mathbf{D}^{\alpha}_{0/t}u-(\Delta_{\mathbb{H}})^{\delta/2}(|u|^m)=|\ u\ |^p \tag{1.5}$$

and we show under certain initial conditions that the equation (NLFWE) admits no solution defined in $\mathbb{H}^N$ if

$$1<p<m+\frac{\delta(m+1)}{\alpha Q-\delta} \tag{1.6}$$

2010 *Mathematics Subject Classification.* 35A01; 35B33; 35R03; 35R11; 35R45.

Key words and phrases. Critical exponent; fractional derivative; evolution equation; test function method; Heisenberg group .

2. Main results

Definition 2.1. *A weak solution u of the problem $(NLFWE)$ in $Q_T = \mathbb{R}^{2N+1} \times (0,T)$ with positive initial data $u_0, u_1 \in L^1_{loc}(\mathbb{R}^{2N+1})$, is a locally integrable function u such that $u \in L^{max(p,m)}(Q_T)$ satisfying*

$$\int_{Q_T} \left(-u D^{\alpha}_{t/T}\varphi + u\Delta_{\mathbb{H}}^{\frac{\delta}{2}}\varphi + |u|^p\varphi + u_1(\eta) D^{\alpha-1}_{t/T}\varphi \right) d\eta dt + \int_{\mathbb{R}^{2N+1}} u_0(\eta) D^{\alpha-1}_{t/T}\varphi(0) d\eta = 0. \tag{2.1}$$

for any nonnegative test function $\varphi \in C^2\left((0,T]; H^{\delta}(\mathbb{R}^{2N+1})\right) \cap C^1\left((0,T]; H^{\delta}(\mathbb{R}^{2N+1})\right)$ such that $\varphi(.,T) = D^{\alpha-1}_{t/T}\varphi(.,T) = 0$.

Theorem 2.2. *Assume that*

$$1 < p < m + \frac{\delta(m+1)}{\alpha Q - \delta}$$

Then , there is no weak nontrivial solution u of the problem $(NLFWE)$.

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Faedo-Galerkin method for time-fractional reaction-diffusion problem

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Abstract: In this paper, we deal with the reaction-diffusion equation involving Caputo's time-fractional derivative of order $\alpha \in (0, 1)$, which can be used to simulate anomalous diffusion in fractal media. We start by giving a definition of weak solution of the time-fractional reaction-diffusion problem. Then, we establish the existence and uniqueness of a weak solution of the proposed model by using the Faedo-Galerkin method and compactness arguments.

Keywords: Faedo-Galerkin method, fractional derivative, reaction-diffusion equation, weak solution.

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Real and imaginary least rank solutions of the matrix equation $AXB = C$

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Abstract. This work is devoted to establish the extremal ranks of the two real matrices X_0 and X_1 in the least rank solution $X = X_0 + iX_1$ of the matrix equation $AXB = C$ over the field $\mathbb{C}$ of complex numbers, As consequences from these rank formulas we derive necessary and sufficient conditions of the matrix equation $AXB = C$ to have only real least rank solution or only pure imaginary least rank solution.

MSC [2020]: 15A24, 15A03, 15A09.
Key words: Matrix equation, Rank formulas, Moore-Penrose generalized inverse, Least-rank solution.

1 Introduction

Throughout this paper $\mathbb{C}^{m\times n}$ stand for the set of all $m \times n$ complex matrices, the symbols, A^*, $r(A)$ stand for the conjugate transpose, the rank of the matrix A respectively. I_m denotes the identity matrix of order m. The Moore-Penrose generalized inverse of a matrix $A \in \mathbb{C}^{m\times n}$, denoted by A^+, is defined to be the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four matrix equations:

$$(1)\ AXA = A,\ (2)\ XAX = X,\ \ (3)\ \ (AX)^* = AX,\ \ (4)\ \ (XA)^* = XA$$

The Moore-Penrose generalized inverse has been the subject of several papers, see [1].

Further, R_A and L_A stand for the two orthogonal projectors $R_A = I_m - AA^+$, $L_A = I_n - A^+A$ induced by $A \in \mathbb{C}^{m\times n}$.

We consider the linear matrix equation

$$AXB = C$$

where A, B, C are given and X is unknown.

This equation is the best known matrix equation in matrix theory and its applications, many studies and researchs about this equation were the subject of many authors. For instance,in [4] Mitra gave a representation of the general common solution of the system $A_1X_1B_1 = C_1$, $A_2X_2B_2 = C_2$. In case where the matrix equation is not consistent, the authors try to find its approximation solutions which they satisfy optimal criteria. such as the least squares solution and the least rank solution, In literature [6] Tian studied the relation between least squares and least-rank solutions of the matrix equation $AXB = C$ where he established necessary and sufficient conditions for the two types of solutions to coincide, in [2] Guerarra derived necessary and sufficient conditions for the matrix equation $AXB = C$ to have a Hermitian Re-positive or Re-negative definite solution.

To develope the content of this paper we need these importants Lemmas

Lemma 1 *[3]Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, $D \in \mathbb{C}^{l\times k}$. Then,*

$$r\begin{bmatrix} A & B \end{bmatrix} = r(A) + r(E_AB) = r(B) + r(E_BA), \qquad r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C),$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_BAF_C), \qquad r\begin{bmatrix} M & AF_P \\ N & BF_P \end{bmatrix} = r\begin{bmatrix} M & A \\ N & B \\ O & P \end{bmatrix} - r(P),$$

$$r\begin{bmatrix} A & BF_P \\ E_QC & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & Q \\ 0 & P & 0 \end{bmatrix} - r(P) - r(Q), \qquad r\begin{bmatrix} M & N \\ E_PA & E_PB \end{bmatrix} = r\begin{bmatrix} M & N & 0 \\ A & B & P \end{bmatrix} - r(P).$$

Lemma 2 *[7]Let* $A \in \mathbb{C}^{m\times n}$*,* $B \in \mathbb{C}^{m\times k}$ *and* $C \in \mathbb{C}^{l\times n}$ *be given. Then*

$$\max_{X\in\mathbb{C}^{k\times n},\ Y\in\mathbb{C}^{m\times l}} r(A+BX+YC) = \left\{m,\ n,\ r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\right\},$$

$$\min_{X\in\mathbb{C}^{k\times n},\ Y\in\mathbb{C}^{m\times l}} r(A+BX+YC) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r(B) - r(C).$$

2 Real and imaginary least rank solutions of $AXB = C$

Following the work of Y. Tian in [5], in which the author derived necessary and sufficient conditions for the two real matrices X_0 and X_1 in the least squares solution $X = X_0 + iX_1$ of the matrix equation $AXB = C$ to be real or imaginary matrices, in this work we derive these results on the other solution of this equation which is the least rank solution.

We consider the linear matrix equation

$$AXB = C \tag{1}$$

Where $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times n}$, $C \in \mathbb{C}^{m\times n}$, are given and $X \in \mathbb{C}^{p\times q}$ is unknown matrix.

Lemma 3 *[6] Let* $A \in \mathbb{C}^{m\times p}$*,* $B \in \mathbb{C}^{q\times n}$*,* $C \in \mathbb{C}^{m\times n}$ *are given. The least-rank solution of* $AXB = C$ *can be written as*

$$X = -TM^{+}S + T_1U + VS_1 \tag{2}$$

where $M = \begin{bmatrix} C & A \\ B & 0 \end{bmatrix}$*,* $T = \begin{bmatrix} 0 & I_p \end{bmatrix}$*,* $S = \begin{bmatrix} 0 \\ I_q \end{bmatrix}$*,* $T_1 = TF_M$*,* $S_1 = E_MS$*, and* U*,* V *are arbitrary matrices with appropriate sizes.*

Lemma 4 *[5] Let* $A = A_0 + iA_1 \in \mathbb{C}^{m\times p}$*,* $B = B_0 + iB_1 \in \mathbb{C}^{q\times n}$ *and* $C = C_0 + iC_1 \in \mathbb{C}^{m\times n}$ *be given. Then, the matrix equation* $AXB = C$ *is consistent over* $\mathbb{C}$ *if and only if the matrix equation*

$$\begin{bmatrix} A_0 & -A_1 \\ A_1 & A_0 \end{bmatrix}\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}\begin{bmatrix} B_0 & -B_1 \\ B_1 & B_0 \end{bmatrix} = \begin{bmatrix} C_0 & -C_1 \\ C_1 & C_0 \end{bmatrix} \tag{3}$$

is consistent over the field $\mathbb{R}$ *of real numbers. In this case the general solution of the matrix equation* $AXB = C$ *can be written as*

$$X = X_0 + iX_1 = \frac{1}{2}(Y_1 + Y_4) + \frac{i}{2}(Y_3 - Y_2), \tag{4}$$

where

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} = \Phi^{-}(A)\Phi(C)\Phi^{-}(B) + 2F_{\Phi(A)}\begin{bmatrix} V_1 & V_2 \end{bmatrix} + 2\begin{bmatrix} W_1 \\ W_2 \end{bmatrix}E_{\Phi(B)}.$$

is the general solution of (3), with V_1*,* $V_2 \in \mathbb{C}^{2p\times q}$*,* W_1*,* $W_2 \in \mathbb{C}^{p\times 2q}$*.*
Hence $Y_1, ..., Y_4$ *are the general solution of equation (3) over* $\mathbb{R}$*. Written in an explicit form,* X_0 *and* X_1 *in equation (4) are*

$$\begin{aligned} X_0 &= \frac{1}{2}P_1\Phi^{-}(A)\Phi(C)\Phi^{-}(B)Q_1 + \frac{1}{2}P_2\Phi^{-}(A)\Phi(C)\Phi^{-}(B)Q_2 \\ &\quad + \begin{bmatrix} P_1F_{\Phi(A)} & P_2F_{\Phi(A)} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} W_1 & W_2 \end{bmatrix}\begin{bmatrix} E_{\Phi(B)}Q_1 \\ E_{\Phi(B)}Q_2 \end{bmatrix}, \\ X_1 &= \frac{1}{2}P_2\Phi^{-}(A)\Phi(C)\Phi^{-}(B)Q_1 - \frac{1}{2}P_1\Phi^{-}(A)\Phi(C)\Phi^{-}(B)Q_2 \\ &\quad + \begin{bmatrix} P_2F_{\Phi(A)} & -P_1F_{\Phi(A)} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} W_1 & W_2 \end{bmatrix}\begin{bmatrix} -E_{\Phi(B)}Q_2 \\ E_{\Phi(B)}Q_1 \end{bmatrix}, \end{aligned}$$

where

$$\Phi(M) = \Phi(M_0 + iM_1) = \begin{bmatrix} M_0 & -M_1 \\ M_1 & M_0 \end{bmatrix}, \text{ for } M = A, B, C,$$

$$P_1 = \begin{bmatrix} I_p & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & I_p \end{bmatrix}, Q_1 = \begin{bmatrix} I_q \\ 0 \end{bmatrix}, Q_2 = \begin{bmatrix} 0 \\ I_q \end{bmatrix},$$

V_1, V_2, W_1 and W_2 are arbitrary over $\mathbb{R}$.

From Lemma 4 we derive the expressions of the two real matrices X_0 and X_1 in the least rank solution $X = X_0 + iX_1$ of the matrix equation (1) as

$$X_0 = -\frac{1}{2}P_1\widehat{T}\Phi^+(M)\widehat{S}Q_1 + \frac{1}{2}P_2\widehat{T}\Phi^+(M)\widehat{S}Q_2 + \begin{bmatrix} P_1\widehat{T}F_{\Phi(M)} & P_2\widehat{T}F_{\Phi(M)} \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + \begin{bmatrix} V_1 & V_2 \end{bmatrix}\begin{bmatrix} E_{\Phi(M)}\widehat{S}Q_1 \\ E_{\Phi(M)}\widehat{S}Q_2 \end{bmatrix} \tag{5}$$

$$X_1 = -\frac{1}{2}P_2\widehat{T}\Phi^+(M)\widehat{S}Q_1 + \frac{1}{2}P_1\widehat{T}\Phi^+(M)\widehat{S}Q_2 + \begin{bmatrix} P_2\widehat{T}F_{\Phi(M)} & -P_1\widehat{T}F_{\Phi(M)} \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + \begin{bmatrix} V_1 & V_2 \end{bmatrix}\begin{bmatrix} -E_{\Phi(M)}\widehat{S}Q_2 \\ E_{\Phi(M)}\widehat{S}Q_1 \end{bmatrix} \tag{6}$$

Theorem 5 *Suppose that the matrix equation (1) is consistent, we denote*

$$S_0 = \left\{X_0 \in \mathbb{C}^{p\times q} \mid AXB = A(X_0 + iX_1)B = C, \ r(C - AXB) = \min\right\},$$
$$S_1 = \left\{X_1 \in \mathbb{C}^{p\times q} \mid AXB = A(X_0 + iX_1)B = C, \ r(C - AXB) = \min\right\},$$

and

$$K = \begin{bmatrix} C_0 & A_0 & -C_1 & -A_1 \\ C_1 & A_1 & C_0 & A_0 \\ B_1 & 0 & B_0 & 0 \end{bmatrix}, L = \begin{bmatrix} C_0 & -C_1 & -A_1 \\ -B_0 & B_1 & 0 \\ C_1 & C_0 & A_0 \\ B_1 & B_0 & 0 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} C_0 & A_0 & -C_1 & -A_1 \\ C_1 & A_1 & C_0 & A_0 \\ B_0 & 0 & -B_1 & 0 \end{bmatrix}, L_1 = \begin{bmatrix} C_0 & -C_1 & -A_1 \\ B_0 & -B_1 & 0 \\ C_1 & C_0 & A_0 \\ B_1 & B_0 & 0 \end{bmatrix}$$

Then,
a)

$$\max_{X_0\in S_0} r(X_0) = \left\{p, \ q, \ \ p + q + r\begin{bmatrix} \Phi(M) & L \\ K & 0 \end{bmatrix} - 4r(M)\right\}, \tag{7}$$

$$\min_{X_0\in S_0} r(X_0) = r\begin{bmatrix} \Phi(M) & L \\ K & 0 \end{bmatrix} - r(L) - r(K). \tag{8}$$

b)

$$\max_{X_1\in S_1} r(X_1) = \left\{p, \ q, \ \ p + q + r\begin{bmatrix} \Phi(M) & L_1 \\ K_1 & 0 \end{bmatrix} - 4r(M)\right\} \tag{9}$$

$$\min_{X_1\in S_1} r(X_1) = r\begin{bmatrix} \Phi(M) & L_1 \\ K_1 & 0 \end{bmatrix} - r(L_1) - r(K_1). \tag{10}$$

Proof. Applying the Lemme 2 to (5) yields

$$\max_{X_0\in S_0} r(X_0) = \{p, \ q, \ r(H)\} \tag{11}$$

$$\min_{X_0\in S_0} r(X_0) = r(H) - r\begin{bmatrix} P_1\widehat{T}F_{\Phi(M)} & P_2\widehat{T}F_{\Phi(M)} \end{bmatrix} - r\begin{bmatrix} E_{\Phi(M)}\widehat{S}Q_1 \\ E_{\Phi(M)}\widehat{S}Q_2 \end{bmatrix} \tag{12}$$

Where

$$H=\begin{bmatrix} -\frac{1}{2}P_1\widehat{T}\Phi^{+}(M)\widehat{S}Q_1+\frac{1}{2}P_2\widehat{T}\Phi^{+}(M)\widehat{S}Q_2 & P_1\widehat{T}F_{\Phi(M)} & P_2\widehat{T}F_{\Phi(M)} \\ E_{\Phi(M)}\widehat{S}Q_1 & 0 & 0 \\ E_{\Phi(M)}\widehat{S}Q_2 & 0 & 0 \end{bmatrix}$$

Applying Lemma 1 and three types of elemantary bloc matrix operations we get,

$$r(H)=r\begin{bmatrix} -\frac{1}{2}P_1\widehat{T}\Phi^{+}(M)\widehat{S}Q_1+\frac{1}{2}P_2\widehat{T}\Phi^{+}(M)\widehat{S}Q_2 & P_1\widehat{T}F_{\Phi(M)} & P_2\widehat{T}F_{\Phi(M)} \\ E_{\Phi(M)}\widehat{S}Q_1 & 0 & 0 \\ E_{\Phi(M)}\widehat{S}Q_2 & 0 & 0 \end{bmatrix}$$

$$=r\begin{bmatrix} -\frac{1}{2}P_1\widehat{T}\Phi^{+}(M)\widehat{S}Q_1+\frac{1}{2}P_2\widehat{T}\Phi^{+}(M)\widehat{S}Q_2 & P_1\widehat{T} & P_2\widehat{T} & 0 & 0 \\ \widehat{S}Q_1 & 0 & 0 & \Phi(M) & 0 \\ \widehat{S}Q_2 & 0 & 0 & 0 & \Phi(M) \\ 0 & \Phi(M) & 0 & 0 & 0 \\ 0 & 0 & \Phi(M) & 0 & 0 \end{bmatrix}-4r(\Phi(M))$$

$$=p+q+r\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & C_0 & A_0 & -C_1 & -A_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & C_1 & A_1 & C_0 & A_0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & B_1 & 0 & B_0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_0 & A_0 & -C_1 & -A_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_0 & 0 & -B_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_1 & A_1 & C_0 & A_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & B_0 & 0 & -B_1 & 0 & -B_1 & 0 & -B_0 & 0 \\
C_0 & -C_1 & -A_1 & 0 & 0 & 0 & A_0 & -\frac{1}{2}C_0 & -\frac{1}{2}A_0 & \frac{1}{2}C_1 & \frac{1}{2}A_1 & 0 & 0 & 0 & 0 \\
B_0 & -B_1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}B_0 & 0 & \frac{1}{2}B_1 & 0 & 0 & 0 & 0 & 0 \\
C_1 & C_0 & A_0 & 0 & 0 & 0 & A_1 & -\frac{1}{2}C_1 & -\frac{1}{2}A_1 & -\frac{1}{2}C_0 & -\frac{1}{2}A_0 & 0 & 0 & 0 & 0 \\
B_1 & B_0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}B_1 & 0 & -\frac{1}{2}B_0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & C_0 & A_0 & -C_1 & A_1 & 0 & 0 & 0 & 0 & -\frac{1}{2}C_0 & -\frac{1}{2}A_0 & \frac{1}{2}C_1 & \frac{1}{2}A_1 \\
0 & 0 & 0 & B_0 & 0 & -B_1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}B_0 & 0 & \frac{1}{2}B_1 & 0 \\
0 & 0 & 0 & C_1 & A_1 & C_0 & -A_0 & 0 & 0 & 0 & 0 & -\frac{1}{2}C_1 & -\frac{1}{2}A_1 & -\frac{1}{2}C_0 & -\frac{1}{2}A_0 \\
0 & 0 & 0 & B_1 & 0 & B_0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}B_1 & 0 & -\frac{1}{2}B_0 & 0
\end{bmatrix}-4r(\Phi(M))$$

$$
= p + q + r \begin{bmatrix} 0 & 0 & 0 & 0 & C_0 & A_0 & -C_1 & -A_1 \\ 0 & 0 & 0 & 0 & B_0 & 0 & -B_1 & 0 \\ 0 & 0 & 0 & 0 & C_1 & A_1 & C_0 & A_0 \\ 0 & 0 & 0 & 0 & B_1 & 0 & B_0 & 0 \\ C_0 & A_0 & -C_1 & -A_1 & \frac{1}{2}C_0 & \frac{1}{2}A_0 & -\frac{1}{2}C_1 & -\frac{1}{2}A_1 \\ B_0 & 0 & -B_1 & 0 & \frac{1}{2}B_0 & 0 & -\frac{1}{2}B_1 & 0 \\ C_1 & A_1 & C_0 & A_0 & \frac{1}{2}C_1 & \frac{1}{2}A_1 & \frac{1}{2}C_0 & \frac{1}{2}A_0 \\ B_1 & 0 & B_0 & 0 & \frac{1}{2}B_1 & 0 & \frac{1}{2}B_0 & 0 \end{bmatrix}
$$

$$
+ r \begin{bmatrix} 0 & 0 & 0 & C_0 & A_0 & -C_1 & -A_1 \\ 0 & 0 & 0 & C_1 & A_1 & C_0 & A_0 \\ 0 & 0 & 0 & B_1 & 0 & B_0 & 0 \\ C_0 & -C_1 & -A_1 & \frac{1}{2}C_0 & \frac{1}{2}A_0 & -\frac{1}{2}C_1 & -\frac{1}{2}A_1 \\ -B_0 & B_1 & 0 & \frac{1}{2}B_0 & 0 & -\frac{1}{2}B_1 & 0 \\ C_1 & C_0 & A_0 & \frac{1}{2}C_1 & \frac{1}{2}A_1 & \frac{1}{2}C_0 & \frac{1}{2}A_0 \\ B_1 & B_0 & 0 & \frac{1}{2}B_1 & 0 & \frac{1}{2}B_0 & 0 \end{bmatrix} - 4r\left(\Phi\left(M\right)\right)
$$

$$
= p + q + r \begin{bmatrix} \Phi\left(M\right) & L \\ K & 0 \end{bmatrix} - 4r\left(M\right) \tag{13}
$$

Now we will compute,

$$
r \begin{bmatrix} P_1 \widehat{T} F_{\Phi(M)} & P_2 \widehat{T} F_{\Phi(M)} \end{bmatrix} = r \begin{bmatrix} P_1 \widehat{T} & P_2 \widehat{T} \\ \Phi\left(M\right) & 0 \\ 0 & \Phi\left(M\right) \end{bmatrix} - 2r\left(\Phi\left(M\right)\right)
$$

$$
= r \begin{bmatrix} I_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_0 & -C_1 & -A_1 & 0 & 0 & 0 & 0 \\ 0 & -B_0 & B_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_1 & C_0 & A_0 & 0 & 0 & 0 & 0 \\ 0 & B_1 & B_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_0 & A_0 & -C_1 & A_1 \\ 0 & 0 & 0 & 0 & B_0 & 0 & -B_1 & 0 \\ 0 & 0 & 0 & 0 & C_1 & A_1 & C_0 & -A_0 \\ 0 & 0 & 0 & 0 & B_1 & 0 & B_0 & 0 \end{bmatrix} - 2r\left(\Phi\left(M\right)\right)
$$

$$
= p + r\left(L\right) - 2r\left(M\right) \tag{14}
$$

$$
r \begin{bmatrix} E_{\Phi(M)} \widehat{S} Q_1 \\ E_{\Phi(M)} \widehat{S} Q_2 \end{bmatrix} = r \begin{bmatrix} \widehat{S} Q_1 & \Phi\left(M\right) & 0 \\ \widehat{S} Q_2 & 0 & \Phi\left(M\right) \end{bmatrix}
$$

$$
= r \begin{bmatrix} I_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_0 & A_0 & -C_1 & -A_1 & 0 & 0 & 0 & 0 \\ 0 & C_1 & A_1 & C_0 & A_0 & 0 & 0 & 0 & 0 \\ 0 & B_1 & 0 & B_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_0 & A_0 & -C_1 & -A_1 \\ 0 & 0 & 0 & 0 & 0 & B_0 & 0 & -B_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_1 & A_1 & C_0 & A_0 \\ 0 & 0 & 0 & 0 & 0 & B_1 & 0 & B_0 & 0 \end{bmatrix} - 2r\left(\Phi\left(M\right)\right)
$$

$$
= q + r\left(K\right) - 2r\left(M\right) \tag{15}
$$

By the substitution of (13), (14) and (15) into (11) and (12) we get (7) and (8). Similarly applying Lemma 2 to (6) yields (9) and (10). ■

Corollary 6 *Assume that the matrix equation $AXB = C$ is consistent. Then,*
a) There is a real least rank solution to $AXB = C$ if and only if

$$
r \begin{bmatrix} \Phi\left(M\right) & L_1 \\ K_1 & 0 \end{bmatrix} = r\left(L_1\right) + r\left(K_1\right).
$$

b) There is a pure imaginary least rank solution to $AXB = C$ if and only if

$$r\begin{bmatrix} \Phi(M) & L \\ K & 0 \end{bmatrix} = r(L) + r(K).$$

c) All least rank solutions of $AXB = C$ are real if and only if

$$r\begin{bmatrix} \Phi(M) & L_1 \\ K_1 & 0 \end{bmatrix} = 4r(M) - p - q.$$

d) All least rank solutions of $AXB = C$ are pure imaginary if and only if

$$r\begin{bmatrix} \Phi(M) & L \\ K & 0 \end{bmatrix} = 4r(M) - p - q$$

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On the exponential stability of a porous thermoelastic system with thermal dissipation given by Gurtin-Pipkin law

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Abstract

In this work we consider a porous thermoelastic system with one dissipation generated by the heat flux modeled by Gurtin-Pipkin thermal law. We use the semigroup approach and prove the existence of a unique solution. We introduce a stability number χ_g depends on the coefficients of the system and establish the exponential stability of the solution provided that $\chi_g = 0$. Our result improves the previous results obtained with Fourier's or Cattaneo's law of thermal conductivity.

Key words: Porous material; Gurtin-Pipkin law, exponential stability; stability number; well-posedness.

1 Introduction

We consider the following problem,

$$\begin{cases} \rho u_{tt} = a u_{xx} + b\phi_x & in \quad (0,\infty)\times(0,\pi), \\ J\phi_{tt} = \alpha\phi_{xx} - bu_x - \xi\phi - \beta\theta_x & in \quad (0,\infty)\times(0,\pi), \\ c\theta_t = -q_x - \beta\phi_{xt} & in \quad (0,\infty)\times(0,\pi), \\ q = -\displaystyle\int_{-\infty}^{0} g(t-s)\theta(x,s)ds, & in \quad (0,\infty)\times(0,\pi). \end{cases} \tag{1}$$

where u, ϕ, θ and q are respectively, the transversal displacement, the volume fraction, the difference of temperature from an equilibrium reference value and the heat flux of a one dimensional porous elastic material of length π. The coefficients $\rho, J, a, c, a, b, \alpha, \xi$ and κ are positive constitutive constants such that $a\xi > b^2$. The coefficient β is a coupling constant that is different from zero but its sign does not matter in the analysis.

The system is endowed with the boundary and the initial conditions

$$\begin{cases} u(x,0) = u_0(x),\ \phi(x,0) = \phi_0(x),\ \theta(x,0) = \theta_0(x), \\ u_t(x,0) = u_1(x),\ \phi_t(x,0) = \phi_1(x), \end{cases} \tag{2}$$

$$u(0,t) = u(\pi,t) = \phi_x(0,t) = \phi_x(\pi,t) = \theta(0,t) = \theta(\pi,t) = 0. \tag{3}$$

We introduce the new variables

$$\theta^t(x,s) := \theta(x,t-s),\ s \geq 0,$$

and

$$\eta(x,s) = \eta^t(x,s) := \int_0^s \theta^t(x,\tau)d\tau,\ s \geq 0,$$

The system (1) becomes

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x & in\ \ (0,\pi)\times(0,\infty), \\ J\phi_{tt} = \alpha\phi_{xx} - bu_x - \xi\phi - \beta\theta_x & in\ \ (0,\pi)\times(0,\infty), \\ c\theta_t = -\beta\phi_{xt} + \displaystyle\int_0^{+\infty} \kappa(s)\eta^t_{xx}(x,s)ds & in\ \ (0,\pi)\times(0,\infty), \\ \eta^t_t = \theta - \eta^t_s & in\ \ (0,\pi)\times(0,\infty), \end{cases} \tag{4}$$

We assume that the kernel satisfies:

(h1) $\kappa \in C(R^+) \cap L^1(R^+)$.

(h2) $\kappa(s) > 0, \kappa'(s) \leq 0, \forall s \geq 0$.

(h3) $\int_0^\infty \kappa(s)ds = g(0)$,

(h4) there exists $\delta > 0$ such that $\kappa'(s) \leq -\delta\kappa(s), \forall s \geq 0$.

The energy of system (4) is defined by

$$\begin{aligned} E(t) \ \ : \ &= \frac{1}{2}\int_0^\pi \left[\rho u_t^2 + J\phi_t^2 + \mu u_x^2 + \xi\phi^2 + 2bu_x\phi + \alpha\phi_x^2 + c\theta^2\right] dx \\ &+ \int_0^\infty \kappa(s)\int_0^\pi \eta_x^2(s)dxds. \end{aligned} \tag{5}$$

2 Well-posedness

Introducing the new variables $u_t = v$ and $\phi_t = \psi$ the problem can be written

$$\begin{cases} U_t + \mathcal{A}U = 0, \\ U(0) = U_0, \end{cases} \tag{6}$$

where $\mathcal{A}$ is the operator defined on

$$\mathcal{H} := H_0^1(0,\pi) \times L^2(0,\pi) \times H_*^1(0,\pi) \times L^2(0,\pi) \times L^2(0,\pi) \times \mathcal{V}.$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{c} U \in \mathcal{H} : u, \phi \in H^2(0,\pi), v, \theta \in H_0^1(0,\pi), \psi \in H_*^1(0,\pi), \\ \eta \in H_\kappa^1((0,+\infty); H_0^1), \\ \int_0^{+\infty} \kappa(s)\eta_{xx}(s)ds \in L^2(0,\pi), \eta(0) = 0. \end{array} \right\}$$

Such that

$$\mathcal{V} = L_\kappa^2((0,+\infty); H_0^1(0,\pi)).$$

Theorem 2.1. *Suppose that κ satisfies the hypothesis (h1)-(h4), then for any $U_0 = (u_0, u_1, \phi_0, \phi_1, \theta_0, \eta_0)^T \in \mathcal{H}$ the problem (6) has a unique solution $U \in C((0,+\infty); \mathcal{H})$. Moreover, if $U_0 \in D(\mathcal{A})$, then the solution U satisfies*

$$U \in C((0,+\infty); D(\mathcal{A})) \cap C^1((0,+\infty); \mathcal{H}).$$

The proof of Theorem 2.1 is based on the Hille-Yosida Theorem.

3 Exponential stability

In this section we state and prove the stability result of our problem.
First, define

$$\gamma_g = c\mu - \rho g(0) \tag{7}$$

and for $\gamma_g \neq 0$, we introduce the stability number

$$\chi_g = \frac{\rho}{\mu} - \frac{J}{\alpha} + \frac{\rho\beta^2}{\alpha\gamma_g}. \tag{8}$$

The main result reads as follow:

Theorem 3.1. *Let (u, ϕ, θ, η) be the solution of (4) subjected to the initial and boundary conditions (2),(3) respectively. Assume that $\gamma_g \neq 0$ and $\chi_g = 0$, then, the energy $E(t)$ satisfies*

$$E(t) \leq \sigma e^{-\omega t}, \; \forall t \geq 0,$$

where σ, ω are positive constants.

The proof of Theorem 3.1 will be done by the multipliers method.

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Application an iterative method for fractional differential problem under $\Psi-$ RL operators

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Abstract:, In this work we present the existence and uniqueness of the solution for a fractional boundary value problem involving the $\Psi-$ Riemann-Liouville operators where we used Banach's fixed point theorem, after that the algorithm of the considered iterative method together with the definition of the operator T, we can obtain an approximate solution of our problem. Finally we present some examples illustrating our method.

Introduction: A large class of problems in the physical, chemical and biological sciences are expressed in their mathematical framework by linear or non-linear integral equations, which has attracted the attention of several researchers to develop numerical methods among these methods we will present in this work an iterative method to a large class of boundary value problems of integer or non-integer order , the advantage of this method is rapid, avoid extre accounts and giving high accuracy in the calculation allowing to solve the following fractional differential problem of nonlinear functional equation.

$$\begin{cases} \mathcal{D}_{0^+}^{\alpha;\Psi} u(\mathfrak{t}) = f\big(\mathfrak{t}, u(\mathfrak{t})\big), \quad 0 \le \mathfrak{t} \le 1 \\ u(0) = 0, \qquad u(1) = p\mathcal{I}_{0^+}^{\mu;\Psi} g_1(\xi, u(\xi)) + q\mathcal{I}_{0^+}^{\nu;\Psi} g_2(\eta, u(\eta)), \end{cases} \tag{0.1}$$

where,

• $1 < \alpha < 2, 0 < \xi, \eta \le 1$,$\mu, \nu, p, q > 0$

$f, g_i : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+ \quad (i = 1, 2)$ are continuous functions ,ψ is an increasing function.

The next sections of this paper are arranged as follows: is reserved for some definitions and properties of ψ -Riemann-Liouville operators,then proved the existence and the uniqueness of solution to our main problem.then the formulation and the analysis of an iterative method with an illustrative example of an approximate solution it is obtained and plotted.

Keywords: Fractional differentail,Iterative method,Existence and uniqueness.

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ON THE VIBRATIONS OF AXIALLY MOVING STRINGS WITH SMALL INTERIOR DAMPING

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Abstract. We consider small vibrations of axially moving strings subject to small damping. We formulate the problem as a one-dimensional wave equation in a bounded interval with two moving endpoints in the same direction. We give the exact solution by a Fourier series formula for the undamped case. Then, using a multiple scales method to construct an asymptotic approximation for the solution of the damped equation. We may also discuss a nonlinear case of damping.

Keywords: Damped wave equation; non-cylindrical domain; energy estimates; multiple scales.

1. Introduction

The present work deals with small transverse vibrations of an infinite string moving axially with a constant speed $0 < v < 1$, as represented in Figure 1.

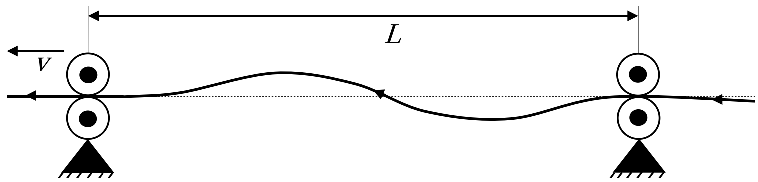

Figure 1. A string travelling to the left with a speed v.

A simplified model describing the free small transverse vibrations of the string is given by the wave equation,

$$\begin{cases} u_{tt} - u_{xx} + \varepsilon\sigma(u_t) = 0, & \text{for } x \in (vt, L+vt), t \geq 0, \\ u(vt,t) = u(L+vt,t) = 0, & \text{for } t \geq 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), & \text{for } x \in (0,L), \end{cases} \tag{WP}$$

where $f(x)$ is the initial shape of the string and $g(x)$ is its initial transverse speed. The function $\sigma(u_t)$ represents a damping term.

The simplicity of this model is only apparent and we should mention that the method of separation of variables cannot be applied to this problem. In [4], Miranker obtained the solution for the undamped case by a series formulas

$$u(t,x) = \sum_{n\in\mathbb{Z}^*} a_n e^{i\lambda_n(t-vx)} \sin\lambda_n(x-vt), \quad \text{for } x \in (vt, L+vt) \text{ and } t \geq 0.$$

where $\lambda_n := n\pi/L$ and a_n can be computed in function of the initial data f and g.

Using this series formula, we show that:

- For the undamped case, the functional

$$\mathcal{E}_v(t) = \frac{1}{2}\int_{vt}^{L+vt} (u_t + vu_x)^2 + \left(1 - v^2\right) u_x^2 dx, \qquad \text{for } t \geq 0,$$

  is conserved in time.
- Using the multiple scales method, we construct an asymptotic solution for the linear damped case $\sigma(u_t) = u_t$.
- We may also discuss a case of non-linear damping.

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Rearrangement of functions and steady vortex rings

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Abstract

We prove an existence theorem for a steady vortex ring in Poiseuille flow of an ideal fluid, using an approach based on a variational principle for of the vorticity. We show that the kinetic energy can be maximised subject to a quantity related to the vorticity belonging to a class of rearrangements of a prescribed function ζ_0 and subject to another functional (called the generalised impulse due to vorticity) having a prescribed value. Additionally, we prove that the maximisers are in fact rearrangements of ζ_0 if this prescribed value is large enough.

Introduction

we prove an existence theorem for a steady 3-dimensional ideal fluid flow containing axisymmetric steady vortex rings in Poiseuille flow. The flow is written in terms of a Stokes stream function $\Psi : \mathbb{R}^3 \to \mathbb{R}$ with respect to cylindrical coordinates $(r, \theta, z) \in \mathbb{R}^3$, which is symmetric about the z-axis direction and approaches at infinity -$\frac{\lambda}{4}r^4$, which represents a flow of velocity field $\mathrm{V} = \big(0, 0, -\lambda r^2\big)$, where λ is a parameter corresponding to the strength of the background flow at infinity; hence the velocity is increasing with respect to r along the z-direction. The magnitude of the vorticity ω is given then in terms of the Stokes stream function by $\omega/r = \mathcal{L}\Psi$, where

$$\mathcal{L} := -\left(\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial z^2}\right). \tag{0.1}$$

The function $\zeta := \omega/r$ is called potential vorticity. The vorticity in the region $r > 0$ is non-negative and Ψ satisfies the equation

$$\mathcal{L}\Psi = \phi \circ (\Psi - \frac{\lambda}{4}r^4) \tag{0.2}$$

almost everywhere on $\Pi = \{(r, z) \in \mathbb{R}^2 \mid r > 0\}$, where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing function and λ is a positive constant, hence for Poiseuille flow we obtain $\zeta = \lambda$. Equation (0.2) for arbitrary ϕ represents the relationship that should exist between the vorticity and the Stokes stream function, when the flow is in a steady state, see Lamb [11, page 245]. By using Burton's method [5], we prove that the energy E can be maximised subject to the function $\zeta \in \mathcal{R}(\zeta_0)$ and subject to another functional (called the generalised impulse due to vorticity) $\mathcal{I}_4$ is prescribed, moreover, we show that if $\zeta = \mathcal{L}\Psi$ is a maximiser, then there exists a positive number λ and an increasing function $\phi : \mathbb{R} \to \mathbb{R}$ for which $(\Psi,\ \phi,\ \lambda)$ is a solution for (0.2), where Ψ satisfies the boundary conditions $\Psi(0, z) = 0$ and $\Psi(r, z) \to 0$ as $r^2 + z^2 \to \infty$. Therefore, we prove the existence of a steady vortex ring in Poiseuille flow.

Main result

We are present our main result as follows

Theorem 1. *Let $I > 0$, let $p > \frac{5}{2}$ and let $\zeta_0 \in L^p(\Pi, \nu)$ be a non-negative function with compact support. Then*

1. *the functional E attains a maximum value subject to $\zeta \in \mathcal{W}(\zeta_0)$ and $\mathcal{I}_4(\zeta) = I$,*
2. *all maximisers are Steiner-symmetric elements of $\mathcal{RC}(\zeta_0)$,*
3. *for any maximiser ζ, there exist a positive λ and an increasing function ϕ such that the function $\Psi := K\zeta$, λ and ϕ satisfy Equation (0.2) almost everywhere in Π.*
4. *There exists a number $I_* > 0$ such that if $I > I_*$ and ζ a maximiser, then $\zeta \in \mathcal{R}(\zeta_0)$.*

In this theorem, the number λ arises as Lagrange multiplier for the constraint $\mathcal{I}_4(\zeta) = I$. Note that in this paper and Rebah [13], we construct a solution for the equation (0.2). In Theorem 1, the maximiser ζ will be shown to give rise to a solution Ψ of the boundary-value problem for axisymmetric steady vortex rings.

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Existence and stability of Ulam for Some classes of random fractional differential equations in Banach spaces

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Abstract

This article deals with the existence and the Ulam-Hyers stability results in a class of fractional random problems in Banach spaces. Here two results are discussed, the first is based of random solutions of Caputo-Fabrizio random fractional dierential equations.The second is based on random fractional differential equations of Katugampola. Two illustrative examples are presented in the last section.

Key words and phrases: Fractional differential equation; random solution; Ulam stability; fixed point.
AMS (MOS) Subject Classifications: 26A33.

1 Introduction

Fractional calculus and fractional differential equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering, and other applied sciences. In recent years, several works and development of fractional differential equation and inclusions are cited to the monographs [1, 2].

We establish the existence and the Ulam-Hyers stability results in a class of fractional random problems in Banach spaces.

Here two results are discussed, the first is based on the existence of random solutions and the stability of Ulam results for a class of Caputo-Fabrizio random fractional dierential equations with boundary conditions in the form

$$({}^{CF}D_0^{\alpha}u)(t,w) = f(t,u(t,w),w);\ t \in I := [0,T],\ w \in \Omega, \tag{1}$$

with the boundary conditions

$$au(0,w) + bu(T,w) = c(w);\ w \in \Omega, \tag{2}$$

where $T > 0,\ f : I \times E \times \Omega \to E$ is a given function, $a, b \in \mathbb{R},\ ,c : \Omega \to E$, with $a + b \neq 0,\ {}^{CF}D_0^{\alpha}$ is the Caputo–Fabrizio fractional derivative of order $\alpha \in (0,1)$, and Ω

is the sample space in a probability space (Ω, F), and E is a real (or complex) Banach space with a norm $\|\cdot\|$.

The second is based on the existence of random solutions and the stability Ulam for a class of random fractional differential equations of Katugampola

$$({}^{\rho}D_0^{\varsigma}x)(\xi, w) = f(\xi, x(\xi, w), w);\ \xi \in I = [0, T],\ w \in \Omega, \tag{3}$$

with the terminal condition

$$x(T, w) = x_T(w);\ w \in \Omega, \tag{4}$$

where $x_T : \Omega \to E$ is a measurable function, $\varsigma \in (0, 1],\ T > 0,\ f : I \times E \times \Omega \to E,\ {}^{\rho}D_0^{\varsigma}$ is the Katugampola operator of order ς, and Ω is the sample space in a probability space.

Our results are based on the theory of the fixed point and random operators. Illustrative examples are presented in each section.

2 Preliminaries

Let $\mathcal{C} := C(I, E)$ be the Banach space of all continuous functions from I into E with the norm

$$\|u\|_{\infty} = \sup\{\|u(t)\| : t \in I\}.$$

In the sequel, we will use the following fixed point Theorem:

Theorem 2.1 *[5] Let X be a nonempty, closed convex bounded subset of the separable Banach space X and let $N : \Omega \times X \to X$ be a compact and continuous random operator. Then the random equation $N(w)u = u$ has a random solution.*

3 Existence of solutions

Theorem 3.1 *Assume that the hypotheses $(H_1) - (H_2)$ hold. If*

$$\left(a_{\alpha} + Tb_{\alpha} + T\frac{bb_{\alpha}}{a+b}\right) p_2^*(w) < 1, \tag{5}$$

then the problem (1)-(2) has at least one random solution defined on I.

Theorem 3.2 *If (H_1) and (H_2) hold, and*

$$\frac{\rho^{-\varsigma}T^{\rho}}{\Gamma(1+\varsigma)} l_2^*(w) < 1, \tag{6}$$

then there exists a random solution for (3)-(4).

4 Ulam stability results

Theorem 4.1 *Assume that the hypotheses* $(H_1), (H_3), (H_4)$ *and the condition (5) hold. Then the problem (1)-(2) has at least one solution on* I *and it is generalized Ulam-Hyers-Rassias stable.*

Theorem 4.2 *If* $(H_1),\ (H_3),\ (H_4)$ *and*

$$\frac{\rho^{-\varsigma}T^{\rho}}{\Gamma(1+\varsigma)}\Phi^{*}(w)q^{*}(w) < 1, \tag{7}$$

hold. Then the problem (3)-(4) has random solutions defined on I, *and it is generalized Ulam-Hyers-Rassias stable.*

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Fractional Sobolev Space via Loiuville operator

Saadi Abderachid

Abstract

This work is devoted to Liouville fractional Sobolev spaces. A novel form of these spaces is well proposed and related properties are also proved.

2010 AMS Classification: 26A33, 34A08, 34Bxx.
Keywords: fractional derivative, Sobolev space, Liouville operator.

1 Introduction

A classical Sobolev space $W^{1,p}(a,b)$ where $(a,b) \subset \mathbb{R}$ is defined by (cf. [1, 4])

$$W^{1,p}(a,b) = \left\{ u \in L^p(a,b), \exists g \in L^p(a,b); \int_a^b u.\varphi' = -\int_a^b g.\varphi \right\}, \forall \varphi \in C_c^\infty(a,b), \tag{1.1}$$

where $C_c^\infty(a,b)$ is the set of infinitely differentiable functions on (a,b) with compact support in (a,b).

We can define the Sobolev space $W^{1,p}(a,b)$ of order $n > 1$ by

$$W^{n,p}(a,b) = \{u \in W^{n-1,p}(a,b), u' \in W^{n-1,p}(a,b)\}. \tag{1.2}$$

However, a Sobolev space $W^{n,p}(a,b)$ can be described (cf. [8]) as follows $W^{n,p}(a,b) = AC^{n,p}(a,b)$, where $AC^{n,p}(a,b)$ is space of functions f from $[a,b]$ to $\mathbb{R}$ such that there exist $c_0, c_1, \cdots c_{n-1}$ and $\varphi \in L^p(a,b)$ such that

$$f(x) = \sum_{k=0}^{n-1} \frac{c_k}{k!}(x-a)^k + \int_a^x \varphi(t)dt, x \in [a,b] a.e. \tag{1.3}$$

This space has a generalization to the space $AC^\alpha(a,b); (n-1 < \alpha < n)$ of functions f from $[a,b]$ to $\mathbb{R}$ such that there exist $c_0, c_1, \cdots c_{n-1}$ and $\varphi \in L^p(a,b)$ such that

$$f(x) = \sum_{k=0}^{n-1} \frac{c_k}{\Gamma(\alpha - n + 1 + k)}(x-a)^{\alpha-n+k} + I_{a^+}^\alpha \varphi, x \in [a,b] a.e, \tag{1.4}$$

where $I_{a^+}^{\alpha}$ denote the left Riemann-Liouville integral.

The work was initially focused on discovering containment relationships between spaces $AC^{\alpha}(a,b)$ (inclusion, embedding, integration by parts... c.f [6, 7]), and later a set of definitions of fractional Sobolev spaces was proposed (cf. [8]). Other properties and results are treated in the articles [3, 5, 6, 7].

As part of our effort, we will create fractional Sobolev space ${}^{L}W_{a^+}^{\alpha,p}(\mathbb{R})$ where $0 < \alpha < 1$ and $1 \leq p < +\infty$. We present some of this space' properties that are analogous to classical properties.

2 Preliminaries

Let $1 \leq p < +\infty,\ 0 < \alpha \leq 1$.

Definition 2.1 *[10] Let* $\alpha > 0$. $\Gamma(.)$ *denote the Euler Gamma function.*

The left and right Liouville fractional integrals of order α *of* $f \in L^p(\mathbb{R})$ *are respectively defined by*

$$\left(I_{+}^{\alpha}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-t)^{\alpha-1}f(t)\,dt. \tag{2.1}$$

$$\left(I_{-}^{\alpha}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{+\infty}(t-x)^{\alpha-1}f(t)\,dt, \tag{2.2}$$

Proposition 2.1 *[10] Let* $\alpha,\beta > 0$ *and* p *such that* $\alpha,\beta < \frac{1}{p}$. *Then, eor all* $f \in L^p(a,b)$, *we have*
$I_{+}^{\alpha}I_{+}^{\beta}f = I_{+}^{\alpha+\beta}f$ *and* $I_{-}^{\alpha}I_{-}^{\beta}f = I_{-}^{\alpha+\beta}f$

Proposition 2.2 *[10] Let* $1 \leq p,q \leq +\infty$ *be such thet* $\frac{1}{p}+\frac{1}{q} \leq 1+\alpha$. *Then, for all* $f \in L^p(a,b)$, $g \in L^p(a,b)$ *we have*

$$\int_{-\infty}^{+\infty} f(x)\left(I_{-}^{\alpha}g\right)(x)\,dx = \int_{-\infty}^{+\infty} g(x)\,I_{+}^{1-\alpha}f(x)\,dx. \tag{2.3}$$

Definition 2.2 *[10] Let* $0 < \alpha < 1$, *the left and right Liouville fractional derivatives of order* α *of* $f \in L^p(\mathbb{R})$ *are respectively defined by*

$$\begin{aligned}\left(D_{+}^{\alpha}f\right)(x) &= \frac{d}{dx}\left(I_{+}^{1-\alpha}f\right)(x)\\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\infty}^{x}(x-t)^{-\alpha}f(t)\frac{dt}{t}dt. \end{aligned}\tag{2.4}$$

$$\begin{aligned}\left(D_{-}^{\alpha}f\right)(x) &= -\frac{d}{dx}\left(I_{-}^{1-\alpha}f\right)(x)\\ &= -\frac{1}{\Gamma(n-\alpha)}\frac{d}{dx}\int_{x}^{+\infty}(t-x)^{-\alpha}f(t)\,dt. \end{aligned}\tag{2.5}$$

3 Fractional Sobolev spaces via Liouville operator

Let $n-1<\alpha<n,\ (n\in\mathbb{N}),\ 1\leq p<+\infty$. As in the classical Sobolev spaces, and following the definitions in [8], we establish the following space

Definition 3.1 *A fractional Sobolev space via Liouville operator (for $n=1$) is given by*

$$ {}^{L}W_{+}^{\alpha,p}(\mathbb{R})=\left\{\begin{array}{l} u\in L^{p}(\mathbb{R})\ /\ \exists g\in L^{p}(\mathbb{R});\\ \displaystyle\int_{-\infty}^{+\infty}u(x)\left(D_{-}^{\alpha}\varphi\right)(x)\,dx=\int_{-\infty}^{+\infty}g(x)\,\varphi(x)\,dx,\ \forall\varphi\in C_{c}^{\infty}(\mathbb{R}) \end{array}\right\}. \tag{3.1} $$

For $n\geq 2$, a fractional Sobolev space is given by recurrence as follow

$$ {}^{L}W_{+}^{\alpha,p}(\mathbb{R})=\left\{\ u\in{}^{L}W_{+}^{\alpha-1,p}(\mathbb{R})\ /\ {}^{L}D_{+}^{\alpha-n-1}u\in W_{+}^{n-1,p}(\mathbb{R})\ \right\}. \tag{3.2} $$

Proposition 3.1 *The function g coincides with $D_{+}^{\alpha}u$ in $\mathbb{R}$.*

Definition 3.2 *we define in ${}^{L}W_{+}^{\alpha,p}(\mathbb{R})$, the norm*

$$ \|u\|_{{}^{L}W_{+}^{\alpha,p}(\mathbb{R})}^{p}=\|u\|_{L^{p}(\mathbb{R})}^{p}+\sum_{k=0}^{n-1}\left\|{}^{L}D_{+}^{\alpha-n+1+k}u\right\|_{L^{p}(\mathbb{R})}^{p}. \tag{3.3} $$

Theorem 3.1 *The space ${}^{L}W_{+}^{\alpha,p}(a,b)$ is complet respect to each norm $\|.\|_{{}^{L}W_{+}^{\alpha,p}(\mathbb{R})}$.*

Proof. Using the same arguments from the proof of Theorem 25 of [8], ■

The following two theorems are established in the same way as the theorem 3.5 in [1] (see also theroem 8.1 in [4], theorem 26 and theorem 27 in [8]).

Theorem 3.2 *The space ${}^{L}W_{+}^{\alpha,p}(\mathbb{R})$ is reflexive for $1<p<\infty$ and separable for $1\leq p<\infty$. The space ${}^{L}W_{+}^{\alpha,2}(\mathbb{R})$ is a separable Hilbert space.*

Theorem 3.3 *The space $C_{c}^{\infty}(\mathbb{R})$ is dense in the space ${}^{L}W_{+}^{\alpha,p}(\mathbb{R})$ with respect to the norm $\|.\|_{{}^{L}W_{+}^{\alpha,p}(\mathbb{R})}$.*

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GROWTH OF SOLUTIONS OF COMPLEX DIFFERENTIAL EQUATIONS WITH COEFFICIENTS MEROMORPHIC

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Abstract: In this paper we investigate the n-iterated exponent of convergence of $f^{(i)} - \varphi$ where $f \not\equiv 0$ is a solution of linear differential equation with analytic or meromorphic coefficients in the unit disc and φ is a small function of f. This work is an extension and counterpart of recent results in the complex plane by Xu et al. [9] and Tu et al. [8] to the unit disc

Key Words: *Meromorphic, functions, logarithmic.*

1. Introduction and statement of results

We study the growth of solutions to a class of linear lane. Differential equations around an isolated essential singularity point. By Nevanlinna value distribution theory of meromorphic function on the complex plane $\mathbb{C}$ and in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ (see [4], [10]). In addition, for $n \in \mathbb{N} - \{0\}$, the n-iterated order of meromorphic function $f(z)$ in D is defined by

$$\sigma_n(f) = \limsup_{r \to 1^-} \frac{\log_n^+ T(r,f)}{-\log(1-r)},$$

where $\log_1^+(x) = \log^+(x) = \max\{\log x, 0\}$, $\log_{n+1}^+(x) = \log^+ \log_n^+(x)$ and $T(r,f)$ is the Nevanlinna characteristic function of f. For an analytic function $f(z)$ in D, we have also

$$\sigma_{M,n}(f) = \limsup_{r \to 1^-} \frac{\log_{n+1}^+ M(r,f)}{-\log(1-r)},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$. If f is analytic in D, Tsuji [[?], p.205] gives that

$$\sigma_1(f) \le \sigma_{M,1}(f) \le \sigma_1(f) + 1. \tag{1}$$

For example, the function $f(z) = \exp\left\{\frac{1}{(1-z)^\mu}\right\}$, $(\mu \ge 1)$, satisfies $\sigma_1(f) = \mu - 1$ and $\sigma_{M,1}(f) = \mu$.

Obviously, we have

$$\sigma(f) < \infty \text{ if and only if } \sigma_M(f) < \infty.$$

The inequalities (1) are the best possible in the sense that there are analytic functions g and h such that $\sigma_{M,1}(g) = \sigma_1(g)$ and $\sigma_{M,1}(h) = \sigma_1(h) + 1$, see

[2]. However, it follows by Proposition 2.2.2 in [7] that $\sigma_{M,n}(f) = \sigma_n(f)$ for $n \geq 2$. The $n-$type of a meromorphic function $f(z)$ in D with $0 < \sigma_n(f) = \sigma_n < \infty$ is defined by

$$\tau_n(f) = \limsup_{r\to 1^-} (1-r)^{\sigma_n} \log^+_{n-1} T(r,f);$$

and if f is an analytic function f in D with $0 < \sigma_{M,n}(f) = \sigma_n < \infty$ we have also

$$\tau_{M,n}(f) = \limsup_{r\to 1^-} (1-r)^{\sigma_n} \log^+_n M(r,f).$$

We signal that also by Proposition 2.2.2 in [7], we have $\tau_n(f) = \tau_{M,n}(f)$ for $n \geq 3$.

Definition 1.1. [5] *A meromorphic function f in the unit disc D is called admissible if*

$$\limsup_{r\to 1^-} \frac{T(r,f)}{-\log(1-r)} = \infty$$

and nonadmissible if

$$\limsup_{r\to 1^-} \frac{T(r,f)}{-\log(1-r)} < \infty.$$

The growth index of the iterated order of a meromorphic function $f(z)$ in D is defined by

$$i(f) = \begin{cases} 0 & \text{if } f \text{ is nonadmissible,} \\ \min\{n \in \mathbb{N} : \sigma_n(f) < \infty\} & \text{if } f \text{ is admissible,} \\ \infty & \text{if } \sigma_n(f) = \infty \text{ for all } n \in \mathbb{N}. \end{cases}$$

We will use the notation $\lambda_n(f)$ to denote the n-iterated exponent of convergence of the zero-sequence of meromorphic function $f(z)$ and $\overline{\lambda}_n(f)$ to denote the n-iterated exponent of convergence of distinct zero-sequence of $f(z)$, which are defined as the following:

$$\lambda_n(f) = \limsup_{r\to 1^-} \frac{\log N\left(r, \frac{1}{f}\right)}{-\log(1-r)} \quad \text{and} \quad \overline{\lambda}_n(f) = \limsup_{r\to 1^-} \frac{\log \overline{N}\left(r, \frac{1}{f}\right)}{-\log(1-r)}.$$

The growth index of the exponent of convergence of the zero-sequence of meromorphic function $f(z)$ is defined by

$$i_\lambda(f) = \begin{cases} 0 & \text{if } N\left(r, \frac{1}{f}\right) = O\left(\log \frac{1}{1-r}\right) \\ \min\{n \in \mathbb{N} : \lambda_n(f) < \infty\} & \text{if } \log \frac{1}{1-r} = o\left(N\left(r, \frac{1}{f}\right)\right) \\ \infty & \text{if } \lambda_n(f) = \infty \text{ for all } n \in \mathbb{N}. \end{cases}$$

We can define $i_{\overline{\lambda}}(f)$ by the same method of $i_\lambda(f)$.

Recently, Xu, Tu and Zheng investigated the relationship between small functions and derivatives of solutions of higher order differential equations:

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = 0, \qquad (2)$$

where $A_j(z)$ are entire or meromorphic functions in the complex plane, and obtained the following result.

Theorem 1.2. *Let $A_j(z)$ $j = 0, 1, ..., k-1$ be entire functions with finite order and satisfy one of the following conditions:*
(i) $\max\{\sigma(A_j) : j = 1, 2, ..., k-1\} < \sigma(A_0) < \infty$*;*
(ii) $0 < \sigma(A_{k-1}) = ... = \sigma(A_1) = \sigma(A_0) < \infty$ *and* $\max\{\tau(A_j) : j = 1, 2, ..., k-1\} = \tau_1 < \tau(A_0) = \tau$*;*
then for every solution $f \not\equiv 0$ of (2) and for any entire function $\varphi(z) \not\equiv 0$ satisfying $\sigma_2(\varphi) < \sigma(A_0)$, we have

$$\overline{\lambda}_2(f - \varphi) = \overline{\lambda}_2(f' - \varphi) = \overline{\lambda}_2(f'' - \varphi) = \overline{\lambda}_2\left(f^{(i)} - \varphi\right) = \sigma_2(f) = \sigma(A_0) \quad (i \in \mathbb{N}).$$

Thereafter, Tu, Xuan and Xu improved this result from entire coefficients of finite order to entire coefficients of finite iterated order of the second order linear differential equation

$$f'' + A(z) f' + B(z) f = 0 \qquad (3)$$

and obtained the following results.

Theorem 1.3. *Let $A(z)$ and $B(z)$ be entire functions of finite iterated order satisfying $\sigma_n(A) < \sigma_n(B)$ or $0 < \sigma_n(A) = \sigma_n(B) < \infty$ and $0 \leq \tau_n(A) < \tau_n(B) \leq \infty$. Then, for every solution $f \not\equiv 0$ of (3) and for any entire function $\varphi(z) \not\equiv 0$ satisfying $\sigma_{n+1}(\varphi) < \sigma_n(B)$, we have*

$$\overline{\lambda}_{n+1}(f - \varphi) = \overline{\lambda}_{n+1}\left(f^{(i)} - \varphi\right) = \sigma_{n+1}(f) = \sigma_n(B), \ i \in \mathbb{N}.$$

Theorem 1.4. *Let $A(z)$ and $B(z)$ be entire functions of finite iterated order satisfying $i(A) < i(B) = n$. Then for every solution $f \not\equiv 0$ of (3) and for any entire function $\varphi(z)$ satisfying $i(\varphi) \leq n$, we have*
(i) $i_{\overline{\lambda}}\left(f^{(i)} - \varphi\right) = i_{\lambda}\left(f^{(i)} - \varphi\right) = i\left(f^{(i)} - \varphi\right) = n + 1 \ (i = 0, 1, 2, ...)$;
(ii) $\overline{\lambda}_{n+1}\left(f^{(i)} - \varphi\right) = \lambda_{n+1}\left(f^{(i)} - \varphi\right) = \sigma_{n+1}\left(f^{(i)} - \varphi\right) = \sigma_n(B), \ (i = 0, 1, 2, ...)$.

In this paper, we will investigate the analogous of these results in the unit disc for higher linear differential equations with analytic or meromorphic coefficients as the following.

.

2. Preliminaries lemmas

We use the following notations that are not necessarily the same at each occurrence:

$E \subset (0,1)$ is a set of finite logarithmic measure, that is $\int_E \frac{dr}{1-r} < \infty$.

$F \subset (0,1)$ is a set of infinite logarithmic measure, that is $\int_F \frac{dr}{1-r} = \infty$.

$c > 0, \ \varepsilon > 0, \ \sigma \geq 0, \ \sigma_1 \geq 0, \ \tau \geq 0, \ \tau_1 \geq 0$, are real constants.

Lemma 2.1. [9] *Assume that* $f \not\equiv 0$ *is a solution of (??). Set* $g = f - \varphi$*; then* g *satisfies the equation*

$$g^{(k)} + A_{k-1} g^{(k-1)} + ... + A_0 g = -\left[\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + ... + A_0 \varphi\right]. \quad (1)$$

Lemma 2.2. [9] *Assume that* $f \not\equiv 0$ *is a solution of (??). Set* $g_i = f^{(i)} - \varphi,\ (i \in \mathbb{N} - \{0\})$*; then* g_i *satisfies the equation*

$$g_i^{(k)} + U_{k-1}^i g_i^{(k-1)} + ... + U_0^i g_i = -\left[\varphi^{(k)} + U_{k-1}^i \varphi^{(k-1)} + ... + U_0^i \varphi\right], \quad (2)$$

where

$$U_j^i = \left(U_{j+1}^{i-1}\right)' + U_j^{i-1} - \frac{\left(U_0^{i-1}\right)'}{U_0^{i-1}} U_{j+1}^{i-1}, \quad (3)$$

$j = 0, 1, ..., k-1,\ U_j^0 = A_j$ *and* $U_k^i \equiv 1$.

Lemma 2.3. *Let* $h : (0,1) \to (c, \infty)$ *be monotone increasing function such that*

$$\limsup_{r \to 1^-} \frac{\log_n^+ h(r)}{-\log(1-r)} = \alpha, \quad (4)$$

(α *is finite or infinite value); then there exists a set* $F \subset (0,1)$ *with infinite logarithmic measure such that for all* $r \in F$*, we have*

$$\lim_{r \to 1^-} \frac{\log_n^+ h(r)}{-\log(1-r)} = \alpha.$$

Lemma 2.4. *Let* f *be a meromorphic function in the unit disc* D *such that* $f^{(j)}$ *does not vanish identically. Let* $\varepsilon > 0$ *be a constant;* k *and* j *be integers satisfying* $k > j \geq 0$ *and* $d \in (0,1)$*. Then we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\left(\frac{1}{1-|z|}\right)^{(2+\varepsilon)} \max\left\{\log\frac{1}{1-|z|}, T(s(|z|), f)\right\}\right)^{k-j}, \ |z| \notin E,$$

where $s(|z|) = 1 - d(1-|z|)$*. As a particular cases: if* $\sigma_1(f) < \infty$*, then*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\frac{1}{1-|z|}\right)^{(k-j)(\sigma_1 + 2 + \varepsilon)}, \ |z| \notin E;$$

and if $\sigma_n(f) < \infty$ *for some* $n \geq 2$*, then*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \exp_{n-1}\left\{\left(\frac{1}{1-|z|}\right)^{\sigma_n + \varepsilon}\right\}, \ |z| \notin E;$$

where $\exp_1(x) = \exp(x)$ *and* $\exp_{n+1}(x) = \exp\{\exp_n(x)\}$.

Lemma 2.5. [3] *Let* $f(z)$ *be an analytic function in the unit disc* D *with* $\sigma_{M,n}(f) = \sigma_n,\ \tau_{M,n}(f) = \tau_n,\ 0 < \sigma_n < \infty,\ 0 < \tau_n < \infty$*, then for any given* $0 < \beta < \tau_n$*, there exists a set* $F \subset (0,1)$ *that has infinite logarithmic measure such that for all* $r \in F$ *we have*

$$\log_n^+ M(r, f) > \frac{\beta}{(1-r)^{\sigma_n}}.$$

By the same method of the proof of Lemma 2.5, we can get the following two lemmas.

Lemma 2.6. *Let $f(z)$ be an analytic function in the unit disc D with $\sigma_{M,n}(f) = \sigma_n$, $0 < \sigma_n < \infty$, then for any given $0 < \beta < \sigma_n$, there exists a set $F \subset (0,1)$ that has infinite logarithmic measure such that for all $r \in F$ we have*

$$\log_n^+ M(r,f) > \frac{1}{(1-r)^{\beta}}.$$

Lemma 2.7. *Let $f(z)$ be meromorphic function in the unit disc D with $\sigma_n(f) = \sigma_n$, $\tau_n(f) = \tau_n$, $0 < \sigma_n < \infty$, $0 < \tau_n < \infty$, then for any given $0 < \beta < \tau_n$, there exists a set $F \subset (0,1)$ that has infinite logarithmic measure such that for all $r \in F$ we have*

$$\log_{n-1}^+ T(r,f) > \frac{\beta}{(1-r)^{\sigma_n}}.$$

Lemma 2.8. *Let $A_j(z)$ $j = 0, 1, ..., k-1$ be analytic functions in the unit disc D such that $i(A_0) = n$, $0 < \sigma_n(A_0) = \sigma < \infty$, $0 < \tau_{M,n}(A_0) = \tau < \infty$ $(n \geq 2,\ n \in \mathbb{N})$ and for every $j \in \{1, ..., k-1\}$, $A_j(z)$ satisfies one of the following conditions:*
(1) $i(A_j) < i(A_0)$;
(2) $i(A_j) = i(A_0)$ and $\sigma_n(A_j) < \sigma_n(A_0)$;
(3) $i(A_j) = i(A_0)$, $\sigma_n(A_j) = \sigma_n(A_0)$ and $\tau_{M,n}(A_j) < \tau_{M,n}(A_0)$.
and U_j^i $(j = 0, 1, ..., k)$ $(i \in \mathbb{N})$ be stated as in (3). Then, for any given ε $(0 < 2\varepsilon < \tau - \tau_1)$, there exists a set F of infinite logarithmic measure such that for $|z| = r \in F$ and $|A_0(z)| = M(r, A_0)$ we have

$$\left|U_0^i\right| \geq \exp_n\left\{\frac{\tau - \varepsilon}{(1-r)^{\sigma}}\right\} \quad \text{and} \quad \left|U_j^i\right| \leq \exp_n\left\{\frac{\tau_1 + \varepsilon}{(1-r)^{\sigma}}\right\},\ (j \neq 0) \tag{5}$$

where $\tau_1 = \max\{\tau_{M,n}(A_j) : j \neq 0\}$. If there is no coefficient that satisfies the condition (3), then we put $\tau_1 = 0$.

Lemma 2.9. *Let $A_j(z)$ $j = 0, 1, ..., k-1$ be analytic functions in the unit disc D such that $i(A_0) = n$, $0 < \sigma_n(A_0) = \sigma < \infty$, $(n \geq 2,\ n \in \mathbb{N})$ and for every $j \in \{1, ..., k-1\}$, $A_j(z)$ satisfies one of the following conditions:*
(1) $i(A_j) < i(A_0)$;
(2) $i(A_j) = i(A_0)$ and $\sigma_n(A_j) < \sigma_n(A_0)$;
and U_j^i $(j = 0, 1, ..., k)$ $(i \in \mathbb{N})$ be stated as in (3). Then, there exists a set F of infinite logarithmic measure such that for $|z| = r \in F$ and $|A_0(z)| = M(r, A_0)$ we have

$$\left|U_0^i\right| \geq \exp_n\left\{\frac{1}{(1-r)^{\sigma-\varepsilon}}\right\} \quad \text{and} \quad \left|U_j^i\right| \leq \exp_n\left\{\frac{1}{(1-r)^{\sigma-2\varepsilon}}\right\}\ (j \neq 0),$$

where $\varepsilon > 0$ is small enough.

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GLOBAL SOLVABILITY OF SOME IMPULSIVE FRACTIONAL ORDER COUPLED SYSTEMS WITH NONLOCAL CONDITIONS ON THE HALF LINE

KHADIDJA NISSE

ABSTRACT. In this work, we deal with initial value problems for coupled systems of nonlinear fractional differential equations, subject to coupled nonlocal initial and impulsive conditions on the half line. Global existence-uniqueness results are obtained under weak conditions allowing the reaction part of the problem to increase indefinitely with time. Our approach relies mainly to some fixed point theorem of Perov's type in generalized gauge spaces. The obtained results improve, generalize and complement many existing results in the literature. An example illustrating our main finding is also given.

1. SETTING OF THE PROBLEM AND RESULTS

Banach's contractive principle is one of the most useful tools in nonlinear functional analysis that ensures the existence and uniqueness of a fixed point on complete metric spaces. One of the extensions of this principle for contractive mappings on spaces endowed with vector valued metrics, was done by Perov in [6] and Perov and Kibento in [7]. Many other generalizations in this direction have been investigated. In [5], Precup established the extension in Perov's sens of some fixed point theorem in spaces endowed with a family of pseudo-metrics. Many authors applied the vector version's fixed point theorems in the study of the existence of solutions for systems of differential and integral equations, see for example [1, 2, 3, 4, 8] and the references therein. In this line of research, we consider in this work, the following nonlinear coupled system of fractional differential equations:

$$\begin{cases} {}^C D_{0^+}^{\alpha} u(t) = f(t, u(t), v(t)), & t \in \mathcal{I}_i = \left]t_i, t_{i+1}\right], \, i \in \mathbb{N} \\ {}^C D_{0^+}^{\beta} v(t) = g(t, u(t), v(t)), & t \in \mathcal{I}_i = \left]t_i, t_{i+1}\right], \, i \in \mathbb{N} \end{cases} \tag{1.1}$$

with coupled nonlocal initial conditions:

$$\begin{cases} u(0) = \varphi(u, v), \\ v(0) = \psi(u, v), \end{cases} \tag{1.2}$$

and subject to coupled impulsive conditions:

$$\begin{cases} \Delta u(t_i) = I_i(u(t_i), v(t_i)), & i \in \mathbb{N}^* \\ \Delta v(t_i) = J_i(u(t_i), v(t_i)), & i \in \mathbb{N}^* \end{cases} \tag{1.3}$$

where ${}^C D_{0^+}^{\alpha}$ and ${}^C D_{0^+}^{\beta}$ denote the Caputo fractional derivative operators with the fixed lower limit equals zero, of order α and β in $]0, 1[$ respectively, $f, g : \mathbb{R}_+ \times \mathbb{R}^2 \longrightarrow$

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$\mathbb{R}$ are nonlinear continuous functions, $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$, where $u(t_i^+)$ and $u(t_i^-)$ represent the right and left limits of u at $t = t_i$ and $\{t_i\}_{i\in\mathbb{N}^*}$ is a sequence of points in $\mathbb{R}_+$ such that $t_i < t_{i+1}$ for $i \in \mathbb{N}^*$, $I_i, J_i : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are nonlinear continuous functions, $\phi, \psi : X \longrightarrow \mathbb{R}$ are nonlinear continuous functional where X is a generalized complete gauge space, which will be defined later.

It should be noted that the coupled nonlocal initial conditions (1.2) generalizes many other types of initial conditions considered in the literature, such as: classical initial conditions, multi-point conditions and integral conditions.

After converting (1.1)- (1.3) into an equivalent fixed point problem in generalized gauge space, we apply some fixed point theorem of Perov's type, established in [5]. Using this approach, we obtain a global existence-uniqueness results for (1.1)- (1.3) under weak conditions allowing the nonlinearity to increase indefinitely with time, which is not the case in many earlier results in the literature. This study allows us also, to improve and generalize some other existence results in the literature for systems of fractional differential equations without impulses.

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A QUASISTATIC CONTACT PROBLEM WITH COULOMB FRICTION IN ELECTRO-VISCOELASTICITY WITH LONG-TERM MEMORY BODY

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Abstract. We consider a quasistatic contact problem with coulomb friction in electro-viscoelasticity with long-term memory body. The contact is modelled with normal compliance. The adhesion of the contact surfaces is taken into account and modelled by a surface variable. We derive variational formulation for the model which is in the form of a system involving the displacement field, the electric potential field, the damage field and the adhesion field. We prove the existence of a unique weak solution to the problem. The proof is based on arguments of time-dependent variational inequalities, parabolic inequalities, differential equations and fixed points.

Keywords. Electro-viscoelastic Materials; Friction contact with adhesion, Fixed point; Variational inequality; Differential equation.

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1. Introduction

In this paper, we study a mathematical model which describes the adhesive contact problem with damage for an electro viscoelastic with long-term memory body, when the frictional tangential traction with the traction due to adhesion. We derive a variational formulation of the model and prove its unique solvability, which provides the existence of a unique weak solution to the adhesive contact problem.

The piezoelectric effect is the apparition of electric charges on surfaces of particular crystals after deformation. Its reverse effect consists of the generation of stress and strain in crystals under the action of the electric field on the boundary. Materials undergoing piezoelectric materials effects are called piezoelectric materials, and their study requires techniques and results from electromagnetic theory and continuum mechanics. Piezoelectric materials are used extensively as switches and, actually, in many engineering systems in radioelectronics, electroacoustics and measuring equipment. However, there

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are very few mathematical results concerning contact problems involving piezoelectric materials and therefore there is a need to extend the results on models for contact with deformable bodies which include coupling between mechanical and electrical properties. General models for elastic materials with piezoelectric effects can be found in [1, 2, 3, 4, 6, 7, 8] and the references therein.

Process of adhesion are important in industry where parts, usually non metallic, are glued together. Recently, composite materials reached prominence, since they are very strong and light, and therefore, of considerable importance in aviation, space exploration and in the automotive industry. However, composite materials my undergo delamination under stress, in which different layers debond and move relative to each other. To model the process when bonding is not permanent, and debonding may take place, we need to describe the adhesion together with the contact. A number of recent publications deal with such models, see, e.g., [9, 10, 11, 12] and the references therein.

The subject of damage is extremely important in design engineering since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. General novel models for damage were derived in [13, 14] from the virtual power principle. The mathematical analysis of one-dimensional problems can be found in [15]. In all these results, the damage of the material is described by a damage function α restricted to have values between zero and one. If $\alpha = 1$, there is no damage in the material. If $\alpha = 0$, then the material is completely damaged. If $0 < \alpha < 1$, there is a partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [5, 16, 17, 18] and the references therein. In this paper, the inclusion describing the evolution of damage field is

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_K\,(\alpha) \ni S(\varepsilon(\mathbf{u}), \alpha),$$

where K denotes the set of admissible damage functions defined by

$$K = \{\zeta \in H^1(\Omega) \mid 0 \leq \zeta \leq 1 \text{ a.e. in } \Omega\},$$

k is a positive coefficient, $\partial\varphi_K$ represents the subdifferential of the indicator function of set K and S is a given constitutive function which describes the sources of the damage in the system.

We use an electro viscoelastic constitutive law with long-term memory given by

$$\sigma = \mathscr{A}\varepsilon(\dot{\mathbf{u}}) + \mathscr{G}(\varepsilon(\mathbf{u}), \alpha) + \int_0^t \mathscr{M}(t-s)\varepsilon(\mathbf{u}(s))ds + \mathscr{E}^*\nabla(\varphi),$$
$$\mathbf{D} = -B\nabla(\varphi) + \mathscr{E}\varepsilon(\mathbf{u}),$$

where $\mathscr{A}$ is a given nonlinear function, $\mathscr{M}$ is the relaxation tensor, and $\mathscr{G}$ represents the elasticity operator where α is an internal variable describing the damage of the material caused by elastic deformations. $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathscr{E} = (e_{ijk})$ represents the third order piezoelectric tensor, $\mathscr{E}^*$ is its transposition and $\mathbf{B}$ denotes the electric permittivity tensor.

The paper is organized as follows. In Section 2, we introduce some essential preliminaries. In Section 3, we present the mechanical problem, list the assumptions on the data, and give the variational formulation of the problem. In Section 4, the last section, we state our main existence and uniqueness result, Theorem 4.1. The proof of the theorem is based on the theory of evolution equations with monotone operators, and a classical existence-uniqueness result for parabolic inequalities.

2. Preliminaries

In this section, we present some essential tools for our main results. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ ($d = 2,3$), while $(\cdot,\cdot)$ and $\|\cdot\|$ represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively. We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\mathbf{u}\cdot\mathbf{v} = u_i v_i, \qquad \|\mathbf{v}\| = (\mathbf{v}\cdot\mathbf{v})^{1/2}, \quad \forall \mathbf{u},\mathbf{v}\in\mathbb{R}^d$$
$$\sigma\cdot\tau = \sigma_{ij}\tau_{ij}, \qquad \|\tau\| = (\tau\cdot\tau)^{1/2}, \quad \forall \sigma,\tau\in\mathbb{S}^d.$$

respectively. Here and below, the indices i and j run from 1 to d, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable.

Let $\Omega\subset\mathbb{R}^d$ be a bounded domain with a regular boundary Γ and let ν denote the unit outer normal on Γ. We shall use the notation

$$H = L^2(\Omega)^d = \{\mathbf{u} = (u_i) \mid u_i \in L^2(\Omega)\}, \qquad H_1(\Omega)^d = \{\mathbf{u} = (u_i) \mid u_i \in H^1(\Omega)\}$$
$$\mathscr{H} = \{\sigma = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}, \qquad \mathscr{H}_1 = \{\sigma\in\mathscr{H} \mid Div\sigma \in H\}.$$

we consider that $\varepsilon : H_1(\Omega)^d \to \mathscr{H}$ and $\mathrm{Div} : \mathscr{H}_1 \to H$ are the deformation and the divergence operators, respectively, defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j}+u_{j,i}), \quad \mathrm{Div}\,\sigma = (\sigma_{ij,i}).$$

The spaces $H, H^1(\Omega)^d$, $\mathscr{H}$ and $\mathscr{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u},\mathbf{v})_H = \int_\Omega \mathbf{u}\cdot\mathbf{v}dx, \quad \forall\mathbf{u},\mathbf{v}\in H, \qquad (\mathbf{u},\mathbf{v})_{H^1(\Omega)^d} = (\mathbf{u},\mathbf{v})_H + (\nabla\mathbf{u},\nabla\mathbf{v})_H,$$

where $\nabla\mathbf{v} = (v_{i,j})$, $\forall\mathbf{v}\in H^1(\Omega)^d$

$$(\sigma,\tau)_{\mathscr{H}} = \int_\Omega \sigma\cdot\tau dx, \quad \forall\sigma,\tau\in\mathscr{H},$$
$$(\sigma,\tau)_{\mathscr{H}_1} = (\sigma,\tau)_{\mathscr{H}} + (\mathrm{Div}\,\sigma,\mathrm{Div}\,\tau)_H, \quad \forall\sigma,\tau\in\mathscr{H}_1.$$

The associated norms on H, $H^1(\Omega)^d$, $\mathscr{H}$ and $\mathscr{H}_1$ are denoted by $\|\cdot\|_H$, $\|\cdot\|_{H^1(\Omega)^d}$, $\|\cdot\|_{\mathscr{H}}$, and $\|\cdot\|_{\mathscr{H}_1}$, respectively. Let $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ and let $\gamma : H^1(\Omega)^d \to H_\Gamma$ be the trace map. For every element $\mathbf{v}\in H^1(\Omega)^d$, we also write $\mathbf{v}$ for the trace $\gamma\mathbf{v}$ of $\mathbf{v}$ on Γ and we denote by v_ν and $\mathbf{v}_\tau$ the normal and tangential components of $\mathbf{v}$ on Γ given by

$$v_\nu = \mathbf{v}\cdot\nu, \qquad \mathbf{v}_\tau = \mathbf{v} - v_\nu\nu. \tag{2.1}$$

Similarly, for a regular tensor field $\sigma : \Omega\to\mathbb{S}^d$, we define its normal and tangential components by

$$\sigma_\nu = (\sigma\nu).\nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu\nu. \tag{2.2}$$

We recall that the following Green's formula holds

$$(\sigma,\varepsilon(\mathbf{v}))_{\mathscr{H}} + (\mathrm{Div}\,\sigma,\mathbf{v})_H = \int_\Gamma \sigma\nu.\mathbf{v}da, \quad \forall\mathbf{v}\in H_1. \tag{2.3}$$

Finally, for the convenience of the reader, we recall the following version of the classical theorem of Cauchy-Lipschitz (see, e.g.,[5] ,p.48)

Theorem 2.1. *Assume that* $(X, \|\cdot\|_X)$ *is a real Banach space and* $T > 0$. *Let* $F(t,\cdot) : X \to X$ *be an operator defined* a.e. *on* $(0,T)$ *satisfying the following conditions*

$$\begin{cases} \text{There exists } L_F > 0 \text{ such that} \\ \|F(t,x) - F(t,y)\|_X \le L_F\|x-y\|_X \qquad \forall x,y \in X, \text{a.e.} t \in (0,T), \end{cases}$$

and there exists $1 \le p \le \infty$ *such that* $F(\cdot,x) \in L^p(0,T;X)$, $\forall x \in X$. *Then, for any* $x_0 \in X$, *there exists a unique function* $x \in W^{1,p}(0,T;X)$ *such that*

$$\begin{cases} \dot{x}(t) = F(t,x(t)) \quad \text{a.e. } t \in (0,T), \\ x(0) = x_0. \end{cases}$$

Theorem 2.1 will be used in Section 4 to prove the unique solvability of the intermediate problem involving the bonding field. The following existence, uniqueness and regularity result is carried out in the next theorem and is based on the following abstract result for evolutionary variational inequalities

Theorem 2.2. *Let* X *be a Hilbert space. Assume that the operator* $A : X \to X$ *satisfies*

$$\begin{cases} \text{(a) } A : X \to X \text{ is strongly monotone and Lipschitz continuous, i.e.,} \\ \text{(b)there exists } m_A > 0 \text{ such that} \\ \qquad (Au_1 - Au_2, u_1 - u_2)_X \ge m_A\|u_1 - u_2\|_X^2 \quad \forall u_1,u_2 \in X; \\ \text{(c)there exists } L_A \text{ such that} \\ \qquad \|Au_1 - Au_2\|_X \le L_A\|u_1 - u_2\|_X \quad \forall u_1,u_2 \in X, \end{cases} \tag{2.4}$$

and that $j : X \to \bar{\mathbb{R}}$ *is a proper, convex, and lower semi continuous functional*

$$\begin{cases} \text{(a) } j(u,\cdot) \text{ is convex and lower semi continuous on } X \text{ for all } u \in X. \\ \text{(b) There exists } m_A > 0 \text{ such that} \\ \qquad j(u_1,v_2) - j(u_1,v_1) + j(u_2,u_1) - j(u_2,u_2) \\ \qquad \le m_A \|u_1 - u_2\|_X \|v_1 - v_2\|_X \quad \forall u_1,u_2,v_1,v_2 \in X. \end{cases} \tag{2.5}$$

Then, for each $f \in X$, *the elliptic variational inequality of the second kind,*

$$(Au, v-u)_X + j(v) - j(u) \ge (f(t), v - u(t))_X, \tag{2.6}$$

has a unique solution. Moreover, the solution depends Lipschitz continuously on f.

Theorem 2.3. *Let* $V \subset H \subset V'$ *be a Gelfand triple. Let* K *be a nonempty closed, and convex set of* V. *Assume that* $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ *is a continuous and symmetric bilinear form such that, for some constants* C_0 *and* $C_1 > 0$,

$$a(\mathbf{v},\mathbf{v}) + C_0|\mathbf{v}|_H^2 \ge C_1|\mathbf{v}|_V^2 \quad \forall \mathbf{v} \in V.$$

Then, for every $\mathbf{u}_0 \in \mathrm{K}$ *and* $\mathbf{f} \in L^2(0,T;H)$, *there exists a unique function* $\mathbf{u} \in W^{1,2}(0,T;H) \cap L^2(0,T;V)$ *such that* $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{u}(t) \in \mathrm{K} \quad \forall t \in [0,T]$, *and for almost all* $t \in (0,T)$,

$$(\dot{\mathbf{u}}(t), \mathbf{v} - \mathbf{u}(t))_{\acute{V}\times V} + a(\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t)) \ge (\mathbf{f}(t), \mathbf{v} - \mathbf{u}(t))_H, \quad \forall \mathbf{v} \in \mathrm{K}.$$

3. MECHANICAL AND VARIATIONAL FORMULATIONS

We describe the model for the process and present its variational formulation. The physical setting is the following. An electro viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$, $(d = 2,3)$ with outer Lipschitz surface Γ. The body submitted to the action of body forces of density f_0 and volume electric charges of density q_0. It is also submitted the mechanical and electric constraint on the boundary. We consider a partition of Γ into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, on one hand, and in two measurable parts Γ_a and Γ_b, on the other hand, such that meas$(\Gamma_1) > 0$, meas$(\Gamma_a) > 0$ and $\Gamma_3 \subset \Gamma_b$. Let $T > 0$ and let $[0,T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0,T)$, so the displacement field vanishes there. A surface traction of density f_2 acts on $\Gamma_2 \times (0,T)$ and a body force of density f_0 acts in $\Omega \times (0,T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0,T)$ and a surface electric charge of density q_2 is prescribed on $\Gamma_b \times (0,T)$. The body is in adhesive contact with an obstacle, or foundation, over the contact surface Γ_3. Moreover, the process is quasistatic, and thus the inertial terms are neglected in the equation of motion. We denote by $\mathbf{u}$ the displacement field, by σ the stress tensor field and by $\varepsilon(\mathbf{u})$ the linearized strain tensor.

To simplify the notations, we do not indicate explicitly the dependence of various functions on the variables $x \in \Omega \cup \Gamma$ and $t \in [0,T]$. Then, the classical formulation of the mechanical problem of electro-viscoelastic material, frictional, adhesive contact may be stated as follows.

Problem P. Find a displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0,T] \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$, an electric displacement field $\mathbf{D} : \Omega \times [0,T] \to \mathbb{R}^d$, a damage field β : $\Omega \times [0,T] \to \mathbb{R}$, and a bonding field $\alpha : \Gamma_3 \times [0,T] \to \mathbb{R}$ such that

$$\begin{aligned}\sigma(t) = \mathscr{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathscr{G}(\varepsilon(\mathbf{u}(t)),\alpha) \\ + \int_0^t \mathscr{M}(t-s)\varepsilon(\mathbf{u}(s))ds + \mathscr{E}^*\nabla\varphi(t)\end{aligned} \quad \text{in } \Omega \times (0,T), \tag{3.1}$$

$$\mathbf{D} = \mathscr{E}\varepsilon(\mathbf{u}) - B\nabla(\varphi) \quad \text{in } \Omega \times (0,T), \tag{3.2}$$

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_K(\alpha) \ni S(\varepsilon(\mathbf{u}),\alpha), \quad \text{in } \Omega \times (0,T), \tag{3.3}$$

$$\operatorname{Div}\sigma + \mathbf{f}_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{3.4}$$

$$\operatorname{div}\mathbf{D} - q_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{3.5}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{3.6}$$

$$\sigma\nu = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{3.7}$$

$$\begin{cases} -\sigma_\nu = p_\nu(u_\nu - g), \\ \|\sigma_\tau\| \le p_\tau(u_\nu - g), \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{3.8}$$

$$\dot{u}_\tau \neq 0 \Rightarrow \sigma_\tau = -p_\tau(u_\nu - g)\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{on } \Gamma_3 \times (0,T), \tag{3.9}$$

$$\dot{\beta} = -(\beta(\gamma_\nu R_\nu(u_\nu))^2 + \gamma_\tau\|\mathbf{R}_\tau(\mathbf{u}_\tau)\|^2) - \varepsilon_a)_+ \quad \text{on } \Gamma_3 \times (0,T), \tag{3.10}$$

$$\frac{\partial\alpha}{\partial\nu} = 0 \quad \text{on } \Gamma \times (0,T), \tag{3.11}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T), \tag{3.12}$$

$$\mathbf{D}\cdot\nu = q_2 \quad \text{on } \Gamma_b \times (0,T), \tag{3.13}$$

$$\mathbf{D}\cdot\nu = \psi(u_\nu - g)\phi_l(\varphi-\varphi_0) \qquad \text{on } \Gamma_3\times(0,T), \tag{3.14}$$

$$\mathbf{u}(0)=\mathbf{u}_0, \qquad \alpha(0)=\alpha_0 \qquad \text{in } \Omega, \tag{3.15}$$

$$\beta(0)=\beta_0 \qquad \text{on } \Gamma_3. \tag{3.16}$$

(3.1) and (3.2) represent the electro-viscoelastic constitutive law with long term-memory and damage. The evolution of the damage field is governed by the inclusion of parabolic type given by relation (3.3), where S is the mechanical source of the damage growth. $\partial\varphi_K$ is the subdifferential of the indicator function of the admissible damage functions set K. Equations (3.4) and (3.5) represent the equilibriumequations for the stress and electric displacement fields while (3.6) and (3.7) are the displacement and traction boundary condition, respectively.

We turn to boundary condition (3.8)-(3.9)and (3.14) which describe the mechanical conditions on the potential contact surface Γ_3. The normal compliance function p_ν in (3.8) is described below, and g represents the gap in the reference configuration between Γ_3 and the foundation, and measured along the direction of ν. When positive, $u_\nu - g$ represents the interpenetration of the surface asperities into those of the foundation. Conditions (3.9) is the associated friction law where p_τ is a given function. According to (3.9), the tangential shear cannot exceed the maximum frictional resistance $p_\tau(u_\nu - g)$, the so-called friction bound. Moreover, when sliding commences, the tangential shear reaches the friction bound and opposes the motion.

Equation (3.10) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [19] (see also [5] for more details). In addition to γ_ν, two new adhesion coefficients are involved, γ_τ and ε_a. The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu\beta^2R_\nu(u_\nu)$. The adhesive traction is tensile and proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu L$. R_ν is the truncation operator defined by

$$R_\nu(s)=\begin{cases} L, & \text{if} \quad s<-L,\\ -s, & \text{if} \quad -L\le s\le 0,\\ 0, & \text{if} \quad s>0.\end{cases}$$

Here $L>0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of operator R_ν, together with operator $\mathbf{R}_\tau$ defined below, is motivated by mathematical arguments but it is not restrictive from the physical point of view, since no restriction on the size of the parameter L is made in what follows. Condition (3.10) represents the adhesive contact condition on the tangential plane, in which p_τ is a given function and $\mathbf{R}_\tau$ is the truncation operator given by

$$\mathbf{R}_\tau(\mathbf{v})=\begin{cases} \mathbf{v}, & \text{if} \quad |\mathbf{v}|\le L,\\ L\dfrac{\mathbf{v}}{|\mathbf{v}|}, & \text{if} \quad |\mathbf{v}|>L.\end{cases}$$

This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L. The frictional tangential traction is assumed to be much smaller than the adhesive one and, therefore, omitted. The introduction of operator R_ν, together with operator R_τ defined above, is motivated by mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of L is made in what follows. Relation (3.11) describes a homogeneous Neumann boundary condition, where $\partial\alpha/\partial\nu$ is the normal derivative of β. (3.12) and (3.13) represent the electric boundary conditions. Next, (3.14) is the electrical contact condition on Γ_3, introduced in [9, 20]. It may be obtained as follows.

We assume that the foundation is electrically conductive and its potential is maintained at φ_0. When there is no contact at a point on the surface (i.e., $u_\nu < g$), the gap is assumed to be an insulator (say, it is filled with air), there are no free electrical charges on the surface and the normal component of the electric displacement field vanishes. Thus,

$$u_\nu < g \;\Rightarrow\; \mathbf{D}\cdot\nu = 0. \tag{3.17}$$

During the process of the contact (i.e., when $u_\nu \geq g$) the normal component of the electric displacement field or the free charge is assumed to be proportional to the difference between the potential of the foundation and the body's surface potential, with k as the proportionality factor. Thus,

$$u_\nu \geq g \;\Rightarrow\; \mathbf{D}\cdot\nu = k\,(\varphi-\varphi_0). \tag{3.18}$$

From (3.17) and (3.18), we obtain

$$\mathbf{D}\cdot\nu = k\,\chi_{[0,\infty)}(u_\nu - g)\,(\varphi-\varphi_0), \tag{3.19}$$

where $\chi_{[0,\infty)}$ is the characteristic function of $[0,\infty)$, that is,

$$\chi_{[0,\infty)}(r) = \begin{cases} 0, & \text{if } r<0,\\ 1, & \text{if } r\geq 0. \end{cases}$$

Condition (3.19) describes perfect electrical contact and is somewhat similar to the well-known Signorini contact condition. Both conditions may be over-idealizations in many applications. To make it more realistic, we regularize condition (3.19) and write it as (3.14) in which $k\,\chi_{[0,\infty)}(u_\nu - g)$ is replaced with ψ, which is a regular function and ϕ_l is the truncation function

$$\phi_l(s) = \begin{cases} -l, & \text{if } s<-l,\\ s, & \text{if } -l\leq s\leq l,\\ l, & \text{if } s>l, \end{cases}$$

where l is a large positive constant. We note that this truncation does not pose any practical limitations on the applicability of the model, since l may be arbitrarily large, higher than any possible peak voltage in the system, and therefore in applications $\phi_l(\varphi-\varphi_0) = \varphi-\varphi_0$. The reasons for regularization (3.14) of (3.19) are mathematical. First, we need to avoid the discontinuity in the free electric charge when the contact is established and, therefore, we regularize function $k\,\chi_{[0,\infty)}$ in (3.19) with a Lipschitz continuous function ψ. A possible choice is

$$\psi(r) = \begin{cases} 0, & \text{if } r<0,\\ k\delta r, & \text{if } 0\leq r\leq 1/\delta,\\ k, & \text{if } r>\delta, \end{cases} \tag{3.20}$$

where $\delta>0$ is a small parameter. This choice means that during the process of the contact the electrical conductivity increases as the contact among the surface asperities improves, and stabilizes when $u_\nu - g$

reaches δ. Second, we need $\phi_l(\varphi-\varphi_0)$ to control the boundednes of $\varphi-\varphi_0$. If $\psi\equiv 0$ in (3.14), then

$$\mathbf{D}\cdot\nu=0,\quad \text{on } \Gamma_3\times(0,T), \tag{3.21}$$

which decouples the electrical and mechanical problems on the contact surface. Condition (3.21) models the case when the obstacle is a perfect insulator and was used in [21, 22]. Condition (3.14), instead of (3.21), introduces strong coupling between the mechanical and the electric boundary conditions and leads to a new and nonstandard mathematical model. Because of friction condition (3.9), which is non-smooth, we do not expect the problem to have, in general, any classical solutions. In equation (3.15), $\mathbf{u}_0$ is the initial displacement, and β_0 is the initial damage. Finally, in equation (3.16), α_0 denotes the initial bonding.

To obtain the variational formulation of (3.1)-(3.16), we introduce

$$\mathscr{Z}=\left\{\ \theta\in L^\infty(0,T;L^2(\Gamma_3))\quad: 0\le\theta(t)\le 1,\quad \text{a.e. } t\in[0,T],\quad \text{on } \Gamma_3\ \right\},$$

and for the displacement field we need the closed subspace of $H^1(W)^d$ defined by

$$V=\left\{v\in H^1(\Omega)^d \mid v=0 \text{ on } \Gamma_1\right\}.$$

Since meas$(\Gamma_1)>0$, Korn's inequality holds and there exists a constant $C_k>0$, that depends only on W and Γ_1 such that

$$\|\varepsilon(v)\|_{\mathscr{H}}\ge C_k\|v\|_{H^1(\Omega)^d},\quad \forall v\in V.$$

On V, we consider the inner product and the associated norm given by

$$(\mathbf{u},\mathbf{v})_V=(\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{\mathscr{H}},\quad \|\mathbf{v}\|_V=\|\varepsilon(\mathbf{v})\|_{\mathscr{H}},\quad \forall\mathbf{u},\mathbf{v}\in V. \tag{3.22}$$

It follows that $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent norms on V and therefore $(V,(\cdot,\cdot)_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant C_0, depending only on Ω, Γ_1 and Γ_3, such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d}\le\tilde{C}_0\|\mathbf{v}\|_V,\quad \forall\mathbf{v}\in V. \tag{3.23}$$

We also introduce the spaces

$$W=\left\{\phi\in H^1(\Omega)\mid \phi=0 \text{ on } \Gamma_a\right\},$$

$$\mathscr{W}=\{\ \mathbf{D}=(D_i)\mid D_i\in L^2(\Omega),\ \operatorname{div}\mathbf{D}\in L^2(\Omega)\},$$

where $\operatorname{div}D=(D_{i,i})$. The spaces W and $\mathscr{W}$ are real Hilbert spaces with the inner products given by

$$(\varphi,\phi)_W=\int_\Omega\nabla\varphi\cdot\nabla\phi dx,$$

$$(\mathbf{D},\mathbf{E})_{\mathscr{W}}=(\mathbf{D},\mathbf{E})_H+(\operatorname{div}\mathbf{D},\operatorname{div}\mathbf{E})_{L^2(\Omega)},$$

The associated norms will be denoted by $\|\cdot\|_W$ and $\|\cdot\|_{\mathscr{W}}$, respectively. Moreover, by the Sobolev trace theorem, there exists a constant c_0, depending only on Ω, Γ_a and Γ_3 such that

$$\|\phi\|_{L^2(\Gamma_3)}\le c_0\|\phi\|_H, \tag{3.24}$$

If $\mathbf{D}\in\mathscr{W}$ is a regular function, then the following Green's type formula holds

$$(\mathbf{D},\nabla\zeta)_H+(\operatorname{div}\mathbf{D},\zeta)_{L^2(\Omega)}=\int_\Gamma\mathbf{D}\cdot\nu\zeta da,\quad \forall\zeta\in H^1(\Omega). \tag{3.25}$$

Notice also that, since meas$(\Gamma_a) > 0$, the following Friedrichs-Poincar inequality holds

$$\|\nabla\zeta\|_H \geq C_F \|\zeta\|_{H^1(\Omega)}, \quad \forall \zeta \in W, \tag{3.26}$$

where $C_F > 0$ is a constant which depends only on W and Γ_a.

In the study of mechanical problem (3.1)-(3.16), we assume that viscosity function $\mathscr{A} : \Omega \times \mathscr{S}^d \to \mathscr{S}^d$ satisfies

$$\begin{cases} \text{(a) There exists constants } \mathscr{C}_1^{\mathscr{A}}, \mathscr{C}_2^{\mathscr{A}} > 0 \text{ such that} \\ \quad \|\mathscr{A}(\mathbf{x},\varepsilon)\| \leq \mathscr{C}_1^{\mathscr{A}}\|\varepsilon\| + \mathscr{C}_2^{\mathscr{A}} \qquad \forall \varepsilon \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(b) There exists } m_{\mathscr{A}} > 0 \text{ such that} \\ \quad (\mathscr{A}(\mathbf{x},\varepsilon_1) - \mathscr{A}(\mathbf{x},\varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_{\mathscr{A}}\|\varepsilon_1 - \varepsilon_2\|^2 \\ \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(c) The mapping } \mathbf{x} \mapsto \mathscr{A}(\mathbf{x},\varepsilon) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \text{for any } \varepsilon \in \mathbb{S}^d. \\ \text{(d) The mapping } \varepsilon \mapsto \mathscr{A}(\mathbf{x},\varepsilon) \text{ is continuous on } \mathbb{S}^d \text{a.e. } \mathbf{x} \in \Omega. \end{cases} \tag{3.27}$$

The elasticity operator $\mathscr{G} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(a) There exists } L_{\mathscr{G}} > 0 \text{ such that} \\ \quad \|\mathscr{G}(\mathbf{x},\xi_1,\alpha_1) - \mathscr{G}(\mathbf{x},\xi_2,\alpha_2)\| \leq L_{\mathscr{G}}(\|\xi_1 - \xi_2\| + \|\alpha_1 - \alpha_2\|) \\ \quad \forall \xi_1, \xi_2 \in \mathbb{S}^d, \forall \alpha_1, \alpha_2 \in \mathbb{R} \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(b) The mapping } \mathbf{x} \mapsto \mathscr{G}(\mathbf{x},\xi,\alpha) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \text{for any } \xi \in \mathbb{S}^d \text{ and } \alpha \in \mathbb{R}. \\ \text{(c) The mapping } \mathbf{x} \mapsto \mathscr{G}(\mathbf{x},0,0) \text{ belongs to } \mathscr{H}. \end{cases} \tag{3.28}$$

The damage source function $S : \Omega \times S^d \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) There exists a constant } L_S > 0 \text{ such that} \\ \quad \|S(\mathbf{x},\varepsilon_1,\alpha_1) - S(\mathbf{x},\varepsilon_2,\alpha_2)\| \leq L_S(\|\varepsilon_1 - \varepsilon_2\| + \|\alpha_1 - \alpha_2\|) \\ \quad \forall \varepsilon_1, \varepsilon_2 \in S^d, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}, \text{ a.e.} \mathbf{x} \in \Omega. \\ \text{(b) } \forall \varepsilon \in \mathbb{S}^d, \text{ and } \alpha \in \mathbb{R}, \quad S(\mathbf{x},\varepsilon,\alpha) \text{ the function is measurable in } \Omega. \\ \text{(c) The mapping } \mathbf{x} \mapsto S(\mathbf{x},0,0) \text{ belongs to } L^2(\Omega). \end{cases} \tag{3.29}$$

The electric permittivity operator $B = (B_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} \text{(a) } B(x,E) = (B_{ij}(x)E_j) \;\; \forall E = (E_i) \in \mathbb{R}^d, \text{ p.p.} x \in \Omega. \\ \text{(b) } B_{ij} = B_{ji} \in L^\infty(\Omega), \; 1 \leq i,j \leq d. \\ \text{(c) There exists a constant } M_B > 0 \text{ such that} \\ \qquad BE.E \geq M_B|E|^2 \;\; \forall E = (E_i) \in \mathbb{R}^d, \text{ a.a.in } \Omega. \end{cases} \tag{3.30}$$

The piezoelectric operator $\mathscr{E} : \Omega \times S^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} \text{(a) } \mathscr{E} = (e_{ijk}), \; e_{ijk} \in L^\infty(\Omega), \; 1 \leq i,j,k \leq d. \\ \text{(b) } \mathscr{E}(\mathbf{x})\sigma.\tau = \sigma.\mathscr{E}^*\tau, \quad \forall \sigma \in S^d, \quad \forall \tau \in \mathbb{R}^d. \end{cases} \tag{3.31}$$

The normal compliance functions $p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$, $(r = \nu, \tau)$ satisfy

$$\begin{cases} \text{(a) } \exists L_r > 0 \text{ such that } \|p_r(\mathbf{x}, u_1) - p_r(\mathbf{x}, u_2)\| \leq L_r \|u_1 - u_2\| \\ \quad \forall u_1, u_2 \in \mathbb{R}, \text{a.e.} \mathbf{x} \in \Gamma_3. \\ \text{(b) } \mathbf{x} \mapsto p_r(\mathbf{x}, u) \text{ is measurable on } \Gamma_3, \text{ for all } u \in \mathbb{R}. \\ \text{(c) } \mathbf{x} \mapsto p_r(\mathbf{x}, u) = 0, \text{ for all } u \leq 0. \end{cases} \tag{3.32}$$

The surface electrical conductivity function $\psi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

$$\begin{cases} \text{(a) } \exists L_\psi > 0 \text{ such that } \|\psi(\mathbf{x}, u_1) - \psi(\mathbf{x}, u_2)\| \leq L_\psi \|u_1 - u_2\| \\ \quad \forall u_1, u_2 \in \mathbb{R}, \text{ a.e.} \mathbf{x} \in \Gamma_3. \\ \text{(b) } \exists M_\psi > 0 \text{ such that } \|\psi(\mathbf{x}, u)\| \leq M_\psi \ \forall u \in \mathbb{R}, \text{ a.e.} \mathbf{x} \in \Gamma_3. \\ \text{(c) } \mathbf{x} \mapsto \psi(\mathbf{x}, u) \text{ is measurable on } \Gamma_3, \text{ for all } u \in \mathbb{R}. \\ \text{(d) } \mathbf{x} \mapsto \psi(\mathbf{x}, u) = 0, \text{ for all } u \leq 0. \end{cases} \tag{3.33}$$

The relaxation tensor $\mathscr{M}$ satisfies

$$M \in C(0, T; \mathscr{H}). \tag{3.34}$$

The adhesion coefficients and the limit bound satisfies

$$\gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3), \quad \varepsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu, \gamma_\tau, \varepsilon_a \geq 0, \text{ a.e. on } \Gamma_3. \tag{3.35}$$

The initial bonding field satisfies

$$\beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1, \text{ a.e. on } \Gamma_3 \tag{3.36}$$

and the initial damage field satisfies

$$\alpha_0 \in K. \tag{3.37}$$

Finally, we assume that the gap function, the given potential and the initial displacement satisfy

$$g \in L^2(\Gamma_3), \quad g \geq 0, \quad \text{a.e. on } \Gamma_3, \tag{3.38}$$

$$\varphi_0 \in L^2(\Gamma_3), \tag{3.39}$$

$$\mathbf{u}_0 \in V. \tag{3.40}$$

The forces, tractions, volume and surface free charge densities satisfy

$$\mathbf{f}_0 \in C(0, T; L^2(\Omega)^d), \qquad \mathbf{f}_2 \in C(0, T; L^2(\Gamma_2)^d), \tag{3.41}$$

$$q_0 \in C(0, T; L^2(\Omega)), \qquad q_2 \in C(0, T; L^2(\Gamma_b)). \tag{3.42}$$

Here, $1 \leq p \leq \infty$. We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a(\xi, \varphi) = k \int_\Omega \nabla\xi . \nabla\varphi dx, \tag{3.43}$$

and the microcrack diffusion coefficient verifies

$$k > 0. \tag{3.44}$$

Next, we use the Riesz representation theorem to define $\mathbf{f} : [0, T] \to V$

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0(t) \cdot \mathbf{v} \, dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot \mathbf{v} \, da, \tag{3.45}$$

for all $v \in V$, $t \in [0, T]$. Then conditions (3.41) and (3.45) imply

$$\mathbf{f} \in C(0, T; V), \tag{3.46}$$

and we denote by $q : [0,T] \to W$ the function defined by

$$(q(t),\zeta)_W = \int_\Omega q_0(t)\zeta\, dx - \int_{\Gamma_b} q_2(t)\zeta\, da, \tag{3.47}$$

for all $\zeta \in W$, $t \in [0,T]$. Then conditions (3.42) and (3.47) imply

$$\mathbf{q} \in C(0,T;W). \tag{3.48}$$

Next, we denote by $j : V \times V \to \mathbb{R}$ the functional

$$j(\mathbf{u},\mathbf{v}) = \int_{\Gamma_3} (p_\nu(u_\nu - g)v_\nu + p_\tau(u_\nu - g)\|\mathbf{v}_\tau\|)da. \tag{3.49}$$

By the assumptions on p_ν and p_τ, we obtain that, for $v \in V$,

$$p_\nu(u_\nu - g),\ p_\tau(u_\nu - g) \in L^2(\Gamma_3), \tag{3.50}$$

and, thus, $j(\cdot,\cdot)$ is well defined on $V \times V$.

Next, we define the mapping $h : V \times W \to W$ by

$$(h(\mathbf{u},\varphi),\zeta)_W = \int_{\Gamma_3} \psi(u_\nu - g)\phi_l(\varphi - \varphi_0)\zeta\, da. \tag{3.51}$$

Using standard arguments, we obtain the variational formulation of (3.1)-(3.16).

Problem PV. Find a displacement field $\mathbf{u} : [0,T] \to V$, a stress field $\sigma : [0,T] \to \mathscr{H}$, an electric potential field $\varphi : [0,T] \to W$, a damage field $\alpha : [0,T] \to H^1(\Omega)$ and a bonding field $\beta : [0,T] \to L^\infty(\Gamma_3)$ such that

$$\sigma(t) = \mathscr{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathscr{G}\varepsilon(\mathbf{u}(t),\alpha) + \int_0^t \mathscr{M}(t-s)\varepsilon(\mathbf{u}(s))ds + \mathscr{E}^*\nabla\varphi(t), \tag{3.52}$$

$$(\sigma(t),\varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}(t))_{\mathscr{H}} + j(\mathbf{u}(t),\mathbf{v}) - j(\mathbf{u}(t),\dot{\mathbf{u}}(t)) \geq (\mathbf{f}(t),\mathbf{v} - \dot{\mathbf{u}}(t))_V, \tag{3.53}$$

for all $\mathbf{v} \in V$ and $t \in [0,T]$,

$$(B\nabla\varphi(t),\nabla\zeta)_H - (\mathscr{E}\varepsilon(\mathbf{u}(t)),\nabla\zeta)_H + (h(\mathbf{u}(t),\varphi(t)),\zeta)_W = (q(t),\zeta)_W, \tag{3.54}$$

for all $\zeta \in W$ and $t \in [0,T]$,

$$\begin{aligned} \alpha(t) \in K,\ \ (\dot{\alpha}(t),\xi - \alpha(t))_{L^2(\Omega)} &+ a(\alpha(t),\xi - \alpha(t)) \\ &\geq (S(\varepsilon(\mathbf{u}(t)),\alpha(t)),\xi - \alpha(t))_{L^2(\Omega)}, \end{aligned} \tag{3.55}$$

for all $\xi \in K$ and $t \in [0,T]$,

$$\dot{\beta} = -(\beta(\gamma_\nu R_\nu(u_\nu))^2 + \gamma_\tau\|\mathbf{R}_\tau(\mathbf{u}_\tau)\|^2) - \varepsilon_a)_+ \ \text{ on } \Gamma_3 \times (0,T), \tag{3.56}$$

and

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \beta(0) = \beta_0, \qquad \alpha(0) = \alpha_0. \tag{3.57}$$

To study problem **PV**, we make the following assumption

$$M_\psi < \frac{m_{\mathbf{B}}}{c_0^2}, \tag{3.58}$$

where M_ψ, c_0 and $m_{\mathbf{B}}$ are given in (3.33), (3.23) and (3.29), respectively. We note that only the trace constant, the coercivity constant of $\mathbf{B}$ and the bound of are involved in (3.58). Therefore, this smallness assumption involves only the geometry and the electrical part, and does not depend on the mechanical data of the problem. Moreover, it is satisfied when the obstacle is insulated. So, $M_\psi = 0$. We notice that variational problem **PV** is formulated in terms of displacement field, an electrical potential field, damage field and bonding field. The existence of the unique solution of problem **PV** is stated and proved in the next section. To this end, we consider the following remark which is used in different places of the paper.

Remark 3.1. We note that, in problem **P** and in problem **PV**, we do not need to impose explicitly the restriction $0 < \beta < 1$. Indeed, equation (3.57) guarantees that $\beta(x,t) \leq \beta_0(x)$ and, therefore, assumption (3.36) shows that $\beta(x,t) \leq 1$ for $t \geq 0$, a.e. $x \in \Gamma_3$. On the other hand, if $\beta(x,t_0) = 0$ at time t_0, then it follows from (3.57) that $\dot{\beta}(x,t) = 0$ for all $t \geq t_0$ and therefore, $\beta(x,t) = 0$ for all $t \geq t_0$, a.e. $x \in \Gamma_3$. We conclude that $0 \leq \beta(x,t) \leq 1$ for all $t \in [0,T]$, a.e. $x \in \Gamma_3$.

4. Existence and Uniqueness

Our main existence and uniqueness result for Problem $\mathscr{PV}$ is the following.

Theorem 4.1. *Assume that* (3.7)-(3.23) *hold. Then there exists* $\mu_0 > 0$ *depending only on* Ω, Γ_1, $\mathscr{A}$ *and* p_r *such that, if* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, *then Problem* $\mathscr{PV}$ *has a unique solution* $(\mathbf{u},\varphi)$. *Moreover, the solution satisfies*

$$\mathbf{u} \in C^1(0,T;V), \tag{4.1}$$

$$\sigma \in C(0,T;\mathscr{H}_1), \tag{4.2}$$

$$\varphi \in C(0,T;W), \tag{4.3}$$

$$\alpha \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{4.4}$$

$$\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathscr{Z}. \tag{4.5}$$

The functions $u,\sigma,\varphi,\mathbf{D},\alpha$ and β which satisfy (3.1)-(3.2) and (3.50)-(3.55) are called weak solutions of contact problem **P**. We conclude that, under the assumptions (3.27)-(3.40), mechanical problem (3.1)-(3.16) has a unique weak solution satisfying (4.1)-(4.5). The regularity of the weak solution is given by (4.1)-(4.5) and, in term of electric displacement,

$$D \in C(0,T;\mathscr{W}). \tag{4.6}$$

Indeed, it follows from (3.53) that $\operatorname{div} D = q_0(t)$ for all $t \in [0,T]$. Therefore the regularity (4.1) and (4.2) of φ combined with (3.30), (3.31) and (3.42) implies (4.6).

The proof of Theorem 4.1 will be split into several steps. From now on, in this section, we always suppose that the assumptions of Theorem 4.1 hold, and we always assume that C is a generic positive constant which depends on Ω, Γ_1, Γ_3, p_ν, p_τ, γ_ν, γ_τ and L may change from place to place. Let $\eta \in C([0,T];\mathscr{H})$ and $\theta \in C([0,T];L^2(\Omega))$ be given.

Problem $\mathscr{PV}^u_\eta$. Find a displacement field $\mathbf{u}_\eta : [0,T] \to V$ and a stress field $\sigma_\eta : [0,T] \to \mathscr{H}_1$ such that for all $t \in [0,T]$,

$$\sigma_\eta = \mathscr{A}\varepsilon(\dot{\mathbf{u}}_\eta) + \eta, \tag{4.7}$$

$$(\sigma_\eta, \varepsilon(\omega - \dot{\mathbf{u}}_\eta)_{\mathscr{H}} + j(\mathbf{u}_\eta, \omega) - j(\mathbf{u}_\eta, \dot{\mathbf{u}}_\eta) \geq (\mathbf{f}, \omega - \dot{\mathbf{u}}_\eta)_V, \quad \forall \omega \in V, \tag{4.8}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0. \tag{4.9}$$

To study Problem $\mathscr{PV}^u_\eta$, we need the following lemma.

Lemma 4.2. *Let* $\mathbf{g} \in C(0,T;V)$. *Then there exists a unique function* $\mathbf{v}_{\eta g} \in C(0,T;V)$ *such that, for all* $t \in [0,T]$,

$$(\mathscr{A}\varepsilon(v_{\eta\mathbf{g}}), \varepsilon(\omega - v_{\eta\mathbf{g}}))_{\mathscr{H}} + j(\mathbf{g},\omega) - j(\mathbf{g}, v_{\eta g}) \\ \geq (\mathbf{f}, \omega - v_{\eta\mathbf{g}})_V - (\eta, \varepsilon(\omega - v_{\eta\mathbf{g}}))_H \quad \forall \omega \in V. \tag{4.10}$$

Proof. It follows from Theorem 2.2 that there exists a unique function $v_{\eta\mathbf{g}} : [0,T] \to V$ solving the elliptic variational inequality (4.10). To establish its regularity by showing that $v_{\eta\mathbf{g}} \in C([0,T];V)$, we let $t_1, t_2 \in [0,T]$ and denote by $\eta_i = \eta(t_i)$, $\mathbf{g}_i = \mathbf{g}(t_i)$, $f_i = f(t_i)$, and $v_i = v_{\eta\mathbf{g}}(t_i)$, $i = 1,2$. We choose $\omega = v_2$ in (4.10) at $t = t_1$, $\omega = v_1$ in (4.10) at $t = t_2$, and add the two inequalities to obtain

$$(\mathscr{A}\varepsilon(\mathbf{v}_1) - \mathscr{A}\varepsilon(\mathbf{v}_2), \varepsilon(v_1 - v_2))_{\mathscr{H}} \leq (f_1 - f_2, v_1 - v_2)_V + \\ + (\eta_1 - \eta_2, \varepsilon(v_1 - v_2))_{\mathscr{H}} + j(\mathbf{g}_1, v_2) - j(\mathbf{g}_1, v_1) + j(\mathbf{g}_2, v_1) - j(\mathbf{g}_2, v_2). \tag{4.11}$$

The left-hand side is bounded from below by (3.27). Thus,

$$(\mathscr{A}\varepsilon(\mathbf{v}_1) - \mathscr{A}\varepsilon(\mathbf{v}_2), \varepsilon(v_1 - v_2))_{\mathscr{H}} \geq m_{\mathscr{A}} \|v_1 - v_2\|_V^2. \tag{4.12}$$

The last line of (4.12) is bounded by the property (3.32) as follows

$$j(\mathbf{g}_1, v_2) - j(\mathbf{g}_1, v_1) + j(\mathbf{g}_2, v_1) - j(\mathbf{g}_2, v_2) \leq c\|\mathbf{g}_1 - \mathbf{g}_2\|_V \|v_1 - v_2\|_V. \tag{4.13}$$

Using these bounds in (4.12), we obtain

$$\|v_1 - v_2\|_V \leq c\Big(\|f_1 - f_2\|_V + \|\eta_1 - \eta_2\|_{\mathscr{H}} + \|g_1 - g_2\|_V\Big). \tag{4.14}$$

Then the conclusion that $v_{\eta\mathbf{g}} \in C([0,T];V)$ follows from the continuity of f, η and $\mathbf{g}$ in their respective spaces V, $\mathscr{H}$ and V. □

With the help of Lemma 4.2, we are in a position to show the following existence and uniqueness result for Problem $\mathscr{P}\mathscr{V}_\eta^1$.

Lemma 4.3. *There exists a unique solution to Problem* $\mathscr{P}\mathscr{V}_\eta^1$ *such that* $\mathbf{u}_\eta \in C^1(0,T;V)$ *and* $\sigma_\eta \in C(0,T;\mathscr{H}_1)$.

Proof. We consider an operator $\Lambda_\eta : C(0,T;V) \to C(0,T;V)$ defined by

$$\Lambda_\eta \mathbf{g}(t) = u_0 + \int_0^t v_{\eta\mathbf{g}}(s)ds, \qquad \mathbf{g} \in C(0,T;V),\ t \in [0,T], \tag{4.15}$$

where $v_{\eta\mathbf{g}}$ is the solution of (4.10). We will show that this operator has a unique fixed point $\mathbf{g}_\eta \in C([0,T];V)$. To this end, let $\mathbf{g}_1, \mathbf{g}_2 \in C([0,T];V)$ and denote by $v_i = v_{\eta\mathbf{g}_i}$, $i = 1,2$, the corresponding solutions of (4.10). Using the definition (4.15), we obtain

$$\|\Lambda_\eta \mathbf{g}_1(t) - \Lambda_\eta \mathbf{g}_2(t)\|_V \leq \int_0^t \|v_1(s) - v_2(s)\|_V ds, \quad \forall t \in [0,T]. \tag{4.16}$$

Moreover, using estimates similar to those leading to (4.14) in the proof of Lemma 4.2, we have

$$\|v_1(s) - v_2(s)\|_V \leq c\|\mathbf{g}_1(s) - \mathbf{g}_2(s)\|_V, \quad s \in [0,T].$$

It follows from (4.16) that

$$\|\Lambda_\eta \mathbf{g}_1(t) - \Lambda_\eta \mathbf{g}_2(t)\|_V \leq \int_0^t \|\mathbf{g}_1(s) - \mathbf{g}_2(s)\|_V ds, \quad \forall t \in [0,T]. \tag{4.17}$$

Reiterating this inequality m times, we obtain

$$\|\Lambda_\eta^m \mathbf{g}_1 - \Lambda_\eta^m \mathbf{g}_2\|_{C(0,T;V)} \leq \frac{c^m T^m}{m!} \|\mathbf{g}_1 - \mathbf{g}_2\|_{C(0,T;V)} ds, \quad \forall t \in [0,T].$$

This shows that for m large enough Λ_η^m is a contraction in space $C([0,T],V)$. Thus, Λ_η has a unique fixed point $\mathbf{g}_\eta \in C([0,T],V)$.

Next, let $v_\eta \in C([0,T];V)$, $u_\eta \in C^1([0,T];V)$ and $\sigma_\eta \in C([0,T];\mathscr{H})$ be given by

$$v_\eta = v_{\eta g_\eta}, \tag{4.18}$$

$$\mathbf{u}_\eta(t) = u_0 + \int_0^t v_\eta(s) ds, \qquad \forall t \in [0,T], \tag{4.19}$$

$$\sigma_\eta = \mathscr{A}\varepsilon(v_\eta) + \eta. \tag{4.20}$$

Clearly, (4.7) and (4.9) are satisfied. Moreover, by (4.19), (4.18) and (4.15), it follows that $u_\eta = \mathbf{g}_\eta$ and $\dot{u}_\eta = v_\eta$. Therefore, if $\mathbf{g} = \mathbf{g}_\eta$ in (4.10), then we obtain (4.8).

To prove the regularity of σ_η, we choose $\omega = \dot{u}_\eta \pm \varphi$ in ((4.8) with $\varphi \in C_0^\infty(\Omega)^d$ to obtain

$$(\sigma_\eta, \varepsilon(\varphi-))_{\mathscr{H}} = (\mathbf{f}, \varphi)_V, \quad \forall \varphi \in C_0^\infty(\Omega)^d, \text{ on } [0,T]. \tag{4.21}$$

From the definition of $(f,\varphi)_V$ in (3.44), we find

$$\operatorname{Div} \sigma_\eta + \mathbf{f}_0 = 0, \qquad \text{on } [0,T]. \tag{4.22}$$

Now, assumption (3.41) and equation (4.22) imply that $\sigma_\eta \in C([0,T];\mathscr{H}_1)$. This establishes the existence part in Lemma 4.3. From (3.27), (3.33) and Gronwall's inequality, we find the uniqueness of the solution follows from (4.7) immediately. □

Next, we use $\mathbf{u}_\eta \in C^1([0,T],V)$, which is obtained in Lemma 4.2, to construct the following variational problem for the electrical potential.

Problem $\mathscr{PV}_\eta^\varphi$. Find an electrical potential $\varphi_\eta : [0,T] \to W$ such that

$$(B\nabla\varphi_\eta(t), \nabla\zeta)_H - (\mathscr{E}\varepsilon(\mathbf{u}_\eta(t)), \nabla\zeta)_H + (h(\mathbf{u}_\eta(t), \varphi_\eta(t)), \zeta)_W = (q(t), \zeta)_W, \tag{4.23}$$

for all $\zeta \in W$, $t \in [0,T]$.

The well-posedness of problem $\mathscr{PV}_\eta^\varphi$ follows.

Lemma 4.4. *There exists a unique solution $\varphi_\eta \in C(0,T;W)$ satisfying (4.23). Moreover, if φ_{η_1} and φ_{η_2} are the solutions of (4.23) corresponding to η_1, $\eta_2 \in C([0,T];\mathscr{H})$, then there exists $c > 0$ such that*

$$\|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)\|_W \leq c\,\|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\|_V \quad \forall t \in [0,T]. \tag{4.24}$$

We use an abstract existence and unique result which may be found in [20].

In the third step, we let $\theta \in L^2(0,T;L^2(\Omega))$ be given and consider the following variational problem for the damage filed.

Problem $\mathscr{PV}_\theta$. Find the damage field $\alpha_\theta : [0,T] \to H^1(\Omega)$ such that

$$\begin{aligned} \alpha_\theta(t) \in K, \quad & (\dot\alpha_\theta(t), \xi - \alpha_\theta)_{L^2(\Omega)} + a(\alpha_\theta(t), \xi - \alpha_\theta(t)) \\ & \geq (\theta(t), \xi - \alpha_\theta(t))_{L^2(\Omega)} \quad \forall \xi \in K, \text{ a.e. } t \in (0,T), \end{aligned} \tag{4.25}$$

$$\alpha_\theta(0) = \alpha_0. \tag{4.26}$$

We apply Theorem 2.3 to problem PV_θ.

Lemma 4.5. *There exists a unique solution β_θ to the auxiliary problem $\mathscr{PV}_\theta$ satisfying* (4.4).

To solve $\mathscr{PV}_\theta$, we recall the following standard result for parabolic variational inequalities (see, e.g.,[5]).

In the fourth step, we use the displacement field $\mathbf{u}_\eta$ obtained in Lemma 4.2 and consider the following initial-value problem.

Problem $\mathscr{PV}^\beta$. Find the adhesion field $\beta_\eta : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot\beta_\eta = -(\beta_\eta(\gamma_\nu R_\nu(u_{\eta\nu})^2 + \gamma_\tau \|\mathbf{R}_\tau(\mathbf{u}_{\eta\tau})\|^2) - \varepsilon_a)_+, \tag{4.27}$$

$$\beta_\eta(0) = \beta_0 \in \Omega. \tag{4.28}$$

We have the following result.

Lemma 4.6. *There exists a unique solution $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathscr{Z}$ to Problem $\mathscr{PV}^\beta$.*

Proof. . For the sake of simplicity, we suppress the dependence of various functions on Γ_3, and note that the equalities and inequalities below are valid a.e. on Γ_3. Consider the mapping $F_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_\eta(t,\beta) = -(\beta(\gamma_\nu R_\nu(u_{\eta\nu}(t))^2 + \gamma_\tau \|\mathbf{R}_\tau(\mathbf{u}_{\eta\tau})(t)\|^2) - \varepsilon_a)_+, \ \forall t \in [0,T]. \tag{4.29}$$

It follows from the properties of R_ν and $\mathbf{R}_\tau$ that F_η is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\beta \in L^2(\Gamma_3)$, mapping $t \to F_\eta(t,\beta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using a version of the classical Cauchy-Lipschitz theorem 2.1, we deduce that there exists a unique function $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution to Problem$\mathscr{PV}^\beta$. Also, the arguments used in Remark 3.1 show that $0 \leq \beta_\eta(t) \leq 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set $\mathscr{Z}$, we find that $\beta_\eta \in \mathscr{Z}$, which concludes the proof of Lemma 4.6. □

Finally, as a consequence of these results and using the properties of operator $\mathscr{G}$ operator $\mathscr{E}$, and function S, we consider the operator

$$\Lambda : C(0,T;\mathscr{H} \times L^2(\Omega)) \to C(0,T;\mathscr{H} \times L^2(\Omega)), \tag{4.30}$$

which maps every element $(\eta,\theta) \in C(0,T;\mathscr{H} \times L^2(\Omega))$ to $\Lambda(\eta,\theta) \in C(0,T;\mathscr{H} \times L^2(\Omega))$ defined by

$$\Lambda(\eta,\theta)(t) = (\Lambda^1(\eta,\theta)(t), \Lambda^2(\eta,\theta)(t)) \in \mathscr{H} \times L^2(\Omega), \tag{4.31}$$

and

$$\begin{aligned} (\Lambda^1(\eta,\theta)(t), v)_{\mathscr{H}\times V} = {} & (\mathscr{G}(\varepsilon(\mathbf{u}_\eta(t)), \alpha_\theta(t)), \varepsilon(v))_{\mathscr{H}} + (\mathscr{E}^* \nabla\varphi_\eta(t), \varepsilon(v))_{\mathscr{H}} \\ & + \Big(\int_0^t \mathscr{M}(t-s), \varepsilon(\mathbf{u}_\eta(s)) ds, \varepsilon(v) \Big)_{\mathscr{H}}, \quad \forall \mathbf{v} \in V. \end{aligned} \tag{4.32}$$

$$\Lambda^2(\eta,\theta)(t) = S\big(\varepsilon(\mathbf{u}_\eta(t)), \alpha_\theta(t)\big). \tag{4.33}$$

Here, for every $(\eta,\theta) \in C(0,T;\mathscr{H}\times L^2(\Omega))$ u_η, φ_η,β_η and α_θ represent the displacement field, the potential electric field, the bonding field and the damage field obtained in Lemmas 4.2, 4.4, 4.5 and 4.6 respectively. We have the following result.

Lemma 4.7. *Λ has a unique fixed point $(\eta^*,\theta^*) \in C(0,T;\mathscr{H}\times L^2(\Omega))$ such that $\Lambda(\eta^*,\theta^*) = (\eta^*,\theta^*)$.*

Proof. Let $t \in (0,T)$ and $(\eta_1,\theta_1),(\eta_2,\theta_2) \in C(0,T;\mathscr{H}\times L^2(\Omega))$. We use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\dot{\mathbf{u}}_{\eta_i} = \dot{\mathbf{u}}_i$, $\beta_{\eta_i} = \beta_i$, $\varphi_{\eta_i} = \varphi_i$ et $\alpha_{\theta_i} = \alpha_i$, for $i = 1,2$. Using (3.28) (3.31), (3.32), (3.34), the definition of R_ν, $\mathbf{R}_\tau$ and Remark 3.1, we have

$$\begin{aligned}
&\|\Lambda^1(\eta_1,\theta_1)(t) - \Lambda^1(\eta_2,\theta_2)(t)\|^2_{\mathscr{H}} \\
&\qquad \le \|\mathscr{G}(\varepsilon(\mathbf{u}_1(t)),\alpha_1(t)) - \mathscr{G}(\varepsilon(\mathbf{u}_2(t)),\alpha_2(t))\|^2_{\mathscr{H}} \\
&\qquad + \|\mathscr{E}^*\nabla\varphi_1(t) - \mathscr{E}^*\nabla\varphi_2(t)\|^2_{\mathscr{H}} \\
&\qquad + \int_0^t \|\mathscr{M}(t-s)\varepsilon(u_1(s)-u_2(s))\|^2_{\mathscr{H}} ds \\
&\qquad \le C\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|^2_V + \int_0^t \|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|^2_V ds \\
&\qquad + \|\varphi_1(t)-\varphi_2(t)\|^2_{\mathscr{H}} + \|\alpha_1 - \alpha_2(t))\|^2_{L^2(\Gamma_3)}\Big).
\end{aligned} \tag{4.34}$$

By a similar argument, we conclude from (4.33) and (3.29) that

$$\begin{aligned}
&\|\Lambda^2(\eta_1,\theta_1)(t) - \Lambda^2(\eta_2,\theta_2)(t)\|^2_{\mathscr{H}} \\
&\qquad \le C\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|^2_V + \|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)}\Big).
\end{aligned} \tag{4.35}$$

Therefore,

$$\begin{aligned}
&\|\Lambda(\eta_1,\theta_1)(t) - \Lambda(\eta_2,\theta_2)(t)\|^2_{\mathscr{H}\times L^2(\Omega)} \le C\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|^2_V \\
&+ \int_0^t \|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|^2_V ds + \|\varphi_1(t)-\varphi_2(t)\|^2_W + \|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)}\Big).
\end{aligned} \tag{4.36}$$

Moreover, from (4.19), we obtain

$$\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V \le \int_0^t \|v_1(s)-v_2(s)\|_V ds. \tag{4.37}$$

Using (4.7), (4.8), and estimates similar to those in the proof of Lemma 4.2 (see (4.14)), we find that, for $s \in [0,T]$,

$$\|v_1(s)-v_2(s)\|_V \le c\Big(\|\eta_1(s)-\eta_2(s)\|_{\mathscr{H}} + \|u_1(s)-u_2(s)\|_V\Big). \tag{4.38}$$

Combining (4.36) and (4.37), and using the Gronwall's inequality, we have

$$\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V \le c\int_0^t \|\eta_1(s)-\eta_2(s)\|_{\mathscr{H}} ds, \quad t \in [0,T],$$

which implies that

$$\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|^2_V \le c\int_0^t \|\eta_1(s)-\eta_2(s)\|^2_{\mathscr{H}} ds, \quad t \in [0,T]. \tag{4.39}$$

From (4.25), we obtain that, a.e. on $(0,T)$,

$$(\dot{\alpha}_1 - \dot{\alpha}_2, \alpha_1 - \alpha_2)_{L^2(\Omega)} + a(\alpha_1-\alpha_2,\alpha_1-\alpha_2) \le (\theta_1-\theta_2,\beta_1-\alpha_2)_{L^2(\Omega)} \quad \forall t \in [0,T].$$

Integrating the previous inequality with respect to time, and using the initial conditions $\alpha_1(0) = \alpha_2(0) = \alpha_0$, one finds that

$$\frac{1}{2}\|\alpha_1(t) - \alpha_2(t)\|^2_{L^2(\Omega)} \le \int_0^t (\theta_1(s) - \theta_2(s), \alpha_1(s) - \alpha_2(s))_{L^2(\Omega)} ds, \quad \forall t \in [0,T],$$

which in turn implies that

$$\|\alpha_1(t) - \alpha_2(t)\|^2_{L^2(\Omega)} \le \int_0^t \|\theta_1(s) - \theta_2(s)\|^2_{L^2(\Omega)} ds + \int_0^t \|\alpha_1(s) - \alpha_2(s)\|^2_{L^2(\Omega)} ds,$$

for all $t \in [0,T]$. This inequality combined with the Gronwall's inequality leads to

$$\|\alpha_1(t) - \alpha_2(t)\|^2_{L^2(\Omega)} \le C \int_0^t \|\theta_1(s) - \theta_2(s)\|^2_{L^2(\Omega)} ds, \quad \forall t \in [0,T]. \tag{4.40}$$

We use now (4.23), (3.30), (3.31) and (3.26) to find

$$\|\varphi_1(t) - \varphi_2(t)\|^2_W \le C\|u_1(t) - u_2(t)\|^2_V ds. \tag{4.41}$$

Using (4.36), (4.39), (4.40) and (4.42), we find that, for all $t \in [0,T]$,

$$\|\Lambda(\eta_1,\theta_1)(t) - \Lambda(\eta_2,\theta_2)(t)\|^2_{\mathscr{H}\times L^2(\Omega)} \le C \int_0^t \|(\eta_1,\theta_1)(s) - (\eta_2,\theta_2)(s)\|^2_{\mathscr{H}\times L^2(\Omega)} ds.$$

Then, as in the proof of Lemma 4.7, we obtain

$$\|\Lambda^n(\eta_1,\theta_1) - \Lambda^n(\eta_2,\theta_2)\|_{C(0,T;\mathscr{H}\times L^2(\Omega))} \le \left(\frac{C^n T^n}{n!}\right)^{\frac{1}{2}} \|(\eta_1,\theta_1) - (\eta_2,\theta_2)\|_{C(0,T;\mathscr{H}\times L^2(\Omega))},$$

for all $n \in \mathbb{N}$. This inequality and the Banach fixed-point theorem imply that Λ has a unique fixed point. □

Now, we have all the ingredients to prove Theorem (4.1).

Existence. Let $(\mathbf{u}_\eta, \sigma_\eta)$ be the solution of problem $\mathbf{PV}^u_\eta$. Let φ_η be the solution of problem $\mathbf{PV}^\varphi_\eta$. Let β_η be the solution of problem $\mathbf{PV}^\beta_\eta$ for $\eta = \eta^*$, and let α^*_θ be the solution of problem $\mathbf{PV}_\theta$ for $\theta = \theta^*$. Since $\eta^* = \mathscr{G}\varepsilon(\mathbf{u}_{\eta^*}, \alpha_{\theta^*}) + \int_0^t \mathscr{M}(t-s)\varepsilon(\mathbf{u}_{\eta^*}(s))ds + \mathscr{E}^*\nabla\varphi_{\eta^*}$ and $\theta^* = S(\varepsilon(\mathbf{u}_{\eta^*}), \alpha_{\theta^*})$, we see that $(\mathbf{u}_{\eta^*}, \sigma_{\eta^*}, \varphi_{\eta^*}, \beta_{\eta^*}, \alpha_{\theta^*})$ is a solution of problem (3.52) through (3.57) and it satisfies (4.1) through (4.5)

Finally, we conclude that the weak solution $(\mathbf{u}_{\eta^*}, \sigma_{\eta^*}, \varphi_{\eta^*}, \mathbf{D}_{\eta^*}, \beta_{\eta^*}, \alpha_{\theta^*})$ of piezoelectric contact problem $\mathbf{P}$ has the regularity (4.1)-(4.6), which concludes the existence part of Theorem 4.1.

Uniqueness. Let $(\mathbf{u}_{\eta^*}, \sigma_{\eta^*}, \varphi_{\eta^*}, \beta_{\eta^*}, \alpha_{\theta^*})$ be the solution of (3.52)-(3.57) obtained above and let ($\mathbf{u}$, σ, φ, β, α) be another solution of the problem, which satisfies (4.1)-(4.5). We denote by $\eta \in C([0,T],\mathscr{H})$ and $\theta \in C([0,T];L^2(\Omega))$ the functions

$$\eta^*(t) = \mathscr{G}\varepsilon((\mathbf{u}(t)),\alpha) + \int_0^t \mathscr{M}(t-s)\varepsilon(\mathbf{u}(s))ds + \mathscr{E}^*\nabla\varphi(t), \tag{4.42}$$

$$\theta^*(t) = S(\varepsilon(\mathbf{u}(t)),\alpha). \tag{4.43}$$

Now, (3.51)-(3.54), (3.56) and (3.57) imply that $(\mathbf{u},\sigma,\varphi,\beta)$ is a solution of Problems $\mathbf{PV}^u_\eta$, $\mathbf{PV}^\varphi_\eta$ and $\mathbf{PV}^\beta_\eta$. From Lemma 4.3, it follows that this problem has a unique solution $\mathbf{u}_\eta \in C^1([0,T];V)$, $\varphi_\eta C([0,T];W)$ and $\sigma_\eta C([0,T];\mathscr{H}_1)$. It follows that

$$\mathbf{u} = \mathbf{u}_\eta, \quad \sigma = \sigma_\eta, \quad \varphi = \varphi_\eta, \qquad \beta = \beta_\eta \tag{4.44}$$

From (3.55) and (3.57), we can obtain that

$$\alpha = \alpha_\theta. \tag{4.45}$$

Using (4.32), (4.44) , (4.45) and (4.42)-(4.43), we obtain $\Lambda(\eta,\theta) = (\eta,\theta)$. By the uniqueness of the fixed point of the operator Λ, guaranteed by Lemma 4.7, it follows that

$$\eta = \eta^*, \qquad \theta = \theta^*. \tag{4.46}$$

The solution uniqueness is now a consequence of (4.44).

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Positive Cohen weakly nuclear m-linear operators

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Abstract: In this talk, we introduce and study a new class of multilinear operators on Banach lattices spaces, called positive Cohen weakly p-nuclear multilinear operators. We establish a Pietsch domination theorem for this new class of multilinear operators. As an application, we show that each positive Cohen weakly p-nuclear multilinear operator is positive Dimant-strongly p-summing and Cohen strongly positive p-summing. Finally, we finish with the tensorial representation of our class.

1 Introduction

For a Banach space X, X^* will denote its topological dual and B_X will denote its closed unit ball. The letters $X, X_1, \ldots, X_m, Y$ will be used for Banach spaces and $E, E_1, \ldots, E_m, F$ for Banach lattices. We will consider $\mathbb{R}$ as the scalar field. All Banach lattices will be considered Archimedean along the paper. An operator $T : E \to F$ is called a lattice homomorphism if it is linear and satisfies $|T(x)| = T(|x|)$ for all $x \in E$. An m-linear operator $T : E_1 \times \cdots \times E_m \to F$ satisfying $|T(x_1, \ldots, x_m)| = T(|x_1|, \ldots, |x_n|)$ for all $x_k \in E_k$, $k = 1, \ldots, m$ is called a lattice m-morphism.

2 Positive Cohen weakly nuclear multilinear operators

In this section, we give our new definition with some proprieties This definition and properties are inspired by the work done by Q. Bu et al. in [6].

Definition 1 *Let $1 \leq p < +\infty$. An m-linear operator $T \in \mathcal{L}(E_1, ..., E_m; F)$ is called positive Cohen weakly p-nuclear if there exists a positive constant C such that for any $\left(x_i^j\right)_{i=1}^n \subset E_j^+$, with $1 \leq j \leq m$, and $(y_i^*)_{i=1}^n \subset F^{*+}$, we have*

$$\sum_{i=1}^n \left\langle T(x_i^1, ..., x_i^m), y_i^* \right\rangle \leq C \left(\sup_{\varphi \in B^+_{\mathcal{L}^r(E_1,\ldots,E_m;\mathbb{R})}} \sum_{i=1}^n \left(\varphi(x_i^1, ..., x_i^m)\right)^p \right)^{\frac{1}{p}} w_{p^*}\left((y_i^*)_{i=1}^n\right) \quad (1)$$

Moreover, the class of all positive Cohen weakly p-nuclear m-linear operators from $E_1 \times \cdots \times E_m$ into F, is denoted by $\mathcal{N}^{m+}_{w,p}(E_1, ..., E_m; F)$. Our space is a Banach space with the norm $n^{m+}_{w,p}(.)$, which is the smallest constant C such that (1) holds. Obviously $\mathcal{N}^{1+}_{w,p}(E; F) = \mathcal{N}^{+}_{p}(E; F)$.

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On existence results of solutions for boundary value problems of fractional differential equations involving Caputo-Hadamrd derivative

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Abstract: In this study, we discuss the existence of the solution for a Caputo-Hadamard boundary value problem of differential equation with non separated integral boundary conditions, by using of fixed point. An example is provided to demonstrate the validity of our findings
Keywords: fixed point, partial metric space, Geraghty contraction, Caputo derivative, fractional differential inclusions.
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1 Introduction

we prove the existence of the solutions for the following boundary value problem of fractional differential equations:

$$\begin{cases} D_1^q x(t) \in F(t, x(t)),\ 1 < t < e, \\ x(1) = \mu \int_\alpha^\beta g(s, x(s))ds, x(e) = \lambda \int_\gamma^\delta h(s, x(s))ds, 0 < \alpha < \beta < \gamma < \delta < 1 \end{cases} \tag{1.1}$$

where D_1 is the Caputo-Hadamard derivative, $1 < q < 2$, $\lambda, \mu \in \mathbb{R}$, $F : [1, e] \times \mathbb{R} \to P(\mathbb{R})$ and $g, h : [1, e] \times \mathbb{R} \to \mathbb{R}$ are given continuous function.
Let $X = C[1, e]$ the space of continuous functions on $[1, e]$ with real values, equipped with uniform norm

$$\|x\|_\infty = \sup |x(t)|$$

This space is a Banach space.
Let (X, d) be a metric space induced from the normed space $(X, \|.\|)$. The function $H_d : P(X) \times P(X) \to \mathbb{R} \cup \{\infty\}$, given by

$$H_d(A, B) = max\{\sup_{a\in A} d(a, B),\quad \sup_{b\in B} d(A, b)\}$$

is known as the Hausdorff-Pompeiu metric.

2 Main results

Lemma 2.1. *A function x is a solution of the fractional integral equation*

$$x(t) = \int_1^e G(t,s)v(t)ds + \lambda \log t \int_\gamma^\delta h(s,x(s))ds + (\mu - \mu \log t) \int_\alpha^\beta g(s,x(s))ds \tag{2.1}$$

if and only if y is a solution of the fractional boundary value problem

$$x(t) =_c^H D^q x(t) = v(t) \quad \forall t \in [1,e] \tag{2.2}$$

$$x(1) = \mu \int_\alpha^\beta g(s,x(s))ds \tag{2.3}$$

$$x(e) = \lambda \int_\gamma^\delta h(s,x(s))ds \tag{2.4}$$

where

$$G(t,s) = \frac{1}{s\Gamma(q)} \begin{cases} \left(\log \frac{t}{s}\right)^{q-1} - \log t(1-\log s)^{q-1}, 1 \le s \le t \le e, \\ -\log t(1-\log s)^{q-1}, 1 \le t \le s \le e \end{cases} \tag{2.5}$$

Theorem 2.1. *Assume the following hypotheses hold:*

- $(H_1) : F : [1,e] \times \mathbb{R} \to P(\mathbb{R})$ *is a Caratheodory multi-valued map.*
- (H_2) *There exists $k_1 > 0$ such that*

$$\|F(t,u)\|_\infty < k_1, \quad \forall t \in [1,e] \text{ and } \forall u \in \mathbb{R}$$

- (H_3) *There exist $k_2 > 0$ and $k_3 > 0$ such that*

$$|g(t,u_1) - g(t,u_2)| < k_2|u_1 - u_2| \quad \forall t \in [1,e] \text{ and } \forall u_1,u_2 \in \mathbb{R}$$

$$|h(t,u_1) - h(t,u_2)| < k_3|u_1 - u_2| \quad \forall t \in [1,e] \text{ and } \forall u_1,u_2 \in \mathbb{R}$$

- (H_4) *There exists a number $k_4 > 0$ such that*

$$H_d(F(t,u),F(t,\bar{u})) \le k_4|u-\bar{u}| \quad \text{for every } u,\bar{u} \in \mathbb{R},\ \forall t \in I.$$

- (H_5) *There exists a number $M > 0$ such that*

$$\frac{M}{\lambda(\delta-\gamma)(k_3A+h_0) + G^*(e-1)k_1} > 1 \tag{2.6}$$

  where

$$G^* = \sup_{t,s\in[1,e]} |G(t,s)|, \quad h_0 = \sup_{t\in[1,e]} |h(t,0)|$$

  and

$$A = \sup_{s\in[1,e]} |x(s)|$$

Then the problem (1.1) has at least one solution on $[1,e]$.

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NULL CONTROLLABILITY FOR HYPERBOLIC PROBLEMS WITH NONLOCAL INTEGRAL TERMS

IMEN BENABBAS AND DJAMEL EDDINE TENIOU

ABSTRACT. Of concern is the internal controllability of linear wave equations with additive integral terms. On the boundary, we have mixed conditions Dirichlet's on one part and dynamical Ventcel conditions on the other part. First, we prove the associated local problem (that without the integral terms) to be controllable in finite time. We accomplish this using multipliers and a contradiction reasoning. Subsequently, by means of unique continuation results we deduce the controllability for the nonlocal problem.

1. INTRODUCTION

Let Ω be a bounded open domain of $\mathbb{R}^n$, $n \geq 2$, with boundary $\Gamma = \Gamma^1 \cup \Gamma^2$ and let $T > 0$. We denote $\Omega_T = \Omega \times (0,T)$, $\Gamma_T^1 = \Gamma^1 \times (0,T)$, $\Gamma_T^2 = \Gamma^2 \times (0,T)$. Consider the following nonlocal hyperbolic problem

$$(1)\quad \begin{cases} \partial_t^2 v - \Delta v + \displaystyle\int_\Omega K_\Omega(x,y)v(y,t)\,dy = w_1 \quad \text{in } \Omega_T \\ \partial_t^2 v_\Gamma + \partial_\nu v - \Delta_\Gamma v_\Gamma + \displaystyle\int_{\Gamma^1} K_\Gamma(\xi,\zeta)v_\Gamma(\zeta,t)\,d\Gamma = w_2,\ v = v_\Gamma \quad \text{on } \Gamma_T^1 \\ v = 0 \quad \text{on } \Gamma_T^2 \\ (v(0), v_\Gamma(0)) = (v^0, v_\Gamma^0),\ (\partial_t v(0), \partial_t v_\Gamma(0)) = (v^1, v_\Gamma^1) \quad \text{in } \Omega \times \Gamma^1 \end{cases}$$

We study the controllability of system (1) in two settings. the first is where Ω is a bounded domain with a smooth boundary $\Gamma = \Gamma^1 \cup \Gamma^2$ such that Γ^1, Γ^2 are nonempty, closed and $\Gamma^1 \cap \Gamma^2 = \emptyset$. The second is where Ω is a rectangular domain in $\mathbb{R}^n$. We denote by ∂_ν the normal derivative on Γ where $\nu = (\nu_1, \cdots, \nu_n)$ is the outward unit normal vector to Γ, and by Δ_Γ the Laplace-Beltrami operator on Γ. Moreover, the kernel functions K_Ω, K_Γ are assumed to belong respectively in $L^2(\Omega \times \Omega)$ and $L^2(\Gamma \times \Gamma)$, and no connection is required between these two functions.

System (1) is said to be null-controllable in time $T > 0$, if we can find control functions (w_1, w_2) that will steer the solution from the initial state $(v^0, v_\Gamma^0, v^1, v_\Gamma^1)$ to the equilibrium

$$(2)\quad \begin{aligned} &(v(T), v_\Gamma(T)) = (0,0) \text{ in } \Omega, \\ &(\partial_t v(T), \partial_t v_\Gamma(T)) = (0,0) \text{ on } \Gamma^1. \end{aligned}$$

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Following Hilbert Uniqueness Method [5], proving system (1) to be null-controllable is equivalent to proving the corresponding adjoint system to be observable in finite time $T > 0$

(3)

$$\begin{cases} \partial_t^2 u - \Delta u + \displaystyle\int_\Omega K_\Omega(x,y) u(x,t)\ dx = 0 \quad \text{in} \quad \Omega_T \\ \partial_t^2 u_\Gamma + \partial_\nu u - \Delta_\Gamma u_\Gamma + \displaystyle\int_{\Gamma^1} K_\Gamma(\xi,\zeta) u_\Gamma(\xi,t)\ d\Gamma = 0,\ u = u_\Gamma \quad \text{on} \quad \Gamma_T^1 \\ u = 0 \quad \text{on} \quad \Gamma_T^2 \\ (u(T), u_\Gamma(T)) = (u_T^0, u_{T,\Gamma}^0),\ (\partial_t u(T), \partial_t u_\Gamma(T)) = (u_T^1, u_{T,\Gamma}^1) \quad \text{in} \quad \Omega \times \Gamma^1, \end{cases}$$

This nonlocal system will be essentialy viewed as a perturbation of its local counterpart

$$(4) \qquad \begin{cases} \partial_t^2 u - \Delta u = 0 \quad \text{in} \quad \Omega_T \\ \partial_t^2 u_\Gamma + \partial_\nu u - \Delta_\Gamma u_\Gamma = 0,\ u = u_\Gamma \quad \text{on} \quad \Gamma_T^1 \\ u = 0 \quad \text{on} \quad \Gamma_T^2 \\ (u(0), u_\Gamma(0)) = (u^0, u_\Gamma^0),\ (\partial_t u(0), \partial_t u_\Gamma(0)) = (u^1, u_\Gamma^1) \quad \text{in} \quad \Omega \times \Gamma^1, \end{cases}$$

We first establish some auxiliary energy estimates [1, 3]. Then, using contradiction arguments and taking account of unique continuation results for wave equations with constant coefficients, we shall achieve two new observability results for system (3).

2. Well-posedness

We introduce the functional spaces $\mathcal{H},\ \mathcal{V},\ \mathcal{V}'$

$$\mathcal{H} = L^2(\Omega) \times L^2(\Gamma^1),$$

$$\mathcal{V} = \left\{(u,v) \in H^1_{\Gamma^2}(\Omega) \times H^1(\Gamma^1);\ v = u|_{\Gamma^1}\right\}$$

where $H^1_{\Gamma^2}(\Omega) = \left\{u \in H^1(\Omega); u|_{\Gamma^2} = 0\right\}$ and $\mathcal{V}'$ is the dual space of $\mathcal{V}$.

These are Hilbert spaces endowed with the norms

$$||(u,v)||^2_{\mathcal{H}} = \int_\Omega |u|^2 dx + \int_{\Gamma^1} |v|^2 d\Gamma,\ ||(u,v)||^2_{\mathcal{V}} = \int_\Omega |\nabla u|^2 dx + \int_{\Gamma^1} |\nabla_\Gamma v|^2 d\Gamma$$

where ∇_Γ denotes the surface gradient on Γ.

Applying results from semigroups theory [2], we get that system (1) is well-posed in the following sense

1. Given $(v^0, v^0_\Gamma, v^1, v^1_\Gamma) \in \mathcal{V} \times \mathcal{H}$ and $(w_1, w_2) \in L^2(0,T;\mathcal{H})$, there exists a unique weak solution to (1) such that

$$(v, v_\Gamma) \in C(0,T;\mathcal{V}) \cap C^1(0,T;\mathcal{H}).$$

2. Moreover, for all $(v^0, v^0_\Gamma, v^1, v^1_\Gamma) \in \mathcal{H} \times \mathcal{V}'$ and $(w_1, w_2) \in L^2(0,T;\mathcal{H})$, we have a unique solution to (1)

$$(v, v_\Gamma) \in C(0,T;\mathcal{H}) \cap C^1(0,T;\mathcal{V}').$$

3. Observability results

3.1. **Local system (4).** First, we present our result in the case where Ω is an open bounded domain of $\mathbb{R}^n$ with a smooth boundary as mentioned in the introduction. We assume the following geometric condition on the boundary Γ : there exists $x_0 \notin \overline{\Omega}$ such that

$$\begin{cases} (x - x_0) \cdot \nu(x) \leq 0 \text{ for all } x \in \Gamma^1, \\ (x - x_0) \cdot \nu(x) \geq 0 \text{ for all } x \in \Gamma^2. \end{cases} \tag{5}$$

Now, let $\omega \subset \Omega$ be a neighborhood of Γ^2 in Ω; namely, there exits a neighborhood $\mathcal{O} \subset \mathbb{R}^n$ of Γ^2 such that $\omega = \mathcal{O} \cap \Omega$. Setting $T_0 = \max\{|x - x_0|, x \in \overline{\Omega}\}$ (see [3]), under the geometric hypothesis (5), our first result is the following

Theorem 3.1. *Let $T > T_0$. Let $\omega \subset \Omega$ be a neighborhood of Γ^2 in Ω. Then, there exists $c > 0$ such that*

$$||(u^0, u^0_\Gamma)||^2_{\mathcal{H}} + ||(u^1, u^1_\Gamma)||^2_{\mathcal{V}'} \leq c(\int_{\omega_T} |u|^2 \; dxdt + \int_{\Gamma^1_T} |u_\Gamma|^2 \; d\Gamma dt) \tag{6}$$

for all solutions to (4) corresponding to initial data $(u^0, u^0_\Gamma) \in \mathcal{H},\ (u^1, u^1_\Gamma) \in \mathcal{V}'$.

Now, we deal with system (4) posed in a rectangular domain. Precisely, we consider problem (4) in a two-dimensional rectangle $\Omega = (0, l_1) \times (0, l_2)$. What follows could be generalized to high-dimensional rectangular domains. In this case, we have Ventcel's condition on $\Gamma^1 = (0, l_1) \times \{l_2\}$ and Dirichlet's condition on the remainder of the boundary $\Gamma^2 = \Gamma^{2,1} \cup \Gamma^{2,2} \cup \Gamma^{2,3}$ where $\Gamma^{2,1} = \{0\} \times (0, l_2),\ \Gamma^{2,2} = (0, l_1) \times \{0\},\ \Gamma^{2,3} = \{l_1\} \times (0, l_2)$.

Let $\mathcal{O} \subset \mathbb{R}^2$ be a neighborhood of the observed region $\Gamma^{2,1} \cup \Gamma^{2,2} \cup \{(0, l_2)\}$. We denote ω^1, ω^2, respectively, the intersections $\mathcal{O} \cap \Gamma^1,\ \mathcal{O} \cap \Omega$; so that $\omega = \omega^1 \cup \omega^2$ is a neighborhood of $\Gamma^{2,1} \cup \Gamma^{2,2} \cup \{(0, l_2)\}$ in $\Omega \cup \Gamma^1$.

Denote $T_{R,0} = 2(\sqrt{2} + 1)\sqrt{l_1^2 + 4l_2^2}$ (see [1]), then we obtain that system (4) observable in any time $T > T_{R,0}$.

Theorem 3.2. *Given $(u^0, u^0_\Gamma) \in \mathcal{H},\ (u^1, u^1_\Gamma) \in \mathcal{V}'$. Then, for $T > T_{R,0}$, we have the corresponding solution of (4) satisfying*

$$||(u^0, u^0_\Gamma)||^2_{\mathcal{H}} + ||(u^1, u^1_\Gamma)||^2_{\mathcal{V}'} \leq c(\int_{\omega^2_T} |u|^2 dxdt + \int_{\omega^1_T} |u_\Gamma|^2 d\Gamma dt) \tag{7}$$

where $\omega = \omega^1 \cup \omega^2$ is a neighborhood of $\{(0, l_2)\} \cup \Gamma^{2,1} \cup \Gamma^{2,2}$ in $\Omega \cup \Gamma^1$ and $c > 0$ is a positive constant.

3.2. **Nonlocal system (3).** Next, we prove the counterparts of the results presented above for the perturbed system (3). Regarding the integral spacial terms as perturbations of system (4), the observability results of the previous section combined with compactness-uniqueness arguments give the following two observability estimates.

Theorem 3.3. *Let $T > T_0$. For all solutions to (3) associated to final data $(u^0_T, u^0_{T,\Gamma}) \in \mathcal{H},\ (u^1_T, u^1_{T,\Gamma}) \in \mathcal{V}'$, we have the estimate*

$$||(u, u_\Gamma)(0)||^2_{\mathcal{H}} + ||(\partial_t u, \partial_t u_\Gamma)(0)||^2_{\mathcal{V}'} \leq c(\int_{\omega_T} |u|^2 dxdt + \int_{\Gamma^1_T} |u_\Gamma|^2 d\Gamma dt) \tag{8}$$

Theorem 3.4. *Let $T > T_{R,0}$. Then, there exists a constant $c > 0$ such that the solution to problem (3) fulfills*

$$||(u, u_\Gamma)(0)||^2_{\mathcal{H}} + ||(\partial_t u, \partial_t u_\Gamma)(0)||^2_{\mathcal{V}'} \leq c\left(\int_{\omega_T^2} |u|^2 dx dt + \int_{\omega_T^1} |u_\Gamma|^2 d\Gamma dt\right) \tag{9}$$

for all final data $(u_T^0, u_{T,\Gamma}^0) \in \mathcal{H}$, $(u_T^1, u_{T,\Gamma}^1) \in \mathcal{V}'$.

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On the class $\delta(\mathcal{H})$, binormal operators and λ-Aluthge transform

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Abstract: Let $T \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator on a Hilbert space $\mathcal{H}$, and let $T = U|T|$ be its polar decomposition. For every $\lambda \in [0,1]$ the λ-Aluthge transform of T is defined by $\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}$. If $U^2|T| = |T|U^2$, then T will be said to be in the class $\delta(\mathcal{H})$. In this paper, we will investigate on some relations between this class and other usual classes of operators via λ-Aluthge Transform .
Keywords: Hilbert space , Binormal operator, Invertible operator, quasinormal, λ- Aluthge Transform .

1 Introduction

let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For an arbitrary operator $T \in \mathcal{B}(\mathcal{H})$, we denote by $\mathcal{R}(T)$, $\mathcal{N}(T)$ and T^* for the range, the null subspace and the adjoint operator of T, respectively. Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be normal if $TT^* = T^*T$, if T commutes with T^*T and binormal if TT^* and T^*T commute. Binormality of operators was defined by Campbell in [2], It is easy to see that normal $\Longrightarrow$ quasinormal $\Longrightarrow$ binormal and the inverse implications do not hold.
It is well known that for every operator $T \in \mathcal{B}(\mathcal{H})$, there is a unique factorization $T = U|T|$, where $\mathcal{N}(U) = \mathcal{N}(T) = \mathcal{N}(|T|)$, U is a partial isometry, i.e. $UU^*U = U$ and $|T| = (T^*T)^{\frac{1}{2}}$ is the modulus of T. This factorization is called the polar decomposition of T. From the polar decomposition, the λ-Aluthge transform of T is defined for any $\lambda \in [0,1]$, by $\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}$. In particular, for $\lambda = \frac{1}{2}$, $\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is called the Aluthge transform of T. This transform was introduced in [1] by Aluthge, in order to study p-hyponormal and log-hyponormal operators. Also, $\Delta_1(T) = |T|U$ is known as Duggal's transform. Throughout the remainder of this paper, we denote by $\delta(\mathcal{H})$ the class of operator $T \in \mathcal{B}(\mathcal{H})$ which satisfies $U^2|T| = |T|U^2$. Clearly, quasinormal operators belong to $\delta(\mathcal{H})$ but the converse is not true in genaral. In this paper, firstly, we provide a condition under which an operator in $\delta(\mathcal{H})$ becomes quasinormal. Secondly, we give a new characterization of invertible operators in $\delta(\mathcal{H})$ via Duggal transform. Finally, we discuss how the class $\delta(\mathcal{H})$ is distinct from the class of binormal operators.

2 Main results

In this section, first we give a condition under which an operator in $\delta(\mathcal{H})$ becomes quasi-normal.

Proposition 2.1. *Let n be a positive integer and $T \in \delta(\mathcal{H})$, with polar decomposition $T = U|T|$. If $U^{2n+1} = I$, then T is quasinormal.*

The following is a characterization of invertible operators in $\delta(\mathcal{H})$ via Duggal transform.

Proposition 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be invertible. Then*

$$T \in \delta(\mathcal{H}) \iff \Delta_1(T^{-1}) = (\Delta_1(T))^{-1}.$$

Example 2.3. *Proposition 2.2 is not valid when the Duggal transform is replaced by the Aluthge transform. To see this let $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, where A and B are invertible positive operators such that $AB \neq BA$. Then T is invertible and*

$$T = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} = U|T|$$

is the polar decomposition of T. Since $U^2 = I$, it follows that $U^2|T| = |T|U^2$ and so $T \in \delta(\mathcal{H} \oplus \mathcal{H})$. On the other hand, since

$$\Delta(T) = \begin{pmatrix} 0 & B^{\frac{1}{2}}A^{\frac{1}{2}} \\ A^{\frac{1}{2}}B^{\frac{1}{2}} & 0 \end{pmatrix}, \text{ we obtain } (\Delta(T))^{-1} = \begin{pmatrix} 0 & B^{-\frac{1}{2}}A^{-\frac{1}{2}} \\ A^{-\frac{1}{2}}B^{-\frac{1}{2}} & 0 \end{pmatrix}.$$

Using $P(3)$ (i) and (ii), we have

$$\Delta(T^{-1}) = |T^{-1}|^{\frac{1}{2}}U^*|T^{-1}|^{\frac{1}{2}} = |T^*|^{-\frac{1}{2}}U^*|T^*|^{-\frac{1}{2}} = \begin{pmatrix} 0 & A^{-\frac{1}{2}}B^{-\frac{1}{2}} \\ B^{-\frac{1}{2}}A^{-\frac{1}{2}} & 0 \end{pmatrix}.$$

Hence $\Delta(T^{-1}) \neq (\Delta(T))^{-1}$.

It is well known that every quasi-normal operator is binormal. Hence one might expect that there is a relationship between $\delta(\mathcal{H})$ and binormal operators. But in the example 2.3, $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \delta(\mathcal{H} \oplus \mathcal{H})$ and T is not binormal because $AB \neq BA$.
Next, we shall show the following result on the binormality of an invertible operator T in $\delta(\mathcal{H})$.

Theorem 2.4. *Let $T \in \delta(\mathcal{H})$ be an invertible operator. Then the following statements are equivalent.*

(1) *T is binormal.*

(2) $\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$ *for all* $\lambda \in]0,1[$.

(3) $\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$ *for some* $\lambda \in]0,1[$.

The following corollary generalizes one implication of [4, Theorem 3.7] to infinite-dimensional Hilbert space.

Corollary 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be invertible. Then*

$$\Delta(T) = T \implies \Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}, \quad \text{for all } \lambda \in]0,1[.$$

Remark 2.6. *In Corollary 2.5, the reverse implication is false in infinite-dimensional Hilbert space as shown by the following example.*

Example 2.7. *Let $T = \begin{pmatrix} 0 & I \\ P & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, where $P \geq 0$ and $P \neq I$ is invertible. The polar decomposition of T is $T = U|T|$, where*

$$|T| = (T^*T)^{\frac{1}{2}} = \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad U = T|T|^{-1} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

For any $\lambda \in]0,1[$, we have

$$\begin{aligned} \Delta_\lambda(T) &= |T|^\lambda U |T|^{1-\lambda} \\ &= \begin{pmatrix} P^\lambda & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} P^{1-\lambda} & 0 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} 0 & P^\lambda \\ P^{1-\lambda} & 0 \end{pmatrix}. \end{aligned}$$

It follows that

$$(\Delta_\lambda(T))^{-1} = \begin{pmatrix} 0 & P^{-(1-\lambda)} \\ P^{-\lambda} & 0 \end{pmatrix}.$$

Also we have

$$\begin{aligned} \Delta_\lambda(T^{-1}) &= |T^*|^{-\lambda} U^* |T^*|^{-(1-\lambda)} \\ &= \begin{pmatrix} I & 0 \\ 0 & P^{-\lambda} \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & P^{-(1-\lambda)} \end{pmatrix} \\ &= \begin{pmatrix} 0 & P^{-(1-\lambda)} \\ P^{-\lambda} & 0 \end{pmatrix}. \end{aligned}$$

Hence, $\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$, while $\Delta_\lambda(T) \neq T$.

The following is an example of a binormal operator which is not in $\delta(\mathcal{H})$.

Example 2.8. *Consider* $T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{C}^4$. *Then* T *is invertible and binormal since*

$$TT^*T^*T = T^*TTT^* = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

By a direct calculation, we have

$$|T| = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ |T^*| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \ \text{and} \ U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that $U^2|T| \neq |T|U^2$, *then* $T \notin \delta(\mathcal{H})$. *Moreover, since*

$$\Delta_\lambda(T) = \begin{pmatrix} 0 & 0 & 2^\lambda & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 2^{1-\lambda} & 0 & 0 & 0 \end{pmatrix}, \ \text{then} \ (\Delta_\lambda(T))^{-1} = \begin{pmatrix} 0 & 0 & 0 & 2^{-(1-\lambda)} \\ 0 & 0 & 1 & 0 \\ 2^{-\lambda} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Also, we have

$$\Delta_\lambda(T^{-1}) = |T^*|^{-\lambda}U^*|T^*|^{-(1-\lambda)} = \begin{pmatrix} 0 & 0 & 0 & 2^{-(1-\lambda)} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2^{-\lambda} & 0 & 0 \end{pmatrix}.$$

Hence, $\Delta_\lambda(T^{-1}) \neq (\Delta_\lambda(T))^{-1}$ *for any* $\lambda \in]0,1[$.

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Blow up of solutions to a logarithmic nonlinear Schrödinger equation with in finite memory and delay term.

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Abstract: In this work, we consider a logarithmic nonlinear Schrödinger condition with delay term. We obtain a blow-up result of solutions under suitable conditions.

keywords : Schrödinger equation, Logarithmic source, blow up, negative initial energy, delay term.

1 Introduction

In this work, we are concerned with the blow-up in finite-time of solutions for the initial boundary value problem:

$$\begin{gathered} u_t - \mathbf{i}\Delta u = \int_0^\infty g(t-s)\Delta u(x,s)ds - \mu_1 u(x,t) \\ -\mu_2 u(x,t-\tau) + u|u|^{p-2}\ln|u|^k, \quad \text{in } \Omega\times(0,\infty), \\ u(x,t) = 0, \quad x\in\partial\Omega, \end{gathered}$$

and the initial conditions

$$\begin{gathered} u(x,t-\tau) = f_0(x,t-\tau), \quad \text{in } (0,\tau), \\ u(x,0) = u_0(x), \quad x\in\Omega. \end{gathered}$$

where $u = u(x,t)$, $t \geq 0$, $x \in \Omega$, Δ means the Laplacian administrator regarding the x variable, Ω is an ordinary and limited area of $\mathbb{R}^n$, $n \geq 1$, $p \geq 2$, k, μ_1, are positive constants, μ_2 is a genuine number, $\tau > 0$ speak to the time delay. The capacity $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function, the unwinding capacity exposed to conditions to be determined and u_0, f_0 are given capacities having a place with reasonable spaces.
The stability analysis of control systems governed by ordinary differential equation subjected to constant or time-varying delay has been one of the main interests for many researchers in systems theory (see [**?**],). Two methods were proposed to derive delay dependent or delay independent stability conditions; one is based on Lyapunov-Razumikhin functionals whereas the other uses Lyapunov-Krasovskii functionals. This type of problems is encountered in many branches of physics such as Nuclear Physics, Optics and Geophysics. It is well known, from the Quantum Field Theory, that such kind of logarithmic nonlinearity appears naturally in inflation cosmology and in supersymmetric field theories In [23] the nonappearance of the viscoelastic term ($g = 0$), the issue has been widely examined and numerous outcomes concerning neighborhood presence result has been set up utilizing the semigroup hypothesis. Likewise, for negative introductory energy, a limited time explode result is demonstrated. For example, for the condition.

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = u|u|^{p-2}\ln|u|^k, \quad \text{in} \ \ \Omega\times(0,\infty).$$

In [21], Han studied the global existence of weak solutions for the initial boundary value problem

$$\begin{gathered} u_{tt} - \Delta u + u - u\ln|u|^2 + u_t + u|u|^2 = 0, \quad in\ \Omega\times(0,T), \\ u(x,t) = 0, \quad x\in\partial\Omega, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x\in\Omega. \end{gathered}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$. The model (1) is closely related to the following equation with logarithmic nonlinearity

$$\begin{gathered} u_{tt} - u_{xx} + u - \varepsilon u\ln|u|^2 + u_t = 0, \text{ in } O\times(0,T), \\ u(x,t) = 0, \quad x\in\partial O, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x\in O. \end{gathered}$$

where $O=[a,b]$, the parameter $\varepsilon \in [0,1]$ [21].
On the other hand, the problem of the uniform stabilization of the Schrödinger equation by a control feedback has been considered by Machtiyngier et Zuazua in the case of type Neumman control. In the case of type Dirichlet control, it has been considered by Lasiecka and Triggiani.
More recently, in [2] has been studied the boundary stabilization of Schrödinger equations with variable coefficient memory feedback.
In this work, we consider blow up of solutions to a logarithmic nonlinear Schrödinger equation with in finite memory and delay term.
We give notations, hypotheses, $(.,.)$ and $\|.\|_p$ denote the inner prodution in the space $L(\Omega)$ and the norm of the space $L^p(\Omega)$, respectively. For breviy, we denote $\|.\|_2$ by $\|.\|$.
For the relaxation function g we assume the following (G) : We assume that the function $g:\mathbb{R}^+\to\mathbb{R}$ is of class C^1 satisfying:

$$1-\int_0^\infty g(s)ds = l>0,\quad g(t)\geq 0,\quad g'(t)\leq 0.$$

and under the assumption

$$\mu_1 \geq |\mu_2|\,.$$

By using the direct calculations, we have

$$\int_0^\infty g(t-s)\left(\nabla u_t(t),\nabla u(s)\right)ds = -\frac{1}{2}g(t)\left\|u(t)\right\|_2^2+\frac{1}{2}(g'\circ\nabla u)(t) -\frac{1}{2}\frac{d}{dt}\left[(g\circ\nabla u)(t)-\left(\int_0^\infty g(s)ds\right)\left\|\nabla u(t)\right\|_2^2\right]$$

where

$$(g\circ u)(t)=\int_0^\infty g(t-s)\left\|u(t)-u(s)\right\|_2^2 ds.$$

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Multiplicity of Solutions for Fourth-Order Elliptic Equations with p-Laplacian and Mixed Nonlinearity

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Abstract. In the paper, we study the multiplicity of solutions for a class of fourth-order elliptic equations with p-Laplacian and mixed nonlinearity of the form:

$$\begin{cases} \Delta^2 u - \Delta_p u + \lambda V(x)u = f(x,u) + \mu\xi(x)|u|^{q-2}u, & x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N). \end{cases}$$

Unlike most other works, we replace the Laplacian with a p-Laplacian. Using the mountain pass theorem and Ekeland's variational principle, we establish the existence of two nontrivial solutions. To overcome the difficulty of the convergence of the subsequences for the Palais–Smale sequences of the Euler–Lagrange functional, we consider Cerami sequences. Our results extend the recent results of Zhang et al. (Electron J Differ Equ 2017(250):1–15, 2017).

Mathematics Subject Classification. 35J35, 35J60, 35J92.

Keywords. Fourth-order elliptic equations, Variational methods, p-Laplacian, mixed nonlinearity, Gagliardo–Nirenberg inequality.

1. Introduction and Statement of Main Result

We consider the following biharmonic equation:

$$\begin{cases} \Delta^2 u - \Delta_p u + \lambda V(x)u = f(x,u) + \mu\xi(x)|u|^{q-2}u, & x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 5$, $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator, $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ with $p > 2$, $V \in \mathcal{C}(\mathbb{R}^N)$, $f \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R})$, $\xi \in L^{\frac{2}{2-q}}(\mathbb{R}^N, \mathbb{R}^+)$, $\lambda > 0, \mu > 0$ and $1 < q < 2$.

The study of fourth-order elliptic equations appears to be important in many areas. Problem (1.1) arises in the study of traveling waves in suspension bridge and the study of the static deflection of an elastic plate in a fluid,

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see [1,2] and the references therein. The solvability of fourth-order elliptic problems has been widely investigated in recent years, and numerous results are obtained on existence and multiplicity of the positive solutions, see ([3–11]), and the references therein.

Many authors have considered the problem with $\xi(x) = 0$ and $p = 2$:

$$\begin{cases} \Delta^2 u - \Delta u + \lambda V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where $V \in \mathcal{C}(\mathbb{R}^N)$ and $f \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R})$. Liu et al. [4] have proved the existence and multiplicity of nontrivial solutions via variational methods. Ye and Tang [5] have established existence and multiplicity results without compact embeddings, which unify and sharply improve the Liu and Wu results. In [6], Zhang et al established the existence of infinitely many solutions for problem (1.2) using the genus properties in critical point theory. By the symmetric mountain pass theorem, Cheng [11] obtained a new existence result of high-energy solutions for problem (1.2).

In [12], the authors studied the following biharmonic equation with p-Laplacian:

$$\begin{cases} \Delta^2 u - \beta\Delta_p u + \lambda V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{1.3}$$

where $N \geq 1, p \geq 2, \beta \in \mathbb{R}$ and $\lambda > 0$ are parameters, using the Gagliardo–Nirenberg inequality and Mountain Pass Lemma, they proved existence and multiplicity of nontrivial solutions. In particular, the authors used an Ambrosetti–Rabinowitz [(AR) for short]-type condition of the form:

(AR) There exist a constant $1 < l < 2$ and a nonnegative function $d \in L^{\frac{2}{2-l}}(\mathbb{R}^N)$, such that: $pF(x,u) - uf(x,u) \leq d(x)|u|^l$ for every $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$, where $F(x,u) = \int_0^u f(x,t)\mathrm{d}t$.

In the problem (1.1), if $V(x) = 0,\ p = 2$ and $\xi(x) = 0$, we obtain the problem:

$$\begin{cases} \Delta^2 u + c\Delta u = f(x,u), & \text{in} \quad \Omega, \quad \text{smooth bounded of } \mathbb{R}^N, \\ u = \Delta u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{1.4}$$

studied by some authors (for example, see [7,8]). In [7], Zhang and Wei obtained the existence of infinitely many solutions via variant fountain theorem. By variational techniques, Zhou and Wu [8] have obtained the existence and multiplicity of sign-changing solutions provided that $f(x,t)$ is odd in t and satisfies some additional requirements. In [9], An and Liu have used the mountain pass theorem to get some existence results. In [10], using linking approaches, Wang et al. have obtained at least three nontrivial solutions of the problem (1.4).

We make the following assumptions:

(A1) $V(x) \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}^+)$, and there exists a constant $b > 0$, the set $\{V < b\} := \left\{x \in \mathbb{R}^N | V(x) < b\right\}$ has finite positive Lebesgue measure.

(A2) $\Omega = \text{int } V^{-1}(0)$ is nonempty and has smooth boundary with $\bar{\Omega} = V^{-1}(0)$.

(A3) $f \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R})$ and $|f(x,u)| \le c(1+|u|^{r-1})$ for some $r \in (2, 2^*)$, where $2^* = \frac{2N}{N-4}$.
(A4) $f(x,u) = o(|u|)$ as $|u| \to 0$ uniformly for $x \in \mathbb{R}^N$.
(A5) There exists $\theta > p$, such that $0 < \theta F(x,u) \le u f(x,u)$ for every $x \in \mathbb{R}^N$ and $u \neq 0$, where $F(x,u) = \int_0^u f(x,t)dt$.

Here is the main result of this work.

Theorem 1.1. *Assume that conditions* (A1)–(A5) *hold,* $2 < p \le \frac{2N}{N-2}$*, and* $\xi \in L^{\frac{2}{2-q}}(\mathbb{R}^N, \mathbb{R}^+)$ $(1 < q < 2)$*. Then, there exist two positive constants* Λ_0 *and* $\mu_0 > 0$*, such that for every* $\lambda > \Lambda_0$ *and* $0 < \mu < \mu_0$*, problem* (1.1) *has at least two nontrivial solutions* $u_\lambda^i (i = 1, 2)$*.*

Under some mild assumptions and $p = 2$ in problem (1.1), Zhang et al. [3] have established the existence of two nontrivial solutions.

2. Preliminaries

Below by $\|\cdot\|_s$, we denote the usual L^s-norm for $2 \le s \le 2^*$, c_i, C, C_i stand for different positive constants.

Let

$$E = \left\{ u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\Delta u|^2 + V(x)u^2)\mathrm{d}x < \infty \right\},$$

be the Hilbert space equipped with the inner product:

$$(u,v) = \int_{\mathbb{R}^N} (\Delta u \Delta v + V(x)uv)\mathrm{d}x, \quad u,v \in E,$$

and the corresponding norm:

$$\|u\| = \Big(\int_{\mathbb{R}^N} (|\Delta u|^2 + V(x)u^2)\mathrm{d}x \Big)^{\frac{1}{2}}, \quad u \in E.$$

For $\lambda > 0$, we also define the inner product:

$$(u,v)_\lambda = \int_{\mathbb{R}^N} (\Delta u \Delta v + \lambda V(x)uv)\mathrm{d}x, \quad u,v \in E,$$

and the corresponding norm $\|u\|_\lambda^2 = (u,u)_\lambda$. It is clear that $\|u\| \le \|u\|_\lambda$, for $\lambda \ge 1$.

Moreover, $E_\lambda = (E, \|u\|_\lambda)$ is a Hilbert space. By Gagliardo–Nirenberg inequality, there exists $C_0 > 0$ such that:

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x &\le C_0^2 \left(\int_{\mathbb{R}^N} |\Delta u|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} u^2 \mathrm{d}x \right)^{\frac{1}{2}} \\ &\le \frac{C_0^2}{2} \left(\int_{\mathbb{R}^N} |\Delta u|^2 \mathrm{d}x + \int_{\mathbb{R}^N} u^2 \mathrm{d}x \right), \end{aligned} \tag{2.1}$$

which indicates that:

$$\int_{\mathbb{R}^N} \left(|\Delta u|^2 + u^2\right) \mathrm{d}x \le \|u\|_{H^2}^2 \le \left(1 + \frac{C_0^2}{2}\right) \int_{\mathbb{R}^N} \left(|\Delta u|^2 + u^2\right) \mathrm{d}x. \tag{2.2}$$

It follows from conditions (A1)–(A2), Hölder, and Gagliardo–Nirenberg inequalities that there exists $C_1 > 0$, such that:

$$\begin{aligned}\int_{\mathbb{R}^N} u^2 \mathrm{d}x &= \int_{\{V \geq b\}} u^2 \mathrm{d}x + \int_{\{V<b\}} u^2 \mathrm{d}x \\ &\leq \frac{1}{b} \int_{\{V \geq b\}} V(x) u^2 \mathrm{d}x + |\{V<b\}|^{\frac{4}{N}} \Big(\int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x \Big)^{\frac{2}{2^*}} \\ &\leq \frac{1}{b} \int_{\mathbb{R}^N} V(x) u^2 \mathrm{d}x + C_1^2 |\{V<b\}|^{\frac{4}{N}} \int_{\mathbb{R}^N} |\Delta u|^2 \mathrm{d}x.\end{aligned}$$

Combining the above inequality with (2.2) yields:

$$\|u\|_{H^2}^2 \leq \Big(1 + \frac{C_0^2}{2}\Big) \max\Big\{1 + C_1^2 |\{V<b\}|^{\frac{4}{N}}, \frac{1}{b}\Big\} \|u\|^2 = \alpha_N \|u\|^2, \quad (2.3)$$

where

$$\alpha_N = \Big(1 + \frac{C_0^2}{2}\Big) \max\Big\{1 + C_1^2 |\{V<b\}|^{\frac{4}{N}}, \frac{1}{b}\Big\},$$

which implies that the imbedding $E \hookrightarrow H^2(\mathbb{R}^N)$ is continuous.
Similar to the inequality (2.3), we also obtain:

$$\|u\|_{H^2}^2 \leq \Big(1 + \frac{C_0^2}{2}\Big) \Big(1 + C_1^2 |\{V<b\}|^{\frac{4}{N}}\Big) \|u\|_\lambda^2, \quad (2.4)$$

for $\lambda \geq \dfrac{1}{b\Big(1 + C_1^2 |\{V<b\}|^{\frac{4}{N}}\Big)}$.

By conditions (A1)–(A2), (2.4) and Hölder and Gagliardo–Nirenberg inequalities, for any $s \in [2, 2^*]$, one has:

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^s \mathrm{d}x &\leq \Big(\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x \Big)^{\frac{2N - s(N-4)}{8}} \Big(\int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x \Big)^{\frac{(s-2)(N-4)}{8}} \\ &\leq C_1^{\frac{N(s-2)}{4}} \Big(\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x \Big)^{\frac{2N - s(N-4)}{8}} \Big(\int_{\mathbb{R}^N} |\Delta u|^2 \mathrm{d}x \Big)^{\frac{N(s-2)}{8}} \\ &\leq C_1^{\frac{N(s-2)}{4}} \Big(1 + \frac{C_0^2}{2}\Big)^{\frac{s}{2}} \Big(1 + C_1^2 |\{V<b\}|^{\frac{4}{N}}\Big)^{\frac{s}{2}} \|u\|_\lambda^s, \quad (2.5)\end{aligned}$$

for $\lambda \geq \dfrac{1}{b\Big(1 + C_1^2 |\{V<b\}|^{\frac{4}{N}}\Big)}$.

Let:

$$\Theta_s = C_1^{\frac{N(s-2)}{4}} \Big(1 + \frac{C_0^2}{2}\Big)^{\frac{s}{2}} \Big(1 + C_1^2 |\{V<b\}|^{\frac{4}{N}}\Big)^{\frac{s}{2}} \quad \text{and} \quad \Lambda = \frac{1}{b\Big(1 + C_1^2 |\{V<b\}|^{\frac{4}{N}}\Big)}.$$

From (2.5), for any $s \in [2, 2^*]$ and $\lambda \geq \Lambda$, we have:

$$\int_{\mathbb{R}^N} |u|^s \mathrm{d}x \leq \Theta_s \|u\|_\lambda^s. \quad (2.6)$$

Let

$$\Phi_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\Delta u|^2 + \lambda V(x)u^2\right)\mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p\mathrm{d}x - \Psi(u), \quad (2.7)$$

where

$$\Psi(u) = \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x + \frac{\mu}{q}\int_{\mathbb{R}^N}\xi(x)|u|^q\mathrm{d}x.$$

By a standard argument and Hölder inequality, it is easy to verify that $\Phi_\lambda \in C^1(E_\lambda, \mathbb{R})$ and:

$$\langle \Phi'_\lambda(u), v\rangle = \int_{\mathbb{R}^N}\left(\Delta u\Delta v + \lambda V(x)uv\right)\mathrm{d}x + \int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u.\nabla v\mathrm{d}x - \langle \Psi'(u), v\rangle, \quad (2.8)$$

for all $u, v \in E_\lambda$, where:

$$\langle \Psi'(u), v\rangle = \int_{\mathbb{R}^N} f(x,u)v\mathrm{d}x + \mu\int_{\mathbb{R}^N}\xi(x)|u|^{q-2}uv\mathrm{d}x.$$

Lemma 2.1. *Assume that* (A5) *is satisfied, and* $\xi \in L^{\frac{2}{2-q}}(\mathbb{R}^N, \mathbb{R}^+)$. *If* u_λ *is nontrivial solution of* (1.1), *then for every* $\lambda \geq \Lambda$ *and* $\mu > 0$, *we have:*

$$\Phi_\lambda(u_\lambda) \geq K := -\frac{(p-2)(2-q)}{2pq}\left(\frac{\mu(p-q)\|\xi\|_{\frac{2}{2-q}}\Theta_2^q}{p-2}\right)^{\frac{2}{2-q}}.$$

Proof. Let u_λ be a nontrivial solution of (1.1). Then:

$$\|u_\lambda\|_\lambda^2 + \int_{\mathbb{R}^N}|\nabla u|^p = \int_{\mathbb{R}^N} f(x,u_\lambda)u_\lambda\mathrm{d}x + \mu\int_{\mathbb{R}^N}\xi(x)|u_\lambda|^q\mathrm{d}x.$$

Combining this with (A5) and (2.7) yields:

$$\begin{aligned}
\Phi_\lambda(u_\lambda) &= \frac{1}{2}\|u_\lambda\|_\lambda^2 + \frac{1}{p}\left(\int_{\mathbb{R}^N} f(x,u_\lambda)u_\lambda\mathrm{d}x + \mu\int_{\mathbb{R}^N}\xi(x)|u_\lambda|^q\mathrm{d}x - \|u_\lambda\|_\lambda^2\right)\\
&\quad - \int_{\mathbb{R}^N}F(x,u_\lambda)\mathrm{d}x - \frac{\mu}{q}\int_{\mathbb{R}^N}\xi(x)|u_\lambda|^q\mathrm{d}x\\
&\geq \frac{p-2}{2p}\|u_\lambda\|_\lambda^2 + \left(\int_{\mathbb{R}^N}\frac{1}{p}f(x,u_\lambda)u_\lambda - F(x,u_\lambda)\right) + \mu\frac{q-p}{pq}\int_{\mathbb{R}^N}\xi(x)|u_\lambda|^q\mathrm{d}x\\
&\geq \frac{p-2}{2p}\|u_\lambda\|_\lambda^2 - \mu\frac{p-q}{pq}\|\xi\|_{\frac{2}{2-q}}\Theta_2^{\frac{q}{2}}\|u_\lambda\|_\lambda^q\\
&\geq -\frac{(p-2)(2-q)}{2pq}\left(\frac{\mu(p-q)\|\xi\|_{\frac{2}{2-q}}\Theta_2^{\frac{q}{2}}}{p-2}\right)^{\frac{2}{2-q}}.
\end{aligned}$$

□

3. Proof of the Main Results

To prove our result, we need the following Mountain Pass Theorem under the $(\mathrm{C})_{\mathrm{c}}$ condition; the readers can refer to [13,14].

Theorem 3.1. *Let* $(X, \|\cdot\|_X)$ *be a Banach space; suppose that* $\varphi \in C^1(X, \mathbb{R})$ *satisfies* $\varphi(0) = 0$ *and:*

(i) *There are constants* $\rho, \eta > 0$*, such that* $\varphi_{\partial B_\rho(0)} \geq \eta$*.*
(ii) *There is a constant* $e \in X \setminus \bar{B}_\rho(0)$*, such that* $\varphi(e) \leq 0$*.*
(iii) φ *satisfies the* $(\mathrm{C})_{\mathrm{c}}$ *condition; that is, for* $c \in \mathbb{R}$*, every sequence* $\{u_n\} \subset X$*, such that:*

$$\varphi(u_n) \to c, \quad \|\varphi'(u_n)\|(1 + \|u_n\|) \to 0$$

has a convergent subsequence.

Then, $c := \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} \varphi(\gamma(s))$ *is a critical value of* φ*, where:*

$$\Gamma := \{\gamma \in C([0,1], X); \gamma(0) = 0, \gamma(1) = e\}.$$

Lemma 3.2. *Assume that* (A3)–(A4) *are satisfied , and* $\xi \in L^{\frac{2}{2-q}}(\mathbb{R}^N, \mathbb{R}^+)$*. Then,there exist three positive constants* μ_0, ρ *and* η*,such that* $\Phi_\lambda(u)|_{\|u\|_\lambda = \rho} \geq \eta > 0$ *for all* $\mu \in (0, \mu_0)$*.*

Proof. For any $\varepsilon > 0$, it follows from conditions (A3) and (A4) that there exists $C_\varepsilon > 0$, such that:

$$f(x, u) \leq \varepsilon |u| + C_\varepsilon |u|^{r-1} \quad \text{for all} \quad u \in E_\lambda; \tag{3.1}$$

then

$$F(x, u) \leq \frac{\varepsilon}{2} |u|^2 + \frac{C_\varepsilon}{r} |u|^r \quad \text{for all} \quad u \in E_\lambda. \tag{3.2}$$

Thus, from (2.6), (3.2) and the Sobolev inequality, we have that for all $u \in E_\lambda$:

$$\begin{aligned} \int_{\mathbb{R}^N} F(x, u) \mathrm{d}x &\leq \frac{\varepsilon}{2} \int_{\mathbb{R}^N} u^2 \mathrm{d}x + \frac{C_\varepsilon}{r} \int_{\mathbb{R}^N} |u|^r \mathrm{d}x \\ &\leq \frac{\varepsilon \Theta_2}{2} \|u\|_\lambda^2 + \frac{C_\varepsilon \Theta_r}{r} \|u\|_\lambda^r, \end{aligned}$$

which implies that:

$$\begin{aligned} \Phi_\lambda(u) &= \frac{1}{2}\|u\|_\lambda^2 + \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x - \int_{\mathbb{R}^N} F(x, u) \mathrm{d}x - \frac{\mu}{q} \int_{\mathbb{R}^N} \xi(x) |u|^q \mathrm{d}x \\ &\geq \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}^N} F(x, u) \mathrm{d}x - \frac{\mu}{q} \int_{\mathbb{R}^N} \xi(x) |u|^q \mathrm{d}x \\ &\geq \frac{1}{2}\|u\|_\lambda^2 - \frac{\varepsilon \Theta_2}{2} \|u\|_\lambda^2 - \frac{C_\varepsilon \Theta_r}{r} \|u\|_\lambda^r - \frac{\mu \Theta_2^{\frac{q}{2}}}{q} \|\xi\|_{\frac{2}{2-q}} \|u\|_\lambda^q \\ &= \|u\|_\lambda^q \left[\frac{1}{2}(1 - \varepsilon \Theta_2) \|u\|_\lambda^{2-q} - \frac{C_\varepsilon \Theta_r}{r} \|u\|_\lambda^{r-q} - \frac{\mu \Theta_2^{\frac{q}{2}}}{q} \|\xi\|_{\frac{2}{2-q}} \right]. \end{aligned} \tag{3.3}$$

Let $\varepsilon = \frac{1}{2\Theta_2}$ and define:

$$g(t) = \frac{1}{4}t^{2-q} - \frac{C_\varepsilon \Theta_r}{r} t^{r-q}, \quad \text{for} \quad t \geq 0.$$

It is easy to prove that there exists $\rho > 0$, such that:

$$\max_{t\geq 0} g(t) = g(\rho) = \frac{r-2}{4(r-q)} \left[\frac{(2-q)r}{4C_\varepsilon \Theta_r (r-q)} \right]^{\frac{2-q}{r-2}}.$$

Then, it follows from (3.3) that there exist positive constants μ_0 and η, such that $\Phi_\lambda(u)|_{\|u\|_\lambda=\rho} \geq \eta > 0$ for all $\mu \in (0, \mu_0)$. □

Lemma 3.3. *Assume that* (A3)–(A5) *are satisfied, and* $\xi \in L^{\frac{2}{2-q}}(\mathbb{R}^N, \mathbb{R}^+)$. *Let* ρ *be as in Lemma* 3.2. *Then, there exists* $e \in E_\lambda$ *with* $\|e\|_\lambda > \rho$, *such that* $\Phi_\lambda(e) < 0$ *for all* $\mu \geq 0$.

Proof. By (A4) and (A5), there exists $c_0 > 0$, such that:

$$F(x, u) \geq c_0(|u|^\theta - |u|^2), \forall (x, u) \in \mathbb{R}^N \times \mathbb{R}.$$

Thus, for $t > 0, u \in E_\lambda$ and $u \not\equiv 0$, we have:

$$\begin{aligned}
\Phi(tu) &= \frac{t^2}{2}\|u\|_\lambda^2 + \frac{t^p}{p}\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x - \int_{\mathbb{R}^N} F(x, tu)\mathrm{d}x - \frac{\mu}{q}\int_{\mathbb{R}^N} \xi(x)|tu|^q \mathrm{d}x \\
&\leq \frac{t^2}{2}\|u\|_\lambda^2 + \frac{t^p}{p}\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x - c_0 t^\theta \int_{\mathbb{R}^N} |u|^\theta \mathrm{d}x + c_0 t^2 \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x \\
&\quad - \frac{\mu}{q} t^q \int_{\mathbb{R}^N} \xi(x)|u|^q \mathrm{d}x,
\end{aligned}$$

which implies that $\Phi_\lambda(tu) \to -\infty$ as $t \to \infty$. Therefore, there exist $t_0 > 0$ and $e := t_0 u$ with $\|e\|_\lambda > \rho$, such that $\Phi_\lambda(e) < 0$. This completes the proof. □

Lemma 3.4. *Assume that* (A 1)–(A 2), (A5) *are satisfied. Let* $\{u_n\}$ *be a* $(C)_c$-*sequence. Then,* $\{u_n\}$ *is bounded in* E_λ *for each* $\lambda \geq \Lambda$.

Proof. For n large enough, by (A5), (2.7) and (2.8), we have:

$$\begin{aligned}
1 + c &\geq \Phi_\lambda(u_n) - \frac{1}{p}\langle \Phi'_\lambda(u_n), u_n \rangle \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_\lambda^2 + \int_{\mathbb{R}^N} \left[\frac{1}{p}u_n f(x, u_n) - F(x, u_n)\right] \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \left(\frac{1}{p} - \frac{1}{q}\right) \mu \xi(x)|u_n|^q \mathrm{d}x.
\end{aligned}$$

Hence:

$$\begin{aligned}
1 + c &+ \int_{\mathbb{R}^N} \left(\frac{1}{q} - \frac{1}{p}\right) \mu \xi(x)|u_n|^q \mathrm{d}x \\
&\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_\lambda^2 + \int_{\mathbb{R}^N} \left[\frac{1}{p}u_n f(x, u_n) - F(x, u_n)\right] \mathrm{d}x.
\end{aligned}$$

Since

$$\begin{aligned}\left(\frac{1}{q}-\frac{1}{p}\right)\mu\int_{\mathbb{R}^N}\xi(x)|u_n|^q\mathrm{d}x &\le \left(\frac{1}{q}-\frac{1}{p}\right)\mu\|\xi\|_{\frac{2}{2-q}}\|u_n\|_2^q\\ &\le \left(\frac{1}{q}-\frac{1}{p}\right)\mu\Theta_2^{\frac{q}{2}}\|\xi\|_{\frac{2}{2-q}}\|u_n\|_\lambda^q,\end{aligned}$$

then

$$\begin{aligned}&1+c+\left(\frac{1}{q}-\frac{1}{p}\right)\mu\Theta_2^{\frac{q}{2}}\|\xi\|_{\frac{2}{2-q}}\|u_n\|_\lambda^q\\ &\quad\ge \left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|_\lambda^2+\int_{\mathbb{R}^N}\left[\frac{1}{p}u_nf(x,u_n)-F(x,u_n)\right]\mathrm{d}x\\ &\quad\ge \left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|_\lambda^2.\end{aligned}$$

This proves that $\{u_n\}$ is bounded in E_λ. □

Lemma 3.5. *Assume that* (A1)–(A2), (A3)–(A5) *are satisfied, and* $2<p\le\frac{2N}{N-2}$. *Then, for each* $D>0$, *there exists* $\Lambda_0=\Lambda(D)\ge\Lambda$, *such that* Φ_λ *satisfies the* $(C)_c$*-sequence in* E_λ *for all* $c<D$ *and* $\lambda>\Lambda_0$.

Proof. Let $\{u_n\}$ be a $(C)_c$-sequence with $c<D$. By Lemma (3.4), $\{u_n\}$ is bounded in E_λ, and there exists $D_0>0$, such that $\|u_n\|_\lambda\le D_0$. We can assume that there exist a subsequence $\{u_n\}$ and u_0 in E_λ, such that:

$$\begin{aligned}&u_n \rightharpoonup u_0 \quad \text{weakly in} \quad E_\lambda,\\ &u_n \longrightarrow u_0 \quad \text{strongly in} \quad L^s_{\mathrm{loc}}(\mathbb{R}^N) \ \text{ for } s\in[2,2^*),\\ &u_n \longrightarrow u_0 \quad \text{a.e. in} \quad \mathbb{R}^N,\end{aligned}$$

and $\Phi'_\lambda(u_0)=0$.
Denote:

$$A=\{x\in\mathbb{R}^N : |x|\le R,\ \xi(x)\le M\}.$$

Then:

$$\begin{aligned}\int_{\mathbb{R}^N}\xi(x)|u_n-u_0|^q\mathrm{d}x &= \int_{\mathbb{R}^N\setminus A}\xi(x)|u_n-u_0|^q\mathrm{d}x+\int_A\xi(x)|u_n-u_0|^q\mathrm{d}x\\ &\le \left(\int_{\mathbb{R}^N\setminus A}|\xi(x)|^{\frac{2}{2-q}}\mathrm{d}x\right)^{\frac{2-q}{2}}\left(\int_{\mathbb{R}^N\setminus A}|u_n-u_0|^2\mathrm{d}x\right)^{\frac{q}{2}}\\ &\quad+\left(\int_A|\xi(x)|^{\frac{2}{2-q}}\mathrm{d}x\right)^{\frac{2-q}{2}}\left(\int_A|u_n-u_0|^2\mathrm{d}x\right)^{\frac{q}{2}}\\ &\le c_1\left(\int_{\mathbb{R}^N\setminus A}|\xi(x)|^{\frac{2}{2-q}}\mathrm{d}x\right)^{\frac{2-q}{2}}+\|\xi(x)\|_{\frac{2}{2-q}}\left(\int_A|u_n-u_0|^2\mathrm{d}x\right)^{\frac{q}{2}}.\end{aligned} \tag{3.4}$$

For R and M large enough, and $\left(\int_{\mathbb{R}^N\setminus A}|\xi(x)|^{\frac{2}{2-q}}\mathrm{d}x\right)^{\frac{2-q}{2}}<\varepsilon$. Hence:

$$\int_{\mathbb{R}^N}\xi(x)|u_n-u_0|^q\mathrm{d}x\le c_1\varepsilon+\|\xi(x)\|_{\frac{2}{2-q}}\left(\int_A|u_n-u_0|^2\mathrm{d}x\right)^{\frac{q}{2}}.$$

By Sobolev's embedding theorem, $u_n \rightharpoonup u_0$ in E_λ implies $u_n \to u_0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, and we have:

$$\int_{\mathbb{R}^N} \xi(x)|u_n - u_0|^q \mathrm{d}x = o(1). \tag{3.5}$$

From the Gagliardo–Nirenberg inequality, there exists $C = C(p) > 0$, such that:

$$\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x \le C^p \left(\int_{\mathbb{R}^N} |\Delta u|^2 \mathrm{d}x\right)^{\frac{p}{4}} \left(\int_{\mathbb{R}^N} |u|^{\frac{2p}{4-p}} \mathrm{d}x\right)^{\frac{4-p}{4}}. \tag{3.6}$$

Using (2.6) and (3.6), the imbedding $E_\lambda \hookrightarrow W^{1,p}(\mathbb{R}^N)$ is continuous, which shows that $u_n \rightharpoonup u_0$ in $W^{1,p}(\mathbb{R}^N)$.
Similar to the proof of [[15], Lemma 4.4] (or Lemma 4.1 in [16]), one can easily obtain that:

$$\nabla u_n(x) \longrightarrow \nabla u_0(x) \quad \text{a.e. in} \quad \mathbb{R}^N.$$

Thus, it follows from Brezis–Lieb Lemma [17] that:

$$\int_{\mathbb{R}^N} |\nabla u_n - \nabla u_0|^p \mathrm{d}x = \int_{\mathbb{R}^N} |\nabla u_n|^p \mathrm{d}x - \int_{\mathbb{R}^N} |\nabla u_0|^p \mathrm{d}x + o(1). \tag{3.7}$$

Now, we prove that $u_n \longrightarrow u_0$ strongly in E_λ. Let $w_n = u_n - u_0$. Then, $w_n \rightharpoonup 0$ in E_λ. Similar to the argument of (2.5), for any $\lambda > \Lambda$, we have that:

$$\int_{\mathbb{R}^N} |w_n|^s \mathrm{d}x \le \Pi_\lambda \|w_n\|_\lambda^s + o(1), \tag{3.8}$$

where $2 \le s \le 2^*$ and $\Pi_\lambda = C_1^{\frac{2^*(s-2)}{2^*-2}} \left(\frac{1}{\lambda b}\right)^{\frac{2^*-s}{2^*-2}}$. Similar to the proof of [[3], Lemma 2.4], one can easily obtain that:

$$\int_{\mathbb{R}^N} F(x, w_n)\mathrm{d}x = \int_{\mathbb{R}^N} F(x, u_n)\mathrm{d}x - \int_{\mathbb{R}^N} F(x, u_0)\mathrm{d}x + o(1). \tag{3.9}$$

$$\int_{\mathbb{R}^N} \xi(x)|w_n|^q \mathrm{d}x = \int_{\mathbb{R}^N} \xi(x)|u_n|^q \mathrm{d}x - \int_{\mathbb{R}^N} \xi(x)|u_0|^q \mathrm{d}x + o(1). \tag{3.10}$$

Thus, using (3.7), (3.9), (3.10), and Brezis–Lieb Lemma [17] gives:

$$\Phi_\lambda(w_n) = \Phi_\lambda(u_n) - \Phi_\lambda(u_0) + o(1) \quad \text{and} \quad \Phi'_\lambda(w_n) = o(1). \tag{3.11}$$

Consequently, by (3.11) and Lemma 2.1, one has:

$$\begin{aligned} D - K &\ge c - \Phi_\lambda(u_0) \\ &\ge \Phi_\lambda(w_n) - \frac{1}{p}\langle \Phi'_\lambda(w_n), w_n\rangle + o(1) \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|w_n\|_\lambda^2 + \int_{\mathbb{R}^N}\left[\frac{1}{p} w_n f(x, w_n) - F(x, w_n)\right]\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N}\left(\frac{1}{p} - \frac{1}{q}\right)\mu\xi(x)|w_n|^q \mathrm{d}x + o(1) \\ &\ge \left(\frac{1}{2} - \frac{1}{p}\right)\|w_n\|_\lambda^2 + \left(\frac{1}{p} - \frac{1}{q}\right)\mu\Theta_2^{\frac{q}{2}}\|\xi\|_{\frac{2}{2-q}}\|w_n\|_\lambda^q + o(1). \end{aligned} \tag{3.12}$$

Since $1 < q < 2$, it follows from (3.12) that there exists a constant $D_1 > 0$, such that:

$$\|w_n\|_\lambda \leq D_1 + o(1) \quad \text{for every} \quad \lambda > \Lambda. \tag{3.13}$$

Then, by the conditions (A3–A4) and combining (3.1),(3.13) with (3.8) and (3.5), we obtain:

$$\begin{aligned} o(1) &= \langle \Phi'_\lambda(w_n), w_n\rangle \\ &= \|w_n\|_\lambda^2 + \int_{\mathbb{R}^N} |\nabla w_n|^p \mathrm{d}x - \int_{\mathbb{R}^N} f(x, w_n) w_n \mathrm{d}x - \mu \int_{\mathbb{R}^N} \xi(x)|w_n|^q \mathrm{d}x \\ &\geq \|w_n\|_\lambda^2 - \varepsilon \int_{\mathbb{R}^N} |w_n|^2 \mathrm{d}x - C_\varepsilon \int_{\mathbb{R}^N} |w_n|^r \mathrm{d}x + o(1) \\ &\geq \|w_n\|_\lambda^2 - \frac{\varepsilon}{\lambda b}\|w_n\|_\lambda^2 - C_\varepsilon \Pi_\lambda \|w_n\|_\lambda^r + o(1). \end{aligned}$$

Since $\Pi_\lambda \to 0$ as $\lambda \to \infty$, there exists $\Lambda_0 = \Lambda(D) \geq \Lambda$, such that for $\lambda > \Lambda_0$, $w_n \to 0$ strongly in E_λ, thus $u_n \to u_0$ strongly in E_λ. This completes the proof. □

Define:

$$c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_\lambda(\gamma(t)) \quad \text{and} \quad c_0(\Omega) := \inf_{\gamma\in\overline{\Gamma}} \max_{t\in[0,1]} \Phi_\lambda|_{H_0^1(\Omega)\cap H^2(\mathbb{R}^N)}(\gamma(t)),$$

where $\Phi_\lambda|_{H_0^1(\Omega)\cap H^2(\mathbb{R}^N)}$ is a restriction of Φ_λ on $H_0^1(\Omega) \cap H^2(\mathbb{R}^N)$:

$$\Gamma := \{\gamma \in C([0,1], E_\lambda); \gamma(0) = 0, \gamma(1) = e\},$$

and

$$\overline{\Gamma}(\Omega) := \{\gamma \in C([0,1], H_0^1(\Omega) \cap H^2(\mathbb{R}^N)); \gamma(0) = 0, \gamma(1) = e_1\},$$

where the function $e_1 \in H_0^1(\Omega) \cap H^2(\mathbb{R}^N)$ which can be found using a similar argument of Lemma 3.3.

Note that:

$$\Phi_\lambda|_{H_0^1(\Omega)\cap H^2(\mathbb{R}^N)}(u) = \frac{1}{2}\|u\|_\lambda^2 + \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x - \frac{\mu}{q}\int_{\mathbb{R}^N} \xi(x)|u|^q \mathrm{d}x,$$

and $c_0(\Omega)$ are independent of λ. Moreover, similar to the proofs of Lemmas 3.2 and 3.3, we can conclude that $\Phi_\lambda|_{H_0^1(\Omega)\cap H^2(\mathbb{R}^N)}$ satisfies the mountain pass hypothesis.

Since $(H_0^1(\Omega) \cap H^2(\mathbb{R}^N)) \subset E_\lambda$ for all $\lambda > 0$, one can see that $0 < \eta \leq c_\lambda \leq c_0(\Omega)$ for all $\lambda \geq \Lambda$ and $0 < \mu < \mu_0$. Taking $D_0 > c_0(\Omega)$. Then, we have:

$$0 < \eta \leq c_\lambda \leq c_0(\Omega) < D_0 \quad \text{for all} \quad \lambda \geq \Lambda \quad \text{and} \quad 0 < \mu < \mu_0.$$

Proof of Theorem 1.1. By Theorem 3.1, and Lemmas 3.2 and 3.3, we obtain that, for each $\lambda \geq \Lambda$, $0 < \mu < \mu_0$, there exists $(C)_c$-sequence $\{u_n\} \subset E_\lambda$ for Φ_λ on E_λ.

Then, by Lemma 3.5 and $0 < \eta \leq c_\lambda \leq c_0(\Omega)$ for all $\lambda \geq \Lambda$, there exist $\Lambda_0 > \Lambda$ and $u_\lambda^1 \in E_\lambda$, such that for every $\lambda > \Lambda_0$:

$$u_n \to u_\lambda^1 \quad \text{in} \quad E_\lambda.$$

Moreover, $\Phi_\lambda(u_\lambda^1) = c_\lambda \geq \eta > 0$. Thus, u_λ^1 is a nontrivial solution of problem (1.1).

The second solution of problem (1.1) will be constructed through the local minimization. Since $\xi \in L^{\frac{2}{2-q}}(\mathbb{R}^N, \mathbb{R}^+)$, we can choose a function $\phi \in E_\lambda$, such that:

$$\int_{\mathbb{R}^N} \xi(x)|\phi|^q \mathrm{d}x > 0. \tag{3.14}$$

Thus, by (A5), we have:

$$\begin{aligned}\Phi_\lambda(t\phi) &= \frac{t^2}{2}\|\phi\|_\lambda^2 + \frac{t^p}{p}\int_{\mathbb{R}^N} |\nabla\phi|^p \mathrm{d}x - \int_{\mathbb{R}^N} F(x, t\phi)\mathrm{d}x - \frac{\mu t^q}{q}\int_{\mathbb{R}^N} \xi(x)|\phi|^q \mathrm{d}x \\ &\leq \frac{t^2}{2}\|\phi\|_\lambda^2 + \frac{t^p}{p}\int_{\mathbb{R}^N} |\nabla\phi|^p \mathrm{d}x - \frac{\mu t^q}{q}\int_{\mathbb{R}^N} \xi(x)|\phi|^q \mathrm{d}x < 0, \end{aligned} \tag{3.15}$$

for $t > 0$ small enough. Hence, there exists $\rho_1 > 0$, such that $\beta := \inf\left\{\Phi_\lambda(u) \ : \ u \in \bar{B}_{\rho_1}\right\} < 0$. By the Ekeland's variational principle, there exists a minimizing sequence $\{u_n\} \subset \bar{B}_{\rho_1}$, such that $\Phi_\lambda(u_n) \to \beta$ and $\Phi'_\lambda(u_n) \to 0$ as $n \to \infty$. Hence, Lemma 3.5 implies that there exists a nontrivial solution u_λ^2 of problem (1.1) satisfying:

$$\Phi_\lambda(u_\lambda^2) < 0 \quad \text{and} \quad \|u_\lambda^2\|_\lambda < \rho_1.$$

This completes the proof. □

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Modèle de compétition entre des organismes porteurs et non porteurs des plasmides dans un chémostat

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Abstract. Dans ce travail, nous considérons un modèle de compétition entre des organismes porteurs des plasmides et des organismes sans plasmides dans un chémostat en présence d'un inhibiteur externe et avec différents taux de dilution. Ce modèle a été introduit précédemment dans le cas où les fonctions de taux de croissance et le taux d'absorption de l'inhibiteur suivent la cinétique de Monod et les taux d'élimination sont les mêmes que le taux de dilution. Dans ce travail, nous considérons le cas général de fonctions monotones et des taux de dilution différents. Grâce aux trois paramètres opératoires du système représentés par le taux de dilution et les concentrations d'entrée du substrat et de l'inhibiteur, nous donnons des conditions nécessaires et suffisantes pour l'existence et la stabilité de tous les équilibres. Au moyen de diagrammes opératoire, nous décrivons le comportement asymptotique du modèle par rapport à ces paramètres opératoires. Quelques exemples sont donnés pour illustrer les résultats mathématiques.

Keywords: Chémostat, compétition, plasmide, inhibiteur, Diagramme opératoire.
Mathematics Subject Classification: 34C23, 34D20, 92B05, 92D25.

1 Introduction

Le Chémostat est un appareil de laboratoire qui permet la culture et l'étude des espèces de microorganismes, il peut être utilisé pour étudier la croissance microbienne car les paramètres sont facilement mesurables et par conséquent, les résultats mathématiques sont facilement testables. La première introduction du chémostat date de 1950 par Novick et Szilard [1] Monod [2].

La possibilité de fabriquer les produits souhaités en utilisant des organismes génétiquement modifiés représente l'une des principales développements de la biotechnologie. L'altération génétique se fait généralement par l'insertion d'ADN dans la cellule sous la forme d'un plasmide.

Comme décrit précédemment, les plasmides étant utilisés pour coder la fabrication d'un produit, malheureusement, le plasmide peut être perdu lors de la reproduction et la perte du plasmide entraîne l'apparition d'un organisme sans plasmide qui est un meilleur compétiteur. Pour compenser, on utilise une toxine qui inhibe l'organisme sans plasmide, et un élément de matériel génétique est ajouté au plasmide, qui code pour la résistance à un inhibiteur (un antibiotique), alors que l'organisme porteur du plasmide n'est pas affecté, et l'inhibiteur est introduit dans le chémostat à une concentration constante et au même débit que le nutriment.

Dans ce travail, nous considérons le modèle introduit par S. B. Hsu, T. K. Luo, et P. Waltman [5] avec différents taux de dilution et des fonctions de croissance monotones générales, où un organisme x sans plasmide et un organisme y porteur de plasmide sont en compétition pour une seule ressource limitante S en présence d'un inhibiteur externe p, qui inhibe la croissance de l'organisme sans plasmide. De plus, l'organisme sans plasmide est capable de détoxifier l'environnement, c'est-à-dire d'éliminer l'inhibiteur de l'environnement.

2 Le modèle

Le modèle s'écrit

$$\left\{\begin{array}{lcl} S' & = & (S^0 - S)D - f(p)f_1(S)\dfrac{x}{\beta} - f_2(S)\dfrac{y}{\beta} \\ x' & = & [f(p)f_1(S) - D - e_1]x + qf_2(S)y \\ y' & = & [(1-q)f_2(S) - D - e_2]y \\ p' & = & (p^0 - p)D - g(p)y \end{array}\right. \tag{2.1}$$

Où S^0: concentration du substrat à l'entrée du chémostat, p^0: la concentration de l'inhibiteur à l'entrée du chémostat, D: taux de dilution, β la coefficient de rendement de x et y, e_1 et e_2 sont les taux de mortalité de x et y, respectivement, q est la fraction des plasmides perdus, $f_i, i = 1, 2$ le taux de croissance de x et y, respectivement, g le taux d'absorption de l'inhibiteur externe par y, f représente le degré d'inhibition de p sur le taux de croissance de x.
Ce modèle ont été étudié par [5] avec $e_i = 0$ et

$$f_1 = \frac{m_1 S}{K_1 + S}, \quad f_2 = \frac{m_2 S}{K_2 + S}, \quad g = \frac{\delta S}{K + S}, \quad f = exp(-\mu p)$$

où m_i, K_i, $i = 1, 2$, δ, K et μ sont des paramètres constants positifs. Ici, à l'exception des trois paramètres opératoire (ou de contrôle), qui sont l'entrée de l'inhibiteur p^0, le taux de dilution D et le substrat entrant S^0, tous les autres paramètres sont des paramètres biologiques qui dépendent des organismes, du substrat et de l'inhibiteur considérés.
Dans ce travail nous considérons le modèle général (2.1) sans nous restreindre au cas particulier des fonctions de Monod des taux de croissance des compétiteurs f_i et du taux d'absorption de l'inhibiteur g et f mentionné précédemment. Nous supposons seulement que f_i, $i = 1, 2$, f et g dans le système (2.1) sont des fonctions $\mathcal{C}^1$ satisfaisant les conditions suivantes :

(C1) Pour $i = 1, 2$, $f_i(0) = 0$ et $f_i'(S) > 0$ pour tout $S \geq 0$.

(C2) $g(0) = 0$ et $g'(p) > 0$ pour tout $p \geq 0$.

(C3) $f(0) = 1$ et $f'(p) < 0$ pour tout $p \geq 0$.

3 Principaux résultats

Les points d'équilibre sont calculés, et les conditions de existence et stabilité sont données, le diagramme opératoire est construit en fixant l'un des paramètres opératoires (par exemple D), et représenter les régions d'existence et de stabilité des équilibres dans le plan (p^0, S^0). Enfin, des simulations numériques sont proposées pour illustrer les résultats mathématiques.

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THE ATOMIC DECOMPOSITION OF HERZ-MORREY TYPE HARDY SPACES WITH MIXED NORM AND ITS APPLICATION

AISSA DJERIOU

Abstract. In this communication, mixed Herz-Morrey-Hardy space is presented and its atom-decomposition theory is established. This result generalize and extend the corresponding results of XU and Yang. Also, the author study the boundedness of fractional integral operators on mixed Herz-Morrey-Hardy spaces by using the atomic decomposition.

1. Introduction

The Lebesgue spaces with mixed norms are a particular case of Banach function spaces, these spaces initially appeared in the paper of Benedek and Panzone [1] in 1961, where they gave general properties of these spaces and a generalization of well-known interpolation theorems.

In recent years, function spaces with mixed norms have been intensively studied by a significant number of authors, for instance, see [3, 6, 7, 8, 9, 10] and the references therein.

In [17], the authors obtained the boundedness of the Hardy-Littlewood maximal operators on mixed Lebesgue spaces. Also, Littlewood–Paley g-function characterization of mixed Hardy spaces and the dual theorem on homogeneous mixed-norm Triebel–Lizorkin are established in [9].

Nogayama [13, 14] studied mixed Morrey spaces $M_{\vec{q}}^{p}(\mathbb{R}^n)$, which covers mixed Lebesgue spaces $L^{\vec{q}}(\mathbb{R}^n)$, where the boundedness of the maximal operator, the fractional integral operator and singular integral operators are obtained.

In [4], the authors introduce and explore Hardy spaces defined by mixed Lebesgue norms and anisotropic dilations, also they gave the relation between anisotropic mixed-norm Hardy spaces and mixed-norm Lebesgue spaces.

Anisotropic homogeneous mixed-norm Besov and Triebel–Lizorkin spaces and the boundedness of Fourier multipliers and pseudodifferential operators with suitable adapted homogeneous symbols are presented in [5].

Inspired by [4, 5, 13, 14], we introduce mixed Herz-Morrey-Hardy spaces which covers the classical Herz-Morrey-Hardy spaces were considered in [16], and establish the atomic decompositions, which make it convenient to study the boundedness of fractional integral operators on these spaces.

2. Preliminaries

As usual, we denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The symbol $\mathbb{Z}$ stands for the set of all integer

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numbers. For a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$. As usual for any $x \in \mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to x. We use c as a generic positive constant, i.e. a constant whose value may change from appearance to appearance.

We use the letters $\vec{s}, \vec{q}$ will denote n-tuples of the numbers in $(0, \infty]$ $(n \geqslant 1)$, $\vec{s} = (s_1, ..., s_n)$, $\vec{q} = (q_1, ..., q_n)$. By definiton, the inequality, for example, $0 < \vec{q} < \infty$ means that $0 < q_i < \infty$ for each i. Furthermore, for $\vec{s} = (s_1, ..., s_n)$ and $r \in \mathbb{R}$, let

$$r\vec{s} = (rs_1, ..., rs_n), \quad \frac{1}{\vec{s}} = (\frac{1}{s_1}, ..., \frac{1}{s_n}) \quad \text{and} \quad \vec{s}\,' = (s_1', ..., s_n'),$$

where s_j' is the conjugate exponent of s_j defined by $\frac{1}{s_j} + \frac{1}{s_j'} = 1$.

The expression $A \lesssim B$ means that $A \leqslant cB$ for some independent constant c (and non-negative functions A and B), and $A \approx B$ means $A \lesssim B \lesssim A$.

The notation $X \hookrightarrow Y$ stands for continuous embeddings from X to Y, where X and Y are quasi-normed spaces. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E and χ_E denotes its characteristic function. The symbol $\mathcal{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions on $\mathbb{R}^n$ and we denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$.

By $\ell^p(\mathbb{Z})$, $0 < p \leqslant \infty$, we denote the space of all (complex) sequences $\{d_k\}_{k\in\mathbb{Z}}$ equipped with the quasi-norm

$$\big\|\{d_k\}_{k\in\mathbb{Z}}\big\|_{\ell^p(\mathbb{Z})} = \Big(\sum_{k\in\mathbb{Z}} |d_k|^p\Big)^{1/p}$$

(with the usual modification if $q = \infty$).

The mixed Lebesgue space $L^{\vec{q}}(\mathbb{R}^n) = L^{(q_1,\dots,q_n)}(\mathbb{R}^n)$ is the class of all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that the norm

$$\|f\|_{\vec{q}} = \Big(\int_{\mathbb{R}} \cdots \Big(\int_{\mathbb{R}} \big(\int_{\mathbb{R}} |f(x_1, ..., x_n)|^{q_1}\, \mathrm{d}x_1\big)^{\frac{q_2}{q_1}} \mathrm{d}x_2\Big)^{\frac{q_3}{q_2}} \cdots \mathrm{d}x_n\Big)^{\frac{1}{q_n}} < \infty$$

is finite. If $q_i = \infty$, then we have to make appropriate modifications and if each $q_i = q$, then $L^{\vec{q}}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$, so mixed Lebesgue spaces generalize classical spaces. This space has properties similar to classical Lebesgue space.

For any given $0 < \vec{q} \leqslant \infty$, $(L^{\vec{q}}(\mathbb{R}^n), \|\cdot\|_{\vec{q}})$ is a quasi-Banach space and, for any $1 \leqslant \vec{q} \leqslant \infty$, $(L^{\vec{q}}(\mathbb{R}^n), \|\cdot\|_{\vec{q}})$ becomes a Banach space (see [1, p. 304, Theorem 1.a]).

For $\vec{p} = (p_1, ..., p_n)$ and $\vec{q} = (q_1, ..., q_n)$ with $1 \leqslant \vec{p}, \vec{q} \leqslant \infty$, Hölder's inequality takes the form

$$\|f \cdot g\|_{\vec{s}} \leqslant \|f\|_{\vec{p}} \, \|g\|_{\vec{q}} \, ,$$

where $\vec{s} = (s_1, ..., s_n)$ is defined by $\frac{1}{s_i} = \frac{1}{p_i} + \frac{1}{q_i}$ for any $i \in \{1, ..., n\}$. Often we use the particular case $\vec{s} = (1, ..., 1)$ corresponding to the situation when $\vec{q} = \vec{p}\,'$ is the conjugate exponent of $\vec{p}$. Furthermore, the monotone convergence theorem, Fatou's lemma and the Lebesgue convergence theorem also holds, see for instance [13]. It is known that for Q be a cube and for $0 < \vec{q} \leqslant \infty$, we have

$$\|\chi_Q\|_{\vec{q}} = |Q|^{\frac{1}{n}\big(\sum_{i=1}^n \frac{1}{q_i}\big)}. \tag{2.1}$$

The Hardy-Littlewood maximal operator $\mathcal{M}$ is defined on L^1_{loc} by

$$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)| dy.$$

It was shown in [13, Theorem 4.5] that $\mathcal{M} : L^{\vec{q}}(\mathbb{R}^n) \to L^{\vec{q}}(\mathbb{R}^n)$ is bounded if $1 < \vec{q} \leqslant \infty$.

A detailed discussion of the properties of $L^{\vec{q}}(\mathbb{R}^n)$ may be found in the papers [1, 2, 6, 15] , and references therein.

Let $\Omega \in L^s(S^{n-1})$ with $1 < s \leqslant \infty$ be homogeneous of degree zero, where S^{n-1} denotes the unit sphere of $\mathbb{R}^n$.

Suppose that T_Ω represents a linear or a sublinear operator such that, for any $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \notin \operatorname{supp} f$, we have

$$T_\Omega f(x) \leqslant c \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|} |f(y)|\, dy \tag{2.2}$$

where c is independent of f and x.

Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} \varphi \subseteq Q_0$, $\int_{\mathbb{R}^n} \varphi(x)dx \neq 0$ and $\varphi_t(\cdot) = t^{-n}\varphi\left(\frac{\cdot}{t}\right)$ for any $t > 0$. Let $M_\varphi(f)$ be the grand maximal function of f defined by

$$M_\varphi(f)(x) = \sup_{t>0} |\varphi_t * f(x)|.$$

Let $\alpha \in \mathbb{R}, 1 < p \leqslant \infty, 1 < \vec{q} < \infty$ and $\gamma \geqslant 0$. The *homogeneous mixed Herz-Morrey type Hardy space* $HM\dot{K}^{\alpha,\gamma}_{p,\vec{q}}(\mathbb{R}^n)$ is defined by

$$\|f\|_{HM\dot{K}^{\alpha,\gamma}_{p,\vec{q}}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\gamma} \Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|M_\varphi(f)\chi_k\|^p_{\vec{q}} \Big)^{1/p} < \infty.$$

3. Main results

Now, we consider the characterizations of the spaces $HM\dot{K}^{\alpha,\gamma}_{p,\vec{q}}(\mathbb{R}^n)$ in terms of central atomic decompositions which make it convenient to study the boundedness of the fractional integral operator on these spaces. The following result generalizes the result of XU and Yang [16] by taking $\alpha(\cdot) = \alpha$ and $, q(\cdot) = q = q_i$ for each $i \in \{1, ..., n\}$.

Theorem 3.1. *Let $\alpha \in \mathbb{R}, 1 < p, \vec{q} < \infty, \gamma \geqslant 0, \alpha \geqslant \max(2\gamma, n - \sum_{i=1}^n)$ and $m \geqslant \left[\alpha - n + \sum_{i=1}^n \frac{1}{q_i}\right]$. For any $f \in HM\dot{K}^{\alpha,\gamma}_{p,\vec{q}}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k,$$

where the series converges in the sense of distributions, $\lambda_k \geqslant 0$, each a_k is a central $(\alpha, \vec{q})$-atom with support contained in Q_k and

$$\sup_{k_0 \in \mathbb{Z}} 2^{-k_0\gamma p} \Big(\sum_{k=-\infty}^{k_0} \lambda_k^p \Big) \lesssim \|f\|_{HM\dot{K}^{\alpha,\gamma}_{p,\vec{q}}(\mathbb{R}^n)}. \tag{3.2}$$

Moreaver

$$\|f\|_{HM\dot{K}^{\alpha,\gamma}_{p,\vec{q}}(\mathbb{R}^n)} \approx \inf \Big(\sup_{k_0 \in \mathbb{Z}} 2^{-k_0\gamma} \Big(\sum_{k=-\infty}^{k_0} \lambda_k^p \Big)^{1/p} \Big),$$

where the infimum is taken over all the decompositions of f as above.

As an application of the atomic decomposition, we will prove the folowing result.

Theorem 3.3. *Suppose that $1 < p_1, p_2, \vec{q}, \vec{s} < \infty, p_1 \leqslant p_2$ and $q_j \sum_{i=1}^n \frac{1}{q_i} = s_j \sum_{i=1}^n \frac{1}{s_i}$ (for each j). Let $\gamma \geqslant 0$ and $\alpha \geqslant \max(2\gamma, n - \sum_{i=1}^n)$. Then T_Ω is bounded from $HM\dot{K}^{\alpha,\gamma}_{p_1,\vec{s}}(\mathbb{R}^n)$ to $HM\dot{K}^{\alpha,\gamma}_{p_2,\vec{q}}(\mathbb{R}^n)$.*

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MATHEMATICAL ANALYSIS OF THE NEW MONKYPOX MATHEMATICAL MODEL VIA THE FRACTAL FRACTIONAL DERIVATIVE

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Abstract. Monkeypox is an epidemic disease caused by infection with the monkeypox virus that appeared in the 1970s and spread in Central and West African countries, But it became a concern starting in May 2022 as it began to spread in several non-endemic countries, such as Belgium, Canada, and France. Therefore, it has become necessary to increase the accuracy of the mathematical models that describe its dynamics in order to help limit its spread. The use of fractional derivatives in mathematical modeling of infectious diseases gives more realistic results and thus accuracy in describing, predicting, and finally controlling the spread of epidemics in society. In recent years, new derivatives related to the fractal dimension and fractional order have been proposed that transcend the constraints of prior fractional-order derivatives, called fractal-fractional derivatives [1].

In this work, we used these new derivatives with the power law kernel to study the dynamics of the spread of monkeypox disease in humans and rodents. We studied the stability of the modal solutions through the concept of Hyers-Ulam stability, and then, for the numerical simulation of the system, we created his numerical scheme using the new numerical method that depends on the newly constructed numerical scheme based on Newton polynomials, and by changing the values of the fractal dimension and fractional order, we analyze the stability of the solutions graphically.

We found that the fractal fractional Monkeypox model is Hyers-Ulam stable in some conditions, and the numerical simulation shows the stability of solutions, and at smaller fractal dimensions, in the susceptible and infected groups, decomposition and growth rates are slower, and vice versa at smaller fractal dimensions. The exact results of fraction operators in mathematical modeling motivate us to use them in different models.

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La théorie de monotonie pour un système non-linéaire

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Abstract: Dans ce travail, j'ai étudié l'existence d'une solution d'un système non linéaire elliptique-parabolique modélisant le couplage entre écoulements Forchheimer et déformations mécaniques dans l'extraction d'hydrocarbures. Le modèle est obtenu sous l'hypothèse que l'écoulement est incompressible ayant lieu à grande vitesse exigeant donc l'utilisation de la loi de Darcy-Forchheimer reliant le gradient de la pression p et le débit massique unitaire $\mathbf{q}$, tandis que les déplacements $\mathbf{u}$ du milieux sont supposés du type poro-élastiques régis par la loi de Biot qui relie la variation des déplacements du milieu à la variation de sa porosité ϕ :

$$\frac{\partial \phi}{\partial t} = \operatorname{div} \frac{\partial \mathbf{u}}{\partial t}.$$

Il résulte un système de deux équations (dont la premiere est non linéaire) liant les inconnues principales, $\mathbf{u}$ et p:

$$\begin{cases} \operatorname{div} \dfrac{\partial \mathbf{u}}{\partial t} - \operatorname{div} F(\nabla p) &= 0, \\ - \operatorname{div} \underline{\sigma}(\mathbf{u}) &= \mathbf{X}. \end{cases}$$

Ici F est une fonction non linéaire, $\underline{\sigma}$ étant le tenseur des contraintes totales et $\mathbf{X}$ exprime les forces exercées. Au système précédent sont associées des conditions initiale et aux limites du type Dirichlet et/ou Neumann. J'ai prouvé l'existence d'une solution forte pour $\mathbf{u}$ et faible pour la pression p. Mon approche consiste à utiliser la méthode de Rothe pour obtenir une famille de systèmes d'équations non linéaires elliptiques dont la résolution nous fournit des solutions approchées. Il s'agit ensuite de prouver des estimations uniformes sur ces solutions qui me permet de passer à la limite, par le biais d'arguments de stricte monotonie, pour obtenir une solution du système initial.
Keywords: Système non linéaire parabolique / elliptique, Stricte monotonie, Loi de Biot, Loi de Forchheimer.
2010 Mathematics Subject Classification: 4F10, 35M32, 76S05.

1 Introduction

Dans ce travail, je propose un modèle mathématique pour étudier les couplages entre écoulement Forchheimer et déformations mécaniques dans l'extraction d'hydrocarbure. Mon but est de forger un outil pour simuler l'extraction d'hydrocarbures en présence des déformations de la matrice poreuse en supposant que le fluide est incompressible et que l'écoulement a lieu à vitesse importante. Utilisant les lois physiques des écoulements dans les milieux poreux (équation de continuitée, équation de Darcy-Forchheimer, ...) et la loi de Biot, elle aboutit au système non linéaire suivant

$$
(S) \qquad \left\{
\begin{array}{rcl}
\dfrac{\partial \phi}{\partial t} - \operatorname{div} \dfrac{\partial \mathbf{u}}{\partial t} & = & 0 \quad \text{sur} \quad \Omega_T = \Omega \times]0, T[, \\
\dfrac{\partial \phi}{\partial t} - \operatorname{div} F(\nabla p) & = & 0 \quad \text{dans} \quad \Omega_T, \\
p(\mathbf{x}, \mathbf{t}) & = & 0 \quad \text{sur} \quad \Gamma_{p1} \times]0, T[, \\
F(\nabla p) \cdot \mathbf{n} & = & 0 \quad \text{sur} \quad \Gamma_{p2} \times]0, T[, \\
-\mathbf{div}\, \underline{\sigma}(\mathbf{u}) & = & \mathbf{X} \quad \text{dans} \quad \Omega_T, \\
\mathbf{u} & = & 0 \quad \text{sur} \quad \Gamma_{u1} \times [0, T], \\
\underline{\sigma}(\mathbf{u})\, \mathbf{n} & = & \mathbf{h} \quad \text{sur} \quad \Gamma_{u2} \times [0, T],
\end{array}
\right.
$$

où $\mathbf{u}$ représente le vecteur déplacement de la matrice poreuse, ϕ sa porosité, p la pression du liquide, $\underline{\sigma}(\mathbf{u})$ est le tenseur des contraintes totales, $\mathbf{X}$ désigne la force exercée sur Ω et $\mathbf{h}$ une force surfacique morte (poids) sur le bord, avec

$$
\frac{\partial \mathbf{u}}{\partial t} = (\partial_t u_1, \partial_t u_2, \partial_t u_3) \qquad \text{et} \qquad F(v) = K \frac{(1 + \eta |v|)^{\frac{1}{2}} - 1}{\eta |v|} v, \; v \in \mathbb{R}^3.
$$

Ici Ω est un domaine borné de $\mathbb{R}^3$ représentant le milieu poreux. La frontière de Ω, notée Γ, est supposée lipchitzienne et est décomposée en deux parties (selon $\mathbf{u}$ et p) $\Gamma = \overline{\Gamma}_{p1} \cup \overline{\Gamma}_{p2} = \overline{\Gamma}_{u1} \cup \overline{\Gamma}_{u2}$ avec

$$
\Gamma_{p1} \cap \Gamma_{p2} = \emptyset \quad \text{et} \quad \Gamma_{u1} \cap \Gamma_{u2} = \emptyset \quad \text{(de mesure superficielle} > 0).
$$

Le cadre théorique de travail seront les espaces de Sobolev $\mathbf{V}$ et W définis par

$$
\mathbf{V} = \{\mathbf{v} \in (H^1(\Omega))^3 \mid \mathbf{v} = 0 \;\; \text{sur} \;\; \Gamma_{u1}\}; \qquad W = \{q \in W^{1,\frac{3}{2}}(\Omega) \mid q = 0 \;\; \text{sur} \;\; \Gamma_{p1}\}.
$$

Quant aux hypothèses de travail elles seront les suivantes :
$(\mathcal{H}_1)$ $\mathbf{X} \in H^1([0,T]; \mathbf{L}^2(\Omega))$ et $\mathbf{h} \in H^1([0,T]; (\mathbf{L}^2(\Gamma_{u2}))$.
$(\mathcal{H}_2)$ $p(\cdot, 0) = p_0 \in W$, $\mathbf{u}_0$ vérifie l'identité (2.1) ci-dessous, pour $\mathbf{X}(t=0) = \mathbf{X}_0 \in \mathbf{L}^2(\Omega)$ et $\mathbf{h}(t=0) = \mathbf{h}_0 \in \mathbf{L}^2(\Omega)$.
$(\mathcal{H}_3)$ $\mathbf{u}_0 \in \mathbf{V}$.
Ici $\mathbf{L}^2(\Omega) = (L^2(\Omega))^3$ et $\mathbf{L}^2(\Gamma_{u2}) = (L^2(\Gamma_{u2}))^3$. Voici ce que l'on entend par solution du système (S) :

2 Main results

Définition de solution de (S) — On dit d'un couple $(p, \mathbf{u}) \in L^{3/2}(0,T; W) \times H^1(0,T; \mathbf{V})$ qu'il est solution du système (S) si $\mathbf{u}(t=0) = \mathbf{u}_0$ et, pour presque tout $t \in (0,T)$, tout $q \in W$ et tout $\mathbf{v} \in \mathbf{V}$, l'on a:

$$
\int_\Omega q \operatorname{div} \frac{\partial \mathbf{u}}{\partial t}\, dx + \int_\Omega F(\nabla p) \cdot \nabla q \, dx = 0, \tag{2.1}
$$

$$
\int_\Omega \sigma(\mathbf{u}) \colon \varepsilon(\mathbf{v})\, dx - \int_\Omega p \operatorname{div} \mathbf{v}\, dx = \int_\Omega \mathbf{X} \cdot \mathbf{v}\, dx + \int_{\Gamma u2} \mathbf{h} \cdot \mathbf{v}\, ds. \tag{2.2}
$$

But principal de la thèse — Le but principal est de montrer que le système (S) possède une solution unique au sens de la définition précédente. Plus précisément, montrer l'existence, l'unicité et la stabilité de la solution forte en temps du déplacement $\mathbf{u} \in H^1(0,T; \mathbf{V})$ et l'existence et l'unicité d'une solution faible de la pression $p \in L^{3/2}(0,T; W)$.

Méthode de résolution du système (S) — Pour résoudre le système d'évolution non linéaire, j'ai utilisé la méthode de Rothe (semi-dicrétisation en temps) couplée avec la théorie des opérateurs maximaux monotones.

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Existence and uniqueness of solutions for the nonlinear retarded Value Problems For First-Order Dynamic Equations On Time Scales

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Abstract: In This paper , is concerned with a class of first-order dynamic equations nonlinear retarded on time scales with nonlocal initial conditions. Qualitative and quantitative results are discussed. Through an application of a fixed point theorem due to O'Regan, the existence of solutions is investigated. Under suitable assumptions, we deduce the existence result for nonlocal dynamic Cauchy problem. We also examine the continuous dependency of solutions on initial conditions.

Keywords: Differential equations, stability theory, Hyers–Ulam–Rassias stability, fixed point theory, generalized metric spaces.

2010 Mathematics Subject Classification: Primary 34K20; Secondary 34K30, 34k40..

1 Introduction

The advantage of working on dynamic equations on time scales is that, under one framework, we can describe continuous-discrete hybrid processes. The results obtained, in the context of time scales, are more general and includes various other results as a particular case. Thus The first-order dynamic equations on time scales have been studied extensively, covering a variety of different problems; see for instance ([8, 16]) and the references therein. Moreover, there has been significant growth in the study of initial and boundary value dynamic problems, see for instance ([9, 10, 11]), but none of them considers nonlocal conditions. The study of nonlocal initial value problems constitutes a very interesting and important class of problems; because, in many physical systems, the measurements by a nonlocal condition may be more precise than the measurement given by a local initial condition. In the literature, a great deal of attention has been given to nonlocal problems for differential equations rather than for difference or dynamic equations, see ([6, 7, 13, 14]) and references therein.
In this paper, we are interested in the analysis of qualitative theory of the problems of the existence of solutions to the first-order dynamic equation of the form

$$x^{\Delta}(t) + p(t)\, x^{\sigma}(t) = f(t, x(t), x(t-\tau)), \ t \in [0,T]_{\mathbb{T}} \tag{1.1}$$

subject to the condition

$$x(t) + (\Phi x)(t) = \psi(t), \ t \in [-\tau, 0]_{\mathbb{T}} \tag{1.2}$$

where $T \in \mathbb{R}^{+}$, $f : [0,T]_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is rd-continuous, $\Phi : C_{rd}([-\tau,T]_{\mathbb{T}}, \mathbb{R}) \to C_{rd}([-\tau,0]_{\mathbb{T}}, \mathbb{R})$, and $p : [0,T]_{\mathbb{T}} \to \mathbb{R}$, is a function which is regressive and rd-continuous on $\mathbb{T}$.if $\mathbb{T} = \mathbb{R}$,
Also, we discuss herein, the existence of solutions to the adjoint equation of (1.1)

$$x^{\Delta}(t) + q(t)\, x(t) = g(t, x(t), x(t-\tau)), \ t \in [0,T]_{\mathbb{T}} = [0,T] \cap \mathbb{T}, \tag{1.3}$$

subject to (1.2), where $q : [0,T]_{\mathbb{T}} \to \mathbb{R}$ is regressive and rd-continuous function, $g : [0,T]_T \to \mathbb{R}$ is rd-continuous function.

Among various techniques available in the analysis that deals with the existence of solutions of dynamic problems, the method of fixed points is more elegant and powerful. Many authors have been employed several fixed point theorems in their investigation of the existence of solutions
In the present paper, we have applied a fixed point theorem of O'Regan

2 Main results

In this section, we establish a result concerning the existence of solutions to $(1)-(2)$.

Lemma 2.1. *Let $p:\mathbb{T}\to\mathbb{R}$ be rd-continuous and regressive. Suppose $f:[0,T]_{\mathbb{T}}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is rd-continuous, $t_0\in\mathbb{T}$, and $\Phi: C_{rd}([-\tau,T]_{\mathbb{T}},\mathbb{R})\to C_{rd}([-\tau,0]_{\mathbb{T}},\mathbb{R})$. Then x is the unique solution of the nonlocal dynamic problem $(1)-(2)$ on $\mathbb{T}$ if and only if*

$$x(t)=\begin{cases} e_{\ominus p}(t,0)(\psi(0)-(\Phi x)(0))+\int_{t_0}^{t} e_{\ominus p}(t,s) f(t,x(t),x(t-\tau))\Delta s,\ t\in[0,T]_{\mathbb{T}} \\ \psi(t)-(\Phi x)(t),\ t\in[-\tau,0]_{\mathbb{T}} \end{cases} \tag{2.1}$$

for all $t\in\mathbb{T}$.

Theorem 2.1. *Assume that $f:[0,T]_{\mathbb{T}}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is rd-continuous function such that for all $t\in[0,T]_{\mathbb{T}}$ and for each $x\in B_r$, $|f(t,x,x_\tau)|\le h_r(t)$, where $h_r:[0,T]_{\mathbb{T}}\to[0,\infty)$ is such that*

$$\lim_{r\to\infty}\left(\frac{1}{r}\int_0^T h_r(t)\Delta t\right)=\beta<\frac{1}{E}<\infty \tag{2.2}$$

Also, suppose that there is a $K\in(0,1)$ such that

$$|\Phi(x)-\Phi(y)|\le K(\|x-y\|) \text{ for all } x,y\in X. \tag{2.3}$$

Then the dynamic nonlocal problem $(1)-(2)$ has at least one solution.

Now, we shall turn our attention to the existence of solutions to the dynamic nonlocal problem (1.3)-(1.2). Notice that the simple useful formula transform (1.3) into (1.1).

Theorem 2.2. *Consider the dynamic nonlocal problem (1.3)-(1.2). Let q be rd-continuous, regressive function such that $1-\mu(t)q(t)>0$ for all $t\in[0,T]_{\mathbb{T}}$. Assume that $g:[0,T]_{\mathbb{T}}\times\mathbb{R}\to\mathbb{R}$ is rd-continuous function such that for all $t\in[0,T]_{\mathbb{T}}$ and for each $x\in B_r$,*

$$|g(t,x)|\le h_r(t)(1-\mu q(t)), \tag{2.4}$$

where the functions h_r and Φ are defined Then the dynamic nonlocal problem (1.3)-(1.2) has at least one solution

Next, again will be used to guarantee the existence of at least one solution to the nonlocal dynamic Cauchy problem of the type

$$x^{\Delta}(t)=F(t,x(t),x(t-\tau)),\ t\in[0,T]_{\mathbb{T}} \tag{2.5}$$

subject to (1.2), where $F:[0,T]_{\mathbb{T}}\times\mathbb{R}\to\mathbb{R}$ is rd-continuous and possibly nonlinear function

Theorem 2.3. *Consider the dynamic nonlocal Cauchy problem (2.5)-(1.2). Let q be rd-continuous, regressive function such that $1-\mu(t)q(t)>0$ for all $t\in[0,T]_{\mathbb{T}}$. Assume that $F:[0,T]_{\mathbb{T}}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is rd-continuous function such that for all $t\in[0,T]_{\mathbb{T}}$ and for each $x\in B_r$,*

$$|F(t,x(t),x(t-\tau))+q(t)x|\le h_r(t)(1-q(t)x),$$

where the functions h_r and Φ are defned . Then the dynamic nonlocal Cauchy problem (2.5)- has at least one solution

The dependency of solutions to (1.1)-(1.2) with respect to the initial conditions can be obtained as follows.

Theorem 2.4. *Suppose* $f : [0;T]_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $\Phi_1, \Phi_2 : X \to \mathbb{R}$ *satisfy the hypotheses of Theorem (2.1). Also, suppose for all* $t \in [0;T]_{\mathbb{T}}$ *and for all* $x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$|f(t, x_2, y_2) - f(t, x_1, y_1)| \leq \frac{1}{E} (\|x_2 - x_1\| + \|y_2 - y_1\|) \tag{2.6}$$

Then for the corresponding solutions x_1, x_2 *of the dynamic equation (1.1) subject to the initial condition*

$$x_i(t) + (\Phi x_i)(t) = \psi(t), \ t \in [-\tau, 0]_{\mathbb{T}}, \ \ (i = 1, 2), \tag{2.7}$$

the inequality

$$\|x_1 - x_2\| \leq E_0 |(\Phi x_1) - (\Phi x_2)| \sup_{t\in[0,T]_{\mathbb{T}}} |e_1(t, 0)|.$$

holds. Additionally, if $|(\Phi_1 x_1)(t_1) - (\Phi_2 x_2)(t_2)| \leq \delta$ for some $\delta > 0$, then we have

$$\|x_1 - x_2\| \leq E_0 \delta \sup_{t\in[0,T]_{\mathbb{T}}} |e_1(t, 0)|. \tag{2.8}$$

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Hidden Modalities of Spirals of Chaotic Attractor via Saturated Function Series and numerical results

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Abstract

This paper investigates the hidden modalities of an odd number of spirals in the multispiral chaotic attractor generated by saturated function series. The system considered in this work allows modeling of all modalities (patterns) governing from 1 to $p + q + 2$ spirals of this attractor. Next, having given the system $1 - D$ of the multispiral chaotic attractor via saturated function series, the procedure for uncovering hidden bifurcations presented by Menacer et al, such hidden bifurcations are governed by a homotopy parameter ε while p and q are kept constant. This additional parameter, which is absent from the initial problem, is perfectly adapted to unfold the structure of the multispiral chaotic attractor. Following this new procedure, at the integration operation, before reaching the asymptotical attractor which possesses an even number of spirals, these latter are generated one after one until they reach their maximum number matching the value fixed by ε, uncovering modalities of an odd number of spirals.
Keywords: saturated function series, modality of an odd number of spirals, hidden bifurcations

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EXISTENCE RESULTS FOR A CLASS OF FOURTH ORDER BOUNDARY VALUE PROBLEMS

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Abstract. In this talk, we investigate the existence of positive solutions for a class of fourth order boundary value problem with mixed integral and multi-point boundary conditions. we will study both superlinear and sublinear growth cases, two corollaries regarding superlinear and sublinear cases are deduced. As applications, three examples are given. Our analysis relies on a fixed point result in a cone.

Keywords: Existence, monotone positive solution, fixed point theorem, boundary value problem, cone.

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The Growth of Solutions of Linear Differential Equations with Fast Growing Coefficients

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Abstract. In this Article, we study the gowth of solutions of higher order linear differential equations with fast growing coefficients. We extend some previous results due to Zemirni and Belaïdi in [11].

Key words and phrases: Complex Differential Equations, iterated p-order, iterated p-type.

1 Introduction and main results

Consider the differential equation

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + ... + a_1(z)f' + a_0(z)f = 0 \tag{1.1}$$

where $k \geq 1$ is an integer, $\{a_j(z)\}_{0\leq j\leq k-1}$ are entire functions and $a_0(z) \not\equiv 0$. The aim of this paper is to study the iterated p-order and iterated p-type of solutions of equation (1.1) by giving new conditions on the coefficients $\{a_j(z)\}_{0\leq j\leq k-1}$ and mainly using the fundamental results and basic notation of Nevanlinna theory see [7, 9, 10]. Let $\mathbb{N} = \{0, 1, 2, ...\}$ denote the set of natural numbers. Let us define inductively for $r \in [0, +\infty)$, $\exp_0 r = r$, $\exp_1 r = e^r$, and $\exp_{n+1} r = \exp(\exp_n r)$, $n \in \mathbb{N}$. For all r sufficiently large, we define $\log_0 r = r$, $\log_1 r = e^r$, and $\log_{n+1} r = \log(\log_n r)$, $n \in \mathbb{N}$. Moreover, we denote by $\exp_{-1} r = \log r$ and $\log_{-1} r = \exp_1 r$.

Definition 1.1 ([2, 8])**.** *For $p \in \mathbb{N} - \{0\}$, the iterated p-order $\sigma_p(f)$ of a meromorphic functions f is defined by*

$$\sigma_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r, f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna characteristic function of f. If f is an entire function, then the iterated p-order of f is defined by

$$\sigma_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r, f)}{\log r} = \limsup_{r\to+\infty} \frac{\log_{p+1} M(r, f)}{\log r},$$

where $M(r, f) = max\{|f(z)|, |z| = r\}$.

Definition 1.2 ([3])**.** *For $p \in \mathbb{N} - \{0\}$, the iterated p-type of a meromorphic function f with iterated p-order $0 < \sigma_p(f) < \infty$ is defined by*

$$\tau_p(f) = \limsup_{r\to+\infty} \frac{\log_{p-1} T(r, f)}{r^{\sigma_p(f)}}.$$

If f is an entire function, then the iterated p-type of f is given by

$$\tau_{M,p}(f) = \limsup_{r\to+\infty} \frac{\log_p M(r,f)}{r^{\sigma_p(f)}}.$$

Definition 1.3 ([8])**.** *For $p \in \mathbb{N} - \{0\}$, the iterated p-exposant of convergence of sequence of the zeros of a meromorphic function f is defined by*

$$\lambda_p(f) = \limsup_{r\to+\infty} \frac{\log_p N(r,\frac{1}{f})}{\log r},$$

where $N(r,\frac{1}{f})$ is the integrated counting function of zeros of f in $\{z : |z| \le r\}$. Similarly, the iterated p-exposant of convergence of sequence of the poles of f is defined by

$$\lambda_p(\frac{1}{f}) = \limsup_{r\to+\infty} \frac{\log_p N(r,f)}{\log r},$$

where $N(r,f)$ is the integrated counting function of poles of f in $\{z : |z| \le r\}$.

Definition 1.4 ([10])**.** *For $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the deficiency of a with respect to a meromorphic function f is defined as*

$$\delta(a,f) = \liminf_{r\to+\infty} \frac{m\left(r,\frac{1}{f-a}\right)}{T(r,f)} = 1 - \limsup_{r\to+\infty} \frac{N\left(r,\frac{1}{f-a}\right)}{T(r,f)}, \quad a \neq +\infty,$$

$$\delta(\infty,f) = \liminf_{r\to+\infty} \frac{m(r,f)}{T(r,f)} = 1 - \limsup_{r\to+\infty} \frac{N(r,f)}{T(r,f)}.$$

Recently, Zemirni and Belaïdi studied equation (1.1) and proved the following results :

Theorem 1.1 ([11])**.** *Let $\{a_j(z)\}_{0\le j\le k-1}$ be entire functions satisfying* $\max\{\sigma_p(a_j) : j = 1, ..., k-1\} \le \sigma_p(a_0) = \sigma$ *such that* $0 < \sigma < +\infty$ *and* $max\{\tau_p(a_j) : j = 1, ..., k-1\} \le \tau_p(a_0) = \tau$ *such that* $0 < \tau < +\infty$ *for* $p \in \mathbb{N} - \{0,1\}$. *Supppose that there exist two positive real numbers α and β with $0 \le \beta < \alpha$, such that*

$$|a_0(z)| \ge \exp_{p-1}(\alpha e^{\tau r^{\sigma}}) \tag{1.2}$$

and

$$|a_j(z)| \le \exp_{p-1}(\beta e^{\tau r^{\sigma}}),\ j = 1......, k-1 \tag{1.3}$$

as $|z| = r \to +\infty$ for $r \in E$ (E is infinite logarithmique measure). Then, every solution $f \not\equiv 0$ of equation (1.1) satisfies $\sigma_{p+1}(f) = \sigma$ and $\tau_{p+1}(f) = \tau$.

Theorem 1.2 ([11])**.** *Let $\{a_j(z)\}_{0\le j\le k-1}$ be entire functions satisfying* $\max\{\sigma_p(a_j) : j = 1, ..., k-1\} \le \sigma_p(a_0) = \sigma$ *such that* $0 < \sigma < +\infty$ *and* $max\{\tau_p(a_j) : j = 1, ..., k-1\} \le \tau_p(a_0) = \tau$ *such that* $0 < \tau < +\infty$ *for* $p \in \mathbb{N} - \{0,1\}$. *Supppose that there exist two positive real numbers α and β with $0 \le \beta < \alpha$, such that*

$$m(r,a_0) \ge \exp_{p-2}(\alpha e^{\tau r^{\sigma}}) \tag{1.4}$$

and

$$m(r,a_j) \le \exp_{p-2}(\beta e^{\tau r^{\sigma}}),\ j = 1......, k-1 \tag{1.5}$$

as $|z| = r \to +\infty$ for $r \in E$ (E is infinite logarithmique measure). Then, every solution $f \not\equiv 0$ of equation (1.1) satisfies $\sigma_{p+1}(f) = \sigma$ and $\tau_{p+1}(f) = \tau$.

We continue to consider the above results by considering the coefficient $a_s(z)(1 \le s \le k-1)$. We will prove the following results :

Theorem 1.3. *Let $\{a_j(z)\}_{0\leq j\leq k-1}$ be entire functions such that there exists $s \in \{1,......,k-1\}$ satisfying $\max\{\sigma_p(a_j), j \neq s\} \leq \sigma_p(a_s) = \sigma$ such that $0 < \sigma < +\infty$ and $max\{\tau_p(a_j), j \neq s\} \leq \tau_p(a_s) = \tau$ such that $0 < \tau < +\infty$ for $p \in \mathbb{N} - \{0,1\}$. Supppose that there exist two positive real numbers α and β with $0 \leq \beta < \alpha$, such that*

$$|a_s(z)| \geq \exp_{p-1}(\alpha e^{\tau r^{\sigma}}) \tag{1.6}$$

and

$$|a_j(z)| \leq \exp_{p-1}(\beta e^{\tau r^{\sigma}}), j \neq s \tag{1.7}$$

as $|z| = r \to +\infty$ for $r \in E$ (E is infinite logarithmique measure). Then, every transcendental entire solution f of equation (1.1) satisfies $\sigma_{p+1}(f) = \sigma$ and $\tau_{p+1}(f) = \tau$.

Theorem 1.4. *Let $\{a_j(z)\}_{0\leq j\leq k-1}$ be entire functions such that there exists $s \in \{1,......,k-1\}$ satisfying $max\{\sigma_p(a_j), j \neq s\} \leq \sigma_p(a_s) = \sigma$ such that $0 < \sigma < +\infty$ and $max\{\tau_p(a_j), j \neq s\} \leq \tau_p(a_s) = \tau$ such that $0 < \tau < +\infty$ for $p \in \mathbb{N} - \{0,1\}$. Supppose that there exist two positive real numbers α and β with $0 \leq \beta < \alpha$, such that*

$$m(r, a_s) \geq \exp_{p-1}(\alpha e^{\tau r^{\sigma}}) \tag{1.8}$$

and

$$m(r, a_j) \leq \exp_{p-1}(\beta e^{\tau r^{\sigma}}), j \neq s \tag{1.9}$$

as $|z| = r \to +\infty$ for $r \in E$ (E is infinite logarithmique meseare). Then, every transcendental entire solution f of equation (1.1) satisfies $\sigma_{p+1}(f) = \sigma$ and $\tau_{p+1}(f) = \tau$.

2 Some lemmas

To avoid some problems of the exceptional sets, we need the following lemma :

Lemma 2.1 ([1, 6])**.** *Let $\varphi : [0,+\infty) \longmapsto \mathbb{R}$ and $\psi : [0,+\infty) \longmapsto \mathbb{R}$ be monotone non-decreasing functions such that $\varphi(r) \leq \psi(r)$ for all $r \notin F_1 \cup [0,1]$, where $F_1 \subset (1,+\infty)$ is a set of logarithmic measure. Let $\gamma > 1$ be a given constant. Then, there exists $R = R(\gamma) > 0$ such that $\varphi(r) \leq \psi(\gamma r)$ for all $r \geq R$.*

Lemma 2.2 ([11])**.** *Let f be a transcendental meromorphic function with $\sigma_p(f) = \sigma < +\infty$ for some $p \in \mathbb{N} - \{0\}$, and let $\varepsilon > 0$ be a given constant. Then, there exists a set $F_2 \subset (1.+\infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin F_2 \cup [0,1]$ and for all integer $j \geq 1$, we have :*
1. If $p = 1$, then

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq r^{i(\sigma-1+\varepsilon)}$$

2. If $p \geq 2$, then

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \exp_{p-1}(r^{\sigma+\varepsilon}).$$

Lemma 2.3 ([5])**.** *Let f be a transcendental meromorphic function, and let $\mu > 1$ be a given constant. Then there exists a set $F_3 \subset (1,+\infty)$ with finite logarithmic measure and constant $B > 0$ that depends only on μ and i, $j (0 \leq i < j)$, such that for all z satisfiying $|z| = r \notin F_3 \cup [0,1]$, we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B\left[\frac{T(\mu r, f)}{r}(log^{\mu} r)\log T(\mu r, f)\right]^{j-i}.$$

Lemma 2.4 ([2])**.** *Let $\{a_j(z)\}_{0\leq j\leq k-1}$ be entire functions such that $0 < p < \infty$ and $max\{\sigma_p(a_j) : j = 0,1,...,k-1\} \leq \sigma < \infty$. Then, every solution $f \not\equiv 0$ of equation (1.1) satisfies $\sigma_{p+1}(f) \leq \sigma$.*

Lemma 2.5. *Let $\{a_j(z)\}_{0\leq j\leq k-1}$ be entire functions such that $1 < p < \infty$. Suppose that there exists $s \in \{1,...,k-1\}$ such that $max\{\sigma_p(a_j) : j \neq s\} \leq \sigma_p(a_s) = \sigma < \infty$ and $max\{\tau_p(a_j) : j \neq s\} \leq \tau_p(a_s) = \tau < \infty$. Then, every transcendental solution f of equation (1.1) with $\sigma_{p+1}(f) = \sigma$ satisfies $\tau_{p+1}(f) \leq \tau$.*

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Existence of positive solutions for fractional integral boundary value problem of fractional differential equation on infinite interval

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Abstract

This work discusses the existence of positive solutions for a fractional differential equations with nonlocal fractional integro-differential boundary conditions set on an unbounded domain. Using properties of the Green's function and the fixed point theory, an existence result was obtained. An example illustrate the existence theorem.

Keywords. Boundary value problem; fractional differential equation; positive solution; infinite interval; nonlocal conditions; fixed point theorem.

1 Introduction

We investigate in this paper the questions of existence of solutions for the following bvp of fractional differential equation with homogeneous fractional integral boundary condition at the left end-point and nonlocal fractional integral boundary condition at the right end-point:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + \varphi(t) f(t, u(t)) = 0, & t > 0, \\ I_{0^+}^{2-\alpha} u(0) = 0, & \lim_{t \to +\infty} D_{0^+}^{\alpha-1} u(t) = \gamma I_{0^+}^{\beta} u(\eta), \end{cases} \tag{1.1}$$

where $1 < \alpha \leq 2$ and $\beta,\ \gamma,\ \eta > 0$. The function $f : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is continuous and $\varphi : [0, +\infty) \to [0, +\infty)$ is continuous, not identically zero on any closed subinterval of $[0, +\infty)$, and $\varphi \in L^1[0, +\infty)$. Here $D_{0^+}^{\alpha}$ refers to the standard Riemann-Liouville fractional derivative while $I_{0^+}^{\beta}$ is the standard Riemann-Liouville fractional integral. We will assume throughout this work that

$$(H0) \quad 0 < \gamma \eta^{\alpha+\beta-1} < \Gamma(\alpha + \beta).$$

More precisely, we establish some sufficient conditions for the existence of at least one, two, and three positive solutions for bvp (**??**). Our approach is motivated by works [**?**], [**?**], and [**?**]. New and general existence results are obtained by making use of some fixed point theorems in suitable Banach spaces.

Firstly, we study the corresponding Green's function associated with bvp (**??**) and describe some of its properties. Section 2 is devoted to presenting some definitions and lemmas which are crucial in our discussion. In Section 3, we prove some technical lemmas which are needed later. Section 4 contains our main results of existence of positive solutions. It's based on Krasnosel'skii's fixed point theorem, that we recall here:

Theorem 1.1. *[**?**] (Krasnosel'skii's fixed point theorem) Let X be a Banach space and let $C \subset X$ be a cone. Assume that Ω_1, Ω_2 are open bounded subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : C \cap (\overline{\Omega}_2 \setminus \Omega_1) \to C$ be a completely continuous operator such that either one of the following conditions hold:*

(i) $\|Tu\|_X \leq \|u\|_X$, for $u \in C \cap \partial\Omega_1$ and $\|Tu\|_X \geq \|u\|_X$, for $u \in C \cap \partial\Omega_2$, or

(ii) $\|Tu\|_X \geq \|u\|_X$, *for* $u \in C \cap \partial\Omega_1$ *and* $\|Tu\|_X \leq \|u\|_X$, *for* $u \in C \cap \partial\Omega_2$.

Then T *has a fixed point in* $C \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

In Section 5, an example of applications is included to illustrate our results.

2 Preliminaries

We first collect some definitions and basic lemmas from fractional calculus (see [**?**], [**?**] for further details). The Gamma function extends the factorial to positive real numbers (and even to complex numbers with positive real parts).

Definition 2.1. *For* $\alpha > 0$, *the Euler Gamma function is defined by*

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt.$$

Definition 2.2. *For some* $p > 0,\ q > 0$, *the Euler Beta function is defined by*

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt.$$

Proposition 2.1. *Let* $\alpha > 0,\ p > 0,\ q > 0$ *and* n *a positive integer. Then*

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2n+1)}{2^{2n}\Gamma(n+1)}, \quad B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Hence

$$\Gamma(\alpha+n) = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)\Gamma(\alpha).$$

In particular

$$\Gamma(1) = \int_0^{+\infty} e^{-t} dt = 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

$$\Gamma(n+1) = n!, \quad \Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}(2n)!}{2^{2n} n!}.$$

Definition 2.3. *The fractional integral of order* $\alpha > 0$ *for a function* h *is defined by*

$$I_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

provided the right side is point-wise defined on $(0, +\infty)$.

Definition 2.4. *For a function h given on the interval $[0, +\infty)$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by*

$$D_{0+}^{\alpha} h(t) = \left(\frac{d}{dt}\right)^n I_{0+}^{n-\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{h(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$.

Lemma 2.1. *[?] Let $\alpha > 0$, then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R},\ i = 1, 2, \ldots, n,\ n = [\alpha] + 1$.

Proposition 2.2. *[?] The following composition relations hold:*

(a) $D_{0+}^{\alpha} I_{0+}^{\alpha} h(t) = h(t), \quad \alpha > 0, \quad h \in L^1[0, +\infty)$.

(b) $D_{0+}^{\alpha} I_{0+}^{\gamma} h(t) = I_{0+}^{\gamma-\alpha} h(t), \quad \gamma > \alpha > 0, \quad h \in L^1[0, +\infty)$.

(c) $I_{0+}^{\alpha} I_{0+}^{\gamma} h(t) = I_{0+}^{\alpha+\gamma} h(t), \quad \alpha > 0, \quad \gamma > 0 \quad h \in L^1[0, +\infty)$.

(d) $D_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}$, *for $\lambda > -1$, in particular for $D_{0+}^{\alpha} t^{\alpha-m} = 0$, $m = 1, 2, \ldots, N$, where N is the smallest integer greater than or equal to α.*

(e) $I_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda+1)} t^{\alpha+\lambda}, \quad \alpha > 0, \quad \lambda > -1$.

3 Related Lemmas

Consider the Banach space X defined by

$$X = \left\{ u \in C([0,\, +\infty), \mathbb{R}) : \quad \sup_{t \geq 0} \frac{|u(t)|}{1 + t^{\alpha-1}} < +\infty \right\}$$

with the norm

$$\|u\|_X = \sup_{t \geq 0} \frac{|u(t)|}{1 + t^{\alpha-1}}.$$

We have

Lemma 3.1. *Let $e \in L^1[0,+\infty)$ and $0 < \gamma\eta^{\alpha+\beta-1} < \Gamma(\alpha+\beta)$. Then bvp*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + e(t) = 0, & t > 0, \\ I_{0^+}^{2-\alpha} u(0) = 0, & \lim_{t\to+\infty} D_{0^+}^{\alpha-1} u(t) = \gamma I_{0^+}^{\beta} u(\eta), \end{cases} \tag{3.1}$$

has a unique solution given by

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e(s)ds + \frac{\Gamma(\alpha+\beta)t^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(\alpha+\beta) - \gamma\eta^{\alpha+\beta-1}\right)} \int_0^{+\infty} e(s)ds \\ &\quad -\frac{\gamma t^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(\alpha+\beta) - \gamma\eta^{\alpha+\beta-1}\right)} \int_0^{\eta} (\eta-s)^{\alpha+\beta-1} e(s)ds. \end{aligned}$$

Proof. By Lemma **??**, Proposition **??**, and since $D_{0^+}^{\alpha} u(t) + e(t) = 0$, we have

$$u(t) = -I_{0^+}^{\alpha} e(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \quad \text{for some constants } \; c_1,\, c_2 \in \mathbb{R}.$$

So the solution of (**??**) can be written as

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}.$$

Furthermore

$$\begin{aligned} I_{0^+}^{2-\alpha} u(t) &= -I_{0^+}^{2-\alpha} I_{0^+}^{\alpha} e(t) + c_1 I_{0^+}^{2-\alpha}(t^{\alpha-1}) + c_2 I_{0^+}^{2-\alpha}(t^{\alpha-2}) \\ &= -I_{0^+}^{2} e(t) + c_1 \Gamma(\alpha) t + c_2 \Gamma(\alpha-1). \end{aligned}$$

From $I_{0^+}^{2-\alpha} u(0) = 0$, we infer that $c_2 = 0$. In addition

$$\begin{aligned} D_{0^+}^{\alpha-1} u(t) &= -D_{0^+}^{\alpha-1} I_{0^+}^{\alpha} e(t) + c_1 D_{0^+}^{\alpha-1}(t^{\alpha-1}) \\ &= -I_{0^+}^{1} e(t) + c_1 \Gamma(\alpha) \\ &= -\int_0^t e(s)ds + c_1 \Gamma(\alpha). \end{aligned}$$

So

$$\lim_{t\to+\infty} D_{0^+}^{\alpha-1} u(t) = -\int_0^{+\infty} e(s)ds + c_1 \Gamma(\alpha).$$

Moreover

$$
\begin{aligned}
I_{0^+}^{\beta} u(t) &= -I_{0^+}^{\beta} I_{0^+}^{\alpha} e(t) + c_1 I_{0^+}^{\beta}(t^{\alpha-1}) \\
&= -I_{0^+}^{\alpha+\beta} e(t) + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1} \\
&= -\frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} e(s) ds + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1}.
\end{aligned}
$$

Hence

$$
I_{0^+}^{\beta} u(\eta) = -\frac{1}{\Gamma(\alpha+\beta)} \int_0^{\eta} (\eta-s)^{\alpha+\beta-1} e(s) ds + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \eta^{\alpha+\beta-1},
$$

$$
\lim_{t\to+\infty} D_{0^+}^{\alpha-1} u(t) = \gamma I_{0^+}^{\beta} u(\eta),
$$

and

$$
\begin{aligned}
c_1 &= \tfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\right)} \int_0^{+\infty} e(s) ds \\
&\quad - \tfrac{\gamma}{\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\right)} \int_0^{\eta} (\eta-s)^{\alpha+\beta-1} e(s) ds.
\end{aligned}
$$

Therefore, the unique solution of fractional boundary value problem (**??**) can be rewritten as

$$
\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e(s) ds + \frac{\Gamma(\alpha+\beta) t^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\right)} \int_0^{+\infty} e(s) ds \\
&\quad - \frac{\gamma t^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\right)} \int_0^{\eta} (\eta-s)^{\alpha+\beta-1} e(s) ds.
\end{aligned}
$$

□

Lemma 3.2. *In case* $0 < \gamma\eta^{\alpha+\beta-1} < \Gamma(\alpha+\beta)$, *then a second integral representation is given by*

$$
u(t) = \int_0^{+\infty} G(t,s) e(s) ds,
$$

where $G(t,s)$ *is the Green's function*

$$
G(t,s) = G_1(t,s) + G_2(t,s)
$$

with

$$G_1(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}-(t-s)^{\alpha-1}, & 0\le s\le t<+\infty,\\ t^{\alpha-1}, & 0\le t\le s<+\infty\end{cases}$$

and

$$G_2(t,s)=\frac{\gamma t^{\alpha-1}}{\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}}\times\frac{1}{\Gamma(\alpha)}\begin{cases} \eta^{\alpha+\beta-1}-(\eta-s)^{\alpha+\beta-1}, & 0\le s\le \eta<+\infty,\\ \eta^{\alpha+\beta-1}, & 0\le \eta\le s<+\infty.\end{cases}$$

Proof. The solution of problem (**??**) is

$$\begin{aligned}
u(t) &= -\tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e(s)ds+\tfrac{1}{\Gamma(\alpha)}\int_0^{+\infty} t^{\alpha-1}e(s)ds\\
&\quad -\tfrac{1}{\Gamma(\alpha)}\int_0^{+\infty} t^{\alpha-1}e(s)ds+\tfrac{\Gamma(\alpha+\beta)t^{\alpha-1}}{\Gamma(\alpha)\big(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\big)}\int_0^{+\infty} e(s)ds\\
&\quad -\tfrac{\gamma t^{\alpha-1}}{\Gamma(\alpha)\big(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\big)}\int_0^{\eta}(\eta-s)^{\alpha+\beta-1}e(s)ds\\
&= -\tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e(s)ds+\tfrac{1}{\Gamma(\alpha)}\int_0^{t} t^{\alpha-1}e(s)ds\\
&\quad +\tfrac{1}{\Gamma(\alpha)}\int_t^{+\infty} t^{\alpha-1}e(s)ds-\tfrac{1}{\Gamma(\alpha)}\int_0^{+\infty} t^{\alpha-1}e(s)ds\\
&\quad +\tfrac{\Gamma(\alpha+\beta)t^{\alpha-1}}{\Gamma(\alpha)\big(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\big)}\int_0^{+\infty} e(s)ds\\
&\quad -\tfrac{\gamma t^{\alpha-1}}{\Gamma(\alpha)\big(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\big)}\int_0^{\eta}(\eta-s)^{\alpha+\beta-1}e(s)ds\\
&= \tfrac{1}{\Gamma(\alpha)}\int_0^t \big(t^{\alpha-1}-(t-s)^{\alpha-1}\big)e(s)ds+\tfrac{1}{\Gamma(\alpha)}\int_t^{+\infty} t^{\alpha-1}e(s)ds\\
&\quad +\tfrac{\gamma\eta^{\alpha+\beta-1}}{\Gamma(\alpha)\big(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\big)}\int_0^{+\infty} t^{\alpha-1}e(s)ds\\
&\quad -\tfrac{\gamma t^{\alpha-1}}{\Gamma(\alpha)\big(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\big)}\int_0^{\eta}(\eta-s)^{\alpha+\beta-1}e(s)ds\\
&= \tfrac{1}{\Gamma(\alpha)}\int_0^t \big(t^{\alpha-1}-(t-s)^{\alpha-1}\big)e(s)ds+\tfrac{1}{\Gamma(\alpha)}\int_t^{+\infty} t^{\alpha-1}e(s)ds\\
&\quad +\tfrac{\gamma t^{\alpha-1}}{\Gamma(\alpha)\big(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\big)}\int_0^{\eta}\big(\eta^{\alpha+\beta-1}-(\eta-s)^{\alpha+\beta-1}\big)e(s)ds\\
&\quad +\tfrac{\gamma t^{\alpha-1}}{\Gamma(\alpha)\big(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\big)}\int_\eta^{+\infty}\eta^{\alpha+\beta-1}e(s)ds\\
&= \int_0^{+\infty} G_1(t,s)e(s)ds+\int_0^{+\infty} G_2(t,s)e(s)ds.
\end{aligned}$$

□

Remark 3.1. *Since* $0<\gamma\eta^{\alpha+\beta-1}<\Gamma(\alpha+\beta)$*, then from the definition of functions* $G_1(t,s)$*,* $G_2(t,s)$*, we can see that* *(a)* G_1 *and* G_2 *are continuous and nonnegative functions on* $[0,+\infty)\times[0,+\infty)$.

(b) $G_1(t,s)$ *is increasing function with respect to the first variable.*

(c) $\frac{G_1(t,s)}{1+t^{\alpha-1}} \le \frac{1}{\Gamma(\alpha)}$ *for* $(t,s)\in[0,+\infty)\times[0,+\infty)$.

(d) $\frac{G_2(t,s)}{1+t^{\alpha-1}} \le \frac{\gamma\eta^{\alpha+\beta-1}}{\Gamma(\alpha)\big(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\big)}$ *for* $(t,s)\in[0,+\infty)\times[0,+\infty)$.

Also, another property for the function $G_1(t,s)$, can be found in [**?**] as follows

Lemma 3.3. *[**?**] For fixed* $k>1$*, then* $G_1(t,s)$ *satisfies the following propertie*

$$\min_{\frac{1}{k}\le t\le k}\frac{G_1(t,s)}{1+t^{\alpha-1}} \ge \frac{1}{4k^2(1+k^{\alpha-1})}\sup_{t\ge 0}\frac{G_1(t,s)}{1+t^{\alpha-1}}.$$

Lemma 3.4. *For fixed* $k>1$*, the function* $G_2(t,s)$ *satisfies:*

$$\min_{\frac{1}{k}\le t\le k}\frac{G_2(t,s)}{1+t^{\alpha-1}} \ge \frac{1}{1+k^{\alpha-1}}\sup_{t\ge 0}\frac{G_2(t,s)}{1+t^{\alpha-1}}.$$

Proof. From Remark **??**-(d), we have

$$\begin{aligned}
\min_{\frac{1}{k}\le t\le k}\frac{G_2(t,s)}{1+t^{\alpha-1}} &= \min_{\frac{1}{k}\le t\le k}\left(\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\right)\frac{\gamma}{\Gamma(\alpha)(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1})}\times \\
&\quad \begin{cases} \eta^{\alpha+\beta-1}-(\eta-s)^{\alpha+\beta-1}, & 0\le s\le\eta<+\infty, \\ \eta^{\alpha+\beta-1}, & 0\le\eta\le s<+\infty. \end{cases} \\
&\ge \frac{1}{1+k^{\alpha-1}}\times\frac{\gamma\eta^{\alpha+\beta-1}}{\Gamma(\alpha)(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1})} \\
&\ge \frac{1}{1+k^{\alpha-1}}\sup_{t\ge 0}\frac{G_2(t,s)}{1+t^{\alpha-1}}.
\end{aligned}$$

□

Remark 3.2. *For fixed* $k>1$*, denote*

$$\begin{aligned}
\lambda(k) &:= \min\left\{\frac{1}{4k^2(1+k^{\alpha-1})},\frac{1}{1+k^{\alpha-1}}\right\} \\
&= \frac{1}{4k^2(1+k^{\alpha-1})}<1.
\end{aligned}$$

From Lemma ?? and Lemma ??, we have

$$\min_{\frac{1}{k}\leq t\leq k}\frac{G(t,s)}{1+t^{\alpha-1}}\geq\lambda(k)\sup_{t\geq 0}\frac{G(t,s)}{1+t^{\alpha-1}}.$$

Define the cone $C\subset X$ by

$$C=\left\{u\in X:\ u(t)\geq 0\ \text{ on }\ [0,+\infty)\ \text{ and }\ \min_{\frac{1}{k}\leq t\leq k}\frac{u(t)}{1+t^{\alpha-1}}\geq\lambda(k)\|u\|_X\right\}.$$

Let the operator $T:C\to X$,

$$(Tu)(t)=\int_0^{+\infty}G(t,s)\varphi(s)f(s,u(s))ds,\quad t\geq 0.$$

Before we proceed with the existence theorems, we list some conditions.

Suppose that there exist constants $0<a<b<r$ and $M=\frac{k^{\alpha-1}}{\lambda(k)}$ such that

$(H1)$ $f(t,(1+t^{\alpha-1})u)<ma$, for $t\in[0,+\infty)$, $u\leq a$ with

$$m=\frac{\Gamma(\alpha)(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1})}{\Gamma(\alpha+\beta)\int_0^{+\infty}\varphi(s)ds}.$$

$(H2)$ $f(t,(1+t^{\alpha-1})u)>\frac{b}{m'}$, for $t\in[\frac{1}{k},k]$, $b\leq u\leq Mb$ with

$$m'=\frac{\lambda(k)\int_{\frac{1}{k}}^{k}\varphi(s)ds}{\Gamma(\alpha)k^{\alpha-1}}.$$

$(H3)$ $f(t,(1+t^{\alpha-1})u)\leq mr$ for $t\in[0,+\infty)$, $u\leq r$.

Since the Arzela-Ascoli theorem fails to work in the space X, we need a modified compactness criterion to prove that the operator T is compact.

Lemma 3.5. *[?] Let $Z\subseteq X$ be bounded set. Then Z is relatively compact on X if the set $Z_1=\left\{\frac{u(t)}{1+t^{\alpha-1}},\ u\in Z\right\}$ is equicontinuous on any compact interval of $[0,+\infty)$ and Z_1 is equiconvergent at infinity.*

Definition 3.1. *Z_1 is called equiconvergent at infinity if for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\left| \frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}} \right| < \varepsilon,$$

for any $t_1,\ t_2 > \delta$ and $u \in Z$.

Equivalently, Z_1 is equiconvergent at infinity if for all $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that

$$\left| \frac{u(t)}{1+t^{\alpha-1}} - \lim_{t\to+\infty} \frac{u(t)}{1+t^{\alpha-1}} \right| < \epsilon, \quad \text{for all } t > \delta \text{ and } u \in Z.$$

4 Main results

Lemma 4.1. *Under Assumptions $(H0)$ and $(H3)$, the operator $T : C \to C$ is completely continuous.*

Proof. Let

$$C_r = \{u \in C : \ \|u\|_X < r\}$$

be the open ball of radius r in C.

Claim 1. $T(C) \subset C$. Let $u \in C$. From $(H0)$ and Remark **??**-(a), $(Tu)(t) \geq 0$ for $t \in [0, +\infty)$. From Remark **??**, we get for all $u \in C$ and $t \in [0, +\infty)$

$$\begin{aligned} \min_{\frac{1}{k}\leq t\leq k} \frac{Tu(t)}{1+t^{\alpha-1}} &= \min_{\frac{1}{k}\leq t\leq k} \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} \varphi(s) f(s,u(s))ds \\ &\geq \lambda(k) \sup_{t\geq 0} \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} \varphi(s) f(s,u(s))ds \\ &= \lambda(k)\|Tu\|_X, \end{aligned}$$

proving our claim.

Claim 2. $T(\overline{C}_r)$ is a bounded set. From $(H3)$ and Remark **??**-(c), (d), we have

$$\begin{aligned}
\left|\frac{Tu(t)}{1+t^{\alpha-1}}\right| &= \int_0^{+\infty}\frac{G(t,s)}{1+t^{\alpha-1}}\varphi(s)f(s,u(s))ds \\
&\leq \left(\frac{1}{\Gamma(\alpha)}+\frac{\gamma\eta^{\alpha+\beta-1}}{\Gamma(\alpha)(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1})}\right)\int_0^{+\infty}\varphi(s)f(s,u(s))ds \\
&\leq mr\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1})}\int_0^{+\infty}\varphi(s)ds \\
&= r.
\end{aligned}$$

By taking the supremum over t, we get:

$$\sup_{t\geq 0}\left|\frac{Tu(t)}{1+t^{\alpha-1}}\right|\leq r,$$

i.e., $Tu\in X$ and $\|Tu\|_X\leq r$.

Claim 3. $T: C\to C$ is continuous. Let $u_n\to u$, as $n\to+\infty$ in $\overline{C}_r$. From $(H0),(H3)$, and Remark **??**, we have

$$\begin{aligned}
\left|\frac{Tu_n(t)}{1+t^{\alpha-1}}-\frac{Tu(t)}{1+t^{\alpha-1}}\right| &= \left|\int_0^{+\infty}\frac{G(t,s)}{1+t^{\alpha-1}}\varphi(s)(f(s,u_n(s))-f(s,u(s)))ds\right| \\
&\leq \left(\frac{1}{\Gamma(\alpha)}+\frac{\gamma\eta^{\alpha+\beta-1}}{\Gamma(\alpha)(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1})}\right) \\
&\quad \int_0^{+\infty}\varphi(s)|f(s,u_n(s))-f(s,u(s))|ds \\
&\leq \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1})}\int_0^{+\infty}\varphi(s)(|f(s,u_n(s))|+|f(s,u(s))|)ds \\
&\leq 2mr\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1})}\int_0^{+\infty}\varphi(s)ds \\
&\leq 2r.
\end{aligned}$$

Thus

$$\left|\frac{Tu_n(t)}{1+t^{\alpha-1}}-\frac{Tu(t)}{1+t^{\alpha-1}}\right|<+\infty.$$

Using the Lebesgue dominated convergence theorem and the continuity of f, we get

$$\|Tu_n-Tu\|_X=\sup_{t\geq 0}\int_0^{+\infty}\frac{G(t,s)}{1+t^{\alpha-1}}\varphi(s)|f(s,u_n(s))-f(s,u(s))|ds\to 0,$$

as $n\to+\infty$, proving our claim.

Claim 4. $T:\overline{C}_r\to C$ is relatively compact. According to Claim 2, the set $T(\overline{C}_r)$ is uniformly bounded. To show that $T(\overline{C}_r)$ is equicontinuous on any compact interval $[0,d]$, $(d>0)$ of $[0,+\infty)$. Let $t_1,\ t_2\in[0,d],\ (t_1<t_2)$ and $u\in\overline{C}_r$. Then

$$\begin{aligned}
\left|\frac{(Tu)(t_2)}{1+t_2^{\alpha-1}}-\frac{(Tu)(t_1)}{1+t_1^{\alpha-1}}\right| &\leq \int_0^{+\infty}\left|\frac{G_1(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G_1(t_1,s)}{1+t_1^{\alpha-1}}\right|\varphi(s)|f(s,u(s))|ds\\
&\quad+\frac{\gamma\eta^{\alpha+\beta-1}}{\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\right)}\\
&\quad\times\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\int_0^{+\infty}\varphi(s)|f(s,u(s))|ds\\
&\leq \int_0^{+\infty}\left|\frac{G_1(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G_1(t_1,s)}{1+t_2^{\alpha-1}}\right|\varphi(s)|f(s,u(s))|ds\\
&\quad+\int_0^{+\infty}\left|\frac{G_1(t_1,s)}{1+t_2^{\alpha-1}}-\frac{G_1(t_1,s)}{1+t_1^{\alpha-1}}\right|\varphi(s)|f(s,u(s))|ds\\
&\quad+\frac{\gamma\eta^{\alpha+\beta-1}}{\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\right)}\\
&\quad\times\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\int_0^{+\infty}\varphi(s)|f(s,u(s))|ds.
\end{aligned}$$

In addition

$$\begin{aligned}
&\int_0^{+\infty}\left|\frac{G_1(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G_1(t_1,s)}{1+t_2^{\alpha-1}}\right|\varphi(s)|f(s,u(s))|ds\\
\leq\ &\int_0^{t_1}\left|\frac{G_1(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G_1(t_1,s)}{1+t_2^{\alpha-1}}\right|\varphi(s)|f(s,u(s))|ds\\
&+\int_{t_1}^{t_2}\left|\frac{G_1(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G_1(t_1,s)}{1+t_2^{\alpha-1}}\right|\varphi(s)|f(s,u(s))|ds\\
&+\int_{t_2}^{+\infty}\left|\frac{G_1(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G_1(t_1,s)}{1+t_2^{\alpha-1}}\right|\varphi(s)|f(s,u(s))|ds\\
\leq\ &mr\int_0^{t_1}\left|\frac{t_2^{\alpha-1}-t_1^{\alpha-1}+(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|\varphi(s)ds\\
&+mr\int_{t_1}^{t_2}\left|\frac{t_2^{\alpha-1}-t_1^{\alpha-1}-(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|\varphi(s)ds\\
&+mr\int_{t_2}^{+\infty}\left|\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{1+t_2^{\alpha-1}}\right|\varphi(s)ds\to 0,\quad \text{uniformly as}\quad |t_1-t_2|\to 0
\end{aligned}$$

and

$$
\begin{aligned}
& \int_0^{+\infty} \left| \frac{G_1(t_1,s)}{1+t_2^{\alpha-1}} - \frac{G_1(t_1,s)}{1+t_1^{\alpha-1}} \right| \varphi(s)|f(s,u(s))|ds \\
\leq \; & \frac{mr}{1+t_2^{\alpha-1}} \int_0^{+\infty} \left| t_2^{\alpha-1} - t_1^{\alpha-1} \right| \left| \frac{G_1(t_1,s)}{1+t_1^{\alpha-1}} \right| \varphi(s)ds \\
\leq \; & \frac{mr}{\Gamma(\alpha)(1+t_2^{\alpha-1})} \int_0^{+\infty} \left| t_2^{\alpha-1} - t_1^{\alpha-1} \right| \varphi(s)ds \to 0, \quad \text{uniformly as} \quad |t_1 - t_2| \to 0.
\end{aligned}
$$

Also

$$
\begin{aligned}
\frac{\gamma\eta^{\alpha+\beta-1}}{\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\right)} \quad \times \quad & \left| \frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}} \right| \int_0^{+\infty} \varphi(s)|f(s,u(s))|ds \\
\leq \quad & \frac{\gamma\eta^{\alpha+\beta-1}}{\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\right)} \\
& \times mr \left| \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})} \right| \int_0^{+\infty} \varphi(s)ds \\
& \to 0, \quad \text{uniformly as} \quad |t_1 - t_2| \to 0.
\end{aligned}
$$

Hence

$$
\left| \frac{(Tu)(t_2)}{1+t_2^{\alpha-1}} - \frac{(Tu)(t_1)}{1+t_1^{\alpha-1}} \right| \to 0, \quad \text{as } |t_1 - t_2| \to 0, \quad \text{for all } u \in \overline{C}_r.
$$

Thus $T(\overline{C}_r)$ is equicontinuous on $[0, +\infty)$.

Claim 5. $T(\overline{C}_r)$ is equiconvergent at infinity. For any $u \in \overline{C}_r$,

$$
\int_0^{+\infty} \varphi(s) f(s,u(s))ds \leq mr \int_0^{+\infty} \varphi(s)ds < +\infty.
$$

As a consequence

$$
\begin{aligned}
\lim_{t\to+\infty} \int_0^{+\infty} \frac{G_1(t,s)}{1+t^{\alpha-1}} \varphi(s) f(s,u(s))ds \quad = \quad & \lim_{t\to+\infty} \frac{1}{\Gamma(\alpha)} \left(\int_0^{+\infty} \frac{t^{\alpha-1}}{1+t^{\alpha-1}} \varphi(s) f(s,u(s))ds \right. \\
& \left. - \int_0^{t} \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}} \varphi(s) f(s,u(s))ds \right) \\
= \quad & 0.
\end{aligned}
$$

We have the estimates

$$\begin{aligned}
\lim_{t\to+\infty}\frac{(Tu)(t)}{1+t^{\alpha-1}} &= \lim_{t\to+\infty}\int_0^{+\infty}\frac{G(t,s)}{1+t^{\alpha-1}}\varphi(s)f(s,u(s))ds\\
&= \lim_{t\to+\infty}\Big(\int_0^{+\infty}\frac{G_1(t,s)}{1+t^{\alpha-1}}\varphi(s)f(s,u(s))ds\\
&\quad+\frac{\gamma}{\Gamma(\alpha)\big(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\big)}\times\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\int_0^{+\infty}\eta^{\alpha+\beta-1}\varphi(s)f(s,u(s))ds\\
&\quad-\frac{\gamma}{\Gamma(\alpha)\big(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\big)}\\
&\quad\times\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\int_0^{\eta}(\eta-s)^{\alpha+\beta-1}\varphi(s)f(s,u(s))ds\Big)\\
&= \frac{\gamma\eta^{\alpha+\beta-1}}{\Gamma(\alpha)\big(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\big)}\int_0^{+\infty}\varphi(s)f(s,u(s))ds\\
&\quad-\frac{\gamma}{\Gamma(\alpha)\big(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\big)}\int_0^{\eta}(\eta-s)^{\alpha+\beta-1}\varphi(s)f(s,u(s))ds<+\infty.
\end{aligned}$$

Hence

$$\begin{aligned}
\left|\frac{(Tu)(t)}{1+t^{\alpha-1}}-\lim_{t\to+\infty}\frac{(Tu)(t)}{1+t^{\alpha-1}}\right| &= \Big|\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\varphi(s)f(s,u(s))ds\\
&\quad-\frac{1}{\Gamma(\alpha)}\int_0^{t}\frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}\varphi(s)f(s,u(s))ds\\
&\quad+\frac{\gamma}{\Gamma(\alpha)\big(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\big)}\\
&\quad\times\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\int_0^{+\infty}\eta^{\alpha+\beta-1}\varphi(s)f(s,u(s))ds\\
&\quad-\frac{\gamma}{\Gamma(\alpha)\big(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\big)}\\
&\quad\times\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\int_0^{\eta}(\eta-s)^{\alpha+\beta-1}\varphi(s)f(s,u(s))ds\\
&\quad-\frac{\gamma\eta^{\alpha+\beta-1}}{\Gamma(\alpha)\big(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\big)}\int_0^{+\infty}\varphi(s)f(s,u(s))ds\\
&\quad+\frac{\gamma}{\Gamma(\alpha)\big(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1}\big)}\int_0^{\eta}(\eta-s)^{\alpha+\beta-1}\varphi(s)f(s,u(s))ds\Big|.
\end{aligned}$$

So

$$\left|\frac{(Tu)(t)}{1+t^{\alpha-1}}-\lim_{t\to+\infty}\frac{(Tu)(t)}{1+t^{\alpha-1}}\right|\to 0,\quad\text{as}\quad t\to+\infty.$$

Then $T(\overline{C}_r)$ is equiconvergent at infinity. By Lemma **??**, we conclude that $T(\overline{C}_r)$ is relatively compact and then $T:C\to C$ is completely continuous operator. □

Theorem 4.1. *If $(H0),(H2),(H3)$ hold, then, bvp (**??**) has at least one positive solution.*

Proof. Consider the open bounded subsets of X

$$C_b = \{u \in X,\ \|u\|_X < b\} \text{ and } C_r = \{u \in X,\ \|u\|_X < r\}.$$

We check the hypotheses in Theorem **??**. First, we have $\overline{C}_b \subset C_r$ and by Lemma **??**, $T : C \cap \left(\overline{C}_r \setminus C_b\right) \to C$ is completely continuous. If $u \in \partial C_r$, then $\|Tu\|_X \leq \|u\|_X$. Indeed, from $(H3)$ and Remark **??**, for all $u \in \partial C_r$ and $t \in [0, +\infty)$, we have

$$\begin{aligned}
\|Tu\|_X &= \sup_{t\geq 0}\left|\frac{Tu(t)}{1+t^{\alpha-1}}\right| \\
&= \int_0^{+\infty} \sup_{t\geq 0}\frac{G(t,s)}{1+t^{\alpha-1}}\varphi(s)f(s,u(s))ds \\
&\leq mr\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)(\Gamma(\alpha+\beta)-\gamma\eta^{\alpha+\beta-1})}\int_0^{+\infty}\varphi(s)ds = r.
\end{aligned}$$

Hence

$$\|Tu\|_X \leq \|u\|_X,\ u \in C \cap \partial C_r.$$

If $u \in \partial C_b$, then $\|Tu\|_X \geq \|u\|_X$. Indeed, from $(H2)$ we have for any $u \in \partial C_b$ and $t \in [\frac{1}{k}, k]$,

$$\begin{aligned}
\|Tu\|_X &\geq \min_{\frac{1}{k}\leq t\leq k}\left|\frac{Tu(t)}{1+t^{\alpha-1}}\right| \\
&\geq \int_0^{+\infty} \min_{\frac{1}{k}\leq t\leq k}\frac{G(t,s)}{1+t^{\alpha-1}}\varphi(s)f(s,u(s))ds \\
&\geq \int_{\frac{1}{k}}^{k} \min_{\frac{1}{k}\leq t\leq k}\frac{G_1(t,s)}{1+t^{\alpha-1}}\varphi(s)f(s,u(s))ds \\
&> \frac{\lambda(k)b}{\Gamma(\alpha)k^{\alpha-1}m'}\int_{\frac{1}{k}}^{k}\varphi(s)ds = b.
\end{aligned}$$

Hence

$$\|Tu\|_X \geq \|u\|_X,\ u \in C \cap \partial C_b.$$

By the second part of Theorem **??**, T has a fixed point in $C \cap \left(\overline{C}_r \setminus C_b\right)$. Hence bvp (**??**) has at least one positive solution u_0 with $b \leq \|u_0\|_X \leq r$. □

5 Example

Each of Theorem **??**, Theorem **??**, and Theorem **??** is now illustrated by means of a concrete example on the positive half-line.

Example 5.1. *Consider the bvp:*

$$\begin{cases} D_{0^+}^{\frac{3}{2}}u(t) + e^{-\frac{3}{2}t}\left(\left(\frac{u(t)}{10^3(1+\sqrt{t})}\right)^2 + 10^2\right) = 0, \qquad t > 0 \\ I_{0^+}^{\frac{1}{2}}u(0) = 0, \qquad \lim_{t\to+\infty} D_{0^+}^{\frac{1}{2}}u(t) = \frac{1}{2}I_{0^+}^{\frac{1}{2}}u(1). \end{cases} \tag{5.1}$$

Let

$$f(t,u) = e^{-\frac{t}{2}}\left(\left(\frac{u(t)}{10^3\left(1+\sqrt{t}\right)}\right)^2 + 10^2\right),\ \varphi(t) = e^{-t}.$$

In this case $\alpha = \frac{3}{2},\ \beta = \gamma = \frac{1}{2},\ \eta = 1$ *and choose* $b = 10^{-2},\ r = 10^3,\ k = 2$. *So*

$$m = \frac{\Gamma(\alpha)\left(\Gamma(\alpha+\beta) - \gamma\eta^{\alpha+\beta-1}\right)}{\Gamma(\alpha+\beta)\int_0^{+\infty}\varphi(s)ds} = \frac{\sqrt{\pi}}{4} \approx 0.44311346,$$

$$m' = \frac{\lambda(k)\int_{\frac{1}{k}}^{k}\varphi(s)ds}{\Gamma(\alpha)k^{\alpha-1}} = \frac{e^{-\frac{1}{2}} - e^{-2}}{8\sqrt{\pi}\left(2+\sqrt{2}\right)} \approx 0.00973297.$$

Then $f : [0,+\infty)\times[0,+\infty) \to [0,+\infty)$, $\varphi : [0,+\infty) \to (0,+\infty)$ *are continuous and* $\varphi \in L^1[0,+\infty)$. *Moreover*

(H0) $0 < \gamma\eta^{\alpha+\beta-1} < \Gamma(\alpha+\beta)$ *satisfies,* $(0 < \frac{1}{2} < 1)$.

(H2) $f(t,(1+\sqrt{t})u) \geq e^{-1}\left(\left(\frac{1}{10^5}\right)^2 + 10^2\right) > \frac{b}{m'}$ *for* $t \in [\frac{1}{2}, 2]$, $u \geq 10^{-2}$.

(H3) $f(t,(1+\sqrt{t})u) \leq 1 + 10^2 \leq mr$ *for* $t \in [0,+\infty)$, $0 \leq u \leq 10^3$.

Hence all conditions of Theorem **??** *hold, which implies that problem (*__??__*) has at least one positive solution* u_0 *such that* $10^{-2} \leq \|u_0\|_X \leq 10^3$.

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Abstract differential equations of elliptic type with general Robin boundary conditions in Hölder spaces: non commutative cases

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Abstract: we prove some new results on second-order abstract differential equation problems of the elliptic type with operator coefficients with general Robin boundary conditions in a non-commutative setting, i.e. the unlimited linear operator in the equation does not commute with that appearing in the boundary conditions, and space $C^{\theta}([0,1];X)$, such as $0<\theta<1$. The originality of this work lies in the fact that we consider the case where two spectral complex parameters appear in the equation and in the Robin abstract boundar conditions. such that operator H is principal. The study is devloped in Hölder spaces under some new natural assumptions generalizing those in [2]. We give necessary and sufficient conditions on the data to obtain a unique strict solution satisfying the maximal regularity property, see theorem below :This article supplements, in a certain sense, the results of Haoua et al in [10] and can also be seen as a continuation of the results of the authors in [7]. In fact, it is a generalization of what is in [2]. Finally, we will support this work with a concrete example, where the obtained Theorem below is applied.

Keywords: second-order abstract differential equation problems; Robin boundary conditions; analytic semigroup.

1 Introduction and hypotheses

Consider the following abstract second-order differential equation

$$u''(x)+Au(x)-\omega u(x)=f(x), \quad x\in]0,1[, \tag{1}$$

together with the abstract boundary conditions of Robin's type

$$\begin{cases} u'(0)-Hu(0)-\mu u(0)=d_0, \\ u(1)=u_1, \end{cases} \tag{2}$$

where A, H are closed linear operators in a complex Banach space X with domain $D(A), D(H)$ respectively, d_0 , u_1 are given elements in X, ω, μ are complex parametres and the second membre f belongs to $C^{\theta}([0,1];X)$, $0<\theta<1$. We will seek for a strict solution u to (1)−(2), that is a fonction u such that:

$$\begin{cases} u \in C^2([0,1];X) \cap C([0,1];D(A)) \\ u(0) \in D(H_\mu). \end{cases}$$

The methode is essentially based on Dunford calculus, interpolation spaces, the semigroup theory and some techniques as in [2], [7] and [10]. For our study of the problem (1)−(2), we will use the following notation: for $\varphi \in (0,\pi)$, we set

$$S_\varphi = \{z \in \mathbb{C} \setminus \{0\} : \ |\arg(z)| \leqslant \varphi\} \cup \{0\}.$$

We will seek for a strict solution u to this problem, that is a fonction u such that:

$$\begin{cases} u \in C^2([0,1];X) \cap C([0,1];D(A)) \\ u(0) \in D(H_\mu). \end{cases}$$

We now set for $\omega \in S_{\varphi_0}$ and $\mu \in \mathbb{C}$ set:

$$A_\omega = A - \omega I \text{ and } H_\mu = H + \mu I.$$

Our main assumption on the A_ω operator is the following

$$\begin{cases} \exists \varphi_0 \in (0,\pi) : S_{\varphi_0} \subset \rho(A) \text{ and } \exists C_A > 0 : \\ \forall \omega \in S_{\varphi_0}, \|(A-\omega I)^{-1}\|_{\mathcal{L}(X)} \leqslant \dfrac{C_A}{1+|\omega|}. \end{cases} \tag{3}$$

It is well known that assumption (3) implies the same properties, for all $\omega \geqslant \omega_0$, on the other hand it is well known that the square roots $Q = -\sqrt{-A}$ and $Q_\omega = -\sqrt{-A+\omega I}$ are well definied and generate analytic semigroups not strongly contiuous at zero, see Martinez [?]. For non dense domains, note that, $\overline{D(A)} = \overline{D(Q)}$.

since in daily life we encounter many boundary problems with a spectral parameter appearing in the equation and in the boundary conditions appearing in different concrete problems, and since our problem that we are going to study is one of these problems, we We will work to develop an approach slightly different from those used until now, which allow to give an easier verification of the hypotheses and their application to concrete problems. But before that we will mention some of the previous work that is related to this research. In one of his later works. In [2], Cheggag et *al*, they studied the problem (1)−(2) in a commutative framework, when $f \in L^p(0,1;X)$ with $1 < p < \infty$, They considered that the spectral parameter which appears in the boundary conditions is zero, and give interesting results for this problem when X is an UMD space. Where they proved that the problem has a unique classical solution $u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A))$, such as $u(0) \in H$ if and only if d_0, u_1 are in the interpolation space $(D(A),X)_{\frac{1}{2p}+\frac{1}{2},p}$, $(D(A),X)_{\frac{1}{2p},p}$respectively. The same authors, in [3] studied the problem (1)−(2) but this time, in the absence of the two spectral parameters $(\omega,\ \mu = 0)$, in the same commutative frame, in the same commutative setting and in a Hölder space. In other words, they assume that f belongs to $C^{\theta}([0,1];X)$ with $\theta \in]0,1[$, and under certain assumptions about the operator A, the authors studied existence, uniqueness and maximal regularity and then give some positive results for this problem, where they show that the problem $(1)-(2)$ has a unique strict solution $u \in C^{2}([0,1];X) \cap C([0,1];D(A))$ such as $u(0) \in H$, satisfying the maximal regularity property u'', $Au \in C^{\theta}([0,1];X)$, if and only if $u_1 \in D(A)$, $d_0 \in D(Q)$ and Qd_0, $f(0), -Au_1 + f(1)$ are in the interpolation space $(D(Q), X)_{1-\theta,\infty}$, with $Q = -\sqrt{-A}$. In [8] The authors, studied the problem $(1)-(2)$ in non-commutative setting, when $f \in L^{p}(0,1;X)$ with $1 < p < \infty$. and when X is an UMD space, they studied the problem in two cases:

First case: $D(H) \subset D(A)$ and X is an UMD space. the authors they found important resultas as for the proposed problem, as they proved that under these conditions, the problem $(1)-(2)$ has a unique classical solution $u \in W^{2,p}(0,1;X) \cap L^{p}(0,1;D(A))$, such as $u(0) \in H_{\mu}$ if and only if u_1 is in the interpolation space $(D(A), X)_{\frac{1}{2p},p}$, moreover this solution u satisfying the maximal regularity property $(1+|\omega|)u$, u'', $Q_{\omega}^{2}u \in L^{p}(0,1;X)$ with $1 < p < \infty$.

Second case: $D(Q) \subset D(H)$. In this case, the authors also found very interesting results, where they proved that the problem $(1)-(2)$ has a unique classical solution if and only if $u_{1,}$ $(Q_{\omega} - H_{\mu})^{-1} d_0$ are in the interpolation space $(D(A), X)_{\frac{1}{2p},p}$, moreover, on the same assumption, this solution u satisfying the maximal regularity property $(1+|\omega|)u$, u'', $Q_{\omega}^{2}u \in L^{p}(0,1;X)$ with $1 < p < \infty$. The authors, in [7] have studied this problem when (2) reduces to the Dirichlet boundary conditions $u(0) = u_0$ and $u(1) = u_1$. they also found positive results for the problem $(1)-(2)$ which, the problem $(1)-(2)$ has a unique classical solution if and only if $u_0, u_{1,}$ are in the interpolation space $(D(A), X)_{\frac{1}{2p},p}$, and this solution u satisfying the maximal regularity property $(1+|\omega|)u$, u'', $Q_{\omega}^{2}u \in L^{p}(0,1;X)$ with $1 < p < \infty$. In this case, in [10], Haoua et *al.* they also found good results regarding the problem $(1)-(2)$ for $\mu = 0$, $f \in C^{\theta}([0,1];X)$ with $\theta \in]0,1[$ and under some hypotheses on operators A and H where they said that this problem has a unique strict solution $u \in C^{2}([0,1];X) \cap C([0,1];D(A))$ such as $u(0) \in H$, if and only if

$$\begin{cases} (Q_{\omega} - H)^{-1} d_0 \in D(Q^2) \\ Q^2 (Q_{\omega} - H)^{-1} (d_0 - HQ^{-2} f(0)) \in \overline{D(Q)} \\ u_1 \in D(Q^2) \\ Q^2 u_1 + f(1) \in \overline{D(Q)}. \end{cases}$$

And this solution u satisfying the maximal regularity property u'', $A_{\omega}u \in C^{\theta}([0,1];X)$,if and only if

$$\begin{cases} (Q_{\omega} - H)^{-1} d_0 \in D(Q^2) \\ Q^2 (Q_{\omega} - H)^{-1} (d_0 - HQ^{-2} f(0)) \in D_Q(\theta, +\infty) \\ u_1 \in D(Q^2) \\ Q^2 u_1 + f(1) \in D_Q(\theta, +\infty). \end{cases}$$

2 Main Result

We obtain the following theorem:
Theorem: Assume (3). Let $f \in C^{\theta}([0,1];X)$, with $0 < \theta < 1$ and $d_0, u_1 \in X$. Then, for any $\omega \geqslant \omega_1^*$, we have

1. Problem (1)−(2) has a unique strict solution u if and only if

$$\begin{cases} (Q_\omega - H_\mu)^{-1}\left[d_0 - Q_\omega^{-1} f(0)\right] \in D(Q^2) \\ Q_\omega^2 (Q_\omega - H_\mu)^{-1}\left[d_0 - Q_\omega^{-1} f(0)\right] + f(0) \in \overline{D(Q)} \\ u_1 \in D(Q^2) \\ Q_\omega^2 u_1 + f(1) \in \overline{D(Q)} \end{cases}$$

2. Problem (1)−(2) has a unique strict solution u satisfying the maximal regularity property u'', $A_\omega u \in C^{\theta}([0,1];X)$ if and only if

$$\begin{cases} (Q_\omega - H_\mu)^{-1}\left[d_0 - Q_\omega^{-1} f(0)\right] \in D(Q^2) \\ Q_\omega^2 (Q_\omega - H_\mu)^{-1}\left[d_0 - Q_\omega^{-1} f(0)\right] + f(0) \in D_Q(\theta; +\infty) \\ u_1 \in D(Q^2) \\ Q_\omega^2 u_1 + f(1) \in D_Q(\theta; +\infty) \end{cases}$$

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NEWTON'S BINOMIAL THEOREM APPLICATION FOR COMPUTE A SQUARE ROOT OF TRIANGULAR TOEPLITZ MATRICES

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Abstract In this paper, we present with the proof a formulat to compute the entries of matrix square root of triangular Toeplitz matrix by using the Newton's Binomial Theorem. Also, we show that the square root of this type of matrices has the same form. Some applications are given to illustrate.

Key words and phrases: Triangular matrix, triangular Toeplitz matrix, root, Newton's Binomial Theorem.

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The stability and stabilization of some generic linear second order time-invariant retarded system with single delay.

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Abstract

We focuses on the stability and stabilization of some generic linear second order time-invariant retarded system with single delay. It provides an appropriate stability criterion based on the manifold defined by the coexistence of the maximal number of negative spectral values.

In the particular case of equidistributed real roots the argument principle is applied to prove the dominancy of such real spectral values.

Keywords: Time-delay systems, Stability, Spectral abscissa.

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Some results on higher orders quasi-symmetric

1 Abstract

The purpose of the present paper is to pursue further study of a class of linear bounded operators, known as n-quasi-m-symmetric operators acting on an infinite complex separable Hilbert space H.We give an equivalent condition for any T to be n-quasi-m-symmetric operator. Using this result we prove that any power of an n-quasi-m-symmetric operator is also an n-quasi-m-symmetric operator. In general the converse is not true.We study the sum of an n-quasi-m-symmetric operator with a nilpotent operator. We also study the product and tensor product of two n-quasi-m symmetrics.

Further, we define n-quasi strict m-symmetric operators and prove their basic properties.

Keywords :m-symmetric, strict m-symmetric, n-quasi-m-symmetric

2 Introduction

Throughout this paper, $\mathbb{N}$ denotes the set of non negative integers, $\mathcal{H}$ stands for an infinite separable complex Hilbert space with inner product $\langle .|. \rangle$,$\mathcal{L}(\mathcal{H})$ is the Banach algebra of all bounded linear operators on $\mathcal{H}$ and $I = I_{\mathcal{H}}$the identity operator. For every $T \in \mathcal{L}(\mathcal{H})$ we denote by$R(T)$,$N(T)$and T^* the range, the null space and the adjoint of T, respectively. A closed subspace $M \subset \mathcal{H}$ is invariant for T (or T-invariant) if $TM \subset M$. As usual, the orthogonal complement and the closure of M are denoted $M^{\perp}$and $\overline{M}$, respectively. We denote by$P_M$the orthogonal projection on M.

In [2] Helton initiated the study of m-symmetric operator, for a positive integer m, an operator $T \in\mathcal{L}(\mathcal{H})$ is said to be m-symmetric if

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j} T^{j} = 0$$

Hence T is 1-symmetric if and only if T is selfadjoint. It is well known that if T is m-symmetric, then T is n-symmetric for all$n \geq m$.

Recently, Fei ZUO and Salah MECHERI [5] introduced the class of n-quasi-m-symmetric operators which generalizes the class of$m$-symmetric operators . For positive integers m and n, an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an n-quasi-m-symmetric operator if

$$T^{*n} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j} T^{j} \right) T^n = 0$$

3 Some properties of n-quasi-m-symmetric operators

After an introduction on the subject and some connection with known facts in this context, the results of the paper are briefly described.We study several properties of n-quasi-m-symmetrics.In particular ,we prove that if T and S are doubly commuting such that T is an n_1-quasi-m-symmetric and S is an n_2-quasi-l-symmetric, then TS is a $n_0 = \max\{n_1, n_2\}$-quasi-$(m + l - 1)$-symmetric. It has also been proved that the sum of an n-quasi-m-symmetric and a commuting nilpotent operator of degree p is a $(n + p)$-quasi-$(m + 2p - 2)$-symmetric.

4 n-quasi strict-m-symmetric

In this section , we recall the definition of n-quasi strict-m-symmetric and we give some of their properties which are similar to those of n-quasi-m-symmetrics.

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Random Solution For Non-local Generalized Caputo Periodic Value Problem

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Abstract. In this article, we present some results on existence and uniqueness of a class of problems for nonlinear implicit fractional differential equations with non-local and periodic conditions. Our results are based on Banach and Krasnoselskii fixed point theorems. We apply our results to a numerical problem for more clarity.

Keywords: ....
Mathematics Subject Classification: 34A08, 34K05.

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Topic: Applied mathematics

Synchronization of a novel fractional order system and the fractional Lotka-Volterra system using FSHPS

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Abstract: In this work a new $3-D$ fractional chaotic system is introduced. Basics dynamical characteristics are studied such as Lyapunov exponent spectrum, Kaplan-Yorke dimension, equilibrium points and their stability. Furthermore, the necessary conditions for existing chaos for the proposed system in the commensurate ordre are given. Also, in ordre to synchronize the proposed fractional chaotic system and the fractional Lotka-Volterra system, Full State Hybrid Projective Synchronization (FSHPS) is successfully applied. The results are validated by numerical simulation using Matlab.
Keywords: Fractional order system, Lyapunov exponent, FSHPS, Synchronization.

1 Introduction

Synchronization is a phenomenon that is so abundant in nature that we can find between two or more systems. The fractional chaotic systems have applications in different fields, including cryptosystems , encryption, secure communication schemes.
In this research, we present a novel chaotic system and its elementary properties. Also, FSHPS between the novel fractional system and the fractional Lotka-voltera system are implemented. Using Matlab, a numerical simulation are presented to show the effectiveness of the theoretical results.

2 Dynamical behavior

2.1 Description of the Novel 3-D Autonomous fractional Chaotic System

The novel 3D autonomous fractional chaotic system expressed as follows:

$$\begin{cases} {}_0^C D_t^{q_1} x(t) = a(y - x), \\ {}_0^C D_t^{q_2} y(t) = cx - y - xz, \\ {}_0^C D_t^{q_3} z(t) = xy + b(y - z). \end{cases} \tag{1}$$

where x, y, z are the state variables and a, b, c are positive reals paramaters , and D^q is the Caputo fractional derivative.
The strange attractor is depicted in the figure 1.

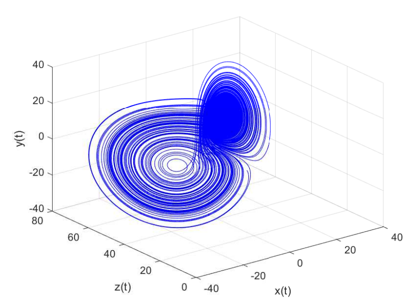

Figure 1: The strange attractor of the new system (1) when $a = 10, b = 2, c = 50$

2.2 Lyapunov exponent of the novel system

Lyapunov exponents are used to measure the exponential rates of divergence and convergence of nearby trajectories, which is an important characteristics to judge if the system exhibit chaotic behavior or not. The existence of at least one positive Lyapunov exponent implies that the system is chaotic.
For $a = 10, b = 2, c = 50$ the novel system have three Lyapunov exponent:

$$L_1 = 0.9853, L_2 = -0.0052 \simeq 0 \text{ and } L_3 = -14.2468 \tag{2}$$

We have $L_1 = 0.9853 > 0$ and $L_1 + L_2 + L_3 < 0$ then the new system is chaotic.

2.3 Kaplan-York dimension :

Among the characteristics of chaotic systems the dimension of the attractor is fractal , so the Kaplan-Yorke dimension of the novel system is calculated as:

$$D_{KL} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.06879. \tag{3}$$

which is fractal.

2.4 Equilibrium points stability

For $a = 10, b = 2, c = 50$ the fixed points of the novel system are:

$$P_1 = \Big(0;0;0\Big), P_2 = \Big(3\sqrt{11}-1;3\sqrt{11}-1;49\Big);$$
$$\text{and } P_3 = \Big(-3\sqrt{11}-1;-3\sqrt{11}-1;49\Big). \tag{4}$$

For P_0, we obtain the eigenvalues:

$$\lambda_1 = -2, \lambda_2 = 17.309, \lambda_3 = -28.309 \tag{5}$$

This implies that P_0 is an unstable saddle point. For P_1, we obtain the eigenvalues:

$$\lambda_1 = -13.717, \lambda_2 = 0.35858 - 11.389i, \lambda_3 = 0.35858 + 11.389i \tag{6}$$

For P_2 , we obtain the eigenvalues:

$$\lambda_1 = -14.822, \lambda_2 = 0.9111 - 12.09i, \lambda_3 = 0.9111 + 12.09i \tag{7}$$

Then P_1 and P_2 are two unstable saddle-focus points because λ_1 and λ_2 are complex and none of the eigenvalues have real part zero.

3 Problem Formulation

We consider the drive system given by:

$$\dot{x}_i(t) = f_i(X(t)), i = 1,\dots,n$$

Where: $X(t) = (x_1, x_2, \dots, x_n)^T$ is the state vector of the system (1), $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ for $i = 1,..,n$ are nonlinear functions, and as response system the system given by:

$$D_t^{q_i} y_i(t) = \sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)) + V_i, i = 1,\dots,n$$

Where: $Y(t) = (y_1, y_2, \dots, y_n)^T$ is the state vector of the system (2), $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ for $i = 1,..,n$ are nonlinear functions, $0 < q_i < 1, D_t^{q_i}$ is the Caputo fractionnal derivative of order q_i for $i = 1,..,n, V_i$ are controllers to be designed such as the system (1) and the system (2) to be synchronized.

4 Synchronization between the novel fractional order system and the fractional Lotka-Volterra system using FSHPS

In this section the synchronization behavior between the novel fractional system and the fractional Lotka-Voltera system is made using a suitable control. We assume that the fractional Lotka-Voltera system derives the novel fractional system.
Consider the master (drive) system given by:

$$\begin{cases} {}_0^C D_t^{q_1} x(t) = ax - bxy + ex^2 - szx^2, \\ {}_0^C D_t^{q_2} y(t) = -cy + dxy, \\ {}_0^C D_t^{q_3} z(t) = -pz + szx^2. \end{cases} \tag{8}$$

Now, we consider the system (9) system as a slave system as follows:

$$\begin{cases} {}_0^C D_t^{q_1} x(t) = a(y - x), \\ {}_0^C D_t^{q_2} y(t) = cx - y - xz, \\ {}_0^C D_t^{q_3} z(t) = xy + b(y - z). \end{cases} \tag{9}$$

After applying the algorithm of full state hybrid projective syncronization we obtain the figure of error system given by:

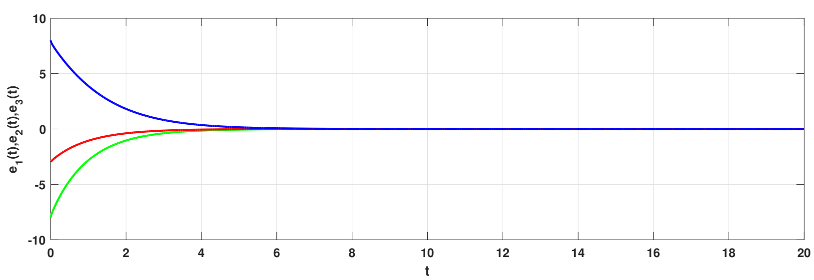

Figure 2: The evolution of the error functions $e_1(t), e_2(t), e_3(t)$

5 Conclusion

In this work, a new fractional chaotic system is introduced. Basic properties of this system are studied, namely, the equilibrum points and their stability, the Lyapunov exponent and the Kaplan-Yorke dimension, commensurate order for existing chaos. Moreover, a suitable control schemes have been applied to synchronize the proposed fractional chaotic system and the fractional Lotka-Voltera system by FSHPS. Numerical simulations using MATLAB have been made to illustrate our results.

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Global Uniqueness Results for Partial Hadamard Fractional Integral Equations

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Abstract In this work we investigate the existence and uniqueness of solutions of partial Integral Equations via Hadamard's fractional integral, by applying a nonlinear alternative of Leray-Schauder due to Frigon and Granas for contraction maps on Fréchet spaces.

Key words and phrases: Partial functional integral equation, fractional order, solution, Hadamard's fractional integral, fixed point, Fréchet spaces.

1 Introduction

In this work we provide sufficient conditions for the global existence and uniqueness to the following Hadamard fractional partial integral equation of the form

$$u(t,x) = z(t,x)$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^t\int_1^x\left(\log\frac{t}{s}\right)^{r_1-1}\left(\log\frac{x}{\tau}\right)^{r_2-1}\frac{f(s,\tau,u(s,\tau))}{s\tau}d\tau ds \quad \text{if } (t,x)\in J, \tag{1}$$

where $J := [1,\infty)\times[1,\infty)$, $(r_1,r_2)\in\,]0,1]\times]0,1]$, $z : J\to\mathbb{R}$, $f : J\times\mathbb{R}\to\mathbb{R}$ are given continuous functions.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.
Let $n\in\mathbb{N}$ and $J_0 = [1,n]\times[1,n]$. By $C(J_0,\mathbb{R}^n)$ we denote the Banach space of all continuous functions from J_0 into $\mathbb{R}^n$ with the norm

$$\|u\|_\infty = \sup_{(t,x)\in J_0}\|u(t,x)\|,$$

where $\|\cdot\|$ denotes a suitable complete norm on $\mathbb{R}^n$.
As usual, by $AC(J_0,\mathbb{R}^n)$ we denote the space of absolutely continuous functions from J_0 into $\mathbb{R}^n$ and $L^1(J_0,\mathbb{R}^n)$ is the space of Lebesgue-integrable functions $u : J_0\to\mathbb{R}^n$ with the norm

$$\|u\|_{L^1} = \int_1^n\int_1^n\|u(t,x)\|dtdx.$$

Definition. [4] The Hadamard fractional integral of order $r > 0$ for a function $u\in L^1([1,n],\mathbb{R}^n)$, is defined as

$$({}^HI_1^r u)(x) = \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\frac{u(s)}{s}ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition. [4] Let $r_1, r_2 \geq 0, \sigma = (1,1)$ and $r = (r_1, r_2)$. For $u \in L^1(J, \mathbb{R}^n)$.

define the Hadamard partial fractional integral of order r by the expression

$$({}^H I_1^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{\tau}\right)^{r_2-1} \frac{u(s,\tau)}{s\tau} d\tau ds.$$

In the sequel we will make use of the following variant of the inequality for two independent variables due to Pachpatte.

Lemma. [5] Let $\omega \in C(J, \mathbb{R}_+), q, D_1 q, D_2 q, D_1 D_2 q \in C(J_1, \mathbb{R}_+)$ and $c > 0$ be a constant. If

$$\omega(t,x) \leq c + \int_1^t \int_1^x q(t,x,s,\tau)\omega(s,\tau) d\tau ds,$$

for $(t,x) \in J$, then

$$\omega(t,x) \leq c \exp\left(\int_1^t \int_1^x B(s,\tau) d\tau ds\right),$$

where

$$B(t,x) = q(t,x,t,x) + \int_1^t D_1 q(t,x,s,x) ds$$
$$+ \int_1^x D_2 q(t,x,t,\tau) d\tau + \int_1^t \int_1^x D_1 D_2 q(t,x,s,\tau) d\tau ds.$$

Definition. [3] Let X be a Fréchet space. A function $N : X \longrightarrow X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in [0,1)$ such that

$$\|N(u) - N(v)\|_n \leq k_n \|u - v\|_n \quad \text{for all } u, v \in X.$$

Theorem. (Nonlinear alternative of Leray-Schauder due to Frigon-Granas type) [3] Let X be a Fréchet space and $Y \subset X$ a closed subset in X. Let $N : Y \longrightarrow X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements holds :

(a) the operator N has a unique fixed point ;

(b) there exists $\lambda \in [0,1)$, $n \in \mathbb{N}$ and $u \in \partial_n Y^n$ such that $\|u - \lambda N(u)\|_n = 0$.

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Interesting solution in the form of numerical series in Hilbert spaces of Sylvester operator equation $AXB - CXD = E$

Abstract: It's well know the solution and some properties of the linear equation $AXB - CXD = E$ is developed in the finite dimensional case. In this paper, we will extend it to infinite dimension by using a similar technique developed in the finite dimension case, the infinite dimension case of the Sylvester equation, the convergence of the solution under the conditions $A = \sum_{n=1}^{\infty} \lambda_n P_n$ and $\sigma(A, C) \cap \sigma(B, D) = \phi$.
Keywords: Solve the operator equation as a series, the spectrum $\sigma(A, C) \cap \sigma(B, D)$, infinite matrix.

1 Orthogonal projectors

Theorem 1.1 *(see [4]) If M is a closed linear subspace of $\mathcal{H}$ and $h \in \mathcal{H}$, let Ph be the unique point in M such that $h - Ph \perp M$. Then*

(a) P is a linear transformation on $\mathcal{H}$,

(b) $\| Ph \| \leq \| h \|$ for every h in $\mathcal{H}$,

(c) $P^2 = P$ (here P^2 means the composition of P with itself),

(d) $\ker P = M^{\perp}$ and $\operatorname{ran} P = M$.

Definition 1.1 *(see [18]) A sequence $\{X_n : n \in \mathbb{N}\}$ in $B(\mathcal{H})$ is said to converge in the **strong operator topology** (SOT) if $\{X_n x : n \in \mathbb{N}\}$ converges in the norm of $\mathcal{H}$ for every $x \in \mathcal{H}$. It is a consequence of the " uniform boundedness principle" that in this case, the equation*

$$Xx = \lim_{n \to +\infty} X_n x$$

defines a bounded operator $X \in B(\mathcal{H})$. We shall abbreviate all this by writing

$$X_n \xrightarrow{SOT} X.$$

We record a couple of simple but very useful facts concerning SOT convergence.

Lemma 1.1 *1. The following conditions on a sequence $\{X_n : n \in \mathbb{N}\} \subset B(\mathcal{H})$ are equivalent:*

- *$X_n \xrightarrow{SOT} X$ for some $X \in B(\mathcal{H})$;*

- $\sup_n \|Xn\| < \infty$, *and there exists some total set* $S \subset \mathcal{H}$ *such that*

$$X_n x \to Xx \quad \forall x \in S;$$

- $\sup_n \|X_n\| < \infty$, *and there exists a dense subspace* $M \subset \mathcal{H}$ *such that*

$$X_n x \to Xx \quad \forall x \in M.$$

2. *If sequences* $X_n \xrightarrow{SOT} X$ *and* $Y_n \xrightarrow{SOT} Y$ *, then also* $X_n Y_n \xrightarrow{SOT} XY$*.*

2 The spectrum of an operator

Definition 2.1 *(see [7]) Let* $\mathcal{H}$ *be a vector space on* $\mathbb{C}$ *and* T *an endomorphism of* $\mathcal{H}$*. We call Spectrum of* T *the set* $\sigma(T) = \{\lambda \in \mathbb{C} / \ T - \lambda I \ \text{ is not invertible}\}$*.*

Definition 2.2 *(see [14]) For a pair* (A, B) *of operators in* $B(\mathcal{H})$*, the spectrum* $\sigma(A, B)$ *of the linear operator pencil* $(A - \lambda B)$*, or of the pair* (A, B)*, is defined by:*

$$\begin{aligned} \sigma(A, B) &= \{\lambda \in \mathbb{C} \text{ such that } (A - \lambda B) \ \text{ is not invertible}\} \\ &= \{\lambda \in \mathbb{C} \text{ such that } 0 \in \sigma(A - \lambda B)\}. \end{aligned}$$

Theorem 2.1 *(see [4]) Let* $T \in B(\mathcal{H})$ *be a compact self-adjoint operator. Then there exists a sequence* (λ_n) *of real numbers tending to 0, and an orthonoraml family* (e_n) *in* $\mathcal{H}$ *such that, if* P_n *denotes the projection on Vect* e_n*,*

$$T = \sum_{n=1}^{\infty} \lambda_n p_n,$$

the convergence taking place in the sense of the operator norm.

For proving Theorem 2.1, we need some general results.

Theorem 2.2 *If* A*,* B*,* C *and* D *are operators in* $B(\mathcal{H})$ *if* $\sigma(A, C) \cap \sigma(B, D) = \phi$ *then the equation*

$$AXB - CXD = E$$

has a unique solution X*, for every operator* E*.*

3 Main results

Theorem 3.1 *Let* A*,* B*,* C*,* D *and* E *be bounded operators on the complex Hilbert space* $\mathcal{H}$*. Let* A *be a compact normal operators on* $\mathcal{H}$*. Let* B*,* C *and* D *be a normal operators on* $\mathcal{H}$ *such that* $A = \sum_{i=1}^{\infty} \alpha_i p_i$*,* $B = \sum_{j=1}^{\infty} \beta_j q_j$*,* $C = \sum_{k=1}^{\infty} \gamma_k f_k$

and $D = \sum_{l=1}^{\infty} \xi_l h_l$ *are orthogonal sums of projections such that* $\sigma(A, C) \bigcap \sigma(B, D) = \emptyset$ *then*

$$X = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{f_k p_i E q_j h_l}{\alpha_i \beta_j - \gamma_k \xi_l}$$

is the unique solution of equation $AXB - CXD = E$*. Here the convergence in the sum is in SOT.*

proof. Write $\sigma(A) = \{\lambda_0, \lambda_1, ...\}$, with $\lambda_0 = 0$, and $A = \sum_{i=1}^{\infty} \alpha_i p_i$. Let $p_0 = I - \sum_{i=1}^{\infty} p_i$ is the projection of onto to kernel of A.

$$AXB - CXD = E \tag{3.1}$$

Then Eq (3.1) implies

$$(\sum_{i=1}^{\infty} \alpha_i p_i)X\big(\sum_{j=1}^{\infty} \beta_j q_j\big) - (\sum_{k=1}^{\infty} \gamma_k f_k)X\big(\sum_{l=1}^{\infty} \xi_l h_l) = E$$

Multiplying from the left by p_v and multiplying from the right by q_s

$$\alpha_v \beta_s p_v X q_s - p_v(\sum_{k=1}^{\infty} \gamma_k f_k)X\big(\sum_{l=1}^{\infty} \xi_l h_l)q_s = p_v E q_s$$

Using the fact that if two matrices commute, we can be written as

$$\alpha_v \beta_s p_v X q_s - \sum_{k=1}^{\infty} \gamma_k f_k p_v X q_s \sum_{l=1}^{\infty} \xi_l h_l = p_v E q_s$$

Now multiply from the left by f_u and multiply from the right by h_r which results in

$$\alpha_v \beta_s f_u p_v X q_s h_r - \gamma_u \xi_r f_u p_v X q_s h_r = f_u p_v E q_s h_r \tag{3.2}$$

Then equation (3.2) implies

$$(\alpha_v \beta_s - \gamma_u \xi_r) f_u p_v X q_s h_r = f_u p_v E q_s h_r$$

If $\alpha_v \beta_s - \gamma_u \xi_r = 0$ implied $\alpha_v \beta_s = \gamma_u \xi_r$, this gave $f_u p_v E q_s h_r = 0$ so

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(\alpha_i \beta_j - \gamma_k \xi_l) f_k p_i X q_j h_l = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} f_k p_i E q_j h_l$$

If $\sigma(A,C)\bigcap\sigma(B,D)=\emptyset$ there is a unique solution

$$X=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}f_kp_iXq_jh_l=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{f_kp_iEq_jh_l}{\alpha_i\beta_j-\gamma_k\xi_l}$$

We have

$$X=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{f_kp_iEq_jh_l}{\alpha_i\beta_j-\gamma_k\xi_l}$$

then

$$\begin{aligned}\| X \| &\leq \sup_{i,j,k}\left|\frac{1}{\alpha_i\beta_j-\gamma_k\xi_l}\right|\|f_kp_iEq_jh_l\|\\ &\leq \frac{1}{\delta}\|f_kp_iEq_jh_l\|\\ &= \frac{1}{\delta}\|E\|\end{aligned}$$

where $\delta=\inf\{|\alpha\beta-\gamma\xi|:\ \alpha\in\sigma(A),\ \beta\in\sigma(B),\ \gamma\in\sigma(C)$ and $\xi\in\sigma(D)\}$ ■

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Fixed Point Theory in Complete Ordered Semi-Normed Space

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Operator theory and EDP: foundations and applications

Key words: Order, semi-normed space, monotonic function.

Abstract

The fixed point theory is one of the most useful tools in analysis, which is the subject of much research. Indeed, it offer a very effective set of tools in the study of existence and uniqueness of problems formulated by a system of differential or integral equations, resulting from the modelling of various research works in physic, chemistry, biology, ect(see [2], [3], [6]).

The most famous of those theorems is Banach contraction principle, which can only have meaning in a metric space and the function must be contractive. But the both conditions don't verify in many problems; For that, the research generalize it to a non metric topological space. One of them based on the order in Banach space. In a recent paper, Dhage [1] proved some common fixed point theorems for pairs of condensing mappings in an ordered Banach space. More recently, Hussain et al. [4] extended the results of Dhage to 1-set contractive mappings. After that, P.Agarwal et Hussain proved some theorem under weaker assumptions [5].

In this work, we tries extend some theory generally defined in Banach ordered space to spaces don't have a Banach structure (complete semi normed spaces), where we depend on the monotony of function, with the weakness of conditions.The use of the concepts of weak compactness and non-metric increases the usefulness of our results in many practical situations especially when we work in non-reflexive complete semi-normed spaces.

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A new study on the mathematical modelling of coupled system of fractional differential equation with Caputo-Fabrizio fractional derivative

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Abstract: This work is concerned a coupled system of linear fractional differential equations of Caputo-Fabrizio type conformable fractional derivation with boundary conditions. In order to prove the existence and uniqueness of solution, the problem is transformed into an equivalent linear Volterra-Fredholm integral equations of the second kind, and by using the Banach's fixed-point theory the existence and uniqueness of solutions is obtained. Finally, the analytical results are supported by numerical results to illustrate of obtained results.

Keywords: Caputo-Fabrizio Fractional Derivative, Fractional integral, Coupled system of fractional differential equations, Fixed point theorems, Adomian decomposition method (ADM).

Introduction

In this paper, We study a coupled system of linear fractional differential equations, as follows:

$$\begin{cases} \mathscr{D}^{(\rho)}\, u(x) = c_1 u(x) + c_2 v(x) + f(x), & x \in I := [0,1] \\ \mathscr{D}^{(\rho)}\, v(x) = c_3 u(x) + c_4 v(x) + g(x), & x \in I := [0,1] \\ u(0) = u(1) = 0,\ v(0) = v(1) = 0 \end{cases} \tag{0.1}$$

where $1 < \rho < 2$ is a real number, $\mathscr{D}^{(\rho)}$ is the new fractional derivative of Caputo-Fabrizio, $f, g : [0,1] \to \mathbb{R}$ are continuous function, and c_i real constants and $i = 1, 2, 3, 4$, and we present some basic materials needed to prove our main results, and obtain the equivalent linear Volterra-Fredholm integral equation of the second kind by using the fractional integral in problem (0.1). Finally, the existence and uniqueness of solution is established by applying Banach's contraction mapping principle, Adomin method and algorithm are introduced to solve the numerical solution of this class of problem.

1 Preliminaries

First, we give the necessary definitions and lemmas from fractional calculus theory.

Definition 1.1. [4] Let $\Omega = [a,b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The left-sided CF-FD ${}^{CF}\mathscr{D}^{\alpha}_{a^+}h$ of order $\alpha \in [0,1[$ of a function h is defined as follows:

$$\mathscr{D}^{(\alpha)}\, h(x) = \frac{M(\alpha)}{1-\alpha} \int_a^x h'(s) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s \tag{1.1}$$

where $\alpha \in [0,1]$ and $a \in]-\infty, x)$, $h \in H^1(a,b)$, $b > a$, and $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$.

Definition 1.2. [4] Let $n \geq 1$, and $\alpha \in [0,1]$ the fractional derivative $\mathscr{D}^{(\alpha+n)}h$ of order $(n+\alpha)$ is defined by

$$\mathscr{D}^{(\alpha+n)}h(x) := \mathscr{D}^{(\alpha)}_x\left(\mathscr{D}^{(n)}h(x)\right) = \frac{M(\alpha)}{1-\alpha} \int_a^x h^{(n+1)}(s) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s.$$

Such that

$$\mathscr{D}^{(\alpha+n)}h(t) = \frac{M(\alpha)}{1-\alpha} \int_a^x h^{(n+1)}(s) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s. \tag{1.2}$$

Definition 1.3. [6] Let $n \geq 1,\ \alpha \in [0,1]$, and $h \in \mathcal{C}^1[a,b]$. The formula:

$$I_a^{n+\alpha} h(x) = \frac{1}{M(\alpha)\cdot n!}\int_a^x (x-s)^{n-1}[\alpha(x-s)+n(1-\alpha)]h(s)\,\mathrm{d}s$$

where $M(\alpha)$, is a normalization function such that $M(0) = M(1) = 1$ is a new fractional integral of order $(n+\alpha)$, and it's as an inverse of the conformable fractional derivative of Caputo of order $(n+\alpha)$.

Lemma 1.1. *[6] Let $\rho \in (n, n+1), n = [\rho] \geqslant 0$. Assume that $h \in \mathcal{C}^n[a,b]$, then those statements hold:*

1. *if $h(a) = 0$, then $\mathscr{D}^{(\rho)}\ (I_a^{\rho} h(x)) = h(x)$.*
2. $I_a^{\rho}\left(\mathscr{D}^{(\rho)} h(x)\right) = h(x) + \sum_{i=0}^{n} a_i x^i,\ a_i \in \mathbb{R}\ i = 0,1,\ldots,n.$

2 Analytic Study

In the following, we suppose the function $M(\alpha) = 1$.

Lemma 2.1. *Let $1 < \rho < 2,\ u,v \in \mathcal{C}^1[0,1],\ f,g : [0,1] \to \mathbb{R}$ are continuous function, and c_i real constants and $i = 1,2,3,4$. Then the solution of (0.1) satisfies the following linear Volterra integral equations of the second kind*

$$u(x) = \int_0^x L(x,s)(c_1u(s)+c_2v(s))\,\mathrm{d}s + \int_0^1 F(x,s)(c_1u(s)+c_2v(s))\,\mathrm{d}s + K(x) \tag{2.1}$$

$$v(x) = \int_0^x L(x,s)(c_3u(s)+c_4v(s))\,\mathrm{d}s + \int_0^1 F(x,s)(c_3u(s)+c_4v(s))\,\mathrm{d}s + G(x). \tag{2.2}$$

where $L(x,s) = \alpha(x-s)+1-\alpha,\ F(x,s) = x(\alpha s - 1),\ K(x) = \int_0^x L(x,s)f(s)\,\mathrm{d}s + \int_0^1 F(x,s)f(s)\,\mathrm{d}s$, and $G(x) = \int_0^x L(x,s)g(s)\,\mathrm{d}s + \int_0^1 F(x,s)g(s)\,\mathrm{d}s$.

Existence and uniqueness of the solution

Let us introduce the space $\mathcal{C}([0,1],\mathbb{R})$ endowed with the norm $\parallel u \parallel = \sup_{x\in[0,1]}|u(x)|$.
Obviously $(\mathcal{C}([0,1],\mathbb{R}), \parallel . \parallel)$ is a Banach space. Denote by $\Lambda = \mathcal{C}([0,1],\mathbb{R}) \times \mathcal{C}([0,1],\mathbb{R})$. Then, the product space $(\Lambda, \parallel . \parallel)$ is also a Banach space endowed with the norm $\parallel (u,v) \parallel = \parallel u \parallel + \parallel v \parallel = \sup_{x\in[0,1]}|u(x)| + \sup_{x\in[0,1]}|v(x)|$, for $(u,v) \in \Lambda$.
In view of Lemma 2.1, we introduce an operator $T : \Lambda \longrightarrow \Lambda$ associated with the problem (0.1) as follows

$$T(u,v)(x) := (T_1(u,v)(x), T_2(u,v)(x)), \tag{2.3}$$

$$T_1(u,v)(x) = \int_0^x L(x,s)(c_1u(s)+c_2v(s))\,\mathrm{d}s + \int_0^1 F(x,s)(c_1u(s)+c_2v(s))\,\mathrm{d}s + K(x), \tag{2.4}$$

and

$$T_2(u,v)(x) = \int_0^x L(x,s)(c_3u(s)+c_4v(s))\,\mathrm{d}s + \int_0^1 F(x,s)(c_3u(s)+c_4v(s))\,\mathrm{d}s + G(x). \tag{2.5}$$

Here we establish the existence of the solutions for the boundary value problem (0.1) by using Banach's contraction mapping principle.

Theorem 2.1. *Let $f,g : I \to \mathbb{R}$ are jointly continuous function. Then the problem (0.1) has a unique solution on I if*

$$\lambda + \varepsilon < \frac{1}{2-\alpha}, \tag{2.6}$$

where $\lambda = \max(|c_1|,|c_2|),\ \varepsilon = \max(|c_3|,|c_4|)$.

3 Numerical study

In this section, we introduce an algorithm for finding a numerical solution of linear Volterra integral equations of the second kind, the methods based the Adomian Decomposition.
In the decomposition method we usually express the solution $u(x)$ and $v(x)$ of the integral equation (2.1) and (2.2) in a series form defined by

$$u(x) = \sum_{i=0}^{\infty} u_i(x) \quad \textit{and} \quad v(x) = \sum_{i=0}^{\infty} v_i(x).$$

So the solution $u(x)$ and $v(x)$ of Eq.(2.1)-(2.2) respectively can be written in a recursive manner by

$$\begin{aligned} u_0(x) &= K(x), \\ u_{n+1}(x) &= \int_0^x L(x,s)\left(c_1 u_n(s) + c_2 v_n(s)\right) + \int_0^1 F(x,s)\left(c_1 u_n(s) + c_2 v_n(s)\right) \mathrm{d}s, \quad n \geqslant 0. \end{aligned}$$

And

$$\begin{aligned} v_0(x) &= G(x), \\ v_{n+1}(x) &= \int_0^x L(x,s)\left(c_3 u_n(s) + c_4 v_n(s)\right) + \int_0^1 F(x,s)\left(c_3 u_n(s) + c_4 v_n(s)\right) \mathrm{d}s, \quad n \geqslant 0. \end{aligned}$$

4 Numerical result

In this section, we give numerical example to illustrate the above methods for solve the linear Volterra integral equations of the second kind.
The exact solution is known and used to justify the numerical solution obtained with our method is correct. We used MATLAB to solve these examples.

Example 4.1. *Consider the following fractional boundary value problem:*

$$\begin{cases} \mathscr{D}^{(\rho)}\, u(x) = c_1 u(x) + c_2 v(x) + f(x), & x \in I := [0,1] \\ \mathscr{D}^{(\rho)}\, v(x) = c_3 u(x) + c_4 v(x) + g(x), & x \in I := [0,1] \\ u(0) = u(1) = 0, \; v(0) = v(1) = 0 \end{cases} \tag{4.1}$$

where $\rho = 1.75, \quad c_1 = -\frac{1}{5},\; c_2 = \frac{1}{6},\; c_3 = \frac{1}{5},\; c_4 = -\frac{1}{3}$,
$f(x) = 8 - 8e^{-3x} + \frac{3x(x-1)}{5} - \frac{x(e^x - e^1)}{42},\; g(x) = \frac{e^x - e^{-3x}}{4} + \frac{x(4e^x - e^1)}{21} - \dfrac{3x(x-1)}{5}$ *with the exact solution* $u(x) = 3x(x-1)$ *and* $v(x) = \frac{x}{7}(e^x - e^1)$.

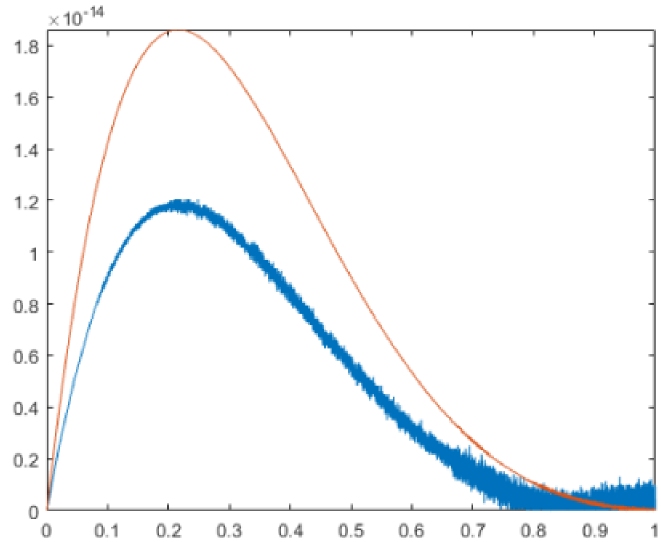

Figure 1: The Absolute Error of test Example (4.1) with $N = 7$.

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Some results for stability of Van Kármán beam

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Abstract: In this paper we introduce and study system of the type full von Kármán beam by a thermal effect and distributed delay and damping . Such that, We prove the global well-posedness of the system by using the C0-semigroup theory of linear operators and we establish exponential energy decay of this system
Keywords: stability, von Kármán, thermal effect .
2010 Mathematics Subject Classification: 35B35, 49K20, 70K20.

1 Introduction

In many fields of engineering, most structures are formed by a single or a large number of beams. So , many basic models (such as , the Raleigh model, the Timoshenko model.....) have been developed.
These structures of the beams depending on their nature are the subject of many problems of partial differential equations and ordinary differential equations.
The full von Kármán model is suitable when taking into account transversal displacements as well as longitudinal displacements for vibrating slender bodies with large deflection. Horn and Lasiecka studied the controllability and stabilization of the von Kármán system.
There are a large range of studies on this model, addressing the problems of existence, uniqueness, and asymptotic behavior in time when some damping effects are considered, as well as some other important properties .
Djebabla and Tatar considered the following one-dimensional full von Kármán beam by coupling the system (namely, the longitudinal component) with only one heat equation according to the theory of Green and Naghdi see ([**?**]) and one light damping for the other component:

$$\begin{cases} u_{tt} - D_1[u_x + \frac{1}{2}(w_x)^2]_x + \gamma\theta_{tx} = 0 \\ w_{tt} - D_1[(u_x + \frac{1}{2}(w_x)^2)w_x]_x + D_2 w_{xxxx} + \delta w_t = 0 \\ \theta_{tt} - \theta_{xx} + \mu_1\theta_t + \gamma u_{tx} = 0 \\ (x,t) \in (0,L) \times (0,\infty) \end{cases}$$

Where D_1, D_2, δ, μ_1, l, and γ are positive constants. For the above full von Kármán system, they obtained an exponential decay result of problem above.
The stability of the wave equation with delay has recently become an active area of research, and many authors have shown that delays Silva, and Zuazua and other showed how the so-called von Kármán model can be obtained as a singular limit of a modified Mindlin-Timoshenko system when the modulus of elasticity in shear k tends to infinity, provided a regularizing term through a fourth order dispersive operator is added. Introducing damping mechanisms, the authors also showed that the energy of solutions for this modified Mindlin-Timoshenko system decays exponentially, uniformly with respect to the parameter k. As $k \longrightarrow \infty$, the authors obtained the damped von Kármán model with associated energy exponentially

decaying to zero as well, (see system in Ref [**?**]) So in this problem the authors obtained a decay rate for the total energy of the solutions of the von Kármán system (as $t \longrightarrow \infty$) as a singular limit of the uniform (with respect to k) decay rate of the energy of the Mindlin-Timoshenko system.
In this works we will prove that solutions decay to zero exponentially of the following system :

$$\begin{cases} w_{tt} - d_1[(u_x + \frac{1}{2}(w_x)^2)w_x]_x + d_2 w_{xxxx} + \mu_1 w_t + \int_{\tau_1}^{\tau_2} \mu_2(s) w_t(x, t-s) ds = 0 \\ u_{tt} - d_1[(u_x + \frac{1}{2}(w_x)^2)]_x + \delta\theta_x = 0 \\ \theta_t + q_x + \delta u_{tx} = 0 \\ q_t + \gamma q + \theta_x = 0 \end{cases} \tag{1.1}$$

Where $\Omega = [0, L]$ and d_1,d_2,δ,l,μ_1 and γ are positive constant. and the following boundary conditions

$$\begin{cases} u = 0, w = 0, \theta_x = 0 \ at \ x = 0, L \ for \ any \ t > 0, \\ w_x = 0 \ at \ x = 0, \ for \ all \ t > 0 \end{cases} \tag{1.2}$$

and we used the initial data :

$$\begin{cases} u(0, .) = u_0, \ u_t(0, .) = u_1 \quad w(0, .) = w_0, \ w_t(0, .) = w_1 \\ \theta(0, .) = \theta_0, \ \theta_t(0, .) = \theta_1 \\ w_t(x, t) = f_0(x, t) \in (0, L) \times (0, \tau_2) \end{cases} \tag{1.3}$$

Such that τ_1,τ_2 are real numbers and we have the assumption :

$$\mu_1 \geq \int_{\tau_1}^{\tau_2} \mu_2(s)$$

Where $\mu_2 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ is an L^∞ function and $\mu_2 \geq 0$ almost everywhere.
Concerning the distributed delay, Nicaise and all considered a wave equation with linear frictional damping and internal distributed delay

$$u_{tt} - \Delta u + \mu_1 u_t + a(x) \int_{\tau_1}^{\tau_2} \mu_2(s) u_t(x, t-s) ds = 0$$

In $\Omega \times (0, \infty)$, with initial and mixed Dirichlet-Neumann boundary conditions, and a is a function chosen in an appropriate space. They established exponential stability of the solution under the assumption that

$$\|a\|_\infty \int_{\tau_1}^{\tau_2} \mu_2(s) ds < \mu_1$$

2 well-posednes

in this section we using the semigroup theory for prove existence and uniqueness of local solutions of problem given above .

3 ewponential decay

In this section, we state and prove our result on the exponential decay.

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SOLVABILITY OF THE OPERATORS EQUATIONS $AX - XB = C$ AND $AX - YB = C$.

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Abstract. This paper provides new necessary and sufficient conditions for the solvability to the operators equations $AX - XB = C$ and $AX - YB = C$, where A and B are group invertible operators defined on an infinite dimensional Hilbert spaces. In addition the general solutions to the equation $AX - YB = C$, are derived in terms of group inverse of A and B. As a consequence, new necessary and sufficient conditions for the solvability to the operator equation $AYB - Y = C$, are derived.

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1991 *Mathematics Subject Classification.* Primary 47A62; Secondary 15A09.
Key words and phrases. Operators equations, group inverse, inner inverse, pseudo-similarity, pseudo-equivalence.

Title: Existence and asymptotic behavior of positive continuous solutions for some nonlinear diagonal parabolic systems

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Abstract. The aim of this work is to show that positive continuous solutions exist for the nonlinear parabolic system by order 2. Our techniques of proof are based on semigroup methods Green function and Lyapunov functional. we show that global solutions exist. Our investigation applied for a wide class of the nonlinear term of parabolic system.

Keywords : Global solution, semigroups, local solution, reaction-diffusion systems, Lyapunov functional, Parabolic system, positive solutions, Green function, parabolic Kato class.

2010 Mathematics Subject Classification. 35K57, 35K40, 35K55, 35K45, 34B27, 35K10.

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Positive solution for some nonlocal elliptic problem

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absrtact In this work, we investigate the existence of a positive solution to a one-dimensional nonlocal elliptic problem under weak conditions on the reaction terms.

Keywords: Existence, Nonlocal elliptic problem, Positive solution .

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Exponential stabilization of a Euler-Bernoulli beam under boundary control

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Abstract. In this work investigate the free vibration of Euler-Bernoulli beam with tension. A boundary control is applied at the free end of the beam to suppress the undesirable vibration based on Lyapunov's method. With the suggested boundary control, the exponential stability under free vibration can be attained.

Keywords: Free vibration, exponential stability, Lyapunov method, Euler-Bernoulli beam.

1 Introduction

Axially moving systems are studied by means of the beam model with fixed ends, where the axial transport of mass and the nonlinear coupling between longitudinal and transversal displacement. For this model the following equations of motion were obtained

$$\begin{cases} \rho\left(v_{tt}+2\gamma v_{tx}+\gamma^2 v_{xx}\right)+c_v\left(v_t+\gamma v_x\right)+EIv_{xxxx} \\ \quad -\left\{Pv_x+EAv_x\left(u_x+\frac{1}{2}v_x^2\right)\right\}_x=0, \\ \rho\left(u_{tt}+2\gamma u_{tx}+\gamma^2 u_{xx}\right)+c_u\left(u_t+\gamma u_x\right)-EA\left(u_x+\frac{1}{2}v_x^2\right)_x=0, \\ \forall\,(x,t)\in(0,L)\times\mathbb{R}_+, \end{cases} \tag{1.1}$$

subject to the following boundary conditions

$$\begin{cases} mv_{tt}(L,t)=U_v(t)+EIv_{xxx}(L,t)-\left\{Pv_x+EAv_x\left(u_x+\frac{1}{2}v_x^2\right)\right\}(L,t), \\ v(0,t)=v_x(0,t)=v_{xx}(L,t)=0, \\ mu_{tt}(L,t)=U_u(t)-EA\left(u_x+\frac{1}{2}v_x^2\right)(L,t), \\ u(0,t)=0, \qquad \forall t\in\mathbb{R}_+, \end{cases} \tag{1.2}$$

and initial conditions

$$\begin{cases} v(x,0)=v^0(x), \quad v_t(x,0)=v^1(x), \\ u(x,0)=u^0(x), \quad u_t(x,0)=u^1(x), \quad \forall x\in(0,L). \end{cases} \tag{1.3}$$

where x and t represent the independent spatial and time variables, respectively, $v(x,t)$ and $u(x,t)$ are the displacements in the transversal and longitudinal direction, respectively, of

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the beam at the position x for time t, ρ are the uniform mass per unit length of the beam, L is the length of the beam, γ is the axial speed of the string, c_u and c_v are the structural distributed transverse and longitudinal damping coefficients. P the axial tension, EI is the bending stiffness and EA is the axial stiffness, m is the mass of the actuator, $U_v(t)$ and $U_u(t)$ are the control at the position L for time t.

Remark 1. *For clarity, notations* $(.)(x,t) = (.)$, $(.)_x = \frac{\partial(.)}{\partial x}$, $(.)_t = \frac{\partial(.)}{\partial t}$ *and* $\int_0^L (.)(x,t)dx = \int (.)dx$, *are used throughout this paper.*

Recently, boundary control has received great attention in a number of research areas such as vibration control of flexible structures and fluid dynamics. The relevant applications of this approach consist of second-order equations (strings, rotor and cables) and fourth-order equations (beams and plates). As well as axially moving structures and immobile structures. In many results concerned the existence and stability of solutions have been established for axially moving system (see [2, 3, 4, 7, 11]). For an axially moving beam system, see [1, 6, 14, 13], and for a problem in axially moving continua, see [12].

In [10], The solutions of an axially moving string of Kirchhoff type are stabilized by a viscoelastic boundary control. They have shown that the dissipation produced by the viscoelastic term is sufficient to suppress the transversal vibrations that occur during the axial motion of the string. For high gain adaptive output feedback type, we can refer to the work in [8], for a distributed delay in internal feedback to [9]. There are different models that can describe the dynamics of a system in axial motion with linear and non-linear tension, the nonlinearity in the tension requires some further manipulations.

2 Preliminary

In this section, we introduce the energy of system (1.1)-(1.3) and propose the control law.

The kinetic energy of the system E_k can be represented as

$$E_k(t) = \frac{\rho}{2}\left[\|v_t + \gamma v_x\|_2^2 + \|u_t + \gamma u_x\|_2^2\right] + E_m(t), \tag{2.1}$$

where $E_m(t) = \frac{m}{2}\left[u_t^2 + v_t^2\right](L,t)$.

The potential energy E_p due to the bending and the axial deformation can be given as

$$E_p(t) = \frac{EA}{2}\left\|u_x + \frac{1}{2}v_x^2\right\|_2^2 + \frac{EI}{2}\|v_{xx}\|_2^2 + \frac{P}{2}\|v_x\|_2^2. \tag{2.2}$$

We define the the total mechanical energy of problem (1.1)-(1.3) by

$$E(t) = E_k(t) + E_p(t). \tag{2.3}$$

2.1 Control

The control objective is to reduce the vibrations of the beam, i.e., $u(x,t)$ and $v(x,t)$, under the time-varying, Lyapunov's direct method is used to construct boundary control laws $U_v(t)$ and $U_u(t)$ at the free boundary of the beam and to analyze the stability of the system.

To stabilize the system given by (1.1)-(1.3), we propose the following control law:

$$\begin{cases} U_v(t) = -k_1 v_t(L,t) - k_2 v_{xt}(L,t) - k_3 v_x(L,t) - k_4 v(L,t), \\ U_u(t) = -k_5 u_t(L,t) - k_6 u_{xt}(L,t) - k_7 u_x(L,t) - k_8 u(L,t), \end{cases} \tag{2.4}$$

where $k_i, i = 1, ..., 8$, are positive constants.

Because the beam is moving with a constant speed γ, the total derivative operator with respect to time is defined by

$$\frac{d}{dt} = (\dot{\ }) = \frac{\partial}{\partial t} + \gamma \frac{\partial}{\partial x}. \tag{2.5}$$

For more details see [15].

3 Stability

Lemma 2. *The energy $E(t)$ satisfies along solutions of system (1.1)-(1.3), for all $t \geq 0$,*

$$\begin{aligned} \frac{d}{dt}E(t) &= U_v(t)\,(v_t + \gamma v_x)\,(L,t) - c_v \left\| v_t + \gamma v_x \right\|_2^2 - \gamma m v_{tt} v_x(L,t) - \gamma EI v_{xx}^2\,(0,t) \\ &+ U_u(t)\,(u_t + \gamma u_x)\,(L,t) - c_u \left\| u_t + \gamma u_x \right\|_2^2 - \gamma m u_{tt} u_x(L,t) - \gamma EA u_x^2(0,t) \end{aligned} \tag{3.1}$$

In order to prove the decay of energy, we define the Lyapunov candidate function by

$$\mathcal{L}(t) = \beta E(t) + \sum_{i=1}^{6} V_i(t), \tag{3.2}$$

where β is a positive constant, $E(t)$ is the energy given by (2.3) and

$$V_1(t) = \rho \int \frac{1}{2} v\,(v_t + \gamma v_x) + u\,(u_t + \gamma u_x)\,dx + \int \frac{c_v}{4} v^2 + \frac{c_u}{2} u^2 dx, \tag{3.3}$$

$$V_2(t) = m\left[\frac{1}{2} v_t v + u_t u\right](L,t) \tag{3.4}$$

$$V_3(t) = m\beta\gamma\,[v_t v_x + u_t u_x]\,(L,t), \tag{3.5}$$

$$V_4(t) = \frac{\beta\gamma}{2}\left[k_2 v_x^2 + k_6 u_x^2\right](L,t), \tag{3.6}$$

$$V_5(t) = \left[\frac{k_1/2 + \beta k_4}{2} v^2 + \frac{k_5 + \beta k_8}{2} u^2\right](L,t), \tag{3.7}$$

$$V_6(t) = [\frac{k_2}{2} v v_x + k_6 u u_x](L,t). \tag{3.8}$$

It is our intention to prove that new functional $\mathcal{L}(t)$ satisfies a differential inequality of the form $\frac{d}{dt}\mathcal{L}(t) \leq -\alpha \mathcal{L}(t)$ for some positive constant α to derive the uniform decay of $\mathcal{L}(t)$. To pass to $E(t)$, we will need some 'equivalence' between the two functionals.

Proposition 3. *Let $E(t)$ and $V_i(t), i = 2, ..., 6$, be the functionals defined by (2.3) and (3.4)-(3.8), respectively. Then, if $k_2 = \gamma m$, $k_6 = \gamma m$, $\frac{k_1/2+\beta k_4}{m} \geq \frac{1}{2\beta}$ and $\frac{k_5+\beta k_8}{m} \geq \frac{1}{\beta}$, we have*

$$\beta E(t) + \sum_{i=2}^{6} V_i(t) \geq 0, \quad t \geq 0. \tag{3.9}$$

Proposition 4. *Let $E(t)$ and $\mathcal{L}(t)$ be the functionals defined by (2.3) and (3.2), respectively. Then, there exist $\alpha_i > 0$, $i = 1, 2$, such that*

$$\alpha_1 \left(\beta E(t) + \sum_{i=2}^{6} V_i(t) \right) \leq \mathcal{L}(t) \leq \alpha_2 \left(\beta E(t) + \sum_{i=2}^{6} V_i(t) \right), \quad \forall t \geq 0. \tag{3.10}$$

Remark 5 ([5])**.** *From the practical points of view, the slope of the beam v_x in the vibration never goes to infinity. Hence, we assume that there exists $c \in \mathbb{R}_+$ such that $\forall t \geq 0$ and $x \in [0, L]$, $|v_x| \leq c$.*

Lemma 6. *The total time derivative of the Lyapunov function (3.2) satisfies*

$$\frac{d}{dt}\mathcal{L}(t) \leq -\alpha \mathcal{L}(t), \quad \forall t \geq 0. \tag{3.11}$$

where α is positive constant.

Theorem 7. *Let $u^0, v^0 \in H^4(0,\ L)$ and $u^1, v^1 \in H^2(0,\ L)$. Then, there exists tow positive constants A and α such that the energy $E(t)$ of system (1.1)-(1.3), satisfies*

$$E(t) \leq Ae^{-\alpha t}, \forall t \geq 0.$$

4 Conclusion

The reduction of the vibrations of a coupled axially moving beam with nonlinear tension is studied. The free boundary is controlled, and the Lyapunov method is used to design a boundary control law ensuring an exponential stabilization result.

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Positive periodic solutions of Lotka-Volterra dynamical systems on a time scale

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Let $\mathbb{T}$ be a time scale. The purpose of this paper is to use Gustafson and Schmitt version of fixed point theorem in a cone to show the existence of solutions for Lotka-Volterra dynamical type ecological models with discrete and distributed delays on time scales. As an application we show the existence of solutions for a particular Lotka-Volterra system of n-predator and m-prey

Keywords: Fixed points, positive solutions, time scales, Lotka-Volterra systems,discret and distributed delays.

1. Introduction

The Lotka–Volterra systems, also known as systems of predator–prey, are a pair of first order nonlinear differential equations, used to describe the dynamics of biological systems, especially interaction between two species, one as a predator and the other as prey. Smita Pati, John R. Graef and Seshadev Padhi [3] studied a very comprehensive system wich is more general and include a lot of spetial cases of competition models that was studied in previous works as [17], [18], [19], [20], [21], [22].

We present here the same problem by [3], but with a generalization that makes it even include discrete cases of the form

$$\begin{cases} x_i^{\Delta}(t) = a_i(t)x_i(t) - f_i(t,x(t),y(t))x_i(t), \quad i = 1,...,n, \\ y_j^{\Delta}(t) = -b_j(t)y_j^{\sigma}(t) + g_j(t,x(t),y(t))y_j(t), \quad j = 1,...,m, \end{cases} \tag{1.1}$$

where $t \in \mathbb{T}$, $x(t) = (x_1(t),x_1(t),...,x_n(t))^T$, $y(t) = (y_1(t),y_1(t),...,y_m(t))^T$, $a_i, b_i : \mathbb{T} \to \mathbb{R}_+$ are asumed to be rd-continuous, and periodic for the same period $T > 0$, $a,b \neq 0$, $f_i, g_j : \mathbb{T} * \mathbb{R}_+^n * \mathbb{R}_+^m \to \mathbb{R}_+$ are rd-continuous and T-periodic in their first variable with $T > 0$ for every $(x,y) \in \mathbb{R}_+^n * \mathbb{R}_+^m$ and all $i = 1,...,n$, $j = 1,...,m$

$$\begin{aligned} a_i(t+T) &= a_i(t), \;\; b_j(t+T) = b_j(t), \\ g_j(t+T,x.y) &= g_j(t,x.y), \quad f_i(t+T,x.y) = f_i(t,x.y). \\ \text{for all } i &= 1,...,n, \; j = 1,...,m. \end{aligned} \tag{1.2}$$

The work is organized as follows. In Section 2, we present some preliminary results on the time scales calculus and we state the fixed point version in a cone duo to Gustafson and Schmitt [15], [16].

In Section 3, we transform (1.1) into an integral system. In Section 4, we explore the existence of periodic solutions. the last section present a particular model as an application.

2. Preliminaries

Define the forward jump operator σ by

$$\sigma(t) = \inf\{s > t : s \in \mathbb{T}\},$$

and the graniness function μ by $\mu(t) = \sigma(t) - t$. A point t in a time scale is called right scattered if $\sigma(t) > t$. Hearefter, we denote by x^{σ} the composit function $x \circ \sigma$.

Definition 1 *We say that a time scale is periodic if there exists $p > 0$ such that if $t \in \mathbb{T}$ then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is colled the period of the time scale.*

Example 1 *The following time scales are periodic*

1. $\mathbb{T} = \mathbb{R}$.
2. $\mathbb{T} = \mathbb{Z}$, *with period* $p = 1$.
3. $\mathbb{T} = h\mathbb{Z}$, *with period* $p = h$.
4. $\mathbb{T} = \cup_{i=-\infty}^{+\infty} [(2i-1)h, 2ih], h > 0$ *with period* $p = 2h$.

Remark 1 *All periodic time scales are unbounded above and below.*
In a periodic time scale $\mathbb{T}$ with period P the inequality $0 \leq \mu(t) \leq P$ hold for all $t \in \mathbb{T}$.

Definition 2 *Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = np$, $f(t+T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t+T) = f(t)$.*
If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t+T) = f(t)$ for all $t \in \mathbb{T}$.

Definition 3 *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at every rightdense point $t \in \mathbb{T}$ and its left-sided limits exist, and is finite at every left-dense point $t \in \mathbb{T}$ The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by*

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R})$$

Definition 4 *For $f : \mathbb{T} \to \mathbb{R}$, we define $f^{\Delta}(t)$ to be the number (if it exists) with the property that for any given $\varepsilon > 0$, there exists a neighborhood U of t such that*

$$(f(\sigma(t)) - f(s)) - f^{\Delta}(t)(\sigma(t) - s) < \varepsilon |\sigma(t) - s| \text{ for all } s \in U.$$

The function $f^{\Delta} : \mathbb{T}^{\kappa} \to \mathbb{R}$ is called the delta (or Hilger) derivative of f on $\mathbb{T}^{\kappa}$.

Definition 5 *A function $h : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1+\mu(t)h(t) \neq 0$ for all $t \in \mathbb{T}^k$, where $\mu(t) = \sigma(t) - t$. The set of all regressive rd-continuous functions $\varphi : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathscr{R}$ while the set $\mathscr{R}^+$is given by $\mathscr{R}^+ = \{h \in \mathscr{R} : 1+\mu(t)h(t) > 0 \, for \, all \, t \in T\}$.*

Let $\varphi \in \mathscr{R}$ and $\mu(t) > 0$ for all $t \in \mathbb{T}$, the exponential function on $\mathbb{T}$ is defined by

$$e_\varphi(t,s) = \exp\left(\int_s^t \zeta_{\mu_{(r)}}(\varphi(r)) \triangle r\right),$$

where $\zeta_{\mu_{(r)}}$ is the cylinder transformation given by

$$\zeta_{\mu_{(r)}} = \begin{cases} \frac{1}{\mu(r)} Log(1+\mu(r)\varphi(r)) & \text{if } \mu(r) > 0 \\ \varphi(r) & \text{if } \mu(r) = 0 \end{cases}.$$

It is well known that if $p \in \mathscr{R}^+$ then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Also the exponential function $y(t) = e_p(t,s)$ is the solution of the initial value problem $y^\triangle(t) = p(t)y(t)$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma.

Let p, $q \in \mathscr{R}$ Then
i. $e_0(t,s) \equiv 1$ and $e_p(t,t) \equiv 1$,
ii. $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$,
iii.$\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$ where, $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$,
iv.$e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$,
v.$e_p(t,s)e_p(s,r) = e_p(t,r)$,
vi.$\left(\frac{1}{e_p(s,t)}\right)^\triangle = -\frac{p(t)}{e_p^\sigma(.,s)}$.
By the notation $[a,b]$ we mean

$$[a,b] = \{t \in \mathbb{T} : a \leq t \leq b\},$$

and similarly we defined $(a,b]$, $[a,b)$, (a,b).

We assume that readers are familiar with the calculus of time scales and for more details we refer to [1],[2].

Theorem 1 *(Fixed point technique in a cone [15], [16]) Let $(X, \|.\|)$ be a Banach space, $K \subset X$ a cone, $0 < r < R$,*

$$K_{r,R} = \{u \in K : r \leq \|u\| \leq R\},$$

and $\phi : K_{r,R} \to K$ be a compact continuous operator such that

(C_1) *$u \in K_{r,R}$, $\mu > 1$, and $u = \mu\phi u$ implies $\|u\| \neq r$,*
(C_2) *$u \in K_{r,R}$, $\mu \in (0,1)$, and $u = \mu\phi u$ implies $\|u\| \neq R$,*
(C_3) $\inf_{\|u\|=r} \|\phi u\| \neq 0$.

Then ϕ has a fixed point in $K_{r,R}$.

Theorem 2 *(Fixed point version in a cone [15], [16]) Let $(X, \|.\|)$ be a Banach space, $K \subset X$ a cone in X. Let r and R be real numbers with $0 < r < R$,*

$$K_{r,R} = \{u \in K : r \le \|u\| \le R\},$$

and let and $\phi : K_{r,R} \to K$ a compact continuous operator such that

(C_1) $u \in K_{r,R}$, $\mu > 1$, *and* $u = \mu\phi u$ *implies* $\|u\| \neq R$,
(C_2) $u \in K_{r,R}$, $\mu < 1$, *and* $u = \mu\phi u$ *implies* $\|u\| \neq r$,
(C_3) $\inf_{\|u\|=R} \|\phi u\| > 0$.

Then ϕ has a fixed point in $K_{r,R}$.

3. Periodic solutions

Let $T > 0$ be a period of $\mathbb{T}$, $t \in \mathbb{T}$ be fixed, if $\mathbb{T} \neq \mathbb{R}$,

Let $\varkappa(t) = (x_1(t), x_2(t), ..., x_n(t), y_1(t), y_2(t), ..., y_m(t))^T$. Define

$$P_T = \left\{\varkappa \in C_{rd}(\mathbb{T}, \mathbb{R}_+^{n+m}) : \varkappa(t+T) = \varkappa(t)\right\}, \tag{3.1}$$

then P_T endowed with the norm

$$\|\varkappa\| = \max_{1 \le i \le n+m} |x_i|_0, \text{ where } |x_i|_0 = \sup_{t \in [0,T]} \{|x_i(t)|\}, i = 1, ..., n+m$$

is a Banach space.

Throughout this paper, we assume that $a(t), b(t) \in \mathscr{R}^+$. Also, we assume that

$$e_{\ominus a_i}(T,0) \neq 1, \;\; e_{b_j}(T,0) \neq 1 \tag{3.2}$$

Assume (1.2) and (3.2)hold. If $\varkappa \in P_T$, then $\varkappa$ is a solution of system (1.1) if and only if for all $i = 1, ..., n$, $j = 1, ..., m$

$$x_i(t) = \int_t^{t+T} \frac{e_{\ominus a_i}(\sigma(s), t)}{1 - e_{\ominus a_i}(T,0)} f_i(s, x(s), y(s)) x_i(s) \Delta s, \tag{3.3}$$

and

$$y_j(t) = \int_t^{t+T} \frac{e_{b_j}(s,t)}{e_{b_j}(T,0) - 1} g_j(s, x(s), y(s)) y_j(s) \Delta s. \tag{3.4}$$

Proof Let $\varkappa \in P_T$ be a solution of (1.1).We put the i^{th} equation in (1.1) as the form

$$x_i^{\Delta}(t) - a_i(t)x(t) = -f_i(t, x(t), y(t)) x_i(t).$$

Multiplying both side by $e_{\ominus a_i}(\sigma(t), v)$ and then integrate from t to $t+T$ to obtain

$$\int_t^{t+T} [x_i(s) e_{\ominus a_i}(s, v)]^{\Delta} \Delta s = -\int_t^{t+T} e_{\ominus a_i}(\sigma(s), v) f_i(t, x(t), y(t)) x_i(t) \Delta s$$

$$x_i(t+T)e_{\ominus a_i}(t+T,v)-x_i(t)e_{\ominus a_i}(t,v)=-\int_t^{t+T} e_{\ominus a_i}(\sigma(s),v)f_i\left(t,x(t),y(t)\right)x_i(t)\Delta s$$

periodicity of $x(t)$ gives

$$x_i(t)\left(e_{\ominus a_i}(t+T,v)-e_{\ominus a_i}(t,v)\right)=-\int_t^{t+T} e_{\ominus a_i}(\sigma(s),v)f_i\left(t,x(t),y(t)\right)x_i(t)\Delta s$$

Divide both sides of the above equation by $e_{\ominus a_i}(t,v)$ and use Lemma (2)

$$x_i(t)\left(e_{\ominus a_i}(t+T,t)-1\right)=-\int_t^{t+T} e_{\ominus a_i}(\sigma(s),t)f_i\left(t,x(t),y(t)\right)x_i(t)\Delta s$$

$$x_i(t)=\int_t^{t+T}\frac{e_{\ominus a_i}(\sigma(s),t)}{(1-e_{\ominus a_i}(T,0))}f_i\left(t,x(t),y(t)\right)x_i(t)\Delta s.$$

Similarly, we put the j^{th} equation in (1.1) as the form

$$y_j^{\Delta}(t)+b_j(t)y_j^{\sigma}(t)=g_j(t,x(t),y(t))y_j(t)$$

Multiplying both side by $e_{b_j}(t,v)$ and then integrate from t to $t+T$ to obtain

$$\int_t^{t+T}\left[y_j(s)e_{b_j}(s,v)\right]^{\Delta}\Delta s=\int_t^{t+T} e_{b_j}(s,v)g_j(s,x(s),y(s))y_j(s)\Delta s$$

Divide both sides by $e_{b_j}(t,v)$ and use Lemma (2)

$$y_j(t)\left[e_{b_j}(T,0)-1\right]=\int_t^{t+T} e_{b_j}(s,t)g_j(s,x(s),y(s))y_j(s)\Delta s$$

$$y_j(t)=\int_t^{t+T}\frac{e_{b_j}(s,t)}{\left(e_{b_j}(T,0)-1\right)}g_j(s,x(s),y(s))y_j(s)\Delta s.$$

□

4. Main results

Let

$$\alpha_i=e_{\ominus a_i}(T,0),\ \beta_j=e_{b_j}(T,0)\ \textit{for all}\ i=1,\dots,n,\ j=1,\dots,m,$$

due to the periodicity of a_i and b_j It is easy to see that

$$\frac{\alpha_i}{1-\alpha_i}\leq\frac{e_{\ominus a_i}(\sigma(s),t)}{(1-e_{\ominus a_i}(T,0))}\leq\frac{1}{1-\alpha_i}, \tag{4.1}$$

$$\frac{1}{\beta_j-1}\leq\frac{e_{b_j}(s,t)}{\left(e_{b_j}(T,0)-1\right)}\leq\frac{\beta_j}{\beta_j-1}. \tag{4.2}$$

Using (4.1) and (4.2) we define the cone K by

$$K=\left\{\varkappa\in P_T: x_i(t)\geq\alpha_i\left|x_i\right|_0,\ y_j(t)\geq\frac{1}{\beta_j}\left|y_j\right|_0\ \textit{for every}\ t\in\mathbb{T},\ i=1,\dots,n,\ j=1,\dots,m\right\}. \tag{4.3}$$

Defined the operator E on P_T by

$$(E\varkappa)(t) = ((E_1\varkappa)(t), (E_2\varkappa)(t), ..., (E_n\varkappa)(t), (\hat{E}_1\varkappa)(t), (\hat{E}_2\varkappa)(t), ..., (\hat{E}_n\varkappa)(t))$$

where

$$E_i(\varkappa)(t) = \int_t^{t+T} \frac{e_{\ominus a_i}(\sigma(s),t)}{(1-e_{\ominus a_i}(T,0))} f_i(s,x(s),y(s))x_i(s)\Delta s, \quad i = 1,2,...,n,$$

and

$$\hat{E}_j(\varkappa)(t) = \int_t^{t+T} \frac{e_{b_j}(s,t)}{\left(e_{b_j}(T,0)-1\right)} g_j(s,x(s),y(s))y_j(s)\Delta s, \quad j = 1,2,...,m.$$

Theorem 3 *Assume that there exist constants* $0 < r < R$ *such that*

(T1) $\int_0^T f_i(t,x(t),y(t))x_i(t)\Delta s \le (1-\alpha_i)\|\varkappa\|$ *for* $0 < \|\varkappa\| < R$, $i = 1,2,...,n$,

(T2) $\int_0^T g_j(t,x(t),y(t))y_j(t)\Delta s \le \frac{\beta_j - 1}{\beta_j}\|\varkappa\|$ *for* $0 < \|\varkappa\| < R$, $j = 1,2,...,m$,

(T3) $\int_0^T f_i(t,x(t),y(t))x_i(t)\Delta s \ge \frac{(1-\alpha_i)}{\alpha_i}\|\varkappa\|$ *for* $\|\varkappa\| = r$, $i = 1,2,...,n$,

(T4) $\int_0^T g_j(t,x(t),y(t))y_j(t)\Delta s \le (\beta_j - 1)\|\varkappa\|$ *for* $\|\varkappa\| = r$, $j = 1,2,...,m$.

Then system (1.1) has a positive T-periodic solution.

Now, we consider the set

$$K_{r,R} = \{u \in K, r \le \|u\| \le R\}.$$

Proof We first show that E is periodic

$$\begin{aligned}
E_i(\varkappa)(t+T) &= \int_{t+T}^{t+2T} \frac{e_{\ominus a_i}(\sigma(s),t+T)}{1-e_{\ominus a_i}(T,0)} f_i(s,x(s),y(s))x_i(s)\Delta s,\\
&\overset{u=s-T}{=} \int_t^{t+T} \frac{e_{\ominus a}(\sigma(u+T),t+T)}{(1-e_{\ominus a}(T,0))} f_i(s,x(u+T),y(u+T))x_i(u+T)\Delta u\\
&= \int_t^{t+T} \frac{e_{\ominus a}(\sigma(u),t)}{1-e_{\ominus a}(T,0)} f_i(s,x(u),y(u))x_i(u)\Delta u\\
&= E_i(\varkappa)(t) \text{ for all } i = 1,...,n,
\end{aligned}$$

similarly, $\hat{E}_j(\varkappa)(t+T) = \hat{E}_j(\varkappa)(t)$ for all $j = 1,...,m$. Therefore, $E(\varkappa)(t+T) = E(\varkappa)(t)$. Hence, E maps P_T into itself, now we shall prove that $E : K_{r,R} \to K$ is compact and continuous.

Show that E is uniformly bounded in $K_{r,R}$. Since (4.1) and (4.2) hold. Let $\varkappa \in K_{r,R}$

$$\begin{aligned}
|E_i(\varkappa)(t)| &\le \sup_{t\in[0,T]} \left| \int_t^{t+T} \frac{e_{\ominus a_i}(\sigma(s),t)}{(1-e_{\ominus a_i}(T,0))} f_i(s,x(s),y(s))x_i(s)\Delta s \right|\\
&\le \frac{1}{1-\alpha_i}\int_0^T f_i(s,x(s),y(s))x_i(s)\Delta s\\
&\le \frac{1}{1-\alpha_i}(1-\alpha_i)\|\varkappa\|\\
&\le R,
\end{aligned}$$

similarly

$$\begin{aligned}
\hat{E}_j(\varkappa)(t) &\leq \sup_{t\in[0,T]}\int_t^{t+T}\left|\frac{e_{b_j}(s,t)}{e_{b_j}(T,0)-1}g_j(s,x(s),y(s))y_j(s)\right|\Delta s \\
&\leq \frac{\beta_j}{\beta_j-1}\int_0^T g_j(s,x(s),y(s))\,y_j(s)\Delta s \\
&\leq \frac{\beta_j}{\beta_j-1}*\frac{\beta_j-1}{\beta_j}\|\varkappa\| \\
&\leq R.
\end{aligned}$$

So E is uniformly bounded.

Next we calculate $E(\varkappa)^{\Delta}(t)$, and show that it is uniformly bounded. Since a_i,b_j, f_i, g_j, are rd-continuous and T-periodic with (1.2) hold we can assure the existence of positive constants F, G, $\hat{a}$, $\hat{b}$ such that

$$E_i(\varkappa)^{\Delta}(t) = -f_i(t,x(t),y(t))\,x_i(t)+a_i\int_t^{t+T}\frac{e_{\ominus a_i}(\sigma(s),t)}{(1-e_{\ominus a_i}(T,0))}f_i(s,x(s),y(s))\,x_i(s)\Delta s.$$

$$\begin{aligned}
\left|E_i(\varkappa)^{\Delta}(t)\right| &\leq \sup_{i=1,2,..,n}\{|f_i(t,x(t),y(t))\,x_i(t)|+ \\
&\qquad \left|a_i\int_t^{t+T}\frac{e_{\ominus a_i}(\sigma(s),t)}{(1-e_{\ominus a_i}(T,0))}f_i(s,x(s),y(s))\,x_i(s)\Delta s\right|\Big\} \\
&\leq FR+\hat{a}\frac{1}{1-\alpha_i}FR \\
&\leq D_1.
\end{aligned}$$

$$\hat{E}_j^{\Delta}(\varkappa)(t) = e_{b_j}(t,\sigma(t))g_j(t,x(t),y(t))\,y_j(t)-b_j\int_t^{t+T}\frac{e_{b_j}(s,\sigma(t))}{\left(e_{b_j}(T,0)-1\right)}g_j(s,x(s),y(s))y_j(s)\Delta s.$$

$$\begin{aligned}
\left|\hat{E}_j^{\Delta}(\varkappa)(t)\right| &\leq \sup_{j=1,2,..,m}\Big\{e_{b_j}(t,\sigma(t))g_j(t,x(t),y(t))\,y_j(t) \\
&\qquad +b_j\int_t^{t+T}\frac{e_{b_j}(s,\sigma(t))}{\left(e_{b_j}(T,0)-1\right)}g_j(s,x(s),y(s))y_j(s)\Delta s\Big\}. \\
&\leq \hat{b}GR+\frac{\beta_j^2}{\beta_j-1}GR \\
&\leq D_2.
\end{aligned}$$

This mean $|E(\varkappa)(t)|\leq D$, where $D=\max\{D_1,D_2\}$. Therefore, the set $\{E(\varkappa)\}$ is equicentinuous. Hence by the Arzela-Ascoli theorem $E(K_{r,R})$ is compact.

Now we shall prove that. E is continuous. Let $\varkappa^l(t) = (x_1^l(t), x_2^l(t), \ldots x_n^l(t), y_1^l(t), y_2^l(t), \ldots, y_m^l(t))^T$, be a sequence in P_T such that

$$\lim_{l\to\infty} \left\| \varkappa^l - \varkappa \right\| = 0.$$

$$\begin{aligned}
\left| E_i(\varkappa^l) - E_i(\varkappa) \right| &= \left| \int_t^{t+T} \frac{e_{\ominus a_i}(\sigma(s),t)}{(1-e_{\ominus a_i}(T,0))} f_i\left(s, x^l(s), y^l(s)\right) x_i^l(s)\Delta s \right.\\
&\quad \left. - \int_t^{t+T} \frac{e_{\ominus a_i}(\sigma(s),t)}{(1-e_{\ominus a_i}(T,0))} f_i\left(s,x(s),y(s)\right) x_i(s)\Delta s \right| \\
&\le \frac{1}{1-\alpha_i} \left| \int_t^{t+T} f_i\left(s, x^l(s), y^l(s)\right) x_i^l(s) - f_i\left(s,x(s),y(s)\right) x_i(s)\Delta s \right| \\
&= \frac{1}{1-\alpha_i} \left| \int_t^{t+T} f_i\left(s, x^l(s), y^l(s)\right) x_i^l(s) + f_i\left(s,x(s),y(s)\right) x_i^l(s) \right.\\
&\quad \left. - f_i\left(s,x(s),y(s)\right) x_i^l(s) - f_i\left(s,x(s),y(s)\right) x_i(s)\Delta s \right| \\
&= \frac{1}{1-\alpha_i} \left| \left[\int_t^{t+T} f_i\left(s, x^l(s), y^l(s)\right) - f_i\left(s,x(s),y(s)\right) \right] x_i^l(s) \right.\\
&\quad \left. + f_i\left(s,x(s),y(s)\right) \left[x_i^l(s) - x_i(s) \right] \Delta s \right|.
\end{aligned}$$

The continuity of f along with Lebegue dominated convergence theorem implies

$$\lim_{l\to\infty} \left| E_i(\varkappa^l) - E_i(\varkappa) \right|_0 = 0.$$

In similar way we have

$$\lim_{l\to\infty} \left| \hat{E}_j(\varkappa^l) - \hat{E}_j(\varkappa) \right|_0 = 0.$$

Thus

$$\lim_{l\to\infty} \left\| E(\varkappa^l) - E(\varkappa) \right\| = 0.$$

This show that E is a continuous map.

To complete the proof we need to show that conditions of Theorem (1) are satisfied.

Let $\varkappa \in K_{r,R}$ with $\varkappa = \mu E \varkappa$ and $\mu \in (0,1)$. we claim that $\|\varkappa\| \neq R$, if this is not true, then $\|\varkappa\| = R$.,for any $t \in [0,T]$, by conditions (T_1) and (T_2) of Theorem (3) we have

$$\begin{aligned}
|E_i(\varkappa)(t)| &= \int_t^{t+T} \frac{e_{\ominus a_i}(\sigma(s),t)}{(1-e_{\ominus a_i}(T,0))} f_i\left(s,x(s),y(s)\right) x_i(s)\Delta s \\
&\le \frac{1}{1-\alpha_i} \int_t^{t+T} f_i\left(s,x(s),y(s)\right) x_i(s)\Delta s < \|\varkappa\|,
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \left|\hat{E}_j(\varkappa)(t)\right| &= \int_t^{t+T} \frac{e_{b_j}(s,t)}{\left(e_{b_j}(T,0)-1\right)} g_j(s,x(s),y(s))y_j(s)\Delta s \\ &\leq \frac{\beta_j}{\beta_j-1}\int_t^T g_j(s,x(s),y(s))y_j(s)\Delta s < \|\varkappa\|, \end{aligned} \tag{4.5}$$

we obtain

$$R = \|\varkappa\| = \mu \|E\varkappa\| < \|\varkappa\| = R,$$

which is contradiction. Hence $\|\varkappa\| \neq R$. Thus, condition (C_2) of Theorem (1) is satisfied.

Now, let $\varkappa \in K_{r,R}$ with $\varkappa = \mu E\varkappa$ and $\mu > 1$. we claim that $\|\varkappa\| \neq r$, if this is not true, then $\|\varkappa\| = r$, for any $t \in [0,T]$, by (T_3) and (T_4) of Theorem (3) we have

$$\begin{aligned} |E_i(\varkappa)(t)| &= \int_t^{t+T} \frac{e_{\ominus a_i}(\sigma(s),t)}{(1-e_{\ominus a_i}(T,0))} f_i(s,x(s),y(s))x_i(s)\Delta s \\ &\geq \frac{\alpha_i}{1-\alpha_i}\int_0^T f_i(s,x(s),y(s))x_i(s)\Delta s > \|\varkappa\|, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \left|\hat{E}_j(\varkappa)(t)\right| &= \int_t^{t+T} \frac{e_{b_j}(s,t)}{\left(e_{b_j}(T,0)-1\right)} g_j(s,x(s),y(s))y_j(s)\Delta s \\ &\geq \frac{1}{\beta_j-1}\int_t^T g_j(s,x(s),y(s))y_j(s)\Delta s > \|\varkappa\|, \end{aligned} \tag{4.7}$$

we obtain

$$r = \|\varkappa\| = \mu \|E\varkappa\| > \|\varkappa\| = r$$

which is contradiction. Hence $\|\varkappa\| \neq r$. Thus, condition (C_1) of Theorem (1) is satisfied.

Furthermore, (4.6) and (4.7) imply $\inf_{\|\varkappa\|=r} \|E\varkappa\| > r \neq 0$, so condition (C_3) of Theorem (1) is satisfied. Therefore, system (1.1) has a positive T-periodic solution. □

5. Particular model duo to Lotka

In this parte we deal with the following ecological models with discrete and distributed time delays

$$\begin{cases} x_i^{\Delta}(t) = p_i(t)x_i(t) - F_i(t,x(t),y(t))x_i(t), & i = 1,\dots,n, \\ y_j^{\Delta}(t) = -h_j(t)y_j^{\sigma}(t) + G_j(t,x(t),y(t))y_j(t), & j = 1,\dots,m, \end{cases} \tag{5.1}$$

where $x(t)=(x_1(t),x_1(t),...,x_n(t))^T$, $y(t)=(y_1(t),y_1(t),...,y_m(t))^T$,

$$\begin{aligned} F_i(t) \quad = \quad & \sum_{k=1}^{n} a_{ik}(t)x_k(t-\tau_{ik}(t))+\sum_{l=1}^{m} b_{il}(t)y_l(t-\vartheta_{il}(t)) \\ & +\sum_{k=1}^{n} c_{ik}(t)\int_{-\infty}^{0} M_{ik}(s)x_k(t+s)\Delta s \\ & +\sum_{l=1}^{m} d_{il}(t)\int_{-\infty}^{0} L_{il}(s)y_l(t+s))\Delta s, \end{aligned} \tag{5.2}$$

$$\begin{aligned} G_i(t) \quad = \quad & \sum_{k=1}^{n} \hat{a}_{kj}(t)x_k(t-\hat{\tau}_{kj}(t))+\sum_{l=1}^{m} \hat{b}_{lj}(t)y_l(t-\hat{\vartheta}_{lj}(t)) \\ & +\sum_{k=1}^{n} \hat{c}_{kj}(t)\int_{-\infty}^{0} \hat{M}_{kj}(s)x_k(t+s)\Delta s \\ & +\sum_{l=1}^{m} \hat{d}_{lj}(t)\int_{-\infty}^{0} \hat{L}_{lj}(s)y_l(t+s)\Delta s. \end{aligned} \tag{5.3}$$

Here p_i, h_i, a_{ik}, $\hat{a}_{kj}$, b_{il}, $\hat{b}_{lj}$, c_{ik}, $\hat{c}_{kj}$, d_{il}, $\hat{d}_{lj}\in C_{rd}(\mathbb{T},(0,\infty))$ *and* τ_{ik}, $\hat{\tau}_{kj}$, ϑ_{il}, $\hat{\vartheta}_{lj}\in C_{rd}(\mathbb{T},\mathbb{R})$ *for* $i,k=1,...,n$, $j,l=1,...,m$ *are all* T*-periodic functions, and* M_{ik}, L_{il}, $\hat{M}_{kj}$, $\hat{L}_{lj}\in C_{rd}(\mathbb{T}\cap(-\infty,0],(0,\infty))$ *with*

$$\int_{-\infty}^{0} M_{ik}(s)\Delta s=\int_{-\infty}^{0} L_{il}(s)\Delta s=\int_{-\infty}^{0} \hat{M}_{kj}(s)\Delta s=\int_{-\infty}^{0} \hat{L}_{lj}(s)\Delta s=1.$$

Set the following notations

$$\theta_i=\frac{1}{1-\alpha_i}\int_0^T\left[\sum_{k=1}^{n}(a_{ik}(s)+c_{ik}(s))+\sum_{l=1}^{m}(b_{il}(s)+d_{il}(t))\right]\Delta s,$$

$$\hat{\theta}_j=\frac{\beta_j}{\beta_j-1}\int_0^T\left[\sum_{k=1}^{n}\left(\hat{a}_{kj}(s)+\hat{c}_{kj}(s)\right)+\sum_{l=1}^{m}\left(\hat{b}_{lj}(s)+\hat{d}_{lj}(s)\right)\right]\Delta s,$$

$$\theta=\min\left\{\frac{1}{\theta_1},\frac{1}{\theta_2},...,\frac{1}{\theta_n},\frac{1}{\hat{\theta}_1},\frac{1}{\hat{\theta}_2},...,\frac{1}{\hat{\theta}_m}\right\},$$

$$\pi_i=\frac{\alpha_i^2}{1-\alpha_i}\int_0^T\left[\sum_{k=1}^{n}(a_{ik}(s)+c_{ik}(s))+\sum_{l=1}^{m}(b_{il}(s)+d_{il}(s))\right]\Delta s,$$

$$\hat{\pi}_j=\frac{1}{\beta_j(\beta_j-1)}\int_0^T\left[\sum_{k=1}^{n}\left(\hat{a}_{kj}(s)+\hat{c}_{kj}(s)\right)+\sum_{l=1}^{m}\left(\hat{b}_{lj}(s)+\hat{d}_{lj}(s)\right)\right]\Delta s,$$

$$\pi = \max\left\{\frac{1}{\pi_1}, \frac{1}{\pi_2}, ..., \frac{1}{\pi_n}, \frac{1}{\hat{\pi}_1}, \frac{1}{\hat{\pi}_2}, ..., \frac{1}{\hat{\pi}_m}\right\},$$

choose tow positive constants $\hat{r}$ *and* $\hat{R}$ *with* $0 < \hat{r} < \theta$ *and* $\hat{R} > \pi$, *since* $\theta_i > \pi_i$ *and* $\hat{\theta}_j > \hat{\pi}_j$, *we have* $0 < \hat{r} < \theta < \pi < \hat{R}$. *Let*

$$D_{\hat{r},\hat{R}} = \left\{u \in K : \hat{r} \le \|u\| \le \hat{R}\right\}.$$

Let P_T *be a Banach space, K be the cone defined in (3.1) and (4.3) respectively, define the operator* Ψ *on* P_T *by* $\Psi = \left(\Psi_1, \Psi_2, ..., \Psi_n, \hat{\Psi}_1, \hat{\Psi}_2, ..., \hat{\Psi}_m\right)^T$, *where*

$$\Psi_i(x)(t) = \int_t^{t+T} \frac{e_{\ominus a_i}(\sigma(s),t)}{(1-e_{\ominus a_i}(T,0))} F_i(s,x(s),y(s))\,x_i(s)\Delta s, \quad i = 1,2,...,n \tag{5.4}$$

$$\hat{\Psi}_j(x)(t) = \int_t^{t+T} \frac{e_{b_j}(s,t)}{\left(e_{b_j}(T,0)-1\right)} G_j(s,x(s),y(s))y_j(s)\Delta s, \quad j = 1,2,...,m \tag{5.5}$$

F_i *and* G_j *are defined in (5.2) and (5.3).*

Theorem 4 *System (5.1) has a positive T-periodic solution.*

Proof It is similar to the proof of Theorem (3) to see that $\Psi : D_{\hat{r},\hat{R}} \to K$ is compact and continuous. Now we need to show that conditions of Thorem (2) are satisfied.

Supose that $x \in D_{\hat{r},\hat{R}}$ with $x = \mu\Psi x$ and $\mu \in (0,1)$, we claim that $\|\varkappa\| \ne \hat{r}$, if this is not true, then $\|\varkappa\| = \hat{r}$, for any $t \in [0,T]$, we have

$$\begin{aligned}
|\Psi_i(x)(t)| &= \int_t^{t+T} \frac{e_{\ominus a_i}(\sigma(s),t)}{(1-e_{\ominus a_i}(T,0))} F_i(s,x(s),y(s))\,x_i(s)\Delta s \\
&\le \frac{1}{1-\alpha_i}\int_0^T \Bigg[x_i(s)\sum_{k=1}^{n} a_{ik}(s)x_k(s-\tau_{ik}(t)) + \sum_{l=1}^{m} b_{il}(s)y_l(s-\vartheta_{il}(t)) \\
&\quad + \sum_{k=1}^{n} c_{ik}(s)\int_{-\infty}^{0} M_{ik}(w)x_k(w+s)\Delta w \\
&\quad + \sum_{l=1}^{m} d_{il}(s)\int_{-\infty}^{0} L_{il}(w)y_l(w+s))\Delta w\Bigg]\Delta s \\
&\le \frac{|x|_0}{1-\alpha_i}\int_0^T \Bigg[x_i(s)\sum_{k=1}^{n} a_{ik}(s)\,|x_k|_0 + \sum_{l=1}^{m} b_{il}(s)\,|y_l|_0 \\
&\quad + \sum_{k=1}^{n} c_{ik}(s)\int_{-\infty}^{0} M_{ik}(w)\,|x_k|_0\,\Delta w + \sum_{l=1}^{m} d_{il}(s)\int_{-\infty}^{0} L_{il}(w)\,|y_l|\,\Delta w\Bigg]\Delta s \\
&\le \frac{\|x\|^2}{1-\alpha_i}\int_0^T \left[\sum_{k=1}^{n}(a_{ik}(s)+c_{ik}(s)) + \sum_{k=1}^{n}(b_{il}(s)+d_{il}(s))\right]\Delta s \\
&\le \theta_i\,\|x\|^2 = \theta_i\hat{r}\,\|x\| < \theta_i\theta\,\|x\| < \|x\|
\end{aligned}$$

$$\begin{aligned}
\left|\hat{\Psi}_j(x)(t)\right| &= \int_t^{t+T} \frac{e_{b_j}(s,t)}{\left(e_{b_j}(T,0)-1\right)} G_j(s,x(s),y(s))y_j(s)\Delta s \\
&\leq \frac{\beta_j}{\beta_j-1}\int_0^T y_j(s)\left[\sum_{k=1}^{n}\hat{a}_{kj}(s)x_k(s-\hat{\tau}_{kj}(t))+\sum_{l=1}^{m}\hat{b}_{lj}(s)y_l(s-\hat{\vartheta}_{lj}(t))\right. \\
&\quad +\sum_{k=1}^{n}\hat{c}_{kj}(s)\int_{-\infty}^{0}\hat{M}_{kj}(w)x_k(w+s)\Delta w \\
&\quad \left.+\sum_{l=1}^{m}\hat{d}_{lj}(s)\int_{-\infty}^{0}\hat{L}_{lj}(w)y_l(w+s)\Delta w\right]\Delta s \\
&\leq \frac{\beta_j\left|y_j\right|_0}{\beta_j-1}\int_0^T y_j(s)\left[\sum_{k=1}^{n}\hat{a}_{kj}(s)\left|x_k\right|_0+\sum_{l=1}^{m}\hat{b}_{lj}(s)\left|y_l\right|_0\right. \\
&\quad +\sum_{k=1}^{n}\hat{c}_{kj}(s)\int_{-\infty}^{0}\hat{M}_{kj}(w)\left|x_k\right|_0\Delta w \\
&\quad \left.+\sum_{l=1}^{m}\hat{d}_{lj}(s)\int_{-\infty}^{0}\hat{L}_{lj}(w)\left|y_l\right|_0\Delta w\right]\Delta s \\
&\leq \hat{\theta}_j\left\|x\right\|^2=\hat{\theta}_j\hat{r}\left\|x\right\|<\hat{\theta}_j\theta\left\|x\right\|<\left\|x\right\|.
\end{aligned}$$

So we see that $\hat{r}=\|x\|=\mu\|\Psi x\|<\|x\|=\hat{r}$ which is contradiction. Hence condition $(C_2)$of Theorem (2) is satisfied.

Now, let $x \in D_{\hat{r},\hat{R}}$ with $x = \mu\Psi x$ and $\mu > 1$, we claim that $\|\varkappa\| \neq \hat{R}$, if this is not true, then $\|\varkappa\| = \hat{R}$, for any $t \in [0,T]$, we have

$$
\begin{aligned}
|\Psi_i(x)(t)| &= \int_t^{t+T} \frac{e_{\ominus a_i}(\sigma(s),t)}{(1-e_{\ominus a_i}(T,0))} F_i(s,x(s),y(s))\, x_i(s)\Delta s \\
&\geq \frac{\alpha_i}{1-\alpha_i}\int_0^T \Bigg[x_i(s)\sum_{k=1}^{n} a_{ik}(s)x_k(s-\tau_{ik}(t)) + \sum_{l=1}^{m} b_{il}(s)y_l(s-\vartheta_{il}(t)) \\
&\quad + \sum_{k=1}^{n} c_{ik}(s)\int_{-\infty}^{0} M_{ik}(w)x_k(w+s)\Delta w \\
&\quad + \sum_{l=1}^{m} d_{il}(s)\int_{-\infty}^{0} L_{il}(w)y_l(w+s))\Delta w \Bigg] \Delta s \\
&\leq \frac{\alpha_i^2 |x_i|_0}{1-\alpha_i}\int_0^T \Bigg[x_i(s)\sum_{k=1}^{n} a_{ik}(s)|x_k|_0 + \sum_{l=1}^{m} b_{il}(s)|y_l|_0 \\
&\quad + \sum_{k=1}^{n} c_{ik}(s)\int_{-\infty}^{0} M_{ik}(w)|x_k|_0 \Delta w + \sum_{l=1}^{m} d_{il}(s)\int_{-\infty}^{0} L_{il}(w)|y_l|\Delta w \Bigg] \Delta s \\
&\leq \frac{\alpha_i^2 \|x\|^2}{1-\alpha_i}\int_0^T \Bigg[\sum_{k=1}^{n}(a_{ik}(s)+c_{ik}(s)) + \sum_{k=1}^{n}(b_{il}(s)+d_{il}(s)) \Bigg] \Delta s \\
&\leq \pi_i \|x\|^2 = \pi_i \hat{R}\|x\| > \pi_i \pi \|x\| > \|x\|
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\hat{\Psi}_j(x)(t)\right| &= \int_t^{t+T} \frac{e_{b_j}(s,t)}{\left(e_{b_j}(T,0)-1\right)} G_j(s,x(s),y(s))y_j(s)\Delta s \\
&\geq \frac{1}{\beta_j-1}\int_0^T y_j(s)\left[\sum_{k=1}^{n}\hat{a}_{kj}(s)x_k(s-\hat{\tau}_{kj}(t))+\sum_{l=1}^{m}\hat{b}_{lj}(s)y_l(s-\hat{\vartheta}_{lj}(t))\right. \\
&\quad +\sum_{k=1}^{n}\hat{c}_{kj}(s)\int_{-\infty}^{0}\hat{M}_{kj}(w)x_k(w+s)\Delta w \\
&\quad \left. +\sum_{l=1}^{m}\hat{d}_{lj}(s)\int_{-\infty}^{0}\hat{L}_{lj}(w)y_l(w+s)\Delta w\right]\Delta s \qquad (5.6)\\
&\leq \frac{|y_j|_0}{\beta_j(\beta_j-1)}\int_0^T\left[\sum_{k=1}^{n}\hat{a}_{kj}(s)\,|x_k|_0+\sum_{l=1}^{m}\hat{b}_{lj}(s)\,|y_l|_0\right. \\
&\quad +\sum_{k=1}^{n}\hat{c}_{kj}(s)\int_{-\infty}^{0}\hat{M}_{kj}(w)\,|x_k|_0\,\Delta w \\
&\quad \left. +\sum_{l=1}^{m}\hat{d}_{lj}(s)\int_{-\infty}^{0}\hat{L}_{lj}(w)\,|y_l|_0\,\Delta w\right]\Delta s \\
&\leq \hat{\pi}_j\left\|x\right\|^2=\hat{\pi}_j\hat{R}\left\|x\right\|>\hat{\pi}_j\pi\left\|x\right\|>\left\|x\right\|.
\end{aligned}
$$

So, we see that $\hat{R}=\|x\|=\mu\|\Psi x\|>\|x\|=\hat{R}$ which is contradiction, so sondition is contradiction, so condition (C_1) of Theorem (2) hold. In addition it is easy to verify that $\inf_{\|x\|=r}\|\Psi x\|=r\neq 0$. Thus, all conditions of Theorem (2) are satisfied. Hence system (5.1) has a positive T-periodic solution. This completes the proof. □

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On the decay of a porous thermoelasticity type III with constant delay

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Abstract: This work focuses on the well-posedness and asymptotic stability of solutions of a delayed porous thermoelastic system of type III, where the delay acts on the heat equation. We investigate the cases of equal and nonequal wave speeds. In the first case, we establish an exponential rate of decay provided that the weight of delay is strictly less than the weight of the thermal dissipation. In the case of nonequal wave speeds, we obtain a polynomial decay rate.

Keywords: porous thermoelasticity, type III thermoelasticity, well-posedness, exponential decay, lack of exponential decay.

1 Introduction

In this work we consider the following problem

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, & \text{in } (0,\pi)\times(0,\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \beta\theta_x = 0, & \text{in } (0,\pi)\times(0,\infty), \\ \alpha\theta_{tt} - \kappa\theta_{xx} + \beta\phi_{ttx} - k_1\theta_{txx} - k_2\theta_{txx}(x,t-\tau) = 0, & \text{in } (0,\pi)\times(0,\infty). \end{cases} \tag{1}$$

where u, ϕ are the displacement and volume fraction, θ is the temperature difference.
We supplement system (1) by the following initial and boundary conditions

$$u(0,t) = u(\pi,t) = \phi_x(0,t) = \phi_x(\pi,t) = \theta(0,t) = \theta(\pi,t) = 0, \quad \text{in } (0,\infty), \tag{2}$$

$$\begin{aligned} &(u(x,0), \phi(x,0), \theta(x,0)) = (u_0(x), \phi_0(x), \theta_0(x)), && \text{in } (0,\pi), \\ &(u_t(x,0), \phi_t(x,0), \theta_t(x,0)) = (u_1(x), \phi_1(x), \theta_1(x)), && \text{in } (0,\pi). \end{aligned} \tag{3}$$

2 Well-posedness

Introducing the new variables $w = \varphi_t$, $v = \psi_t$,$q = \theta_t$ and

$$z(x,\omega,t) = \theta_t(x, t-\omega\tau), \quad \text{for } 0 \le \omega \le 1,$$

the problem can be written

$$(\varphi_t, w_t, \psi_t, v_t, \theta_t, q_t, z_t)^T = A(\varphi, w, \psi, v, \theta, q, z)^T$$

where A is the operator defined on

$$\mathcal{H} = H_0^1(0,\pi) \times L^2(0,\pi) \times H_*^1(0,\pi) \times L_*^2(0,\pi) \times H_0^1(0,\pi) \times L^2(0,\pi) \times L^2((0,1); H_0^1(0,\pi)),$$

such that

$$H_*^m(0,\pi) := \left\{\varphi \in H^m(0,\pi); \int_0^\pi \varphi dx = 0\right\}, m = 0,1 \text{ et } H^0 = L^2$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{lll} U \in \mathcal{H}: & \varphi \in H^2(0,\pi) \cap H_0^1(0,\pi), & \psi \in H_*^2(0,\pi) \cap H_*^1(0,\pi), \\ & w, q \in H_0^1(0,\pi), v \in H_*^1(0,\pi), & z \in H^1((0,1); H_0^1(0,\pi)), \\ & (\kappa\theta + k_1 q + k_2 z(\cdot,1)) \in H^2(0,\pi) & \end{array} \right\}.$$

The energy of system (1) is defined by

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^\pi & \left(\rho\varphi_t^2 + J\psi_t^2 + \alpha\theta_t^2 + \mu\varphi_x^2 + \xi\psi^2 + 2b\varphi_x\psi + \delta\psi_x^2 + \kappa\theta_x^2\right) dx \\ & + \frac{k_1\tau}{2}\int_0^\pi \int_0^1 z_x^2(x,\omega,t)\, d\omega dx, \end{aligned} \tag{4}$$

it satisfies

$$\frac{dE(t)}{dt} = -k_0\left(\int_0^\pi \theta_{tx}^2 dx + \int_0^\pi z_x^2(x,1,t)\, dx\right) \le 0. \tag{5}$$

where $k_0 := \dfrac{k_1 - |k_2|}{2} > 0$.

Theorem 2.1. *For any $(\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0, f_0)^T \in \mathcal{H}$, the problem (1)-(2-(3) has a unique mild solution $(\varphi, w, \psi, v, \theta, q, z)^T$ such that*

$$(\varphi, w, \psi, v, \theta, q, z)^T \in C([0,+\infty[; \mathcal{H}).$$

Moreover, if $(\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0, f_0)^T \in D(A)$, then the solution $(\varphi, w, \psi, v, \theta, q, z)^T$ satisfies

$$(\varphi, w, \psi, v, \theta, q, z)^T \in C([0,+\infty[; D(A)) \cap C^1([0,+\infty[; \mathcal{H}).$$

Proof. The proof of this theorem will be given by the use of the semigroup approach and the HILLE-YOSIDA theorem. □

3 Exponential Decay

Theorem 3.1. *Suppose that*

$$\chi = \frac{\mu}{\rho} - \frac{\delta}{\mathrm{J}} = 0 \text{ and } |\mathrm{k}_2| < \mathrm{k}_1.$$

Then there exist two positive constants A, λ, for which the energy functional (4) satisfies the estimate

$$E(t) \le Ae^{-\lambda t}, \ \forall t \ge 0. \tag{6}$$

The proof of this theorem is based on Multiplier method.

4 Polynomial stability

we introduce the second-order energy defined by

$$E_* (t) = \frac{1}{2} \int_0^\pi \left(\rho \varphi_{tt}^2 + J \psi_{tt}^2 + \alpha \theta_{tt}^2 + \delta \psi_{tx}^2 + \mu \varphi_{tx}^2 + \xi \psi_t^2 + 2b \varphi_{tx} \psi_t + \kappa \theta_{tx}^2 \right) dx$$
$$+ \frac{k_1 \tau}{2} \int_0^\pi \int_0^1 z_{xt}^2 (x, 1, t) \, d\omega dx.$$

$E_* (t)$ satisfies

$$E_*' (t) \leq -k_0 \left(\int_0^\pi \theta_{ttx}^2 dx + \int_0^\pi z_{tx}^2 (x, 1, t) \, dx \right) \leq 0. \quad (7)$$

Theorem 4.1. *Suppose that*

$$\chi = \frac{\mu}{\rho} - \frac{\delta}{J} \neq 0,$$

and let (φ, ψ, θ) *be the strong solution of (1). Then there exists a positive constant* λ *such that the energy* $E_*(t)$ *satisfies,*

$$E_* (t) \leq \frac{\lambda}{t}, \forall t > 0. \quad (8)$$

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A DYNAMIC CONTACT PROBLEM BETWEEN VISCOELASTIC PIEZOELECTRIC BODIES WITH FRICTION AND DAMAGE

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1. Introduction

In this paper we study a frictional bilateral contact problem involving the piezoelectric effect. The piezoelectricity can be described as follows: when mechanical pressure is applied to a certain classes of crystalline materials (e.g ceramics $BaTiO_3$, $BiFeO_3$), the crystalline structure produces a voltage proportional to the pressure. Conversely, when an electric field is applied, the structure changes his shape producing dimensional modifications in the material. Actually, there is a big interest into the study of piezoelectric materials, this type of materials being used in radioelectronics, electroacoustics and measuring equipments. In the same time, due to the fact that the parts of the equipments are in contact, the interest for the contact problems is increasing. Different models have been developed to describe the interaction between the electrical and mechanical fields see for example [3, 10] and the references therein. General models for elastic materials with piezoelectric effect, called electro-elastic materials, can be found in [3]. A static frictional contact problem for electro-elastic materials was considered in [4], under the assumption that the foundation is insulated. Contact problems involving elasto-piezoelectric materials [4, 11], viscoelastic piezoelectric materials [5, 2] have been studied, contact problem for electro-elastio-viscoplastic materials was studied in [9].

The damage is an extremely important topic in engineering, since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. General models for damage were derived in [7] from the virtual power principle. The models of mechanical damage, which were derived from thermodynamical considerations and the principle of virtual work, can be found in [1, 6]. The new idea of [8, 7] was the introduction of the *damage function*$\zeta^\alpha = \zeta^\alpha(x,t)$, which is the ratio between the elastic moduli of the damage and damage-free materials. In an isotropic and homogeneous elastic material, let E_Y^α be the Young modulus of the original material and E_{eff}^α be the current modulus, then the damage function is defined by $\zeta^\alpha = E_{eff}^\alpha / E_Y^\alpha$. Clearly, it follows from this definition that the damage function ζ^α is restricted to have values between zero and one. When $\zeta^\alpha = 1$, there is no damage in the material, when $\zeta^\alpha = 0$, the material is completely damaged, when $0 < \zeta^\alpha < 1$ there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [12]. The differential inclusion used for the evolution of the damage field is

$$\dot{\zeta}^\alpha - \Delta\zeta^\alpha + k^\alpha \partial\chi_{K^\alpha}(\zeta^\alpha) \ni S^\alpha(\varepsilon(u^\alpha), \zeta^\alpha) \quad \text{in } \Omega^\alpha \times (0,T), \tag{1}$$

where k^α is a positive coefficient and K^α the set of admissible damage defined by

$$K^\alpha = \{\xi \in H^1(\Omega^\alpha);\ 0 \le \xi \le 1,\ \text{a.e. in } \Omega^\alpha\}. \tag{2}$$

2. Problem statement and variational formulation

Let us consider two electro-viscoelastic bodies, occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d (d = 2, 3)$. For each domain Ω^α, the boundary Γ^α is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1^α, Γ_2^α and Γ_3^α, on one hand, and on two measurable parts Γ_a^α and Γ_b^α, on the other hand, such that $meas\Gamma_1^\alpha > 0$, $meas\Gamma_a^\alpha > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The Ω^α body is submitted to f_0^α forces and volume electric charges of density q_0^α. The bodies are assumed to be clamped on Γ_1^α. The surface tractions f_2^α act on Γ_2^α. We also assume that the electrical potential vanishes on Γ_a^α and a surface electric charge of density q_2^α is prescribed on Γ_b^α. The two bodies can enter in contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. The classical form of bilateral contact with Tresca's friction and damage between two electro-viscoelastic bodies is given by:

Problem P. For $\alpha = 1, 2$, find a displacement field $u^\alpha : \Omega^\alpha \times (0, T) \to \mathbb{R}^d$, a stress field $\sigma^\alpha : \Omega^\alpha \times (0, T) \to \mathbb{S}^d$, an electric potential $\varphi^\alpha : \Omega^\alpha \times (0, T) \to \mathbb{R}$, an electric displacement field $D^\alpha : \Omega^\alpha \times (0, T) \to \mathbb{R}^d$, and a damage $\zeta^\alpha : \Omega^\alpha \times (0, T) \to \mathbb{R}$ such that

$$\sigma^\alpha = \mathcal{A}^\alpha \varepsilon(\dot{u}^\alpha) + \mathcal{B}^\alpha(\varepsilon(u^\alpha), \zeta^\alpha) + (\mathcal{E}^\alpha)^* \nabla \varphi^\alpha \ \text{in}\ \ \Omega^\alpha \times (0, T), \tag{3}$$

$$D^\alpha = \mathcal{E}^\alpha \varepsilon(u^\alpha) + \mathcal{C}^\alpha E(\varphi^\alpha) \quad \text{in}\ \Omega^\alpha \times (0, T), \tag{4}$$

$$\dot{\zeta}^\alpha - \Delta \zeta^\alpha + k^\alpha \partial \chi_{K^\alpha}(\zeta^\alpha) \ni S^\alpha(\varepsilon(u^\alpha), \zeta^\alpha) \quad \text{in}\ \Omega^\alpha \times (0, T), \tag{5}$$

$$\operatorname{Div} \sigma^\alpha + f_0^\alpha = \rho^\alpha \ddot{u}^\alpha \quad \text{in}\ \Omega^\alpha \times (0, T), \tag{6}$$

$$\operatorname{div} D^\alpha - q_0^\alpha = 0 \quad \text{in}\ \Omega^\alpha \times (0, T), \tag{7}$$

$$u^\alpha = 0 \quad \text{on}\ \Gamma_1^\alpha \times (0, T), \tag{8}$$

$$\sigma^\alpha \nu^\alpha = f_2^\alpha \quad \text{on}\ \Gamma_2^\alpha \times (0, T), \tag{9}$$

$$\begin{cases} [u_\nu] = 0\ ,\ \sigma_\tau^1 = -\sigma_\tau^2 \equiv \sigma_\tau, \quad \|\sigma_\tau\| \le g \\ \|\sigma_\tau\| < g \Rightarrow [\dot{u}_\tau] = 0 \\ \|\sigma_\tau\| = g \Rightarrow \exists \delta \ge 0 \text{ such that } \sigma_\tau = -\delta\, [\dot{u}_\tau] \end{cases} \quad \text{on}\ \Gamma_3 \times (0, T), \tag{10}$$

$$\frac{\partial \zeta^\alpha}{\partial \nu^\alpha} = 0 \quad \text{on}\ \Gamma^\alpha \times (0, T), \tag{11}$$

$$\varphi^\alpha = 0 \quad \text{on}\ \Gamma_a^\alpha \times (0, T), \tag{12}$$

$$D^\alpha \cdot \nu^\alpha = q_2^\alpha \quad \text{on}\ \Gamma_b^\alpha \times (0, T), \tag{13}$$

$$u^\alpha(0) = u_0^\alpha,\ \dot{u}^\alpha(0) = v_0^\alpha,\ \zeta^\alpha(0) = \zeta_0^\alpha \quad \text{in}\ \Omega^\alpha. \tag{14}$$

Here, Eqs (3) and (4) represent the electro-viscoelastic constitutive law. The evolution of the damage field is governed by the inclusion given by the relation (5). Next, Eqs (6) and (7) are the equations of motion written for the stress field and of balance written for the electric displacement field, respectively, in which Div and div denote the divergence operators for tensor and vector valued functions. Conditions (8) and (9) are the displacement and traction boundary conditions, respectively. The relation (11) represents a homogeneous Neumann boundary condition, (12) and (13) represent the electric boundary conditions, and (14) are the initial conditions. Conditions (10) represent the bilateral contact condition with Tresca's friction law where $[u_\nu] = u^1_\nu + u^2_\nu$ is the stands for the displacements in normal direction, and where the friction yield limit is g which is assumed to depend only on each point of Γ_3, where $[u_\tau] = u^1_\tau - u^2_\tau$ stands for the jump of the displacements in tangential direction.

Now, to proceed with the variational formulation, we need the following function spaces:

$$E_0 = L^2(\Omega^1) \times L^2(\Omega^2),\ H^\alpha = [L^2(\Omega^\alpha)]^d,\ \mathcal{H}^\alpha = [L^2(\Omega^\alpha)]^{d\times d}_s,$$

$$E_1 = H^1(\Omega^1) \times H^1(\Omega^2),\ H = H^1 \times H^2,\ \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2,$$

and define the following spaces:

$$V^\alpha = \{v^\alpha \in [H^1(\Omega^\alpha)]^d;\ \ v^\alpha\big|_{\Gamma^\alpha_1} = 0\},\quad W^\alpha = \{\psi^\alpha \in H^1(\Omega^\alpha);\ \psi^\alpha\big|_{\Gamma^\alpha_a} = 0\},$$

$$\mathcal{W}^\alpha = \{\mathbf{D}^\alpha \in H^\alpha;\ \operatorname{div}\mathbf{D}^\alpha \in L^2(\Omega^\alpha)\},\quad W = W^1 \times W^2,$$

$$\mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2,\quad \mathbf{V} = \{v \in V^1 \times V^2;\ \ [v_\nu]\big|_{\Gamma_3} = 0\}.$$

Since $meas\Gamma^\alpha_1 > 0$, Korn's inequality holds and there exists a constant $C_K > 0$ depending only on Ω^α and Γ^α_1, such that

$$\|\varepsilon(v^\alpha)\|_{\mathcal{H}^\alpha} \geq C_K\|v^\alpha\|_{H^\alpha_1},\quad \forall v^\alpha \in V^\alpha, \tag{15}$$

On the space V^α, we consider the inner product and the associated norm given by

$$(u^\alpha, v^\alpha)_{V^\alpha} = (\varepsilon(u^\alpha), \varepsilon(v^\alpha))_{\mathcal{H}^\alpha},\qquad \forall u^\alpha, v^\alpha \in V^\alpha, \tag{16}$$

and let $\|v^\alpha\|_{V^\alpha}$ the associated norm given by

$$\|v^\alpha\|_{V^\alpha} = \|\varepsilon(v^\alpha)\|_{\mathcal{H}^\alpha},\quad \forall v^\alpha \in V^\alpha. \tag{17}$$

Notice also that, since $meas\Gamma^\alpha_a > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\zeta^\alpha\|_{L^2(\Omega^\alpha)^d} \geq C_F\,\|\zeta^\alpha\|_{H^1(\Omega^\alpha)},\quad \forall \zeta^\alpha \in W^\alpha, \tag{18}$$

where $C_F > 0$ is a constant which depends only on Ω^α and Γ^α_a and $\nabla\xi^\alpha = (\xi^\alpha_{,i})$.

Further, we denote by X' the dual space of X, and we use the notation $\langle .,.\rangle_{X'\times X}$ to represent the duality pairing between X' and X.
We now list assumptions on the data. Assume the operators $\mathcal{A}^\alpha, \mathcal{B}^\alpha, \mathcal{E}^\alpha, \mathcal{C}^\alpha$ and S^α satisfy the following conditions ($L_{\mathcal{A}^\alpha}, m_{\mathcal{A}^\alpha}, L_{\mathcal{B}^\alpha}, m_{\mathcal{C}^\alpha}$ and L_{S^α} being positive

constants).

$$\left.\begin{array}{l}\text{(a) } \mathcal{A}^\alpha : \Omega^\alpha \times \mathbb{S}^d \to \mathbb{S}^d,\\ \text{(b) } \|\mathcal{A}^\alpha(x,\varepsilon_1) - \mathcal{A}^\alpha(x,\varepsilon_2)\| \le L_{\mathcal{A}^\alpha}\|\varepsilon_1-\varepsilon_2\| \ \ \forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x\in\Omega^\alpha,\\ \text{(c) } (\mathcal{A}^\alpha(x,\varepsilon_1) - \mathcal{A}^\alpha(x,\varepsilon_2)).(\varepsilon_1-\varepsilon_2) \ge m_{\mathcal{A}^\alpha}\|\varepsilon_1-\varepsilon_2\|^2\\ \qquad \forall \varepsilon_1,\varepsilon_2\in\mathbb{S}^d \text{ a.e } x\in\Omega^\alpha,\\ \text{(d) For any } \xi\in\mathbb{S}^d, x\mapsto \mathcal{A}^\alpha(x,\xi) \text{ is measurable on } \Omega^\alpha,\\ \text{(e) The mapping } x\mapsto \mathcal{A}^\alpha(x,0)\in\mathcal{H}^\alpha.\end{array}\right\} \tag{19}$$

$$\left.\begin{array}{l}\text{(a) } \mathcal{B}^\alpha : \Omega^\alpha \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d,\\ \text{(b) } \|\mathcal{B}^\alpha(x,\varepsilon_1,\zeta_1) - \mathcal{B}^\alpha(x,\varepsilon_2,\zeta_2)\| \le L_{\mathcal{B}^\alpha}(\|\varepsilon_1-\varepsilon_2\| + |\zeta_1-\zeta_2|)\\ \qquad \forall\varepsilon_1,\ \varepsilon_2\in\mathbb{S}^d, \forall\zeta_1,\ \zeta_2\in\mathbb{R}, \text{ a.e. } x\in\Omega^\alpha,\\ \text{(c) The mapping } x\mapsto\mathcal{B}^\alpha(x,\varepsilon,\zeta) \text{ is measurable in } \Omega^\alpha \ \ \forall\varepsilon\in\mathbb{S}^d, \forall\zeta\in\mathbb{R},\\ \text{(d) The mapping } x\mapsto\mathcal{B}^\alpha(x,0,0)\in\mathcal{H}^\alpha.\end{array}\right\} \tag{20}$$

$$\left.\begin{array}{l}\text{(a) } \mathcal{E}^\alpha : \Omega^\alpha\times\mathbb{S}^d\to\mathbb{R}^d,\\ \text{(b) } \mathcal{E}^\alpha(x,\tau) = (e^\alpha_{ijk}(x)\tau_{jk}) \quad \text{where } e^\alpha_{ijk} = e^\alpha_{ikj}\in L^\infty(\Omega^\alpha).\end{array}\right\} \tag{21}$$

$$\left.\begin{array}{l}\text{(a) } \mathcal{C}^\alpha : \Omega^\alpha\times\mathbb{R}^d\to\mathbb{R}^d,\\ \text{(b) } \mathcal{C}^\alpha(x,E) = (c^\alpha_{ij}(x)E_j) \quad \forall E=(E_i)\in\mathbb{R}^d, \quad \text{a.e. } x\in\Omega^\alpha,\\ \text{(c) } c^\alpha_{ij} = c^\alpha_{ji},\ c^\alpha_{ij}\in L^\infty(\Omega^\alpha), \quad 1\le i,j\le d,\\ \text{(d) } \mathcal{C}^\alpha E.E \ge m_{\mathcal{C}^\alpha}|E|^2, \quad \forall E=(E_i)\in\mathbb{R}^d, \text{ a.e. } x\in\Omega^\alpha.\end{array}\right\} \tag{22}$$

$$\left.\begin{array}{l}\text{(a) } S^\alpha : \Omega^\alpha\times\mathbb{S}^d\times\mathbb{R}\to\mathbb{R},\\ \text{(b) } |S^\alpha(x,\xi_1,d_1) - S^\alpha(x,\xi_2,d_2)| \le L_{S^\alpha}\big(|\xi_1-\xi_2| + |d_1-d_2|\big),\\ \qquad \forall\,\xi_1,\xi_2\in\mathbb{S}^d, \forall d_1,d_2\in\mathbb{R} \text{ a.e. } \mathbf{x}\in\Omega^\alpha,\\ \text{(c) For any } \xi\in\mathbb{S}^d, d\in\mathbb{R}, \quad x\mapsto S^\alpha(x,\xi,d) \text{ is measurable on } \Omega^\alpha,\\ \text{(d) The mapping } x\mapsto S^\alpha(x,0,0) \text{ belongs to } L^2(\Omega^\alpha).\end{array}\right\} \tag{23}$$

The mass density and the friction yield limit g satisfies

$$\rho^\alpha\in L^\infty(\Omega^\alpha), \quad \min_{\alpha=1,2}\inf_{x\in\Omega^\alpha}\rho^\alpha(x) = \rho^* > 0, \tag{24}$$

$$g\in L^\infty(\Gamma_3), \quad g\ge 0 \ \text{ on } \Gamma_3. \tag{25}$$

The forces, tractions, volume and surface free charge densities have the regularity

$$f^\alpha_0\in L^2(0,T;H^\alpha), \quad f^\alpha_2\in L^2(0,T;L^2(\Gamma^\alpha_2)^d), \tag{26}$$

$$q^\alpha_0\in C(0,T;L^2(\Omega^\alpha)), \quad q^\alpha_2\in C(0,T;L^2(\Gamma^\alpha_b)), \tag{27}$$

$$q^\alpha_2(t) = 0 \quad \text{on } \Gamma_3 \ \ \forall t\in[0,T]. \tag{28}$$

Finally, we assume that initial data satisfy the regularity

$$u^\alpha_0\in V^\alpha, \ \ v^\alpha_0\in V^\alpha, \ \ \zeta^\alpha_0\in K^\alpha. \tag{29}$$

We define the mappings $F:[0,T]\to\mathbf{V}'$, $q:[0,T]\to W$, $a:E_1\times E_1\to\mathbb{R}$ and $j:\mathbf{V}\to\mathbb{R}$ respectively, by

$$\left.\begin{array}{l}\langle F(t),v\rangle_{V'\times V} = \sum\limits_{\alpha=1}^{2}\int\limits_{\Omega^\alpha} f^\alpha_0(t)v^\alpha dx + \sum\limits_{\alpha=1}^{2}\int\limits_{\Gamma^\alpha_2} f^\alpha_2(t)v^\alpha da, \quad \forall v\in V,\\ (q(t),\phi)_W = \sum\limits_{\alpha=1}^{2}\int\limits_{\Omega^\alpha} q^\alpha_0(t)\phi^\alpha dx - \sum\limits_{\alpha=1}^{2}\int\limits_{\Gamma^\alpha_b} q^\alpha_2(t)\phi^\alpha da, \quad \forall\phi\in W,\\ a(\zeta,\xi) = \sum\limits_{\alpha=1}^{2} k^\alpha\int\limits_{\Omega^\alpha}\nabla\zeta^\alpha.\nabla\xi^\alpha dx \ \text{ and } \ j(v) = \int\limits_{\Gamma_3} g\|\,[v_\tau]\,\|_{L^2(\Gamma_3)^d}da.\end{array}\right\} \tag{30}$$

From the assumptions (26) and (27) it follows that

$$F \in L^2(0,T;V'), \quad q \in C(0,T;W). \tag{31}$$

We use a modified inner product on H given by

$$((u,v))_H = \sum_{\alpha=1}^{2} (\rho^\alpha u^\alpha, v^\alpha)_{H^\alpha}, \quad \forall u, v \in H, \tag{32}$$

that is, it is weighted with ρ^α. We let $|||.|||_H$ be the associated norm, i.e.,

$$|||v|||_H = ((v,v))_H^{\frac{1}{2}}, \quad \forall v \in H. \tag{33}$$

It follows from assumption (24), that $|||.|||_H$ and $\|.\|_H$ are equivalent norms on H, and also the inclusion mapping of V into H is continuous and dense. Identifying H with its own dual, we can write the Gelfand triple $V \subset H \subset V'$, so we have

$$\langle u, v\rangle_{V'\times V} = ((u,v))_H, \quad \forall u \in H, \ \forall v \in V. \tag{34}$$

By a standard procedure based on integration by parts and Green's formula, we obtain the following weak formulation of the piezoelectric contact problem P.

Problem PV. Find $u : [0,T] \to V$, $\varphi : [0,T] \to W$ and $\zeta : [0,T] \to E_1$ such that

$$\left.\begin{array}{l} \langle \ddot{u}(t), w - \dot{u}(t)\rangle_{V'\times V} + \sum\limits_{\alpha=1}^{2} (\mathcal{A}^\alpha \varepsilon(\dot{u}^\alpha) + \mathcal{B}^\alpha(\varepsilon(u^\alpha), \zeta^\alpha), \varepsilon(w^\alpha - \dot{u}^\alpha(t)))_{\mathcal{H}^\alpha} + \\ \sum\limits_{\alpha=1}^{2} ((\mathcal{E}^\alpha)^* \nabla\varphi^\alpha, \varepsilon(w^\alpha - \dot{u}^\alpha(t)))_{\mathcal{H}^\alpha} + j(w) - j(\dot{u}(t)) \geq \langle F(t), w - \dot{u}(t)\rangle_{V'\times V} \\ \forall w \in V, \quad \text{a.e. } t \in (0,T), \end{array}\right\} \tag{35}$$

$$\sum_{\alpha=1}^{2} (\mathcal{C}^\alpha \nabla\varphi^\alpha(t) - \mathcal{E}^\alpha \varepsilon(u^\alpha(t)), \nabla\phi^\alpha)_{H^\alpha} = (q(t), \phi)_W, \quad \forall \phi \in W, \ \text{a.e. } t \in (0,T), \tag{36}$$

$$\left.\begin{array}{l} \zeta(t) \in K = K^1 \times K^2, \quad \sum\limits_{\alpha=1}^{2} (\dot{\zeta}^\alpha(t), \xi^\alpha - \zeta^\alpha(t))_{L^2(\Omega^\alpha)} + a(\zeta(t), \xi - \zeta(t)) \\ \geq \sum\limits_{\alpha=1}^{2} (S^\alpha(\varepsilon(u^\alpha(t)), \zeta^\alpha(t)), \xi^\alpha - \zeta^\alpha(t))_{L^2(\Omega^\alpha)}, \quad \forall \xi \in K, \quad \text{a.e. } t \in (0,T), \end{array}\right\} \tag{37}$$

$$u(0) = u_0, \ \dot{u}(0) = w_0, \ \zeta(0) = \zeta_0. \tag{38}$$

The existence of a unique solution to Problem **PV** will be presented in the next section.

3. Main existence and uniqueness result

Now, we propose our existence and uniqueness result.

Theoreme 3.1. *Under the assumptions (19)–(29). Then there exists a unique solution* $\{u, \varphi, \zeta\}$ *to problem* **PV**. *Moreover, the solution satisfies*

$$u \in W^{1,2}(0,T;V) \cap C^1(0,T;H) \cap W^{2,2}(0,T;V'), \tag{39}$$

$$\varphi \in C(0,T;W), \tag{40}$$

$$\zeta \in W^{1,2}(0,T;E_0) \cap L^2(0,T;E_1). \tag{41}$$

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Analysis and applications for Caputo generalized hybrid Langevin differential systems

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Abstract

This research inscription gets to grips with two novel varieties of boundary value problems. One of them is a hybrid Langevin fractional differential equation. Whilst the other is a coupled system of hybrid Langevin differential equation encapsuling a collective fractional derivative known as the ψ-Caputo fractional operator. Such operators are generated by iterating a local integral of a function with respect to another increasing positive function Y.The existence of the solutions of the aforehanded equations is tackled by using Dhage fixed point theorem while their uniqueness is handled capitalizing on the Banach fixed point theorem. On the top of this, the stability within the scope of Ulam-Hyers of solutions to these systems are also considered. Two pertinent examples are presented to corroborate the reported results.

KEY WORDS: ψ-Caputo; Coupled system; Ulam-Hyers stability; Langevin; Hybrid; Existence and uniqueness.

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Multiple solutions for subcritical and critical p-fractional elliptic equations via Nehari manifold method

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Abstract: This paper deals with the existence and multiplicity of nonnegative solutions to the following p-fractional Laplacian problem

$$\begin{cases} (-\Delta)_p^s u(x) = \lambda |u|^{p-2} u + f(x,u) + \mu g(x,u) \text{ in } \Omega, u > 0, \\ u = 0 \text{ on } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{E}$$

where $\Omega \subset \mathbb{R}^n (n > ps)$, is a bounded smooth domain, $s \in (0,1), \lambda,\ \mu$ are positive parameters, and $f, g : \overline{\Omega} \times [0,\infty) \longrightarrow \mathbb{R}$, are continuous functions. Using variational methods, especially, fibering maps and Nehari manifold, we obtain existence results for either, subcritical and critical cases. The results of the present paper, extend previous works which have recently appeared in the literature.
Keywords: Nehari manifold , fibering maps, multiplicity of solutions..
2010 Mathematics Subject Classification: 35P30, 35J35, 35J60.

1 Introduction

Our first result about the sub-critical and concave case is the following.

Theorem 1.1. *Let $s \in (0,1)$. Assume that the nonlinearities f, g are continuous satisfying homogenous conditions. If*

$$0 < r < 1 < p < q < p_s^* - 1, \text{ and } n > ps.$$

Then, for all $\lambda \in (0, \lambda_1)$, there exists $\mu_* (\lambda) > 0$, such that, for all $\mu \in (0, \mu_* (\lambda))$, problem (E) *has at least two positive solutions.*

The second main result of this paper is devoted to the critical case $(q = p_s^* - 1)$. Since the embedding $X_0 \hookrightarrow L^{p_s^*}(\mathbb{R}^n)$, is not compact, then the energy functional does not satisfy the Palais-Smale condition globally, but it is true for the energy functional in a suitable range related to the best fractional critical Sobolev constant.

Theorem 1.2. *Assume that $s \in (0,1), n > ps$ and $0 < r < 1 < p < q = p_s^* - 1$. If there exist $t_0 > 0$ and $u_0 \in X_0 \backslash \{0\}$, with $u_0 > 0$ in $\mathbb{R}^n$, such that*

$$\left(\frac{1}{p} A(u_0) t_0^p - t_0^{p_s^*} B(u_0) \right) < \frac{s}{n} \left(p_s^* \gamma_1\right)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{sp}}. \tag{1.1}$$

Then, for all $\lambda \in (0, \lambda_1)$, there exists $\mu^* (\lambda) > 0$, such that, for all $\mu \in (0, \mu^* (\lambda))$, problem (E) *has at least two positive solutions.*

2 main results

Proof of Theorem (1.1) In order to prove Theorem (1.1), we need to present several results.

Proposition 2.1. *There exists a minimizer $u_{\lambda,\mu}$ in $\mathcal{N}_{\lambda,\mu}^+$ for $J_{\lambda,\mu}$ satisfying:*
(1) $J_{\lambda,\mu}(u_{\lambda,\mu}) = \alpha_{\lambda,\mu}^+ < 0$.
(2) $u_{\lambda,\mu}$ is a solution of problem (E).

Proposition 2.2. *If $0 < r < 1 < q < p_s^* - 1$. Then, $J_{\lambda,\mu}$ has a minimizer $v_{\lambda,\mu}$ in $\mathcal{N}_{\lambda,\mu}^-$ satisfying*
(1) $J_{\lambda,\mu}(v_{\lambda,\mu}) = \alpha_{\lambda,\mu}^- > 0$.
(2) $v_{\lambda,\mu}$ is a solution of problem (E).

Proof [**Proof of Theorem (1.1)**] By Propositions (2.1), (2.2) and Lemma (**??**), we get that problem (E) has two solutions $u_{\lambda,\mu} \in \mathcal{N}_{\lambda,\mu}^+$ and $v_{\lambda,\mu} \in \mathcal{N}_{\lambda,\mu}^-$on X_0. Since $\mathcal{N}_{\lambda,\mu}^+ \cap \mathcal{N}_{\lambda,\mu}^- = \emptyset$, then, $u_{\lambda,\mu}$ and $v_{\lambda,\mu}$ are distinct. This completes the proof of Theorem 1.1. ■ **Proof of Theorem (1.2)** Remark: The number

of references should not exceed five.

Proposition 2.3. *Assume that $0 < r < 1 < q = p_s^* - 1$. Then, every Palais smail sequence $\{u_k\} \subset X_0$ for $J_{\lambda,\mu}$ at level c, with*

$$c < \frac{s}{n} \left(p_s^* \gamma_1\right)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} \left(1 - \frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} - M \left(1 - \frac{\lambda}{\lambda_1}\right)^{-\frac{r+1}{p-r-1}} \mu^{\frac{p}{p-r-1}}, \tag{2.1}$$

has a convergent subsequence, where S_p is the Sobolev best emmbedding constant .

Proposition 2.4. *There exist $\mu^*(\lambda) > 0$, $\tilde{t}_1$ and $\tilde{u}_1 \in X_0$, such that, for all $(\lambda, \mu) \in (0, \lambda_1) \times (0, \mu^*(\lambda))$, we have*

$$J_{\lambda,\mu}(\tilde{t}_1 \tilde{u}_1) \leq \frac{s}{n} \left(p_s^* \gamma_1\right)^{-\frac{n}{sp_s^*}} S_p^{\frac{n}{ps}} \left(1 - \frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} - M \left(1 - \frac{\lambda}{\lambda_1}\right)^{-\frac{r+1}{p-r-1}} \mu^{\frac{p}{p-r-1}}. \tag{2.2}$$

In particular,

$$\alpha_{\lambda,\mu}^- < \frac{s}{n} \left(p_s^* \gamma_1\right)^{-\frac{n}{sp_s^*}} S_p^{\frac{n}{ps}} \left(1 - \frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} - M \left(1 - \frac{\lambda}{\lambda_1}\right)^{-\frac{r+1}{p-r-1}} \mu^{\frac{p}{p-r-1}}. \tag{2.3}$$

Proof of Theorem (1.2) By Propositions (2.3) and (2.4), there exists two sequences $\{u_k^+\}$ and $\{u_k^-\}$ in X_0, such that

$$\begin{aligned} J_{\lambda,\mu}(u_k^+) &\longrightarrow \alpha_{\lambda,\mu}^+, J'_{\lambda,\mu}(u_k^+) \longrightarrow 0, \\ &\text{and} \\ J_{\lambda,\mu}(u_k^-) &\longrightarrow \alpha_{\lambda,\mu}^-, J'_{\lambda,\mu}(u_k^-) \longrightarrow 0. \end{aligned}$$

as $k \longrightarrow \infty$. We observe that from the analysis of fibering maps $\varphi_u(t)$, we have $\alpha_{\lambda,\mu}^+ < 0$. Similar to the proof of Propositions (2.1) and (2.2) and Theorem (1.1), problem (E) has two solutions $u_{\lambda,\mu} \in \mathcal{N}_{\lambda,\mu}^+$ and $v_{\lambda,\mu} \in \mathcal{N}_{\lambda,\mu}^-$ in X_0.Since $\mathcal{N}_{\lambda,\mu}^+ \cap \mathcal{N}_{\lambda,\mu}^- = \emptyset$, then these two solutions are distinct. This finishes the proof.

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Modeling of the reinforcement of a Kirchhoff-Love plate with a thin layer of varying thickness.

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1 Abstract

In this paper, we deal with the asymptotic modeling of the behavior of an elastic plate reinforced with a thin layer of varying thickness $\delta(x) = \delta f(x)$, where δ is a small positive parameter.
More precisely, an extension of the results obtained in a previous work, where the case of a layer with constant thickness was studied, is given. More general approximate boundary conditions are derived, valid for a larger class of layers, having a thickness variation as a function of geometry coordinates. Optimal Error estimates between the exact and the approximate solutions of the reinforced problem are proved.

2 Keywords

Asymptotic modeling, Asymptotic expansion method, Reinforcement, Approximate boundary conditions, Kirchhoff-Love plate, Thin layer with variable thickness.

3 Introduction

In this work, we consider the situation where the thin layer is of varying thickness $\delta(x) = \delta f(x)$, where δ is a small positive parameter and $x \longmapsto f(x)$ is a smooth function with respect to the variable x.
In this work, we handle the special case where the rigidities of both the plate and the layer are independent of the small parameter δ. In this situation, the stiffness of the thin layer does not compensate its small thickness and the convergence approach used before leads to a limit model into which the

effect of the layer is ignored. As it was already pointed out, we use an other asymptotic method to identify a more precise model that takes into account the effect of the thin layer. This method, called asymptotic expansions method consists in constructing an asymptotic expansion of the solution of the scaled problem and defining approximate models by truncating this expansion at a given order. To ensure the construction of the terms of this asymptotic expansion (which requires more regularity of the domain), we consider here a curved plate surrounded with a thin layer of varying thickness. Making an approximation of order 0 (by keeping just the first term in the asymptotic expansion), we obtain a model into which the effect of the thin layer is neglected. A better approximation is obtained by truncating the expansion at order one. It leads to a model into which the effect of the thin layer is incorporated via new approximate boundary conditions. Let us emphasize that the derivation of these approximate boundary conditions for a layer with varying thickness induces more technical difficulties comparatively with layers of constant thickness. Moreover, we give error estimates between the exact and the approximate solution.

4 The Kirchhoff-Love problem for a reinforced plate with layer of varying thickness

We consider a bi-dimensional elastic plate occupying the set $\Omega_+ =]0,1[\times]0,1[$, with boundary $\partial\Omega_+ = \Sigma \cup \Gamma_+$, where $\Sigma =]0,1[\times \{0\}$. The plate is clamped on the portion Γ_+ and reinforced by a thin layer on the part Σ. This last one occupies the set

$$\Omega_-^\delta = \Big\{ (x,y) \in \mathbb{R}^2,\ 0 < x < 1,\ \ -\delta f(x) < y < 0 \Big\},$$

where δ is a small positive parameter and the function $x \longmapsto f(x)$ is supposed to be sufficiently derivable with respect to the variable x. The boundary of Ω_-^δ is given by $\partial\Omega_-^\delta = \Sigma_-^\delta \cup \Sigma \cup \Gamma_-^\delta$, where :

$$\Sigma_-^\delta = \Big\{ (x,y) \in \mathbb{R}^2,\ 0 < x < 1,\ y = \delta f(x) \Big\}.$$

We denote by $\Omega^\delta = \Omega_+ \cup \Sigma \cup \Omega_-^\delta$ the complete domain occupied by the whole structure.
The Kirchhoff-Love model for this structure is given by :

- **Equilibrium equations :**

$$\begin{aligned} D_+\Delta^2 w_+ &= g_+ \qquad \text{in} \quad \Omega_+, \\ D_-\Delta^2 w_- &= 0 \qquad \text{in} \quad \Omega_-^\delta. \end{aligned}$$

- **Clamped boundary conditions :**

$$w = 0, \quad \partial_n w = 0 \quad \text{on} \quad \Gamma_+ \cup \Gamma_-^\delta.$$

- **Transmission conditions :**

$$\begin{aligned} [[T(w)]] = h_1, \quad [[M(w)]] = h_2 \qquad &\text{on} \quad \Sigma, \\ [[w]] = 0, \quad [[\partial_n w]] = 0 \qquad &\text{on} \quad \Sigma. \end{aligned}$$

- **Free boundary conditions :**

$$T_-(w) = k_1, \quad M_-(w_-) = k_2 \quad \text{on} \quad \Sigma_-^\delta.$$

The function w models the transverse displacement of the structure (w_+ and w denote respectively the restriction of w to Ω_+ and Ω_-^δ). We denote by $n = (n_1, n_2)$ the unit normal to Σ oriented outwardly of Ω_+ and by [[]] the jump through Σ.
Furthermore, the indices "+" and "-" stand for the restriction to Ω_+ and Ω_-^δ, respectively. The trace operators T and M designate respectively the shear forces and the bending moment and are defined as follows :

$$\begin{aligned} M &= D\Big[\Delta + (1-\nu)\left(2n_1n_2\partial_{xy} - n_1^2\partial_y^2 - n_2^2\partial_x^2\right)\Big], \\ T &= D\Big[\partial_n\Delta + (1-\nu)\,\partial_\tau\Big(\left(n_1^2 - n_2^2\right)\partial_{xy} + n_1n_2\left(\partial_y^2 - \partial_x^2\right)\Big)\Big], \end{aligned}$$

where ∂_τ represents the tangential derivative. The coefficient $D = \dfrac{E}{(1-\nu^2)}$ is the flexural rigidity of the plate having a Young modulus E and a Poisson's ratio ν. We assume that $E > 0$, $0 < \nu < \dfrac{1}{2}$ and that these coefficients are piecewise constant : $E = E_+$ in Ω_+ and E_-^δ in Ω_-^δ ; $\nu = \nu_+$ in Ω_+ and $\nu = \nu_-$ in Ω_-^δ. Consequently, we set $D = D_+$ in Ω_+ and D_-^δ in Ω_-^δ. In the following study, only the layer's Young modulus E_-^δ may depend on the parameter δ.

In this work, our objectif is to construct an asymptotic expansion of the solution w in powers of δ, of the form :

$$w_+^\delta = w_+^0 + \delta w_+^1 + \delta^2 w_+^2 + \cdots + \delta^n w_+^n, \tag{1}$$
$$w_-^\delta = w_-^0 + \delta w_-^1 + \delta^2 w_-^2 + \cdots + \delta^n w_-^n, \tag{2}$$

where the terms w_+^n and w_-^n do not depend on the parameter δ.

Approximate boundary condition of order 0

The approximate problem of order 0, is given by :

$$\begin{cases} D_+\Delta^2 w_+^0 = g_+ & \text{dans } \Omega_+, \\ M_+\left(w_+^0\right) = 0, \quad T_+\left(w_+^0\right) = 0 & \text{sur } \Sigma, \\ w_+^0 = 0, \quad \partial_n w_+^0 = 0 & \text{sur } \Gamma_+. \end{cases}$$

This approximation neglects completely the effect of the thin layer.

Approximate boundary condition of order 1

The approximate model of order 1 reads :

$$\begin{cases} D_+\Delta^2 w_+^{[1]} = g_+ & \text{dans } \Omega_+, \\ M_+\left(w_+^{[1]}\right) + \delta Q_0\left(w_+^{[1]}\right) = 0 & \text{sur } \Sigma, \\ T_+\left(w_+^{[1]}\right) + \delta P_0\left(w_+^{[1]}\right) = 0 & \text{sur } \Sigma, \\ w_+^{[1]} = 0, \quad \partial_n w_+^{[1]} = 0 & \text{sur } \Gamma_+, \end{cases}$$

This approximate problem of order 1 differs from that of order 0 by the appearance of the operators $\tilde{P}_0$ and $\tilde{Q}_0$ in the right hand sides of the boundary conditions on Σ.

5 Conclusion

In this work, we have derived approximate models for the problem of reinforcement of an elastic Kirchhoff-Love plate with a thin layer of varying thickness. We have used the asymptotic expansion method to handle the case where the rigidity of the layer doesn't depend on . Indeed, naturally, the limit behavior in this situation corresponds to a model into which the effect of the layer is ignored. However, one of the merits of the asymptotic expansion method is the possibility of identifying a more precise model that incorporates this effect. In this fashion, we have derived approximate boundary conditions for this structure and given optimal error estimate.

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SOLVABILITY OF SINGULAR AND DEGENERATE FRACTIONAL NONLINEAR PARABOLIC DIRICHLET PROBLEMS

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Abstract: we establish the existence and uniqueness of the weak solution in functional weighted Sobolev space for a class of initial-boundary value degenerate and singular fractional semi-linear parabolic problems.
Keywords: fractional differential equations, fractional,Existence of solutions, boundary value problems.
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Characterization of invertible operators in $\delta(\mathcal{H})$ via Duggal transform

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Abstract

Let $T \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator on a Hilbert space $\mathcal{H}$, and $T = U|T|$ be its polar decomposition. We say that T belongs to the class $\delta(\mathcal{H})$ if $U^2|T| = |T|U^2$. The Duggal transform of T is defined by $\Delta_1(T) = |T|U$. It is well known that T is invertible if and only if $\Delta_1(T)$ is invertible. In general $\Delta_1(T^{-1}) \neq (\Delta_1(T))^{-1}$. In this paper, we show that an invertible operator T belongs to $\delta(\mathcal{H})$ if and only if $\Delta_1(T^{-1}) = (\Delta_1(T))^{-1}$.

Key Words: *polar decomposition, Duggal transform, invertible operator , quasinormal operator.*

1. Introduction and preliminaries

let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For an arbitrary operator $T \in \mathcal{B}(\mathcal{H})$, we denote by $\mathcal{R}(T)$, $\mathcal{N}(T)$ and T^* for the range, the null subspace and the adjoint operator of T, respectively. For any closed subspace M of $\mathcal{H}$, let P_M denote the orthogonal projection onto M.
Recall that for $T \in \mathcal{B}(\mathcal{H})$, there is a unique factorization $T = U|T|$, where $\mathcal{N}(U) = \mathcal{N}(T) = \mathcal{N}(|T|)$, U is a partial isometry, i.e. $UU^*U = U$ and $|T| = (T^*T)^{\frac{1}{2}}$ is the modulus of T. This factorization is called the polar decomposition of T. It is known that if T is invertible then U is unitary and $|T|$ is also invertible. From the polar decomposition, the Aluthge transform of T is defined by

$$\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}, \quad T \in \mathcal{B}(\mathcal{H}).$$

This transform was introduced in [1] by Aluthge, in order to study p-hyponormal and log-hyponormal operators. In [3], Okubo introduced a more general notion called λ-Aluthge transform which has later been studied also in detail. This is defined for any $\lambda \in [0,1]$ by

$$\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}, \quad T \in \mathcal{B}(\mathcal{H}).$$

Clearly, for $\lambda = \frac{1}{2}$ we obtain the usual Aluthge transform. Also, $\Delta_1(T) = |T|U$ is known as Duggal's transform.
These transforms have been studied in many different contexts and considered by a number of authors (see for instance, [1, 2, 3, 4]). One of the interests of the Aluthge transform lies in the fact that it respects many properties of the original operator. It would be certainly interesting to know which invertible operators in $\mathcal{B}(\mathcal{H})$ satisfy $\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$. Recently, the answer to this problem in the case of matrices was given in [4]. In this paper we solve this problem for bounded linear operators in case of $\lambda = 1$.
Throughout the remainder of this paper, we denote by $\delta(\mathcal{H})$ the class of operator $T \in \mathcal{B}(\mathcal{H})$ which satisfies $U^2|T| = |T|U^2$. In this paper, firstly, we provide a condition under which an operator in $\delta(\mathcal{H})$ becomes quasinormal. Secondly, we show that an invertible operator T belongs to the class $\delta(\mathcal{H})$ if and only if $\Delta_1(T^{-1}) = (\Delta_1(T))^{-1}$. Finally, we give example under which we prove that this result is not valid for Aluthge transform.
Now we state some known properties of the polar decomposition, needed in the sequel. If $T = U|T|$ is the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$, then

$$UU^* = P_{\overline{R(T)}} = P_{\overline{R(|T^*|)}} \quad \text{and} \quad U^*U = P_{\overline{R(T^*)}} = P_{\overline{R(|T|)}}.$$

Moreover, we have

$P(1)$ $T^* = U^*|T^*|$ is the polar decomposition of T^*;

$P(2)$ $U|T|^\alpha = |T^*|^\alpha U$, for any $\alpha \geq 0$. Indeed let (q_n) be a sequence of polynomials such that $q_n(t) \to t^{\frac{1}{\alpha}}$ uniformly on $\sigma(|T|) \cup \sigma(|T^*|)$ as $n \to 0$. From $P(1)$, we have $U|T| = |T^*|U$ and so $Uq_n(|T|) = q_n(|T^*|)U$. Hence $U|T|^\alpha = |T^*|^\alpha U$. this property is trivial in case $\alpha = 0$.

$P(3)$ If T is invertible,
- (i) $T^{-1} = U^*|T^{-1}|$ is the polar decomposition of T^{-1};
- (ii) $|T^{-1}| = |T^*|^{-1}$;
- (iii) $|T|^{-\alpha} = U^*|T^{-1}|^\alpha U$, for $\alpha > 0$, (it follows from $P(2)$ and $P(3)$ (ii)).

2. Main results

First we give a condition under which an operator in $\delta(\mathcal{H})$ becomes quasinormal.

Proposition 2.1 *[5] Let n be a positive integer and $T \in \delta(\mathcal{H})$, with polar decomposition $T = U|T|$. If $U^{2n+1} = I$, then T is quasinormal.*

proof 1 *From $U^2|T| = |T|U^2$, we get $U^{2n}|T| = |T|U^{2n}$. This implies $U^{2n+1}|T|U = U|T|U^{2n+1}$. If $U^{2n+1} = I$, then $U|T| = |T|U$. Hence, T is quasinormal.*

The following is a characterization of invertible operators in $\delta(\mathcal{H})$ via Duggal transform.

Theorem 2.2 *[5] Let $T \in \mathcal{B}(\mathcal{H})$ be invertible. Then*

$$T \in \delta(\mathcal{H}) \iff \Delta_1(T^{-1}) = (\Delta_1(T))^{-1}.$$

proof 2 *Suppose that $T = U|T|$ is the polar decomposition of T. Since T is invertible, it follows that*

$$T \in \delta(\mathcal{H}) \iff U^2|T| = |T|U^2 \iff U^2|T|^{-1} = |T|^{-1}U^2.$$

By $P(3)$ (iii), $U^2|T|^{-1} = U^2U^*|T^{-1}|U$. Since U is unitary, then

$$\begin{aligned} T \in \delta(\mathcal{H}) &\iff U^2U^*|T^{-1}|U = |T|^{-1}U^2 \\ &\iff U|T^{-1}|U = |T|^{-1}U^2 \\ &\iff U|T^{-1}| = |T|^{-1}U \\ &\iff |T^{-1}|U^* = U^*|T|^{-1} \\ &\iff \Delta_1(T^{-1}) = (\Delta_1(T))^{-1}. \end{aligned}$$

Example 2.3 *Theorem 2.2 is not valid when the Duggal transform is replaced by the Aluthge transform. To see this let $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, where A and B are invertible positive operators such that $AB \neq BA$. Then T is invertible and*

$$T = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} = U|T|$$

is the polar decomposition of T. Since $U^2 = I$, it follows that $U^2|T| = |T|U^2$ and so $T \in \delta(\mathcal{H} \oplus \mathcal{H})$. On the other hand, since

$$\Delta(T) = \begin{pmatrix} 0 & B^{\frac{1}{2}}A^{\frac{1}{2}} \\ A^{\frac{1}{2}}B^{\frac{1}{2}} & 0 \end{pmatrix}, \ \ we\ obtain \ \ (\Delta(T))^{-1} = \begin{pmatrix} 0 & B^{-\frac{1}{2}}A^{-\frac{1}{2}} \\ A^{-\frac{1}{2}}B^{-\frac{1}{2}} & 0 \end{pmatrix}.$$

Using $P(3)$ (i) and (ii), we have

$$\Delta(T^{-1}) = |T^{-1}|^{\frac{1}{2}}U^*|T^{-1}|^{\frac{1}{2}} = |T^*|^{-\frac{1}{2}}U^*|T^*|^{-\frac{1}{2}} = \begin{pmatrix} 0 & A^{-\frac{1}{2}}B^{-\frac{1}{2}} \\ B^{-\frac{1}{2}}A^{-\frac{1}{2}} & 0 \end{pmatrix}.$$

Hence $\Delta(T^{-1}) \neq (\Delta(T))^{-1}$.

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MATRIX BOUNDARY-VALUE PROBLEMS AND DRAZIN INVERTIBLE OPERATORS

MILOUD HOCINE KOUIDER

Abstract. Let A and B be given linear operators on Banach spaces X and Y, we denote by M_C the operator defined on $X \oplus Y$ by $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. In this paper, we study an abstract matrix-boundary value problems with a spectral parameter described by Drazin invertibile operators of the form

$$\begin{cases} U_L = \lambda M_C w + F \\ \Gamma w = \Phi \end{cases}$$

where U_L, M_C are upper triangular operators matrices (2×2) acting in Banach spaces, Γ is boundary operator, F and Φ are given vectors and λ is a complex spectral parameter. We introduce the concept of initial boundary operators adapted to the Drazin invertibility and we present a spectral approach for solving the problem. It can be shown that the considered boundary-value problems are uniquely solvable and that their solutions are explicitly calculated. As an application we give an example to illustrate our results.

1. Overview on the communication

Many linear boundary-value problems in mathematical physics can be written as the abstract equation

$$\begin{cases} U_L = \lambda M_C w + F \\ \Gamma w = \Phi \end{cases} \tag{1.1}$$

where U_L, M_C are upper triangular operators matrices (2×2) acting in Banach spaces, Γ is boundary operator, F and Φ are given vectors and λ is a complex spectral parameter.

The boundary-value problems have been studied by numerous authors, see for example [1, 3, 5, 6, 7, 9] and the references cited therein. This work is devoted to solving problem (1.1) in the case where the operator U_L is Drazin invertible.

Let X, Y, E and Z be complex Banach spaces. Let $\mathcal{C}(X,Y)$ and $\mathcal{B}(X,Y)$ denote the set of closed linear operators and bounded linear operators from X into Y, respectively. When $X = Y$, we write $\mathcal{C}(X,X) = \mathcal{C}(X)$ and $\mathcal{B}(X,X) = \mathcal{B}(X)$. The identity operator on a Banach space X is denoted by I_X. The domain of an operator A defined from X into Y is denoted by $\mathcal{D}(A)$, the null space and range of A will be denoted by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively.

The product AB of two operators A, B defined from X into X is given by

$$BA(x) = B(Ax) \text{ for } x \in \mathcal{D}(BA)$$

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where

$$\mathcal{D}(BA) = \{x \in \mathcal{D}(A) : \ Ax \in \mathcal{D}(B)\}.$$

For all $n \in \mathbb{N}$, the domain, the null space and the range of power operator A^n are defined by :

If $n \geq 1$:

$$\mathcal{D}(A^n) := \{x \in \mathcal{D}(A) : \ A^k x \in \mathcal{D}(A), \ k = 1, \ldots, n-1\},$$
$$\mathcal{N}(A^n) := \{x \in \mathcal{D}(A^n) : \ A^n x = 0\},$$
$$\mathcal{R}(A^n) := \{A^n x : \ x \in \mathcal{D}(A^n)\}.$$

If $n = 0$:

$$A^0 = I, \quad \mathcal{D}(A^0) = X, \quad \mathcal{N}(A^0) = \{0\}.$$

The ascent $a(A)$ and the descent $d(A)$ of A are given by $a(A) = \inf\{n \geq 0 : \mathcal{N}(A^n) = \mathcal{N}(A^{n+1})\}$ and $d(A) = \inf\{n \geq 0 : \mathcal{R}(A^n) = \mathcal{R}(A^{n+1})\}$. An operator $A \in \mathcal{C}(X)$ is said to be Drazin invertible, if there exists an operator $S \in \mathcal{B}(X)$ such that

$$SA = AS \quad SAS = S \text{ and } ASA = A + U \text{ where } U \text{ is a nilpotent operator.} \tag{1.2}$$

The operator S is called a Drazin inverse of A , denoted by A^D.

In this work we generalize the results of [7] to the matrix case by studying boundary-value problem (1.1) described by an upper triangular operator matrices (2×2) acting in Banach spaces.

Let U_1 and U_2 be linear operators defined on X and Y, respectively. We denote by U_L the matrix operator defined on $X \oplus Y$ by

$$U_L = \begin{pmatrix} U_1 & L \\ 0 & U_2 \end{pmatrix}.$$

for a given linear operator $L : Y \longrightarrow X$.

To identify explicitly the unique solution of (1.1), we first define the initial boundary operators corresponding to Drazin invertible operators and we construct the adapted boundary
operator Γ of U_L.

If A^D is the Drazin inverse of the operator A, then

$$\mathcal{D}(A) = \mathcal{R}(A^m) \oplus \mathcal{N}(A^m), \quad \text{with } \ d(A) = a(A) = m < \infty. \tag{1.3}$$

Definition 1.1 ([7]). The operator $\Gamma : X \to E$ is said to be an initial boundary operator for a Drazin invertible operator A corresponding to its Drazin inverse A^D if

(i) $\Gamma A^D = 0$ on $\mathcal{D}(A^D)$;
(ii) There exists an operator $\Pi : E \to X$ such that $\Gamma\Pi = I_E$ and $\mathcal{R}(\Pi) = \mathcal{N}(A^m)$ with $m = a(A) = d(A)$.

Lemma 1.2 ([7], Theorem 3). *Let $A \in \mathcal{C}(X)$ be Drazin invertible operator with $A^D \in \mathcal{B}(X)$. Then there exists an $\varepsilon > 0$ such that $(I - \lambda A^D)$ is invertible for $|\lambda^{-1}| < \varepsilon$ and the boundary-value problem*

$$\begin{cases} (A - \lambda I)x = f \\ \Gamma x = \varphi \end{cases}$$

has the unique solution

$$x_\lambda^{f,\phi} = A^D(I-\lambda A^D)^{-1}f + (I-\lambda A^D)^{-1}\Pi\varphi$$

for every $f \in \mathcal{R}(A^m)$, *with* $a(A) = d(A) = m$.

Theorem 1.3 ([6]). *Let* A, B *be two linear operators on* X *such that* $\mathcal{R}(A) \subset \mathcal{D}(B)$ *and* $\mathcal{R}(B) \subset \mathcal{D}(A)$, *then*

$I - \lambda AB$ *is invertible if and only if* $I - \lambda BA$ *is invertible for all* $\lambda \neq 0$.

In this case, we have

$$(I-\lambda BA)^{-1} = I + \lambda B(I-\lambda AB)^{-1}A, \tag{1.4}$$

and

$$(I-\lambda AB)^{-1} = I + \lambda A(I-\lambda BA)^{-1}B. \tag{1.5}$$

Corollary 1.4. *Let* A, B *be two linear operators on* X *such that* $\mathcal{R}(A) \subset \mathcal{D}(B)$ *and* $\mathcal{R}(B) \subset \mathcal{D}(A)$. *If* $\lambda^{-1} \in \rho(AB)$ *then*

$$(I_X - \lambda AB)^{-1}A = A(I_X - \lambda BA)^{-1}$$

In the following proposition, we construct the boundary operator for Drazin invertible upper triangular matrix operator.

Proposition 1.5. *Let* $U_L = \begin{pmatrix} U_1 & L \\ 0 & U_2 \end{pmatrix}$ *defined on* $X \oplus Y$. *Assume that* U_1^D *and* U_2^D *are Drazin inverses of* U_1 *and* U_2 *respectively.* Γ_1 *and* Γ_2 *are boundary operators for* U_1 *and* U_2 *with the boundary spaces* E *and* Z, *respectively. If* $\mathcal{N}(U_2^m) \subset \mathcal{N}(L)$ *with* $m = a(U_L) = d(U_L)$ *then the operator* $\Gamma = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}$ *from* $X \oplus Y$ *into* $E \oplus Z$ *is a boundary operator for* U_L.

Let A and B be given linear operators on Banach spaces X and Y, and consider the operator M_C defined on $X \oplus Y$ by

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

where C is a linear operator from Y into X such that $\mathcal{D}(U_L) \subset \mathcal{D}(M_C)$.
According to Proposition 1.5, we define the following spectral matrix boundary-value problem for unknown $w \in \mathcal{D}(U_L)$ by

$$(\mathcal{P}) \begin{cases} (U_L - \lambda M_C)w = F \\ \Gamma w = \Phi \end{cases},$$

where $F \in X \times Y$, $\Phi \in E \times Z$ and $\lambda \in \mathbb{C}$ is a spectral parameter. We denote $\mathbf{R}_\lambda[U_1^D A] = (I_X - \lambda U_1^D A)^{-1}$ and $\mathbf{R}_\lambda[U_2^D A] = (I_Y - \lambda U_2^D B)^{-1}$, U_1^D and U_2^D are Drazin inverses of U_1 and U_2, respectively.

Our main objective is to establish the existence and uniqueness of solutions for the boundary-value problem $(\mathcal{P})$.

As an application, we consider the problem

$$\begin{cases} \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = f(x,y), \\ u(0,y) = u_0, \\ u(x,0) = v_0. \end{cases} \tag{1.6}$$

in $UCB(\mathbb{R})$ the space of all bounded, uniformly continuous complexvalued functions on $\mathbb{R}$ equipped with the uniform norm $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$.

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THE EXISTENCE OF SOLUTIONS OF A THIRD ORDER BOUNDARY VALUE PROBLEM

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ABSTRACT. In this presentation we prove the existence of the solutions of the Falkner-Skan equation using nonstandard analysis techniques.

1. MAIN RESULTS

We proved that the Falkner-Skan equation given by:

$$y''' + yy'' + r\left(1 - {y'}^2\right) = 0 \tag{1.1}$$

where r is a strictly negative constant.
with the conditions :

$$y(0) = 0 \quad , \quad y'(0) = 0 \quad , \quad \lim_{T \to +\infty} y'(T) = 1 \tag{1.2}$$

has a solution.
We used some methods of the nonstandard analysis, for find certain results more geometric.

Proposition. *For any $T \geqslant 0$, the problem (1.1), (1.2) has a solution.*

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Key words and phrases. Differential equations; Falkner-Skan equation; Nonstandard analysis.

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Results and Existence for Differential Equation involving $\theta-$Laplacian operator

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Abstract: In this paper, we have studied existence and uniqueness of solutions for a coupled system of multi-point boundary value problems via Phi−Hilfer derivative. By applying fixed point theorem new existence results have been obtained.
Key Words: $\theta-$Laplacian operator; Fixed point theorem.
Mathematics Subject Classification : 34A08

1 Introduction

The subject of fractional calculus has received significant attention from researchers, There exist several definitions of fractional derivatives and fractional integrals such as the Riemann-Liouville, Caputo and Hilfer...

In this paper, we are concerned with the following coupled system:

$$\begin{cases} {}^{H}\mathcal{D}_{0^+}^{\alpha_1,\beta_1;\varphi}\psi_\theta\left({}^{H}\mathcal{D}_{0^+}^{\overline{\alpha_1},\overline{\beta_1};\varphi}u_1\right)(t) = f_1(t,u_1(t)), \quad t\in J, \\ {}^{H}\mathcal{D}_{0^+}^{\alpha_2,\beta_2;\varphi}\psi_\theta\left({}^{H}\mathcal{D}_{0^+}^{\overline{\alpha_2},\overline{\beta_2};\varphi}u_2\right)(t) = f_2(t,u_2(t)), \quad t\in J, \end{cases} \tag{1}$$

subject to periodic boundary conditions

$$\begin{cases} \psi_\theta\left({}^{H}\mathcal{D}_{0^+}^{\overline{\alpha_1},\overline{\beta_1};\varphi}u_1\right)(T) = \sum\limits_{i=1}^{n} u_1(t_i), \\ \psi_\theta\left({}^{H}\mathcal{D}_{0^+}^{\overline{\alpha_1},\overline{\beta_1};\varphi}u_1\right)(0) = u_1(0) = 0, \\ \psi_\theta\left({}^{H}\mathcal{D}_{0^+}^{\overline{\alpha_2},\overline{\beta_2};\varphi}u_2\right)(T) = \sum\limits_{j=1}^{m} u_2(t_j), \\ \psi_\theta\left({}^{H}\mathcal{D}_{0^+}^{\overline{\alpha_2},\overline{\beta_2};\varphi}u_2\right)(0) = u_2(0) = 0, \end{cases} \tag{2}$$

also

$$\begin{cases} I_{0^+}^{\delta;\varphi}u_1(T) = \sum\limits_{i=1}^{n} I_{0^+}^{\delta;\varphi}u_1(\omega_i), \quad 0<\omega_i<T, \\ I_{0^+}^{\delta;\varphi}u_2(T) = \sum\limits_{j=1}^{m} I_{0^+}^{\delta;\varphi}u_2(\omega_j), \quad 0<\omega_j<T, \\ \omega_i,\omega_j \in \mathbb{R}, \quad i=\overline{1,n}, \quad j=\overline{1,m}, \ \delta>0, \\ 0<t_1<t_2<...<t_n \le T, \\ 0<t_1<t_2<...<t_m \le T, \\ 0 \le \beta_k, \overline{\beta_k} \le 1, \ 1<\alpha_k, \ \overline{\alpha_k}<2, \\ k=\overline{1,2}, \ J=[0,T], \end{cases} \tag{3}$$

where ${}^{H}\mathcal{D}_{0^+}^{\alpha_1,\beta_1;\varphi}$, ${}^{H}\mathcal{D}_{0^+}^{\alpha_2,\beta_2;\varphi}$, ${}^{H}\mathcal{D}_{0^+}^{\overline{\alpha_1},\overline{\beta_1};\varphi}$ and ${}^{H}\mathcal{D}_{0^+}^{\overline{\alpha_2},\overline{\beta_2};\varphi}$ are the $\varphi-$Hilfer fractional derivative of orders $\alpha_k, \overline{\alpha_k}$, $k=\overline{1,2}$ respectively and β_1, β_2 two parameters. $f_1, f_2 : J\times\mathbb{R}\to\mathbb{R}$ are given continuous functions and $I_{0^+}^{\delta;\varphi}$ the left-sided $\varphi-$Riemann Liouville fractional integral of order $\delta>0$ such that $\varphi : J\to\mathbb{R}$ be an increasing function such that $\varphi'(t)\neq 0$, for all $t\in J$. ψ_θ denotes the $\theta-$Laplacian operator satisfies:

$$\exists\theta'>0, \quad \frac{\theta+\theta'}{\theta\theta'}=1, \quad (\psi_\theta)^{-1}-\psi_{\theta'}=0.$$

We assume that the following conditions hold

$$\begin{cases} \exists L_0>0: \ |f_k(t,x)-f_k(t,y)| \le L_0\,|x-y|, \quad x,y\in\mathbb{R}. \\ \gamma_i=\alpha_i+\beta_i(n-\alpha_i), \ i=\overline{1,2}, \\ \overline{\gamma_i}=\overline{\alpha_i}+\overline{\beta_i}(n-\overline{\alpha_i}), \quad i=\overline{1,2}. \end{cases} \tag{4}$$

2 Preliminaries

Definition 1 *Let φ be a positive increasing function on $(0,T]$, which has a continuous derivative $\varphi'(t)$ on $(0,T)$. The $\varphi-$Riemann–Liouville fractional integral of a function u with respect to another function φ on $[0,T]$ is defined by:*

$$I_{0^+}^{\alpha;\varphi}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t \varphi'(s)\varphi_{\alpha-1}(t,s)f(s)ds,$$

where

$$\Gamma(\theta)=\int_0^\infty \exp(-u)u^{\theta-1}du.$$

Definition 2 *Let $\varphi, f\in C^n([0,T])$ be two functions such that φ is increasing and $\varphi'(t)\neq 0$, for all $t\in(0,T]$. The left-sided $\varphi-$Riemann Liouville fractional* *derivative of a function f of order α is defined by:*

$$\begin{aligned}
&\mathcal{D}_{0^+}^{\alpha;\varphi} f(t) \\
= &\left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^m I_{0^+}^{m-\alpha;\varphi} f(t) \\
= &\frac{1}{\Gamma(m-\alpha)}\left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^m \int_0^t \varphi'(s)\varphi_{m-\alpha-1}(t,s) f(s)ds,
\end{aligned}$$

3 Existence of solutions

Let us prove the following very important lemma.

Lemma 3 *Let*

$$\begin{cases}
(\varphi(T)-\varphi(0))^{\delta+\overline{\gamma_k}-1} - \sum\limits_{i=1}^{n}(\varphi(\omega_i)-\varphi(0))^{\delta+\overline{\gamma_k}-1} \neq 0, \\
0 \leq \beta_k, \overline{\beta_k} \leq 1, \quad 1 < \alpha_k, \overline{\alpha_k} < 2.
\end{cases}$$

For any $\phi_k \in C([0,T])$, $k = 1,2$, the problem:

$$\begin{cases}
{}^H\mathcal{D}_{0^+}^{\alpha_k,\beta_k;\varphi}\psi_\theta\left({}^H\mathcal{D}_{0^+}^{\overline{\alpha_k},\overline{\beta_k};\varphi} u_k\right)(t) = \phi_k(t), \quad t \in J, \\
\psi_\theta\left({}^H\mathcal{D}_{0^+}^{\overline{\alpha_k},\overline{\beta_k};\varphi} u_k\right)(T) = \sum\limits_{i=1}^{n} u_i(t_i), \\
\psi_\theta\left({}^H\mathcal{D}_{0^+}^{\overline{\alpha_k},\overline{\beta_k};\varphi} u_k\right)(0) = u_k(0) = 0, \\
I_{0^+}^{\delta;\varphi} u_k(T) = \sum\limits_{i=1}^{n} I_{0^+}^{\delta;\varphi} u_k(\omega_i), \quad 0 < \omega_i < T, \\
0 < t_1 < t_2 < ... < t_n \leq T,
\end{cases} \tag{5}$$

admits the following integral solution:

$$\begin{aligned}
&u_k(t) \\
= \;& I_{0^+}^{\overline{\alpha_k};\varphi}\left(\psi_{\theta'}\left[I_{0^+}^{\alpha_k;\varphi}\phi_k(t) + \frac{(\varphi(t)-\varphi(0))^{\gamma_k-1}}{(\varphi(T)-\varphi(0))^{\gamma_k-1}}\left(\sum_{i=1}^{n} u_i(t_i) - I_{0^+}^{\alpha_k;\varphi}\phi_k(T)\right)\right]\right) \qquad (6)\\
&+\frac{\Gamma(\overline{\gamma_k}+\delta)}{\Gamma(\overline{\gamma_k})\left[(\varphi(T)-\varphi(0))^{\delta+\overline{\gamma_k}-1} - \sum\limits_{i=1}^{n}(\varphi(\omega_i)-\varphi(0))^{\delta+\overline{\gamma_k}-1}\right]}(\varphi(t)-\varphi(0))^{\overline{\gamma_k}-1} \\
&\times\sum_{i=1}^{n} I_{0^+}^{\delta+\overline{\alpha_k};\varphi}\left(\psi_{\theta'}\left[I_{0^+}^{\alpha_k;\varphi}\phi_k(\omega_i) + \frac{(\varphi(\omega_i)-\varphi(0))^{\gamma_k-1}}{(\varphi(T)-\varphi(0))^{\gamma_k-1}}\left(\sum_{i=1}^{n} u_i(t_i) - I_{0^+}^{\alpha_k;\varphi}\phi_k(T)\right)\right]\right) \qquad (7)\\
&-\frac{\Gamma(\overline{\gamma_k}+\delta)}{\Gamma(\overline{\gamma_k})\left[(\varphi(T)-\varphi(0))^{\delta+\overline{\gamma_k}-1} - \sum\limits_{i=1}^{n}(\varphi(\omega_i)-\varphi(0))^{\delta+\overline{\gamma_k}-1}\right]}(\varphi(t)-\varphi(0))^{\overline{\gamma_k}-1} \\
&\times I_{0^+}^{\delta+\overline{\alpha_k};\varphi}\left(\psi_{\theta'}\left[I_{0^+}^{\alpha_k;\varphi}\phi_k(T) + \left(\sum_{i=1}^{n} u_i(t_i) - I_{0^+}^{\alpha_k;\varphi}\phi_k(T)\right)\right]\right).
\end{aligned}$$

Theorem 4 *Let L be a closed, bounded, convex and nonempty subset of a Banach space X. Let M, N be operators such that:*

1. $Mx + Ny \in L, \quad x, y \in L.$
2. *M is compact and continuous.*
3. *N is a contraction mapping.*

   Then there exists $z \in L$ such that $z = Mz + Nz$ (see [15]).

Theorem 5 *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (4). In addition, assume that:*

$$(\varphi(T) - \varphi(0))^{\delta+\overline{\gamma_k}-1} - \sum_{i=1}^{n}(\varphi(\omega_i) - \varphi(0))^{\delta+\overline{\gamma_k}-1} \neq 0,$$

then the boundary value problem (1)-(3) has at least one solution on $[0, T]$.

Example 6 *Consider the following:*

$$\begin{cases} {}^{H}\mathcal{D}_{0^+}^{\sqrt{3}/3,3/5;\varphi}\psi_\theta\left({}^{H}\mathcal{D}_{0^+}^{17/11,2/5;\varphi}u_1\right)(t) = \dfrac{u_1(t)}{e+\sqrt{3}}, \quad t \in \left[0, \frac{1}{\sqrt{2}}\right], \\ {}^{H}\mathcal{D}_{0^+}^{\sqrt{2}/3,2/5;\varphi}\psi_\theta\left({}^{H}\mathcal{D}_{0^+}^{16/11,1/5;\varphi}u_2\right)(t) = \dfrac{1}{t^2(1+u_2^2(t))}, \quad t \in \left[0, \frac{1}{\sqrt{2}}\right], \\ \psi_\theta\left({}^{H}\mathcal{D}_{0^+}^{17/11,2/5;\varphi}u_1\right)\left(\frac{1}{\sqrt{2}}\right) = \psi_\theta\left({}^{H}\mathcal{D}_{0^+}^{16/11,1/5;\varphi}u_2\right)\left(\frac{1}{\sqrt{2}}\right) = \sum_{i=1}^{5} u_k\left(\dfrac{i}{\sqrt{3}}\right), \\ u_k(0) = 0, \ k = \overline{1,2}. \end{cases}$$

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Study of Semilinear wave models with classical and visco-elastic damping.

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Abstract

In this work, We study the global (in time) existence of small data solutions to the Cauchy problem for the semilinear wave models with friction and visco-elastic and a power nonlinearity, that is

$$u_{tt} - \Delta u + u_t - \Delta u_t = |u_t|^p \text{ for } (t,x) \in (0,\infty) \times \mathbb{R}^n,$$
$$u(0,x) = u_0(x), \; u_t(0,x) = u_1(x) \text{ for } x \in \mathbb{R}^n,$$

where the data u_0 and u_1 are given Cauchy data.
Under certain assumptions for the data and the dimension n. Our main goals are to study the influence of regularity parameters $s_1 > s_2 > \frac{n}{2}$ and additional regularity parameter $m \in [1,2)$ for the data (u_0, u_1), that is,

$$(u_0, u_1) \in (H^{s_1} \cap L^m) \times (H^{s_2} \cap L^m)$$

on the admissible range of exponents p which allow to prove the global (in time) existence of small data Sobolev solutions or energy solutions with suitable regularity.

AMS Mathematics Subject Classification 2010 : 91Gxx, 26A33, 34A08.

Keywords: global in time existence; small data solutions; visco-elastic damping; wave equation; frictional damping; power non-linearity; higher regularity of data; fractional chain rule

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Asymptotic stability of periodic solutions of generalized van der Pol oscillator

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Abstract

In this paper, we study the behavior of the solution of a certain class of nonlinear differential equations representing the Van Der Pol oscillator in a generalized form. After using the appropriate variables, the first Levinson converts the equations into a system with two equations, and the second converts these systems into lipschitzian systems. Our main result is obtained by applying the second Bogolubov's theorem. We establish some integral identities, which are used to compute the average function of these systems, and we arrive at a new general condition for the existence of an asymptotically stable unique periodic solution.

Keywords

Asymptotic Stability, Periodic Solutions, Second Bogolubov's theorem, Van Der Pol equations in their general form.

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Deconvolving the Hazard Rate Function from Associated Data Corrupted by Additive Noise: The Asymptotic Normality

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Abstract
In reliability theory or survival analyses, it is common to observe data that are not only contaminated but weakly dependent too. The goal of this paper is to estimate the hazard rate from contaminated data in such way the underlying data are assumed to come from strict stationary sequences satisfying association conditions. We obtained the quadratic-mean convergence and the asymptotic normality for the estimator. The estimation performance is illustrated in a simulation study.

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Key Words and Phrases: *Hazard rate, Positively Associated, Deconvolution, Asymptotic Normality.*

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A common fixed point for multi-valued contractions and an application to integral inclusions.

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Abstract

In this work, we present a common fixed point theorem for two pairs of single and set valued mappings via subsequential continuity and δ-compatibility. To illustrate the validity of our results, an example is provided and we also give an application for a system of integral inclusions of Volterra type.

Introduction

Starting from Banach principle. Jungck et al. [4] have furnished an extension to compatible mappings notion, called weak compatibility in the setting of single-valued and multi-valued mappings. Recently, Bouhadjera and Godet Tobie [2] introduced subsequential continuity which is weaker than the reciprocal continuity introduced by Pant [5]. Quite recently, Beloul et al. [1] extended the notion of subsequential continuity to the context of set value maps in order to establish a common fixed point by using Hausdorff distance.
In this paper we will utilize a θ-contraction introduced by Jleli and Samet [3] and δ-distance to establish a strict coincidence and a strict common fixed point of a δ-compatible and subsequentially hybrid pair of mappings.

Preliminaries

Let (X, d) be a metric space, $B(X)$ is the set of all non-empty bounded subsets of X. For all $A, B \in B(X)$ we define the two functions: $D, \delta : B(X) \times B(X) \to \mathbb{R}_+$ such that

$$D(A, B) = \inf\{d(a, b); a \in A, b \in B\},$$
$$\delta(A, B) = \sup\{d(a, b); a \in A, b \in B\}.$$

If A consists of a single point a, we write $\delta(A, B) = \delta(a, B)$ and $D(A, B) = D(a, B)$, also if $B = \{b\}$ is a singleton we write

$$\delta(A, B) = D(A, B) = d(a, b).$$

Clearly that δ satisfies the following properties:

$$\delta(A, B) = \delta(B, A) \geq 0,$$
$$\delta(A, B) \leq \delta(A, C) + \delta(C, B),$$
$$\delta(A, A) = \text{diam}A,$$
$$\delta(A, B) = 0 \text{ implies } A = B = \{a\},$$

for all $A, B, C \in B(X)$.
Notice that for all $a \in A$ and $b \in B$ we have

$$D(A, B) \leq d(a, b) \leq \delta(A, B),$$

where $A, B \in B(X)$.

Definition 1. *[2] The pair (f, g) of self mappings is said to be subsequentially continuous if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, for some $z \in X$ and $\lim_{n\to\infty} fgx_n = fz$, $\lim_{n\to\infty} gfx_n = gz$.*

Definition 2. *[1] Let $f : X \to X$ and $S : X \to CB(X)$ two single and multi-valued mappings respectively, the hybrid pair (f, S) is to be subsequentially continuous if there exists a sequence $\{x_n\}$ such that*

$$\lim_{n\to\infty} Sx_n = M \in CB(X) \text{ and } \lim_{n\to\infty} fx_n = z \in M,$$

for some $z \in X$ and $\lim_{n\to\infty} fSx_n = fM$, $\lim_{n\to\infty} Sfx_n = Sz$.

Notice that continuity or reciprocal continuity implies subsequential continuity, but the converse may be not.
Let Θ be the set of all functions $\theta : (0, +\infty) \to (1, +\infty)$ be a function satisfying:

(θ_1) : θ is non decreasing,

(θ_2) : for each sequence $\{t_n\}$ in $(0, +\infty)$, $\lim_{n\to\infty} t_n = 1$ if and only if $\lim_{n\to\infty} t_n = 0$,

(θ_3) : there exists $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t) - 1}{t^r} = l$.

Definition 3. *[3] Let (X, d) be a metric space and $T : X \to X$ be a mapping. For $\theta \in \Theta$, we say T is θ-contraction, if there exists $k \in [0, 1]$ such that for $x, y \in X$,*

$$d(Tx, Ty) > 0 \Longrightarrow \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k.$$

Theorem 1. *[3] Let (X, d) be a complete metric space and let $T : X \to X$ be an θ-contraction. Then T has a unique fixed point in X.*

Main results

In this section, we introduce a multivalued θ_δ-contraction and prove a common fixed point theorem for hybrid pair mappings with δ-distance.

Definition 4. *Let (X, d) be a metric space and $T : X \to B(X)$ be a mapping. For $\theta \in \Theta$, we say T is θ_δ-contraction, if there exists $k \in [0, 1]$ such that for $x, y \in X$, $\delta(Tx, Ty) > 0$ implies $\theta(\delta(Tx, Ty)) \leq [\theta(d(x, y))]^k$.*

Definition 5. *Let f be a self mapping on a metric space (X, d) and let $T : X \to B(X)$ be a multivalued mapping. Then T is called generalized multivalued $(f, \theta_\delta$- contraction if for all $x, y \in X$ there exists $k \in [0, 1]$ such that,*

$$\delta(Tx, Ty) > 0 \quad \text{implies} \quad \theta(\delta(Tx, Ty)) \leq [\theta(R(x, y))]^k,$$

where $\theta \in \Theta$

$$R(x, y) = \max\{d(fx, fy), D(fx, Tx), D(fy, Ty), \frac{1}{2}[D(fx, Ty) + D(fy, Tx)]\}.$$

Now we extend the last definition for two pairs of hybrid pair, in order to establish a common fixed point for set valued and single valued mapping in metric space, without continuity and completeness of space, we use only subsequential continuity with δ-compatibility.

Theorem 2. *Let $f, g : X \to X$ be single valued mappings and $S, T : X \to B(X)$ be multi-valued mappings of metric space (X, d). If the two pairs (f, S) and (g, T) are subsequentially continuous and δ-compatible. Then the pair (f, S) as well as (g, T) has a strict coincidence point. Moreover, f, g, S and T have a common strict fixed point provided that there exists $k \in (0, 1)$ such that for all x, y in X we have:*

$$\delta(Sx, Ty) > 0 \text{ implies } \theta(\delta(Tx, Ty)) \leq [\theta(R(x, y))]^k, \tag{1}$$

where $\theta \in \Theta$. and

$$R(x, y) = \max\{d(fx, fy), D(fx, Tx), D(gy, Ty), \frac{1}{2}[D(fx, Ty) + D(fy, Tx)]\}.$$

Application to integral inclusions

In this section, we apply the obtained results to assert the existence of solution for a system of integral inclusions.
Consider the following integral inclusions system's.

$$x_i(t) \in g(t) + \int_0^t K_i(t, s, x_i(s))ds, i = 1, 2 \tag{2}$$

where g is a continuous function on $[0, 1]$, i,e., $f \in C([0, 1], \mathbb{R})$ and $K_i : [0, 1] \times [0, 1] \times \mathbb{R} \to CB(\mathbb{R})$ are a set valued functions.
Clearly $X = C([0, 1])$ with convergence uniform metric's $d_\infty(x, y) = \sup_{x\in X} |x(t) - y(t)|$ is a complete metric space. Define two set valued mappings:

$$Sx_1(t) = \{z \in X, z(t) \in f(t) + \int_0^t K_1(t, s, x_1(s))ds\},$$

$$Tx_2(t) = \{z \in X, z(t) \in f(t) + \int_0^t K_2(t, s, x_2(s))ds\}.$$

Assume that;

A_1 : The function $K_i : (t, s) \mapsto K_i(t, s, x_i(s))$ are continuous on $[0, 1] \times (0, 1]$ for all $x \in C((0, 1])$;

A_2 : For all $x_i \in X$ and $k_i \in K_i$ $(i = 1, 2)$, there exists a function $\varphi : [0, 1] \times [0, 1] \to [0, +\infty)$ such that

$$|k_1(t, s, x_1(s)) - k_2(t, s, x_2(s))| \leq \varphi(t, s)|x_1 - x_2|;$$

A_3 : There exists $\tau > 0$ such that

$$\sup_{t\in[0,1]} \int_0^t \varphi(t, s)ds \leq e^{-\tau};$$

A_4 : There exist two sequences $\{x_n\}, \{y_n\}$ and two elements x, y in X such that

$$\lim_{n\to\infty} Sx_n = M \in B(X),$$

$$\lim_{n\to\infty} x_n = x \in M$$

and

$$\lim_{n\to\infty} Ty_n = N \in B(X),$$
$$\lim_{n\to\infty} y_n = y \in N.$$

Theorem 3. *Under assumptions $(A_1) - (A_4)$ the system of integral inclusions (2) has a solution in $C((0, 1]) \times C([0, 1])$.*

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A quadrature method for solving the general nonlinear Fredholm integral equations of the second kind

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Abstract

In this work, a numerical method for solving a class of nonlinear Fredholm integral equations of the second kind is presented. By using a numerical integration based on the Weddle quadrature rule together with Newton's method. Finally, some numerical results are presented to illustrate the efficiency and accuracy of the numerical method in comparison to other known results.

keywords

- Nonlinear Fredholm integral equations.
- Weddle's quadrature rule.
- Newton's method.

Introduction

Integral equations represent an important field in the area of applied mathematics and arise naturally in many problems of the real world which include physical phenomena and engineering problems, such as mechanics, astronomy, inverse problems, potential theory, mathematical biology and epidemiology. Also, many initial and boundary value problems associated with ordinary and partial differential equations can be reformulated as integral equations. This study is focused on the numerical solutions of the nonlinear Fredholm integral equations of the second kind

$$u(x) = f(x) + \lambda \int_a^b k(x,t,u(t))dt, \quad I := [a,b], \tag{1}$$

where $k(x,t,\varphi)$ and $f(x)$ are known functions and $u(x)$ is an unknown function to be determined, and $\lambda \in \mathbb{R}$ is a non-zero constant. The existence and the uniqueness of solutions to this type of integral equations have been investigated in the literature by many authors (see e.g. [1, 2, 4, 3]). Throughout this paper, we assume that the problem (1) has a unique solution. Then the following conditions are assumed:

(i) $f : I \to \mathbb{R}$ is a continuous function;

(ii) $k : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous function in both arguments and satisfy the Lipschitz condition with respect to the third variable: there exists a constant $L > 0$ such that

$$|k(x,t,u_1) - k(x,t,u_2)| \leq L|u_1 - u_2|,$$

where L is Lipschitz constant;

(iii) $|\lambda| \leq \min(\frac{r_1-d_1}{R(b-a)}, \frac{d_2-r_2}{R(b-a)}, \frac{1}{L(b-a)})$ where $r_1 < f(x) < r_2$, $|k(x,t,u(t))| < R$, $d_1 < u(x) < d_2$ and L is Lipschitz constant of k (see [6]).

To obtain the numerical solution of second kind nonlinear Fredholm integral equations of form (1), various numerical approaches have been proposed in the literature. For instance, Saberi-Nadjafi and Heidari [5] used Newton-Kantorovich method combined with a quadrature method to approximate the nonlinear Fredholm integral equations, Katani [6] applied the quadrature methods to study the numerical solutions of (1), Z. Mahmoodia et al. [7] proposed B-spline collocation method to approximate the nonlinear Fredholm integral equations, Brezinski and Redivo-Zaglia [8] explored the extrapolation methods for the numerical solution of nonlinear Fredholm integral equations.

The main idea of this study is to apply the Weddle's quadrature rule to the integral in (1) for approximating the solution of the above problem.

Weddle's rule

Let the interval $[a,b]$ be finite and divided into $6N$ subintervals of equal width $h = \frac{b-a}{6N}$. Then using the uniformly-spaced points $x_i := a + ih$, $i = 0, 1, \ldots, 6N$, $x_{6N} = b$ yields

$$\begin{aligned}\int_a^b k(x,t,u(t))dx \approx{}& \frac{3h}{10}\sum_{j=0}^{N}\left(k(x,x_{6j-6},u(x_{6j-6})) + 5k(x,x_{6j-5},u(x_{6j-5}))\right)\\ &+ \frac{3h}{10}\sum_{j=0}^{N}\left(k(x,x_{6j-4},u(x_{6j-4})) + 6k(x,x_{6j-3},u(x_{6j-3}))\right)\\ &+ \frac{3h}{10}\sum_{j=0}^{N}\left(k(x,x_{6j-2},u(x_{6j-2})) + 5k(x,x_{6j-1},u(x_{6j-1}))\right)\\ &+ \frac{3h}{10}\sum_{j=0}^{N}k(x,x_{6j},u(x_{6j})) = \sum_{i=0}^{6N} w_i k(x,x_i,u(x_i)),\end{aligned} \tag{2}$$

where

$$w_i = \begin{cases} \frac{3h}{5}, & i = 6, 12, \ldots, 6i-6,\\ \frac{3h}{2}, & i = 1, 7, \ldots, 6i-5;\ i = 5, 11, \ldots, 6i-1,\\ \frac{9h}{2}, & i = 3, 9, \ldots, 6i-3,\\ \frac{3h}{10}, & otherwise. \end{cases}$$

Approximate solution using Weddle's quadrature rule

In order to obtain the approximate solution of (1), we apply the Weddle's quadrature rule (2) to the integral on the right-hand side of (1) yields

$$u(x) = f(x) + \sum_{j=0}^{6N} w_j k(x, x_j, u(x_j)), \quad x \in I. \tag{3}$$

Substituting $x = x_i, i = 0, 1, \ldots, 6N$ in the system (3), we get

$$u_i = f_i + \sum_{j=0}^{6N} w_j k(x_i, x_j, u_j), \quad i = 0, 1, \ldots, 6N \tag{4}$$

where u_i is the approximate value of the exact solution in the points x_i and $f_i = f(x_i)$. Thus for $i = 0, 1, \ldots, 6N$ we will have a system of $6N+1$ nonlinear equations with $6N+1$ unknown coefficients $u_0, u_1, \ldots, u_{6N}$. The above nonlinear system of equations can be written in the following vector form

$$U - K(U)W = F, \tag{5}$$

where

$$U = [u(x_0), u(x_1), \ldots, u(x_{6N})]^T, \quad F = [f(x_0), f(x_1), \ldots, u(f_{6N})]^T,$$

and $W = [\omega_0, \omega_1, \ldots, \omega_{6N}]^T$, and the martix $K(U)$ is a square matrix whose elementss are $K(U)_{ij} = k(x_i, x_j, u_j)$. Note that in the case $k(x,t,u(t)) = k(x,t)u^m$, we can evauatee the exact jacobian for running the steps of the iterative nonlinear solver such as the Newton method. For a nonlinear system of equation (5), we define a new function G as

$$G(U) = U - K(U)W - F, \tag{6}$$

and then applying the Newton method for solving nonlinear equation (6) to obtain the value of U as

$$U^{k+1} = U^k - \left[DG(U^k)\right]^{-1} G(U^k),$$

where $DG(U)$ is the Jacobian matrix of $G(U)$, defined by $DG(U) = I - mB$ with $B_{ij} = k_{ij}U_j^{m-1}$, and the initial value (initial guess) $U^{(0)} = f_i$.

Numerical results

Example 1. Consider the following nonlinear Fredholm integral equation:

$$u(x) = \sin(\pi x) + \frac{1}{5}\int_0^1 \cos(\pi x)\sin(\pi t)u^3(t)dt, \quad 0 \leq x, t \leq 1, \tag{7}$$

where the exact solution of the above equation is

$$u(x) = \sin(\pi x) + \frac{1}{3}(20 - \sqrt{391})\cos(\pi x).$$

Table 1 presents the absolute error for **Example 1** in some arbitrary points and compared with the method [5, 9, 10].

Table 1: Absolute error for **Example 1.**

	Ref. [9]	Ref. [5]	Ref. [9]	Ref. [10]	Present method	
x_i	$N=10$	$N=10$	$N=20$	$N=20$	$N=10$	$N=20$
0.0	5.44e−08	4.98e−02	3.19e−16	5.53e−15	5.83e−16	5.98e−16
0.2	4.40e−08	4.03e−02	2.22e−16	4.55e−15	4.44e−16	5.55e−16
0.4	1.86e−08	1.53e−02	1.11e−16	1.77e−15	2.22e−16	2.22e−16
0.6	1.68e−08	1.53e−02	1.11e−16	1.77e−15	2.22e−16	2.22e−16
0.8	4.40e−08	4.03e−02	2.22e−16	4.55e−15	5.55e−16	5.55e−16
1.0	5.44e−08	1.53e−02	3.19e−16	5.53e−15	5.83e−16	5.98e−16

Example 2. Consider the following nonlinear Fredholm integral equation [6]:

$$u(x) = \sin(x) + \frac{1}{2}x(\cos(1)\sin(1) - 1) + \int_0^1 xu^2(t)dt, \quad 0 < x < 1, \tag{8}$$

where the function $f(x)$ is chosen so that the solution $\varphi(x)$ is given by $u(x) = \sin(x)$. **Table 2** presents the absolute error for **Example 2** in some arbitrary points and compared with the method [6].

Table 2: Absolute error for **Example 2.**

	Method of [6]			Present method		
x_i	$N=2$	$N=10$	$N=20$	$N=2$	$N=10$	$N=20$
0.25	7.63e−08	4.73e−12	7.38e−14	3.68e−09	1.05e−13	1.39e−13
0.50	1.52e−07	9.46e−12	1.47e−13	7.36e−09	2.10e−13	2.69e−13
0.75	2.28e−07	1.41e−11	2.22e−13	1.10e−08	3.16e−13	4.04e−13
1.00	3.05e−07	1.89e−11	2.95e−13	1.47e−08	4.21e−13	5.39e−13

Conclusion

Nonlinear integral equations are usually difficult to solve analytically. In many cases, it is required to obtain approximate solutions. In this work, a numerical method for solving a class of nonlinear Fredholm integral equations based on Weddle's quadrature rule and Newton's method is presented. The efficiency of this method is tested by solving some examples for which the exact solution is known. This allows us to estimate the exactness of our numerical results and compare those with other results.

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Variational and numerical analysis of a dynamic contact problem for viscoelastic piezoelectric materials with damage

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1. Introduction

This work studies a mathematical model involving a dynamic contact between two a viscoelastic piezoelectric bodies with damage. The contact is modelled using the normal compliance contact condition and a normal damped response law associated with friction.and the linear electro viscoelastic constitutive law is employed to simulate the piezoelectric effects. We derive a variational formulation for the problem and prove the existence of its unique weak solution. Finally, we investigate a fully discrete approximation using, respectively, Euler scheme and finite element method for the spatial variable and the time derivatives. Some error estimates are then derived, leading to convergence results under suitable additional regularity conditions.

2. Problem statement

Problem P. For $\ell = 1, 2$, find a displacement field $\boldsymbol{u}^\ell : \Omega^\ell \times [0, T] \longrightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^\ell : \Omega^\ell \times [0, T] \longrightarrow \mathbb{S}^d$, an electric potential field $\varphi^\ell : \Omega^\ell \times [0, T] \longrightarrow \mathbb{R}$, a damage field $\beta^\ell : \Omega^\ell \times [0, T] \longrightarrow \mathbb{R}$ and an electric displacement field $\boldsymbol{D}^\ell : \Omega^\ell \times [0, T] \longrightarrow \mathbb{R}^d$ such that

$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell \varepsilon(\dot{\boldsymbol{u}}^\ell) + \mathcal{B}^\ell(\varepsilon(\boldsymbol{u}^\ell), \boldsymbol{\beta}^\ell) + (\mathcal{E}^\ell)^* \nabla \varphi^\ell \quad \text{in } \Omega^\ell \times (0, T), \tag{2.1}$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) - \boldsymbol{B}^\ell \nabla \varphi^\ell \quad \text{in } \Omega^\ell \times (0, T), \tag{2.2}$$

$$\dot{\boldsymbol{\beta}}^\ell - \kappa^\ell \Delta \boldsymbol{\beta}^\ell + \partial_{k^\ell}(\boldsymbol{\beta}^\ell) \ni \boldsymbol{S}^\ell(\varepsilon(\boldsymbol{u}^\ell), \boldsymbol{\beta}^\ell). \tag{2.3}$$

$$\operatorname{Div} \boldsymbol{\sigma}^\ell = \rho^\ell \ddot{\boldsymbol{u}}^\ell - \boldsymbol{f}_0^\ell \quad \text{in } \Omega^\ell \times (0, T), \tag{2.4}$$

$$\operatorname{div} \boldsymbol{D}^\ell = q_0^\ell \quad \text{in } \Omega^\ell \times (0, T), \tag{2.5}$$

$$\boldsymbol{u}^\ell = 0 \quad \text{on } \Gamma_1^\ell \times (0, T), \tag{2.6}$$

$$\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell = \boldsymbol{f}_2^\ell \quad \text{on } \Gamma_2^\ell \times (0, T), \tag{2.7}$$

$$\begin{cases} \sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu \\ -\sigma_\nu = p_\nu([u_\nu] - g) \end{cases} \text{on } \Gamma_3 \times (0, T), \quad \begin{cases} \boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau \\ \boldsymbol{\sigma}_\tau = 0 \end{cases} \text{on } \Gamma_3 \times (0, T), \tag{2.8}$$

$$\frac{\partial \beta^\ell}{\partial \nu^\ell} \quad \text{on } \Gamma^\ell \times (0, T), \tag{2.9}$$

$$\varphi^\ell = 0 \quad \text{on } \Gamma_a^\ell \times (0, T), \tag{2.10}$$

$$\boldsymbol{D}^\ell . \boldsymbol{\nu}^\ell = q_2^\ell \quad \text{on } \Gamma_b^\ell \times (0, T), \tag{2.11}$$

$$\boldsymbol{u}^\ell(0) = \boldsymbol{u}_0^\ell, \quad \dot{\boldsymbol{u}}^\ell(0) = \boldsymbol{v}_0^\ell, \quad \boldsymbol{\beta}^\ell(0) = \boldsymbol{\beta}_0^\ell \quad \text{in } \Omega^\ell. \tag{2.12}$$

First, equations (2.1) and (2.1) represent the electro-viscoelastic constitutive law of the material , equation (2.4) is the equation of motion and (2.5) represents the balance equation for the electric-displacement field. Next equations (2.6) and (2.7) are the displacement and traction boundary conditions , whereas (2.10) and (2.11) represent the electric boundary conditions the electrical conductivity coefficient.The relation (2.9) describes a homogeneous Neumann boundary condition where $\partial \beta^\ell | \partial v^\ell$ is the normal derivative of β^ℓ, condition (2.8) represents the normal compliance contact . Finally, conditions (2.12) represent the initial conditions where $\boldsymbol{u}_0^\ell$ and $\boldsymbol{v}_0^\ell$ denote the initial displacement and the initial velocity, respectively and β_0^ℓ is the initial damage.

3. Variational formulation

Problem $\mathcal{P}_V$. Find a displacement field $\boldsymbol{v}^\ell : [0, T] \to \boldsymbol{V}$, an electric potential field $\varphi^\ell : [0, T] \to W$ and a damages field $\beta^\ell : [0, T] \to H^1(\Omega^\ell)$ such that $\boldsymbol{v}^\ell(0) = \boldsymbol{v}_0^\ell$ and for a.e. $t \in (0, T)$,

$$(\dot{\boldsymbol{v}}(t), \boldsymbol{w})_{\boldsymbol{V}' \times \boldsymbol{V}} + \sum_{\ell=1}^{2} (\mathcal{A}^\ell \boldsymbol{\varepsilon}(\boldsymbol{v}^\ell(t)) + \mathcal{B}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)) + (\mathcal{E}^\ell)^* \nabla \varphi^\ell(t), \boldsymbol{\varepsilon}(\boldsymbol{w}^\ell))_{Q^\ell} +$$
$$j(\boldsymbol{u}(t), \boldsymbol{w}) = (\mathbf{f}(t), \boldsymbol{w})_{\boldsymbol{V}' \times \boldsymbol{V}}, \quad \forall \boldsymbol{w} \in \boldsymbol{V}, \tag{3.1}$$

$$\sum_{\ell=1}^{2} (\mathcal{B}^\ell \nabla \varphi^\ell(t), \nabla \phi^\ell)_{\mathcal{W}^\ell} - \sum_{\ell=1}^{2} (\mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell(t)), \nabla \phi^\ell)_{\mathcal{W}^\ell} = (q(t), \phi)_W, \quad \phi \in W, \tag{3.2}$$

$$\boldsymbol{\beta}(t) \in K, \quad \sum_{\ell=1}^{2} (\dot{\boldsymbol{\beta}}^\ell(t), \xi^\ell - \boldsymbol{\beta}^\ell(t))_{L^2(\Omega^\ell)} + a(\boldsymbol{\beta}(t), \xi - \boldsymbol{\beta}(t)) \geq$$
$$\sum_{\ell=1}^{2} (\mathcal{S}^\ell(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)), \boldsymbol{\beta}(t)), \xi^\ell - \boldsymbol{\beta}^\ell(t))_{L^2(\Omega^\ell)}, \quad \xi \in K, \tag{3.3}$$

where the displacement field $\boldsymbol{u}(t)$ is then defined as $\boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s) ds + \boldsymbol{u}_0$

4. Main Results

Now, we propose our existence and uniqueness result.

Theorem 4.1 *Assume that* f *and* ḟ *lie in* $L^2(0, T; \boldsymbol{V}')$. *There exists a unique solution to Problem* $\mathcal{P}_V$ *with the following regularity :*

$$\boldsymbol{v} \in C^1(0, T; H) \cap H^1(0, T; \boldsymbol{V}), \quad \varphi \in C(0, T; W), \quad \boldsymbol{\beta} \in C^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).$$

5. Fully discrete approximations : error estimates

Using the backward Euler scheme, the fully discrete approximation of Problem $\mathcal{P}_V$ is the following

Problem $\mathcal{P}_V^{hk}$. Find a discrete velocity field $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N \subset \boldsymbol{V}^h$, a discrete electric potential field $\varphi^{hk} = \{\varphi_n^{hk}\}_{n=1}^N \subset W^h$, and a discrete damage field $\boldsymbol{\beta}^{hk} = \{\boldsymbol{\beta}_n^{hk}\}_{n=0}^N \subset K^h$ such that $\boldsymbol{v}_0^{hk} = \boldsymbol{v}_0^h, \ \boldsymbol{\beta}_0^{hk} = \boldsymbol{\beta}_0^h$ and for all $n = 1, ..., N$

$$(\delta \boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_{\boldsymbol{V}' \times \boldsymbol{V}} + \sum_{\ell=1}^{2} (\mathcal{A}^\ell \boldsymbol{\varepsilon}(\boldsymbol{v}_n^{\ell,hk}) + \mathcal{B}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{\ell,hk}) + (\mathcal{E}^\ell)^* \nabla \varphi_n^{\ell,hk}, \boldsymbol{\varepsilon}(\boldsymbol{w}^{\ell,h}))_{Q^\ell} +$$
$$j(\boldsymbol{u}_n^{hk}, \boldsymbol{w}^h) = (\mathbf{f}_n, \boldsymbol{w}^h)_{\boldsymbol{V}' \times \boldsymbol{V}}, \quad \forall \boldsymbol{w}^h \in \boldsymbol{V}^h, \tag{5.1}$$

$$\sum_{\ell=1}^{2} (\mathcal{B}^\ell \nabla \varphi_n^{\ell,hk}, \nabla \phi^{\ell,h})_{\mathcal{W}^\ell} - \sum_{\ell=1}^{2} (\mathcal{E}^\ell \varepsilon(\boldsymbol{u}_n^{\ell,hk}), \nabla \phi \phi^{\ell,h})_{\mathcal{W}^\ell} = (q_n, \phi^h)_W, \quad \phi^h \in W, \tag{5.2}$$

$$\sum_{\ell=1}^{2} (\delta \boldsymbol{\beta}_n^{\ell,hk}, \xi^{\ell,h} - \boldsymbol{\beta}_n^{\ell,hk})_{L^2(\Omega^\ell)} + a(\boldsymbol{\beta}_n^{hk}, \xi^h - \boldsymbol{\beta}_n^{hk}) \geq$$
$$\sum_{\ell=1}^{2} (\mathcal{S}^\ell(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{\ell,hk}), \boldsymbol{\beta}_{n-1}^{\ell,hk}), \xi^{\ell,h} - \boldsymbol{\beta}_n^{\ell,hk})_{L^2(\Omega^\ell)}, \quad \xi^h \in K^h, \tag{5.3}$$

where the discrete displacement field $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=1}^N \subset \boldsymbol{V}^h$ is given by

$$\boldsymbol{u}_n^{hk} = \boldsymbol{u}_0^h + \sum_{j=0}^{n} k \boldsymbol{v}_j^{hk} \tag{5.4}$$

and $\boldsymbol{u}_0^h$, $\boldsymbol{v}_0^h$ and $\boldsymbol{\beta}_0^h$ are appropriate approximations of the initial conditions $\boldsymbol{u}_0$, $\boldsymbol{v}_0$ and $\boldsymbol{\beta}_0$, respectively.

Our interest in this section lies in estimating the numerical errors $\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H$, $\|\varphi_n - \varphi_n^{hk}\|_W$ and $\|\boldsymbol{\beta}_n - \boldsymbol{\beta}_n^{hk}\|_{L^2(\Omega)}$. We have the following main error estimates result.

Theorem 5.1 *Let the assumptions of Theorem* (4.1) *hold. Let* $(\boldsymbol{v}, \varphi, \boldsymbol{\beta})$ *and* $(\boldsymbol{v}^{hk}, \varphi^{hk}, \boldsymbol{\beta}^{hk})$ *denote the solutions to problems* $\mathcal{P}_V$ *and* $\mathcal{P}_V^{hk}$, *respectively. Then, the following error estimates hold for all* $\boldsymbol{w}^h = \{\boldsymbol{w}_n^h\}_{j=1}^N \subset \boldsymbol{V}^h$ *and* $\phi^h = \{\phi_n^h\}_{j=1}^N \subset W^h$ *and* $\xi^{hk} = \{\xi_n^{hk}\}_{j=1}^N \subset K^h$,

$$\max_{1 \leq n \leq N} \left\{ \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\boldsymbol{\beta}_n - \boldsymbol{\beta}_n^{hk}\|_{L^2(\Omega)}^2 \right\} + k \sum_{j=1}^{N} \left(\|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_{\boldsymbol{V}}^2 + \|\nabla(\boldsymbol{\beta}_j - \boldsymbol{\beta}_j^{hk})\|_{L^2(\Omega)^d}^2 \right)$$
$$\leq c \begin{pmatrix} \max\limits_{1 \leq n \leq N} \|\varphi_n - \phi_n^h\|_W^2 + k \sum\limits_{j=1}^{N} \left[\|\dot{\boldsymbol{v}}_j - \delta \boldsymbol{v}_j\|_H^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_{\boldsymbol{V}}^2 + \right] + \max\limits_{1 \leqslant n \leqslant N} \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_H^2 \\ + k^2 + \frac{1}{k} \sum\limits_{j=1}^{N-1} \|\boldsymbol{v}_j - \boldsymbol{w}_j^h - (\boldsymbol{v}_{j+1} - \boldsymbol{w}_{j+1}^h)\|_H^2 + \|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\|_H^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_{\boldsymbol{V}}^2 + \|\boldsymbol{\beta}_0 - \boldsymbol{\beta}_0^h\|_{L^2(\Omega)}^2 \\ + \max\limits_{1 \leq n \leq N} \|\boldsymbol{\beta}_n - \xi_n^h\|_{L^2(\Omega)}^2 + \frac{1}{k} \sum\limits_{j=1}^{N-1} \|(\boldsymbol{\beta}_{j+1} - \xi_{j+1}^h) - (\boldsymbol{\beta}_j - \xi_j^h)\|_{L^2(\Omega)}^2 + k \sum\limits_{j=1}^{N} \|\delta \boldsymbol{\beta}_j - \dot{\boldsymbol{\beta}}_j\|_{L^2(\Omega)}^2 \\ + k \sum\limits_{j=1}^{n} \|(\mathcal{S}(\boldsymbol{\varepsilon}(\boldsymbol{u}_j), \boldsymbol{\beta}_j) - \delta \boldsymbol{\beta}_j + \kappa \Delta \boldsymbol{\beta}_j\|_{L^2(\Omega)} \|\boldsymbol{\beta}_j - \xi_j^h\|_{L^2(\Omega)} + k \sum\limits_{j=1}^{n} \|\boldsymbol{\beta}_j - \xi_j^h\|_{L^2(\Omega)} \end{pmatrix}. \tag{5.5}$$

Finally, we have the following corollary which states the linear convergence of the algorithm under suitable regularity conditions.

Corollaire 5.2 *Let the assumptions of Theorem* (4.1) *hold. Let* $(\boldsymbol{v}, \varphi, \boldsymbol{\beta})$ *and* $(\boldsymbol{v}^{hk}, \varphi^{hk}, \boldsymbol{\beta}^{hk})$ *denote the solutions to problems* $\mathcal{P}_V$ *and* $\mathcal{P}_V^{hk}$, *respectively, and let the discrete initial conditions. Under the following regularity conditions :*

$$\boldsymbol{u} \in C^1(0, T; [H^2(\Omega)]^d) \cap H^3(0, T; \boldsymbol{V}), \quad \varphi \in C(0, T; H^2(\Omega)),$$
$$\boldsymbol{\beta} \in H^2(0, T; L^2(\Omega)) \cap C^1(0, T; H^1(\Omega)) \cap C(0, T; H^2(\Omega)),$$

the linear convergence of the algorithm is achieved ; that is, there exists a positive constant $c > 0$, *independent of the discretization parameters* h *and* k, *such that*

$$\max_{1 \leqslant n \leqslant N} \left\{ \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_{\boldsymbol{V}}^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\boldsymbol{\beta}_n - \boldsymbol{\beta}_n^{hk}\|_{L^2(\Omega)}^2 \right\} + k \sum_{j=1}^{N} \|\nabla(\boldsymbol{\beta}_j - \boldsymbol{\beta}_j^{hk})\|_{L^2(\Omega)^d}^2 \leqslant c(h + k). \tag{5.6}$$

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DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE WITH VARIABLE OPERATORS AND GENERAL ROBIN BOUNDARY CONDITION IN UMD SPACES

RABAH HAOUA

ABSTRACT. In this paper we study an abstract second order differential equation of elliptic type with variable operator coefficients and general Robin boundary conditions, in the framework of UMD spaces. These problems presents for example the linearized stationary case of a model describing information diffusion in online social networks. Existence and regularity results are obtained when the Labbas-Terreni assumption is fulfilled using semigroups theory and interpolation spaces.

1. INTRODUCTION AND HYPOTHESES

This paper is devoted to study the following general problem

$$\begin{cases} u''(x) + A(x)u(x) - \omega u(x) = f(x), \quad x \in (0,1) \\ u'(0) - Hu(0) = d_0 \\ u(1) = u_1, \end{cases} \tag{1.1}$$

with $f \in L^p(0,1,E)$, $1 < p < +\infty$, where E is a complex Banach space, d_0, u_1 are given elements in E and $(A(x))_{x\in[0,1]}$ is a family of closed linear operators whose domains $D(A(x))$ are dense in E. H is a closed linear operator in E, ω is a positive real number. The results proved here in the L^p case complete our recent paper concerning the hölderian case, see [**?**].

For all $x \in [0,1]$, set:

$$A_\omega(x) = A(x) - \omega I.$$

We will seek for a classical solution u to (**??**), i.e. a function u such that

$$\begin{cases} \text{a.e } x \in (0,1), \quad u(x) \in D(A(x)) \text{ and} \\ x \mapsto A(x)u(x) \in L^p(0,1;E) \\ u \in W^{2,p}(0,1;E) \\ u(0) \in D(H), \end{cases} \tag{1.2}$$

The method is essentially based on Dunford calculus, interpolation spaces, the semigroup theory and some techniques as in [**?**], [**?**].

We will assume that

$$E \text{ is a } UMD \text{ space.} \tag{1.3}$$

We suppose that:

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Key words and phrases. Differential equation, Robin boundary conditions, analytic semigroup, maximal regularity, Dore-Venni theorem and UMD spaces.

$\exists\omega_0 > 0, \exists C > 0 : \forall x \in [0,1], \forall z \geq 0, (A_{\omega_0}(x) - zI)^{-1} \in L(E)$ and

$$\left\|(A_{\omega_0}(x) - zI)^{-1}\right\|_{L(E)} \leq \frac{C}{1+z}; \tag{1.4}$$

and setting $Q_\omega(x) = -(-A_\omega(x))^{1/2}$ (see [**?**]), we suppose also that:
$\exists C, \alpha, \mu > 0 : \forall x, \tau \in [0,1], \forall\omega \geq \omega_0 :$

$$\begin{cases} \left\|Q_\omega(x)(Q_\omega(x) - zI)^{-1}\left(Q_\omega(x)^{-1} - Q_\omega(\tau)^{-1}\right)\right\|_{L(E)} \leq \dfrac{C|x-\tau|^\alpha}{|z+\omega|^\mu} \\ \text{with } \alpha + \mu - 2 > 0; \end{cases} \tag{1.5}$$

this hypothesis is well known as Labbas-Terreni assumption.
We obtain the following theorem.

Theorem 1.1. *Assume (**??**)~(**??**). Let $f \in L^p(0,1;E)$, $1 < p < +\infty$ and*

$$(Q_\omega(0) - H)^{-1} d_0 \in (D(A(0)), E)_{\frac{1}{2p},p}, \quad u_1 \in (D(A(1)), E)_{\frac{1}{2p},p}.$$

*Then there exists $\omega^* > 0$ such that for all $\omega \geq \omega^*$, the problem (**??**) has a unique solution $w(\cdot) = Q_\omega(\cdot)^2 u(\cdot)$ verifying*

(1) $Q_\omega(\cdot)^2 u(\cdot) \in L^p(0,1;E)$.
(2) $u'' \in W^{2,p}(0,1;E)$.

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MODELLING THE FREQUENCY OF AUTOMOTIVE INSURANCE CLAIMS.

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Abstract: *In the strongly competitive automotive insurance market, the insurer tries to determine factors that explain the frequency and cost of claims. In this work, we study the factors that explain the number of accidents declared by the responsible insurant to his or her insurer giving consideration to the importance of the number of insurants without an accident over a given year. We use zero-inflated distributions (Poisson and binomial negative). These distributions model count data that have many zeros. For example, the zero-inflated Poisson distribution might be used when the proportion of zero counts is greater than expected on the basis of the mean of the non-zero counts. Specifically, we separate the zero accidents into two groups: those without an accident from those who had an accident but did not declare it. These models have not been used on data for the French automobile insurance market. Empirically, we show that the explanatory variables of the frequency of the disasters are appreciably the same as those with the classic models of counting, with the exception of the choice of contract for which we find adverse selection. The probability that the policyholder does not declare a claim increases with the bonussurcharge coefficient and decreases with the age of the driver and the age of the car.*

Key Words: *Insurance, Frequency, Distribution...*

The numerical range of bounded operator

Sellat Ouafa and Bouzenada Smail

1 Abstract

It is well known that if the numerical range $W(T)$ of a bounded linear operator T on a complex Hilbert space included in the real line then $T^* = T$. In this paper, we generalize this result for all line of the complex plan, exactly we determine the scalars λ and μ such that $T^* = \lambda T + \mu I$ for any operator T whose numerical range is included in a determined line. We also generalize a condition for the selfadjointness of an operator.

2 Introduction

Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $B(H)$ denote the algebra of bounded linear operators defined on the complex Hilbert space H. The numerical range of an operator $T \in B(H)$ is defined by

$$W(T) = \{\langle Tx, x \rangle : x \in H, \|x\| = 1\}.$$

It is well known that $W(T)$ is a non-empty bounded convex subset of $\mathbb{C}$ that contains the set of the eigenvalues of T. In particular, the numerical range of a normal matrix is the convex hull of its eigenvalues. The geometrical properties of $W(T)$ often provide useful information about the analytic and algebraic properties of T. For example $W(T)$ is a point $\{\delta\}$ if and only if $T = \delta I$, $W(T)$ is real if and only if T is self-adjoint [2, p. 7], and if $W(T)$ is a line segment, then T is normal [2, p. 15]. Readers are refer to the the books [2,3] for more detail properties on the numerical range.

Chettouh and Bouzenada [1, th 2.2] shows that the numerical range of a bounded linear operator T on a complex Hilbert space is a line segment if and only if there are scalars λ and μ such that $T^* = \lambda T + \mu I$, where they determines the equation of the straight support of this numerical range in terms of λ and μ. In the present paper we give the converse theorem, we determine λ and μ from the equation of the straight support of the numerical range. We also generalize a result of Toivo Nieminen [4] presented by George H. Orland in [5] for the seltadjointness of a bounded linear operator.

Before starting and providing our results, we begin by introducing some notation used throughout the text. The spectra and resolvent set of operators T are denoted by $\sigma(T)$ and $\rho(T)$, respectively. Also, we let $R_\alpha(T) = (T - \alpha I)^{-1}$ for all $\alpha \in \rho(T)$. An operator T is said to be convexoid if $\overline{W(T)}$ coincide with the convex hull of their spectrum. Here, Similarly ans in reference [1], $\mathbb{S}(H)$ denote the class of operators T in $B(H)$ which satisfy $T^* = \lambda T + \mu I$, for $\lambda, \mu \in \mathbb{C}$.

We make use of the following theorems to prove our results.

Theorem 1 [1] *Let $T \in B(H)$. Then $W(T)$ is a line segment if and only if there are $\lambda, \mu \in \mathbb{C}$ such that $T^* = \lambda T + \mu I$.*

Theorem 2 [1] *Let $T \in \mathbb{S}(H)$ and $\lambda, \mu \in \mathbb{C}$ such that $T^* = \lambda T + \mu I$. Then,*

(1) *If $|\lambda| \neq 1$, then $W(T)$ is the point $\{\delta\}$, where*

$$\delta = \frac{\overline{\lambda}\mu + \overline{\mu}}{1 - |\lambda|^2}.$$

(2) *If $\lambda = 1$, then $W(T)$ is an horizontal line segment whose the equation of their straight support is*

$$Y = \frac{\mu}{2} i, \text{ with } \operatorname{Re} \mu = 0.$$

(3) *If $\lambda = -1$, then $W(T)$ is a vertical line segment whose the equation of their straight support is*

$$X = \frac{\mu}{2}, \text{ with } \mu \in \mathbb{R}.$$

(4) *Otherwise, $W(T)$ is an inclined line segment whose the equation of their straight support is*

$$Y = \left(\frac{-1 + \operatorname{Re} \lambda}{\operatorname{Im} \lambda} \right) X + \frac{\operatorname{Re} \mu}{\operatorname{Im} \lambda}.$$

Theorem 3 [5] *If $\sigma(T)$ is real and $\|R_\alpha(T)\| \leq |\alpha|^{-1}$ for all nonzero purely imaginary α, then T is self-adjoint.*

3 Results

Theorem 4 *Let $T \in B(H)$ a non scalar operator such that $W(T)$ included in a complex line L. Then $T^* = \lambda T + \mu I$ such that:*

(1) *If the equation of L is $y = ax + b$ with $a \neq 0$, then*

$$\lambda = \frac{-(a+i)^2}{1+a^2}, \quad \text{and} \quad \mu = -2b \frac{a+i}{1+a^2}.$$

(2) *If the equation of L is $y = b$, then*

$$\lambda = 1, \text{ and } \mu = -2ib.$$

(3) *If the equation of L is $x = a$, then*

$$\lambda = -1, \text{ and } \mu = 2a.$$

Remark 5 *If $W(T) = \{\delta\}$ such that $\delta \neq 0$, then*

$$T^* = \frac{\overline{\delta}}{\delta} T.$$

Example 6 *Let $P \in B(H)$ an orthogonal projection and let's pose $T = (1+i)P + iI$. Then*

$$W(T) = (1+i)W(P) + i,$$

hence $W(T)$ is the line segment joining the two points i and $1+2i$, identically $W(T)$ included in a complex line of the equation $y = x+1$. Furthermore, we have $a = 1$ and $b = 1$ from the first case of the theorem, then $\lambda = -i$ and $\mu = -1-i$. Therefore

$$T^* = -iT - (1+i)I = -i((1+i)P + iI) - (1+i)I = (1-i)P - iI,$$

which is equal to $((1+i)P + iI)^*$.

Remark 7 *According to the previous theorem, if the operator T is convexoid and $\sigma(T)$ included in the line L of the equation $y = ax$ $(a \neq 0)$, then*

$$T^* = \frac{-(a+i)^2}{1+a^2}T + -2b\frac{a+i}{1+a^2}I.$$

We now give a weaker sufficient condition on the operator T with $T^* = \lambda T + \mu I$, and $\lambda, \mu \in \mathbb{C}$. Similarly, we generalize Theorem 3 for all complex line through the origin.

Theorem 8 *Let $T \in B(H)$. If $\sigma(T)$ included in the line L of the equation $y = ax$, $(a \neq 0)$ and $\|R_\alpha(T)\| \leq |\alpha|^{-1}$ for all*

$$\alpha \in \left(1 - \frac{1}{a}i\right)\mathbb{R}^*,$$

then, $W(T)$ is a segment of the line L and

$$T^* = \frac{-(a+i)^2}{1+a^2}T.$$

Remark 9 *If $\sigma(T) \subset i\mathbb{R}$, and $\|R_\alpha(T)\| \leq |\alpha|^{-1}$ for all $\alpha \in \mathbb{R}^*$, we obtain $W(T) \subset i\mathbb{R}$ and T is anti-self-adjoint.*

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On Cohen weakly nuclear multilinear operators

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Abstract: In this work, we introduce and study a new class of multilinear operators on Banach spaces, called Cohen weakly p-nuclear multilinear operators. We establish a Pietsch domination theorem for this new class of multilinear operators. As an application, we show that every Cohen weakly p-nuclear multilinear operator is strongly p-summing and Cohen strongly p-summing.

Keywords: multiliear operator; nuclear operator; p-summing operator.

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Solvability and M-L-U stability of sequential Caputo-Riemann-Liouville fractional Duffing problem

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Abstract: *In this present manuscript, we discuss the existence, uniqueness and Mittag-Leffler-Ulam stability (M-L-U stability) of solutions for fractional Duffing equations involving Caputo and Riemann-Liouville fractional derivatives. Uniqueness result for solution of the underlying Duffing problem is presented with the aid of Banach's fixed point theorem, while the existence result is derived from Leray-Schauder's alternative. Also the M-L-U stability results are obtained by using generalized singular Gronwall's inequality.*

Keywords: Fractional derivative, fixed point, existence, Duffing equation, Mittag-Leffler-Ulam stability.

1 Introduction

Differential equations involving fractional derivative operators have attracted great attention in the last years, these fractional differential equations arise in the modeling of various problems in sciences and engineering such as economy, control, biology and electrodynamics, etc. For more details, we refer to [5, 8]. Many interesting and important area concerning of research for differential equations involving fractional calculus are devoted to the existence theory and stability analysis of the solutions. Recently, there are several sciffentific researchers have studied the existence, uniqueness and different types of Ulam-stability and Mittag-Leffler-Ulam-stability of solutions for differential equations of fractional order. For more information, see [5, 6, 7]. In recent years, many scholars have exposed attention in the field of theory of nonlinear fractional differential equations, which will be used to describe phenomena of real-world problems, for example see [2, 3]. One of the very important nonlinear differential equations is the Duffing equation [1]

$$D^{2}(t)+\kappa D^{1}(t)+g(t,w(t))=f(t),0\leq t\leq 1,\ \kappa>0,$$

under the conditions:

$$w(0)=B_1,\ D^1w(0)=B_2,\ B_i\in\mathbb{R},i=1,2,$$

where g and f are given continuous functions, this equation is used to model certain driven-damped oscillators, see [2, 3]. Many scholars have discussed the fractional version of the Duffing equation, for instance see [4, 6] and references therein. In this present work, we discuss the existence, uniqueness and Mittag-Leffler-Ulam-stability of solutions for sequential Caputo-Riemann-Liouville fractional Duffing equation

$$\begin{cases} {}^{C}D^{\lambda}\left[{}^{R.L}D^{\omega}w\left(t\right)\right]+\Lambda g\left(t,w\left(t\right),{}^{R.L}D^{\gamma}w\left(t\right)\right)+h\left(t,w\left(t\right),I^{\alpha}w\left(t\right)\right) \\ \qquad =f\left(t\right), \\ w\left(0\right)=0,{}^{RL}D^{\omega}w\left(0\right)=\theta_{1},{}^{RL}D^{\omega}w\left(1\right)=\theta_{2},\theta_{i}\in\mathbb{R},i=1,2, \\ t\in\left[0,1\right],1<\lambda\leq 2,0<\omega<\gamma\leq 1,\alpha>0,\Lambda>0,\theta\in\mathbb{R}, \end{cases} \tag{1}$$

where ${}^{C}D^{\lambda},{}^{RL}D^{\omega}$ denote the Caputo and Riemann-Liouville fractional derivatives, I^{α} denotes the Riemann–Liouville integral of order α, $g,h:[0,1]\times\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ and $f:[0,1]\rightarrow\mathbb{R}$ are given continuous functions. The operator ${}^{C}D^{\lambda}$ is the fractional derivative in the sense of Caputo [9], defined by

$${}^{C}D^{\lambda}w\left(t\right)=\frac{1}{\Gamma\left(n-\lambda\right)}\int_{0}^{t}\left(t-s\right)^{n-\lambda-1}w^{(n)}\left(s\right)ds,n=[\lambda]+1,$$

where $\Gamma\left(.\right)$ is the Euler gamma function. The operator ${}^{RL}D^{\omega}$ is the fractional derivative in the sense of Riemann-Liouville [9], defined by

$${}^{RL}D^{\omega}w\left(t\right)=\frac{1}{\Gamma\left(n-\omega\right)}\left(\frac{d}{dt}\right)^{n}\int_{0}^{t}\left(t-s\right)^{n-\omega-1}w\left(s\right)ds,n=[\omega]+1,$$

and the Riemann-Liouville fractional integral [9] of order $\alpha>0$, defined by

$$I^{\alpha}w\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}\left(t-s\right)^{\alpha-1}w\left(s\right)ds,\ t>0.$$

Let $W=\left\{w:w\in C\left(\left[0,1\right],\mathbb{R}\right)\text{ and }{}^{RL}D^{\omega}w\in C\left(\left[0,1\right],\mathbb{R}\right)\right\}$ denotes the space equipped with the norm $\left\Vert w\right\Vert_{W}=\left\Vert w\right\Vert+\left\Vert {}^{RL}D^{\omega}w\right\Vert$, where

$$\left\Vert w\right\Vert=\sup_{t\in[0,1]}\left\vert w\left(t\right)\right\vert\text{ and }\left\Vert {}^{RL}D^{\omega}w\right\Vert=\sup_{t\in[0,1]}\left\vert {}^{RL}D^{\omega}w\left(t\right)\right\vert.$$

It is clear that $(W,\left\Vert .\right\Vert_{W})$ is a Banach space.

2 Existence and uniqueness of solutions for problem (1)

Lemma 1 *For a given $m\left(t\right)\in C\left(\left[0,1\right],\mathbb{R}\right)$, the fractional problem*

$$\begin{cases} {}^{C}D^{\lambda}\left[{}^{R.L}D^{\omega}w\left(t\right)\right]=m\left(t\right),\ t\in\left[0,1\right], \\ w\left(0\right)=0,{}^{R.L}D^{\omega}w\left(0\right)=\theta_{1},{}^{R.L}D^{\omega}w\left(1\right)=\theta_{2}, \\ 1<\lambda\leq 2,0<\omega<1,0<\eta<1,\theta_{i}\in\mathbb{R},i=1,2, \end{cases} \tag{2}$$

has a unique solution

$$
\begin{aligned}
w(t) &= \frac{1}{\Gamma(\lambda+\omega)}\int_0^t (t-s)^{\lambda+\omega-1} m(s)\,ds \\
&-\frac{t^{\omega+1}}{\Gamma(\omega+2)\Gamma(\lambda)}\int_0^1 (1-s)^{\lambda-1} m(s)\,ds \\
&+\frac{\theta_2-\theta_1}{\Gamma(\omega+2)}t^{\omega+1}+\frac{\theta_1}{\Gamma(\omega+1)}t^{\omega}.
\end{aligned} \tag{3}
$$

In view of Lemma 1 , we define operator $P: W \rightarrow W$ by:

$$
\begin{aligned}
& Pw(t) \\
= & \frac{1}{\Gamma(\lambda+\omega)}\int_0^t (t-s)^{\lambda+\omega-1}\left(f(s)-\Lambda g_w^*(s)-h_w^*(s)\right)ds \\
& -\frac{t^{\omega+1}}{\Gamma(\omega+2)\Gamma(\lambda)}\int_0^1 (1-s)^{\lambda-1}\left(f(s)-\Lambda g_w^*(s)-h_w^*(s)\right)ds \\
& +\frac{\theta_2-\theta_1}{\Gamma(\omega+2)}t^{\omega+1}+\frac{\theta_1}{\Gamma(\omega+1)}t^{\omega}.
\end{aligned} \tag{4}
$$

and ${}^{RL}D^{\omega}Pw(t)$ is given by:

$$
{}^{RL}D^{\omega}Pw(t)=\int_0^t \frac{(t-s)^{-\omega}}{\Gamma(1-\omega)}\left(D^1 Pw\right)(s)\,ds, \tag{5}
$$

where

$$
\begin{aligned}
& D^1 Pw(t) \\
= & \frac{1}{\Gamma(\lambda+\omega-1)}\int_0^t (t-s)^{\lambda+\omega-2}\left(f(s)-\Lambda g_w^*(s)-h_w^*(s)\right)ds \\
& -\frac{t^{\omega}}{\Gamma(\omega+1)\Gamma(\lambda)}\int_0^1 (1-s)^{\lambda-1}\left(f(s)-\Lambda g_w^*(s)-h_w^*(s)\right)ds \\
& +\frac{\theta_2-\theta_1}{\Gamma(\omega+1)}t^{\omega}+\frac{\theta_1}{\Gamma(\omega)}t^{\omega-1}.
\end{aligned} \tag{6}
$$

For computation convenience, we introduce the notations:

$$
\begin{aligned}
\Delta_1 &:= \frac{1}{\Gamma(\lambda+\omega+1)}+\frac{1}{\Gamma(\omega+2)\Gamma(\lambda+1)}, \\
\Delta_2 &:= \frac{1}{\Gamma(\lambda+\omega)}+\frac{1}{\Gamma(\omega+1)\Gamma(\lambda+1)}.
\end{aligned} \tag{7}
$$

By using Banach's fixed point theorem, we will establish find a unique solution of the fractional Duffing problem (1).

Theorem 2 *Let* $g, h : [0,1]\times\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ *and* $f : [0,1]\rightarrow\mathbb{R}$ *be continuous functions. In addition we assume that:*

(C_1) : *There exist a constants* $\vartheta_i > 0, (i = 1, 2)$ *such that for all* $t \in [0, 1]$ *and* $w_j, z_j \in \mathbb{R}, j = 1, 2$,

$$\begin{aligned} |\varphi(t, w_1, w_2) - \varphi(t, z_1, z_2)| &\leq \vartheta_1 (|w_1 - z_1| + |w_2 - z_2|), \\ |\psi(t, w_1, w_2) - \psi(t, z_1, z_2)| &\leq \vartheta_2 (|w_1 - z_1| + |w_2 - z_2|). \end{aligned}$$

If the inequality

$$\left[\Lambda\vartheta_1 + \vartheta_2 \left(1 + \frac{1}{\Gamma(\alpha + 1)}\right)\right] (\Delta_1 + \Delta_2) < 1, \tag{8}$$

is valid, then the fractional Duffing problem (1) has a unique solution on $[0, 1]$.

Theorem 3 *Assume that* $g, h : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $f : [0, 1] \to \mathbb{R}$ *are continuous functions. Suppose that:*

(h_3) : *There exist real constants* $\beta_i, \eta_i \geq 0, i = 1, 2$ *and* $\beta_0 > 0, \eta_0 > 0$ *such that for any* $w_1, w_2 \in \mathbb{R}$, *we have*

$$|\varphi(t, w_1, w_2)| \leq \beta_0 + \beta_1 |w_1| + \beta_2 |w_2|,$$

and

$$|\psi(t, w_1, w_2)| \leq \eta_0 + \eta_1 |w_1| + \eta_2 |w_2|.$$

(h_4) : *There exists constant* $\mu > 0$ *such that*

$$|f(t)| \leq \mu, \quad \text{for all } t \in [0, 1].$$

If

$$[\Lambda(\beta_1 + \beta_2) + (\eta_1 + \eta_2)] (\Delta_1 + \Delta_2) < 1. \tag{9}$$

Then the problem (1) has at least one solution on $[0, 1]$.

3 Mittag-Leffler-Ulam-Hyers-stability of problem (1)

In this part, we will define and study Mittag-Leffler-Ulam-Hyers stability and Mittag-Leffler-Ulam-Hyers-Rassias stability of the fractional Duffing problem (1). For ρ is positive real number and $\psi : [0, 1] \to \mathbb{R}^+$ is continuous function, we give the following fractional inequalities:

$$\left| {}^C D^\lambda \left[{}^{RL} D^\omega z(t) \right] - (f(t) - \Lambda g_z^*(t) - h_z^*(t)) \right| \leq \rho, t \in [0, 1], \tag{10}$$

and

$$\left| {}^C D^\lambda \left[{}^{RL} D^\omega z(t) \right] - (f(t) - \Lambda g_z^*(t) - h_z^*(t)) \right| \leq \rho\psi(t), t \in [0, 1], \tag{11}$$

where $g_z^*(t) = g\left(t, z(t), {}^{RL} D^\beta z(t)\right)$ and $h_z^*(t) = h(t, z(t), I^\alpha z(t))$.

Definition 4 *The fractional Duffing problem (1) is Mittag-Leffler-Ulam-Hyers stable, with respect to* $E_{\lambda+\omega}$ *if there exists a real number* σ *such that for each* $\rho > 0$ *and for each solution* $z \in Z$ *of the inequality (10), there exists a solution* $w \in Z$ *of the problem (1) with*

$$|z(t) - w(t)| \leq \sigma \rho E_{\lambda+\omega}[t], \ t \in [0,1].$$

Definition 5 *The fractional Duffing problem (1) is Mittag-Leffler-Ulam-Hyers-Rassias stable, with respect to* $\psi E_{\lambda+\omega}$ *if there exists a real number* $\sigma_\psi > 0$ *such that for each* $\rho > 0$ *and for each solution* $z \in Z$ *of the inequality (11), there exists a solution* $w \in Z$ *of problem (1) with*

$$|z(t) - w(t)| \leq \sigma_\psi \rho \psi(t) E_{\lambda+\omega\delta}[t], \ t \in [0,1].$$

Remark 6 *A function* $z \in Z$ *is a solution of the inequality (10) if and only if there exists a function* $\varphi : [0,1] \to \mathbb{R}$ *(which depend on z) such that*

$$|\varphi(t)| \leq \varpi, \quad t \in [0,1],$$

and

$${}^{C}D^{\lambda}\left[{}^{RL}D^{\omega}z(t)\right] - (f(t) - \Lambda g_z^*(t) - h_z^*(t)) = \varphi(t), t \in [0,1].$$

Theorem 7 *Assume that* $g, h : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, f : [0,1] \to \mathbb{R}$ *are continuous functions and suppose that* (H_1) *holds. Then the fractional Duffing problem (1) is Mittag-Leffler-Ulam-Hyers stable.*

Theorem 8 *Let* $g, h : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, f : [0,1] \to \mathbb{R}$ *be continuous functions and assume that* (H_1) *holds. Suppose there exists a function* $\chi \in C([0,1], \mathbb{R}_+)$ *is increasing and there exists* $\sigma_\chi > 0$ *such that for any* $t \in [0,1]$

$$\frac{1}{\Gamma(\lambda+\omega)} \int_0^t (t-s)^{\lambda+\omega-1} \chi(s) ds \leq \sigma_\chi \chi(t). \tag{12}$$

Then the fractional Duffing problem (1) is Mittag-Leffler-Ulam-Hyers-Rassias stable with respect to $\psi E_{\lambda+\omega}$.

4 Conclusion

In this paper, we considered a fractional Duffing problem with sequential Caputo-Riemann-Liouville fractional derivatives. The existence and Mittag-Leffer-Ulam stability of solutions have been discussed. The existence results of the solutions for the mentioned problem were investigated by applying contraction mapping principle and Leray-Schauder's alternative. The Mittag-Leffer-Ulam-Hyers stability results have been also proved by using generalized singular Gronwall's inequality.

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GALERKIN METHOD FOR THE BOUSSINESQUE EQUATION WITH VISCOELASTIC MEMORY AND INTEGRAL CONDITION

DRAIFIA ALAEDDINE

Abstract. The present paper deals with the boussinesque equation with viscoelastic memory and integral condition. We'll demonstrate the existence and uniqueness of a solution by applying the Faedo-Galerkin's method.

1. INTRODUCTION

In this paper, we study the existence and uniqueness of solutions of the boussinesque equation with term viscoelastic memory

$$v_{tt} - \alpha^2 \Delta v - \beta^2 \Delta v_{tt} + \int_0^t h(t-s)\, \Delta v(s)\, ds = |v|^{p-2} v, \quad (x,t) \in D_T, \tag{1.1}$$

with initial data

$$v(x,0) = v_0(x), \; v_t(x,0) = v_1(x), \quad x \in \Omega, \tag{1.2}$$

and the integral condition

$$\frac{\partial v}{\partial \eta} = g(x,t) + \int_0^t \int_\Omega v(\xi,\mu)\, d\xi d\mu, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.3}$$

where $D_T := \Omega \times (0,T)$, Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $2 < p \leq \dfrac{2(N-1)}{N-2}$, $N \geq 3$, η is the unit outward normal on $\partial\Omega$, $T < \infty$, $h(.) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are given functions thid paper which will be specified later, and $g(x,t)$, $h(t)$, $v_0(x)$, and $v_1(x)$ are given functions satisfying conditions specified later.

The convolution term $\int_0^t h(t-s)\, \Delta v(s)\, ds$ reflects the memory effects of materials due to viscoelasticity. Here the convolution kernel h satisfies proper conditions exhibiting "memory character" which will be explained later. Under some assumptions on the kernel h, existence and uniqueness of the generalized solution is established by using Galerkin method.

As a special application see Bouziani [4], where the author has considered a nonlocal problem which is proposed in the mathematical modeling of technologic process of external elimination of gas, practices in the refining of impurities of Silicon lamina. The nonlocal condition appearing in this mathematical model represents the total mass of impurities in the lamina. For some hyperbolic non local mixed problems, the reader should consult the works done by Nukushev [5], and Mesloub

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and Messaoudi [8, 9], Mesloub and Bouziani [10], Pulkina [6], Mesloub and Lekrine [11], Muravei and Philinovskii [7], Beilin [12]. Recent works dealing with nonlinear nonlocal mixed problems can be found in Mesloub [13, 14, 15].

The existence and uniqueness of the generalized solution is constructed using the Galerkin method for the boussinesque equation with viscoelastic memory and integral condition with the following formula $\frac{\partial u}{\partial \eta} = g(x,t) + \int_0^t \int_\Omega u(\xi,\mu)\, d\xi d\mu$ where $(1.1)-(1.3)$ is considered a new problem.

However, in $[1-15]$; did not study the existence and uniqueness, of problem $(1.1)-(1.3)$ for integral condition with the following formula $\frac{\partial u}{\partial \eta} = g(x,t) + \int_0^t \int_\Omega u(\xi,\mu)\, d\xi d\mu$. Motivated by the above research, we will consider the existence and uniqueness for integral condition with the following formula $\frac{\partial u}{\partial \eta} = g(x,t) + \int_0^t \int_\Omega u(\xi,\mu)\, d\xi d\mu$ of the model $(1.1)-(1.3)$ in this paper.

The outline of the paper is as follows. In Section 2, we define the function spaces, state some inequalities and we supply an appropriate definition of weak solution of the posed problem. Section 3 is proving the existence of a solution using Faedo–Galerkin's method. Finally, Sect. 4 is devoted to establishing the uniqueness of the generalized solution of the posed problem.

2. PRELIMINARIES

Let $V(D_T)$ and $W(D_T)$ be the set spaces defined respectively by

$$V(D_T) := \left\{ v \in W_2^1(D_T) : v_t \in H^1(D_T) \right\}, \quad W(D_T) := \left\{ u \in V(D_T) : u(x,T) = 0 \right\}.$$

Consider the equation

$$\begin{aligned}
& -(v_t(t), u_t(t))_{L^2(D_T)} + \alpha^2 (\nabla v, \nabla u(t))_{L^2(D_T)} - \beta^2 (\nabla v_t, \nabla u_t)_{L^2(Q_T)} \\
& - \left(\int_0^t h(t-s) \nabla v(s)\, ds, \nabla u(t) \right)_{L^2(D_T)} \\
= \; & (v_t(0), u(0))_{L^2(\Omega)} + \beta^2 (\nabla v_t(0), \nabla u(0))_{L^2(\Omega)} \\
& + \alpha^2 \int_0^T \int_{\partial\Omega} g(x,t) v(t)\, dS_x dt + \alpha^2 \int_0^T \int_{\partial\Omega} \left[\left(\int_0^t \int_\Omega v(\xi,\mu)\, d\xi d\mu \right) u(t) \right] dS_x dt \\
& + \beta^2 \int_0^T \int_{\partial\Omega} g_{tt}(x,t) v(t)\, dS_x dt + \beta^2 \int_0^T \int_{\partial\Omega} \left[\left(\int_\Omega v_t(\xi,t)\, d\xi \right) u(t) \right] dS_x dt \\
& - \beta^2 \int_0^T \int_{\partial\Omega} \left[\left(\int_\Omega v_t(\xi,0)\, d\xi \right) u(t) \right] dS_x dt - \int_0^T \int_{\partial\Omega} \left[\left(\int_0^t h(t-s) g(s,x)\, ds \right) u(t) \right] dS_x dt \\
& - \int_0^T \int_{\partial\Omega} \left[\left(\int_0^t g(t-s) \left\{ \int_0^s \int_\Omega v(\xi,\mu)\, d\xi d\mu \right\} ds \right) u(t) \right] dS_x dt + \left(|v|^{p-2} v(t), u(t) \right)_{L^2(D_T)}.
\end{aligned} \tag{2.1}$$

where $(.,.)_{L^2(D_T)}$ stands for the inner product in $L^2(D_T)$, v is supposed to be a solution of $(1.1)-(1.3)$ and $u \in W(D_T)$.

Definition 1. *A function $v \in V(D_T)$ is called a generalized solution of problem* $(1.1)-(1.3)$ *if it satisfies equation* (2.1) *for each $u \in W(D_T)$.*

We recall the binary notation

$$(h \circ w)(t) := \int_0^t h(t-s) \|w(x,s) - w(x,t)\|^2_{L^2(\Omega)} ds.$$

3. EXISTENCE OF THE GENERALIZED SOLUTION

We make the following assumptions:

$(\mathbf{H}_1)$ $2 < p < \dfrac{2(N-1)}{N-2}$, $N \geq 3$. $(\mathbf{H}_2)$ $h(t) \geq 0$ and $h'(t) \leq 0$ for all $t \geq 0$.

$(\mathbf{H}_3)$ $c^2 - \bar{h} > 0$ where $\bar{h} := \int_0^\infty h(s)\, ds$. $(\mathbf{H}_4)$ $g \in L^2(0,T; L^2(\partial\Omega))$, g', $g'' \in L^2(0,T; L^2(\partial\Omega))$.

We now give the main result on the existence of solution of problem $(1.1)-(1.3)$ and prove it by using the Galerkin method.

Theorem 1. *Assume that the hypotheses* $(\mathbf{H}_1)-(\mathbf{H}_4)$ *hold, the initial data $v_0(x)$, $v_1(x) \in H^1(\Omega)$, then there is at least one generalized solution in $V(D_T)$ to problem* $(1.1)-(1.3)$.

Proof. Let $\{\psi_l(x)\}_{l\geq 1}$ be a fundamental system in $W_2^1(\Omega)$ and assume for convenience the it has been orthonormalized in $L^2(\Omega)$, that is $(\psi_l(x), \psi_k(x))_\Omega = \delta_{l,k}$. We seek an approximate solution $v^m(x)$ in the form

$$v^m(x,t) = \sum_{l=1}^{l=m} f_l(t)\, \psi_l(x),$$

and can be determined from the relations for all $k = 1, ..., m$

$$\begin{aligned}
&(v^m_{tt}, \psi_k(x))_{L^2(\Omega)} + \alpha^2 (\nabla v^m, \nabla\psi_k(x))_{L^2(\Omega)} + \beta^2 (\nabla v^m_{tt}, \nabla\psi_k(x))_{L^2(\Omega)} \\
&- \left(\int_0^t h(t-s) \nabla v^m(s)\, ds, \nabla\psi_k(x)\right)_{L^2(\Omega)} \\
= \;& \alpha^2 \int_{\partial\Omega} g(x,t)\, \psi_k(x)\, dS_x \\
&+ \alpha^2 \int_{\partial\Omega} \left(\int_0^t \int_\Omega v^m(\xi,\mu)\, d\xi d\mu\right) \psi_k(x)\, dS_x \\
&+ \beta^2 \int_{\partial\Omega} g_{tt}(x,t)\, \psi_k(x)\, dS_x \\
&+ \beta^2 \int_{\partial\Omega} \left(\int_0^t \int_\Omega v^m_{\mu\mu}(\xi,\mu)\, d\xi d\mu\right) \psi_k(x)\, dS_x \\
&- \int_{\partial\Omega} \left(\int_0^t h(t-s)\, g(x,s)\, ds\right) \psi_k(x)\, dS_x \\
&- \int_{\partial\Omega} \left(\int_0^t h(t-s) \left\{\left(\int_0^s \int_\Omega v^m(\xi,\mu)\, d\xi d\mu\right)\right\} ds\right) \psi_k(x)\, dS_x \\
&+ \left(|v^m|^{p-2} v^m(t), \psi_k(x)\right)_{L^2(\Omega)}. \qquad (3.1)
\end{aligned}$$

The above system is a system of ordinary differential equations in $f_l(t)$, $l = 1, ..., m$, and initial conditions

$$f_l(0) = (\psi_l, v_0(x))_{L^2(\Omega)}, \; f_l'(0) = (\psi_l, v_1(x))_{L^2(\Omega)}.$$

By Caratheodory theorem [30], there exists solution $f_l(t)$, $l = 1, ..., m$, $t \in [0, t_m)$. We need a priori estimates that permit us to extend the solution to the whole domain $[0, T]$. Thus for every m there exists a function $v^m(x)$ satisfying (3.1). Let us obtain bounds for v^m which do not depend on m.
In the first key estimate, we put $S^m(t) := \|v^m(t)\|^2_{W_2^1(\Omega)} + \|v_t^m(t)\|^2_{H^1(\Omega)}$. To do this, we multiply each equation of (3.1) by the appropriate $f_k'(t)$ add them up from 1 to m and them integrate with respect o t from a to τ, with $\tau \leq T$, we obtain

$$\|v^m(t)\|^2_{W_2^1(D_T)} + \|v_t^m(t)\|^2_{H^1(D_T)} \leq A.$$

Therefore the sequence $\{v^m\}_{m\geq 1}$ is bounded in $V(D_T)$, and we can extract from it a subsequence for which we use the same notation which converges weakly in $V(D_T)$ to a limit function $v(x,t)$. We have to show that $v(x,t)$ is a generalized solution of (1.1). Since $v^m(x,t) \to v(x,t)$ in $L^2(D_T)$ and $v^m(x,0) \to v(x,0)$ in $L^2(\Omega)$. Now to prove that (2.1) holds, we multiply each of the relations (3.1) by a function $p_k(t) \in W_2^1(0,T)$, $p_k(T) = 0$, then add up the obtained equalities ranging from $k = 1$ to $k = m$, and integrate over t on $(0, T)$. Since

$$\|v^m(t) - v(t)\|_{W_2^1(D_T)} \longrightarrow 0, \quad \text{as } m \longrightarrow \infty,$$

and

$$\left(|v^m|^{p-2} v^m(t), \phi^m(t)\right)_{L^2(D_T)} \longrightarrow \left(|v|^{p-2} v(t), \phi(t)\right)_{L^2(D_T)}, \quad \text{as } m \longrightarrow \infty,$$

thus, the limit function v satisfies (3.1) for every $\phi^m(x,t) := \sum_{k=1}^{k=m} p_k(t)\psi_k(x)$. We denote by $\mathbb{Q}_m$ the totality of all functions of the form $\phi^m(x,t) := \sum_{k=1}^{k=m} p_k(t) Z_k(x)$, with $p_k(t) \in W_2^1(0,T)$, $p_k(T) = 0$. But $\cup_{k=1}^{k=m}\mathbb{Q}_k$ is dense in $W(D_T)$, then relation (3.1) holds for all $v \in W(D_T)$. Thus we have shown that the limit function $v(x,t)$ is a generalized solution of problem $(1.1) - (1.3)$ in $V(D_T)$. □

4. UNICITY OF THE GENERALIZED SOLUTION

Theorem 2. *The problem* $(1.1) - (1.3)$ *cannot have more than one generalized solution in* $V(Q_T)$.

Proof. Suppose that $v_1 \in V(D_T)$ and $v_2 \in V(D_T)$ are two solutions of problem $(1.1) - (1.3)$ such that v_1 is different from v_2. Then $v := v_1 - v_2$ solves

$$(4.1) \qquad \begin{cases} v_{tt} - \alpha^2 \Delta v - \beta^2 \Delta v_{tt} + \displaystyle\int_0^t h(t-s)\Delta v(s)\,ds \\ \qquad = |v_1|^{p-2} v_1 - |v_2|^{p-2} v_2, \\ v(x,0) = v_t(x,0) = 0, \\ \dfrac{\partial v}{\partial \eta} = \displaystyle\int_0^t \int_\Omega v(\xi,\tau)\,d\xi d\tau, \quad (x,t) \in \partial\Omega \times (0,T), \end{cases}$$

Define the function $u(x,t)$ by

$$u(x,t) := \begin{cases} \int_t^\tau v(x,s)\,ds, & 0 \le t \le \tau, \\ 0, & \tau \le t \le T. \end{cases} \tag{4.2}$$

It is obvious that $v \in W(D_T)$ and $u_t(x,t) = -v(x,t)$ for all $t \in [0,\tau]$. Now multiply the differential equation in (4.1) by v_t and integrate over $D_\tau := \Omega \times (0,\tau)$, and using Young's inequality, Cauchy-Schwarz inequality, $(\mathbf{T.I})$, and (4.2), we get

$$\begin{aligned} & \frac{\bar{h}}{2}\|\nabla v(\tau)\|^2_{L^2(\Omega)} + \frac{1}{2}\|v_\tau(\tau)\|^2_{L^2(\Omega)} + \frac{\beta^2}{2}\|\nabla v_\tau(\tau)\|^2_{L^2(\Omega)} \\ \le\ & C'_T \int_0^\tau (S(t))^{p-1}\,dt + C_3\int_0^\tau S(t)\,dt, \end{aligned} \tag{4.3}$$

And since τ is arbitrary, we assume that $\left(\frac{\alpha^2}{2} - \tau\left(1 + \left[\alpha^2+\beta^2+1\right]\gamma_\Omega\right)\right) > 0$, we obtain

$$\begin{aligned} & \|v(t)\|^2_{W_2^1(\Omega)} + \|v_t(t)\|^2_{H^1(\Omega)} + \|\nabla\theta(t)\|^2_{L^2(\Omega)} \\ \le\ & 0, \quad \forall\tau \in \left[0, \frac{\alpha^2}{2\left(1+\left[\alpha^2+\beta^2+1\right]\gamma_\Omega\right)}\right]. \end{aligned}$$

Proceeding in the same way for the intervals $\tau \in \left[\frac{(m-1)\alpha^2}{2\left(1+\left[\alpha^2+\beta^2+1\right]\gamma_\Omega\right)}, \frac{m\alpha^2}{2\left(1+\left[\alpha^2+\beta^2+1\right]\gamma_\Omega\right)}\right]$ to cover the whole interval $[0,T]$, and thus proving that $v(x,\tau) = 0$, for all τ in $[0,T]$. □

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SOME GENERALIZATIONS OF NONLINEAR RETARDED INTEGRODIFFERENTIAL INEQUALITIES

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Abstract. The Integral inequalities with a term of delay are utilized a lot in the study and modeling of retarded partial differential equations. A number of researchers have already established their basic properties, namely generalizations in the bidimensional and multidimensional cases, applications to the partial differential equations with delay, and existence and uniqueness of solutions. In this paper, we establish some new nonlinear retarded integrodifferential inequalities in two and n independent variables. Some applications are given as illustration.
Keywords: nonlinear inequalities; retarded integrodifferential inequalities; functions of two or n variables; partial integrodifferential equations.
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1. Introduction

The study of integrodifferential inequalities for functions of two or n variables is very significant in assuming the existence and uniqueness of the solutions of Wendroff-type integrodifferential inequalities and equations as well as the boundedness of solutions of nonlinear retarded hyperbolic partial integrodifferential equations for functions of two or n variables [1,2,3,4,5,7,8,9].
Pachpatte [6] presented some nonlinear integrodifferential inequalities of the Wendroff-type for two-variable functions.

Lemma 1.1. *(see Theorem 1 [6]) Let $\phi(x,y)$ and $c(x,y)$ be nonnegative continuous functions defined for $x \geq 0,\ y \geq 0$, and $\phi(x,0) = \phi(0,y) = 0$ for which the inequality*

$$\phi_{xy}(x,y) \leq a(x) + b(y) + \int_0^x \int_0^y c(s,t)(\phi(s,t) + \phi_{xy}(s,t))dsdt,$$

holds for $x \geq 0,\ y \geq 0$, where $a(x),\ b(y) > 0$; $a'(x),\ b'(y) \geq 0$ are continuous functions defined for $x \geq 0,\ y \geq 0$. Then

$$\begin{aligned}\phi_{xy}(x,y) \quad &\leq \quad a(x) + b(y) + \int_0^x \int_0^y c(s,t) \left[\frac{[a(0)+b(t)][a(s)+b(0)]}{[a(0)+b(0)]} \right. \\ &\quad \left. \times \ \exp\left(\int_0^s \int_0^t [1 + c(m,n)]dmdn \right) \right] dsdt.\end{aligned}$$

2. **Main Results**

In this section, some results of nonlinear retarded integrodifferential inequalities in two independent variables are presented. In what follows, $x_0,\ y_0 \in \mathbb{R}_+$, with $x_0 \leq x,\ y_0 \leq y$.

Theorem 2.1. *Let $u(x,y)$, $c(x,y)$, $a(x,y)$, $Du(x,y)$ and $D_iu(x,y)$ be nonnegative continuous functions for all $i = 1,2$ defined for $x,y \in \mathbb{R}_+$, and $\alpha,\ \beta \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ be nondecreasing functions for each variable, with $\alpha(x) \leq x$ on $\mathbb{R}_+$ and $\beta(y) \leq y$ on $\mathbb{R}_+$. Let $c(x,y)$ be a nondecreasing function for each variable $x,y \in \mathbb{R}_+$, and $u(x_0,y) = u(x,y_0) = 0$. If*

$$Du(x,y) \leq c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t)[u(s,t) + Du(s,t)]dsdt, \tag{2.1}$$

for $x,y \in \mathbb{R}_+$, then

$$\begin{aligned} Du(x,y) \quad &\leq \quad c(x,y)\Bigg[1 + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) \\ &\times \exp\left(\int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} (1 + a(\tau,\sigma))d\tau d\sigma\right)dsdt\Bigg]. \end{aligned} \tag{2.2}$$

Remark 2.2. *It is enough to put $\alpha(x_0) = \beta(y_0) = 0,\ \alpha(x) = x,\ \beta(y) = y$, and $c(x,y) = c_1(x) + c_2(y)$ in Theorem 2.1 so as to obtain Theorem 1 in [6].*

Theorem 2.3. *Let $u(x,y), c(x,y)$, $a(x,y)$, α, and β be defined as in Theorem 2.1, and assuming that $b(x,y)$ is nonnegative continuous function. If*

$$\begin{aligned} u(x,y) \quad &\leq \quad c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t)u(s,t)dsdt \\ &+ \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t)\left(\int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} b(\tau,\sigma)u(\tau,\sigma)d\tau d\sigma\right)dsdt, \end{aligned} \tag{2.3}$$

for $x,y \in \mathbb{R}_+$, then

$$\begin{aligned} u(x,y) \quad &\leq \quad c(x,y)\exp\Bigg[\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t)dsdt \\ &+ \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t)\left(\int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} b(\tau,\sigma)d\tau d\sigma\right)dsdt\Bigg]. \end{aligned} \tag{2.4}$$

Remark 2.4. *(i) It is enough to put $\alpha(x_0) = \beta(y_0) = 0,\ \alpha(x) = x,\ \beta(y) = y$, and $c(x,y) = c_1(x) + c_2(y)$ in Theorem 2.3 so as to obtain Theorem 3 in [6].*

(ii) If $b(x,y)=0$, the bound obtained in (2.4) reduces to

$$u(x,y) \le c(x,y)\exp\left[\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)dsdt\right]. \tag{2.5}$$

Corollary 2.5. *Under the same hypotheses of Theorem 2.3, and if*

$$Du(x,y) \le M + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)u(s,t)dsdt + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)Du(s,t)dsdt, \tag{2.6}$$

for $x,y\in\mathbb{R}_+$, where $M>0$ is constant, then we obtain the following results:

$$\begin{aligned}(1)\ Du(x,y) \quad &\le \quad M\Bigg\{1+\left(\int_{\alpha(x_0)}^{\infty}\int_{\beta(y_0)}^{\infty} a(s,t)p(s,t)dsdt\right)\\ &\times\ \exp\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)p(s,t)dsdt\right)\Bigg\}\exp\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)dsdt\right).\end{aligned}$$

$$\begin{aligned}(2)\ u(x,y) \le M\Bigg\{1 \quad &+ \quad \left(\int_{\alpha(x_0)}^{\infty}\int_{\beta(y_0)}^{\infty} a(s,t)p(s,t)dsdt\right)\\ &\times\exp\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)p(s,t)dsdt\right)\Bigg\}p(x,y),\end{aligned}$$

where $p(x,y)$ is defined by: $p(x,y) = \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}\exp\left(\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} b(\tau,\sigma)d\tau d\sigma\right)dsdt.$

3. Retarded Nonlinear Integrodifferential Inequalities in n Independent Variables

This section is devoted to presenting some results of nonlinear retarded integrodifferential inequalities in n independent variables.
In what follows, $D=D_1D_2...D_n$, where $D_i=\frac{\partial}{\partial x_i}$, for $i=1,2,...,n$. For $x=(x_1,x_2,...,x_n)$, $t=(t_1,t_2,...,t_n)$, $x^0=(x_1^0,x_2^0,...,x_n^0)\in\mathbb{R}_+^n$(where $\mathbb{R}_+^n=[0,\infty)$ is a subset of $\mathbb{R}^n, n\ge 1$), we assume: For $x,t\in\mathbb{R}_+^n$, we write $t\le x$ whenever $t_i\le x_i$, $i=1,2,..,n$, and $x\ge x_0\ge 0\in\mathbb{R}_+^n$. For any $X=(X_1,X_2,...,X_n)\in\mathbb{R}_+^n$, we write $x^0\le x\le X$ whenever $x_i^0\le x_i\le X_i, i=1,2,...,n$.

$\tilde{\alpha}(x)=(\alpha_1(x_1),\alpha_2(x_2),...,\alpha_n(x_n))\in\mathbb{R}_+^n$, *and* $\tilde{\beta}(x)=(\beta_1(x_1),\beta_2(x_2),...,\beta_n(x_n))\in\mathbb{R}_+^n$.

We assume $\tilde{\alpha}(x)\le x$ and $\tilde{\beta}(x)\le x$ whenever $\alpha_i(x_i)\le x_i$ and $\beta_i(x_i)\le x_i$ respectively for $i=1,2,...,n$, and

$$\int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} dt = \int_{\alpha_1(x_1^0)}^{\alpha_1(x_1)}\int_{\alpha_2(x_2^0)}^{\alpha_2(x_2)}\cdots\int_{\alpha_n(x_n^0)}^{\alpha_n(x_n)}...dt_n...dt_1,\ and\ \int_{\tilde{\beta}(x^0)}^{\tilde{\beta}(x)} dt = \int_{\beta_1(x_1^0)}^{\beta_1(x_1)}\int_{\beta_2(x_2^0)}^{\beta_2(x_2)}\cdots\int_{\beta_n(x_n^0)}^{\beta_n(x_n)}...dt_n...dt_1.$$

The main results are established in the following theorems.

Theorem 3.1. *Let $u(x), c(x)$ and $a(x)$ be nonnegative continuous functions defined for $x \in \mathbb{R}^n_+$, and $\tilde{\alpha} \in C^1(\mathbb{R}^n_+, \mathbb{R}^n_+)$ be a nondecreasing function for each variable, with $\tilde{\alpha}(x) \leq x$ on $\mathbb{R}^n_+$. We consider that $c(x)$ is nondecreasing for each variable $x \in \mathbb{R}^n_+$. If*

$$u(x) \leq c(x) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)u(t)dt, \tag{3.1}$$

for $x \in \mathbb{R}^n_+$, then

$$u(x) \leq c(x) \exp\left(\int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)dt \right). \tag{3.2}$$

Remark 3.2. *When $n = 2$, $x \in \mathbb{R}^2_+$, $(x_1^0, x_2^0) = (0,0)$, $\alpha_1(x_1) = x$, $\alpha_2(x_2) = y$, and $c(x) = c_1(x) + c_2(y)$ then Theorem 3.1 reduces to Lemma 1 in [6].*

Theorem 3.3. *Under the same hypotheses of Theorem 2.3, and assuming that $f(x,y)$ is nonnegative continuous and nondecreasing function, let $K(u(x,y))$ be a real-valued, positive, continuous, strictly nondecreasing, sub-additive, and sub-multiplicative function for $u(x,y) \geq 0$, and $H(u(x,y))$ be a real-valued, continuous, positive, and nondecreasing function defined for $x, y \in \mathbb{R}_+$. If*

$$Du(x) \leq c(x) + f(x)H\left(\int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(u(t))dt \right) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} b(t)Du(t)dt, \tag{3.3}$$

for $x \in \mathbb{R}^n_+$, then

$$Du(x) \leq \left\{ c(x) + f(x)H\left(G^{-1}\left[G(\xi) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(f(t)p(t))dt \right] \right) \right\} \exp\left(\int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} b(t)dt \right), \tag{3.4}$$

for all $x \in \mathbb{R}^n_+$, where

$$p(x) = \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} \exp\left(\int_{\tilde{\alpha}(x^0)}^{s} b(\tau)d\tau \right) dt. \tag{3.5}$$

$$\xi = \int_{\tilde{\alpha}(x^0)}^{\infty} a(t)K(c(t)p(t))dt, \ G(r) = \int_{r_0}^{r} \frac{ds}{K(H(s))}, \ r \geq r_0 \geq 0,$$

where G^{-1} is the inverse function of G, and $G(\xi) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(f(t)p(t))dsdt \in$ $dom(G^{-1})$ for $x \in \mathbb{R}^n_+$.

Remark 3.4. *(i) Based on the inequalitie* (3.3) *and the equation* (3.5), *we can obtain the following result:*

$$u(x) \quad \leq \quad \left\{ c(x) + f(x)H\left(G^{-1}\left[G(\xi) + \int\limits_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(f(t)p(t))dt \right] \right) \right\} p(x). \quad (3.6)$$

(ii) It is enough to put $n = 2$, $x \in \mathbb{R}_+^2$, $(x_1^0, x_2^0) = (0,0)$, $\alpha_1(x_1) = x$, $\alpha_2(x_2) = y$, $c(x) = c_1(x) + c_2(y)$, $f(x) = 1$, $H(x) = K(x) = 1$, *and* $a(x) = b(x)$ *so as Theorem* 3.3 *reduces to Theorem 1 in* [6].

4. APPLICATIONS

This section suggests some applications of our results in order to study the boundedness and continuity of solutions of some nonlinear partial integrodifferential equations with delay.

APPLICATION 1: Suppose the following equation for functions of two independent variables

$$Du(x,y) = f(x,y) + \int\limits_{\alpha(x_0)}^{\alpha(x)} \int\limits_{\beta(y_0)}^{\beta(y)} h(x,y,s,t,u(s,t),Du(s,t))dsdt, \quad (4.1)$$

with the boundary conditions $u(x_0, y) = u(x, y_0) = 0$, for $x, y \in \mathbb{R}_+$, where $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and $h : \mathbb{R}_+^2 \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions so that

$$|f(x,y)| \leq M,$$

and

$$|h(x,y,s,t,u(s,t),Du(s,t))| \leq a(s,t)|u(s,t)| + b(s,t)|Du(s,t)|,$$

for $x, y \in \mathbb{R}_+$, where $M > 0$ is constant and $a(x,y)$ and $b(x,y)$ are nonnegative continuous functions defined for $x, y \in \mathbb{R}_+$. If $u(x,y)$ is any solution of Problem (4.1), then

$$|Du(x,y)| \leq M + \int\limits_{\alpha(x_0)}^{\alpha(x)} \int\limits_{\beta(y_0)}^{\beta(y)} a(s,t)|u(s,t)|dsdt + \int\limits_{\alpha(x_0)}^{\alpha(x)} \int\limits_{\beta(y_0)}^{\beta(y)} b(s,t)|Du(s,t)|dsdt.$$

Now, it is possible to obtain the bound on the solution $u(x,y)$ of (4.1) by applying Corollary 2.5 (inequality 2)

$$|u(x,y)| \leq M \left\{ 1 + \left(\int\limits_{\alpha(x_0)}^{\infty} \int\limits_{\beta(y_0)}^{\infty} a(s,t)p(s,t)dsdt \right) \exp\left(\int\limits_{\alpha(x_0)}^{\alpha(x)} \int\limits_{\beta(y_0)}^{\beta(y)} a(s,t)p(s,t)dsdt \right) \right\} p(x,y),$$

for all $x, y \in \mathbb{R}_+$, where $p(x,y)$ is defined in Corollary 2.5.

APPLICATION 2: Suppose the following equation for functions of n independent variables

$$Du(x) = q(x) + f(x)H\left(\int\limits_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} Q(x,t,u(t),K(u(t)))dt \right) + \int\limits_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} W(x,t,u(t),Du(t))dt, \quad (4.2)$$

with the conditions $u(x_1^0, x_2, ..., x_n) = 0$, $u(x_1, ..., x_{i-1}, x_i^0, x_{i+1}, ..., x_n) = 0$ for any $i = 2, ..., n$, where f, K, and H are defined in Theorem 3.3. $q : \mathbb{R}_+^n \to \mathbb{R}$ and $Q, W : \mathbb{R}_+^n \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions so that

$$|q(x)| \leq M,$$

and

$$|Q(x,t,u(t),K(u(t)))| \leq a(t)K(|u(t)|), \ \textit{and} \ |W(x,t,u(t),Du(t))| \leq b(t)|Du(t)|,$$

for $x \in \mathbb{R}_+^n$, where $M > 0$ is constant, $a(x)$ and $b(x)$ are nonnegative continuous functions defined for $x \in \mathbb{R}_+^n$. If $u(x)$ is any solution of Problem (4.2), then

$$|Du(x)| \leq M + f(x)H\left(\int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(|u(t)|)dt\right) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} b(t)|Du(t)|dt.$$

Now, it is possible to obtain the bound on the solution $u(x)$ of (4.2) by applying Theorem 3.3 and Remark 3.4 (inequality 3.6) with $c(x) = M$

$$|u(x)| \quad \leq \quad \left\{M + f(x)H\left(G^{-1}\left[G(\xi) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(f(t)p(t))dt\right]\right)\right\}p(x),$$

for all $x \in \mathbb{R}_+^n$, where $p(x), G$ and ξ are defined in Theorem 3.3.

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APPROXIMATE SOLUTIONS OF WEAKLY NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this work, we study the most useful approximation methods for proving solutions to analytic approximation for solving a weak second-order nonlinear differential equation in a power series with small parameters. We prove the second-order periodic approximation solution and also the best third-order approximation of the weak nonlinear differential equation.

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Hybrid Level set driven by signed pressure force for welding defect Segmentation

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Abstract – *This paper deals with the segmentation of welding defect from radiographic image which is the major interest for the diagnosis and monitoring of defects in the industry field. Our challenge is the implementation of a method that takes ownership of local segmentation geodesic active contours and welding defect contour from radiographic images used for non-destructive control and analysis. The objective of this work is to investigate the robustness of this model on different radiographic images. As a result of this study, we found that the proposed method is also effective and robust.*

Keywords: *Image segmentation, Vocal Tract, Hybrid Level set.*

1. INTRODUCTION

Differences methods of non-destructive control are used to improve the diagnosis of welding defects in the field of industry; one of the most used methods is the X-ray. In this context of the radiograph, extraction and analysis of the parameters (such as the area and perimeter of welding defect ...) is a strong need and provide valuable information for diagnosis. However, the segmentation of defect which is a typically task performed manually by an expert has a subjective interpretation. Tools for automatic segmentation are required to perform this task quickly, objectively and repetitively. It is in this context that our team takes care of the segmentation of radiographic images. So a robust segmentation involves a robust estimate of a model that can detect the cause of speech disorder problem, and help the patient by using an application of visual speech which requires online image segmentation. To reach this purpose we opted for a proposed model which is hybridization between two types of level set segmentation.

3. Segmentation using Level set

The Level Set method was developed by the mathematicians Stanley Osher and James Sethian [1-7], it is an example of a geometric active contour model, such model begins with a contour in the image plane defining an initial segmentation, and then we evolve this contour according to some evolution equation. There are various models based on this idea [8-10], which are slightly different from each other. Two typical models are the following: geodesic active contours model (GAC) and Chan-Vese model (CV).

3.1 The geodesic active contours (GAC) model

This model is based on the contour [11]; its implementation follows the equation form

$$\frac{\partial\phi}{\partial t} = g(|\nabla I|)|\nabla\phi|\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) + \nabla g\nabla\phi \qquad (1)$$

Where $g = \exp(-\alpha.|\nabla(I_0(x) * G)|), \alpha > 0$, $I_0(x) * G$ is the convolution of the original image $I_0(x)$ with a Gaussian function G. For the level set equations, a re-initialization phase is necessary. The purpose of re-initialization is to keep the evolving level set function close to a signed distance function during the evolution. It is a numerical remedy for maintaining stable curve evolution. The re-initialization step is to solve the following evolution equation:

$$\phi_t + \mathrm{sgn}(\phi_0)(|\nabla\phi| - 1) = 0 \qquad (2)$$

Here $\phi(x, t = 0) = \phi_0(x)$

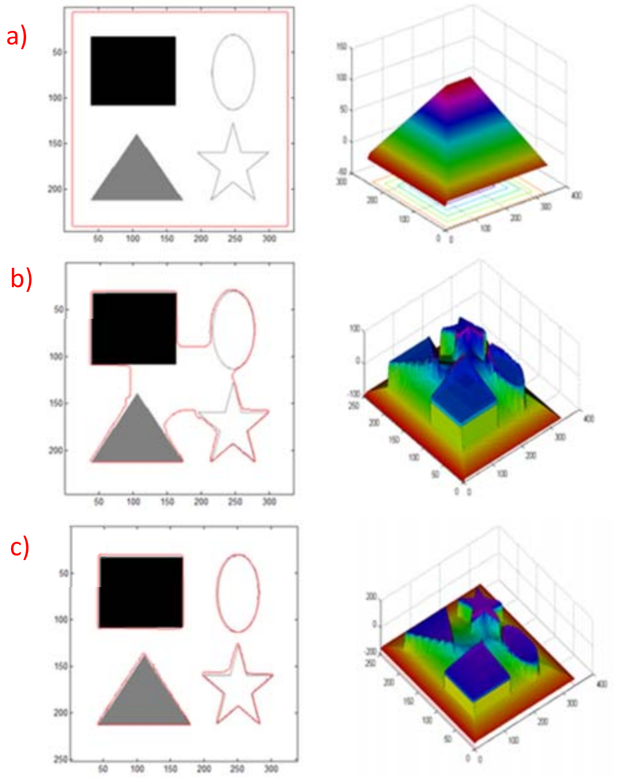


Fig. 2: Synthetic image which has geometrical shapes and without noise: a) initial contour, b) after 175 iterations, c) after 245 iterations.

Fig.2 deals with the segmentation of a synthetic image which has geometrical shapes without noise. The initial level set function is $\Phi_0(x, y)$ representing a rectangle. The curve is reinitialized every 5 steps and $v = -1$. The curve in 3D (second column in figure3) represents the distance function. From the practical point of view, the implementation of the level set equation can be quite complicated, expensive, because of the operation of a re-initialization and also because of numerical schemes used to ensure stability of the solution.

3.2. Chan-Vese (C-V) model

In the work [12], the authors propose a new model for image segmentation based on Mumford-Shah functional and level sets. This model doesn't depend on the gradient of the image as stopping term. Therefore it can be used in images with ambiguous boundaries. The model can be defined as the following minimization problem

$$F(\phi) = \mu \mathrm{Length}(\phi) + v \mathrm{Area}(\mathrm{inside}(\phi)) + \lambda_1 \int_{\mathrm{inside}(c)} |I(x) - c_1|^2 dx + \lambda_2 \int_{\mathrm{outside}(c)} |I(x) - c_2|^2 dx \tag{3}$$

$$c_1(\phi) = \frac{\int_\Omega I(x) H(\phi) dx}{\int_\Omega H(\phi) dx} \tag{4}$$

and

$$c_2(\phi) = \frac{\int_\Omega I(x)(1 - H(\phi)) dx}{\int_\Omega (1 - H(\phi)) dx} \tag{5}$$

Where

$$\mathrm{Length}(\phi) = \int_\Omega \delta(\phi) |\nabla \phi| d\Omega$$

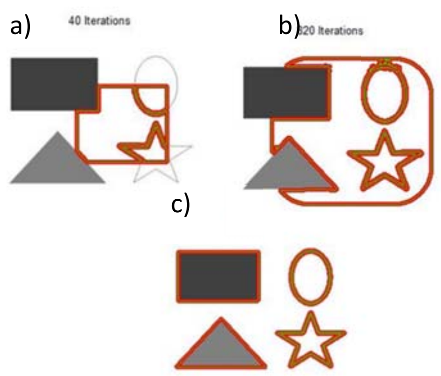
and

$$\mathrm{Area}(\mathrm{inside}(\phi)) = \int_\Omega H(\phi) d\Omega$$

To minimize this functional with respect to φ, parameterize the descent direction by an artificial time $t \geq 0$, the equation in φ is:

$$\frac{\partial \phi}{\partial t} = \delta(\phi)\left[\mu \nabla \left(\frac{\nabla \phi}{|\nabla \phi|}\right) - v - \lambda_1 (I - c_1)^2 + \lambda_2 (I - c_2)^2\right] \tag{6}$$

Fig. 3: Synthetic image segmentation using C-V model: a)



after 40 iterations, b) after 820 iterations, c) after 1420 iterations.

Fig. 3 shows the evolution procedure. This model is much robust; the performance of this model is excellent. When we applied this model to real radiographic images, the situation is much more complicated, because the real radiographic images always contain sub-structures outside weld defect, which will cause a false segmentation.

3.3. The adopted model

This model [15] combines the advantages of both CAG and CV models for its construction. Indeed, it uses statistical information by the average (c1,c2) calculated with the equation (4) and (5) and operates the formulation of the equation (1) to achieve a model by replacing the function g in equation (1) by another function called pressure force signed (FPS). This latter is a force generated by the difference of the average intensities inside and outside of the object (c1 and c2 in equation (4) and (5)). This force applies pressure prompting the contour to shrink or grow depending on its sign. The formula of the SPF is given by the following equation:

$$\mathrm{spf}(I(x)) = \frac{I(x) - \frac{c_1 + c_2}{2}}{\max(|I(x) - \frac{c_1 + c_2}{2}|)} \qquad (7)$$

The representation of the appearance is made by the average of the intensities of the different regions of the image. In the formula (7), the numerator gives the sign of the SPF and manages the direction of change of the contour. The denominator is introduced to give a value to the SPF ranging from [-1, 1]. Once the SPF of the equation (7) is calculated, it will be replaced in equation (1) instead of the function g. This allows us to achieve the following formulation [11]

$$\frac{\partial \phi}{\partial t} = \mathrm{spf}(I(x))|\nabla\phi|(\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) + \alpha) + \nabla \mathrm{spf}(I(x))\nabla\phi \qquad (8)$$

Where α a positive constant, its role is increasing the speed up of convergence. The adopted model has several advantages compared to CAG and CV models, which motivated us to adopt it for our application. In addition, a small comparison between the three models justifies further our choice. Thus, we can summarize these comparisons in these few points:

Contrary to the model CAG, the adopted model uses statistical information to change and stop the curve on the edges of objects. It is, therefore, more robust and resistant to noise. This model can be successfully applied to images with low edges.

Compared to the C-V model, this model allows the extraction of objects with internal inhomogeneous intensity.

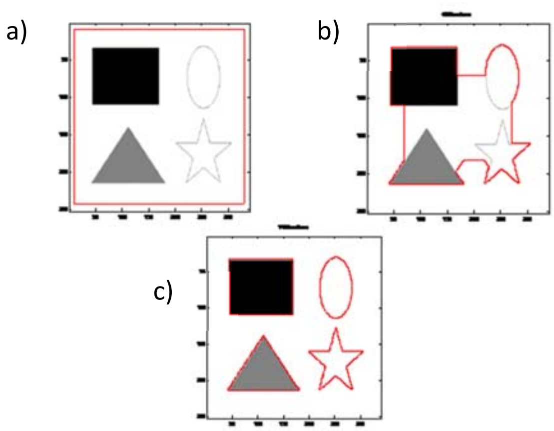


Fig. 4: Synthetic image segmentation using the proposed model: a) initial contour, b) after 60 iterations, c) after 140 iterations.

3.4. Comparison

From the results of the implementation (figures 2, 3 and 4) of the three models (CV, CAG, and SPF), we can make a quantitative comparison in terms of execution time:

Table 1. Comparison between CV, CAG and FSP models

	C.V	**C.A.G**	**F.P.S**
CPU time (s)	14.593	26.654	1.178
Iterations	1420	245	140
Segmentation type	Global	Local	Offers the choice between a global and local.

Performances between models are to the advantage of the FSP model; this performance is due in particular to the fact that the algorithm does not require re-initialization of the level set function which can be costly. The implementation method offers the choice between global and local segmentation. For local segmentation, the algorithm can be initialized by a signed constant function (it is not required to calculate the conventional signed distance function that can take some time). Therefore, only φ close the contour will evolve. In addition, through its global character, where is the initial contour has no importance for global detection of objects.

4. Experiments

For our experiments we have used a disk PC (DELL Optilex 780 Processor intel Core 2 Duo (E7500) 2.93 GHz, a GB RAM 3581 Mo. ATI Radeon 5400 memory bandwidth 12.8 GB/sec. To demonstrate the advantages and the limitations of the adopted approach, we have conducted tests on different radiographic images from multiple welding defect for different position

4.1. Implementation

Similarly to the second term in equation (1) the term $\nabla \mathrm{spf}(I(x))\nabla\phi$ is used to increase the capture of edges. Since the proposed model used statistical information of region this term could be removed because region-based models have a large capture of edges and high capacity of anti-edge leakage. Inspired by [13], and because the evolution of the function level-set with the Laplacian is equivalent to filtering with the Gaussian kernel, the proposed model uses a Gaussian kernel filter in place of the curvature term $\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)$ to regularize the contour, by making a convolution product after each iteration of the convergence process. However, the formulation given in (8) might be reduced to the following one:

$$\frac{\partial\phi}{\partial t} = \alpha\,\mathrm{spf}(I(x))\,|\nabla\phi| \qquad (9)$$

4.2. Algorithm:

The algorithm needs as parameter: the original image, the initial curve from which we compute the binary level set, Δt, α, ρ, σ and number of iterations N:

While $n \leq N$:

1) Compute the average intensities C_1 and C_2.
2) Compute the SPF value.
3) Update the level set using equation (9)
4) Keep the level set as binary function $\phi = \rho$ if $\phi > 0$ otherwise $\phi = -\rho$.
5) Regularize the level set with a Gaussian filter $\phi * G_\sigma$.
6) n=n+1

End while.

4.3. Results

In this paper, we have proposed a Hybrid Level set driven by signed pressure algorithm to extract the defects boundaries. We applied our solution to some of radiographic images weld that include defaults that could happen during the welding operation.Figures4, 5, 6 and 7 representa result the proposed method. The figures 4 and 6showthe results

a) b)

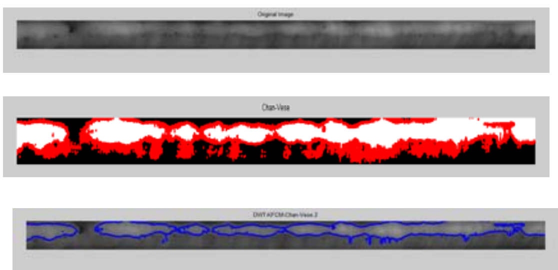


Fig. 5: Evolution contour of the defects boundaries

a)

b)

c)

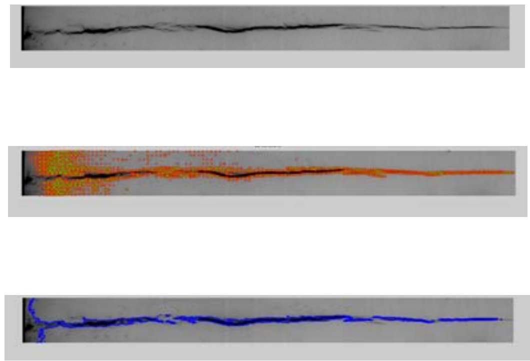


Fig. 6: Welding defect segmentation: a) Initial image, b) segmentation with CV model, c) segmentation with the proposed model.

This result allows us to see the difference between models, only the proposed model can detect the region of interesting (weld defect) with robustness and ignore other regions of the image. From these results we can see that the adopted model in this work does not require a preprocessing step.

5. Conclusion

In this manuscript, we presented a segmentation method that allows robust weld radiographic images segmentation; we described the hybridization interesting properties of the level set and kernel fuzzy c-means. The simulation covers weld radiographic images used in Non Destructive Testing (NDT) to delineate the weld defects. The results were very satisfactory. All objects were surrounded.

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Existence and decay of the energy for a quasilinear hyperbolic equation with the $p-$Laplacian operator.

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May 7, 2022

Abstract: In this work, we study the value problem related to the quasilinear hyperbolic equation with the $p-$Laplacian operator. Firstly, we get the local existence theorem. Secondly, we prove the global existance of solutions for a quasilinear hyperbolic equation with the $p-$laplacian. Finally, we study the decay of the energy by using Nakao's inequality.

Keywords: energy function; $p-$Laplacian operator; decay.

AMS Subject Classification: 81U90, 35B38.

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Haar wavelet quadrature method for a numerical solution of some class of Caputo-Fabrizio derivative Fractional problem

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Abstract

In this paper, we investigate the Haar wavelet quadrature method to solve a class of fractional differential equations with Caputo-Fabrizio fractional derivative. This method is based on Haar wavelet integration quadrature rule to convert the considered problem to an algebraic system of equations. The main advantage of our proposed method is its accuracy compared to classical quadrature methods. Finally, illustative examples are provided to demonstrate the efficiency of our method.

Keywords: Haar wavelet, Caputo-Fabrizio derivative, Quadrature rule.

MSC: 65T60, 26A33, 34A08.

1 Introduction

In recent years, fractional calculus has attracted great interest from many scientific researchers. Various definitions of derivative and fractional integral have been proposed such as: the Riemann-Liouville fractional derivative [1], the Caputo fractional derivative [2], the Caputo-Fabrizio fractional derivative [3] and others. Currently, the Caputo-Fabrizio fractional derivative is the most popular. Continuation to that, in this paper we investigate the Haar wavelet quadrature method to solve the following problem [4]:

$$\begin{cases} {}^{CF}D_0^{(\rho)}u(t) + q(t)u(t) = f(t),\ 0 \leq t \leq 1, \\ u(0) = a,\ u(1) = b, \end{cases} \tag{1}$$

where $1 < \rho < 2$ is a real number, q and f are continuous functions, ${}^{CF}D_0^{(\rho)}$ represents the Caputo-Fabrizio derivative of order ρ.

In fact, M. Moumen et al. [4] have proven the existence and uniqueness of the solution of this problem and they applied the trapezoidal rule to evaluate this solution numerically. However, as it is known, the trapezoidal method has polynomial convergence. To overcome this problem and to enhance the convergence rate, the Haar wavelet quadrature method is suitable and has exponential convergence.

2 Haar wavelet quadrature rule

Generally, the Haar wavelets family on $[0,1)$ [1] is defined by the scaling and wavelets functions as follows:

The scaling function:

$$h_1(t) = \begin{cases} 1,\ if\ t \in [0,1), \\ 0,\ otherwise. \end{cases} \tag{2}$$

The wavelets functions: $\forall i \geq 2$,

$$h_i(t) = \begin{cases} 1,\ if\ t \in [\zeta_1, \zeta_2), \\ -1,\ if\ t \in [\zeta_2, \zeta_3), \\ 0,\ otherwise, \end{cases} \tag{3}$$

where $\zeta_1 = \frac{k}{m}$, $\zeta_2 = \frac{k+0.5}{m}$ and $\zeta_3 = \frac{k+1}{m}$, $m = 2^j$, $j = 0, 1, ..., J$, J is the resolution level of wavelet approximation, $k = 0, ..., m-1$ represents the translation parameter. The relation between i, m and k is given by $i = m + k + 1$. In case of the minimal values of $m = 1$, $k = 0$, then $i = 2$. The maximum value of i is $i = 2M$, $M = 2^J$.

Any function $u \in L^2([0,1))$ can be expanded as:

$$u(t) = \sum_{i=1}^{+\infty} c_i h_i(t), \tag{4}$$

where $c_i = \int_0^1 u(t) h_i(t)\, dt$.

Theorem 1 *Let f be a function of $L^2([a,b))$, then the approximate value of the integral $\int_a^b f(x)dx$ is given by:*

$$\int_a^b f(x)dx \simeq \frac{b-a}{2M} \sum_{k=1}^{2M} f(a + \frac{(b-a)(k-0.5)}{2M}). \tag{5}$$

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Abstract

In this paper we propose a primal-dual interior point method for second order cone optimization based on kernel function focused on the technique of central path, by using the latter and the symmetrizing scheme,called NT scaling scheme ,we obtain a new search direction, furthermore the introduction of kernel function does not only measure the distance between the iterate and the central path but also ameliorate the computationalcomplexity.Finally, we present few numerical results to demonstrate the efficiency of the algorithm.
Keywords: Second order cone optimization,kernel function,Interior point method,complexity bound

Identification of the source term in coupled system using an optimal control framework

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Abstract: In this paper, we study an inverse problem consists of determine the unknown pa- rameters in a coupled dynamic population using an optimal control framework. the main idea is to transform a study of the inverse problem into an optimal control problem. The source term will be characterized by a coupled optimality system.

Keywords:
Inverse problem, source term, optimal control.

Functional data analysis: Semi parametric regression under dependent truncation

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1. INTRODUCTION:

In survival analysis, it is interesting to know how survival time (the variable of interest) is influenced by a number of factors (covariates or variables explanatory). Many models can be used to describe this relationship as linear regression, parabolic regression, sinusoid regression, $\cdots$, etc.

Practically, the data to be processed may are not complete, therefore conventional methods do not apply correctly. This paper focuses on pairs (T, Y) of truncated variables satisfying $Y > T$. The variables are said to be left-truncated if Y is the variable of interest and right-truncated otherwise.

The assumption that the studied data are always independent is little realistic in practice, which is why many mathematicians have concentrated their studies on another type of data which are the dependent ones and sees in recent years, a great concentration on a method consists of modelling dependency between incomplete random variable pairs using a parametric family of copulas.

On the other hand, thanks to the progress made by the computer tool at the level of storage capacity which makes it possible to record increasingly voluminous data, the branch of statistics dedicated to the analysis of functional data experienced a real growth both in terms of theoretical developments and methodological as well as the diversification of fields of application. This makes treatment of the prediction problem for such data very necessary.

The aim of this work is to create a semi parametric estimator of the regression function when the response variable is subject to left-truncation and based on the assumption of quasi-independence between the time of interest and the truncation time where the dependency is modelled by a parametric family of copulas. The covariate is assumed to be of a functional type

2. Model:

Let (Y_i, T_i) for $i = 1, \dots, N$, be a sample of the pair (Y, T). The unknown distribution functions of Y and T are denoted respectively by F and G, while the survival functions are noted by $\bar{F}$ and $\bar{G}$. Since the lifetime Y_i and the truncation r.v. T_i are both observable only when $Y_i \geq T_i$. We denote $(Y_i, T_i), i = 1, 2, \dots, n (n \leq N)$ the actual observed sample. On the other side, let X a covariate which takes its values in a semi metric space $\mathcal{F}$ endowed with a semi metric d.

Using a one-parameter family of Archimedean copulas with generating functions $\phi_\alpha, \alpha \in \mathbb{R}$ such as Clayton's (1978) family or Frank's family (Genest, 1987),gives

$$\pi(y,t) = \mathbb{P}(T \leq t, Y > y / Y > T) = \frac{1}{c}\phi_\alpha^{-1}[\phi_\alpha(\bar{F}(y)) + \phi_\alpha(G(t))], \; y \geq t,$$

where ϕ^{-1} is a Laplace transform and c is a normalising constant. Basing on the idea of Derrar et al. (2015), a proposed estimator of the regression function under left truncation model defined by

$$m^*(x) = E[Y/X = x, Y > T]$$

is given by

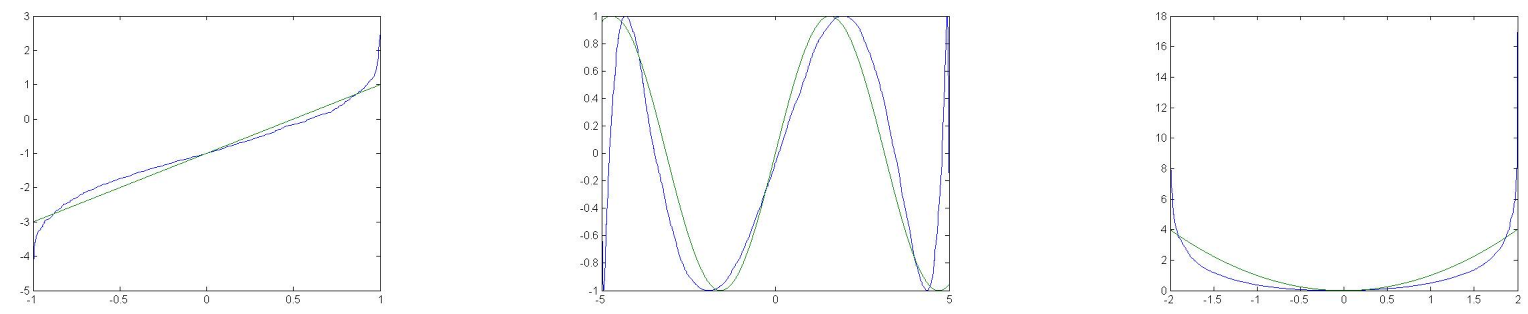
$$m_n(x) = \frac{\sum_{i=1}^n Y_i \hat{G}^{-1}(Y_i) K(h^{-1} d(x, X_i))}{\sum_{i=1}^n \hat{G}^{-1}(Y_i) K(h^{-1} d(x, X_i))}$$

where

$$\hat{G}(x) = \phi_{\hat{\alpha}}^{-1}\left(-\sum_{t_i > x}\left[\phi_{\hat{\alpha}}\{\hat{c}\hat{\pi}(t_i)\} - \phi_{\hat{\alpha}}\{\hat{c}\hat{\pi}(t_i) - \hat{c}\frac{1}{n}\}\right]\right)$$

3. Simulation:

Simulation studies has been carried out for different types of regression (linear, parabolic and sinusoid)



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Well posedness and exponential stability of a thermoelastic Shear beam model

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Abstract. In this paper we consider a thermoelastic shear beam model with thermal dissipation. We prove a well posedness result by the use of Faedo–Galerkin method and an exponential stability by the multiplier method.

Key words: Shear beam model, well–posedness, exponential stability, Faedo–Galerkin method.

1 Introduction

In this work we consider the system

$$\begin{cases} \rho\varphi_{tt} - \kappa(\varphi_x + \psi)_x + \mu\theta_x = 0, & \text{in } (0, L) \times (0, \infty), \\ -b\psi_{xx} + \kappa(\varphi_x + \psi) = 0, & \text{in } (0, L) \times (0, \infty), \\ c\theta_t - \delta\theta_{xx} + \mu\varphi_{xt} = 0, & \text{in } (0, L) \times (0, \infty), \end{cases} \tag{1}$$

where φ is the transverse displacement ,ψ is the rotation of the neutral axis and θ is the difference of the temperature.

In addition, we endow system (1) with the following initial and boundary conditions

$$\varphi(x,0)=\varphi_0(x),\varphi_t(x,0)=\varphi_1(x),\psi(x,0)=\psi_0(x),\theta(x,0)=\theta_0(x),\ x\in[0,L] \tag{2}$$

$$\varphi(0,t)=\varphi(L,t)=\psi(0,t)=\psi(L,t)=\theta_x(0,t)=\theta_x(L,t)=0,\ t\in(0,+\infty)\,. \tag{3}$$

The energy of the system is defined by

$$E(t):=\frac{\rho}{2}\int_0^L|\varphi_t|^2dx+\frac{b}{2}\int_0^L|\psi_x|^2dx+\frac{c}{2}\int_0^L|\theta|^2dx+\frac{\kappa}{2}\int_0^L|\varphi_x+\psi|^2dx. \tag{4}$$

We have

$$\frac{d}{dt}E(t)=-\delta\int_0^L|\theta_x|^2dx\le 0. \tag{5}$$

2 Well-posdness

We introduce the phase space:

$$\mathcal{H}:=H_0^1(0,L)\times\ L^2(0,L)\times\ H_0^1(0,L)\times H_*^1(0,L) \tag{6}$$

where

$$H_*^1(0,L):=\left\{\phi\in H^1(0,L),\ \int_0^L\phi(x)dx=0\right\}. \tag{7}$$

Definition 2.1. *Let* $(\varphi_0,\varphi_1,\psi_0,\theta_0)\in\mathcal{H}$. *A weak solution of* (1)–(3) *is a function* $U=(\varphi,\psi,\theta)\in\mathcal{H}$ *which satisfies:*

$$\begin{cases}\rho\dfrac{d}{dt}\displaystyle\int_0^L\varphi_t u dx+\kappa\int_0^L(\varphi_x+\psi)\,u_x dx+\mu\int_0^L\theta_x u dx=0, & \forall u\in H_0^1(0,L),\\ b\displaystyle\int_0^L\psi_x w_x dx+\kappa\int_0^L(\varphi_x+\psi)\,w dx=0, & \forall w\in H_0^1(0,L),\\ c\dfrac{d}{dt}\displaystyle\int_0^L\theta\nu dx+\delta\int_0^L\theta_x\nu_x dx+\mu\int_0^L\varphi_t\nu_x dx=0, & \forall\nu\in H_*^1(0,L)\end{cases} \tag{8}$$

for a.e. $t\in(0,T)$ *and*

$$\varphi(0)=\varphi_0,\ \varphi_t(0)=\varphi_1,\ \psi(0)=\psi_0,\ \theta(0)=\theta_0.$$

The well-posedness result reads as follows:

Theorem 2.1. *For any initial data $(\varphi_0, \varphi_1, \psi_0, \theta_0) \in \mathcal{H}$ and any $T > 0$, the problem* (1)–(3) *has a weak solution (φ, ψ, θ) such that*

$$\begin{gathered}\varphi \in L^\infty\left(0, T; H_0^1(0, L)\right), \ \varphi_t \in L^\infty\left(0, T; L^2(0, L)\right), \\ \psi \in L^\infty\left(0, T; H_0^1(0, L)\right), \ \theta \in L^\infty\left(0, T; H_*^1(0, L)\right) \cap L^2(0, T; H_*^1(0, L)).\end{gathered} \tag{9}$$

Proof. The proof to tis theorem will be given by the use of Feado–Galerkin method through four steps. □

3 Exponential decay

Theorem 3.1. *There exists a positive constant $\gamma > 0$, such that the energy $E(t)$ defined by* (4) *satisfies along the solution (φ, ψ, θ) the estimate*

$$E(t) \leq E(0)e^{-\gamma t}, \ \forall t \geq 0. \tag{10}$$

Proof. The proof of Theorem 3.1 will be given by the use of the multiplier method, in fact we construct a Lyapunov functional $\mathcal{L}(t; U(t; x))$ which is positive definite, equivalent to the energy $E(t)$ and $\mathcal{L}_t(t; U(t; x))$ is negative definite.

We write $\mathcal{L}$ as a combination of functionals $\{E(t), \mathcal{F}_1(t); \mathcal{F}_2(t)\}$ where each derivative of $E(t), \mathcal{F}_1(t); \mathcal{F}_2(t)$ satisfies an estimate of a term from the energy with negative sign. We obtain, $\mathcal{L}'(t) \leq -\gamma\mathcal{L}(t)$. Thus, an integration we respect to t and the equivalence between $\mathcal{L}$ and $E(t)$ lead to the exponential stability of $E(t)$. □

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Existence of solutions for second-order integral boundary value problems at resonance with kernel less or equal to n

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Abstract

This paper studies the solvability of nonlinear second-order differential equations with integral boundary conditions at resonance. An interesting point is the effect of the state variable $u \in \mathbb{R}^n$, and the coefficient matrices of the matrices B and C on the solvability of the problem, using the so-called Mawhin coincidence theory. In addition, an example is included to demonstrate the main results.

Key words: Nonlinear second-order differential equations, Integral boundary conditions at resonance, Mawhin coincidence theory.

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Fractional Hilbert space operator: influence of fractional derivatives on the One-dimensional harmonic oscillator

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In this paper we are studying the one-dimensional harmonic oscillator in the base case, Then we use fractional derivatives to formulate the general expression for the one-dimensional harmonic oscillator and we want to compare the two systems in the presence and absence of fractional derivatives , we are using Hermite function in this approximation , then we are studying the fractional Hilbert space deformation and influence of this approximation at energy stability of Schrodinger equation . Finally, We want to estimate the effect of fractional derivatives on the sum of quantum effects in Hilbert space, which accurately describes all possible interactions in quantum mechanics and modern physics.

I. ONE-DIMENSIONAL HARMONIC OSCILLATOR IN QUANTUM MECHANICS

A. One-dimensional harmonic oscillator

The harmonic oscillator is an extremely important physics problem. is given the soloution of many problem . many potentials look like a harmonic oscillator near their minimum. This is the first non-constant potential for which we will solve the Schrödinger Equation.

The harmonic oscillator[1] Hamiltonian is given by.

$$\left(\frac{P_x^2}{2m}+\frac{1}{2}mw^2x^2\right)\psi=E\psi \tag{1}$$

The solution of the Schrodinger equation for the first four energy states gives the normalized wave functions at left. These functions are plotted at left in the above illustration. The probability of finding the oscillator at any given value of x is the square of the wave function, and those squares are shown at right above. Note that the wave functions for higher n have more within the potential well. This corresponds to a shorter wavelength and therefore by the de Broglie relationship they may be seen to have a higher momentum and therefore higher energy.

$$\left(-\frac{\partial^2}{\partial\beta^2}+\frac{\beta^2}{4}\right)\phi(\beta)=\varepsilon_n\phi(\beta) \tag{2}$$

When the Schrodinger equation for the harmonic oscillator is solved by a series method, the solutions contain this set of polynomials, named the Hermite polynomials.the wave functions for the quantum harmonic oscillator contain the Gaussian form which allows them to satisfy the necessary boundary conditions at infinity. In the wave function associated with a given value of the quantum number n, the Gaussian is multiplied by a polynomial of order n (the Hermite polynomials above) and the constants necessary to normalize the wave functions.

$$\psi_n(x)=C_ne^{-\frac{mw}{2\hbar}x^2}H_n\left(\sqrt{\frac{mw}{\hbar}}x\right) \tag{3}$$

$$C_n^2=\frac{\sqrt{\lambda_n}}{2^n\sqrt{\pi}n!} \tag{4}$$

The most probable value of position for the lower states is very different from the classical harmonic oscillator where it spends more time near the end of its motion. But as the quantum number increases, the probability distribution becomes more like that of the classical oscillator , this tendency to approach the classical behavior for high quantum numbers[2] is called the correspondence principle.

$$\frac{\partial^2}{\partial\beta^2}\phi(\beta)+\left\{\varepsilon_n-\frac{\beta^2}{4}\right\}\phi(\beta)=0 \tag{5}$$

This form of the frequency is the same as that for the classical simple harmonic oscillator. The most surprising difference for the quantum case is the so-called "zero-point vibration" of the $(n=0)$ ground state. This implies that molecules are not completely at rest, even at absolute zero temperature

$$\varepsilon_n=\frac{E_n}{\hbar w},\ \beta=\sqrt{\frac{2wm}{\hbar}}x\,\text{and}\,\psi(x)=\phi(\beta), \tag{6}$$

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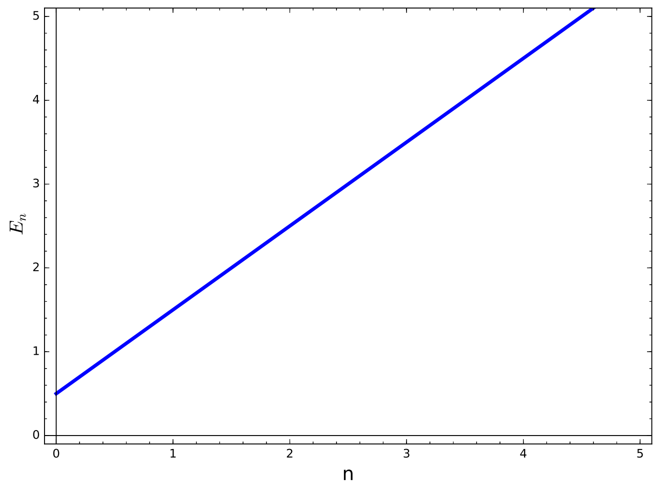


(a) Energy of the one-dimensional harmonic oscillator

After resolving this equation We get the energy spectrum , with n is quantum number .

$$E = \hbar w\left(n + \frac{1}{2}\right) \tag{7}$$

The quantum harmonic oscillator has implications far beyond the simple diatomic molecule. It is the foundation for the understanding of complex modes of vibration in larger molecules, the motion of atoms in a solid lattice, the theory of heat capacity, etc. In real systems, energy spacings are equal only for the lowest levels where the potential is a good approximation of the type harmonic potential. The anharmonic terms which appear in the potential for a diatomic molecule are useful for mapping the detailed potential of such systems.

II. INTRODUCTION TO FRACTIONAL CALCULUS

Fractional calculus is a branch of mathematical analysis that studies the several different possibilities of defining real number powers or complex number powers of the differentiation operator ${}_CD^t_\alpha$,one of the motivations behind the introduction and study of these sorts of extensions of the differentiation operator D is that the sets of operator powers $\{D^\alpha|\alpha \in \mathbb{R}\}$ defined in this way are continuous semigroups with parameter a, of which the original discrete semigroup of $\{D^n|n \in \mathbb{Z}\}$ for integer n is a denumerable subgroup: since continuous semigroups have a well developed mathematical theory, they can be applied to other branches of mathematics.fractional differential equations, also known as extraordinary differential equations, are a generalization of differential equations through the application of fractional calculus.

The Riemann-Louville fractional[3] integral is defined as

$$I^\alpha_{0|x}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\xi)^{\alpha-1}f(\xi)d\xi, x > 0 \tag{8}$$

where (Γ) is the gamma function

Another option for computing fractional derivatives is the Caputo fractional derivative. It was introduced by Michele Caputo in his 1967 in contrast to the Riemann-Liouville fractional derivative, when solving differential equations using Caputo's definition, it is not necessary to define the fractional order initial conditions. Caputo's definition is illustrated as follows, where again $(n = [\alpha])$,where $(0 < \alpha < 1)$ and$f(x)$ is a continuous function. also the Caputo fractional derivative is introduced as

$$D^\alpha_t f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^b dt\varphi(\nu)\left[D^\nu f(t)\right] \tag{9}$$

which has the advantage that is zero when $f(t)$ is constant and its Laplace Transform is expressed by means of the initial values of the function and its derivative. Moreover, there is the Caputo fractional derivative of distributed order defined as

$$D^\alpha_{0|t}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\xi)^{n-\nu-1}\frac{d^n}{d\xi^n}f(\xi)d\xi \tag{10}$$

III. ONE-DIMENSIONAL HARMONIC OSCILLATOR IN FRACTIONAL DERIVATIVES

A. The fractional quantum mechanics postulates

As a natural generalization of the fractional Schrödinger equation, the variable-order fractional Schrödinger equation has been exploited to study fractional quantum phenomena In the fractional quantum mechanics, the time- dependent Schrödinger equation reads

$$i\hbar^\alpha{}_\alpha D^\alpha_t\psi(x,t) = H_\alpha(\hat{x}_\alpha, \hat{P}_\alpha)\psi(x,t) \tag{11}$$

$$= \left(\frac{\hat{P}^2_\alpha}{2m^\alpha} + V_\alpha(\hat{x}_\alpha)\right)\psi(x,t) \tag{12}$$

where the fractional wave function [4] is

$$\phi_n(x) = \frac{1}{2^n n!}\left(\frac{m^\alpha w^\alpha}{\pi\hbar^\alpha}\right)e^{-\frac{m^\alpha w^\alpha}{2\hbar^\alpha}x^2_\alpha}H_n\left(\sqrt{\frac{m^\alpha w^\alpha}{\hbar^\alpha}}x_\alpha\right) \tag{13}$$

For the first few quantum energy levels, one can see little resemblance between the quantum and classical probabilities, but when you reach the value n=10 there begins to be some similarity. to be sure, they don't look the same, but they do agree that the probability is greatest near the ends of the motion. as you proceed to very high n values, they become more and more alike, and the oscillations of the quantum probability become so close together that they are practically smeared together. The fact that the overall picture of probability of finding the oscillator at a given value of x converges for the quantum and classical pictures is called the correspondence principle [5]. where

$$\hat{P}_\alpha = \frac{\hbar^\alpha}{i} D_x^\alpha \tag{14}$$

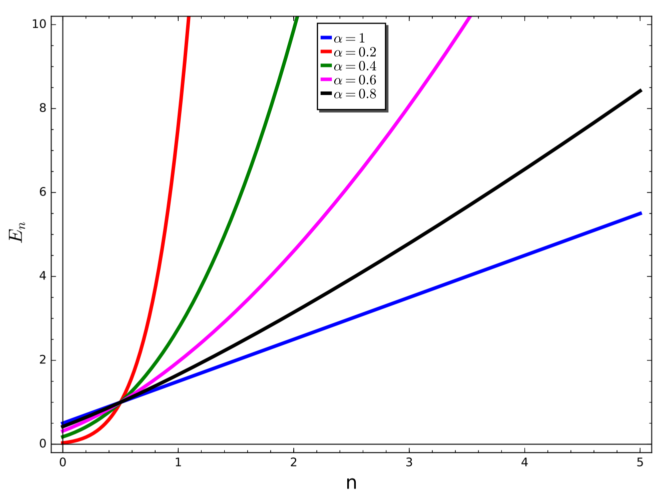
and for a smooth function in x, conformable fractional derivative which is defined by

$$D_x^\alpha f(x) = \lim_{\varepsilon \to 0} \frac{f\left(x + \varepsilon |x|^{1-\alpha}\right) - f(x)}{\varepsilon} \tag{15}$$

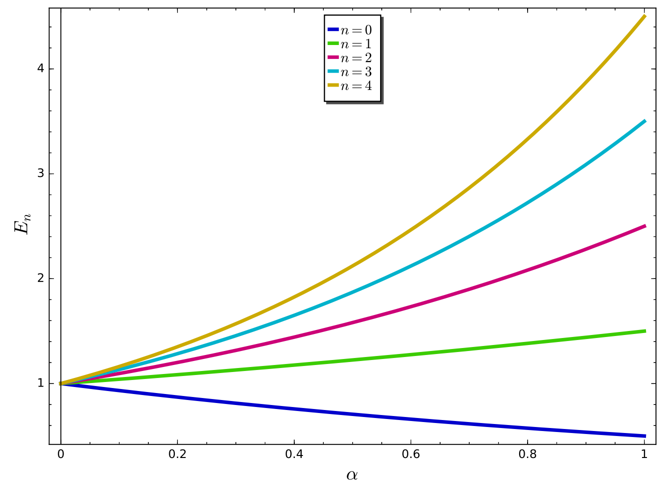
In the one-dimensional fractional harmonic oscillator the spectre of energy give by this relation :

$$E(n, \alpha) = (n + \frac{1}{2})^{\frac{1}{\alpha}} \tag{16}$$

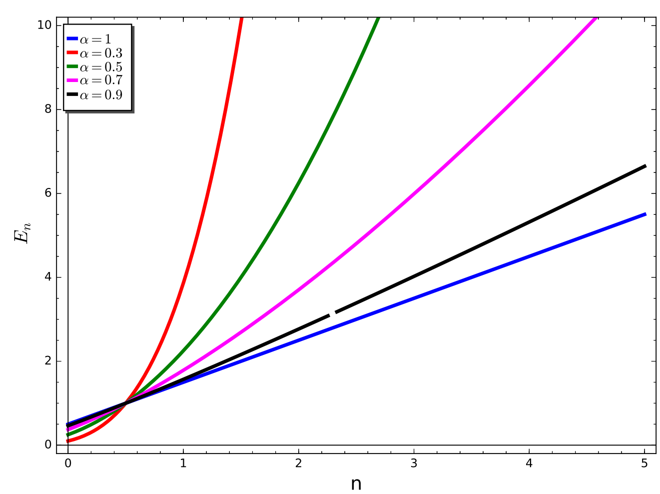
where (α) is the fractional derivative ordre , and we can plot all function of energy specter tradisional and fractional derivatives,in the fractional harmonic derivatives the energy spectre dependant with (n, α) , we can draw with different value



(a) Energy spectre of one-dimensional harmonic oscillator for $(\alpha = 0.2, \alpha = 0.4, \alpha = 0.6, \alpha = 0.8, \alpha = 1)$



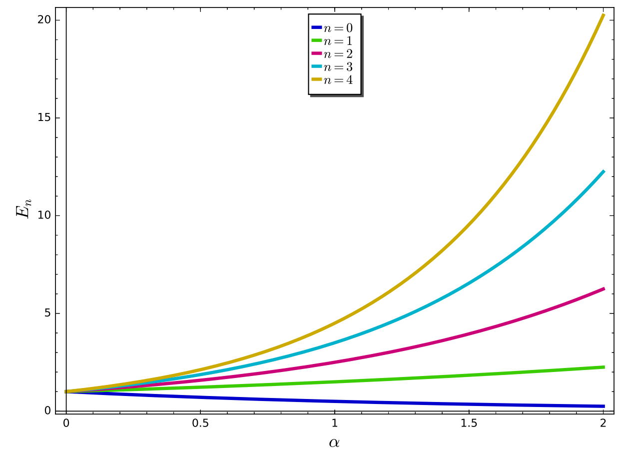
(b) Energy spectre of one-dimensional harmonic oscillator with α for $(n = 0, n = 1, n = 2, n = 3, n = 4)$



(c) Energy spectre of one-dimensional harmonic oscillator for $(\alpha = 1, \alpha = 0.3, \alpha = 0.5, \alpha = 0.7, \alpha = 0.9)$

IV. CONCLUSION

In this study we wanted to make the influenc of the fractional derivative in this system and comparison between the first case(harmonic oscillator in quantum mechanics) ,which studies the harmonic oscillator with traditional and fractional derivatives, and then draw the results.



(d) Energy spectre of one-dimensional harmonic oscillator with α for $(n = 0, n = 1, n = 2, n = 3, n = 4)$

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General Decay of viscoelastic equation with strong p-Laplacian damping and source terms.

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***Abstract*—In this paper, we consider the initial boundary value problem for the p-Laplacian equation with strong p-Laplacian damping and source terms.By using potential well method we prove global existence, also, by introducing suitable energy and perturbed Lyapunov functionals, we get a decay rate result in cases $p > q > 2$.**

***Index Terms*—Wave equation. p-Laplace type. Global existence. General decay.**

I. Introduction

We consider the following initial boundary value problem

$$\begin{cases} u_{tt} - \Delta_p u - div[a(x)\nabla u] \\ + \int_0^t g(t-s) div[a(x)\nabla u] ds + \Delta_p u_t \\ = u(x,t)|u|^{q-2} & in \;\; Q_T \\ u(x,t) = 0, & on \;\; \Gamma \times (0, +\infty) \\ u(0) = u_0(x), u_t(0) = u_1(x), & in \;\; \Omega \end{cases} \tag{I.1}$$

Where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary Γ. $p, q \geq 2$ and g is a positive nonincreasing kernel function satisfies some conditions to be specified later, and u_0, u_1 are given functions belonging to suitable spaces.

The operator Δ_p is the classical p-Laplacian given by $\Delta_p u = div\Big(|\nabla u|^{p-2}\nabla u\Big)$.

The study of the wave equation with p-Laplacian term has an interest that grows overnight thanks to its importance in the study of vibrations of a rod obeying a nonlinear voigt model. This enriches us with several works concerning the Global existence of the solution such as that of Nako and Nambu [5], Biazutti [2], J.M. Greenberg [3], and concerning the geberal decay of solution we mention that of Abbès Benaissa et al. [1], Messaoudi et al. [10], and for more details we refer the readers to the papers [4, 6–9, 11] and the references therein.

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II. Material and assumptions

In this section, we setup some notation, hypothesis and lemmas that will be used throughout this work. We will follow the standard norms notation of Lebesgue and sobolev spaces. Firstly, let $a(x) \in C^1(\Omega)$ be a positive function such that

$$a(x) \geq a_1^2 > 0, \tag{II.1}$$

and we define the Hilbert space

$$H_a := \left\{ u \in L^2(\Omega) : \int_\Omega a(x)|\nabla u|^2 dx \leq +\infty \right\},$$

endowed with the norm

$$\|\nabla u\|_a^2 = \int_\Omega a(x)|\nabla u|^2 dx, \tag{II.2}$$

From II.1 and II.2, we have

$$\|\nabla u\|_2^2 \leq \frac{1}{a_1^2}\|\nabla u\|_a^2 \tag{II.3}$$

Lemma 2.1: (Sobolev-embedding inequality) Let $2 \leq q < \infty$ for $n = 1, 2$, or $2 \leq q < \dfrac{2n}{n-2}$ for $n \geq 3$, then, there exists a constant $c_*(\Omega, q)$ such that

$$\|u\|_q \leq c_* \|\nabla u\|_2 \quad for \quad u \in H_0^1(\Omega), \tag{II.4}$$

Through this paper,we will state that

(A_1)

$$g(s) \geq 0, g'(s) \leq 0, m_0 - \int_0^{+\infty} g(s) ds = l \geq 0.$$

(A_2) There exists a positive differentiable function ξ such that

$$g'(s) \leq -\xi(s) g(s) \quad forall \quad s > 0,$$

(A_3) The constant p satisfies

$$p \geq 2\gamma + 2, \quad if \quad n = 1, 2, \quad and \quad 2\gamma + 2 \leq p \leq \frac{n+2}{n-2} \quad if \quad n \geq 3.$$

Now, we define the energy associated with problem (I.1) by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_a^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{q}\|u\|_q^q. \tag{II.5}$$

Lemma 2.2: Let u be a solution of problem I.1. Then

$$E'(t) \leq -\|\nabla u_t\|_p^p - \frac{1}{2}g(t)\|\nabla u\|_a^2 + \frac{1}{2}(g' \circ \nabla u)(t). \tag{II.6}$$

Proof 1: Multiplying the first equation in I.1 by u_t and integrating over Ω, we get

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\|u\|_q^q\right] = -\|\nabla u_t\|_p^p - \frac{1}{2}g(t)\|\nabla u\|_a^2 + \frac{1}{2}(g' \circ \nabla u)(t). \tag{II.7}$$

III. GLOBAL EXISTENCE

We define the following functionals

$$I(t) = I(u(t)) = \left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_a^2 + \frac{2}{p}\|\nabla u\|_p^p + (g \circ \nabla u)(t) - \|u\|_q^q. \tag{III.1}$$

and

$$J(t) = J(u(t)) = \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{q}\|u\|_q^q. \tag{III.2}$$

Then, it is obvious that

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + J(t). \tag{III.3}$$

Lemma 3.1: Assume that (A_1)-(A_2) hold, and for any $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, such that

$$I(0) > 0, \quad \vartheta = \frac{c_*^q}{l}\left(\frac{2q}{l(q-2)}E(0)\right)^{\frac{q-2}{2}} < 1, \tag{III.4}$$

then,

$$I(t) > 0, forall \;\; t > 0. \tag{III.5}$$

Proof 2: Since $I(0) > 0$, then by continuity of $u(t)$, there exist a time $T_* > 0$ such that

$$I(t) > 0, \forall \;\; t \in (0, T_*). \tag{III.6}$$

Then, we have

$$\begin{aligned} J(u(t)) &= \frac{q-2}{2q}\left[\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_a^2 + \frac{2}{p}\|\nabla u\|_p^p + (g \circ \nabla u)(t)\right] + \frac{1}{q}I(t) \\ &\geq \frac{q-2}{2q}\left[\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_a^2 + \frac{2}{p}\|\nabla u\|_p^p + (g \circ \nabla u)(t)\right]. \end{aligned} \tag{III.7}$$

Using (A_1),III.7 and III.3, we obtain

$$\begin{aligned} l\|\nabla u\|_a^2 &\leq \left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_a^2 \\ &\leq \frac{2q}{q-2}J(u(t)) \\ &\leq \frac{2q}{q-2}E(t) \\ &\leq \frac{2q}{q-2}E(0), \forall 0 < t < T_*. \end{aligned} \tag{III.8}$$

By II.4,III.8, and III.4, we obtain

$$\begin{aligned} \|u\|_q^q &\leq c_*^q\|\nabla u\|_2^q \leq \frac{c_*^q}{a_1^q} \\ &\leq \|\nabla u\|_a^q \frac{c_*^q}{a_1^q l}\left(\frac{2q}{l(q-2)}E(0)\right)^{\frac{q-2}{2}} l\|\nabla u\|_a^q \\ &= \vartheta l\|\nabla u\|_a^2 \\ &< \left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_a^2, \forall \;\; 0 < t < T_*. \end{aligned} \tag{III.9}$$

Hence, we can get

$$I(t) > 0, , \forall \;\; 0 < t < T_*$$

By repeating the procedure, T_* is extended to T. This completes the proof.

Theorem 3.2: Assume that the conditions of Lemma 3.1 hold, then the solution I.1 is global and bounded.

Proof 3: It suffices to show that

$$\|u_t\|_2^2 + \|\nabla u\|_a^2 + \|\nabla u\|_p^p \leq CE(0),$$

so, by using III.3, and III.7, we get

$$\begin{aligned} E(0) &\geq E(t) = \frac{1}{2}\|u_t\|_2^2 + J(t) \\ &\geq \frac{q-2}{2q}\left(l\|\nabla u\|_a^2 + \frac{2}{2p}\|\nabla u\|_p^p\right) + \frac{1}{2}\|u_t\|_2^2. \end{aligned} \tag{III.10}$$

Therefore,

$$\|u_t\|_2^2 + \|\nabla u\|_a^2 + \|\nabla u\|_p^p \leq CE(0), \tag{III.11}$$

where C is a positive constant, which depends only on q, l and p.

IV. General Decay

In this section, we shall study the general decay of energy to problem I.1. For this goal, we set

$$F(t) := E(t) + \varepsilon\phi(t), \tag{IV.1}$$

where ε is a positive constants to be specified later and

$$\phi(t) = \int_\Omega u_t u dx. \tag{IV.2}$$

Then, we have the following lemmas

Lemma 4.1: For ε small enough, we have

$$\beta_1 E(t) \leq F(t) \leq \beta_2 E(t). \tag{IV.3}$$

holds for tow positives constants β_1 and β_2

Proof 4: By using Holder's, Young's and Sobolev-Poincare inequalities.II.4, we get

$$\begin{aligned} |F(t) - E(t)| &\leq \frac{\varepsilon}{2}\|u_t\|_2^2 + \frac{\varepsilon}{2}\|u\|_2^2 \\ &\leq \frac{\varepsilon}{2}\|u_t\|_2^2 + \frac{\varepsilon c_*^2}{2a_1^2}\|\nabla u\|_a^2 \\ &\leq c\varepsilon E(t). \end{aligned} \tag{IV.4}$$

If we take ε to be sufficiently small, then IV.3 follows from IV.4

Lemma 4.2: The functional ϕ defined in IV.2 satisfies

$$\begin{aligned} \phi'(t) &\leq \|u_t\|_2^2 - (l-\eta)\|\nabla u\|_a^2 - \left(1 - \frac{\eta^p}{p}\right)\|\nabla u\|_p^p \\ &+ \frac{p-1}{p}\eta^{-p/p-1}\|\nabla u_t\|_p^p + \frac{1-l}{4\eta}(g\circ\nabla u)(t) + \|u\|_q^q. \end{aligned} \tag{IV.5}$$

Proof 5: Differentiating IV.2 with respect to t and using equation I.1, we get

$$\begin{aligned} \phi'(t) &= \|u_t\|_2^2 - \|\nabla u\|_a^2 - \|\nabla u\|_p^p + \|u\|_q^q \\ &+ \int_\Omega \nabla u \int_0^t g(t-s)a(x)\nabla u(s)dsdx \\ &- \int_\Omega |\nabla u_t|^{p-2}\nabla u_t \nabla u dx \\ &= \|u_t\|_2^2 - \|\nabla u\|_a^2 - \|\nabla u\|_p^p + \|u\|_q^q + I_1 + I_2 + I_3. \end{aligned} \tag{IV.6}$$

By using Holders and Youngs inequalities, we estimate I_1,I_2 and I_3 as follows,

$$\begin{aligned} I_1 &= \left| \int_\Omega \nabla u \int_0^t g(t-s)a(x)(\nabla u(s) - \nabla u(t))dsdx(t) \right| \\ &+ \int_0^t g(s)ds\|\nabla u\|_a^2 \\ &\leq (\eta + (1-l))\|\nabla u\|_a^2 + \frac{1-l}{4\eta}(g\circ\nabla u)(t) \end{aligned} \tag{IV.7}$$

and

$$\begin{aligned} I_2 &= \left| \int_\Omega |\nabla u_t|^{p-2}\nabla u_t \nabla u dx \right| \\ &\leq \frac{1}{p}\eta^p\|\nabla u\|_p^p \\ &+ \frac{p-1}{p}\eta^{-p/p-1}\|\nabla u_t\|_p^p, \end{aligned} \tag{IV.8}$$

A substitution of IV.7-IV.8 into IV.6 yields IV.5.

Lemma 4.3: Assume that (A_1)–(A_2) holds. Let $(u_0, u_1) \in H_0^1 \times L^2(\Omega)$ be given and satisfying III.4. Then, for any t_0, the functional $F(t)$ verifies

$$F'(t) \leq -\delta_1 E(t) + \delta_2(g\circ\nabla u)(t), \tag{IV.9}$$

for some $\delta_i > 0$, $(i = 1, 2)$.

Proof 6: First, since the function g is a positive and continuous with $g(0) \geq 0$, then, for any $t_0 > 0$,

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0.$$

From Lemmas 2.1 and 4.2, we have

$$\begin{aligned} F'(t) &= E'(t) + \varepsilon\phi'(t) \\ &\leq \varepsilon\|u_t\|_2^2 - \varepsilon(l-\eta)\|\nabla u\|_a^2 - \left(1 - \varepsilon\frac{p-1}{p}\eta^{-p/p-1}\right)\|\nabla u_t\|_p^p \\ &-\varepsilon\left(1 - \frac{\eta^p}{p}\right)\|\nabla u\|_p^p + \varepsilon\|u\|_q^q + \frac{1}{2}(g'\circ\nabla u)(t) \\ &+\varepsilon\frac{1-l}{4\eta}(g\circ\nabla u)(t). \end{aligned} \tag{IV.10}$$

By using the Sobolev–Poincaré inequality and Sobolev-injection

$$\|u_t\|_2^2 \leq c_*^2\|\nabla u_t\|_2^2 \leq \gamma c_*^2\|\nabla u_t\|_p^p.$$

Then, inequality IV.10 becomes

$$\begin{aligned} F'(t) &\leq -\varepsilon(l-\eta)\|\nabla u\|_a^2 - \left(1 - \varepsilon\left(\gamma c_*^2 + \frac{p-1}{p}\eta^{-p/p-1}\right)\|\nabla u_t\|_p^p\right. \\ &-\varepsilon\left(1 - \frac{\eta^p}{p}\right)\|\nabla u\|_p^p + \varepsilon\|u\|_q^q + \frac{1}{2}(g'\circ\nabla u)(t) \\ &+\varepsilon\frac{1-l}{4\eta}(g\circ\nabla u)(t) \end{aligned} \tag{IV.11}$$

At this point, we chose η small enough, such that

$$l - \eta > 0, \quad 1 - \frac{\eta^p}{p} > 0$$

When η is fixed, we choose ε so small that

$$1 - \varepsilon(1+\eta) > 0, \quad 1 - \varepsilon\left(\gamma c_*^2 + \frac{p-1}{p}\eta^{-p/p-1}\right) > 0.$$

Therefore, for all $t > 0$, the inequality IV.9 is satisfied

Theorem 4.4: Assume that (A_1)–(A_2) holds. If $(u_0, u_1) \in H_0^1 \times L^2(\Omega)$, then, there exist two positive constants K and k such that the energy of I.1 satisfies

$$E(t) \leq K e^{-k \int_{t_0}^{t} \xi(s) ds}, \ \ t_0 \geq t.$$

Proof 7: Multiplying IV.9 by $\xi(t)$, we have

$$\xi(t)F'(t) \leq -\delta_1 \xi(t)E(t) - \delta_2 \xi(t)(g \circ \nabla u)(t), \tag{IV.12}$$

Since (A_1) and using II.6, we obtain

$$\begin{aligned} \xi(t)F'(t) &\leq -\delta_1 \xi(t)E(t) - \delta_2 (g' \circ \nabla u)(t) \\ &\leq \delta_1 \xi(t)E'(t) - 2\delta_2 E'(t), \forall t_0 \leq t, \end{aligned} \tag{IV.13}$$

That is,

$$L'(t) \leq -\delta_3 \xi E(t) \leq -k\xi(t)L(t), \forall t_0 \leq t, \tag{IV.14}$$

where $L(t) = \xi(t)F(t) + CE(t)$ is equivalent to $E(t)$ because of IV.3, and k is a positive constant. A simple integration of IV.14, leads to

$$L(t) \leq L(t_0) e^{-k \int_{t_0}^{t} \xi(s) ds}, \forall t_0 \leq t, \tag{IV.15}$$

This completes the proof.

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Implementation of Algebraic multigrid method for variational inequality related to HJB equation

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Abstract. We treat multigrid methods for solving one-sided obstacle problems based on restating the variational inequality as HJB-equation. For the discretization, we use adaptive finite element method to get a large sparse linear system. Uniform convergence of the multigrid method has been established which proves that the these methods have a contraction number with respect to the maximum norm. we present a results of numerical experiments to demonstrate the high efficiency of these methods.

Keywords: Variational Inequality · Finite element method · Multigrid Method · HJB Equation.

1 Continuous problem

Let Ω be an open in $\mathbb{R}^N$, with sufficiently smooth boundary $\partial\Omega$ for $u, v \in V$ $\left(V = H_0^1(\Omega)\right)$, Consider the following problem: Find $u \in V$ the unique solution of

$$\begin{cases} a\left(u, v-u\right) \geq \langle f, v-u\rangle & v \in V, \\ u \leq \psi;\ v \leq \psi. \end{cases} \tag{1}$$

2 Discrete problem

The numerical approximation of the VI (1) by finite elements leads to the solution of the discrete VI in finite dimension. Find the unique $u_k \in V_k$ such that

$$\begin{cases} \langle A_k u_k, v_k - u_k\rangle \geq \langle f, v_k - u_k\rangle\,, & \forall v_k \in V_k, \\ u_k \leq \psi_k, \quad v_k \leq \psi_k, \end{cases} \tag{2}$$

where

$$V_k = \left\{ v_k \in C\left(\Omega\right) \cap H^1\left(\Omega\right) \mid v_{k|\Omega_k} \in P_1 \right\}. \tag{3}$$

3 Description of Multigrid Methods for the VI (2)

After the HJB-formulation of the discrete problem (2), the multigrid iteration of the HJB-equation obtained may be described as the following algorithm.

Algorithm 1 Muligrid methods

$u \leftarrow MGM(A_k^\nu, f_k^\nu, u_0)$
$u_0 := smoother(A_k^\nu, b, u_0, \alpha_1)$; % α_1 is a number of iterations performed (pres-moothing)
$d_k := f_k^\nu - A_k^\nu u_0$; % residual computation
$R_k; P_k$; % define the prolongation and the restriction matrices
$A_{k-1}^\nu := R_k A_k^\nu P_k$; % restriction of A_k
$d_{k-1} := R_k d_k$; % restriction of d_k
$e_{k-1} := d_{k-1} \cdot 0$; % start value for coarse grid iteration
if $size(A_k^\nu) \leq \mu$ % coarsest grid Ω_μ **then**
    $e_{k-1} := (A_k^\nu)^{-1} d_{k-1}$; % direct solve on the coarse grid
else
    $e_{k-1} := MGM(A_{k-1}^\nu, d_{k-1}, e_{k-1})$; % solve coarse problem
end if
$e_k := Pe_{k-1}$; % prolongation of e_{k-1}
$u_k := u_0 + e_k$; % add correction to the solution
$u_k := smoother(A_k^\nu, f_k^\nu, u_k, \alpha_2)$; % α_2 is a number of iterations performed (postsmoothing)
return u_k

4 Numerical experiments

After the discussion of the uniform convergence analysis for the multigrid algorithm described in the previous section, we present numerical example, find the solution of the following variational Inequalitie:

$$\begin{cases} Au \leq f, & \text{in } \Omega = \left\{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 1\right\}, \\ \langle Au - f \ , u^\nu - k \rangle = 0, \\ u^\nu \leq k, \\ u = 0, & in \ \ \partial\Omega. \end{cases} \tag{4}$$

Where

$$\begin{aligned} Au &= -\Delta u, \\ f(x) &= x_1 + x_2, \\ k &= 0.001. \end{aligned}$$

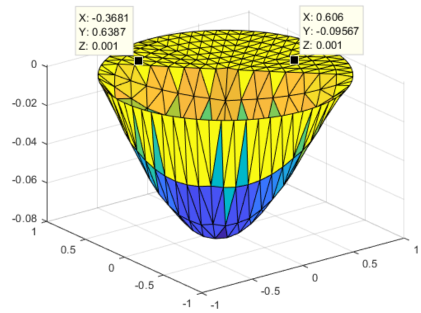

Fig. 1. Solution of the VI (4) by 25 iterations of MGM with k=0.001

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STUDY OF PARABOLIC PROBLEM WITH VARIABLE EXPONENT AND NONLOCAL BOUNDARY CONDITIONS

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Abstract. In this work we study the existence for quasilinear parabolic problem with variable exponent and with nonlocal boundary conditions and L1 data. The main contribution of our work is to prove the existence of a renormalized solution. The results of the problem discussed can be applied to a variety of different fields in applied mathematics for example in elastic mechanics, image processing and electro-rheological fluid dynamics, etc..

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