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Study of some spectral properties of linear operators and their applications

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Abstract

In this thesis, we try to provide some spectral properties of bounded operators on Hilbert spaces, including extended spectrum and provide of some applications.
In the first chapter, we will presente a some of necessary definitions and theories and some results.
In the second chapter, we will presente various definitions of spectrums and some general properties of them.
In the last chapter, we will presente two applied project, the ones in the field of differential equations and the calculation of the extended spectrum of Volterra.

keywords

The operator, adjoint operator, spectrum, point spectrum, continuous spectrum, vector eigenvalues

Resumé

Dans cette thèse, nous essayons de fournir quelques propriétés spectrales des opérateurs borné sur les espaces de Hilbert, y compris le spectre étendu et de fournir de certaines applications.
Dans le premier chapitre, nous avons présenté quelques définitions et théories nécessaires et quelques résultats.
Dans le deuxiéme chapitre, nous avons présenté diverses définitions des spectres et certaines propriétés générales de ceux-ci.
Dans le dernier chapitre, nous avons présenté deux projets appliqués, ceux dans le domaine des équations différentielles et le calcul du spectre étendu de Volterra.

Mots-clés

operateur, adjoint d'operateur, valeur propre, vecteur propre, spectra etendu.

Notations

H	Hilbert space.
$\langle ., . \rangle$	Scalar product.
$\|.\|$	Norm
T	Linear operator.
$R(T)$	Range of operator T.
$\mathcal{B}(H)$	Space of all bounded linear operators.
T^*	Adjoint of operator T.
T^{-1}	Inverse of operator T.
$\sigma(T)$	Spectrum of T.
$\sigma_p(T)$	Point spectrum of T.
$\sigma_c(T)$	Continuous spectrum of T.
$\sigma_r(T)$	Residual spectrum of T.
$\sigma_{ap}(T)$	Approximate spectrum of T.
$\sigma_{com}(T)$	Compression spectrum of T.
$\sigma_{ext}(T)$	Extended spectrum of T
$\rho(T)$	Resolevent of T.
$r(T)$	Spectral radius of T.
$R(\lambda, T)$	Resolevent application of T.

Contents

General introduction

In mathematics, and more particularly in analysis, a spectral theory is a theory extending to operators defined on general functional spaces the elementary theory of eigenvalues and eigenvectors of matrices. Although these ideas come from the development of linear algebra. They are also linked to the study of analytical functions, because the spectral properties of an operator are linked to those of analytical functions on the values of its spectrum.

The name spectral theory was introduced by David Hilbert in his original formulation of Hilbert space theory, which was cast in terms of quadratic forms in infinitely many variables. The original spectral theorem was therefore conceived as a version of the theorem on principal axes of an ellipsoid, in an infinite-dimensional setting. The later discovery in quantum mechanics that spectral theory could explain features of atomic spectra was therefore fortuitous. Hilbert himself was surprised by the unexpected application of this theory, noting that "I developed my theory of infinitely many variables from purely mathematical interests, and even called it 'spectral analysis' without any presentiment that it would later find application to the actual spectrum of physics.

So, we can define the spectrum of a bounded linear operator (or, more generally, an unbounded linear operator) as a generalisation of the set of eigenvalues of a matrix. Specifically, a complex number λ is said to be in the spectrum of a bounded linear operator T if $T-\lambda I$ is not invertible, where I is the identity operator. The study of spectra and related properties is known as spectral theory, which has numerous applications, most notably the mathematical formulation of quantum mechanics.

The spectrum of an operator on a finite-dimensional vector space is precisely the set of eigenvalues. However an operator on an infinite-dimensional space may have additional elements in its spectrum, and may have no eigenvalues.

In first chapter, we will present a set of preliminary concepts and basic theories that we need in this work, this chapter is divided into three sections.

- **In the first section** We will present some basic concepts and theories about Hilbert space and some results related to Bounded operators on Hilbert spaces.

- **In the second section** We will present some classes of operators and related additional results.

- **In third section** We will present an enterance for about unbounded operators, however this work have not interested, but we did because more scientifics research about them.

In second chapter in this chapter we will present some spectral notions and its basic properties, we determine relation of the spectrums with the operators classes (self-adjoint, normal, compact, compact normal) and with numerical range, finally we will present extended spectrum notion and some results.

In third chapter we will present two projects, first project, we provided some results in PDE, we taken Laplacian problem and Dirichlet problem, in second project we calculated the extended eigenvalues of volterra operator and determine its location.

Chapter 1

Preliminaries

1.1 Preliminary notions

1.1.1 Notions and definitions

Definition 1.1. *Let E be a linear space over $\mathbb{C}$. A scalar product is a mapping $\langle .,. \rangle$ from $E \times E$ into $\mathbb{C}$ such that*

(1) $\langle x, y\rangle = \overline{\langle y, x\rangle} \qquad \forall\, x, y \in E$,

(2) $\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$ *and* $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle \quad \forall\, x, y \in E, \lambda \in \mathbb{C}$,

(3) $\forall\ x \in E \quad \langle x, x\rangle \geq 0$ *and* $\langle x, x\rangle = 0$ *if and only if* $x = 0$.

Example 1.1. *Let $E = \mathcal{C}([a, b], \mathbb{C})$, the following function is a scalar product on E*

$$\begin{array}{rccl} \langle .,. \rangle : & E^2 & \longrightarrow & \mathbb{C} \\ & (f, g) & \longmapsto & \langle f, g\rangle = \int_a^b f(t)\overline{g(t)}dt. \end{array}$$

Proposition 1.1. *The scalar product $\langle .,. \rangle$ defines a norm on E, such that*

$$\|x\| = \sqrt{\langle x, x\rangle} \quad \forall x \in E.$$

Definition 1.2. *A complex linear space E having a scalar product is called a scalar product space or a **pre Hilbert space**, and it will be called **Hilbert space** if it is complete.*

Example 1.2. *1. Let $H = \mathbb{C}^n$, then H is Hilbert space with scalar product*

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i\overline{y_i} \qquad \forall x, y \in \mathbb{C}^n,$$

where $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$.

2. Let $H = L^2([a, b])$, then H is Hilbert space with scalar product

$$\langle f, g\rangle = \int_a^b f(t)\overline{g(t)}dt \qquad \forall f, g \in L^2([a, b]).$$

3. Let

$$H = \ell^2 = \{x = (x_n)_{n\in\mathbb{N}^*} \subset \mathbb{C}, \quad \sum_{i=1}^{\infty} |x_i|^2 < \infty\},$$

then H is Hilbert space with scalar product

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i\overline{y_i} \qquad \forall x, y \in \ell^2.$$

Theorem 1.1. *(**Cauchy-Schwarz inequality**) Let H be a pre-Hilbert space. Then*

$$|\langle x, y\rangle| \leqslant \|x\|\|y\| \quad \forall x, y \in H.$$

Definition 1.3. *Two vectors x and y in H are said to be orthogonal if $\langle x, y\rangle = 0$, we denote $x \perp y$.*

Definition 1.4. *Let M be a subset of H, the orthogonal complement of M(denoted by $M^{\perp}$) is defined by*

$$M^{\perp} = \{y \in H; \langle x, y\rangle = 0, \forall x \in M\}.$$

Proposition 1.2. *Let H be a Hilbert space and M be a subset of H, then*

1. *$M^{\perp}$ is a closed subspace of H.*
2. *$\overline{M} \subseteq M^{\perp\perp}$.*
3. *$M^{\perp\perp}$ is the smallest closed subspace containing M.*

Lemma 1.1. *If M is a subspace of a Hilbert space H, then*

$$M^{\perp\perp} = \overline{M}.$$

Lemma 1.2. *If M is a closed subspace of H, then H is the sum orthogonal direct of M and $M^{\perp}$, noted by $H = M \oplus M^{\perp}$.*

Definition 1.5. *The set $M = \{x_i \in H; i \in I \subset N\}$ is called orthogonal system if for all $i, j \in I$ and $i \neq j$ we have $\langle x_i, x_j\rangle = 0$. M is called orthonormal system if it is orthogonal and $\| x_i \| = 1$ for all $i \in I$.*

Theorem 1.2. *Let $x_1, x_2, \ldots x_n$ be orthonormal vectors in a pre-Hilbert space H. For every $x \in H$,*

$$\left\| x - \sum_{k=1}^{n} \langle x, x_k\rangle x_k \right\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x, x_k\rangle|^2 .$$

Hence,

$$\sum_{k=1}^{n} |\langle x, x_k\rangle|^2 \leq \|x\|^2.$$

Theorem 1.3. ***(Gram-Schmidt orthonormalisation)*** *Let $\{x_n\}_{n\geqslant 1}$ be a linearly independent sequence in a scalar product space H. Define*

$$\begin{aligned} &y_1 = x_1, \quad \textit{and} \quad u_1 = \frac{x_1}{\|x_1\|} \\ &y_2 = x_2 - \langle x_2, u_1\rangle u_2 \quad \textit{and} \quad u_2 = \frac{y_2}{\|y_2\|} \\ &\vdots \\ &y_n = x_n - \langle x_n, u_1\rangle u_1 - \langle x_n, u_2\rangle u_2 - \cdots - \langle x_n, u_{n-1}\rangle u_{n-1} \quad \textit{and} \quad u_n = \frac{y_n}{\|y_n\|}. \end{aligned}$$

Then $u_1, u_2, \ldots$ is an orthonormal sequence in H, and for $n = 1, 2, \ldots,$

$$span\{u_1; u_2; \ldots u_n\} = span\{x_2; \ldots; x_n\}.$$

1.1.2 Bounded Linear Operators

Definition 1.6. *A mapping defined over a Hilbert space H is said to be a linear operator if*

$$\forall\ x, y \in H \quad \forall \alpha, \beta \in \mathbb{C} \qquad T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

Definition 1.7. *A linear operator T on a Hilbert space H is said to be bounded (or continuous)if there exists a constant $C > 0$ such that*

$$\|Tx\| \leqslant C\|x\| \qquad \forall x \in H.$$

and we have the norm of T is defined by:

$$\|T\| = inf\{C > 0 : \|Tx\| \leq C\|x\|\forall x \in H\}.$$

Definition 1.8. *An operator $T \in \mathcal{B}(H)$ is said to be bounded from below if there exists a $C > 0$ such that*

$$\|Tx\| \geqslant C\|x\| \quad \textit{for all} \quad x \in H.$$

Lemma 1.3. *Let $T \in \mathcal{B}(H)$ is an operator on H. Then T is not bounded from below if and only if there is a squence $(x_n)_{n\in\mathbb{N}} \subset H$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \|Tx_n\| = 0$.*

Theorem 1.4. *For all linear operator on Hilbert space $T \in \mathcal{B}(H)$, we have*

$$\|T\| = \sup\{\|Tx\|; x \in H, \|x\| = 1\}.$$

Proof. (1) if the linear operator T is bounded, we have

$$\|Tx\| \leq \|T\|\|x\| = \|T\| \quad for \ \|x\| = 1$$

so

$$\sup\{\|Tx\|; x \in H, \|x\| = 1\} \leqslant ||T||.$$

(2) we put $a = \sup\{\|Tx\|; x \in H, \|x\| = 1\}$ for $x \in H$ and $x \neq 0$

$$\|Tx\| = \left\|T\left(\|x\|\frac{x}{\|x\|}\right)\right\| = \left\|T\left(\frac{x}{\|x\|}\right)\right\| \|x\| \leq a\|x\|.$$

$\|T\| \leq a.$

□

Definition 1.9. *Let $T \in \mathcal{B}(H)$ and M a closed subspace of H*

- *M is called invariant by T if $T(M) \subseteq M$.*
- *M is called reducing for T if $T(M) \subseteq M$ and $T(M^{\perp}) \subseteq M^{\perp}$.*

Definition 1.10. *A linear operator T is said to be invertible on Hilbert space if there exists a linear operator S satisfies:*

$$ST = TS = I$$

where I is identity operator on H.

Theorem 1.5. *Let T be a linear operator of $\mathcal{B}(H)$, the following statements are equivalents:*

- *T is invertible .*
- *$R(T)$ is dense in H, and T is bounded from below i.e $\overline{R(T)} = H$ and there exists a canstant $C > 0$ such that $\|Tx\| \geqslant C\|x\|$ for all $x \in H$.*

Proposition 1.3. *Let T be a linear operator on Hilbert space such that $\|T\| < 1$. Then $(I - T)$ is invertible and*

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k.$$

Remark 1.1. *From this propositon we deduce that If $\|I - T\| < 1$, then T is invertible.*

1.1.3 The adjoint operator

Definition 1.11. *For all operator of $\mathcal{B}(H)$, T^* denotes adjoint operator of T defined by*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \qquad \forall x, y \in H.$$

Example 1.3. *Let T be an operator is defined on ℓ^2 as*

$$T((x_n)_{n\geqslant 1}) = (0, x_1, x_2, ...) \quad \textit{for all } (x_n)_{n\geqslant 1} \in \ell^2$$

and we have its adjoint is defined as:

$$T((x_1, x_2, x_3, ...)) = (x_2, x_3, ...) \quad \textit{for all } (x_1, x_2, ...) \in \ell^2$$

$$\begin{aligned}
(x, T^*y) &= \left\langle T\left(\sum_{k=1}^{\infty}\lambda_k e_k\right), \sum_{k=1}^{\infty}\mu_k e_k\right\rangle, \quad \textit{where} \quad x = \sum_{k=1}^{\infty}\lambda_k e_k, y = \sum_{k=1}^{\infty}\mu_k e_k \\
&= \left\langle \sum_{k=1}^{\infty}\lambda_k e_{k+1}, \sum_{k=1}^{\infty}(\mu_k e_k)\right\rangle \\
&= \sum_{k=1}^{\infty}\lambda_k\bar{\mu}_{k+1} \\
&= \left\langle \sum_{k=1}^{\infty}\lambda_k e_k, \sum_{k=1}^{\infty}\mu_{k+1} e_k\right\rangle \\
&= \left\langle \sum_{k=1}^{\infty}\lambda_k e_k, \sum_{k=1}^{\infty}\mu_k e_{k-1}\right\rangle
\end{aligned}$$

As the above equality holds for all $x, y \in H$, it follows that

$$T^*y = \sum_{k=2}^{\infty}\mu_k e_{k-1}, \qquad \textit{where} \qquad y = \sum_{k=1}^{\infty}\mu_k e_k$$

In particular, $T^*e_k = e_{k-1}, k = 1, 2, \ldots,$ where $e_0 = 0$. Thus, the adjoint of T is

$$T^*\left(\sum_{k=1}^{\infty}\mu_k e_k\right) = \sum_{k=2}^{\infty}\mu_k e_{k-1}$$

This can also be described as

$$T^*((x_1, x_2, ...)) = (x_2, x_3, ...) \quad \textit{for all } (x_1, x_2, ...) \in \ell^2.$$

Proposition 1.4. *If T and S are two bounded linear operators, we have also T^* and S^* are bounded linear operators, and the following properties are satisfied*

(1) $\|T\| = \|T^*\|$,

(2) $(S+T)^* = S^* + T^*$,

(3) $(\lambda T)^* = \bar{\lambda} T^*$ *for all* $\lambda \in \mathbb{C}$,

(4) $(T^*)^* = T$,

(5) $(T^*)^{-1} = (T^{-1})^*$ *if* T *is invertible,*

(6) $(ST)^* = T^* S^*$.

Proposition 1.5. *Let* T *be a linear operator on Hilbet space* H*, then we have*

- $R(T)^\perp = N(T^*)$,
- $R(T^*)^\perp = N(T)$,
- $\overline{R(T)} = N(T^*)^\perp$,
- $\overline{R(T^*)} = N(T)^\perp$.

Lemma 1.4. *Let* T *be a linear operator on a Hilbert space* H*, then*

- $R(T)$ *is dense in* H *if and only if* T^* *is injective,*
- $R(T^*)$ *is dense in* H *if and only if* T *is injective.*

Definition 1.12. *An operator* $X \in \mathcal{B}(H)$ *is said to be commutator, if there exists* $A, B \in \mathcal{B}(H)$ *such that* $X = AB - BA$*,denoted by* $[A, B]$ *.*

Lemma 1.5. *If the commutator* $X = [A, B] = 0$*,Then* A *is said to be in the commutant of* B *The commutant of* A *is denoted by* $\{A\}'$ *and is given by the set*

$$\{A\}' = \{X \in \mathcal{B}(H); \quad [A, X] = 0\}.$$

Theorem 1.6. ***(The Riesz Representation Theorem)*** *Let* f *be a bounded linear functional on* H*, then there exists a vector* $z \in H$ *such that*

$$f(x) = \langle x, z \rangle \quad \forall x \in H.$$

Moreover,

$$\|f\| = \|z\|.$$

1.2 Some Classes of Operators

Definition 1.13. *A bounded operator $T \in \mathcal{B}(H)$ is said to be*

(1) ***Compact*** *if $T(B_H)$ has compact closure in H.*

(2) ***Finite rank*** *if the range of T $(R(T))$ is finite dimensional.*

(3) ***Self-adjoint*** *if $T = T^*$.*

(4) ***Positive*** *if $\langle Tx, x\rangle \geq 0$ for all $x \in H$.*

(5) ***Unitary*** *if $T^*T = TT^* = I$.*

(6) ***Isometry*** *if $T^*T = I$.*

(7) ***Normal*** *if $T^*T - TT^* = 0$ or $||Tx|| = ||T^*x||$ for all $x \in H$.*

(8) ***Subnormal*** *if it has a normal extention.*

(9) ***Quasi-normal*** *if swich with TT^*.*

(10) ***Hyponormal*** *$T^*T - TT^* \geqslant 0$ or $\forall\ x \in H$ $||T^*x|| \leq \|Tx\|$.*

(11) ***p-Hyponormal*** *$(T^*T)^p - (TT^*)^p \geqslant 0$.*

(12) ***Paranormal*** *if $\|Tx\|^2 \leqslant \|T^2x\|$.*

(13) ***Nilpotent*** *if $T^n = 0$ for some positive iteger n.*

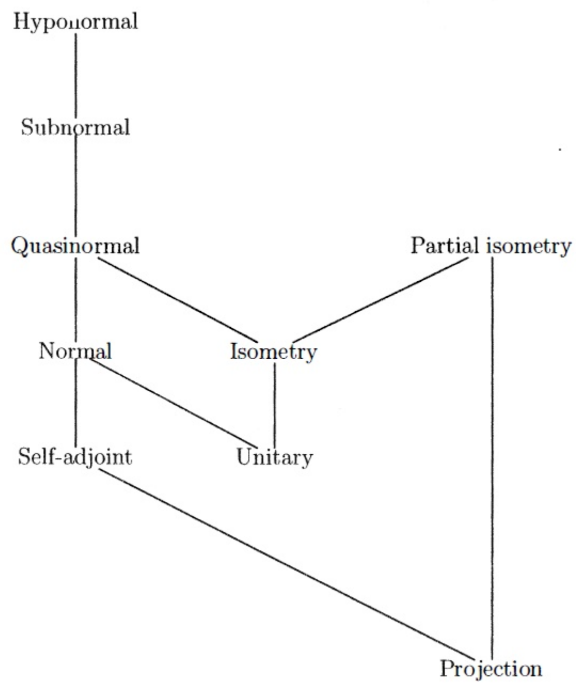
(14) ***Posinormal*** *if there exists a positive operator $P \in \mathcal{B}(H)$ called the interrupter such that $TT^* = T^*PT$.*

(15) ***Partial isometry*** *if $T = TT^*T$.*

(16) ***Contraction*** *if $||T|| \leq 1$.*

(17) ***Projection*** *$T^2 = T$ and $T^* = T$ or equivalently$T = T^*T$.*

we have some inclusions of operators classes as



Theorem 1.7. *Let $T \in \mathcal{B}(H)$ be an operator on a Hilbert space H over the complex scalars $\mathbb{C}$, then the following statements are equivalent:*

- $T = 0$ *i.e* $Tx = 0$ *for all* $x \in H$,
- $\langle Tx, x\rangle = 0$ *for all* $x \in H$,
- $\langle Tx, y\rangle = 0$ *for all* $x, y \in H$.

Theorem 1.8. *Let H be a complex Hilbert space if T is an operator on H, then*

(a) T *is normal if and only if* $\|Tx\| = \|T^*x\|$ *for all* $x \in H$,

(b) T *is unitary if and only if* $\|Tx\| = \|T^*x\| = \|x\|$ *for all* $x \in H$,

(c) T *is hypo-normal if and only if* $\|T^*x\| \leq \|Tx\|$ *for all* $x \in H$.

1.2.1 Self-adjoint operators

Definition 1.14. *An operator $T \in \mathcal{B}(H)$ is said to be self adjoint if*

$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \forall x, y \in H.$$

Example 1.4. *Let $H = L^2([a,b])$, and let h be a real function on $[a,b]$, then T wich is defined as:*

$$T : H \to H$$
$$f \mapsto hf$$

is a self-adjoint operator.

Proof. Let $f, g \in H$, then

$$\begin{aligned}\langle Tg, f\rangle &= \int_a^b h(t)g(t)\overline{f(t)}dt \\ &= \int_a^b g(t)\overline{h(t)f(t)}dt \quad \text{since } h \text{ is real function} \\ &= \langle g, Tf\rangle\end{aligned}$$

□

Definition 1.15. *A sequence $\{T_n\}_{n\geqslant 1}$ of self-adjoint operators is said to be bounded monotone increasing if there exists an operator T such that*

$$T_1 \leqslant T_2 \leqslant \cdots \leqslant T_n \leqslant \cdots \leqslant T.$$

Similarly, a sequence $\{T_n\}_{n\geqslant 1}$ of self-adjoint operators is said to be bounded monotone decreasing if there exists an operator T such that

$$T_1 \geqslant T_2 \geqslant \cdots \geqslant T_n \geqslant \cdots \geqslant T.$$

Theorem 1.9. *Let $T \in \mathcal{B}(H)$, then T is self-adjoint if and only if $\langle Tx, x\rangle$ is real for all $x \in H$.*

Proof. We assume that T is self adjoint operator i,e $T = T^*$, then

$$\langle Tx, x\rangle = \langle T^*x, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}.$$

Thus $\langle Tx, x\rangle$ is real.

Now we assume that $\langle Tx, x\rangle$ is real for all $x \in H$, then for all $x \in H$

$$\langle Tx, x\rangle = \overline{\langle Tx, x\rangle} = \langle x, Tx\rangle = \langle T^*x, x\rangle.$$

Hence it follows $T - T^* = 0$, by (1.7), we deduce that T is self adjoint. □

Theorem 1.10. *Let T be a linear operator on Hilbert space H. Then there exists two selfadjoint operators A and B such that $T = A + iB$.*

Necessarily

$$A = \frac{1}{2}(T^* + T) \quad \textit{and} \quad B = \frac{1}{2}i(T^* - T).$$

Theorem 1.11. *If a sequence $\{T_n\}_{n\geqslant 1}$ of self- adjoint operators is bounded monotone increasing, then there exists a self-adjoint operator T such that $T_n \to T$.*

1.2.2 Positive operators

Definition 1.16. *An operator $T \in B(H)$ is called positive notation $T \geqslant 0$ if*

$$\langle Tx, x\rangle \geqslant 0 \quad \textit{for all} \quad x \in H.$$

Example 1.5. *Let $H = \mathbb{R}^2$, and*

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

for $x = (a, b) \in \mathbb{R}^2$, we have $Tx = (a, 3b)$, so $\langle Tx, x\rangle = a^2 + 3b^2 \geqslant 0$.

Theorem 1.12. *Any positve operator T admits a unique positive operator S such that $T = S^2$.*

Proof.

- **Existence of S** We may suffice to assume that $0 \leq T \leq I$. Let S_k be defined as follows: for $k = 1, 2, \cdots$

$$S_0 = 0 \quad \textit{and} \quad S_{k+1} = S_k + \frac{1}{2}\left(T - S_k^2\right) \tag{1.1}$$

since S_n is written by a polynomial of T , S_n is a self-adjoint such that

$$0 = S_0 \leq S_1 \leq \cdots \leq I \tag{1.2}$$

In fact, (1.1) yields

$$I - S_{k+1} = \frac{1}{2}\left(I - S_k\right)^2 + \frac{1}{2}(I - T) \tag{1.3}$$

(1.3) ensures $I \geq S_{k+1}$ and

$$S_{k+1} - S_k = \frac{1}{2}\left\{\left(I - S_{k-1}\right) + \left(I - S_k\right)\right\}\left(S_k - S_{k-1}\right) \tag{1.4}$$

and we have $S_{k+1} \geq S_k$ by induction, so that (1.2) holds. Also (1.2) implies that $\{S_k\}$ has a limit S by theorem 1.11 If $k \to \infty$ in (1), then

$$S = S + \frac{1}{2}\left(T - S^2\right)$$

that is, $S^2 = T.S \geq 0$

- **Uniqueness of** S Suppose that $0 \leq S_j \leq I$ such that $S_j^2 = T$ for $j = 1, 2$. As $S_2 T = S_2 S_2 S_2 = T S_2$, $S_1 S_2 = S_2 S_1$ holds .

$$\left(S_1 + S_2\right)\left(S_1 - S_2\right) = S_1^2 - S_2^2 - S_1 S_2 + S_2 S_1 = 0 \quad \text{since } S_1 S_2 = S_2 S_1 \tag{1.5}$$

There exist two positive operators R_1 and R_2 such that $R_1^2 = S_1$ and $R_2^2 = S_2$ by the former part. Put $y = \left(S_1 - S_2\right) x$ for any $x \in H$. Then

$$\begin{aligned}
&\left\|R_1 y\right\|^2 + \left\|R_2 y\right\|^2 = \langle R_1^2 y, y\rangle + \langle R_2^2 y, y\rangle \\
&= \langle\left(S_1 + S_2\right) y, y\rangle \\
&= \langle\left(S_1 + S_2\right)\left(S_1 - S_2\right) x, y\rangle \\
&= 0 \quad \text{by (1.5)}
\end{aligned}$$

so that $R_1 y = R_2 y = 0$ and $S_1 y = R_1^2 y = 0$, and similarly $S_2 y = R_2^2 y = 0$. It follows that

$$\left\|\left(S_1 - S_2\right) x\right\|^2 = \langle\left(S_1 - S_2\right)\left(S_1 - S_2\right) x, x\rangle = \langle\left(S_1 - S_2\right) y, x\rangle = 0$$

since $S_1 y = S_2 y = 0$. Thus $S_1 = S_2$ which is the desired relation of the latter half.

□

Proposition 1.6. *(**Generalized Cauchy-Schwarz inequality**) If $T \in \mathcal{B}(H)$ is a positive operator. Then we have*

$$|\langle Tx, y\rangle|^2 \leqslant \langle Tx, x\rangle\langle Ty, y\rangle \quad \forall x, y \in H.$$

1.2.3 Hyponormal operators

Definition 1.17. *An operator T is hyponormal if*

$$T^*T \geqslant TT^*.$$

Proposition 1.7. *Let T be a hyponormal operator. If T is invertible so T^{-1} is hyponormal.*

Proof. This proof uses the fact that if A is an invertible and positive operator and $A \geqslant I$, then $A^{-1} \leqslant I$
Since

$$T^*T \geqslant TT^*$$

and T is invertible then

$$T^{-1}T^*TT^{*-1} \geqslant T^{-1}TT^*T^{*-1} = I$$

Therefore

$$T^*T^{-1}T^{*-1}T \leqslant I.$$

Thus

$$T^{-1}T^{*-1} \leqslant T^{*-1}T^{-1}.$$

Hence T^{-1} is hyponormal. □

Proposition 1.8. *If T is a hyponormal operator then*

$$\| T^n \| = \| T \|^n \quad n \in \mathbb{N}.$$

Proof. If $x \in H$ and $n \geqslant 1$, then

$$\begin{aligned}
\| T^n x \|^2 &= \langle T^n x, T^n x \rangle \\
&= \langle T^*T^n x, T^{n-1} f \rangle \quad \text{by Cauchy schwarz inequality} \\
&\leqslant \| T^*T^n x \| \| T^{n-1} x \| \quad \text{using} \quad \|T^*y\| = \|Ty\| \\
&\leqslant \| T^{n+1} x \| \| T^{n-1} x \| .
\end{aligned}$$

Therefore

$$\| T^n \|^2 \leqslant \| T^{n+1} \| \| T^{n-1} \| .$$

We will now prove equality by induction. Clearly, it is true for $n = 1$, so let's assume that

$$\| T^k \|=\| T \|^k, \quad \text{for} \quad 1 \leqslant k \leqslant n$$

so by assumption

$$(\| T \|^n)^2 =\| T^n \|^2 \leqslant \| T^{n+1} \| \| T^{n-1} \| = \| T^{n+1} \| \| T \|^{n-1}$$

from where

$$\| T \|^{n+1} \leqslant \| T^{n+1} \|$$

we have in otherwise

$$\|T^{n+1}\| \leqslant \|T\|^{n+1}.$$

So for all hyponormal operator T

$$\| T^{n+1} \| = \| T \|^{n+1}$$

So, for each positive integer number n, $\| T^n \| = \| T \|^n$. □

1.2.4 Compact linear operators

Definition 1.18. *Let T be a linear operator, if $\overline{T(B_H)}$ is compact. Then T is said to be compact linear operator(or compeletly continuous), set of the compact operators is denoted by $\mathcal{K}(H)$.*

Example 1.6. .

- *The identity operator on H is compact if and only H is finite dimensional, (using Ries theorem for proof).*
- *Let $H = L^2(a, b)$ and $Tf \mapsto \int_a^b k(x,t)f(t)dt$, then T is a compact operator.*

Theorem 1.13. *Let $\{T_n\}_{n\geqslant 1}$ be a sequence of compact operators of $\mathcal{B}(H)$, such that*

$$\lim_{n\to\infty} \|T_n - T\| = 0.$$

Then T is a compact operator.
Or in other words, we can say that $\mathcal{K}(H)$ is complete subspace of $\mathcal{B}(H)$.

Theorem 1.14. *Let $S, T \in \mathcal{B}(H)$, and S be compact, then Both ST and TS are compact.*

Proof. We have

$$ST(B_H) = S(T(B_H)).$$

Since B_H is bounded and T is a bounded operator, it follows $T(B_H)$ is bounded subset. S being compact implies $\overline{S(T(B_H))}$ is compact.
Consider now $TS(B_H) = S(T(B_H))$. By compactness of S. $\overline{S(B_H)}$ is compact. Since T is continuous(bounded), So $T(\overline{S(B_H)})$ is compact. We know that $T(S(B_H)) \subset T(\overline{S(B_H)})$. □

Lemma 1.6. *The adjoint of a compact operator is compact.*

Proof. Let $\{x_n\}_{n\geq 1}$ be a sequence in H such that $\|x_n\| \leq M, n = 1, 2, \ldots$ and $M > 0$ If $y_n = T^* x_n, n = 1, 2, \ldots$, then $\{y_n\}_{n\geq 1}$ is also a bounded sequence in H. since T is compact, the sequence $\{Ty_n\}_{n\geq 1}$ has a convergent subsequence $\left\{Ty_{n_j}\right\}_{j\geq 1}$, say. For all i, j

$$\begin{aligned}
\left\|y_{n_i} - y_{n_j}\right\|^2 &= \left\|T^* x_{n_i} - T^* x_{n_j}\right\|^2 \\
&= \langle T^* x_{n_i} - T^* x_{n_j}, T^* x_{n_i} - T^* x_{n_j} \rangle \\
&= \langle TT^* \left(x_{n_i} - x_{n_j}\right), \left(x_{n_i} - x_{n_j}\right) \rangle \\
&\leq \left\|TT^* \left(x_{n_i} - x_{n_j}\right)\right\| \left\|x_{n_i} - x_{n_j}\right\| \\
&\leq 2M \left\|Ty_{n_i} - Ty_{n_j}\right\|.
\end{aligned}$$

This implies that the sequence $\left\{y_{n_j}\right\}_{j>1}$ is a Cauchy sequence in H, and since H is complete, it converges in H. Consequently, T^* is a compact operator. □

Theorem 1.15. *Let $T \in \mathcal{B}(H)$ be a compact operator and $\lambda \neq 0$, then we have*

- $N(T - \lambda I)$ *is finite dimentional and* $\dim(N(T - \lambda I)) = \dim\left(N\left(T^* - \bar{\lambda} I\right)\right)$.
- $R(T - \lambda I)$*is closed subspace in H.*
- $R(T - \lambda I) = \overline{R(T - \lambda I)} = N\left(T^* - \bar{\lambda} I\right)^{\perp}$.
- $H = R(T - \lambda I) \Longleftrightarrow N\left(T^* - \bar{\lambda} I\right) = \{0\} \Longleftrightarrow N(T - \lambda I) = \{0\} \Longleftrightarrow R\left(T^* - \bar{\lambda} I\right) = H$.

1.3 Unbouded operators

Definition 1.19. *A linear mapping is defined on a subspace $D \subset H$ into H is said to be unbounded operator denoted by $(T, D(T))$*

$$\begin{aligned} T : D(T) &\longrightarrow H \\ x &\longmapsto Tx. \end{aligned}$$

Definition 1.20. *An unbounded operator $(T, D(T))$ is said to be densely defined if $\overline{D(T)} = H$*

Example 1.7. *we introduce these usuelly examples, Let $I \subset \mathbb{R}$.*

- *Let*
$$\begin{aligned} T : L^2(I) &\longrightarrow L^2(I) \\ f &\longmapsto (Tf)(t) = tf(t). \end{aligned}$$
Then T is densely defined unbounded operator.

- *Let*
$$\begin{aligned} T : L^2(I) &\longrightarrow L^2(I) \\ f &\longmapsto T(f) = f'. \end{aligned}$$
Then T is densely defined unbounded operator.

Definition 1.21. *The operator $(T, D(T))$ is called a closed operator if and only if $G(T)$ is closed in H.*

Proposition 1.9. *The inverse of a closed injective operator is closed.*

Proof. Let $T : D(T) \subset H \to H$ be a closed injective operator, so

$$D(T^{-1}) = \{(y, x) \in H, \quad (x, y) \in D(T)\}.$$

Clearly $D(T^{-1})$ is closed . □

Definition 1.22. *Let T be an unbounded operator in H which is densly defined. The operator T^* is defined by*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \forall x \in D(T), y \in D(T^*).$$

Where

$$D(T^*) = \{y \in H : \textit{there exists } y^* \in H \quad \textit{for wich} \quad \langle Tx, y\rangle = \langle x, y^*\rangle \quad x \in D(T)\}.$$

Definition 1.23. *An operator T in H is said to be symmetric if*

$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \forall x, y \in D(T).$$

Theorem 1.16. *Let T be a linear operator in Hilbert space with domain $D(T) = H$ and satisfies $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$, then T is bounded.*

Proof. we will prove that $G(T)$ is closed, suppose that $(x_n, Tx_n) \to (x, y)$, we need only prove that $(x, y) \in G(T)$. For all $z \in H$

$$\langle z, y\rangle = \lim_{n\to\infty} \langle z, Tx_n\rangle = \lim_{n\to\infty} \langle Tz, x_n\rangle = \langle Tz, x\rangle = \langle z, Tx\rangle$$

Thus $y = Tx$, so $G(T)$ is closed, by graph theorem T is bounded. □

Theorem 1.17. *Let T be a densely defined operator in H and symmetric. Then*

- *If $D(T) = H$, then T is self -adjoint and $T \in \mathcal{B}(H)$,*
- *If T is self-adjoint and injective, then $R(T)$ is dense in H and T^{-1} is self adjoint,*
- *If $\overline{R(T)} = H$, then T is injective,*
- *If $R(T) = H$, then T is self-adjoint, $T^{-1} \in \mathcal{B}(H)$.*

Proof.
- Since T symmetric , it follows that $T \subset T^*$, from $D(T) = H$ therefore, $T = T^*$. Hence, T is closed. It therefore follows that T is continuous.
- $\overline{R(T)} \neq H$, there exists $y \perp \overline{R(T)}$. Then $x \to \langle x, Ty\rangle$ is continuous for all $x \in D(T)$; hence, $y \in D(T) = D(T^*)$ and $\langle x, Ty\rangle = \langle Tx, y\rangle = 0$ for all $x \in D(T)$. Thus, $Ty = 0$, wich implies $y = 0$, since T is assumed injective, this shows that $\overline{R(T)} = H$. T^{-1} is therefore densely defined with $D(T^{-1}) = R(T)$ and $(T^{-1})^*$ exists, it follows that $(T^{-1})^* = (T^*)^{-1} = T^{-1}$ since T is self-adjoint.

- Suppose $Tx = 0$ for $x \in D(T)$. Then $\langle Tx, y\rangle = \langle x, Ty\rangle = 0$ for every $y \in D(T)$. Thus, $x \perp R(T)$ this implies $x = 0$ since $\overline{R(T)} = H$

- Since $R(T) = H$,(c) implies T is injective and $D(T^{-1}) = H$.If $x, y \in H$, then$x = Tz$ and $y = Tw$ for some $x, w \in D(T)$ so that

$$\langle T^{-1}x, y\rangle = \langle z, Tw\rangle = \langle Tz, w\rangle = \langle x, T^{-1}y\rangle.$$

Hence T^{-1} is symmetric.

□

Theorem 1.18. *Let T be densely defined symmetric opetator in Hilbert space H. Then the following statements are equivalent*

(a) T is self adjoint,

(b) T is closed and $N(T^* \pm iI) = \{0\}$,

(c) $R(T \pm iI) = H$.

Proof.
- $(a) \Rightarrow (b)$ Supposes that T is a self-adjoint operator and there is a $y \in D(T^*) = D(T)$ such that $T^*y = -iy$. Then $Ty = iy$ and

$$i\langle y, y\rangle = \langle iy, y\rangle\langle Ty, y\rangle = \langle y, T^*y\rangle = \langle y, iy\rangle = -i\langle y, y\rangle.$$

So $y = 0$. A similar proof shows that $T^*y = -iy$ can have nonzero solution.

- $(b) \Rightarrow (c)$ Since $T^*y = -iy$ has no solutions, $R(T - iI)$ must be dense. Otherwise, if $y \in R(T - iI)^{\perp}$, we would have $\langle (T - iI)x, y\rangle = 0$ for all $x \in D(T)$, so $y \in D(T^*)$ and $(T - iI)^*y = (T^* + iI)y = 0$, wich is impossible since $(T^* + iI)y = 0$ has no solution. Reversing last argument, it can be checked that if $R(T - iI)$ is dense, then $N(T^* + iI) = \{0\}$. Since $R(T - iI) = H$. But for all $x \in D(T)$,

$$\|(T - iI)x\|^2 = \|Tx\|^2 + \|x\|^2.$$

Thus, if $x_n \in D(T)$ and $(T - iI)x_n \to y_0$, we deduce that x_n coverges to some x_0 and Tx_n converges too. Since T is closed, $x_0 \in D(T)$ and $(T - iI)x_0 = y_0$. Thus, $R(T - iI)$ is closed, so $R(T - iI) = H$. Similarly. $R(T + iI) = H$.

- $(c) \Rightarrow (a)$ Let $y \in D(T^*)$. Since $R(T - iI) = H$, there is an $x \in D(T)$ such that $(T - iI)x = (T^* - iI)y$. $D(T) \subset D(T^*)$, so $x - y \in D(T^*)$ and $(T^* - iI)(x - y) = 0$, Since $R(T + iI) = H$, $N(T - iI) = \{0\}$, so $x = y$. This proves that $D(T^*) = D(T)$, so that T is self-adjoint.

$\square$

Chapter 2

Spectral Properties of Linear Operators

2.1 Spectral Notions

Definition 2.1. *Let $T \in \mathcal{B}(H)$, we define **the spectrum** of T to be the set*

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible in } \mathcal{B}(H)\}$$

*and **the resolvent** set of T to be the set*

$$\rho(T) = \mathbb{C} \setminus \sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is invertible in } \mathcal{B}(H)\}$$

***the spectral radius** of T is defined by*

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

Example 2.1. *(a) For the identity operator $I \in \mathcal{B}(H)$, $\sigma(I) = \{1\}, \rho(T) = \mathbb{C} \backslash \{1\}$ and $r(I) = 1$.*

(b) For an $n \times n$ matrix T, $\lambda I - T$ is not invertible if and only if $det(\lambda I - T) = 0$, so

$$\sigma(T) = \{\lambda \in \mathbb{C}, \quad det(\lambda I - T) = 0\}.$$

Definition 2.2. .

*(1) **The point spectrum** (eigenspectrum,eigenvalue) of $T \in \mathcal{B}(H)$ is defined to be the set*

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : N(\lambda I - T) \neq \{0\}\}.$$

*(2) **The continuous spectrum** $\sigma_c(T)$ is the set*

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : N(\lambda I - T) = \{0\} \text{ and } \overline{R(\lambda I - T)} = H \text{ but } (\lambda I - T)^{-1} \text{ is not bounded } \}.$$

*(3) **The residual spectrum** $\sigma_r(T)$ is the set:*

$$\sigma_r(T) = \{\lambda \in \mathbb{C} : \quad N(\lambda I - T) = \{0\} \text{ and } \overline{R(\lambda I - T)} \neq H\}.$$

Corollary 2.1. *For all $T \in \mathcal{B}(H)$, we have*

i) $\sigma(T) \neq \emptyset$

ii) If H is a finite-dimensional Hilbert space.Then we have $\sigma(T) = \sigma_p(T)$

iii)

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

where $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ are mutually dijoint parts of $\sigma(T)$.

we presente the following example, let T be a bounded operator(matix) on $\mathbb{R}^2$ as

$$T = \begin{pmatrix} 0 & -5 \\ 1 & 0 \end{pmatrix}.$$

We have by corollary 2.1

$$\begin{aligned} \sigma(T) &= \{\lambda \in \mathbb{R}, \quad det(\lambda I - T) = 0\} \\ &= \{\lambda \in \mathbb{R} \quad \lambda^2 + 5 = 0\} = \emptyset. \end{aligned}$$

So we have the following remark

Remark 2.1. *If H is a real Hilbert space, can be $\sigma(T) = \emptyset$*

we have another useful division of the spectrums as

Definition 2.3. *Let $T \in \mathcal{B}(H)$ be a linear operator, then*

(a) ***the approximate point spectrum*** *is defined to be the set:*

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \quad (\lambda I - T) \textit{ is not bounded from below}\}.$$

(b) ***the compression spectrum*** *is defined to be the set :*

$$\sigma_{\text{com}}(T) = \{\lambda \in \mathbb{C} : \overline{R(\lambda I - T)} \neq H\}.$$

Proposition 2.1. *Let $T \in \mathcal{B}(H)$ be a linear operator. Then*

(a) $\sigma_p(T) \subset \sigma_{ap}(T)$.

(b) $\sigma_r(T) = \sigma_{com}(T) \setminus \sigma_p(T)$.

(c) $\sigma_p(T) = \overline{\sigma_{com}(T^*)}$.

(d) $\sigma(T) = \sigma_{ap}(T) \cup \overline{\sigma_{ap}(T^*)}$.

Proof. (a) Let $\lambda \in \sigma_p(T)$, so there is a vector $x \in H$ such that $Tx = \lambda x$ and $x \neq 0$, it sufficies taking $x_n = \dfrac{x}{\|x\|}$ for all $n \in \mathbb{N}$, then $\lambda \in \sigma_{ap}(T)$.

(b) If $\lambda \in \sigma_r(T)$, $R(\lambda I - T)$ isn't dense in H, then $\lambda \in \sigma_{com}(T)$, clearly $\lambda \notin \sigma_p(T)$.

(c) By lemma 1.4, we have $N(\lambda I - T) = R(\bar{\lambda} I - T^*)^{\perp}$, then $N(\lambda I - T) \neq \{0\}$, if and only if, $R(\bar{\lambda} I - T^*)^{\perp} \neq \{0\}$, it means that $\lambda \in \sigma_p(T)$ if and only if $\overline{\lambda} \in \sigma_{com}(T^*)$.

(d) The operator $\lambda I - T$ isn't invertible if and only if either $\lambda I - T$ isn't bounded from below or $R(\lambda I - T)$ isn't dense in H, by theorem 1.5 we can say the operator $\lambda I - T$ isn't invertible if and only if either $\lambda I - T$ isn't bounded from below or $\bar{\lambda} I - T^*$ isn't bounded below, in other words $\lambda \in \sigma(T)$ if and only if either $\lambda \in \sigma_{ap}$ or $\bar{\lambda} \in \sigma_{ap}(T^*)$. This means

$$\sigma(T) = \sigma_{ap}(T) \cup \overline{\sigma_{ap}(T^*)}.$$

□

2.1.1 Resolvent equation and spectral radius

Theorem 2.1. *(The resolvent equation)*
For $\lambda_1, \lambda_2 \in \rho(T)$

$$R(\lambda_1, T) - R(\lambda_2, T) = -(\lambda_1 - \lambda_2) R(\lambda_1, T) R(\lambda_2, T)$$

where $R(\lambda, T) = (\lambda I - T)^{-1}$

Proof. We have

$$\begin{aligned} R(\lambda_1, T) - R(\lambda_2, T) &= (\lambda_1 I - T)^{-1} - (\lambda_2 I - T)^{-1} \\ &= (\lambda_1 I - T)^{-1}[(\lambda_2 I - T) - (\lambda_1 I - T)](\lambda_2 I - T)^{-1} \\ &= -(\lambda_1 - \lambda_2) R(\lambda_1, T) R(\lambda_2, T). \end{aligned}$$

□

Remark 2.2. *The familly $\{R(\lambda, T) : \lambda \in \rho(T)\}$ is commuttig with decomposition operatation.*

Theorem 2.2. *Let $T \in \mathcal{B}(H)$, if $\lambda > \|T\|$. Then $\lambda \in \rho(T)$ and*

$$R(\lambda, T) = (\lambda I - T)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} T^n.$$

Proof. If $|\lambda| > \|T\|$, then $\|\lambda^{-1}T\| < 1$, using proposition 1.3. □

Lemma 2.1. *Let $T \in \mathcal{B}(H)$, then we have $r(T) \leqslant \|T\|$.*

Theorem 2.3. *Let $T \in \mathcal{B}(H)$, then the map $\lambda \to \langle R(\lambda I - T)x, y\rangle$ is a holomorphic function on $\rho(T)$. for all $x, y \in H$.*

Proof. By theorem 2.1 we have

$$R(\lambda_1, T) - R(\lambda_2, T) = -(\lambda_1 - \lambda_2) R(\lambda_1, T) R(\lambda_2, T).$$

So

$$\frac{R(\lambda_1, T) - R(\lambda_2, T)}{(\lambda_1 - \lambda_2)} = -R(\lambda_1, T) R(\lambda_2, T).$$

from this relation, we deduce that the map is holomorphic and its derivative $-R(\lambda I - T)^2$. □

Theorem 2.4. *(Gelfond's formula)*
For any $T \in \mathcal{B}(H)$, the following limit exists

$$\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$$

and equals $r(T)$.

For prove this theorem, we need the following corollary

Corollary 2.2. *Let $A, B, C \in \mathcal{B}(H)$ and they satisfies*

$$AB = C \quad \text{and} \quad BA = C$$

where C is invertible. Then both A and B are invertible.

Proof.

- **Firstly**, we show that this limit $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ exists.

  Let

  $$a_n = \log \|T^n\|$$

  . we want to show that $\frac{a_n}{n}$ converges,since $\|T^{n+m}\| \leqslant \|T^n\|\|T^m\|$, we have $a_n \leqslant na_1$ and

  $$a_{m+n} \leqslant a_n + a_m$$

  we write $n = pm + q$ where $0 \leqslant q \leqslant m$. It follows that

  $$\frac{a_n}{n} \leqslant \frac{pa_m}{n} + \frac{a_q}{n}$$

  with m fixed we have $\lim_{n\to\infty} \frac{p}{n} = \frac{1}{m}$, so

  $$\limsup_{n\to\infty} \frac{a_n}{n} \leqslant \liminf_{m\to\infty} \frac{a_m}{m}$$

  which implies that $\frac{a_n}{n}$ is converges.

- **Secondly** We show that $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \geqslant r(T)$ Let $\lambda > \|T^m\|^{\frac{1}{m}}$, so $\lambda^m > \|T^m\|$, it follows that $\lambda \in \rho(T^m)$, since

  $$\begin{aligned} T^m - \lambda^m I &= (T - \lambda I)\left(T^{m-1} + \lambda T^{m-2} + \cdots + \lambda^{m-1} I\right) \\ &= \left(T^{m-1} + \lambda T^{m-2} + \cdots + \lambda^{m-1} I\right)(T - \lambda I) \end{aligned}$$

  By corollary 2.2 $\lambda I - T$ is invertible, it means that $\lambda \in \rho(T)$ we conclude that for all $\lambda > \|T^m\|^{\frac{1}{m}}$ then $\lambda \in \rho(T)$, it follows that $r(T) \leqslant \|T^m\|^{\frac{1}{m}}$ therefore

  $$r(T) \leqslant \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}.$$

- **Thirdly** we show that $r(T) \geqslant \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$.

  If $\lambda \geqslant \|T\|$, then

  $$R(\lambda, T) = \sum_{n=0}^{\infty} \lambda^{-n-1} T^n$$

  which converges in the operator norm. since $|\lambda| > \|T\| \geq r(T)$, by uniqueness of Laurent series, it follows that

  $$R(\lambda, T) = \sum_{n=0}^{\infty} \lambda^{-n-1} T^n \quad \text{if } |\lambda| > r(T)$$

Hence,

$$\lim_{n} \left\|\lambda^{-n-1}T^{n}\right\| = 0 \quad \text{if } |\lambda| > r(T)$$

and so, for any $\varepsilon > 0$, we must have

$$\begin{aligned} \|T^n\| &\leq \varepsilon|\lambda|^{n+1} \quad \text{for large } n \text{ and } |\lambda| > r(T) \\ &\leq (\varepsilon + |\lambda|)^{n+1} \end{aligned}$$

which implies

$$\|T^n\|^{\frac{1}{n}} \leq (\varepsilon + |\lambda|)^{1+\frac{1}{n}} \quad \text{for large } n \text{ and } |\lambda| > r(T)$$

and hence,

$$\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq |\lambda| \quad \text{for } |\lambda| > r(T)$$

Consequently,

$$\lim_{n\to\infty} ||T^n||^{\frac{1}{n}} \leq r(T)$$

Finally

$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}.$$

□

Remark 2.3. *If $T \in \mathcal{B}(H)$*

(i) If T is a normal operator. Then $r(T) = \|T\|$.

(ii) $\sigma(T) = \{0\}$ if and only if $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = 0$.

Corollary 2.3. *Let $T, S \in \mathcal{B}(H)$ be a linear operators, such that $ST = TS$, then*

i) $r(ST) \leqslant r(S)r(T)$,

ii) $r(S + T) \leqslant r(S) + r(T)$.

2.1.2 Spectral Mapping theorem

Let $T \in \mathcal{B}(H)$, for every polynomial $p(z) = \displaystyle\sum_{k=0}^{n} a_k z^k$, we can associate the operator $p(T) \in \mathcal{B}(H)$ defined by $\displaystyle\sum_{k=0}^{n} a_k T^k$. With $f(z) = \bar{z}$ and $f(z) = z^{-1}$, we can associate the operators $f(T) = T^*$ and $f(T) = T^{-1}$.

We let the reader see the following theorem.

Theorem 2.5. *Let $T \in \mathcal{B}(H)$, then*

(a) $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$

(b) if T invertible. Then $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$

(c) if $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial with complex coefficients and if $p(T)$ is defined by $\sum_{k=0}^{n} a_k T^k$, then $\sigma(p(T)) = \{p(\lambda) : \lambda \in \sigma(T)\} = p(\sigma(T))$.

Proof. (a) Suppose $\lambda \notin \sigma(T)$. Then, $(\lambda I - T)^{-1}$ exists, so that $\left(\bar{\lambda} I - T^*\right)^{-1} = [(\lambda I - T)^*]^{-1} = [(\lambda I - T)^{-1}]^*$ exists , thus, $\bar{\lambda} \notin \sigma(T^*)$. We have thus proved $\sigma(T^*) \subseteq \{\bar{\lambda} : \lambda \in \sigma(T)\}$. Applying this argument to T^*, we get $\sigma(T) \subseteq \left\{\bar{\lambda} : \lambda \in \sigma(T^*)\right\}$, that is, $\{\lambda : \lambda \in \sigma(T)\} \subseteq \{\bar{\lambda} : \lambda \in \sigma(T^*)\}$. Taking conjugates, we get $\{\bar{\lambda} : \lambda \in \sigma(T)\} \subseteq \{\lambda : \lambda \in \sigma(T^*)\} = \sigma(T^*)$, so that $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$

(b) If T is invertible, then $0 \notin \sigma(T)$, so that $\{\lambda^{-1} : \lambda \in \sigma(T)\}$ is well defined. If $\lambda \notin \sigma(T)$ and $\lambda \neq 0$, then the equation

$$\left(\lambda^{-1} I - T^{-1}\right) = \lambda^{-1} T^{-1}(T - \lambda I) = -\lambda^{-1} T^{-1}(\lambda I - T)$$

shows $\lambda^{-1} \notin \sigma(T^{-1})$, for if $\lambda^{-1} \in \sigma(T^{-1})$, then either $\lambda \in \sigma(T)$ or $\lambda = 0$. In other words, $\sigma(T^{-1}) \subseteq \{\lambda^{-1} : \lambda \in \sigma(T)\}$. To prove the reverse inclusion, we apply the result to T^{-1}. Thus,

$$\sigma\left(T^{-1}\right) = \left\{\lambda^{-1} : \lambda \in \sigma(T)\right\}$$

(c) When p is the zero polynomial or has degree 1, this is obvious. Let $\lambda \in \sigma(T)$ and p be a polynomial of degree $n > 1$. Then, $p(z) - p(\lambda)$ is a polynomial of degree n with λ as a root and we can factor $p(z) - p(\lambda)$ as $(z - \lambda)q(z)$, where q is a polynomial of degree $n - 1$. Then,

$$p(T) - p(\lambda)I = (T - \lambda I)q(T) = B, \text{ say}$$

If B where invertible, then the equation $BB^{-1} = B^{-1}B = I$ can be written as

$$(T - \lambda I)q(T)B^{-1} = B^{-1}q(T)(T - \lambda I) = I.$$

This would mean $T - \lambda I$ is invertible, which is not possible if $\lambda \in \sigma(T)$. Thus, B is not invertible, i.e. $p(\lambda) \in \sigma(p(T))$. So, $p(\sigma(T)) \subseteq \sigma(p(T))$ Let $\lambda \in \sigma(p(T))$. Factorise the polynomial $p(z) - \lambda$ into linear factors and write

$$p(T) - \lambda I = c\,(T - \lambda_1 I) \cdots (T - \lambda_n I)$$

since $p(T) - \lambda I$ is not invertible, one of the factors $T - \lambda_j I$ is not invertible. Thus, $\lambda_j \in \sigma(T)$, and also, $p(\lambda_j) - \lambda = 0$. This shows that $\lambda = p(\lambda_j)$ for some $\lambda_j \in \sigma(T)$, Hence $\sigma(p(T)) \subseteq p(\sigma(T))$.

□

Theorem 2.6. *Let T be a linear operator, then $\sigma(T)$ is a compact subset of complex plane and $\sigma(T) \subset \overline{B_{\mathbb{C}}(0, \|T\|)}$.*

Proof. Let $\lambda_0 \notin \sigma(T)$, so that $\lambda_0 I - T$ is invertible, then

$$\begin{aligned} \|I - (\lambda_0 I - T)^{-1}(\lambda I - T)\| &= \|(\lambda_0 I - T)^{-1}\{(\lambda_0 I - T) - (\lambda I - T)\}\| \\ &= \|(\lambda I - T)^{-1}\||\lambda - \lambda_0| \end{aligned}$$

since $\lambda_0 \neq \lambda$ so there is an $\varepsilon > 0$ such that $|\lambda - \lambda_0| < \varepsilon$ and $\|I - (\lambda_0 I - T)^{-1}(\lambda I - T)\| < 1$, it follows $(\lambda_0 I - T)^{-1}(\lambda I - T)$ is invertible and $\lambda I - T$ is also invertible whenever $|\lambda - \lambda_0| < \varepsilon$ by remark 1.1, this yields that $B(\lambda_0, \varepsilon) \subset \rho(T)$, so $\rho(T)$ is open, it means that $\sigma(T)$ is cloesed. If $|\lambda| > \|T\|$ then $\frac{\|T\|}{|\lambda|} < 1$ and therefore $I - \frac{T}{\lambda}$ is invertible. It follows that $\lambda \notin \rho(T)$, and hence contraposition asserts that $\lambda \in \sigma(T)$, then $\lambda \leqslant \|T\|$. □

Proposition 2.2. *Let $T \in \mathcal{B}(H)$, then $\sigma_{ap}(T)$ is a closed subset of $\mathbb{C}$.*

Proof. Let $\lambda \notin \sigma_{ap}(T)$. Then $\lambda I - T$ is bounded from below. So

$$\exists \varepsilon > 0 \quad \|(\lambda I - T)x\| \geqslant \varepsilon \|x\|$$

and we have for all λ_1,

$$\|(\lambda I - T)x\| \leqslant \|(\lambda_1 I - T)x\| + \|(\lambda - \lambda_1)x\| \quad \forall x \in H.$$

It follows that

$$(\varepsilon - |\lambda - \lambda_1|)\|x\| \leqslant \|(\lambda_1 I - T)x\| \quad \forall\lambda_1 \quad \forall x \in H.$$

For $|\lambda - \lambda_1|$ sufficietly small, the preceding impllies is bounded from below . Hence, the complement of $\sigma_{ap}(T)$ is open. □

Theorem 2.7. *If $T \in \mathcal{B}(H)$, then $\partial\sigma(T) \subset \sigma_{ap}(T)$.*

Example 2.2. *Let T be an operator is defined on ℓ^2 as*

$$T((x_n)_{n\geqslant 1}) = (0, x_1, x_2, ...) \quad \textit{for all } (x_n)_{n\geqslant 1} \in \ell^2$$

from example 1.3 we have its adjoint is defined as:

$$T^*((x_n)_{n\geqslant 1}) = (x_2, x_3, x_4, ...) \quad \textit{for all } (x_n)_{n\geqslant 1} \in \ell^2$$

since $\|T^*\| = \|T\|$, it follows that $\|T^*\| = 1$ Consequently

$$\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \quad \textit{and} \quad \sigma(T^*) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

(1) Suppose $|\lambda| < 1$. The vector $x_\lambda = (1, \lambda, \lambda^2, \ldots)$ is in ℓ^2 and satisfies $(\lambda I - T^*) x_\lambda = 0$. Thus, all such λ are in the point spectrum of T^*. Thus, $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subseteq \sigma_p(T^*)$ since the spectrum of an operator is a bounded closed subset of $\mathbb{C}$, it follows that $\sigma(T^*) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. So we deduce that $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

(2) From Theorem 2.7, $\partial\sigma(T^*) \subseteq \sigma_{ap}(T^*)$. since $\sigma_p(T^*) \subseteq \sigma_{ap}(T^*)$ by definition, we have

$$\begin{aligned}
\sigma(T^*) &= \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \\
&= \{\lambda \in \mathbb{C} : |\lambda| < 1\} \cup \{\lambda \in \mathbb{C} : |\lambda| = 1\} \\
&= \{\lambda \in \mathbb{C} : |\lambda| < 1\} \cup \partial\sigma(T^*) \\
&\subseteq \sigma_p(T^*) \cup \sigma_{ap}(T^*) \\
&= \sigma_{ap}(T^*) \subseteq \sigma(T^*)
\end{aligned}$$

where we have used (1) for the first inclusion. Thus, we have shown that $\sigma(T^*) = \sigma_{ap}(T^*)$.

(3) It may be remarked that no λ satisfying $|\lambda| = 1$ is in $\sigma_p(T^*)$. Indeed, if $x = \{x_n\}_{n\geq 1}, x \neq 0$, is such that $T^*x = \lambda x$, then $(x_2, x_3, \ldots) = (\lambda x_1, \lambda x_2 \ldots)$, which implies $x_{n+1} = \lambda x_n$ for $n \geq 1$. So, $x_{n+1} = \lambda^n x_1, n \geq 1$. Hence, $x_1(1, \lambda, \lambda^2, \ldots) \in \ell^2$. Since $|\lambda| = 1$, the vector $x_1(1, \lambda, \lambda^2, \ldots) \in \ell^2$ if and only if $x_1 = 0$, which implies $x = 0$, a contradiction.

Now we consider the spectrum of T

(1) $\sigma_p(T) = \emptyset$. Indeed, if $\{x_n\}_{n\geq 1} \in \ell^2$ and $(\lambda I - T)\left((x_n)_{n\geqslant 1}\right) = 0, \lambda \neq 0$, then

$$0 = \lambda x_1, \quad x_1 = \lambda x_2, \quad x_2 = \lambda x_3, \ldots \Longrightarrow x_1 = 0 \quad, x_2 = 0, \ldots$$

(2) $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. If $|\lambda| < 1$, and $x \in \ell^2$, then

$$\|(T - \lambda I)x\| \geq \|Tx\| - |\lambda|\|x\| \geq |(1 - |\lambda|)\|x\| \quad \textit{because} \quad \|Tx\| = \|x\|$$

which implies $\lambda \notin \sigma_{ap}(T)$. Consequently, $\sigma_{ap}(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. It follows that $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ by theorem 2.7

(3) we have $\sigma_{\text{com}}(T) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$,by Proposition 2.1

(4)

$$\begin{aligned} \sigma_c(T) &= \sigma(T) \setminus (\sigma_{com}(T) \cup \sigma_p(T)) \\ &= \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \setminus \{\lambda \in \mathbb{C} : |\lambda| < 1\} \\ &= \{\lambda \in \mathbb{C} : |\lambda| = 1\} \textit{ since } \sigma_p(T)) = \emptyset \end{aligned}$$

(5)

$$\sigma_r(T) = \sigma(T) \setminus (\sigma_c(T) \cup \sigma_p(T)) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$$

we have also:

$$\sigma(T^*) = \sigma_{ap}(T^*) \quad \textit{and} \quad \sigma_{\text{com}}(T^*) = \overline{\sigma_p(T)} = \emptyset$$

. and

$$\sigma_p(T^*) = \{\lambda \in \mathbb{C} : |\lambda| < 1\} \quad \textit{and} \quad \sigma_c(T^*) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \quad \textit{and} \quad \sigma_r(T^*) = \emptyset$$

We summarise below the decomposition of the spectrum of T

$$\begin{aligned} \sigma(T) &= \sigma_{ap}(T) \cup \sigma_{\text{com}}(T) \\ &= \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{\lambda \in \mathbb{C} : |\lambda| < 1\} \end{aligned}$$

and

$$\begin{aligned}\sigma(T) &= \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T) \\ &= \emptyset \cup \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{\lambda \in \mathbb{C} : |\lambda| < 1\}.\end{aligned}$$

2.2 Spectrum of various classes of operators

2.2.1 Normal operator

Theorem 2.8. *Let $T \in \mathcal{B}(H)$, such that T is a normal operator. Then*

(a) $\sigma_p(T) = \overline{\sigma_p(T^*)}$,

(b) For all eigenvectors corresponding to distinct eigenvalues, if any are orthogonal,

(c) $\sigma_r(T) = \emptyset$.

Proof. (a) Since T is normal, for $\lambda \in \mathbb{C}$, $\|(\lambda I - T)x\| = \|(\lambda I - T)^* x\|$, by theorem 1.8 , for all $x \in H$. It follows that $N(\lambda I - T) \neq \{0\}$ if and only if $(\lambda I - T)^* \neq 0$, so $\sigma_p(T) = \overline{\sigma_p(T^*)}$.

(b) Let λ_1, λ_2 distinct eigenvalues of T and $x, y \in H$ corresponding eigenvectors, then $Tx = \lambda_1 x$ and $Ty = \lambda_2 y$, it follows from (a) that

$$\lambda_1 \langle x, y \rangle = \langle \lambda_1 x, y \rangle = \langle Tx, y \rangle = \langle x, T^* y \rangle = \langle x, \overline{\lambda_2} y \rangle = \lambda_2 \langle x, y \rangle.$$

Since $\lambda_1 \neq \lambda_2$, we deduce that $\langle x, y \rangle = 0$.

(c) By (b) and (c) in proposition 2.1, we have $\sigma_r(T) = \overline{\sigma_p(T^*)} \setminus \sigma_p(T)$. Since T is normal we obtain $\sigma_r(T) = \emptyset$.

□

Theorem 2.9. *Let $T \in \mathcal{B}(H)$ be a normal operator, then*

$$\sigma(T) = \sigma_{ap}(T).$$

Proof. If $T \in \mathcal{B}(H)$ be a normal operator, so using proposition 2.1 , we have

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_{com}(T) = \sigma_{ap}(T) \cup \overline{\sigma_p(T^*)}.$$

Since we have shown that $\overline{\sigma_p(T^*)} = \sigma_p(T)$, the above equality leads to

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_p(T).$$

From which we get $\sigma(T) = \sigma_{ap}(T)$, since $\sigma_p(T) \subset \sigma_{ap}(T)$, by proposition 2.1. □

2.2.2 Self-adjoint operator

Theorem 2.10. *Let $T \in \mathcal{B}(H)$ be a self adjoint linear operator, then $\sigma(T) \subset \mathbb{R}$. In addition,if T is positive, then $\sigma(T) \subset \mathbb{R}_+$*

Proof. Let $\lambda = \lambda_1 + i\lambda_2$(where λ_1 and λ_2 are real and $\lambda_2 \neq 0$) be a complex number. If T is self adjoint operator and $x \in H$

$$\begin{aligned}
\|(\lambda I - T)x\|^2 &= \langle(\lambda I - T)x, (\lambda I - T)x\rangle \\
&= \langle(\bar{\lambda} I - T)(\lambda I - T)x, x\rangle \\
&= |\lambda|^2\langle x, x\rangle - 2\lambda_1\langle Tx, x\rangle + \|Tx\|^2 \\
&= \|(\lambda_1 I - T)x\|^2 + \lambda_2^2\|x\|^2 \\
&\geq \lambda_2^2\|x\|^2
\end{aligned}$$

So $(\lambda I - T)$ is bounded from below. This means that $\lambda \notin \sigma_{ap}(T)$, hence $\lambda \notin \sigma(T)$, by theorem 2.9

Assmume that T is positive and $\lambda < 0$, then

$$\begin{aligned}
\|(\lambda I - T)x\|^2 &= \langle(\lambda I - T)x, (\lambda I - T)x\rangle \\
&= \lambda^2\langle x, x\rangle - 2\lambda\langle Tx, x\rangle + \|Tx\|^2 \\
&\geqslant \lambda^2\|x\|^2.
\end{aligned}$$

So $\lambda I - T$ is bounded from below. This means that $\lambda \notin \sigma_{ap}(T)$, hence $\lambda \notin \sigma(T)$, by theorem 2.9. □

Theorem 2.11. *Let $T \in \mathcal{B}(H)$ be a self adjoint operator, if $m = \inf\limits_{\|x\|=1} \langle Tx, x\rangle$ and $M = \sup\limits_{\|x\|=1} \langle Tx, x\rangle$, then $\sigma(T) \subset [m, M]$ and $m, M \in \sigma(T)$.*

Proof. Let T be a self-adjoint operator and $\lambda \in \mathbb{R}$, we define that $d(\lambda)$ is the distance from λ to the interval $[m, M]$. For all $x \in H$ and $x \neq 0$ we have

$$\langle \lambda x - Tx, x\rangle = \|x\|^2(\lambda - \langle Ty, y\rangle)$$

where $y = \dfrac{x}{\|x\|}$. Then by Cauchy-Schwarz inequality we have

$$d(\lambda)\|x\|^2 \leqslant |\langle \lambda x - Tx, x\rangle| \leqslant \|x\|\|\lambda x - Tx\|$$

If $\lambda \notin [m, M]$, it means that T is bounded from below, then $\lambda \notin \sigma_{ap}(T) = \sigma(T)$, it follows that $\lambda \in \rho(T)$.

Let us show that $m \in \sigma(T)$, we put $S = T - m$, then S is positive. Now by definition of m, there is a sequence $(x_n)_{n\in\mathbb{N}}$, such that $\|x_n\| = 1$ and $\lim\limits_{n\to\infty} \langle Sx_n, x_n\rangle = 0$

$$\begin{aligned} \|Sx_n\|^2 &\leq |\langle Sx_n, x_n\rangle|^{\frac{1}{2}} \left|\langle S^2x_n, x_n\rangle\right|^{\frac{1}{2}} && \text{by cauchy- schwarz} \\ &\leq |\langle Sx_n, x_n\rangle|^{\frac{1}{2}} |\langle Sx_n, Sx_n\rangle|^{\frac{1}{2}} && \text{since S is self adjoint} \\ &\leq |\langle Sx_n, x_n\rangle|^{\frac{1}{2}} \|S\|^{\frac{1}{2}} \|Sx_n\| . \end{aligned}$$

Therefore

$$\|Sx_n\|^4 \leqslant \langle Sx_n, x_n\rangle \|S\|^3$$

hence tends to 0, If $m \notin \sigma(T)$, S is invertible and hence x_n tends into 0 which is impossible. We use this method for prove that $M \in \sigma(T)$, but we take $S = M - T$. we will obtain $M \in \sigma(T)$. □

Corollary 2.4. *Let $T \in \mathcal{B}(H)$ be a self adjoint operator such that $\sigma(T) = \{0\}$, then $T = 0$.*

2.2.3 Compact operator

Theorem 2.12. *(spectral theorem of compact operators) For every compact operator T. Then $0 \in \sigma(T)$.*

Proof. If $0 \notin \sigma(T)$, Then T^{-1} exists and is bounded, by theorem 1.14, it follows that $I = T^{-1}T$ is a compact operator. This contradicts with example 1.6. □

Proposition 2.3. *If T is a compact operator and λ is a nonzero scalar. Then $N(T - \lambda I)$ is finite dimensional.*

Proof. Suppose, for $\lambda \neq 0$, $N(T - \lambda I)$ is infinite- dimesional. Then, we can select an infinite collection of linearly independent vectors $x_1, x_2, ...$ from $(T - \lambda I)$. Moreover, in view of the gram-Schmidt orthonormalisation process, this set of vectors can be assumed to be an orthonormal set. For $n \neq m$

$$\|Tx_n - Tx_m\|^2 = \|\lambda x_n - \lambda x_m\|^2 = |\lambda|^2 \|x_n - x_m\| = 2|\lambda|^2$$

wich means that no subsequence of $\{Tx_n\}_{n\geqslant 1}$ can possibly be a Cauchy sequence. This precludes the possibility of T being compact and proves the proposition. □

Theorem 2.13. *Let T be a compact operator on a Hilbert space H and $\lambda \in \sigma_{ap}(T) \setminus \{0\}$. Then, λ is an eigenvalue for T.*

Proof. By assumption, there exists a sequence $\{x_n\}_{n\geqslant 1}$ of vectors such that $\|x_n\| = 1$ for every n and $\|Tx_n - \lambda x_n\| \to 0$ as $n \to \infty$. Since T is compact. we may suppose, after passing to a subsequence, that Tx_n is convergent, say $Tx_n \to y$ as $n \to \infty$. Since

$$\|y - \lambda x_n\| \leqslant \|y - Tx_n\| + \|Tx_n - \lambda x_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

It follows that $\lambda x_n \to y$. Then $T(\lambda x_n) = \lambda T x_n \to Ty$. Thus

$$Ty = \lim_{n\to\infty} T(\lambda x_n) = \lambda y.$$

Also

$$\|y\| = \|\lim_n \lambda x_n\| = \lim_{n\to\infty} |\lambda| \|x_n\| = |\lambda| > 0.$$

Hence, λ is an eigenvalue for T, and y is the corresponding eigenvector. □

2.2.4 Compact Normal operator

Theorem 2.14. *If T is a normal compact operator. Then*

(a) $\sigma_p(T) \setminus \{0\} = \sigma(T) \setminus \{0\}$, *in other words* $\sigma_p(T) \cup \{0\} = \sigma(T)$

(b) $\sigma_p(T) \neq \emptyset$ *and there is some* $\lambda \in \sigma_p(T)$ *such that* $|\lambda| = \|T\| = r(T)$. *Obviouisly,* $\lambda \in \partial\sigma(T)$.

Proof. (a) Since T is normal, we know that Theorem 2.9

$$\sigma_{ap}(T) = \sigma(T).$$

So that

$$\sigma_{ap}(T) \setminus \{0\} = \sigma(T) \setminus \{0\}$$

By Theorem 2.13

$$\sigma_{ap}(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$$

The two equalities above imply that

$$\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$$

(b) If T is nonzero, then $\|T\| > 0$, and since T is normal, $r(T) = \|T\|$,(by remark 2.3), which implies $\sigma(T) \setminus \{0\} \neq \emptyset$. Also, there must be some λ such that $|\lambda| = \|T\|$. Since $\lambda \neq 0$, then $\lambda \in \sigma_p(T)$.

□

Theorem 2.15. *Let $T \in \mathcal{B}(H)$ be a compact normal operator on a complex Hilbert space H. Then $\sigma_p(T)$ is at most countable (it could be empty) and 0 is its only possible limit point.*

Proof. We assume that for $\varepsilon > 0$ there exists an infinite sequence $\{\lambda_k\}_{k\geq 1}$ of distinct proper values of T such that $\varepsilon \leq |\lambda_k| \leq \|T\|$. Passing to a subsequence, we may assume that

$$\lambda_k \to \lambda, \quad \text{where} \quad \varepsilon \leq |\lambda| \leq \|T\|.$$

Let $Tx_k = \lambda_k x_k, \|x_k\| = 1$. since T is compact, $\{Tx_k\}$ has a convergent subsequence. We denote this subsequence by $\{Tx_k\}_{k\geq 1}$ and assume that $Tx_k \to y$ for suitable $y \in H$. Thus, $\lambda_k x_k \to y$, which implies $x_k \to \lambda^{-1}y$. Hence,

$$\|x_n - x_m\| = \|\lambda_n^{-1}y - \lambda_m^{-1}y\| \to 0 \quad \text{as} \quad m, n \to \infty.$$

But $\{x_n\}_{n\geq 1}$ is an orthonormal theorem 2.8 sequence; hence, $\|x_n - x_m\|^2 = 2$ whenever $m \neq n$, This contradiction. □

Corollary 2.5. *If $T \in \mathcal{B}(H)$ is compact and normal, then $\sigma(T)$ is at most countable.*

Theorem 2.16. *Let T be a positive compact operator on H and let λ_k be its eigenvalues, in decreasing order: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0$. We have $\lambda_1 = \sup_{v\in H, v\neq 0} \frac{\langle v,Tv\rangle}{\|v\|^2}$ More generally, we have*

$$\lambda_k = \inf_{u_1,\ldots,u_{k-1}\in H} \sup_{\substack{v\in H, u\neq 0 \\ v\perp Vect(u_1,\ldots,u_{k-1})}} \frac{\langle v, Tv\rangle}{\|v\|^2}.$$

2.3 Numerical Range of an Operator

Definition 2.4. *The numerical range $W(T)$ of an operator $T \in \mathcal{B}(H)$ is defined by :*

$$W(T) = \{\langle Tx, x\rangle, \quad \|x\| = 1\}.$$

Theorem 2.17. *The numerical range $W(T)$ of an operator T is a convex set of $\mathbb{C}$.*

Example 2.3. *Let T is a matrix as :*

$$T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$W(T) = \{z \in \mathbb{C} \quad |z| < \frac{1}{2}\}$

Theorem 2.18. *Let H be a Hilbert space over $\mathbb{C}$ and $T \in \mathcal{B}(H)$. Then, $W(T) = \{(Tx, x) : \|x\| = 1\}$ has the following properties:*

- $\lambda \in W(T)$ *if and only if* $\bar{\lambda} \in W(T^*)$
- $W(T)$ *is a convex subset of* $\mathbb{C}$ (c) $\quad \sigma(T) \subseteq \overline{W(T)}$
- *if T is normal, then the convex hull of the spectrum $\sigma(T)$ of T* $\cot(\sigma(T)) = \overline{W(T)}$

2.4 Extended Spectrum

Definition 2.5. *Let $T \in \mathcal{B}(H)$, we define* ***the extented spectrum*** *of T to be the set*

$$\sigma_{ext}(T) = \{\lambda \in \mathbb{C}, \exists X \in \mathcal{B}(H) \setminus \{0\} \text{ such that } TX = \lambda XT\}.$$

Remark 2.4. *We have $\sigma_{ext}(T) \neq \emptyset$, it suffices taking $X = I$, we get $1 \in \sigma_{ext}(T)$*

Proposition 2.4. *Let $T \in \mathcal{B}(H)$, then $\sigma_{ext}(T) \subset \{\lambda \in \mathbb{C}, \quad \sigma(T) \cap \sigma(\lambda T) \neq \emptyset\}$, in special case, if $\sigma(T) = \{a\}$ with $a \neq 0$, then $\sigma_{ext}(T) = \{1\}$.*

By this proposition, we deduce the following corollary:

Corollary 2.6. *Let $T \in \mathcal{B}(H)$, we assume that T is quasinilpotent, then*

$$\sigma_{ext}(aI + T) = \{1\} \quad \forall a \in \mathbb{C} \setminus \{0\}.$$

2.4.1 Relation with the point spectrum

Proposition 2.5. *Let $T \in \mathcal{B}(H)$, the we have*

$$\{\alpha\bar{\beta}^{-1}; \alpha \in \sigma_p(T), \beta \in \sigma_p(T^*)\} \subset \sigma_{ext}(T).$$

Proof. Assume that $\lambda_1 \in \sigma_p(T)$, $\lambda_2 \in \sigma_p(T^*)$, then there exists $x, y \in H$ such that $Tx = \lambda_1 x$ and $T^*y = \lambda_2 y$, we define an operator as

$$\begin{aligned} X : H &\to H \\ z &\mapsto \langle z, y\rangle x. \end{aligned}$$

Let $z \in H$,

$$\begin{aligned} \bar{\lambda_2}\lambda_1^{-1}TXz &= T(\langle z, y\rangle x) \\ &= \bar{\lambda_2}\lambda_1^{-1}\langle z, y\rangle Tx \\ &= \bar{\lambda_2}\lambda_1^{-1}\langle z, y\rangle \lambda_1 x \\ &= \bar{\lambda_2}\langle z, y\rangle x \\ &= \langle z, T^*y\rangle x \\ &= \langle Tz, y\rangle x. \end{aligned}$$

□

Remark 2.5. .

- *If both* T *and* T^* *aren't injective, then* $\sigma_{ext}(T) = \mathbb{C}$.
- *If* $1 \in \sigma_p(T^*)$, *then* $\sigma_p(T) \subset \sigma_{ext}(T)$.

Theorem 2.19. *Let* $T \in \mathcal{B}(H)$, *if* H *is finite-dimentional, then*

$$\sigma_{ext}(T) = \{\lambda \in \mathbb{C}, \quad \sigma(T) \cap \sigma(\lambda T) \neq \emptyset\}.$$

Corollary 2.7. *Let* $T \in \mathcal{B}(H)$, *and* H *is finite-dimensional, then we have*

- $\sigma_{ext}(T) = \{1\}$ *if and only if* $\sigma(T) = \{\alpha\}$ *with* $\alpha \neq 0$
- $\sigma_{ext}(T) = \mathbb{C}$ *if and only if* T *isn't invertible.*
- *If* T *is invertible, then* $\sigma_{ext}(T) = \{\frac{\lambda_1}{\lambda_2}, \quad \lambda_1, \lambda_2 \in \sigma(T), \lambda_2 \neq 0\}$.

Chapter 3

Applications

3.1 Dirichlet and Neumann problem

In this section we will use some consepts and properties about PDE, for more details the reader can refer to reference [8].

Let Ω be a bounded lipschitzian domain, and let $V \in C(\Omega)$, $f \in L^2(\Omega)$, we Consider the Dirichlet problem

$$\begin{cases} -\Delta u + Vu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

A strong solution of (3.1) is a function $u \in C^2(\Omega)$ satisfying the first PDE pointwise and the Dirichlet condition on the boundary. This implies that $f \in C(\Omega)$. However, we will want to consider (3.1) for less regular f, typically for $f \in L^2(\Omega)$. We introduce definition of the weak solutions of (3.1) as :

Definition 3.1. *Let Ω be a Lipschitzian domain, let $V \in C(\Omega)$, and let $f \in L^2(\Omega)$. A weak solution of (3.1) is a function $u \in H_0^1$ which satisfies*

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega V u \varphi = \int_\Omega f \varphi, \quad \forall \varphi \in H_0^1(\Omega).$$

And we consider the Neumann problem

$$\begin{cases} -\Delta u + Vu = f & \text{in } \Omega \\ \partial_n u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

Definition 3.2. *Let Ω be a Lipschitzian domain, let $V \in C(\Omega)$, and let $f \in L^2(\Omega)$. A weak solution of (3.2) is a function $u \in H^1(\Omega)$ which satisfies :*

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega V u \varphi = \int_\Omega f \varphi, \quad \forall \varphi \in H^1(\Omega).$$

For introduce some results, we need to the following theorem in PDE.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain, $V \in L^\infty(\Omega)$. There exists $C > 0$ such that, if $f \in L^2(\Omega)$ and if u is a weak solution of (3.1) or of (3.2), then $u \in H^2(\Omega)$, and we have*

$$\|u\|_{H^2(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right).$$

Furthermore, if $f, V \in C^\infty(\bar{\Omega})$, then $u \in C^\infty(\bar{\Omega})$

3.1.1 Diagonalisation of the Dirichlet and Neumann Laplacian in a bounded domain

We have the following theorem

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitzian domain*

- *There exists a sequence $0 < \lambda_1^D \leq \lambda_2^D \leq \ldots \leq \lambda_k^D \leq \ldots$ and orthonormal basis $(\varphi_k)_{k\geqslant 1}$ of $L^2(\Omega)$ with $\varphi_k \in C^\infty(\Omega)$ satisfying*

$$\begin{cases} -\Delta\varphi_k = \lambda_k^D \varphi_k & \text{in } \Omega \\ \varphi_k = 0 & \text{on } \partial\Omega \end{cases}$$

  in the weak sense.

- *There exists a sequence $0 < \lambda_1^N \leq \lambda_2^N \leq \ldots \leq \lambda_k^N \leq \ldots$ and orthonormal basis $(\psi_k)_{k\geqslant 1}$ of $L^2(\Omega)$ with $\psi_k \in C^\infty(\Omega)$ satisfying*

$$\begin{cases} -\Delta\psi_k = \lambda_k^N \psi_k & \text{in } \Omega \\ \partial_n\psi_k = 0 & \text{on } \partial\Omega \end{cases}$$

  in the weak sense.

Proof. If $f \in L^2(\Omega)$, let Tf be the unique solution in H_0^1 to the equation $-\Delta(Tf) = f$, then the operator T is bounded, the operator $\tilde{T} : L^2(\Omega) \longrightarrow L^2(\Omega)$ is compact operator. we check that $\tilde{T}$ is self-adjoint, Let $f, g \in L^2(\Omega)$. we have

$$\begin{aligned} \int_\Omega (\tilde{T}f) \cdot g &= \int_\Omega (\nabla \tilde{T}f) \cdot \nabla \tilde{T}g \\ &= \int_\Omega (\tilde{T}g) \cdot f. \end{aligned}$$

The operator $\tilde{T}$ is self-adjoint . Therefore, by Theorem 3.1 , there exists a non-increasing sequence of numbers $(u_k)_{k\geqslant 1}$, going to zero, and an orthonormal basis $(\varphi_k)_{k\geqslant 1}$ of $L^2(\Omega)$, such that $\tilde{T}\varphi_k = u_k\varphi_k$

Now we have

$$\int_\Omega f\tilde{T}f = -\int_\Omega (\Delta\tilde{T}f)\tilde{T}f = \int_\Omega |\nabla \tilde{T}f|^2 \geq 0$$

We deduce from this that the $u_k \geqslant 0$ for all $k \geqslant 1$, Furthermore if $f \neq 0$, then $\int_\Omega f\tilde{T}f > 0$ so that the u_k is strictly positive.
From the equation $\tilde{T}\varphi_k = u_k\varphi_k$, we deduce that

$$-\Delta\varphi_k = \frac{1}{u_k}\varphi_k$$

we thus set $\lambda_k^D = \dfrac{1}{u_k}$, The functions φ_k are in $C^\infty(\Omega)$, by theorem 3.1 $\square$

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitzian domain, We have*

$$\lambda_k^D = \sup_{u_1,\ldots,u_{k-1}\in L^2(\Omega)} \inf_{v\in H_0^1(\Omega), v\neq 0} \frac{\|\nabla v\|_{L^2(\Omega)}^2}{\|v\|_{L^2(\Omega)}^2}$$
$$\lambda_k^N = \sup_{u_1,\ldots,u_{k-1}\in L^2(\Omega)} \inf_{v\in H^1(\Omega), v\neq 0, v\perp V} \frac{\|\nabla v\|_{L^2(\Omega)}^2}{\|v\|_{L^2(\Omega)}^2}.$$

where $Vect\,(u_1,\ldots,u_{k-1})$, and both λ_k^D, λ_k^N are defined as in 3.2.

3.1.2 Dependence of the eigenvalues on the domain

Lemma 3.1. *Let Ω, Ω' be Lipschitzian domains such that $\Omega \subset \Omega'$. Then for every $k \in \mathbb{N}$, we have*

$$\lambda_k^D(\Omega') \leq \lambda_k^D(\Omega)$$

Proof. using theorem 3.3 extending functions in $H_0^1(\Omega)$ to functions $H_0^1(\Omega')$ $\square$

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitzian domain, and let $\Omega_1, \Omega_2 \subset \Omega$ be Lipschitzian domains such that $\bar{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$, and $\Omega_1 \cap \Omega_2 = \emptyset$. Then for every $k \in \mathbb{N}$ we have*

$$\lambda_k^N(\Omega_1 \cup \Omega_2) \leq \lambda_k^N(\Omega)$$

Proof. With the hypotheses, we have $L^2(\Omega) = L^2(\Omega_1 \cup \Omega_2)$ and $H^1(\Omega) \subset H^1(\Omega_1 \cup \Omega_2)$, therefore we have:

$$\begin{aligned}\lambda_k^N(\Omega_1 \cup \Omega_2) &= \sup_{u_1,\ldots,u_{k-1}\in L^2(\Omega_1\cup\Omega_2)} \inf_{v\in H^1(\Omega_1\cup\Omega_2), v\neq 0} \frac{\|\nabla v\|_{L^2(\Omega)}^2}{\|v\|_{L^2(\Omega)}^2(\Omega)} \\ &\leq \sup_{u_1,\ldots,u_{k-1}\in L^2(\Omega)} \inf_{\substack{v\in H^1(\Omega), v\neq 0 \\ v\perp \mathrm{Vect}(u_1,\ldots,u_{k-1})}} \frac{\|\nabla v\|_{L^2(\Omega)}^2}{\|v\|_{L^2(\Omega)}^2} = \lambda_k^N(\Omega)\end{aligned}$$

□

and we have

$$\lambda_k^N(\Omega_1 \cup \Omega_2) \leq \lambda_k^N(\Omega) \leq \lambda_k^D(\Omega) \leq \lambda_k^D(\Omega_1 \cup \Omega_2)$$

Proposition 3.1. *Let $(\Omega_j)_{j\in\mathbb{N}}$ be an increasing sequence of Lipschitzian domains of $\mathbb{R}^n$, and suppose that $\Omega = \cup_{l\in\mathbb{N}^*}\Omega_l$ is a Lipschitzian domain. Then for any $k \in \mathbb{N}$*

$$\lambda_k^D(\Omega_j) \to \lambda_k^D(\Omega)$$

Proof. Lemma 3.1 tells us that, for any $k \in \mathbb{N}$ $\lambda_k^D(\Omega) \leqslant \lambda_k^D(\Omega_j)$, we take $k \in \mathbb{N}$ and $\varepsilon > 0$, Let show that there exists $j_0 \in \mathbb{N}$ such that for all $j \geqslant j_0$, $\lambda_k^D(\Omega_j) \leqslant \lambda_k^D(\Omega) + \varepsilon$
Let $u_1, u_2, ..., u_k \in H_0^1$ denote the mutually orthogonal

first k eigenvalues of the Dirichlet Laplacian on Ω with eigenvalues

$$\lambda_1^D(\Omega), \lambda_2^D(\Omega), ..., \lambda_k^D(\Omega)$$

and $U = Vect(u_1, u_2...u_k)$ we have so for any $u \in U$

$$\|\nabla u\|_{L^2(\omega)} \leq \lambda_k^D(\Omega)\|u\|_{L^2(\Omega)}$$

Using the density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$, we can approximate $u_1, u_2, ..., u_k$ by functions $v_1, v_2, ..., v_k \in C_c^\infty(\Omega)$, such that $v_1, v_2, ..., v_k$ are linearly independent, and $\|\nabla v\|_{L^2(\omega)} \leq \left(\lambda_k^D(\Omega) + \varepsilon\right)\|v\|_{L^2(\Omega)}$ for all $v \in Vect(v_1, v_2, ..., v_k)$, Let K be a compact set containing all the supports of the $v_1, v_2, ..., v_k$.By assumption , there exists j_0 such that for all $j \geqslant j_0$, $K \subset \Omega_j$.
Now $j \geqslant j_0$, and let $w_1, w_2, ..., w_{k-1} \in L^2(\Omega_j)$. There exists a function $v \in Vect(v_1, v_2, ..., v_k)$ which is orthogonal to $w_1, w_2, ..., w_{k-1}$ in $L^2(\Omega_j)$. Therefore we have

$$\inf_{\substack{v\in H_0^1(\Omega_j), v\neq 0\\ \mathrm{Vect}(w_1,\ldots,w_{k-1})}} \frac{\|\nabla v\|^2_{L^2(\Omega_j)}}{\|v\|^2_{L^2(\Omega_j)}} \leq \lambda_k^D(\Omega) + \varepsilon$$

and we may conclude the proof using the max-min principle. □

3.2 Extended Spectrum of Volterra Operator

Before we start to calculate of extended eigenvalues and extended eigenvectors, we presente some notations and some results that we will need.
Let $\mathcal{D} = \{f, \quad f$ is absolutely continuous and $f(0) = 0\}$, and we define an operator D on $\mathcal{D}$ as : $Df = \dfrac{df}{dx}$ for $f \in \mathcal{D}$, Clearly we can see that $DV = VD = I$ on $\mathcal{D}$ where V is Volterra operator.
For each $t \geqslant 0$ we define an operator S_t on $L^2(0,1)$ as :

$$S_t f(x) = \chi_{[t,\infty[\cap[0,1]}(x) f(x-t)$$

We can look that $\{S_t, \quad t \geqslant 0\}$ is a strongly continuous semigroup on $L^2(0,1)$.

Theorem 3.4. *The infinitesimal generator of S_t is $-D$. Moreover, for all $z \in \mathbb{C}$ $(z+D)^{-1}$ exists as a bounded operator and*

$$(z+D)^{-1} = \int_0^1 e^{zt} S_t dt$$

Lemma 3.3. *Let $\lambda > 0$ and let T be an operator on $L^2(0,1)$. Then $VT = \lambda TV$ if and only if $S_t T = T S_{t/\lambda}$, for all $t \geq 0$*

3.2.1 Determine of Extended Eigenvalues Location

We have the following theorem

Theorem 3.5. *Let $\lambda \in \mathbb{C} \backslash]0, \infty[$. Then we have*

$$VT = \lambda TV \Longrightarrow T = 0.$$

Proof. We have known that V is injective, so if $\lambda = 0$ this implication is satisfied, we assume that $\lambda \neq 0$,
Let T an operator on $L^2(0,1)$ satisfing:

$$VT = \lambda TV \tag{3.3}$$

Since $\mathcal{D}$ is the range of V , equation (3.3) implies tha $T(\mathcal{D}) \subset \mathcal{D}$, so equation (3.3) equivalent to

$$TD = \lambda DT \tag{3.4}$$

and $z \in \mathbb{C}$. Then

$$T(z+D) = (z+\lambda D)T = \lambda\left(\frac{z}{\lambda}+D\right)T$$

Since z was arbitrary, the previous equation is equivalent to

$$T(\lambda z + D) = \lambda\,(z+D)\,T.$$

Therefore $(z+D)^{-1}T = \lambda T(\lambda z + D)^{-1}$. Using theorem 3.4 we obtain

$$\int_0^1 e^{zt} S_t T dt = \lambda \int_0^1 e^{\lambda z t} T S_t dt \tag{3.5}$$

Let $u, v \in L^2(0,1)$, we define f, g as

$$f(t) = \langle S_t T u, v\rangle \quad g(t) = \lambda \langle T S_t u, v\rangle.$$

Equation(3.5) implies that

$$\int_0^1 e^{zt} f(t)dt = \int_0^1 e^{\lambda z t} g(t)dt.$$

Next, we expand both sides of this equation as power series in z and we notice that both series converge uniformly in t, for each $z \in \mathbb{C}$. Thus we can integrate the series term by term. Comparing coefficients in these series we obtain

$$\int_0^1 t^n f(t)dt = \int_0^1 \lambda^n t^n g(t)dt \quad \forall n \in \mathbb{N}.$$

It follows that, for every polynomial p, we have

$$\int_0^1 p(t)f(t)dt = \int_0^1 p(\lambda t)g(t)dt. \tag{3.6}$$

Suppose that $\lambda \in \mathbb{C}\backslash]0,\infty[$. Let $S = \{\lambda t, \quad t \in [0,1]\} \cup [0,1]$, define a function h on S as

$$\begin{cases} h(\lambda t) = \overline{g(t)} \quad \text{if} \quad t \in [0,1] \\ 0 \quad \text{if} \quad t \notin [0,1] \end{cases}$$

Then $h \in L^2(S)$. By **Mergelyanś theorem**, there exists a sequence of polynomials p_n on S converging to h in $L^2(S)$. But

$$\int_0^1 p_n(t)f(t)dt \to 0 \quad \text{and} \quad \int_0^1 p_n(\lambda t)g(t)dt \to \|g\|^2 \quad \text{as} \quad n \to \infty.$$

In view of (3.6) this proves that $g = 0$ in $L^2(0,1)$ and, since g is continuous, this means that

$$g(t) = 0, \text{ for all } t \in [0,1].$$

For $t = 0$

$$g(0) = \lambda(Tu, v) = 0.$$

Since $\lambda \neq 0$ and u, v are arbitrary in $L^2(0,1)$ it follows that $T = 0$. □

Theorem 3.6. *Let $\varphi \in L^\infty(0,1)$ and let M_φ be the operator of multiplication by φ in $\mathcal{B}(L^2(0,1))$. If $M_\varphi V = \lambda V M_\varphi$, then $\varphi = 0$ a.e in $]0,1[$.*

Proof. The equation $M_\varphi V = \lambda V M_\varphi$ implies that

$$\varphi(x)\int_0^x f(t)dt = \lambda \int_0^x \varphi(t)f(t)dt, \quad \forall f \in L^2(0,1), \quad \text{for } x \in [0,1] \tag{3.7}$$

Equivalently,

$$\int_0^x f(t)[\varphi(x) - \lambda\varphi(t)]dt = 0$$

we put $k(t) = \varphi(x) - \lambda\varphi(t)$ for all $f \in L^2(0,x)$ and for a. e. $x \in [0,1]$. Hence,

$$\int_0^1 f(t)\chi_{[0,x]}(t)k(t)dt = 0 \tag{3.8}$$

we know that $f \mapsto \int_0^1 f(t)\chi_{[0,x]}(t)k(t)dt$ is a linear functional wich vanishes on $L^2(0,x)$, by (representation riez theorem 1.6), we deduce that

$$k(t) = 0 \quad \text{a,e on } [0,x]$$

it follows that $\varphi(t) = \lambda^{-1}\varphi(x)$ a.e on$(0,x)$. Consequently, φ must be a constant function in $L^\infty(0,1)$. Furthermore, if $\varphi(x) = C$ then $C - \lambda C = 0$ for a. e. t and, therefore, $C = 0$.

□

Theorem 3.7. *Let λ be a positive number and let K be an integral operator with kernel $K(x,y)$. Then $VK = \lambda KV$ if and only if $K(x,y) = g(x-\lambda y)$, for some measurable function g on $\mathbb{R}$ such that $g = 0$ on the interval $[-\lambda, 1-\lambda]$ and also $\int_0^1 \int_0^1 |g(x-\lambda y)| dx dy < \infty$.*

Proof. Let K be an integral operator with kernel $K(x,y)$, and suppose that $VKf = \lambda KVf$, for all $f \in L^2(0,1)$. Then,

$$\int_0^x \int_0^1 K(t,y)f(y)dydt = \lambda \int_0^1 \int_0^t K(x,y)f(y)dydt \quad \forall f \in L^2(0,1) \text{ and a.e } x. \tag{3.9}$$

Changing the order of integration on both sides of (3.9) yields

$$\int_0^1 f(y)dy \left[\int_0^x K(t,y)dt\right] = \lambda \int_0^1 f(y)dy \left[\int_y^1 K(x,t)dt\right]$$

In view of the arbitrary choice of $f \in L^2(0,1)$ it follows that

$$\int_0^x K(t,y)dt = \lambda \int_y^1 K(x,t)dt \tag{3.10}$$

for a.e x and a.e y in $[0,1]$. Conversely, if a kernel $K(x,y)$ satisfies (3.10) for a. e. x and a. e.y in $[0,1]$ then it is easy to see that $VK = \lambda KV$.

Suppose now that K is an integral operator satisfying $VK = \lambda KV$. Then its kernel $K(x,y)$ must satisfy (3.10). Notice that this kernel is indeed an equivalence class of functions that agree a. e. Now we pick any particular representative and denote it again by $K(x,y)$. The advantage of this strategy is that each side of (3.10) is now defined, for all $(x,y) \in [0,1] \times [0,1]$. With this understanding it is easy to see that the left hand side of (3.10), which we denote by $F_1(x,y)$, is a measurable function of y for every x, and that it is a continuous function of x for every $y \in [0,1]$. Since a similar argument shows that the right hand side of (3.10), which we denote by $F_2(x,y)$, we conclude that $F_1(x,y) = F_2(x,y)$ a. e. Furthermore, $F_1(x,y)$ has partial derivative $(F_1)_x$ for every $(x,y) \in [0,1] \times [0,1]$ and $F_2(x,y)$ has partial derivative $(F_2)_y$ for every $(x,y) \in [0,1] \times [0,1]$. Hence, $F_1(x,y)$ has partial derivative $(F_1)_y$ for a. e. point $(x,y) \in [0,1] \times [0,1]$. The conclusion is that we can, and we do denote both sides of (3.10) as $F(x,y)$. A calculation shows that $F_x(x,y) = K(x,y)$ and $F_y(x,y) = -\lambda K(x,y)$. Clearly that both F_x and F_y are measurable functions.

Finally, we have that

$$F_y(x,y) = -\lambda F_x(x,y), \quad \text{for a . e. } (x,y) \in [0,1] \times [0,1] \tag{3.11}$$

Let (a, b) be any point in $[0,1] \times [0,1]$, and let $R = R_{a,b}$ denote the rectangle bounded by $x = 0$, $x = a$, $y = b$, and $y = 1$. Then

$$\iint_R F_x(x,y)dxdy = \int_b^1 dy \int_0^a F_x(x,y)dx = \int_b^1 dy \int_0^a K(x,y)dx = \int_b^1 F(a,y)dy$$

while

$$\iint_R F_y(x,y)dydx = \int_0^a dx \int_b^1 F_y(x,y)dy = \int_0^a dx \int_b^1 -\lambda K(x,y)dy = -\int_0^a F(x,b)dx$$

In view of (3.11), we have

$$\iint_R \left[F_y(x,y) + \lambda F_x(x,y)\right] dxdy = 0$$

Therefore,

$$\lambda \int_b^1 F(a,y)dy - \int_0^a F(x,b)dx = 0$$

or,

$$\int_0^a F(x,b)dx = \lambda \int_b^1 F(a,y)dy \tag{3.12}$$

We emphasize that (3.12) holds for every $(a, b) \in [0,1] \times [0,1]$. Denote either side of it by $G(a, b)$. Then G is a measurable function that satisfies the equation

$$G_y(x,y) = -\lambda G_x(x,y), \qquad \text{for all } (x,y) \in [0,1] \times [0,1] \tag{3.13}$$

Although this equation is similar to (3.11) there is an important difference. The equation (3.13) holds for every$(x, y) \in [0,1] \times [0,1]$. Now we exploit this fact by using a well known argument from classical analysis. Let c be a fixed number. The function $H(x) = G(x,(c+x)/\lambda)$ is a measurable function as a composition of a measurable function G and a continuous map

$$\rho : x \mapsto (x,(c+x)/\lambda).$$

Moreover, the function H is a differentiable function of x as a composition of two differentiable functions. Then

$$H'(x) = \frac{\partial}{\partial x} G(x,(c+x)/\lambda) = G_x(x,(c+x)/\lambda) + G_y(x,(c+x)/\lambda)\frac{1}{\lambda}$$

and so using (3.13), $H'(x) = 0$. We conclude that H is a constant function of x. This implies that $G(x,(c+x)/\lambda) = H(c)$ is a constant function of x, we put $y = (c+x)/\lambda$,

we deduce $G(x,y) = H(x-\lambda y)$. since $G_{xx}(x,y) = K(x,y)$ it follows that there exists a measurable function g such that $K(x,y) = g(x-\lambda y)$. Moreover, the integrability condition on K translates to the required integrability condition on g.
Finally, (3.10) implies that

$$\int_0^x g(t-\lambda y)dt = \lambda \int_y^1 g(x-\lambda t)dt \quad \text{for a. e. x and a. e. y in } [0,1].$$

A change of variable leads to the equality

$$\int_{-\lambda y}^{x-\lambda y} g(s)ds = \int_{x-\lambda}^{x-\lambda y} g(s)ds$$

that holds for a. e. x and a. e. y in $[0,1]$, which in turn means that

$$\int_{-\lambda y}^{x-\lambda} g(s)ds = 0$$

for a. e. x and a. e. y in $[0,1]$, for $y = 1$, $\displaystyle\int_{-\lambda}^{x-\lambda} g(s)ds$ for a.e $x \in [0,1]$. It follows that $g = 0$ on $[-\lambda, 1-\lambda]$. Conversely, if K is an integral operator with kernel $K(x,y) = g(x-\lambda y)$, for some measurable function g that vanishes on $[-\lambda, 1-\lambda]$, then it is easy to verify that K satisfies (3.10). □

Theorem 3.8. *Let C_φ be a composition operator on $L^2(0,1)$ that satisfies $VC_\varphi = \lambda C_\varphi V$ for some $\lambda \in]0,\infty[$. Then $\lambda \geqslant 1$ and $\varphi(x) = \dfrac{x}{\lambda}$.*

Proof. by lemma 3.3 we have that $VC_\varphi = \lambda C_\varphi V$ if and only if

$$S_t C_\varphi = C_\varphi S_{t/\lambda} \quad \forall t \geqslant 0$$

this means that, for any $f \in L^2(0,1)$

$$\chi_{[t,\infty)\cap[0,1]}(x) f(\varphi(x-t)) = \chi_{[t/\lambda,\infty)\cap[0,1]}(\varphi(x)) f(\varphi(x) - t/\lambda) \tag{3.14}$$

If f is canstant, we obtain

$$\chi_{[t,\infty)\cap[0,1]}(x) = \chi_{[t/\lambda,\infty)\cap[0,1]}(\varphi(x)) \tag{3.15}$$

wich shows that $x \geqslant t$ if and if $\varphi(x) \geqslant t/\lambda$. From this we conclude that $\lambda \geqslant 1$ Indeed, if $\lambda < 1$, then we could take $x = t = (\lambda+1)/2$ and obtain $\varphi((\lambda+1)/2) > 1$ which is a contradiction.

Furthermore, it follows from 3.14 and 3.15 that, when $x \geq t$, $\varphi(x-t) = \varphi(x) - t/\lambda$. In particular, taking $x = t$, we get that $\varphi(x) = \varphi(0) + x/\lambda$. However, for such a function φ

$$C_\varphi V f(x) = \int_0^{\varphi(0)+x/\lambda} f(t)dt$$

while

$$VC_\varphi f(x) = \int_0^x f(\varphi(0) + t/\lambda)dt$$

Using the substitution $s = \varphi(0) + t/\lambda$ the last integral becomes

$$\lambda \int_{\varphi(0)}^{\varphi(0)+x/\lambda} f(s)ds$$

and it is easy to see that $\varphi(0) = 0$, it follows that $\varphi(x) = \dfrac{x}{\lambda}$. $\square$

Bibliography

[1] Jean Alexandre Dieudonné, History of functional analysis, Elsevier, (1981)

[2] H. Alkanjo, Spectre etendu des operateurs et applications, thèse de doctorat, Université de Claude Bernard Lyon 1, 2014

[3] V. Antonovich Sadovnichi, The geometry of Hilbert space: the spectral theory of operators. Theory of Operators. Springer, (1991)

[4] W. Arveson, spectral theory and Banach algebras: A short course on spectral theory, Springer, (2002)

[5] C. Badea, Muller. V, Growth conditions and inverse producing extensions J. Op. Th. 54 (2005).

[6] H. Boccara, Analyse fonctionnelle, Edition Marketing, Paris, 1984.

[7] A. Biswas, A. Lambert, S. Petrovic, Extended eigenvalues and the Vollterra operator, Glasgow Mathematical Journal Trust.(2001).

[8] H. Brezis, Analyse Fonctionnelle : Théorie et Applications. Masson, Paris (1983).

[9] G. Cassier, J. Verliat, Stability for some Operator Classes by Aluthge Transform, Operator Theory Live (2010).

[10] N. Dunford, J. T. Shwartz, Linear operator, I, II and III, Interscience, New York, 1958, 1963 and 1971.

[11] J. A. Deddens. Another description of nest algebras. In Hilbert space operators (Proc. Conf., Calif. State Univ., Long Beach, Calif., 1977), volume 693 of Lecture Notes in Math. Springer, Berlin, 1978.

[12] S. S. Dragomir, Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces, Preprint RGMIA Res. Rep. Coll., 11(e) (2008), Art. 15.

[13] T. Furuta, Invitation to linear operators, Taylor and Francis. London,(2001).

[14] A. Hechifa , Quelques Proprietés spectrales de quelques classes d'operateurs dans un espace de Banach,thèse de doctorat, Universite Badji Mokhtar Annaba (2017).

[15] H. Lal Vasudeva, Elements of Hilbert Spaces and Operator Theory, (2017).

[16] Steen, Lynn Arthur, Highlights in the history of spectral theory, AMM, 359–381, 4 (1973).

[17] W. Rudin, Real and complex analysis, Third edition (McGraw-Hill Book Co., New York, 1987).

[18] C. Read. J, Spectrum reducing extension for one operator on a Banach space. Trans. Ames. Math. Soc 308 (1988), 413-429.

[19] T. Yamazaki, On powers of class A(k) operators including p-hyponormal and log-hyponormal operators, Math. Inequal. Appl., 3(2000), 97-104.

[20] D. Wang, A convex operator function, Internat. J . Math. and math. Sci., 11(1988),401-402.

[21] C. R. Putnam, An inequality for the area of hyponormal sepctra, Math. Z., 116 (1970), 323-330.

[22] Pierre Petitcuvot, Problèmes de similarité et spectre étendu d? un àpérateur, 2008.

[23] S. Petrovic. On the extendedeig envalues of some Volterra operators. Integral Equations Operator Theory, 2007.

[24] P.Y. Wu, Numerical Range of Aluthge Transform of Operator, Linear Algebra and Applications.