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Existence and Uniqueness of solutions for fractional differential equations using Fixed-Point Theory

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Abstract

In this paper we have studied the existence and uniqueness of the solution to a type of fractional nonlinear differential equations with conditions that allow finding solutions by the fixed point method of Banach and Krasnoselskii's

1. Introduction

One of the most dynamic area of research of the last 50 years, fixed point theory plays a fundamental role in several theoretical and applied areas, such as nonlinear analysis, integral and differential equations and inclusions, dynamic systems theory, artificial intelligence and web page management, physics, biology, the maps, mathematics of fractals, mathematical economics (game theory, equilibrium problems, optimization problems) and mathematical modeling. This paper will present relevant works related to the theory of fixed points and its various applications to Nonlinear fractional differential equations

2. Definition and Preliminaries

2.1 Banach Fixed Point theorem

Theorem 1 *Let (X, d) be a complet metrique space and $T : X \to X, T$ is lipschitz, then, there exist a positive constant L such that*

$$\forall x, y \in X : d(T(x), T(y)) = |T(x) - T(y)| \leq L|x - y| \, foreach x, y \in X.$$

If $L < 1$, then T is a contraction.

Theorem 2 *Let X be a Banach space and $T : X \to X$ is a contraction, then T has a unique fixed poind such that*

$$\exists! x \in X : \; Tx = x.$$

2.2 Karasnoselkii's fixed point theorem

lemma 1 *Let M be a closed bounded convex and nonempty subset of a Banach space X. Let A, B be the operators such that*

Theorem 3 *a) $Ax + Bx \in M$ whenever $x, y \in M$.*
b) A is compact and continious.
c) B is a contraction mapping.
Then there exist $z \in M$ such that $z = Az + Bz$.

2.3 Schaefer fixed point theorem

lemma 2 *Let X be a Banach space. Assome that $T : X \to X$ is a completely continuous operator and the set*

$$V = \{u \in X / u = \mu T u, \; 0¡\mu¡1\}$$

is bounded, then T has a fixed point in X.

2.4 Brouwer's Fixed Point Theorem

Theorem 4 *Every continuous function from the closed unit disk onto itself has a fixed point. That is, if $f : D^2 \to D^2$ is a continuous then there exists a point $(x, y) \in D^2$ such that $f(x, y) = (x, y)$.*

Theorem 5 ***(General Brouwer's Fixed Point Theorem):*** *If $A \subset \mathbb{R}^n$ is homeomorphic to D^n then every continuous function $f : A \to A$ has a fixed point.*

example 1 *A continous function that maps $[0, 1]$ into itself has a fixed point.*
A continuous function that maps a disk into itself has a fixed point.
A continuous function that maps a spherical ball into itself necessarily has a fixed point

2.5 Schauder's Fixed Point Theorem

Theorem 6 *Let K be a bounded convex and compact nonempty subset of a Banach space X and $T : X \to X$ is continue, then T has a fixed point in K.*

2.6 Ascoli-Arzela Fixed Point Theorem

Theorem 7 *Let $A \subset C(K, \mathbb{R}^n), (K = [a, b] \subset \mathbb{R})$, then, A is relatively compact ($\overline{A}$ is compact) if and only if*
1)A is uniformely bounded. 2) A is equicontinuous.

2.7 Shizuo Kakutani's Fixed Point Theorem

Shizuo Kakutani discovered and proved in 1941 a generalization of Brouwer's Fixed Point Theorem. Brouwer's theorem applies to continuous point-to-point functions. Kakutani dealt with set-valued function; i.e. point-to-set functions.

Theorem 8 *Let M be a compact, convex subset of Euclidean n-space. Let T be a continuous set-valued function on M; i.e., a mapping from M to the set of all subsets of $M, T \to P(M)$. If T is such that $T(x)$ is convex for all x belonging to M then there exists a z such that $T(z)$ contains z.*

3. Application

3.1 Presentation of problem

In this section we study the existence and uniqueness of solution for problem consisting of nonlinear fractional differential equation of capoto type fractional derivative with integral condition.
A variety of fixed point theorem are used Banach's fixed point theorem, Karasnoselkii's fixed point theorem.
The problem of the form

$$\begin{cases} {}^cD^\alpha y(t) = f(x, y(t)) \qquad t \in J = [0, 1] \\ y(0) = \int_0^1 y(s)\, ds \\ y(1) = \int_0^1 (1-s)^{\beta-1} y(s)\, ds \end{cases}, \tag{1}$$

where ${}^cD^\alpha$ is the Caputo derivative defined as $\Gamma(\alpha\Gamma)$

$${}^cD^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{h^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0 \;\; n-1 < \alpha < n$$

such as $y(t) \in C^n([0,1], \mathbb{R})$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $1 < \alpha \leq 2, 0 < \beta \leq 1$

3.2 Preliminary

This section is devoted to some preliminary concepts of fractional integral calculs.

definition 1 *The Riemman-Liouville fractional Integral of order q with the lower limit zero for a function f is defined as*

$$J^q f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-q}}, t > 0, q > 0$$

and $\Gamma(\alpha) = \int_0^\infty t^{q-1} e^{-t} dt$ is a Gamma function

lemma 3 *Let $\alpha > 0$, the differential equation ${}^cD^\alpha h(t) = 0$ admet a solution $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}, i = 1...n, n = [\alpha] + 1$.*

lemma 4 *Let $\alpha > 0$, then*

$$I^\alpha \, {}^cD^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1} \tag{2}$$

lemma 5 *Let $1 < \alpha \leq 2$, and let $h = J \times \mathbb{R} \times \mathbb{R}$ continuous function $y \in C^2([0.1], \mathbb{R})$ is a solution of the linear fractional differential equation ${}^cD^\alpha h(t) = h(t)$, $1 < \alpha \leq 2$, if and only if*

$$y(t) = \frac{1}{p(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds + \int_0^1 \left[\begin{array}{c} \frac{1}{\zeta_1 p(\alpha)} \int_s^1 (1-s)^{\beta-1} (\Gamma - s)^{\alpha-1} h(\Gamma)\, d\Gamma \\ -\frac{(1-s)^{\alpha-1}}{\zeta_1 p(\alpha)} + \left(\frac{2\zeta_2}{\zeta_1 p(\alpha)} - \frac{2t}{\alpha p(\alpha)}\right)(1-s)^\alpha \end{array} \right] h(s)\, ds$$

with $\quad \zeta_1 = \left(1 - \frac{1}{\beta}\right), \zeta_2 = \left(1 - \frac{1}{\beta(\beta+1)}\right)$

3.3 Existance and uniqueness result via Banach's fixed point theorem

Theorem 9 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the lipschitz condition:*

$$(H_1) \qquad |f(t, x) - f(t, y)| < L|x - y|, t \in [0, 1], \;\; x, y \in \mathbb{R}, L > 0,$$

then the boundary value problem (1) has a unique solution on $[0, 1]$ if $L\phi < 1$.With

$$\phi = \frac{1}{p(\alpha+1)} + \frac{B(\beta, \alpha)}{|\zeta_1|(\alpha+\beta)p(\alpha)} + \frac{1}{|\zeta_1| p(\alpha+1)} + \frac{2|\zeta_2|}{|\zeta_1| p(\alpha+2)} + \frac{2}{p(\alpha+2)} \tag{3}$$

example 2 *Consider the following nonlinear Caputo fractional differential equation*

$$\begin{aligned} {}^cD^{\frac{4}{3}} y(t) &= \frac{1}{t^3 + 32} \frac{|x|}{+ |x|} + \ln^2(1+t) \\ y(0) &= \int_0^1 y(s)\, ds \\ y(1) &= \int_0^1 (1-s)^{-\frac{2}{3}} y(s)\, ds \end{aligned}$$

with $\alpha = \frac{4}{3}, \beta = \frac{1}{3}$ and $f(x, t) = \frac{1}{t^3 + 32} \frac{|x|}{+ |x|} + \ln^2(1+t)$, $L = \frac{1}{6}$.
*Using **Matlab** logicial for calcul ϕ, we obtain $\phi = 3.9198$, It's clear that the hypothesis of the theorem (1.10) is satisfied, thus problem has at least one solution on $[0, 1]$.*

3.4 Existence result via Karasnoselkii's fixed point theorem

Theorem 10 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (H_1), in addition we assum that*

$$(H_2) \quad |f(x, t)| \leq \mu(t), \; \forall (t, x) \in [0, 1] \text{ and } \mu \in \mathcal{C}([0, 1], \mathbb{R}^+)$$

then the problem (1) has at least one solution on $[0, 1]$ prouded

$$L\left\{\frac{B(\beta, \alpha)}{|\zeta_1|(\alpha+\beta)p(\alpha)} + \frac{1}{|\zeta_1| p(\alpha+1)} + \frac{2|\zeta_2|}{|\zeta_1| p(\alpha+2)} + \frac{2}{p(\alpha+2)}\right\} < 1 \tag{4}$$

Proof 1 *We define the operator p and φ by*

$$\begin{aligned} (py)(t) &= \frac{1}{p(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))\, ds \quad t \in [0, 1] \\ (\varphi y)(t) &= \frac{1}{\zeta_1 p(\alpha)} \int_0^1 \int_s^1 (1-r)^{\beta-1} (r-s)^{\alpha-1} f(s, y(s))\, ds dr - \frac{1}{|\zeta_1| p(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, y(s))\, ds \\ &+ \frac{2|\zeta_2|}{\zeta_1 \alpha p(\alpha)} \int_0^1 (1-s)^\alpha f(s, y(s))\, ds - \frac{2t}{\alpha p(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, y(s))\, ds \;, \end{aligned}$$

we subdivide the proof into several steps
Step1: Consider $B_r = \{y \in \mathcal{C}([0,1]), \|y\| \leq r\}$ verifiying $px + \varphi y \in B_r$.
Step2: φ is contraction.
Step3: p is compactness operator and uniformely bounded.

Programm matlab

```
alpha=input('alpha=');
betta=input('betta=');
L=input('L=');
r1=abs(1-(1/betta))
r2=abs(1-1/(betta*(betta+1)))
%%% Q1 condition of Ba,ach fixed point theorem
Q1=(1/(gamma(alpha+1)))+(beta(betta,alpha))/(r1*(alpha+betta)*gamma(alpha))+
(1/(r1*gamma(alpha+1)))+(2*r2/(r1*gamma(alpha+1)))+(2/(gamma(alpha+2)))
%Q1 condtion of contraction of Q2 via krasnoselskii fixed point theorem
Q2=(beta(betta,alpha))/(r1*(alpha+betta)*gamma(alpha))+
(1/(r1*gamma(alpha+1)))+(2*r2/(r1*gamma(alpha+1)))+(2/(gamma(alpha+2)))
disp('Q1=');
disp(Q1)
if (L.*Q1¡1)
 disp('admet une solution unique')
else
 disp('aucune solution ')
end
disp('Q2=');
disp(Q2)
if (L.*Q2¡1)
 disp('Q2 is contraction ')
else
 disp('Q2 not contraction ')
end
```

4. Conclusion

Talking about the fixed point in mathematics and its applications in physics, biological phenomena, economics and artificial intelligence is not sufficient to demonstrate its extreme importance in contributing to solving life problems and to study its importance requires specialized lectures separately.

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