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An Asymptotic Model for the Elasticity Problem in Thin Heterogeneous Domain

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General Introduction

The mathematical modeling of mechanical phenomena involving deformable structures has become an important field of research at the intersection of functional analysis, partial differential equations, and continuum mechanics. In particular, the study of viscoelastic materials, damage evolution, and contact interactions leads to complex nonlinear systems that require a rigorous analytical framework in order to ensure the well-posedness of the models and to understand their qualitative behavior.

In this context, variational methods and Sobolev space theory provide a natural setting for the formulation of weak solutions to boundary value problems arising in mechanics. The use of energy methods, monotonicity techniques, and compactness arguments plays a central role in the analysis of existence, uniqueness, and stability of solutions for evolution problems governed by partial differential equations.

The present work is devoted to the mathematical analysis of a quasistatic viscoelastic contact problem with damage evolution posed in a thin three-dimensional domain. The model couples nonlinear viscoelastic constitutive laws, a damage process described by a differential inclusion, and frictional contact conditions of Coulomb type. The presence of a small geometric parameter introduces strong anisotropic effects, which makes the analysis particularly delicate and motivates the use of asymptotic techniques.

The main objective of this study is to establish a rigorous variational formulation of the problem, prove the existence of weak solutions, and derive uniform a priori estimates independent of the small parameter. These estimates are then used to justify the asymptotic passage to the limit as the thickness tends to zero, leading to an effective reduced model that preserves the essential mechanical features of the original problem.

Finally, this work aims to contribute to the mathematical understanding of thin viscoelastic structures with damage and contact, and to provide a solid theoretical basis for future analytical and numerical developments. The mathematical tools and analytical framework used in this work are based on a wide range of classical and modern references in functional analysis, partial differential equations, and continuum mechanics. In particular, the theory of Sobolev spaces and weak formulations is developed in [1, 3, 8], which provide the fundamental functional setting for variational methods.

The study of nonlinear evolution equations and monotone operator theory relies essentially on the works of Lions and Barbu [15, 2, 17], which constitute the main analytical tools for existence and stability results.

From the mechanical point of view, elasticity, contact mechanics, and thin structures are treated in the classical monographs of Ciarlet, Duvaut–Lions, Kikuchi–Oden, and Wriggers [5, 6, 7, 13, 18].

Damage mechanics and viscoelastic modeling are also based on the works of Frémond, Han–Reddy, and related contributions in nonsmooth thermomechanics and inelastic behavior [10, 11, 12].

Finally, asymptotic analysis and dimension reduction techniques for thin structures are inspired by the works of Friesecke–James–Müller and related contributions on plate theories and reduced models [9], as well as recent developments in viscoelastic contact problems [14, 16]. The thesis is organized as follows.

Chapter 1 is devoted to the mechanical preliminaries, where we recall the basic concepts of continuum mechanics, including strain, stress, equilibrium equations, as well as constitutive laws for viscoelastic materials and damage models in three-dimensional elasticity.

Chapter 2 presents the main tools from functional analysis, Sobolev spaces, and variational methods, which provide the mathematical framework of the study.

Chapter 3 introduces the weak formulation of the quasistatic viscoelastic contact problem with damage evolution in a thin domain. The existence framework is established under suitable assumptions on the constitutive operators and boundary data. In this chapter, a rescaling procedure is performed in order to map the thin domain onto a fixed reference domain independent of the small parameter ε. Uniform a priori estimates, independent of

ε, are then derived. These estimates are essential for the asymptotic analysis of the problem as $\varepsilon \to 0$, and they lead to compactness properties and convergence results.

Finally, the thesis concludes with a general discussion of the main results and possible applications in thin viscoelastic structures with damage and contact phenomena.

Basic Notations and Conventions

- Throughout this work, $\Omega \subset \mathbb{R}^d$, with $d = 2$ or 3, denotes an open bounded domain with sufficiently smooth boundary.

- The boundary of Ω is denoted by $\Gamma = \partial\Omega$.

- $L^2(\Omega)$ denotes the space of square-integrable functions on Ω, equipped with the inner product
$$(u, v)_{L^2(\Omega)} = \int_\Omega uv\, dx,$$
and the associated norm
$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u|^2\, dx \right)^{1/2}.$$

- The Sobolev space of first order is denoted by
$$H^1(\Omega) = \left\{ u \in L^2(\Omega) \; ; \; \nabla u \in (L^2(\Omega))^d \right\}.$$

- $H_0^1(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.

- $H^{-1}(\Omega)$ denotes the dual space of $H_0^1(\Omega)$.

- The duality pairing between a space and its dual is denoted by $\langle \cdot, \cdot \rangle$.

- $\mathcal{L}(X, Y)$ denotes the space of continuous linear operators from X into Y.

- The norm in a Banach or Hilbert space X is denoted by $\| \cdot \|_X$.

- The symbol C denotes a generic positive constant whose value may change from line to line.

- The parameter $\varepsilon > 0$ usually represents a small parameter related to the film thickness or an asymptotic scale.

- ∇u denotes the gradient of u, while div denotes the divergence operator.

- We denote by
$$S^d = \mathbb{R}_s^{d\times d} = \left\{\tau = (\tau_{ij})_{1\le i,j\le d}\ ;\ \tau_{ij} = \tau_{ji}\right\}.$$
the space of symmetric second-order tensors (or symmetric matrices).

- For any tensors
$$\sigma = (\sigma_{ij}), \qquad \tau = (\tau_{ij}) \in S^d,$$
the standard scalar product is defined by
$$\sigma : \tau = \sum_{i,j=1}^{d} \sigma_{ij}\tau_{ij}.$$

- The associated Euclidean norm is given by
$$|\tau| = (\tau : \tau)^{1/2}.$$

- For vectors
$$u = (u_1, \dots, u_d), \qquad v = (v_1, \dots, v_d),$$
the usual scalar product in $\mathbb{R}^d$ is denoted by
$$u \cdot v = \sum_{i=1}^{d} u_i v_i,$$
with associated norm
$$|u| = (u \cdot u)^{1/2}.$$
Let $\Gamma = \partial\Omega$ and let ν denote the outward unit normal vector on Γ, defined by
$$\nu(x) = (\nu_1(x), \dots, \nu_d(x)), \qquad |\nu| = 1 \quad \text{on } \Gamma.$$

- For any vector field v defined on Γ, we define its normal component by
$$v_\nu = v \cdot \nu, \qquad v_\nu\, \nu = (v \cdot \nu)\nu.$$

- The tangential component is defined by

$$v_\tau = v - (v \cdot \nu)\nu.$$

Hence, the orthogonal decomposition reads

$$v = v_\nu \, \nu + v_\tau, \qquad v_\tau \cdot \nu = 0.$$

For any tensor field σ defined on the boundary, we define the associated Cauchy traction vector $\sigma\nu$ and introduce the following general decompositions:

- Normal component of a tensor (traction):

$$(\sigma\nu)_\nu = (\sigma\nu) \cdot \nu.$$

- Tangential component of the traction:

$$(\sigma\nu)_\tau = \sigma\nu - \big((\sigma\nu) \cdot \nu\big)\nu.$$

Hence, we have the orthogonal decompositions:

$$\sigma\nu = (\sigma\nu)_\nu \, \nu + (\sigma\nu)_\tau, \qquad v = v_\nu \, \nu + v_\tau.$$

Chapter 1

Basic Equations and Boundary Conditions

The mathematical modeling of deformable bodies relies on a rigorous framework combining functional analysis, partial differential equations, and continuum mechanics. Sobolev spaces, weak formulations, and variational methods provide the natural setting for the analysis of boundary value problems and generalized solutions [1, 3, 8, 15, 2]. Mechanical models in elasticity, viscoelasticity, and thin structures are commonly formulated through partial differential equations associated with constitutive laws and boundary conditions [5, 6]. Contact, friction, and interface phenomena often lead to variational inequalities and non-smooth problems requiring specialized analytical tools [7, 13, 18]. Moreover, the modeling of damage and dissipative processes involves internal variables and nonlinear thermomechanical formulations [10, 11, 12]. The present chapter introduces the governing equations, boundary conditions, and mathematical framework underlying the subsequent analysis [9, 14, 16].

1.1 Physical Setting and Evolution Equations

We use throughout the manuscript the following notation:

$$H = L^2(\Omega)^d, \qquad H_1 = H^1(\Omega)^d,$$

and

$$Q = L^2(\Omega; S^d), \qquad Q_1 = H^1(\Omega; S^d),$$

where $S^d = \mathbb{R}_s^{d\times d}$ denotes the space of symmetric second-order tensors.
The admissible displacement space is defined by

$$V = \{v \in H_1 \; ; \; v = 0 \text{ on } \Gamma_1\}\,.$$

The scalar product on V is given by

$$(u, v)_V = \int_\Omega u \cdot v \, dx + \int_\Omega \nabla u : \nabla v \, dx,$$

with associated norm

$$\|v\|_V^2 = \int_\Omega |v|^2 \, dx + \int_\Omega |\nabla v|^2 \, dx.$$

The space Q is endowed with the scalar product

$$(\sigma, \tau)_Q = \int_\Omega \sigma : \tau \, dx.$$

For every displacement field $u \in V$, the linearized strain tensor is defined by

$$e(u) = \big(e_{ij}(u)\big),$$

where

$$e_{ij}(u) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \qquad i, j = 1, \ldots, d$$

and

$$u_{i,j} = \partial u_i / \partial x_j, \qquad i, j = 1, \ldots, d$$

The divergence operator acting on a tensor field $\sigma \in Q_1$ is defined componentwise by

$$\operatorname{Div}\sigma = (\sigma_{ij,j})_{i=1}^{d}$$

where

$$\sigma_{ij,j} = \sum_{j=1}^{d} \partial\sigma_{ij}/\partial x_j, \qquad i = 1, \dots, d$$

For sufficiently smooth tensor fields, Green's formula takes the form

$$\int_\Omega \sigma : e(v)\,dx + \int_\Omega \operatorname{Div}\sigma \cdot v\,dx = \int_\Gamma \sigma\nu \cdot v\,d\Gamma, \qquad \forall v \in V.$$

In this chapter, we introduce the mathematical model describing the mechanical behavior of a deformable viscoelastic body in contact with a rigid foundation. The notation and functional framework introduced in the "Basic Notations and Conventions" section (pages 1–3) are used throughout this chapter in order to ensure consistency of the mathematical analysis.

Let

$$\Omega \subset \mathbb{R}^d, \qquad d = 2, 3,$$

be a bounded domain with Lipschitz boundary

$$\Gamma = \partial\Omega.$$

The time interval is denoted by

$$[0, T], \qquad T > 0.$$

The boundary Γ is decomposed into three measurable disjoint parts:

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \qquad \Gamma_i \cap \Gamma_j = \emptyset, \quad i \neq j.$$

Here:

- Γ_1 is the clamped part of the boundary;
- Γ_2 is the portion where surface tractions are prescribed;

- Γ_3 represents the potential contact surface.

Moreover, we assume that

$$\mathrm{meas}(\Gamma_1) > 0,$$

which guarantees Korn's inequality and eliminates rigid body motions.

1.1.1 Mechanical Unknowns

The mechanical state of the body is described by the following unknown fields:

- The displacement field

$$u : \Omega \times [0, T] \to \mathbb{R}^d.$$

- The stress tensor

$$\sigma : \Omega \times [0, T] \to S^d.$$

- The damage variable

$$\zeta : \Omega \times [0, T] \to [0, 1].$$

The infinitesimal strain tensor associated with the displacement field is defined by

$$e(u) = \frac{1}{2} \left(\nabla u + \nabla u^T \right).$$

1.1.2 Admissible Spaces

The displacement field satisfies

$$u(t) \in V_0, \qquad \forall t \in [0, T].$$

The stress tensor satisfies

$$\sigma(t) \in Q, \qquad \forall t \in [0,T].$$

For the damage variable, we consider

$$\zeta(t) \in H^1(\Omega), \qquad 0 \le \zeta \le 1 \quad \text{a.e. in } \Omega.$$

1.1.3 Equilibrium Equations

The mechanical evolution is governed by the balance law of linear momentum.

Dynamic Model

When inertial effects are taken into account, the displacement field satisfies

$$\rho \ddot{u} = \operatorname{Div} \sigma + f_0 \qquad \text{in } \Omega \times (0,T),$$

where:

- $\rho > 0$ denotes the mass density;
- f_0 is the density of body forces.

Quasistatic Model

When inertial effects are neglected, the equilibrium equation becomes

$$\operatorname{Div} \sigma + f_0 = 0 \qquad \text{in } \Omega \times (0,T).$$

Example. Consider a one-dimensional elastic bar occupying the interval

$$\Omega = (0,L).$$

Under a distributed load f, the quasistatic equilibrium equation is

$$-\frac{d\sigma}{dx} = f.$$

Assuming Hooke's law

$$\sigma = Eu',$$

we obtain

$$-\frac{d}{dx}(Eu') = f.$$

This elementary example illustrates the relation between equilibrium equations and constitutive laws.

1.1.4 Constitutive Laws

The constitutive law describes the material response.

Elastic Law

For a linear elastic material,

$$\sigma = \mathcal{B}\, e(u),$$

where $\mathcal{B}$ denotes the elasticity operator.

In the isotropic case,

$$\sigma_{ij} = \lambda \,\mathrm{tr}(e(u))\delta_{ij} + 2\mu\, e_{ij}(u),$$

where λ and μ are the Lamé coefficients.

Viscoelastic Law

For Kelvin–Voigt viscoelastic materials,

$$\sigma = \mathcal{A}\, e(\dot{u}) + \mathcal{B}\, e(u),$$

where:

- $\mathcal{A}$ is the viscosity operator;

- $\mathcal{B}$ is the elasticity operator.

Example. In one dimension,

$$\sigma = \eta\, \dot{u}' + E\, u',$$

where $E > 0$ is the Young modulus and $\eta > 0$ is the viscosity coefficient.

1.2 Boundary Conditions

The body is subjected to displacement, traction, and contact conditions.

1.2.1 Dirichlet and Neumann Conditions

The displacement condition on Γ_1 is

$$u = 0 \qquad \text{on } \Gamma_1 \times (0, T).$$

The traction condition on Γ_2 is

$$\sigma\nu = f_2 \qquad \text{on } \Gamma_2 \times (0, T),$$

where ν denotes the unit outward normal vector.

1.2.2 Normal and Tangential Components

For every vector field v, we define

$$v_\nu = v \cdot \nu, \qquad v_\tau = v - v_\nu \nu.$$

Similarly, for the stress tensor,

$$\sigma_\nu = (\sigma\nu) \cdot \nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

1.2.3 Contact Conditions

The interaction with the rigid foundation takes place on Γ_3.

Bilateral Contact

The bilateral contact condition is

$$u_\nu = 0 \qquad \text{on } \Gamma_3 \times (0, T).$$

Normal Compliance

The normal compliance condition is written as

$$-\sigma_\nu = p_\nu(u_\nu - g),$$

where g denotes the initial gap.

A common choice is

$$p_\nu(r) = c_\nu (r_+)^m,$$

with

$$r_+ = \max\{0, r\}, \qquad c_\nu > 0, \qquad m \geq 1.$$

Interpretation. The contact pressure increases progressively with penetration, which makes this model suitable for deformable foundations and rough surfaces.

Signorini Contact Condition

For rigid obstacles, the unilateral Signorini condition is

$$u_\nu \leq g, \qquad \sigma_\nu \leq 0, \qquad \sigma_\nu (u_\nu - g) = 0.$$

1.2.4 Friction Laws

Frictionless Contact

In the absence of friction,

$$\sigma_\tau = 0.$$

Coulomb Friction

The Coulomb friction law is

$$\|\sigma_\tau\| \leq F_b,$$

where

$$F_b = \mu|\sigma_\nu|, \qquad \mu \geq 0.$$

During sliding,

$$-\sigma_\tau = F_b \frac{u_\tau}{\|u_\tau\|} \qquad \text{if } u_\tau \neq 0.$$

1.3 Damage Model

1.3.1 Damage Variable

The internal damage variable satisfies

$$\zeta(x, t) \in [0, 1].$$

Its physical interpretation is:

- $\zeta = 1$: undamaged material;
- $\zeta = 0$: completely damaged material;
- $0 < \zeta < 1$: partially damaged material.

A common interpretation is

$$\zeta = \frac{E_{\text{eff}}}{E_Y},$$

where E_{eff} is the effective Young modulus.

1.3.2 Damage Evolution Equation

The damage evolution is governed by

$$\dot{\zeta} - k\Delta\zeta + \partial\psi_{[0,1]}(\zeta) \ni \phi(e(u), \zeta),$$

where:

- $k > 0$ is the diffusion coefficient;
- $\partial\psi_{[0,1]}$ ensures the constraint
$$0 \leq \zeta \leq 1;$$
- ϕ is the damage source function.

1.3.3 Viscoelastic Law with Damage

The constitutive law with damage is written as

$$\sigma = \mathcal{A}\, e(\dot{u}) + \mathcal{B}(e(u), \zeta).$$

A simple degradation model is

$$\mathcal{B}(e(u), \zeta) = \zeta\, \mathcal{B}_0 e(u),$$

where $\mathcal{B}_0$ denotes the elasticity tensor of the undamaged material.

1.3.4 Mathematical Stability Assumptions

To establish existence and uniqueness results, the source function ϕ is assumed to satisfy suitable continuity and growth conditions.

Singular laws such as

$$\phi(e(u), \zeta) = \lambda_D \frac{1-\zeta}{\zeta} - \frac{1}{2}\lambda_E \|e(u)\|^2 + \lambda_w$$

may become singular as $\zeta \to 0$.

To avoid these difficulties, truncated formulations are introduced:

$$\phi(e(u), \zeta) = \phi_{Fr}\left(\min\{\|e(u)\|, e_*\}, \max\{\zeta, \zeta_*\}\right),$$

where:

- $\zeta_* > 0$ is a lower bound for the damage variable;
- $e_* > 0$ is a strain threshold.

1.3.5 Functional Inequalities

The following fundamental inequalities play a key role in the well-posedness analysis of the variational formulation.

Korn's inequality. Since $\text{meas}(\Gamma_1) > 0$, Korn's inequality holds in V, namely there exists a constant $C_K > 0$ such that

$$\|v\|_{H^1(\Omega)^d} \leq C_K \|e(v)\|_{L^2(\Omega)^{d\times d}}, \qquad \forall v \in V.$$

Poincaré's inequality. There exists a constant $C_P > 0$ such that

$$\|v\|_{L^2(\Omega)^d} \leq C_P \|\nabla v\|_{L^2(\Omega)^{d\times d}}, \qquad \forall v \in V.$$

These inequalities ensure coercivity of the energy functional and are fundamental in the variational analysis of the mechanical problem.

A detailed mathematical treatment of these results will be provided in the next chapter.

Chapter 2

Preliminary Mathematical and Mechanical Framework

This chapter introduces the mathematical framework required for the subsequent analysis. The material recalled here is classical and mainly concerns functional analysis, Sobolev and Bochner spaces, compactness arguments, and nonlinear analysis in Hilbert spaces. Particular emphasis is placed on time-dependent functional spaces, weak convergence, variational tools, and monotonicity methods, which constitute the basic analytical setting for evolution problems. The notation and functional framework introduced previously are systematically employed throughout this work without further repetition.

The notation and functional framework introduced previously are systematically employed throughout this work without repetition. For completeness and further details on Sobolev spaces, partial differential equations, variational methods, and nonlinear analysis, we refer to the classical references [1, 3, 8, 15, 2, 17].

2.1 Time-Dependent Functional Framework

Let $T > 0$ and let X be a real Banach space with norm $\|\cdot\|_X$. We introduce the functional spaces used to describe evolution problems.

2.1.1 Bochner Spaces

For $1 \leq p < \infty$, the space $L^p(0,T;X)$ consists of all measurable functions $u : [0,T] \to X$ such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} < \infty.$$

For $p = \infty$, we define

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{t \in [0,T]} \|u(t)\|_X.$$

These spaces are called *Bochner spaces* and generalize classical Lebesgue spaces.

2.1.2 Space $L^2(0,T;X)$ as a Hilbert space

If X is a Hilbert space, then $L^2(0,T;X)$ is also a Hilbert space with scalar product

$$(u,v)_{L^2(0,T;X)} = \int_0^T (u(t), v(t))_X \, dt.$$

2.1.3 Time Sobolev Space

The Sobolev space in time is defined by

$$H^1(0,T;X) = \left\{ u \in L^2(0,T;X) \; ; \; u' \in L^2(0,T;X) \right\},$$

where u' is the weak time derivative defined by

$$\int_0^T (u'(t), \varphi(t))_X \, dt = -\int_0^T (u(t), \varphi'(t))_X \, dt.$$

2.1.4 Fundamental Embeddings

The following continuous embeddings hold:

$$H^1(0,T;X) \hookrightarrow C([0,T];X),$$

which ensures that every function in $H^1(0,T;X)$ has a continuous representative.

Moreover,

$$L^r(0,T;X) \hookrightarrow L^q(0,T;X), \qquad 1 \leq q \leq r \leq \infty.$$

2.1.5 Energy Estimates and Key Inequalities

Hölder inequality in time. For $u \in L^p(0,T;X)$ and $v \in L^q(0,T;X)$,

$$\int_0^T \|u(t)\|_X \|v(t)\|_X \, dt \le \|u\|_{L^p(0,T;X)} \|v\|_{L^q(0,T;X)}.$$

Young inequality. For $a, b \ge 0$ and $\varepsilon > 0$,

$$ab \le \varepsilon a^2 + \frac{1}{\varepsilon} b^2.$$

Energy structure. These inequalities are systematically used to control nonlinear and coupling terms in evolution problems.

2.1.6 Weak Continuity in Time

If

$$u \in L^2(0,T;X), \quad u' \in L^2(0,T;X),$$

then

$$u \in C([0,T]; X_{\text{weak}}),$$

i.e. $u(t)$ is weakly continuous in time.

Moreover, the mapping

$$t \mapsto \|u(t)\|_X^2$$

is absolutely continuous and satisfies energy identities.

2.1.7 Compactness Results

Theorem 2.1.1 (Aubin–Lions type principle) *Let $X_0 \hookrightarrow X \hookrightarrow X_1$ be Banach spaces with compact embedding $X_0 \hookrightarrow X$. Then the space*

$$\{u \in L^2(0,T;X_0) \ ; \ u' \in L^2(0,T;X_1)\}$$

is compactly embedded in $L^2(0,T;X)$.

Remark. This result is fundamental for passing to the limit in nonlinear evolution problems.

2.1.8 Gronwall Inequalities

If

$$\phi(t) \leq \alpha + \int_0^t n(s)\phi(s)\, ds,$$

then

$$\phi(t) \leq \alpha \exp\left(\int_0^t n(s)\, ds\right).$$

This inequality is the main tool for:

- uniqueness proofs,
- stability estimates,
- continuous dependence on data.

2.2 Nonlinear Analysis in Hilbert Spaces

In this section, we recall several notions from nonlinear functional analysis that are frequently used in variational formulations and evolution problems arising in contact mechanics and damage models.

Let H be a real Hilbert space endowed with the scalar product

$$(\cdot, \cdot)_H$$

and norm

$$\|u\|_H = \sqrt{(u, u)_H}.$$

2.2.1 Weak Convergence

Definition 2.2.1 *A sequence $(u_n) \subset H$ is said to converge weakly to $u \in H$, and we write*

$$u_n \rightharpoonup u \quad \text{in } H,$$

if

$$(u_n, v)_H \to (u, v)_H, \qquad \forall v \in H.$$

Let $(u_n) \subset H$.

1. If
$$u_n \to u \quad \text{strongly in } H,$$
then
$$u_n \rightharpoonup u \quad \text{weakly in } H.$$

2. If
$$u_n \rightharpoonup u \quad \text{in } H,$$
then (u_n) is bounded in H.

3. If
$$u_n \rightharpoonup u \quad \text{in } H,$$
then
$$\|u\|_H \leq \liminf_{n\to\infty} \|u_n\|_H.$$

Theorem 2.2.1 (Banach–Alaoglu) *Every bounded sequence in a Hilbert space admits a weakly convergent subsequence.*

Importance. Weak convergence plays a fundamental role in existence proofs since bounded energy estimates generally provide weak compactness rather than strong compactness.

2.2.2 Convex Functions

Definition 2.2.2 *Let $\varphi : H \to \mathbb{R} \cup \{+\infty\}$.*

The function φ is said to be convex if
$$\varphi(\lambda u + (1-\lambda)v) \leq \lambda\varphi(u) + (1-\lambda)\varphi(v),$$
for all $u, v \in H$ and $\lambda \in [0,1]$.

Definition 2.2.3 *A function $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ is called proper if*
$$\mathrm{dom}(\varphi) = \{u \in H \ ; \ \varphi(u) < +\infty\} \neq \emptyset.$$

Definition 2.2.4 *A convex function φ is lower semicontinuous if*
$$u_n \to u \quad \Longrightarrow \quad \varphi(u) \leq \liminf_{n\to\infty} \varphi(u_n).$$

Remark. Lower semicontinuity is essential in variational methods and minimization problems.

2.2.3 Gâteaux Differentiability

Definition 2.2.5 *A functional $\varphi : H \to \mathbb{R}$ is said to be Gâteaux differentiable at $u \in H$ if there exists an element $\varphi'(u) \in H$ such that*

$$\lim_{\lambda \to 0} \frac{\varphi(u + \lambda v) - \varphi(u)}{\lambda} = (\varphi'(u), v)_H, \qquad \forall v \in H.$$

Proposition 2.2.1 *Let $\varphi : H \to \mathbb{R}$ be convex and Gâteaux differentiable. Then*

$$\varphi(v) - \varphi(u) \geq (\varphi'(u), v - u)_H, \qquad \forall u, v \in H.$$

2.2.4 Monotone Operators

Definition 2.2.6 *An operator*

$$A : H \to H$$

is said to be monotone if

$$(Au - Av, u - v)_H \geq 0, \qquad \forall u, v \in H.$$

It is said to be strongly monotone if there exists $m > 0$ such that

$$(Au - Av, u - v)_H \geq m\|u - v\|_H^2, \qquad \forall u, v \in H.$$

Definition 2.2.7 *An operator $A : H \to H$ is Lipschitz continuous if there exists $L > 0$ such that*

$$\|Au - Av\|_H \leq L\|u - v\|_H, \qquad \forall u, v \in H.$$

Importance. Monotonicity methods are widely used in nonlinear contact problems, damage evolution equations, and variational inequalities.

2.2.5 Subdifferential of a Convex Function

Let $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ be proper, convex, and lower semicontinuous.

Definition 2.2.8 *The subdifferential of φ at $u \in H$ is defined by*

$$\partial\varphi(u) = \{w \in H \;;\; \varphi(v) - \varphi(u) \geq (w, v - u)_H, \quad \forall v \in H\}.$$

If φ is differentiable, then

$$\partial\varphi(u) = \{\varphi'(u)\}.$$

Application. Subdifferential operators naturally appear in unilateral contact conditions, friction laws, and constrained damage models.

2.2.6 Weak Lower Semicontinuity

Theorem 2.2.2 *Let $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ be convex and lower semicontinuous. Then φ is weakly lower semicontinuous, namely*

$$u_n \rightharpoonup u \quad \textit{in } H$$

implies

$$\varphi(u) \leq \liminf_{n\to\infty} \varphi(u_n).$$

Remark. This property is fundamental when passing to the limit in nonlinear variational formulations.

Chapter 3

Mathematical Model and Asymptotic Analysis

3.1 Introduction

This chapter is devoted to the asymptotic analysis of the mathematical model introduced previously in a thin three-dimensional domain. The objective is to investigate the behavior of the mechanical fields as the thickness parameter ε tends to zero and to derive estimates that remain uniform with respect to this small parameter.

The analysis follows a systematic variational approach based on uniform a priori estimates, compactness arguments, and weak convergence methods. Such techniques constitute a classical framework in the mathematical study of thin structures and nonlinear evolution problems [3, 8, 15, 2, 17]. Particular attention is devoted to the anisotropic effects induced by the geometry of thin domains and to the scaling properties governing displacement, stress, and damage variables.

The methodology adopted here is consistent with the classical theories of elasticity and thin structures developed by Ciarlet [5, 6], as well as with variational approaches to contact and nonlinear mechanical problems initiated by Duvaut and Lions [7]. The asymptotic viewpoint is further motivated by the fundamental work of Friesecke, James, and Müller on geometric rigidity and reduced models for thin bodies [9].

Throughout this chapter, the results are presented in a systematic and concise form. The analytical techniques employed are standard scaling and compactness methods adapted to thin domains, while detailed technical developments and related asymptotic arguments may be found in the specialized literature and related works on thin viscoelastic and contact problems [14, 16].

3.2 Physical Setting and Governing Equations

We consider a viscoelastic body occupying a thin three-dimensional domain of thickness of order $\varepsilon \in (0,1)$. The mid-surface of the body is a bounded regular domain $\omega \subset \mathbb{R}^2$ with coordinates $x' = (x_1, x_2)$. The upper surface is described by

$$x_3 = h^\varepsilon(x') = \varepsilon h(x'),$$

where

$$h \in C^1(\overline{\omega}), \qquad 0 < h_* \le h(x'),$$

for all $x' \in \omega$.

The reference configuration of the thin body is therefore

$$\Omega^\varepsilon = \left\{ x = (x', x_3) \in \mathbb{R}^3 \; ; \; x' \in \omega, \; 0 < x_3 < \varepsilon h(x') \right\}.$$

The evolution is studied over the time interval $[0, T]$. We denote by

$$u^\varepsilon = u^\varepsilon(x,t), \qquad \sigma^\varepsilon = \sigma^\varepsilon(x,t), \qquad \zeta^\varepsilon = \zeta^\varepsilon(x,t),$$

the displacement field, stress tensor, and damage variable, respectively.

For any linear space X, we define

$$X^d = \{ v = (v_i) \, ; \, v_i \in X, \; i = 1, \dots, d \},$$

and

$$X_s^{d \times d} = \{ \tau = (\tau_{ij}) \, ; \, \tau_{ij} = \tau_{ji} \in X, \; i,j = 1, \dots, d \}.$$

We further denote by

$$S^d = \mathbb{R}_s^{d \times d}, \qquad d = 2, 3,$$

the space of symmetric tensors endowed with the Euclidean inner product and norm.

The functional spaces used throughout this chapter are

$$H^\varepsilon = L^2(\Omega^\varepsilon)^3, \qquad Q^\varepsilon = L^2_s(\Omega^\varepsilon)^{3\times 3},$$

and

$$H_1^\varepsilon = H^1(\Omega^\varepsilon)^3, \qquad Q_1^\varepsilon = H^1_s(\Omega^\varepsilon)^{3\times 3}.$$

The linearized strain tensor is defined by

$$e(u^\varepsilon) = \left(e_{ij}(u^\varepsilon)\right), \qquad e_{ij}(u^\varepsilon) = \frac{1}{2}\left(u^\varepsilon_{i,j} + u^\varepsilon_{j,i}\right),$$

while the divergence operator acting on stresses is

$$Div\sigma^\varepsilon = \left(\sigma^\varepsilon_{ij,j}\right)_{i=1}^3.$$

The spaces H_1^ε and Q_1^ε admit the equivalent characterizations

$$H_1^\varepsilon = \{v \in H^\varepsilon\,;\, e(v) \in Q^\varepsilon\},$$

and

$$Q_1^\varepsilon = \{\sigma \in Q^\varepsilon\,;\, Div\sigma \in H^\varepsilon\}.$$

They are Hilbert spaces equipped with the inner products

$$(u,v)_{H^\varepsilon} = \int_{\Omega^\varepsilon} u\cdot v\, dx,$$

$$(u,v)_{H_1^\varepsilon} = (u,v)_{H^\varepsilon} + (e(u), e(v))_{Q^\varepsilon},$$

$$(\sigma,\tau)_{Q^\varepsilon} = \int_{\Omega^\varepsilon} \sigma : \tau\, dx,$$

and

$$(\sigma,\tau)_{Q_1^\varepsilon} = (\sigma,\tau)_{Q^\varepsilon} + (Div\sigma, Div\tau)_{H^\varepsilon}.$$

We also introduce the admissible displacement space

$$V^\varepsilon = \left\{v \in H^1(\Omega^\varepsilon)^3\,;\, v = 0 \text{ on } \Gamma_1^\varepsilon, \quad v_\nu = 0 \text{ on } \omega\right\},$$

equipped with

$$(u,v)_{V^\varepsilon} = (e(u), e(v))_{Q^\varepsilon}, \qquad \|v\|_{V^\varepsilon} = \|e(v)\|_{Q^\varepsilon}.$$

The Poincaré and Korn inequalities hold in V^ε: there exists a constant $m_K > 0$, independent of ε, such that

$$\|e(v)\|_{Q^\varepsilon} \geq m_K \|\nabla v\|_{Q^\varepsilon},$$

and

$$\|v\|_{H^\varepsilon} \leq \varepsilon h^* \|\nabla v\|_{Q^\varepsilon}, \qquad \forall v \in V^\varepsilon.$$

For a Banach space X, we use the spaces

$$C([0,T];X), \qquad C^1([0,T];X),$$

with norm

$$\|f\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X + \max_{t\in[0,T]} \|\dot{f}(t)\|_X.$$

The quasistatic damaged viscoelastic contact problem is governed by the following equations.

Constitutive Law

The nonlinear viscoelastic constitutive relation with damage is

$$\sigma^\varepsilon = \mathcal{A}^\varepsilon(e(u^\varepsilon), \zeta^\varepsilon) + \mathcal{B}^\varepsilon(e(u^\varepsilon)), \qquad \text{in } \Omega^\varepsilon \times (0,T), \tag{3.2.1}$$

where $\mathcal{A}^\varepsilon$ and B^ε denote the elasticity and viscosity operators.

Damage Evolution

The damage field satisfies the differential inclusion

$$\dot{\zeta}^\varepsilon - \kappa^\varepsilon \Delta\zeta^\varepsilon + \partial\chi_{[0,1]}(\zeta^\varepsilon) \ni \phi^\varepsilon(e(u^\varepsilon), \zeta^\varepsilon), \tag{3.2.2}$$

in $\Omega^\varepsilon \times (0,T)$.

Here ϕ^ε denotes the damage source function, κ^ε is the diffusion coefficient, and $\partial\chi_{[0,1]}$ is the subdifferential of the indicator function of $[0,1]$.

The damage variable satisfies

$$0 \leq \zeta^\varepsilon \leq 1,$$

where $\zeta^\varepsilon = 1$ corresponds to an undamaged material and $\zeta^\varepsilon = 0$ to complete damage.

Equilibrium Equation

The quasistatic equilibrium equation reads

$$\mathrm{Div}\ \sigma^\varepsilon + f^\varepsilon = 0 \qquad \text{in } \Omega^\varepsilon \times (0,T). \tag{3.2.3}$$

Let ν denote the outward unit normal vector on $\Gamma^\varepsilon = \partial\Omega^\varepsilon$. The normal and tangential components are defined by

$$\sigma^\varepsilon_\nu = (\sigma^\varepsilon \nu)\cdot \nu,$$

$$\sigma^\varepsilon_\tau = \sigma^\varepsilon \nu - \sigma^\varepsilon_\nu \nu,$$

and

$$u^\varepsilon_\nu = u^\varepsilon \cdot \nu, \qquad u^\varepsilon_\tau = u^\varepsilon - u^\varepsilon_\nu \nu.$$

Boundary and Initial Conditions

The upper boundary is clamped:

$$u^\varepsilon = 0 \qquad \text{on } \Gamma^\varepsilon_1 \times (0,T). \tag{3.2.4}$$

Known surface tractions act on the lateral boundary:

$$\sigma^\varepsilon \nu = g^\varepsilon \qquad \text{on } \Gamma^\varepsilon_L \times (0,T). \tag{3.2.5}$$

At the contact surface ω, the Coulomb friction conditions are imposed:

$$u^\varepsilon_\nu = 0, \tag{3.2.6}$$

and

$$|\sigma^\varepsilon_\tau| \le \mu^\varepsilon |R\sigma^\varepsilon_\nu|, \tag{3.2.7}$$

together with the classical stick-slip relations.

The operator

$$R : H^{-1/2}(\omega) \to L^2(\omega)$$

is assumed linear and continuous.

For the damage variable we impose homogeneous conditions

$$\frac{\partial \zeta^\varepsilon}{\partial \nu} = 0 \quad \text{on } (\omega \cup \Gamma^\varepsilon_L) \times (0,T), \tag{3.2.8}$$

and

$$\zeta^\varepsilon = 0 \quad \text{on } \Gamma_1^\varepsilon \times (0,T). \tag{3.2.9}$$

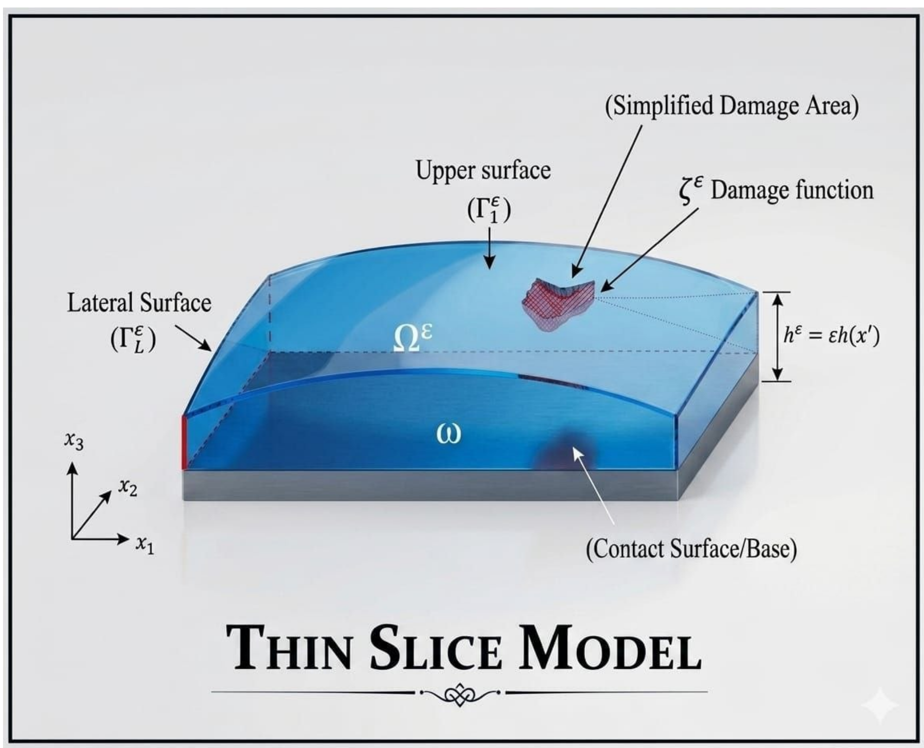
Finally,

$$u^\varepsilon(0) = u_0^\varepsilon, \qquad \zeta^\varepsilon(0) = \zeta_0^\varepsilon \quad \text{in } \Omega^\varepsilon. \tag{3.2.10}$$

The objective of this work is to derive an asymptotic model associated with (3.2.1)–(3.2.10) as

$$\varepsilon \to 0.$$

3.2.1 Geometric description of the thin domain



3.3 Weak Formulation and Main Assumptions

We now introduce the assumptions required for the weak formulation of the problem.

3.3.1 Constitutive Assumptions

The elasticity operator

$$\mathcal{A}^\varepsilon : \Omega^\varepsilon \times S^3 \times \mathbb{R} \to S^3$$

is assumed to satisfy:

(A1) There exists a constant $L_{\mathcal{A}^\varepsilon} > 0$ such that

$$|\mathcal{A}^\varepsilon(x, \eta_1, \zeta_1) - \mathcal{A}^\varepsilon(x, \eta_2, \zeta_2)| \leq L_{\mathcal{A}^\varepsilon} \left(|\eta_1 - \eta_2| + |\zeta_1 - \zeta_2|\right),$$

for all $\eta_1, \eta_2 \in S^3$, $\zeta_1, \zeta_2 \in \mathbb{R}$, a.e. in Ω^ε.

(A2) For every $(\eta, \zeta) \in S^3 \times \mathbb{R}$, the mapping

$$x \mapsto \mathcal{A}^\varepsilon(x, \eta, \zeta)$$

is Lebesgue measurable.

(A3)

$$\mathcal{A}^\varepsilon(\cdot, 0, 0) \in Q^\varepsilon.$$

The viscosity operator

$$B^\varepsilon : \Omega^\varepsilon \times S^3 \to S^3$$

satisfies:

(B1) There exists $L_{\mathcal{B}^\varepsilon} > 0$ such that

$$|\mathcal{B}^\varepsilon(x, \eta_1) - \mathcal{B}^\varepsilon(x, \eta_2)| \leq L_{\mathcal{B}^\varepsilon} |\eta_1 - \eta_2|,$$

for all $\eta_1, \eta_2 \in S^3$.

(B2) There exists $m_{\mathcal{B}^\varepsilon} > 0$ such that

$$\big(\mathcal{B}^\varepsilon(x, \eta_1) - \mathcal{B}^\varepsilon(x, \eta_2)\big) : (\eta_1 - \eta_2) \geq m_{\mathcal{B}^\varepsilon} |\eta_1 - \eta_2|^2,$$

for all $\eta_1, \eta_2 \in S^3$.

(B3) For every $\eta \in S^3$, the mapping

$$x \mapsto \mathcal{B}^\varepsilon(x, \eta)$$

is measurable.

(B4)

$$\mathcal{B}^\varepsilon(\cdot, 0) \in Q^\varepsilon.$$

The damage source function

$$\phi^\varepsilon : \Omega^\varepsilon \times S^3 \times \mathbb{R} \to \mathbb{R}$$

is assumed to satisfy:

(P1) There exists $L_{\phi^\varepsilon} > 0$ such that

$$|\phi^\varepsilon(x, \eta_1, \zeta_1) - \phi^\varepsilon(x, \eta_2, \zeta_2)| \leq L_{\phi^\varepsilon} \left(|\eta_1 - \eta_2| + |\zeta_1 - \zeta_2|\right),$$

for all admissible arguments.

(P2) For every (η, ζ), the mapping

$$x \mapsto \phi^\varepsilon(x, \eta, \zeta)$$

is measurable.

(P3)

$$\phi^\varepsilon(\cdot, 0, 0) \in L^2(\Omega^\varepsilon).$$

Moreover, the external data satisfy

$$f^\varepsilon \in C([0, T]; H^\varepsilon), \qquad g^\varepsilon \in C([0, T]; L^2(\Gamma_L^\varepsilon)^3), \tag{3.3.1}$$

and

$$\mu^\varepsilon \in L^\infty(\omega), \qquad \mu^\varepsilon \geq 0 \quad \text{a.e. on } \omega. \tag{3.3.2}$$

The initial conditions satisfy

$$u_0^\varepsilon \in V^\varepsilon, \qquad \zeta_0^\varepsilon \in K^\varepsilon, \tag{3.3.3}$$

where

$$K^\varepsilon = \left\{\xi \in H^1_{\Gamma_1^\varepsilon}(\Omega^\varepsilon) \; ; \; 0 \leq \xi \leq 1 \text{ a.e.}\right\}$$

denotes the admissible damage set.

3.3.2 Auxiliary Forms

We define the bilinear form

$$a : H^1(\Omega^\varepsilon) \times H^1(\Omega^\varepsilon) \to \mathbb{R}$$

by

$$a(\zeta, \eta) = \kappa^\varepsilon \int_{\Omega^\varepsilon} \nabla\zeta \cdot \nabla\eta \, dx.$$

The friction functional

$$J : K^\varepsilon \times V^\varepsilon \times V^\varepsilon \to \mathbb{R}$$

is defined as

$$J(\zeta; u, v) = \int_\omega \mu^\varepsilon |R\sigma_\nu^\varepsilon(\zeta, u)||v| \, dx',$$

where

$$\sigma_\nu^\varepsilon(\zeta, u) = A_{33}^\varepsilon(e(u), \zeta) + B_{33}^\varepsilon(e(u)).$$

The continuity of the operator R ensures that J is well defined.

3.3.3 Weak Formulation

We are now in a position to state the variational problem.

Problem 3.3.1 *Find*

$$u^\varepsilon : [0, T] \to V^\varepsilon, \qquad \sigma^\varepsilon : [0, T] \to Q_1^\varepsilon, \qquad \zeta^\varepsilon : [0, T] \to H^1_{\Gamma_1^\varepsilon}(\Omega^\varepsilon),$$

such that for every $t \in [0, T]$:

$$\sigma^\varepsilon(t) = \mathcal{A}^\varepsilon(e(u^\varepsilon(t)), \zeta^\varepsilon(t)) + \mathcal{B}^\varepsilon(e(u^\varepsilon(t))), \tag{3.3.4}$$

and

$$\begin{aligned}
&(\sigma^\varepsilon(t), e(w - u^\varepsilon(t)))_{Q^\varepsilon} \\
&\quad + J(\zeta^\varepsilon(t); u^\varepsilon(t), w) - J(\zeta^\varepsilon(t); u^\varepsilon(t), u^\varepsilon(t)) \\
&\geq (f^\varepsilon(t), w - u^\varepsilon(t))_{H^\varepsilon} + \int_{\Gamma_L^\varepsilon} g^\varepsilon(t) \cdot (w - u^\varepsilon(t)) \, d\alpha,
\end{aligned}$$

for all $w \in V^\varepsilon$.

Moreover,

$$\zeta^\varepsilon(t) \in K^\varepsilon,$$

and

$$(\dot{\zeta}^\varepsilon(t), \xi - \zeta^\varepsilon(t))_{L^2} + a(\zeta^\varepsilon(t), \xi - \zeta^\varepsilon(t))$$
$$\geq (\phi^\varepsilon(e(u^\varepsilon(t)), \zeta^\varepsilon(t)), \xi - \zeta^\varepsilon(t))_{L^2},$$

for all $\xi \in K^\varepsilon$ and a.e. $t \in (0,T)$.

Finally,

$$u^\varepsilon(0) = u_0^\varepsilon, \qquad \zeta^\varepsilon(0) = \zeta_0^\varepsilon. \tag{3.3.5}$$

The following result ensures the well-posedness of the weak problem.

Lemma 3.3.1 *Assume that the hypotheses (A1)–(A3), (B1)–(B4), (P1)–(P3), and* (3.3.1)–(3.3.3) *hold. Then, for every fixed $\varepsilon > 0$, there exists a solution*

$$(u^\varepsilon, \sigma^\varepsilon, \zeta^\varepsilon)$$

to Problem 3.1 satisfying

$$u^\varepsilon \in C^1([0,T]; V^\varepsilon),$$

$$\sigma^\varepsilon \in C([0,T]; Q_1^\varepsilon),$$

and

$$\zeta^\varepsilon \in W^{1,2}(0,T; L^2(\Omega^\varepsilon)) \cap L^2(0,T; H^1_{\Gamma_1^\varepsilon}(\Omega^\varepsilon)).$$

Moreover, if the friction coefficient μ^ε is sufficiently small, then the solution is unique.

The existence of a weak solution to the coupled viscoelastic damage contact problem is established using monotonicity methods, compactness arguments, and evolution equation techniques in Banach spaces [16].

3.4 A Priori Estimates

We now derive uniform estimates for the solution of Problem 3.1. These estimates constitute the main tool for the asymptotic analysis as $\varepsilon \to 0$.

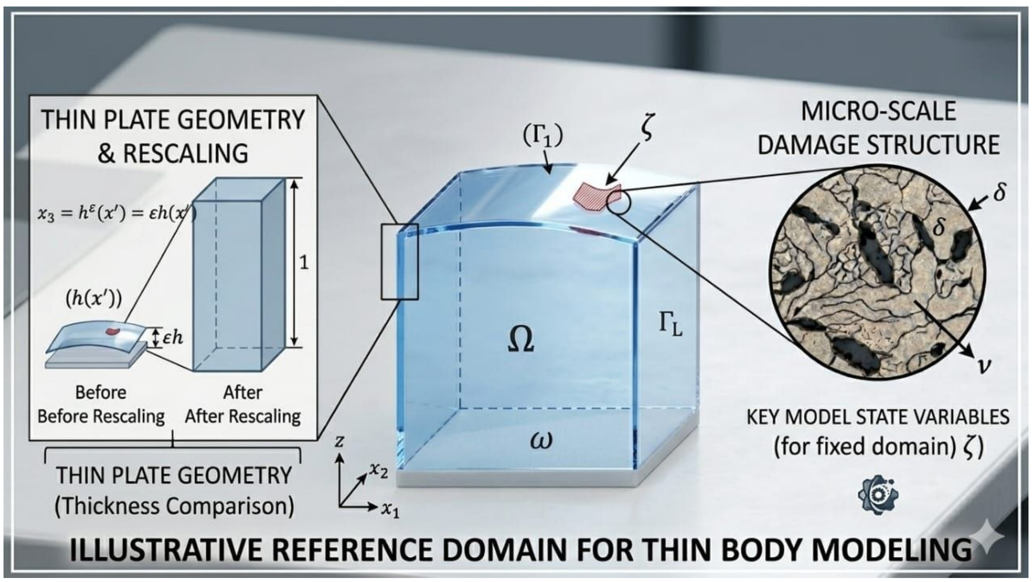
3.4.1 Rescaled Configuration

Following the classical scaling used in thin-film mechanics, we introduce the fixed reference domain

$$\Omega = \left\{(x', z) \in \mathbb{R}^3 \; ; \; x' \in \omega, \quad 0 < z < h(x')\right\},$$

through the change of variables

$$x_3 = \varepsilon z.$$



The unknowns are rescaled according to the anisotropic behavior of thin structures. For $(x', z, t) \in \Omega \times (0, T)$, we define

$$\hat{u}_i^\varepsilon(x', z, t) = \varepsilon^{-1} u_i^\varepsilon(x', \varepsilon z, t), \qquad i = 1, 2,$$

$$\hat{u}_3^\varepsilon(x', z, t) = \varepsilon^{-2} u_3^\varepsilon(x', \varepsilon z, t),$$

$$\hat{\zeta}^\varepsilon(x', z, t) = \zeta^\varepsilon(x', \varepsilon z, t),$$

and

$$\hat{\sigma}_{ij}^\varepsilon(x', z, t) = \sigma_{ij}^\varepsilon(x', \varepsilon z, t), \qquad 1 \leq i, j \leq 3.$$

The associated functional spaces become

$$H = L^2(\Omega)^3, \qquad Q = L_s^2(\Omega)^{3\times 3},$$

$$H_1 = H^1(\Omega)^3, \qquad Q_1 = H_s^1(\Omega)^{3\times 3},$$

and

$$V = \left\{ v \in H^1(\Omega)^3 \; ; \; v = 0 \text{ on } \Gamma_1, \quad v_3 = 0 \text{ on } \omega \right\}.$$

We further introduce the ε-independent operators

$$\mathcal{A} : \Omega \times S^3 \times \mathbb{R} \to S^3,$$

$$\mathcal{B} : \Omega \times S^3 \to S^3,$$

and

$$\phi : \Omega \times S^3 \times \mathbb{R} \to \mathbb{R},$$

defined by

$$A_{ij}(x', z, \cdot, \cdot) = A^{\varepsilon}_{ij}(x', \varepsilon z, \cdot, \cdot),$$

$$B_{ij}(x', z, \cdot) = B^{\varepsilon}_{ij}(x', \varepsilon z, \cdot),$$

and

$$\phi(x', z, \cdot, \cdot) = \phi^{\varepsilon}(x', \varepsilon z, \cdot, \cdot).$$

These operators satisfy assumptions analogous to those imposed on $\mathcal{A}^{\varepsilon}$, $\mathcal{B}^{\varepsilon}$, and ϕ^{ε}, with constants independent of ε.

The data are assumed to satisfy the scaling

$$f = \varepsilon f^{\varepsilon}, \qquad g = \varepsilon g^{\varepsilon},$$

$$u_{0i} = \varepsilon^{-1} u^{\varepsilon}_{0i}, \quad i = 1, 2,$$

$$u_{03} = \varepsilon^{-2} u^{\varepsilon}_{03},$$

$$\zeta_0 = \zeta^{\varepsilon}_0, \qquad \mu = \mu^{\varepsilon}, \qquad \kappa = \varepsilon^{-2} \kappa^{\varepsilon}.$$

We now establish the fundamental estimates.

Theorem 3.4.1 *Let*

$$(u^{\varepsilon}, \sigma^{\varepsilon}, \zeta^{\varepsilon})$$

be the solution of Problem 3.1. Then there exists a constant $C > 0$, independent of ε and t, such that for all $t \in [0, T]$, the following estimates hold:

$$\sum_{i=1}^{2} \|\partial_z \hat{u}^{\varepsilon}_i(t)\|^2_{L^2(\Omega)} + \varepsilon^2 \sum_{i,j=1}^{2} \left\|\partial_{x_j} \hat{u}^{\varepsilon}_i(t)\right\|^2_{L^2(\Omega)} \tag{3.4.1}$$

$$+\varepsilon^2 \|\partial_z \hat{u}_3^\varepsilon(t)\|^2_{L^2(\Omega)} + \varepsilon^4 \sum_{j=1}^{2} \left\|\partial_{x_j} \hat{u}_3^\varepsilon(t)\right\|^2_{L^2(\Omega)} \le C,$$

$$\varepsilon^2 \sum_{i=1}^{2} \left\|\partial_{x_i} \hat{\zeta}^\varepsilon(t)\right\|^2_{L^2(\Omega)} + \left\|\partial_z \hat{\zeta}^\varepsilon(t)\right\|^2_{L^2(\Omega)} \le C, \tag{3.4.2}$$

and

$$\int_0^t \|\hat{\zeta}^\varepsilon(s)\|^2_{L^2(\Omega)}\, ds \le C. \tag{3.4.3}$$

The a priori estimates and energy methods used in this chapter are based on classical results in functional analysis, variational inequalities, and thin domain mechanics. The following references : provide the main theoretical tools and related developments.

Proof. The proof relies on energy arguments coupled with Korn, Poincaré, and Gronwall inequalities [15, 17], under the standing assumptions on the constitutive operators and the data of the problem.

For ease of reading, we present only a brief outline of the main steps of the proof. The full technical details and a complete derivation of similar arguments in the context of thin viscoelastic structures and contact problems can be found in [14].

Step 1: Mechanical estimate.

Choosing $w = 0$ in the variational inequality and using the positivity of the friction functional,

$$J(\zeta^\varepsilon; u^\varepsilon, u^\varepsilon) \ge 0,$$

we obtain

$$(\sigma^\varepsilon, e(u^\varepsilon))_{Q^\varepsilon} \le (f^\varepsilon, u^\varepsilon)_{H^\varepsilon} + \int_{\Gamma_L^\varepsilon} g^\varepsilon \cdot u^\varepsilon\, d\alpha.$$

Using the constitutive law and the monotonicity of $\mathcal{B}^\varepsilon$, we deduce

$$m_{\mathcal{B}^\varepsilon} \|e(u^\varepsilon)\|^2_{Q^\varepsilon} \le C\left(1 + \|e(u^\varepsilon)\|_{Q^\varepsilon} + \|\zeta^\varepsilon\|_{L^2}\right).$$

The Poincaré and Korn inequalities imply

$$\|u^\varepsilon\|_{H^\varepsilon} \le C\|e(u^\varepsilon)\|_{Q^\varepsilon}.$$

Therefore,

$$\|e(u^\varepsilon)\|_{Q^\varepsilon} \le C\left(1 + \|\zeta^\varepsilon\|_{L^2}\right).$$

After rescaling to the fixed domain Ω, we obtain

$$\varepsilon^{-1}\|e(u^\varepsilon)\|_{Q^\varepsilon}^2 \le C\left(1+\varepsilon^{-1}\|\zeta^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2\right). \tag{3.4.4}$$

Step 2: Damage estimate.

Choosing

$$\xi = \zeta^\varepsilon(t)$$

in the variational inequality associated with the damage field yields

$$(\dot{\zeta}^\varepsilon, \zeta^\varepsilon) + a(\zeta^\varepsilon, \zeta^\varepsilon) \le (\phi^\varepsilon(e(u^\varepsilon), \zeta^\varepsilon), \zeta^\varepsilon).$$

Since

$$(\dot{\zeta}^\varepsilon, \zeta^\varepsilon) = \frac{1}{2}\frac{d}{dt}\|\zeta^\varepsilon\|_{L^2}^2,$$

we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\zeta^\varepsilon\|_{L^2}^2 + \kappa^\varepsilon\|\nabla\zeta^\varepsilon\|_{L^2}^2\\ &\le (\phi^\varepsilon(e(u^\varepsilon), \zeta^\varepsilon), \zeta^\varepsilon).\end{aligned}$$

Using the Lipschitz continuity of ϕ^ε, Young's inequality, and estimate (3.4.4), we deduce

$$\begin{aligned}&\frac{d}{dt}\|\zeta^\varepsilon\|_{L^2}^2 + \kappa^\varepsilon\|\nabla\zeta^\varepsilon\|_{L^2}^2\\ &\le C\left(1+\|\zeta^\varepsilon\|_{L^2}^2\right).\end{aligned}$$

Integrating over $(0,t)$ and applying Gronwall's inequality gives

$$\|\zeta^\varepsilon(t)\|_{L^2}^2 + \kappa^\varepsilon\int_0^t \|\nabla\zeta^\varepsilon(s)\|_{L^2}^2 ds \le C.$$

After rescaling, this yields (3.4.2) and (3.4.3).

Step 3: Conclusion.

Combining (3.4.4), the rescaled Korn inequality, and the previous damage estimate, we obtain (3.4.1). This completes the proof. ■

3.5 Convergence Results

In this section we establish the asymptotic behavior of the solution $(u^\varepsilon, \sigma^\varepsilon, \zeta^\varepsilon)$ as $\varepsilon \to 0$.

3.5.1 Functional Setting in the Limit Domain

We introduce the following limit spaces:

$$V_z = \left\{\varphi \in L^2(\Omega)\ ;\ \partial_z\varphi \in L^2(\Omega), \quad \varphi = 0 \text{ on } \Gamma_1\right\},$$

$$H_z = \left\{\varphi \in V_z\ ;\ \partial_{zz}\varphi \in L^2(\Omega)\right\}, \qquad (H_z)^2 = H_z \times H_z,$$

$$H_{\text{div}} = \left\{\varphi \in L^2(\Omega)^3\ ;\ \operatorname{div}\varphi \in L^2(\Omega)\right\},$$

$$H^1_{\text{div}} = \left\{\varphi \in H^1(\Omega)^3\ ;\ \operatorname{div}\varphi \in H^1(\Omega)\right\},$$

$$\Pi(H^1_{\text{div}}) = \left\{(v_1, v_2)\ ;\ (v_1, v_2, v_3) \in H^1_{\text{div}}\right\}.$$

The associated norms are

$$\|\varphi\|_{V_z} = \|\varphi\|_{L^2(\Omega)} + \|\partial_z\varphi\|_{L^2(\Omega)},$$

$$\|\varphi\|_{H_z} = \|\varphi\|_{L^2(\Omega)} + \|\partial_z\varphi\|_{L^2(\Omega)} + \|\partial_{zz}\varphi\|_{L^2(\Omega)}.$$

3.5.2 Main Convergence Result

We now state the main asymptotic result.

Theorem 3.5.1 *Let $(u^\varepsilon, \sigma^\varepsilon, \zeta^\varepsilon)$ be the solution of Problem 3.1. Under the assumptions of Lemma 5.1, there exist limit functions*

$$u^\star = (u_1^\star, u_2^\star) \in C^1(0, T; (H_z)^2),$$

$$\zeta^\star \in W^{1,2}(0, T; V_z) \cap L^2(0, T; H_z),$$

$$\sigma^\star = (\sigma_1^\star, \sigma_2^\star) \in C(0, T; \Pi(H^1_{\text{div}})),$$

such that, as $\varepsilon \to 0$,

$$\hat{u}_i^\varepsilon \to u_i^\star \quad i = 1, 2 \quad \textit{strongly in } C^1(0, T; V_z), \tag{3.5.1}$$

$$\hat{\zeta}^\varepsilon \to \zeta^\star \quad \textit{strongly in } C(0, T; V_z), \tag{3.5.2}$$

$$\hat{\sigma}_{i3}^\varepsilon \to \sigma_i^\star \quad i = 1, 2 \quad \textit{strongly in } C(0, T; V_z). \tag{3.5.3}$$

Moreover, the following scaled quantities vanish strongly in $C(0,T;L^2(\Omega))$:

$$\varepsilon\hat{u}_3^\varepsilon \to 0, \qquad \varepsilon\partial_{x_j}\hat{u}_i^\varepsilon \to 0, \qquad \varepsilon^2\partial_{x_j}\hat{u}_3^\varepsilon \to 0, \qquad \varepsilon\partial_z\hat{u}_3^\varepsilon \to 0, \varepsilon\partial_{x_j}\hat{\zeta}^\varepsilon \to 0, \quad (i,j{=}1,2) \tag{3.5.4}$$

Finally,

$$R\widehat{\sigma}_\nu^\varepsilon \to R\sigma_\nu^\star \quad \textit{strongly in } C(0,T;L^2(\omega)).$$

Proof. The proof is based on the uniform estimates obtained in Theorem 4.1 and compactness arguments.

Step 1: Compactness for the displacement.

From the a priori estimates, we have uniform bounds:

$$\partial_z\hat{u}_i^\varepsilon,\ \partial_{zz}\hat{u}_i^\varepsilon \text{ bounded in } L^2(\Omega), \quad i = 1,2,3.$$

Using the Poincaré inequality in the vertical direction,

$$\|\varphi\|_{L^2(\Omega)} \leq h^*\|\partial_z\varphi\|_{L^2(\Omega)},$$

applied to $\varphi = \partial_z\hat{u}_i^\varepsilon$, we obtain

$$\|\hat{u}_i^\varepsilon\|_{H_z} \leq C, \quad i = 1,2,3.$$

Hence $\{\hat{u}_i^\varepsilon\}$ is bounded in $C^1(0,T;H_z)$. By compact embedding

$$C^1(0,T;H_z) \hookrightarrow C^1(0,T;V_z),$$

we deduce convergence (3.5.1).

Step 2: Compactness for the damage.

From the estimates of Section 4,

$$\|\hat{\zeta}^\varepsilon\|_{H_z}^2 + \int_0^t \|\hat{\zeta}^\varepsilon(s)\|_{V_z}^2 ds \leq C.$$

Thus $\{\hat{\zeta}^\varepsilon\}$ is bounded in

$$W^{1,2}(0,T;V_z) \cap L^2(0,T;H_z).$$

Using the compact embedding

$$W^{1,2}(0,T;H_z) \hookrightarrow C(0,T;H_z) \hookrightarrow C(0,T;V_z),$$

we obtain (3.5.2).

Step 3: Stress convergence.

From the constitutive laws and uniform bounds, we deduce that σ^ε is bounded in $C(0,T;H^1_{\mathrm{div}})$. Hence, up to a subsequence,

$$\hat{\sigma}^\varepsilon \rightharpoonup \sigma^\star \quad \text{in } C(0,T;H^1_{\mathrm{div}}).$$

The strong convergence in (3.5.3) follows by compactness arguments and identification of the limit.

Step 4: Vanishing scaled terms.

The estimates of Theorem 4.1 directly imply that all scaled quantities in (3.5.4) converge strongly to zero in $C(0,T;L^2(\Omega))$.

Step 5: Convergence of contact stress.

The trace operator

$$H_{\mathrm{div}} \to H^{-1/2}(\omega)$$

is continuous. Therefore,

$$R\widehat{\sigma}^\varepsilon_\nu \to R\sigma^\star_\nu \quad \text{in } C(0,T;L^2(\omega)).$$

This completes the proof. ■

Conclusion

This work has been devoted to the mathematical study of a quasistatic viscoelastic contact problem with damage evolution in a thin three-dimensional domain of thickness of order ε. The model combines viscoelastic effects, damage evolution, and Coulomb-type frictional contact conditions, leading to a nonlinear coupled system arising in contact mechanics and continuum mechanics [5, 7, 13, 18].

The analysis was developed within a variational framework based on functional analysis, Sobolev spaces, and nonlinear evolution methods [1, 3, 8, 15, 2, 17]. After establishing the weak formulation and the existence setting, we derived uniform a priori estimates independent of the small parameter ε. These estimates provide the compactness properties required for the asymptotic analysis and reveal the anisotropic effects induced by the thin geometry.

Using compactness and convergence techniques, we proved the convergence of the displacement, stress, and damage fields as $\varepsilon \to 0$. This limiting process rigorously justifies the passage from the original three-dimensional model to an effective lower-dimensional formulation, following the classical asymptotic approach developed for thin structures and reduced models [6, 9]. In addition to the developments presented in Chapter 2, the continuation of the analysis and the study of the derived limit model can be further pursued by following the asymptotic and variational approaches developed in [14], which provide a relevant framework for the investigation of thin viscoelastic structures and related reduced models.

From a mathematical point of view, the asymptotic framework based on uniform energy estimates, compactness arguments, and weak convergence techniques proves to be sufficient to establish the convergence results. In this sense, the analytical strategy adopted in this work can be considered complete, in the sense that it successfully captures the limit behavior of the solution as $\varepsilon \to 0$ and leads to a well-justified reduced model.

Abstract

This work is devoted to the mathematical analysis of a quasistatic viscoelastic contact problem with damage evolution in a thin three-dimensional domain of thickness of order ε. The model combines viscoelastic effects, damage evolution, and frictional contact conditions, leading to a nonlinear coupled system. The analysis is based on a variational framework, Sobolev spaces, and compactness methods. Uniform a priori estimates independent of ε are established, allowing the derivation of convergence results as ε→0 and the justification of a reduced lower-dimensional limit model.

Keywords: viscoelasticity, damage mechanics, contact problems, thin structures, variational methods, asymptotic analysis.

Résumé

Ce travail est consacré à l'analyse mathématique d'un problème de contact quasistatique en viscoélasticité avec évolution de dommage dans un domaine tridimensionnel mince d'épaisseur de l'ordre de ε. Le modèle couple la viscoélasticité non linéaire, l'évolution du dommage et les conditions de contact avec frottement de type Coulomb. L'étude repose sur un cadre variationnel, les espaces de Sobolev et des méthodes de compacité. Des estimations a priori uniformes indépendantes de ε sont établies, permettant de justifier la convergence vers un modèle limite de dimension réduite lorsque ε→0**.**

Mots clés : viscoélasticité, endommagement, contact, structures minces, méthodes variationnelles, analyse asymptotique.

الملخص

تتناول هذه الرسالة الدراسة الرياضية لمسألة تماس شبه ساكن في اللزوجة-المرونة مع تطور التلف داخل مجال ثلاثي الأبعاد رقيق بسماكة من الرتبة ε يجمع النموذج بين اللزوجة-المرونة غير الخطية، وتطور التلف، وشروط التماس مع الاحتكاك من نوع كولوم، مما يؤدي إلى نظام غير خطي مترابط. تعتمد الدراسة على إطار تبايني وفضاءات سوبوليف وتقنيات التراص . تم إثبات تقديرات قبلية موحدة مستقلة عن ε ، مما يسمح بإثبات نتائج التقارب ε يؤول نحو الصفر.

الكلمات المفتاحية: اللزوجة-المرونة، التلف، مسائل التماس، الهياكل الرقيقة، الطرق التباينية، التحليل التقاربي.

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