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بسم الله الرحمن الرحيم

إلى والدي

عبد العزيز جوقة الذي يرقد تحت التراب منذ حوالي العقد من الزمن منتظرا دعاءا يصله أو صدقة جارية تثري عمله

إلى والدتي

منوبية نصبة التي رغم أنها بلغت من الكبر عتيا مازالت تحوي وترعى أولادي بصبر وحكمة

إلى أختي الغالية

فاطمة الزهراء التي تتحمل معي متاعب الحياة وتهون عليا الكثير من صعابها

إلى أخي العزيز

ابن العزيز سعد الياس الذي اعتبره أبي رغم انه يصغرني سننا

إلى زوجي الفاضل

زوزو عبد القادر رفيق دربي في السراء والضراء ومفخرتي وظل بيني وأبو أولادي

عكاشه و**عدي** و**عاصم**

إلى عائلتي الثانية عائلة زوجي:

إلى أمي الثانية

لسود عائشة جدة أولادي التي ترعاهم وتعامل الجميع برحابة صدر وسمو أخلاق

إلى أخواتي اللاتي لم تلدهم أمي

مريم وسعيدة زوزو لا أنسى ما عشت عشرتهما الطيبة ومعاملتهما الراقية

إلى اخو زوجي **عبد الباسط** وعم أبنائي

إلى خالتي وجارتي **خيرة نصبة** انسي وأم الجميع

إلى هؤلاء جميعا اهدي رسالتي

إلى هؤلاء جميعا لن أنسى ما حييت فضلهم ومساعدتهم لي عند أقصى ضغوط العمل والدراسة وتربية الأولاد واصل إلى هذا المستوى العلمي الذي أتمنى من الله أن يصلهم جميعا فضل اجر صدقته

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أسأل الله أن يبارك لي في هذا النجاح، وأن يجعله بداية خير وتوفيق في مستقبلي العلمي والمهني.

وآخرُ دعوانا أنِ الحمدُ للهِ ربِّ العالمين.

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ولا يفوتنا أن نشكر كل من ساعدنا من قريب أو بعيد في انجاز هذا العمل .

وفي الأخير نسال الله أن يجزي الجميع خير جزاء , وان يوفقهم لما فيه خير

Abstract

In this work, we study the stability and controllability of the wave equation with dynamical boundary control. The considered system is governed by a wave equation defined on a bounded domain together with mixed boundary conditions, including a dynamical condition acting .on a part of the boundary

The main objective is to analyze the well-posedness of the system using semigroup theory and operator methods. We first reformulate the partial differential equation into an abstract evolution equation in an appropriate Hilbert space. Then, by applying semigroup techniques and dissipative operator theory, we establish the existence and uniqueness .of solutions

Furthermore, we investigate the asymptotic behavior of the energy and derive stability results for the system. In particular, we discuss strong stability and the decay properties of solutions under suitable .assumptions on the boundary feedback

The second part of this work is devoted to controllability issues. Using observability inequalities and duality arguments, we study the exact controllability of the wave equation with dynamical boundary control. The obtained results show the effectiveness of boundary controls in .stabilizing and steering the system

This study combines tools from functional analysis, partial differential equations, semigroup theory, and control theory

ملخص

تتناول هذه المذكرة دراسة الاستقرار والتحكم في معادلة الموجة مع تحكم ديناميكي على الحدود. يتمثل النظام المدروس في معادلة موجة معرفة على مجال محدود مع شروط حدية مختلطة، حيث يُطبق التحكم الديناميكي على جزء من الحدود.

في البداية، نقوم بإعادة صياغة المسألة التفاضلية الجزئية في إطار مجرد داخل فضاء هلبرت مناسب، وذلك باستعمال نظرية أشباه الزمر والمؤثرات غير المحدودة. ثم ندرس الوجود والوحدانية للحلول من خلال إثبات أن المؤثر المرتبط بالنظام يولد شبه زمرة انكماشية من الصنف

بعد ذلك، نهتم بدراسة استقرار النظام والسلوك طويل المدى للطاقة، حيث يتم إثبات نتائج تتعلق بالاستقرار القوي ومعدل اضمحلال الطاقة باستعمال طرق تحليلية تعتمد على نظرية المؤثرات والطاقة.

كما نخصص جزءًا من هذا العمل لدراسة قابلية التحكم لمعادلة الموجة مع التحكم الحدودي الديناميكي، وذلك بالاعتماد على متراجحات الرصد ومبدأ الازدواجية، بهدف إثبات قابلية التحكم التام للنظام تحت شروط مناسبة.

يعتمد هذا العمل على أدوات من التحليل الدالي، المعادلات التفاضلية الجزئية، نظرية أشباه الزمر، ونظرية التحكم.

Contents

Notations

$W^{m,p}(\Omega)$	The sobolev space of function on Ω derivatives up to order m in L^p
$H^m = W^{m,2}(\Omega)$	Sobolev Spaces ,
∂	The operator of patial differentiation ,
$D(A)$	The domain of the operator A ,
D^α	A partial derivative of multi-order α ,
$\sigma(A)$	The spectrum of operator A ,
$\lvert . \rvert$	The euclidean norm on $\mathbb{R}^d$
$N(A)$	The kernel of the operator A ,
$R(A)$	The range of the operator A ,
$\lVert . \rVert_x$	The norm on a normed space X ,
$\mathcal{L}(X)$	The space of bounded linear operators from X into X ,
X'	The dual space of X ,
$\mathrm{Re}\,\langle ., . \rangle$	The real part of the inner product ,
$C_0^\infty(\Omega)$	The test functions space ,
$(T(t))_{t\geq 0}$	A semigroup
$C(X, Y)$	The space of all continuous function from X into Y ,
$\Omega \subset \mathbb{R}^N$	A domain of $\mathbb{R}^N$,
$\partial\Omega = \Gamma$	The boundary of Ω ,
$L(\Omega)$	The space of measurable functions on the domain Ω
$L^p(\Omega)$	The space of functions whose absolute value raised to the power p is integrable,
$L_p^\infty(\Omega)$	The space of essentilly bounded function with weal-* convergence ,
$C(\Omega)$	The space of continuous function on the domain Ω ,
$C^\infty(\Omega)$	The space of infinitely differentiable functions on Ω ,
$C(\overline{\Omega})$	The space of coninuous functions on the closure of Ω ,
$B_r(x)$	The open ball of radius r centered at point x
$S_r(x)$	The sphere of radius r centered at point x ,
$\langle ., . \rangle$	The inner Product ,
$\rho(A)$	The resolvent set of the operator A ,
$\mathbb{R}^d$	The space of all ordered d-tuples of real numbers ,
$\Gamma(.)$	Fonction Gamma d'Euler.

Introduction

Partial differential equations are among the most important mathematical tools used to model natural, physical, and engineering phenomena. They describe the behavior of systems that evolve in space and time, such as the vibration of strings and membranes, heat and wave propagation, fluid flow, and elasticity. Among the most famous of these equations are the wave equation, the heat equation, and Laplace's equation, each of which possesses distinct physical and mathematical properties.

The solution of any partial differential equation requires the specification of suitable initial and boundary conditions. Often, it is difficult to find exact solutions, which necessitates studying the qualitative behavior of solutions. One of the most important aspects of this behavior is the issue of stability, i.e., does the energy of the system decay over time $t > 0$? If so, what is the rate of decay $\frac{M}{t^\beta} : \beta > 0$? Is it exponential or polynomial? These questions are fundamental in control theory, engineering, and mechanics, because an unstable system may lead to catastrophic outcomes.

Wave equations are among the fundamental models in describing numerous physical and engineering phenomena, such as the vibration of strings and membranes, sound propagation, and elasticity. In many practical applications, classical boundary control is insufficient to achieve the desired behavior of the system, which has led to the emergence of the concept of dynamical boundary control, where an ordinary differential equation is added on a part of the boundary to represent the interaction of the actuator or controller with the system. Such mechanisms are used in modeling pressurized gas tanks, mass-spring dampers, and indirect control systems proposed by Russell [12]. Unlike classical boundary control, which imposes an instantaneous value of the function or its derivative, dynamical control introduces a new degree of freedom on the boundary, allowing for more realistic modeling of systems involving electromechanical actuators or viscoelastic effects.

Our thesis is organized as follows: In the first chapter we recall some basic definitions and the necessary preliminary concepts and results, including Lebesgue and Sobolev spaces, embedding theorems, semigroup theory, and dissipative operators, as well as fundamental stability theorems such as the Lumer-Phillips and Hille-Yosida theorems. In the next chapter we present certain criteria under which the problem proposed of the wave equation system in a bounded multidimensional domain, equipped with a dynamical boundary condition of the form:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \Omega \times \mathbb{R}_+ \\ u = 0 & \text{on } \Gamma_0 \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu} + \eta = 0 & \text{on } \Gamma_1 \times \mathbb{R}_+ \\ \eta_t - u_t = v & \text{on } \Gamma_1 \times \mathbb{R}_+ \end{cases} \tag{.0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ ($\Gamma_0 \neq \emptyset$, $\Gamma_1 \neq \emptyset$), η represents the dynamical boundary variable, and v is the boundary control is polynomial energy decay rate $\frac{1}{t}$ under a geometric condition but is not uniformly stable. Furthermore, we will study the observability and the exact controllability of this system with a dynamical control.
The treatment of such systems represents a mathematical challenge in higher dimensions, because the damping operator resulting from the dynamical control is bounded but non-compact in the energy space, and the generator's resolvent is not compact, which hinders the application of classical methods such as LaSalle's invariance principle [28] or the spectral decomposition theory of Benchimol [11]. In one dimension, the resolvent is often compact or compact perturbation theory can be applied, but in higher dimensions this property fails, requiring more general tools. Therefore, this thesis relies on modern tools:
Our proof is divided into four sections: Firstly, we bigen with the strong stability of the solution of $U(t) = S_{A+B}(t)$ in the absence of the compactness of the infinitesimal $A+B$ via the Arendt-Batty criterion. In section 2, using a criteria of Huang and Prüss, we prove that this solution $U(t)$ of system .0.1 is not uniformly stable in the energy space H. In section 3 we will establish a polynomial energy decay rate for smooth solutions by a multiplier method. Finally, we will use a new adaptation of Hilbert Uniqueness Method (HUM) to show the exact controllability of

$$U_t = AU + V, \text{ with } U(0) = U_0$$

by means of some singular dynamical controls v for all initial data in the usual energy space, whenever v satisfying

$$v(t) = v_0(t) - \frac{d}{dt}v_1(t) \quad \text{on } \Gamma_1 \times [0,T] \quad \text{with } v_0, v_1 \in L^2(0,T; L^2(\Gamma_1)).$$

In the last chapter we consider the stabilization of the wave equation with variable coefficients and a delay in the dissipative internal feedback

$$\begin{cases} u_{tt} + \mathcal{A}u + a(x)\left[\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)\right]) = 0 & \text{in } \Omega \times \mathbb{R}_+ \\ u = 0 & \text{on } \Gamma_2 \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu_A} = 0 & \text{on } \Gamma_1 \times \mathbb{R}_+ \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega \times (0,\tau) \end{cases}$$

where $\mathcal{A}u = -\operatorname{div}(A(x)\nabla u)$ with $A(x)$ symmetric positive definite, $a(x) \geq 0$ is a weight function, $\tau > 0$ is the delay, $\mu_1 > 0$, $\mu_2 \neq 0$, and $\frac{\partial u}{\partial \nu_A} = \langle A(x)\nabla u, \nu\rangle$ is the co-normal derivative. It is well known that even arbitrarily small delays can destabilize a system [39, 41, **?**]; however, under the condition

$$|\mu_2| < \mu_1$$

we prove that exponential stability can be recovered. The analysis uses differential geometric methods introduced by Yao [31] and further developed in [29, 42, 43, 44, 8, 21, 22, 46, 53, 47, 32, 33, 34, 35, 36, 54]. A key assumption is the existence of a vector field H on $\overline{\Omega}$ such that its covariant derivative $D_g H$ (with respect to the Riemannian metric $g = A^{-1}$) satisfies

$$D_g H(X,X) \geq \rho_0 |X|_g^2 \quad \text{for } X \in \mathbb{R}^n_x, x \in \overline{\Omega}. \tag{.0.2}$$

for some $\rho_0 > 0$, and $\langle H, \nu\rangle \leq 0$ on Γ_2. Under this geometric condition, we define an appropriate energy functional:

$$E(t) = \frac{1}{2}\int_\Omega \left(u_t^2 + \sum_{i,j} a_{ij} u_{x_i} u_{x_j} \right) dx + \xi \int_0^1 \int_\Omega a(x) u_t^2(x, t-\rho\tau)\, dx\, d\rho,$$

with ξ satisfying $\tau|\mu_2| < 2\xi < \tau(2\mu_1 - |\mu_2|)$. We then prove the exponential decay estimate:

$$E(t) \leq C_1 e^{-C_2 t} E(0), \qquad \forall t \geq 0,$$

for some constants $C_1, C_2 > 0$. This result extends previous works on constant coefficient wave equations with delay [17, 18, **?**, **?**] to the variable coefficient setting, and complements the boundary controllability results of Chapter 2.
Our purpose in this chapter is study the wave equation with variable coefficients and a delayed internal feedback. It presents the problem formulation, the geometric Assumption (.0.2), well-posedness, and the proof of exponential stability through observability inequalities and multiplier techniques. Consequently, the results rely on differential geometry and extend the constant coefficient theory to variable coefficients. These results contribute to a deeper understanding of the behavior of wave systems with dy-

namical boundary control and with delayed internal damping. They provide a mathematical framework that can be used in engineering and optimal control applications, and open perspectives for studying more complex systems such as plates and elastic structures with memory or time delay.

Chapter I

Preliminaire

This chapter aims to provide a review of the essential mathematical concepts and theories that from the theoretical foundation for the work presented in this thesis. These concepts will be relied upon later in addredding the various topics discussed

Remark I.0.1 *[15]Applying n-times the inequality (**??**) for all $x, y, z_1, z_2, ..., z_n \in X$, we obtain*

Definition I.0.1 *[15]Let (X, d) be space, $x_0 \in X$ and $r > 0$. Then*

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\}$$

is said to be an open ball having radius r with center x_0.

$$\overline{B_r}(x_0) = \{x \in X : d(x, x_0) \leq r\}$$

is said to be a closed ball having radius r with center x_0.

$$S_r(x_0) = \{x \in X : d(x, x_0) = r\}$$

is said to be the boundary ball having radius r with center x_0.

Remark I.0.2 *Open ball, closed ball and the boundary of a ball having radius r with center x_0 are denoted by $B_r(x_0) = B(x_0, r), \overline{B_r}(x_0) = \overline{B_r}(x_0, r)$ and $S_r(x_0) = S(x_0, r)$, respectively.*

Definition I.0.2 *A sequence (x_n) in a metric space (X, d) is said to convergy to a point $x_0 \in X$ if for $\varepsilon > 0$ there exists an integer N such that $n \geq N$ implies that $\mathbb{R}$*

$$d(x_n, x_m) < \varepsilon.$$

We say that the sequence (x_n) is convergent and x_0 is the limit of (x_n). In other words we write

$$x_n \longrightarrow x_0 \quad or \quad \lim_{n \longrightarrow \infty} x_n = x_0.$$

On the other hand, if (x_n) is not convergent, it is divergent.

Definition I.0.3 *A sequence (x_n) in a metric space (X, d) is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists an integer N such that*

$$d(x_n, x_m) < \varepsilon$$

whenever $n \geq N$ and $m \geq M$. In other words, (x_n) is a Cauchy sequence if

$$d(x_n, x_m) \longrightarrow 0 \quad as \quad n, m \longrightarrow \infty$$

or equivalently

$$\lim_{n,m \longrightarrow \infty} d(x_n, x_m) = 0$$

I.1 Normed Space

Definition I.1.1 *[15]Let X be a vector space over the real (or complex) numbers. A function,$\|.\|$, from X into the non-ngative real real numbers is called a norm on X if for all $x, y \in X$ and $\lambda \in \mathbb{R}$, the conditions $\|x\| > 0$ if $x \neq 0$ and $\|x + y\| \leq \|x\| + \|y\|$ are satisfied. The norm of x is denoted by $\|x\|$. Moreover, the vector space X with norm is called normed vector space or normed space .*

i) $\|x\| > o$ *if* $x \neq 0$,

ii) $\|x\| = 0 \Longleftrightarrow x = 0$,

iii) $\|\lambda x\| = |\lambda| \, \|x\|$,

iv) $\|x + y\| \leq \|x\| + \|y\|$,

Remark I.1.1 *$\|.\|$ is said to be a semi norm if only the conditions (n3) and (n4) are satisfied. Notice that for a semi norm the conditions $\|x\| = 0$ may be satisfied for non-zero x .*

Theorem I.1.1 *Let $(X, \|.\|)$ be a normed space. Then for all $x, y \in X$, the function defined by*

$$d(x, y) = \|x - y\|$$

is a metric on X. This metric is called normed metric.

Definition I.1.2 *[15] Let $\|.\|_1$ and $\|.\|_2$ norms be given on the vector space X.*
For every $x \in X$ if there exists $c_1, c_2 > 0$ such that

$$c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|_1 .$$

then $\|.\|_1$and $\|.\|_2$ are called equivalent norms.

Definition I.1.3 *As a result of this definition. We can deduce that different norms can determine the same topology. The following result is based on the fact that any two norms on a finite dimensional vector space determine the same topology.*

Theorem I.1.2 *All the norms defined on finite dimensional normed (vector) spaces are equivalent.*

As a consquence of Theorem, all the norms defined on finite dimensional normed spaces describe the same topology on these spaces. For instance, if a squence (x_n) *in norm space* X *is convergent, bounded or Cauchy sequence with respect to the norm* $\|.\|_1$ $(\|.\|_2)$*, then it is also convergent, bounded or Cauchy sequence with respect to the norm* $\|.\|_1$ $(\|.\|_2)$.

Definition I.1.4 *If every point sequence in* A *has a subsquence that converges to an element of* A *in* X*, then the subset of a normed space* X *is said to be compact.*

Definition I.1.5 *Let* X *and* Y *be normed spaces, and let* $\|.\|_X$ *and* $\|.\|_Y$ *be their respective norms. If every bounded subset of* X *maps into a comparatively compact subset of* Y*, then the continuous linear operator* $L \in \mathcal{L}(X, Y)$ *is considered compact. The Heine–Borel theorem for finite dimensions normed spaces might be seen as leading to the following outcome.*

Theorem I.1.3 *Assume that* X *is a finite dimensional normed space and that* $Y \subset X$*. If and only if* Y *is closed and bounded, then the set is compact.*

Remark I.1.2 *According to the theory of metric spaces, every convegent sequence is Cauchy. There re metric spaces and normed spaces where a Cauchy sequence does not converge, hence the oposite of this statement may not always be true. The completeness property of a normed space, which yields a Banach space, is given in the following. Only the basic characteristics of Bach spaces presented here. Additional information is available in the functional and real analysis books listed in the reference section.*

I.2 The Hölder and Young Inequalities

1) [49]Let (X, Γ, μ) be a measured space and let $1 \leq p, q, r \leq +\infty$ be real numbers such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$$

If $u \in L^p(X)$ and $v \in L^q(X)$ then $uv \in L^r(X)$ and,

$$\|uv\|_{L^r(X)} \leq \|u\|_{L^p(X)} \|v\|_{L^q(X)} .$$

2) Let $1 \leq p, q \leq +\infty$ and let $r \geq 1$ be such that,

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

If $u \in L^p(\mathbb{R}^d)$ then for almost all x in $\mathbb{R}^d$ the integral,

$$(u * v)(x) = \int_{\mathbb{R}^d} u(x-y)v(y)dy$$

is convergent, and $u * v$ belongs to $L^r(\mathbb{R}^d)$. Moreover we have,

$$\|u * v\|_{L^r(\mathbb{R}^d)} \leq \|u\|_{L^r(\mathbb{R}^d)} \|v\|_{L^r(\mathbb{R}^d)} .$$

I.3 Approximation of the Identity

1. [49] Let $\rho \in C^\infty(\mathbb{R}^d)$ be such that

$$\operatorname{supp}(\rho) \subset \left\{x \in \mathbb{R}^d : |x| \leq 1\right\}, \rho \geq 0, \int_{\mathbb{R}^d} \rho(x)dx = 1.$$

For ε in $(0,1]$ set,$\rho_\varepsilon(x) = \varepsilon^{-d}\rho(\frac{x}{\varepsilon})$.

$$\lim_{\varepsilon \to 0}(\rho_\varepsilon * u) = u \quad \text{in} \quad L^p(\mathbb{R}^d).$$

The family $[\rho_\varepsilon]_{\varepsilon \in (0,1]}$ is called an appoximation of the identity. Then we have,

2. for any $1 \leq p < +\infty$, if u belongs to $\in L^p(\mathbb{R}^d)$,then

$$\lim_{\varepsilon \to 0}(\rho_\varepsilon * u) = u \quad \text{in} \quad L^p(\mathbb{R}^d).$$

3. If $u : \mathbb{R}^d \longrightarrow \mathbb{C}$ is a bounded uniformly continuous function, then

$$\lim_{\varepsilon \to 0}(\rho_\varepsilon * u) = u \ \text{ in } \ L^\infty(\mathbb{R}^d).$$

4. For any $1 \leq p \leq +\infty$, if u belongs to $L^p(\mathbb{R}^d)$, then for almost all x in $\mathbb{R}^d$,

$$\lim_{\varepsilon \to 0}(\rho_\varepsilon * u)(x) = u(x).$$

5. For any $1 \leq p \leq +\infty$, if u belongs to $L^p(\mathbb{R}^d)$, then for any $\varepsilon \in (0,1]$, the function $\rho_\varepsilon * u$ belongs to $C^\infty(\mathbb{R}^d)$.

I.4 The Lebesgue Space $L^p(\Omega)$

I.4.1 Definition and Basic Properties of $L^p(\Omega)$

[38] Let p be a positive real number and let Ω be domain in $\mathbb{R}^n$. The class of all measurable functions u defined on Ω in indicated by $L^p(\Omega)$, for which

$$\int_\Omega |u(x)|^p \, dx < \infty. \tag{I.4.1}$$

The members of $L^p(\Omega)$ are therfore equivalence classes of measurable functions satisfying (I.4.1), where two functions are comparable if they are equal a.e.in Ω. We find functions in $L^p(\Omega)$ that are identical nearly everywhere in Ω. We disregard this distinction for simplicity and write $u = 0$ in $L^p(\Omega)$ if $u(x) = 0$ a.e.in Ω, and $u \in L^p(\Omega)$ if u fulfills (I.4.1). If $u \in L^p(\Omega)$ and $c \in \mathbb{C}$, then $cu \in L^p(\Omega)$ is obviously true. We must demonstrate that if $u, v \in L^p(\Omega)$, then $u + v \in L^p(\Omega)$ in order to prove that $L^p(\Omega)$ is a vector space. This is a direct result of the inquity thar follows, which will also be helpful in the future.

Lemma I.4.1 *If $1 \leq p \leq \infty$ and $a, b \geq 0$, then*

$$(a+b)^p \leq 2^{p-1}(a^p + b^p). \tag{I.4.2}$$

Lemma I.4.2 *(**The $\mathbf{L}_p$ Norm**) Now will confirm that the functional $\|.\|_p$ defined*

$$\|u\|_p = \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p}$$

is a standard on $L^p(\Omega)$, given $1 < p < \infty$.(If $0 < p < \infty$, it is not a norm). When there is a possibility of domain confusion, we substitute $\|.\|_{p,\Omega}$ for $\|.\|_p$.

It is evident that if and only if $u = 0$ in $L^p(\Omega)$, then $\|u\|_p \geq 0$ and $\|u\|_p = 0$. Additionally

$$\|cu\|_p = |c| \, \|u\|_p \ , \qquad c \in \mathbb{C}.$$

Therefore, after we have confirmed the triangle inequality, we will have demonstrated that $\|.\|_p$ is norm on $L^p(\Omega)$.

$$\|u+v\|_p \leq \|u\|_p + \|v\|_p \ ,$$

Minkowski's inequality is the name given to it. This requires Holder's inequality first.

Corollary I.4.1 *It is possible to apply Hölder's inquality to products of more than two functions. Assume that where $u_j \in L^q(\Omega), 1 \leq j \leq N$, and $p_j > 0, u = \prod_{j=1}^{N} u_j$. Soppose* *that* $\sum_{j=1}^{N}(1/p_j) = 1/q$, *then* $u \in L^q(\Omega)$ *and* $\|u\|_q \leq \prod_{j=1}^{N} \|u_j\|_{p_j}$. *By induction on* $\mathbb{N}$, *this follows from the prior corollary.*

Theorem I.4.1 ***(A Converse of Hölder's Inequality)*** *A mesurable function* u *is a member of* $L^p(\Omega)$ *if and only if*

$$\sup\left\{\int_\Omega |u(x)|\, v(x)dx : v(x) \geq 0 \text{ on } \Omega, \|v\|_{p'} \leq 1\right\} \tag{I.4.3}$$

is finite, and then that supremum equals $\|u\|_p$.

Theorem I.4.2 ***(Minkowski's Inquality)*** *If* $1 \leq p \leq \infty$, *then*

$$\|u+v\|_p \leq \|u\|_p + \|v\|_p . \tag{I.4.4}$$

Theorem I.4.3 ***(Minkowski's Inequality for Integals)*** *Assume* $1 < p < \infty$. *Assume that the function* $y \to \|f(.,y)\|_{p,\mathbb{R}^n}$ *belongs to* $L^1(\mathbb{R}^n)$. *that* f *is measurable on* $\mathbb{R}^n \times \mathbb{R}^n$, *and that* $f(.,y) \in L^p(\mathbb{R}^m)$ *for almost all* $y \in \mathbb{R}^n$. *Then,* $L^p(\mathbb{R}^m)$ *a contains the function* $x \to \int_{\mathbb{R}^n} f(x,y)dy$.

$$\left(\int_{\mathbb{R}^m}\left|\int_{\mathbb{R}^n} f(x,y)dy\right|^p dx\right)^{1/p} \leq \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m}|f(x,y)|^p\, dx\right)^{1/p} dy.$$

that is,

$$\left\|\int_{\mathbb{R}^n} f(.,y)dy\right\|_{p,\mathbb{R}^m} \leq \int_{\mathbb{R}^n}\|f(.,y)\|_{p,\mathbb{R}^m}\, dy.$$

I.4.2 The Space $L^\infty(\Omega)$

If a constant K exists such that $|u(x)| \leq K$ a.e.on Ω, then a function u that is measurable on Ω is said to be basically limited on Ω. The essential supremum of $|u|$ on Ω, represented by ess $\sup_{x\in\Omega}|u(x)|$, is the maximum lower bound of such constants K. The vector space of all functions u that are fundamentally bounded on Ω is represented by $L^\infty(\Omega)$, with functions being once more recognized if they are equal a.e.on Ω. Verification of the functional $\|.\|_\infty$ defined by

$$\|u\|_\infty = \text{ess}\sup_{x\in\Omega}|u(x)|$$

is a norm on $L^\infty(\Omega)$. Moreover, Hölder's inequality (3) and its corollaries extend to cover the two cases $p = 1, p' = \infty$ and $p = \infty, p' = 1$.

I.5 The Sobolev Space $W^{m,p}(\Omega)$

I.5.1 Definitions and Basic Properties

The Sobolev Norms

We define a functional $\|.\|_{m,p}$ where m is a positive integer and $1 \leq p \leq \infty$, as follows:

$$\|u\|_{m,p} = \Big(\sum_{0\leq|\alpha|\leq m} \|D^{\alpha}u\|_p^p\Big)^{\frac{1}{p}} \quad \text{if} \quad 1 \leq p < \infty, \tag{I.5.1}$$

$$\|u\|_{m,\infty} = \max_{0\leq|\alpha|\leq m} \|D^{\alpha}u\|_{\infty} \quad \text{with } p = \infty \tag{I.5.2}$$

for any function u for which the right side makes sence , $\|.\|_p$ being, of course, the norm in $L^p(\Omega)$. In some situations where confusion of domains may occur we will use $\|u\|_{m,p,\Omega}$ in place of $\|u\|_{m,p}$. Evidently (II.0.1) or (I.5.2) defines a norm on any vector space of functions on which the right side takes finite values provided functions are identified in the space if they are equal almost everywhere in Ω.

Sobolev Spaces

For any positive integer m and $1 \leq p \leq \infty$ we consider three vector spaces on which $\|.\|_{m,p}$ is a norm:

a) $H^{m,p}(\Omega) \equiv$the completion of $\left\{u \in C^m(\Omega) : \|u\|_{m,p} < \infty\right\}$ with respect to the norm $\|.\|_{m,p}$.

b) $W^{m,p}(\Omega) \equiv \{u \in L^p(\Omega) : D^{\alpha}u \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\}$

c) $W_0^{m,p}(\Omega) \equiv$ the closure of $C_0^{\infty}(\Omega)$ in the space $W^{m,p}(\Omega)$.

Equipped with the appropriate norm (1) or (2) thesΩe are called Sobolev Space over Ω. Clearly $W^{0,p}(\Omega) = L^p(\Omega)$, and if $1 \leq p \leq \infty$, $W_0^{0,p}(\Omega) = L^p(\Omega)$. For any m , we have the obvious chain of imbeddings

$$W_0^{m,p}(\Omega) \longrightarrow W^{m,p}(\Omega) \longrightarrow L^p(\Omega).$$

Theorem I.5.1 *[38]$W^{m,p}(\Omega)$ is a Banach space.*
Proof: *Let $\{u_n\}$ be a Cauchy sequence in $W^{m,p}(\Omega)$. Then $\{D^{\alpha}u\}$ is a Chauchy sequence in $L^p(\Omega)$ for $0 \leq |\alpha| \leq m$. Since $L^p(\Omega)$ is complet there exist functions u and $u_{\alpha}, 0 \leq |\alpha| \leq m$, such that $u_n \longrightarrow u$ and $D^{\alpha}u_n \longrightarrow u_{\alpha}$ in $L^p(\Omega)$ as $n \longrightarrow \infty$. Now $L^p(\Omega) \subset L^1_{loc}(\Omega)$ and so u_n determines a distribution $T_{u_n} \in \mathcal{D}'(\Omega)$. For any $\phi \in \mathcal{D}(\Omega)$ we have*

$$|T_{u_n}(\phi) - T_u(\phi)| \leq \int_{\Omega} |u_n(x) - u(x)|\,|\phi(x)|\,dx \leq \|\phi\|_{p'}\,\|u_n - u\|_p$$

by Hölder's inequality, where p' is the exponent conjugate to p. Therfre $T_{u_n}(\phi) \longrightarrow T_{u_n}(\phi)$ for every $\phi \in \mathcal{D}(\Omega)$ as $n \longrightarrow \infty$. Similarly, $T_{D^\alpha u_n}(\phi) \longrightarrow T_{u_\alpha}(\phi)$ for every $\phi \in \mathcal{D}(\Omega)$. It follows that

$$T_{u_\alpha}(\phi) = \lim_{n\to\infty} T_{D^\alpha u_n}(\phi) = \lim_{n\to\infty}(-1)^{|\alpha|}T_{u_n}(D^\alpha \phi) = (-1)^{|\alpha|}T_u(D^\alpha\phi)$$

for every $\phi \in \mathcal{D}(\Omega)$. Thus $u_\alpha = D^\alpha u$ in the distributional sence on Ω for $0 \leq |\alpha| \leq m$, whence $u \in W^{m,p}(\Omega)$. Since $\lim_{n\to\infty}\|u_n - u\|_{m,p} = 0$, the space $W^{m,p}(\Omega)$ is complete. ■

Theorem I.5.2 *[38]$W^{m,p}(\Omega)$ is separable if $1 \leq p \leq \infty$, and is uniformly convex and reflexive if $1 \leq p \leq \infty$. In particular, $W^{m,2}(\Omega)$ is a separable Hilbert space with inner product*

$$(u,v)_m = \sum_{0\leq|\alpha|\leq m}(D^\alpha u, D^\alpha v),$$

where $(u,v) = \int_\Omega u(x)\overline{v(x)}dx$ is the inner product on $L^2(\Omega)$.

I.5.2 Duality and the spaces $W^{-m,p'}(\Omega)$

In this part, the operator P, the spaces $L^p(\Omega^{(m)})$ and W, and the number N for fixed Ω, m and p. Additionally, for any functions u, v for which the right side makes sence,we define

$$\langle u, v\rangle = \int_\Omega u(x)v(x)dx$$

for any functions u, v that make sence on the right side.

Let's agree that p' always represents the conjugate exponent for given p :

$$p' = \begin{cases} \infty & \text{if } p = 1 \\ p/(p-1) & \text{if } 1 < p < \infty \\ 1 & \text{if } p = \infty \end{cases}$$

The Riesz Representation Theorem is first extended too the space $W^{m,p}(\Omega)$. The dual of $W_0^{m,p}(\Omega)$ with a subspace of $\mathcal{D}(\Omega)$ is then found lastly, we demonstrate that the completion of $L^{p'}(\Omega)$ with regard to a norm weaker than the typical $L^{p'}$norm may likewise be associated with the dual of $W_0^{m,p}(\Omega)$ if $1 < p < \infty$.

The Dual of $L^p(\Omega^{(m)})$

A unique $v \in L^{p'}(\Omega^{(m)})$ corresponds to each $L \in (L^p(\Omega^{(m)}))'$, where $1 < p < \infty$, so that for each $u \in L^p(\Omega^{(m)})$

$$L(u) = \int_{\Omega^{(m)}} u(x)v(x)dx = \sum_{|\alpha|\leq m}\int_{\Omega_\alpha} u_\alpha(x)v_\alpha(x)dx = \sum_{|\alpha|\leq m}\langle u_\alpha, v_\alpha\rangle\,,$$

where u_α and v_α are the restrictions of u and v, resectively, to Ω_α. Moreover, $\left\|L;(L^p(\Omega^{(m)}))'\right\| = \left\|v;L^{p'}(\Omega^{(m)}))'\right\|$. Thus $(L^p(\Omega^{(m)}))' = L^{p'}(\Omega^{(m)})$.

This is valid because $L^p(\Omega^{(m)})$ is, after all, an L^p space, albeit one defined on an unusual domain.

Theorem I.5.3 *[38] **(The Dual of $\mathbf{W}^{m,p}(\Omega)$)** Let $1 \le p \le \infty$. Then for every functional $L \in (W^{m,p}(\Omega))'$ there exist a function $v \in L'(W^{m,p}(\Omega))$ such that if he restriction of v to Ω_α is v_α, we have for all $u \in W^{m,p}(\Omega)$*

$$L(u) = \sum_{0\le|\alpha|\le m} \langle D^\alpha u, v_\alpha\rangle \tag{I.5.3}$$

Moreover

$$\|L;(W^{m,p}(\Omega))'\| = \inf\left\|v;L^{p'}(\Omega^{(m)})\right\| = \min\left\|v;L^{p'}(\Omega^{(m)})\right\|, \tag{I.5.4}$$

For any $u \in W^{m,p}(\Omega)$, (I.5.3) holds, and the infimum is taken over and reached on the set of all $v \in L^{p'}(\Omega^{(m)})$. If $1 < p < \infty$, the element $v \in L^{p'}(\Omega^{(m)})$ satisfying (I.5.3) and (I.5.4) is unique.

Theorem I.5.4 ***(The Normed Dual of $W_o^{m,p}(\Omega)$)*** *The dual space $(W_0^{m,p}(\Omega))'$ is isometrically isomorphic to he Banach space $W^{-m,p'}(\Omega)$, which is composed of those distributions $T \in \mathcal{D}(\Omega)$ that satisfy (I.7.1) and have norm*

$$\|T\| = \min\left\{\left\|v;L^{p'}(\Omega^{(m)})\right\| : v \text{ satisfies (I.7.1)}\right\}.$$

The isometric isomiphism results in this space being complete.

If $1 < p < \infty$, then $W_0^{m,p'}(\Omega)$ is clearly separable and reflexive.

Distributions T provided by (I.7.1) do not fully define continuous linear functionals on $W^{m,p}(\Omega)$ when $W_0^{m,p}(\Omega)$ is proper subset of $W^{m,p}(\Omega)$ due to their limits to $C_0(\Omega)$.

I.5.3 Approximation by Smooth Functions on Ω

$\left\{\phi \in C^\infty(\Omega) : \|\phi\|_{m,p} < \infty\right\}$ is dense in $W^{m,p}(\Omega)$ is what we want to demonstrate . The following existence theorem for ndlessly defferentiable partial functions of unity is necessary for his .

Theorem I.5.5 ***(Unity Partitions)*** *Let A be a set of open sets in W that cover A, or $A \subset \cup_{u\in\theta} U$, and let A be an abritrary subset of W then, a set of functions $\ell \in C_0^\infty(\mathbb{R}^n)$ having the following properties;*

i) *For every $\ell \in \Psi$ and every $x \in \mathbb{R}^n$, $o \le \ell(x) \le 1$.*

ii) *If $K \Subset A$, all but finitely many $\ell \in \Psi$ vanish identically on K.*

iii) *For every* $\ell \in \Psi$ *there exists* $U \in \theta$ *such that supp* $(\ell) \subset U$.

iv) *For every* $x \in A$, *we have* $\sum_{\ell \in \Psi} \ell(x) = 1$.

Such a collection Ψ *is called a* C^∞*-partition of unity for* A *subordinate to* θ.

I.5.4 Approximation by Smooth Functions on $\mathbb{R}^n$

Having shown that an element of $W^{m,p}(\Omega)$ can always be approximated by functions smooth on Ω we now ask whether the approximation can in fact be done with bounded functions having bounded derivatives of all orders, or at least of all orders up to and including at least m. That is, we are asking whether, for any values of $k \geq m$, the space $C^k(\overline{\Omega})$ is dence in $W^{m,p}(\Omega)$. The following example shows that the answer mav be negative.

I.6 Sobolev embedding theorems

When examining the regularity of a weak solution to a boundary value issue, for example, Sobolev embedding theorems are crucial.

I.6.1 The Rellich Kondrachov Theorems

Definition I.6.1 *[6] Let* $V \subset W$, V *and* W *be two Banach spaces. We write* $V \hookrightarrow W$ *if the spaces* V *is continuously embedded in* W

$$\|\nu\|_W \leq c\,\|\nu\|_V \qquad \forall \nu \in V. \tag{I.6.1}$$

We say the space V *is compactly embedded in* W *and write* $V \hookrightarrow\hookrightarrow W$, *if (I.6.1) holds and each bounded sequence in* V *has a subsequence cooverging in* W.

Remark I.6.1 *if* $V \hookrightarrow W$, *the functions in* V *are more smooth than the remaining functions in* W. *A simple example is* $H^1(\Omega) \hookrightarrow L^2(\Omega)$ *and* $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$.

Proofs of most parts of the following two theorems can be found in [**?**]. The first theorem is embedding of Sobolev spaces, and the second on compact embedding.

Theorem I.6.1 *(**continuously embedded**) Consider the Lipshitz domain* $\Omega \in \mathbb{R}^d$. *Then, the following claims are true.*

a) *If* $k < d/p$, *then* $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for any* $q \leq p^*$, *where* p^* *is defined by* $1/p^* = 1/p - k/d$.

b) *If* $k = d/p$, *then* $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for any* $q < \infty$.

c) *If $k > d/p$, then*
$$W^{k,p}(\Omega) \hookrightarrow C^{k-[d/p]-1,\beta}(\Omega),$$
where
$$\beta = \begin{cases} [d/p]+1-d/p \ , & \text{if } d/p \neq \text{integer,} \\ \text{any positive number } <1 \ , & \text{if } d/p = \text{integer} \end{cases}$$

In the theorem, denotes the integer part of x, i.e., the largest integer less than or equal to x. We remark that in the one-dimensional case, with $\Omega = (a, b)$ a bounded interval, we have
$$W^{k,p}(a,b) \hookrightarrow C[a,b]$$
for any $k \geq 1, p \geq 1$.

Theorem I.6.2 ***(compactly embedded)*** *Consider the Lipschitz domain $\Omega \in \mathbb{R}^d$. Then, the following claims are true.*

a) *If $k < d/p$, then $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for any $q < p^*$, where p^* is defined by $1/p^* = 1/p - k/d$.*

b) *If $k = d/p$, then $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for any $q < \infty$.*

c) *If $k > d/p$, then*
$$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow C^{k-[d/p]-1,\beta}(\Omega),$$
where $\beta \in (0, [d/p]+1-d/p)$.

How can we recall these outcomes ?. We use the case of theorem I.7.2. The functions from the space $W^{k,p}(\Omega)$ are smoother the larger the product kp. For this product, there is a crucial value d (the domain's dimension) such that a $W^{k,p}(\Omega)$ function is truly continuous if $kp > d$ or more specifically, represents a continuous function a.e.For an exponent p^* greater than p, a $W^{k,p}(\Omega)$ function belongs to $L^{p^*}(\Omega)$ when $kp < d$. Starting with the requirement $kp < d$, expressed as $1/p - k/d > 0$, we calculate the exponent p^*. The difference $1/p - k/d$ is then defined as $1/p^*$. Knowing if a $W^{k,p}(\Omega)$ function has continuous derivatives up to a specific order is typically helpful when $kp > d$. We start with

$$W^{k,p}(\Omega) \hookrightarrow C(\overline{\Omega}) \qquad \text{if } k > d/p.$$

Then we apply this embedding result to derivatives of Sobolev functions, it is easy to see that
$$W^{k,p}(\Omega) \hookrightarrow C^1(\overline{\Omega}) \qquad \text{if } k - l > d/p.$$
As some concrete examples, for a two-dimensional Lipschitz domain Ω, $H^1(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)\ \ \forall\ 1 \leq q \leq \infty$ and $H^2(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$. So in particular, a sequence bounded in $H^1(\Omega)$ has a subsequence that converges in $L^2(\Omega)$, $C(\overline{\Omega})$. For a three-dimensional

Lipschitz domain Ω, $H^1(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)\ \forall\, 1 \leq q < 6$, $H^1(\Omega) \hookrightarrow L^6(\Omega)$, and $H^1(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$.

Theorem I.6.3 *A direct consequence of Theorem I.7.3 is the following compact embedding result.*

I.6.2 Traces theorem

$L^P(\Omega)$ spaces are used to define Sobolev spaces. Sobolev functions are therefore only uniquely defined in Ω. The boundary value of a Sobolev function appears to be poorly defined now that the boundary Γ has measure zero in $\mathbb{R}^d$. However, a Sobolev function's tace on the boundary can be defined so that, for a Sobolev function that is continuous up to the border, its trace and its boundary value coincide.

Theorem I.6.4 *[6] Let $\mathbb{R}^d$, $1 \leq p < \infty$, be a Lipschitz domain with Ω. A continuous linear operator $\gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ then has the following characteristics:*

a) $\gamma v = v|_\Gamma$ *if* $v \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$.

b) *For some constant* $c > 0$, $\|\gamma v\|_{L^p(\Gamma)} \leq c\,\|v\|_{W^{1,p}(\Omega)} \quad \forall v \in W^{1,p}(\Omega)$.

c) *The mapping $\gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ is compact; i.e., for any bounded sequence $\{v_n\}$ in $W^{1,p}(\Omega)$, there is a subsequence $\{v_{n'}\} \subset \{v_n\}$ such that $\{\gamma v_{n'}\}$ is convergent in $L^p(\Gamma)$.*

Remark I.6.2 *i) The trace operator is denoted by γ, and the generalaized boundary value of v is denoted γv. The trace operator from $W^{1,p}(\Omega)$ to $L^p(\Gamma)$ is neither an injection nor a surjection. The range $\gamma(W^{1,p}(\Omega))$ is a positive order Sobolev space over the boundary, namely $W^{1-1/p,p}(\Omega)$, which is smaller than $L^p(\Gamma)$. Typically, the trace of $v \in H^1(\Omega)$ is represented by the same symbol v*
ii) If $v_n \to v$ in $H^1(\Omega)$, then $v_n \to v$ in $L^2(\Gamma)$.

Theorem I.6.5 *When we discuss weak formulations of boundary value problems later in this book, we need to use traces of the $H^1(\Omega)$ functions. These traces form the space $H^{1/2}(\Gamma)$; in other words,*

$$H^{1/2}(\Gamma) = \gamma(H^1(\Omega)).$$

Correspondingly, we can use the following as the norm for $H^{1/2}(\Gamma)$:

$$\|g\|_{H^{1/2}(\Gamma)} = \inf_{v \in H^1(\Omega)} \|v\|_{H^1(\Omega)}. \tag{I.6.2}$$

The ability to appropriately impose necessary boundary conditions in formulations is a prerequisite for understanding boundary value problems. Theorem I.7.3 is enough for the purpose because necessary boundary conditions for second-order boundary value *issues only require function values on the border. The traces of partial derivatives on the border must be used for higher-order boundary value problems. For instance, all boundary requirements containing derivatives of order at most one are regarded as fundamental boundary conditions for fourth-order boundary value issues. Given that By differentiating the function's boundary value, we may get the tangential derivative of a function on the boundary; all we need to do is use the function's traces and normal derivative.Let the outward unit normal to the boundary* Γ *of* Ω *be represented by* $v = (v_1, ..., v_d)^T$. *Remember that the classical normal derivative of* v *on the boundary if it is a member of* $C^1(\Omega)$ *is*

$$\frac{\partial v}{\partial \nu} = \sum_{i=1}^{d} \frac{\partial v}{\partial x_i} \nu_i.$$

I.6.3 Generalized Poincaré Inequality

Theorem I.6.6 *Let* Ω *be a Lipschitz bounded domain in* $\mathbb{R}^n$. *Let* $p \in [1, +\infty[$ *and let* $\mathcal{N}$ *be a continuous seminorm on* $W^{1,p}(\Omega)$*; that is, a norm on the constant functions. Then there exists a constant* $C > 0$ *that depends only on* Ω, N, p*, such that*

$$\|u\|_{W^{1,p}(\Omega)} \leq C \left(\left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p} + \mathcal{N}(u) \right).$$

Proposition I.6.1 *[13] Let* Ω *be a bounded domain of class* C^1*; then there exists a constant* $C_P > 0$ *such that every* u *in* $H_0^1(\Omega)$ *satisfies*

$$\|u\|_{H^1(\Omega)} \leq C_P \|\nabla u\|_2 .$$

I.6.4 The Lax-Milgram Lemma

The Lax-Milgram Lemma is employed frequently in the study of linear elliptic boundary value problems. For a real Banach space V, let us first explore the relation between a linear operator $A : V \to V'$ and a bilinear form $a : V \times V \to \mathbb{R}$ related by

$$\langle Au, v \rangle = a(u, v) \quad \forall u, v \in V. \tag{I.6.3}$$

The bilinear form $a(.,.)$ is continuous if and only if there exixts $M > 0$ such that

$$|a(u, v)| \leq M \|u\| \|v\| \qquad \forall u, v \in V$$

Theorem I.6.7 *[6] there exixts a one-to-one correspondence between linear continuous operators* $A : V \to V'$ *and continuous bilinear forms* $a : V \times V \to \mathbb{R}$*, given by the formula (I.6.3).*

I.7 Weak Derivative

Definition I.7.1 *[15] Let α be a multi-index and the function $u \in L^1_{loc}(\Omega)$ be provided for each $\varphi \in C_0^\infty(\Omega)$ the relationship*

$$T_{v_\alpha}(\phi) = \langle \phi, v_\alpha \rangle = \int_\Omega u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_\Omega \varphi(x) v(x) dx \qquad \text{(I.7.1)}$$

is met, the function $v \in L^1_{loc}(\Omega)$ is known as the $\alpha - th$ weak (generalized) derivative of u, and it is represented by the notation $v = D^\alpha u$.

A number of authors also use $v = v_\alpha = D^\alpha u$ to indicate the weak derivative. Only weak derivatives exist.

I.7.1 Properties of the Weak Derivative

We wrap off the chapter by presenting the following theorem, which allows one to compare the weak derivative with the classical one and determine some of its fundamental characteristics, especially from the first two propositions.

Theorem I.7.1 ***i)*** *A function has a weak derivative if it has a (classical) derivative. The opposite might not always be true, though. In other words, a function with a weak derivative might not have a (classical) derivative.*

ii) *The weak derivative is defined worldwide, whereas the classical derivative is pointwise.*

iii) *The weak derivative is linear. In other words, if $c_1, c_2 \in \mathbb{R}$ and $u_1, u_2 \in L^1_{loc}(\Omega)$, we obtain*

$$D^\alpha(c_1 u_1 + c_2 u_2) = c_1 D^\alpha(u_1) + c_2 D^\alpha(u_2).$$

iv) *If u has a weak derivative $v = D^\alpha u$ and if v has a weak derivative $w = D^\beta v = D^\beta(D^\alpha u)$, then*

$$D^{\beta+\alpha} u = D^\beta(D^\alpha u) = D^\beta v = w.$$

The fact that the sequence of differentiation is irrelevant for mixed derivatives is another essential characteristic of the weak derivative that sets it apart from the classical derivative. Stated otherwise, the weak derivative is independent of the differentiation order.

Remark I.7.1 *In the classical sense, a function must be continuous in order to be differentiable. However, the weak derivative does not need neither continuity nor differentiability of the function; integrability is sufficient.*

I.8 Green's Identities

To keep things simple, we look at $\mathbb{R}^3$ elements. Let $F = (F_1, F_2, F_3)$ be a vector in $\mathbb{R}^3$ and $u = u(x, y, z)$ be a function. The following lists some helpful notations.

- $\nabla u = \operatorname{grad} u = (u_x, u_y, u_z)$,
- $\nabla \cdot F = \operatorname{div} F = F_{1x} + F_{2y} + F_{3z}$,
- $\triangle u = \operatorname{div} \operatorname{grad} u = \nabla \cdot \nabla u = u_{xx} + u_{yy} + u_{zz}$,
- $|\nabla u|^2 = u_x^2 + u_y^2 + u_z^2$,
- $\nabla \cdot \nabla = \nabla^2 = \triangle$.

Theorem I.8.1 ***(Green's First Identity)****[15] For $u \in C^2(\Omega) \cap C(\overline{\Omega})$,*

$$\int_\Omega v \triangle u dx = \int_{\partial\Omega} v \frac{\partial u}{\partial n} ds - \int_\Omega \nabla v \nabla u dx$$

where n is the unit vector in the outward normal direction and $\frac{\partial u}{\partial n} = n \cdot \nabla u$.
(Green's Second Identity).*For $u \in C^2(\Omega) \cap C(\overline{\Omega})$,*

$$\int_\Omega (u\triangle v - v\triangle u)dx = \int_{\partial\Omega} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})ds.$$

I.9 Some Semigroup arguments

Definition I.9.1 *Let $\mathcal{B} \in \mathcal{L}(X)$ be a bounded operator. The resolvent set $\rho(\mathcal{B})$ of $\mathcal{B}$ is the set of all λ in $\mathbb{C}$ for which $(\mathcal{B} - \lambda I)^{-1}$ exists and bounded*

$$\rho(\mathcal{B}) = \left\{\lambda \in \mathbb{C}; (\mathcal{B}-\lambda I)^{-1} \in \mathcal{L}(X)\right\}.$$

The spectrum of $\mathcal{B}$, denoted by $\sigma(\mathcal{B})$, is the complement of the resolvent set, i.e.,

$$\sigma(\mathcal{B}) = \mathbb{C}\backslash\rho(\mathcal{B}).$$

A complex number λ is an eigenvalue of $\mathcal{B}$ if

$$N(\mathcal{B}-\lambda I) \neq \{0\}.$$

Theorem I.9.1 *Let X be a Banach space and $\mathcal{B} \in \mathcal{L}(X)$. If the operator $\mathcal{B}$ satisfies $\|\mathcal{B}\| < 1$, then $I - \mathcal{B}$ is invertible and its inverse is given by*

$$(I - \mathcal{B})^{-1} = \sum_{n=0}^{\infty} \mathcal{B}^n.$$

Definition I.9.2 *[7] Let X be a Banach space. A semigroup of bounded linear operators is a family of bounded linear operators $T(t) \in \mathcal{L}(X)$, which depend on a parameter $0 \leq t \leq \infty$ and that satisfies the following properties*

$T(0) = I$, (I is the identity operator on X).

$T(t+s) = T(t)T(s)$ for every $t, s \geq 0$ (the semigroup property).

A semigroup of bounded linear operators$T(t)$is uniformly continuous if

$$\lim_{t \to 0} \|T(t) - I\| = 0.$$

the infunitesinal generator of a semigroup $T(t)$ is the operator $\mathcal{B}$ defined on

$$D(\mathcal{B}) = \left\{ x \in X : \lim_{t \to o} \frac{T(t)x - x}{t} \quad \textit{exists} \right\}$$

by

$$\mathcal{B}x = \lim_{t \to 0} \frac{T(t)x - x}{t}, \quad \textit{for} \;\; x \in D(\mathcal{B}).$$

Definition I.9.3 ***(Maximal Monotone Operators)*** *[20] Let X be a Hilbert space,an unbounded linear operator $\mathcal{B} : D(\mathcal{B}) \subset X \to X$ is said to be monotone(accretive) if it satisfies*

$$\operatorname{Re} \langle \mathcal{B}v, v \rangle \geq 0, \forall v \in D(\mathcal{B})$$

In addition, the operator is called maximal monotone , if $R(I + \mathcal{B}) = X$ i.e,

$$\forall f \in X; \exists v \in D(\mathcal{B}), \textit{such that } u + \mathcal{B}u = f.$$

Remark I.9.1 *If -$\mathcal{B}$ is monotone, we say that B is dissipative.*

Proposition I.9.1 ***Proposition I.9.2*** *[20] Let $\mathcal{B}$ be a maximal monotone operator on a Hilbert space. Then*

1) $D(\mathcal{B})$ is dense in X.

2) $\mathcal{B}$ is a closed operator.

3) For every $\lambda > 0$, $(I + \lambda\mathcal{B})$ is bijective from $D(\mathcal{B})$ into X, $(I + B)^{-1}$ is a bounded operator, and $\|(I + \lambda\mathcal{B})^{-1}\|_{\mathcal{L}(X)} \leq 1$.

I.9.1 The Lumer Philips Theorem

Definition I.9.4 *[2] If there is a $x^* \in F(x)$ such that $\operatorname{Re} \langle Ax, x^* \rangle \leq 0$ for each $x \in D(A)$,then a linear operator A is dissipative. The following is a helpful description of dissipative operators.*

Theorem I.9.2 ***(Lumer-Phillips)*** *Let A be a linear operator in X with the dense domain $D(A)$.*

1) *If A is dissipative and there is a $\lambda_0 > 0$ such that the range, $R(\lambda_0 I - A)$, of $\lambda_0 I - A$ is X, then A is the infinitsimal generator of a C_0 semigroup of contractions on X.*

2) *If A is the infinitesimal generator of a C_0 semigroup of contractions on X the $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover, for every $x \in D(A)$ and every $x^* \in F(x)$, $R\langle Ax, x^*\rangle \leq 0$.*

I.9.2 The Hille-Yosida Theorem

Consider a C_0 semigroup $T(t)$, for $t > 0$, there are constants $\omega \geq 0$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}$. $T(t)$ is said to be uniformly bounded if $M = 1$, and a C_0 semigroup of contractions if $\omega = 0$. Characterizing the infinitesimal generators of C_0 semigroup of contractions is the focus of this section. In order for an opperator A to be the infinitesimal generator of a C_0 semiggroup of contractions, some conditions on the resolvent's behavior are provided.

Recall that if A is a linear, not necessarily bounded, operator in X, the resolvent set $\rho(A)$ of A is the set of all complex numbers λ for which $\lambda I - A$ is invertible,i.e.,$(\lambda I - A)^{-1}$ is a bounded linear operator in X . The family $R(\lambda : A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ of bounded linear operators is called the resolvent of A.

Theorem I.9.3 *[2] **(Hill-Yosida)** A linear (unbounded) operator A is the infinitesimal generator of a C_0-semigroup of contractions $T(t)$, if and only if*

1) *A is closed and $\overline{D(A)} = X$.*

2) *The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$*

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda}. \tag{I.9.1}$$

I.9.3 Operator m-dissipative

Definition I.9.5 *[2] A dissipative operator A is referred to as m-dissipative if $R(I - A) = X$.*

If A dissipative so is μA for all $\mu > 0$ and therefore if A is m-dissipative then $R(\lambda I - A) = X$ for every $\lambda > 0$. In terms of m-dissipative operators the Lumer-Philips theorem can be restated as: A densely defined operator A is the infinitesimal generator of a C_0 semigroup of contractions if and only if it is m-dissipative.

The main result of this section is the following perturbation theorem for m-dissipative operators.

I.10 Stability and Hyperbolicity for Semigroups

We first examine the stability of strongly continuous semigroups $(T(t))_{t\geq 0}$, one of the many intriguing forms of asymptotic behavior. This means that as $t \to \infty$, the operators $T(t)$ ought to converge to zero. However, we must discriminate between several definitions of convergence, as is to be expected in infinite-dimensional spaces.

I.10.1 Stability concepts

Definition I.10.1 *A strongly continuous semigroup $(T(t))_{t\geq 0}$ is called*
- strongly exponentially stable, provided that $\varepsilon > 0$.

$$\lim_{t\to\infty} e^{\varepsilon t} \|T(t)\| = 0, \tag{I.10.1}$$

- uniformly stable if

$$\lim_{t\to\infty} \|T(t)\| = 0, \tag{I.10.2}$$

- strongly stable if

$$\lim_{t\to\infty} \|T(t)x\| = 0 \qquad \textit{for all } x \in X, \tag{I.10.3}$$

- weakly stable if

$$\lim_{t\to\infty} \langle T(t)x, x'\rangle = 0 \qquad \textit{for all } x \in X \textit{ and } x' \in X'. \tag{I.10.4}$$

In the case A is not compact, then the classical methods such as Lasalle's invariance principle [18] or the spectral decomposition theory of SzNagy-Foias, Foguel and Benchimol [13, 8, 33] are not applicable. in this case. Thus, we will use a Stability Theorem of Arendt-Batty (given below) to study the strong stability of our system

Theorem I.10.1 ***(Stability Theorem)****[?] Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a reflixive space X. Denote by A the generator of $(T(t))_{t\geq 0}$ and by $\sigma(A)$ the spectrum of A. If $\sigma(A)\cap i\mathbb{R}$ is countable and no eigenvalue of A lies on the imaginary axis, then $\lim_{t\to\infty} T(t)x = 0$ for all $x \in X$.*

Theorem I.10.2 *On a Banach space X, let $(T(t))_{t\geq 0}$ be a bounded strongly continuous semigroup with generator A. If*

$P\sigma(A') \cap i\mathbb{R} = \emptyset$ and
$\sigma(A) \cap i\mathbb{R}$ is countable,
then $(T(t))_{t\geq 0}$ is strongly stable, i.e.,

$$\lim_{t\to\infty} T(t)x = 0 \qquad \text{for all } x \in X. \tag{I.10.5}$$

Corollary I.10.1 *On a Banach space X, let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup with generator A. The following claims are identical if $(T(t))_{t\geq 0}$ is substantially compact for the strong operator topology. $(T(t))_{t\geq 0}$ is strongly stable. $P\sigma(A) \cap i\mathbb{R} = \emptyset$.*

I.10.2 Lyapunov direct method

Theorem I.10.3 *Suppose that there exist constant $a, b, c, r > 0, p \geq 1$ and a C^1 function $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that,*

i) $\forall x, \|x\| \leq r, \;\; a\|x\|^p \leq V(x,t) \leq b\|x\|^p, \forall t \geq 0.$

ii) $V_t(t,x) \leq -c\|x\|^p, \forall t \geq 0, \forall x, \|x\| \leq r.$

Then, the equilibrium point of the equation $x_t = f(t,x)$ is exponentially stable.

I.11 Perturbation by Compact Operator

I.11.1 Fredholm Operator

An invertible operator is a basic illustration of a fredholm operator $J : E \to F$; in this case $N(t) = \{0\}, R(T) = F$, and so $\alpha(T) = \beta(T) = 0$. Conversely, $\alpha(T) = \beta(T) = 0$ implies that J is invertible. We shall now charactirize the Fredholm operators, that is, the operators $T \in \Phi$ such that $\alpha(T) = \beta(T), i(T) = 0$.

Definition I.11.1 *A* ***Fredholm operator*** *is one whose kernel is finite-dimensional and whose image has finite codimension. The* ***index*** *of a fredholm operator is the difference*

$$\textit{index}(T) := \dim\ker T - \textit{codim}\,\mathrm{Im}\,T.$$

A Fredholm operator $T : X \to Y$ gives rise to decompositions

$$X = \ker T \oplus M, \qquad Y = im\ T \oplus N,$$

for some closed linear subspace M, N. The restricted operator $R : M \to im\ T, x \longmapsto Tx$ is then bijective and continuous, and thus an isomorphism by the open mapping theorem.

Theorem I.11.1 *[26] **(Riesz-Schauder)** If $T = J + V$, where J is invertible and V is compact, then T is fredholm, that is $\alpha(T) = \beta(T) < \infty$.*

I.12 Hilbert Uniqueness Method (HUM)

I.12.1 The Hilbert Uniqueness Method (HUM)

Introduced by J.-L. Lions, is one of the most important methods in the controllability theory of partial differential equations. It is based on the duality between controllability and observability and provides an effective framework for proving exact controllability results for distributed parameter systems.

The main idea of the HUM method is to transform the controllability problem into an observability problem associated with the corresponding adjoint system. Instead of constructing the control directly, the method studies the adjoint equation and establishes suitable observability estimates. These estimates allow the energy of the solution to be determined through measurements performed on a suitable part of the domain or its boundary.

A fundamental ingredient of the HUM method is the observability inequality, which ensures that the initial energy of the adjoint system can be estimated by means of boundary or internal observations over a finite time interval. Using this property, a linear operator, called the HUM operator, is defined between appropriate Hilbert spaces. The observability inequality guarantees that this operator is an isomorphism, which leads to the existence and uniqueness of a control driving the system from a given initial state to a desired final state.

Consequently, the HUM method establishes a strong relationship between exact controllability and observability of the adjoint system. This relationship can be summarized by the principle:

$$.\text{ExactControllability} \Leftrightarrow \text{ObservabilityoftheAdjointSystem}$$

Due to its effectiveness and generality, the HUM method has been successfully applied to a large class of evolution equations, including wave equations, heat equations, elasticity systems, and many other models arising in mathematical physics and control theory

Chapter II

Stability and controllability of a wave equations with dynamical boundary control

Contents

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with boundary Γ of class C^2, consisting of a clamped part $\Gamma_0 \neq \emptyset$ and a rimmed part Γ_1 such that $\Gamma_0 \cap \Gamma_1 = \emptyset$.Consider the following system:

$$u_{tt} - \Delta u = 0 \quad \text{in } \Omega \times [0,T], \tag{II.0.1}$$
$$u = 0 \quad \text{on } \Gamma_0 \times [0,T], \tag{II.0.2}$$
$$\frac{\partial u}{\partial \nu} + \eta = 0 \quad \text{on } \Gamma_1 \times [0,T], \tag{II.0.3}$$
$$\eta_t - u_t = v \quad \text{on } \Gamma_1 \times [0,T], \tag{II.0.4}$$

with the initial condition:

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \quad \eta(x,0) = \eta_0(x), \quad x \in \Gamma_1. \tag{II.0.5}$$

where v denotes the dynamical boundary control. A physical implementation of the dynamic control may be used in pressurized gas tanks with servo controlled actors, as well as in standard mass-spring dampers (see [37] and the references hereby). We mention that the dynamical controls form part of indirect mechanisms proposed by Russell in[12].

Let us first consider the boundary stabilization by taking

$$v = -\eta \quad \text{on } \Gamma_1 \times [0, T]. \tag{II.0.6}$$

Denoting by $U = (u, u_t, \eta)$ the state of system (II.0.1)-(II.0.5), we formulate system (II.0.1)-(II.0.5) into an abstract problem:

$$U_t = (A + B)U, \quad U(0) = U_0. \tag{II.0.7}$$

where the operator A is m-dissipative and the operator B is dissipative on an appropriate Hilbert space H. Therefore $A + B$ generates a C_0-semigroup $S_{A+B}(t)$ of contractions on the energy space H and equation (II.0.7) is well-posed in the sense of semigroups.In the one-dimensional case, the stabilization and the exact controllability of system (II.0.1)-(II.0.5) were studied in ([3], [4]). But the problem is very different in the higher-dimensional case. First, the damping operator B is bounded but not compact in the energy space H. So, the theory of compact perturbation does not apply here. On the other hand, the resolvent of $A + B$ is not compact, therefore classical methods such as LaSalle's invariance principle [28] or the theory of spectral decomposition of Benchimol [11] are not applicable here.

We next consider the exact controllability by a singular dynamical boundary control of the following form

$$v(t) = v_0(t) - \frac{d}{dt}v_1(t) \quad \text{on } \Gamma_1 \times [0, T] \quad \text{with } v_0, v_1 \in L^2(0, T; L^2(\Gamma_1)). \tag{II.0.8}$$

Setting $V = (0, 0, v)$, we get an abstract formulation of the control problem:

$$U_t = AU + V, \quad U(0) = U_0. \tag{II.0.9}$$

Noting that system (II.0.1)-(II.0.5) is composed of a wave equation and an ordinary differential equation of first order with respect to t. So the usual formulation for a sole wave equation is no more convenient for this mixed system. In order to establish the exact controllability, we have to use the formulation (II.0.9) of first order and identify the energy space H with its dual space. At this stage, we identify $L^2(\Omega)$ with its dual space, and simultaneously but independently $H^1(\Omega)$ with its dual space.

Now we give a brief outline of the chapitre. In Section 1, using a general criteria of Arendt in [51], we show the strong stability of the semigroup $S_{A+B}(t)$ in the absence of the compactness of the infinitesimal $A+B$. In Section 3, using a criteria of Huang [16] and Prüss [25], we first prove the non uniform stability of the semigroup $S_{A+B}(t)$. Next by a multiplier method, we establish a polynomial energy decay rate as $\frac{1}{t}$ for smooth solutions. The last section is devoted to the exact controllability. We first give an inequality of observability. Then, by a new adaptation of Hilbert Uniqueness Method (see [23],[?]), we show the exact controllability of (II.0.9) by means of some singular dynamical controls v for all initial data in the usual energy space H.

II.1 Strong stability with non compact resolvent

In this section, we will study the stability of the following wave equation with one dynamical boundary feedback:

$$u_{tt} - \Delta u = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \tag{II.1.1}$$

$$u = 0 \quad \text{on } \Gamma_0 \times \mathbb{R}_+, \tag{II.1.2}$$

$$\frac{\partial u}{\partial \nu} + \eta = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}_+, \tag{II.1.3}$$

$$\eta_t - u_t + \eta = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}_+. \tag{II.1.4}$$

associated with the initial condition:

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \quad \eta(x,0) = \eta_0(x), \quad x \in \Gamma_1. \tag{II.1.5}$$

Let u be a smooth solution of system (II.1.1)-(II.1.5). We define the energy $E(t)$ by

$$E(t) = \frac{1}{2}\left\{\int_\Omega (|u_t|^2 + |\nabla u|^2)\, dx + \int_{\Gamma_1} |\eta|^2\, d\Gamma\right\}. \tag{II.1.6}$$

Then a straightforward computation gives

$$\frac{d}{dt}E(t) = -\int_{\Gamma_1} |\eta|^2\, d\Gamma \leq 0. \tag{II.1.7}$$

Therefore, system (II.1.1)-(II.1.5) is dissipative in the sense that the energy $E(t)$ is non increasing with respect to the time variable t.

Let us introduce the following space:

$$V = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\}$$

and define the energy space $H = V \times L^2(\Omega) \times L^2(\Gamma_1)$ which is endowed with the following inner product:

$$\left\{ \begin{array}{c} (U,\widetilde{U})_H = \int_\Omega \nabla u \cdot \nabla \widetilde{u} dx + \int_\Omega v\widetilde{v} dx + \int \eta\widetilde{\eta} d\Gamma, \\ U = (u,v,\eta), \quad \widetilde{U} = (\widetilde{u},\widetilde{v},\widetilde{\eta}) \in H. \end{array} \right. \tag{II.1.8}$$

We next define the linear unbounded operator A by

$$D(A) = \{(u,v,\eta) \in H : \Delta u \in L^2(\Omega),\ v \in V \text{ and } \partial_\nu u + \eta = 0 \text{ on } \Gamma_1\}, \tag{II.1.9}$$

$$AU = (v, \Delta u, \gamma(v)), \quad \forall U = (u, v, \eta) \in D(A), \tag{II.1.10}$$

where $\gamma : V \to L^2(\Gamma_1)$ is the trace operator $\gamma(\psi) = \psi|_{\Gamma_1}$ and the linear bounded operator B by

$$BU = (0, 0, -\eta), \quad \forall U = (u, v, \eta) \in H. \tag{II.1.11}$$

Then setting $U = (u, u_t, \eta)$ as the state of system (II.1.1)-(II.1.5), we rewrite the problem into an evolution equation

$$U_t = (A + B)U, \quad U(0) = U_0 \in H. \tag{II.1.12}$$

It is easy to see that $A+B$ is m-dissipative on H, therefore it generates a C_0-semigroup $S_{A+B}(t)$ of contractions on the energy space H (see Pazy [1]). But unlike the one-dimensional case, the resolvent of $A + B$ is not compact. Then classical methods such as Lasalle's invariance principle [28] or the spectrum decomposition theory of Benchimol [11] are not applicable to the higher dimensional case. In the following, we will prove the strong stability with a more general criteria of Arendt in [11].

Proposition II.1.1 *The operator $A + B$ has no pure imaginary eigenvalue.*

Proof: Let $\beta \in \mathbb{R}$ and $U = (u, v, \eta) \in H$ such that $(A + B)U = i\beta U$. Then we have

$$\begin{cases} i\beta u - v = 0 \quad \text{in } \Omega, \\ i\beta v - \Delta u = 0 \quad \text{in } \Omega, \\ i\beta\eta - (v - \eta) = 0 \quad \text{on } \Gamma_1. \end{cases} \tag{II.1.13}$$

It follows that

$$v = i\beta u \quad \text{in } \Omega, \qquad \eta = \frac{1}{i\beta + 1} v \quad \text{on } \Gamma_1, \tag{II.1.14}$$

and

$$\begin{cases} \beta^2 u + \Delta u = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma_0, \\ \partial_\nu u + \frac{i\beta}{i\beta+1} u = 0 \quad \text{on } \Gamma_1. \end{cases} \tag{II.1.15}$$

Then by multiplying (II.1.15) by u and by integrating by parts, we get

$$\beta^2 \int_\Omega |u|^2 \, dx - \int_\Omega |\nabla u|^2 \, dx - \frac{i\beta + \beta^2}{\beta^2 + 1} \int_{\Gamma_1} |u|^2 \, d\Gamma = 0. \tag{II.1.16}$$

If $\beta = 0$, we get $\nabla u = 0$, which together with the boundary condition $u = 0$ on Γ_0 implies that $u \equiv 0$. If $\beta \neq 0$, then by taking the imaginary part in (II.1.16), we get $u = 0$ on Γ_1, which together with the boundary condition on Γ_1 implies that $\partial_\nu u = u = 0$ on Γ_1. Then, by Carlman's Theorem (see [30]), we get again $u \equiv 0$. The proof is thus complete. ■

Proposition II.1.2 *The resolvent of $A + B$ is not compact*

Proof: Let $(0,0,h) \in H$ and write the equation $(I-A-B)U = F$ into the following form:

$$\begin{aligned} u - v = 0 \quad & \text{in } \Omega, \\ v - \Delta u = 0 \quad & \text{in } \Omega, \\ \eta - (v - \eta) = h \quad & \text{on } \Gamma_1. \end{aligned} \tag{II.1.17}$$

It follows that

$$v = u \quad \text{in } \Omega, \qquad \eta = \frac{1}{2}(u+h) \quad \text{on } \Gamma_1, \tag{II.1.18}$$

and

$$\begin{aligned} u - \Delta u = 0 \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \Gamma_0, \\ 2\partial_\nu u + u = -h \quad & \text{on } \Gamma_1. \end{aligned} \tag{II.1.19}$$

We see that (II.1.19) admits a unique solution u such that

$$\|u\|_{H^s(\Omega)} \leq c\|h\|_{L^2(\Gamma_1)}, \forall s < \frac{3}{2}. \tag{II.1.20}$$

Therefore, if h is in a bounded set of $L^2(\Gamma_1)$, then the first two components (u,v) of U are in a relatively compact set of $V \times L^2(\Omega)$. But because of the presence of h in the expression (II.1.18) of η, the last component η does not lay in a relatively compact set of $L^2(\Gamma_1)$. The proof is complete. ■

Proposition II.1.3 *For $\beta \in \mathbb{R}$ and any $f \in L^2(\Omega)$, the following problem*

$$\begin{cases} \beta^2 u + \Delta u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma_0, \\ \partial_\nu u + \frac{i\beta}{i\beta+1} u = 0 \quad \text{on } \Gamma_1 \end{cases} \tag{II.1.21}$$

admits a unique solution $u \in V$.

Proof: Let us first consider the following problem

$$\begin{cases} -\Delta u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma_0, \\ \partial_\nu u + \frac{i\beta}{i\beta+1} u = 0 \quad \text{on } \Gamma_1. \end{cases} \tag{II.1.22}$$

Let $u = u_1 + iu_2$, $f = f_1 + if_2$ and we separate the real part and the imaginary part of

(II.1.22) as

$$\begin{cases} -\Delta u_1 = f_1 & \text{in } \Omega, \\ -\Delta u_2 = f_2 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \Gamma_0, \\ \partial_\nu u_1 + \dfrac{1}{\beta^2+1}(\beta^2 u_1 - \beta u_2) = 0 & \text{on } \Gamma_1, \\ \partial_\nu u_1 + \dfrac{1}{\beta^2+1}(\beta u_1 + \beta^2 u_2) = 0 & \text{on } \Gamma_1. \end{cases} \tag{II.1.23}$$

Next we give a variational formation of (II.1.23): Find $u = (u_1, u_2) \in V \times V$ such that

$$a(u, \varphi) = l(\varphi), \quad \forall \varphi = (\varphi_1, \varphi_2) \in V \times V, \tag{II.1.24}$$

where

$$\begin{aligned} a(u,\varphi) &= \int_\Omega (\nabla u_1 \nabla \varphi_1 + \nabla u_2 \nabla \varphi_2) dx \\ &\quad + \frac{\beta}{\beta^2+1} \int_{\Gamma_1} (\beta(u_1\varphi_1 + v_2\varphi_2) + (u_1\varphi_2 - v_2\varphi_1)) d\Gamma, \\ l(\varphi) &= \int_\Omega (f_1\varphi_1 + f_2\varphi_2) dx. \end{aligned}$$

It is clear that the bilinear form a is continuous and coercive on the space $(V \times V)^2$. By Lax-Milgram's theorem, the variational equation (II.1.24) admits a unique solution $u \in V \times V$. Moreover, we have (see [24])

$$\|u\|_{H^2(\Omega)} \leq c \|f\|_{L^2(\Omega)}. \tag{II.1.25}$$

Consequently, the inverse operator $(-\Delta)^{-1}$ defined in (II.1.22) is compact in $L^2(\Omega)$. Then applying $(-\Delta)^{-1}$ to (II.1.21), we get

$$(\beta^2(-\Delta)^{-1} - I)u = (-\Delta)^{-1} f. \tag{II.1.26}$$

The same computation in (II.1.15) shows that

$$\ker(\beta^2(-\Delta)^{-1} - I) = \{0\}.$$

Then following Fredholm's alternative (see [19]), the equation (II.1.26) admits a unique solution. ■

Theorem II.1.1 *The semigroup $S_{\mathcal{A}+\mathcal{B}}(t)$ is strongly stable in $\mathcal{H}$. That means that for all $U_0 \in H$, the energy $E(t)$ of the corresponding solution of (II.1.12) satisfies*

$$E(t) \to 0 \quad as \quad t \to +\infty. \tag{II.1.27}$$

Proof: Following a general criteria of Arendt in [51], the C^0-semigroup $S_{\mathcal{A}+\mathcal{B}}(t)$ of contractions is strongly stable, if $\sigma(\mathcal{A}+\mathcal{B})$ has only a countable number of continuous spectrum on the imaginary axe. By Proposition II.1.1, $\mathcal{A}+\mathcal{B}$ has non pure imaginary eigenvalues. It is sufficient to show that $\mathrm{Im}(i\beta I-(\mathcal{A}+\mathcal{B}))=\mathcal{H}$ for all real number $\beta\in\mathbb{R}$. Then the closed graph theorem of Banach will imply that $\sigma(\mathcal{A}+\mathcal{B})\cap i\mathbb{R}=\{\emptyset\}$.

For any given $F=(f,g,h)\in\mathcal{H}$, we consider the equation $(i\beta I-\mathcal{A}-\mathcal{B})U=F$ which is equivalent to the following system

$$\begin{cases} i\beta u-v=f \text{ in } \Omega,\\ i\beta v-\Delta u=g, \text{ in } \Omega,\\ i\beta\eta-(v-\eta)=h, \text{ on } \Gamma_1. \end{cases} \tag{II.1.28}$$

It follows that

$$v=i\beta u-f \text{ in } \Omega, \quad \eta=\frac{1}{i\beta+1}(v+h) \text{ on } \Gamma_1, \tag{II.1.29}$$

and

$$\begin{cases} \beta^2u+\Delta u=-(i\beta f+g) & \text{in } \Omega,\\ u=0 & \text{on } \Gamma_0,\\ \partial_\nu u+\frac{1}{i\beta+1}u=\frac{1}{i\beta+1}(f-h) & \text{on } \Gamma_1. \end{cases} \tag{II.1.30}$$

In order to finish the proof, we will show that problem (II.1.30) always has a solution $u\in V$.

First, let $\varphi\in V$ be defined by

$$\begin{cases} -\Delta\varphi=0 & \text{in } \Omega,\\ \partial_\nu\varphi=h & \text{on } \Gamma_1. \end{cases} \tag{II.1.31}$$

Then setting

$$\tilde{u}=u+\frac{1}{i\beta+1}\varphi,$$

in (II.1.30), we get

$$\begin{cases} \beta^2\tilde{u}+\Delta\tilde{u}=\frac{\beta^2}{i\beta+1}\varphi-(i\beta f+g) & \text{in } \Omega,\\ \tilde{u}=0 & \text{on } \Gamma_0,\\ \partial_\nu\tilde{u}+\frac{i\beta}{i\beta+1}\tilde{u}=\frac{i\beta}{(i\beta+1)^2}\varphi+\frac{1}{i\beta+1}f & \text{on } \Gamma_1. \end{cases} \tag{II.1.32}$$

Next let $\theta\in H^2(\Omega)\cap V$ such that

$$\theta=0, \quad \partial_\nu\theta=\frac{i\beta}{(i\beta+1)^2}\varphi+\frac{1}{i\beta+1}f\in H^{\frac{1}{2}}(\Gamma_1) \quad \text{on } \Gamma_1.$$

Then setting $w = \tilde{u} - \theta$ in (II.1.32), we get

$$\begin{cases} \beta^2 w + \Delta w = \frac{\beta^2}{i\beta+1}\varphi - (i\beta f + g) - \beta^2\theta - \Delta\theta & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma_0, \\ \partial_\nu w + \frac{i\beta}{i\beta+1} w = 0 & \text{on } \Gamma_1. \end{cases} \tag{II.1.33}$$

By virtue of Proposition II.1.3, equation (II.1.33) has a unique solution. This shows that $\mathrm{Im}(i\beta I - \mathcal{A} - \mathcal{B}) = \mathcal{H}$. The proof is thus complete. ■

Remark II.1.1 *We will give a short proof of Theorem II.1.1 under an additional geometric condition (see Corollary II.3.1)*

II.2 Non uniform stability

Noting that the damping operator $\mathcal{B}$ is bounded but not compact in H. Therefore, we can not apply the classical theory of compact perturbation to get the non uniform stability. We will show the non uniform stability of system (II.1.12) by a criteria of Huang [16] and Prüss's [25] on the boundedness of the resolvent of $\mathcal{A} + \mathcal{B}$ on the imaginary axe. We next establish a polynomial energy decay rate for smooth solutions by a multiplier method in Rao [9].

Theorem II.2.1 *The semigroup $S_{\mathcal{A}+\mathcal{B}}(t)$ of contractions is not uniformly stable in the energy space $\mathcal{H}$.*

Proof: Following Huang [16] and Prüss [25], it is sufficient to show that the resolvent of $\mathcal{A} + \mathcal{B}$ is not uniformly bounded on the imaginary axis.

First let φ_n be an eigenfunction of the following problem

$$\begin{cases} -\Delta\varphi_n = \mu_n^2 \varphi_n & \text{in} \quad \Omega, \\ \varphi_n = 0 & \text{on} \quad \Gamma_0, \\ \partial_\nu \varphi_n + \varphi_n = 0 & \text{on} \quad \Gamma_1 \end{cases} \tag{II.2.1}$$

such that

$$\|\varphi_n\|^2_{H^1(\Omega)} = \int_{\Gamma_1} |\varphi_n|^2 d\Gamma + \int_\Omega |\nabla\varphi_n|^2 dx = 1. \tag{II.2.2}$$

Next let

$$\psi_n = i\mu_n\varphi_n, \quad \xi_n = \varphi_n, \quad \Phi_n = (\varphi_n, \psi_n, \xi_n). \tag{II.2.3}$$

We check easily that $\Phi_n \in D(\mathcal{A})$. Moreover, a straightforward computation gives

$$\|(\mathcal{A} + \mathcal{B} - i\mu_n I)\Phi_n\|^2_{\mathcal{H}} = \int_{\Gamma_1} |\xi_n|^2 d\Gamma = \int_{\Gamma_1} |\varphi_n|^2 d\Gamma. \tag{II.2.4}$$

Now by multiplying equation (II.2.1) with φ_n and by integrating by parts, we get

$$\int_{\Gamma_1} |\varphi_n|^2 d\Gamma + \int_\Omega |\nabla\varphi_n|^2 dx = \int_\Omega |\mu_n\varphi_n|^2 dx. \tag{II.2.5}$$

Then using condition (II.2.2), it follows that

$$\int_\Omega |\mu_n\varphi_n|^2 dx = 1. \tag{II.2.6}$$

On the another hand, using an interpolation inequality (see Theorem 1.4.4 in [52]) and (II.2.6), we have

$$\|\varphi_n\|_{L^2(\Gamma_1)} \leq c\|\varphi_n\|_{H^1(\Omega)}\|\varphi_n\|_{L^2(\Omega)} \leq c\|\varphi_n\|_{L^2(\Omega)} = \frac{c}{\mu_n}. \tag{II.2.7}$$

By inserting (II.2.7) into (II.2.4), it follows that :

$$\|(\mathcal{A}+\mathcal{B}-i\mu_n I)\Phi_n\|^2_{\mathcal{H}} \leq \frac{c}{\mu_n}. \tag{II.2.8}$$

Finally, from (II.2.2) and (II.2.6), for all $n \geq 1$, it follows that

$$\|\Phi_n\|^2_{\mathcal{H}} = \int_\Omega |\nabla\varphi_n|^2 dx + \int_\Omega |\mu_n\varphi_n|^2 dx + \int_{\Gamma_1} |\varphi_n|^2 d\Gamma = 2. \tag{II.2.9}$$

Then from (II.2.8) and (II.2.9), it follows that

$$\|(\mathcal{A}+\mathcal{B}-i\mu_n I)^{-1}\| \geq \frac{\mu_n}{c} \to +\infty \quad \text{as} \quad n \to +\infty.$$

The proof is thus complete. ∎

II.3 Rational energy decay rate

In this section, we will establish the rational energy decay rate for the smooth solution of system (II.1.12). We first recall the following classical result. it will play an important role in this chapter and the next chapter

II.3.1 Some integral inequality

In this section we recall the following simple result

Theorem II.3.1 *(An integral inequality) Let* $E : [0,+\infty) \to [0,+\infty)$ *be a non-*

increasing function and assume that there exists a constant $T > 0$ such that

$$\int_t^{+\infty} E(s)\,ds < TE(t)\,, \quad t \geq 0$$

then,

$$E(t) < E(0)\,e^{1-\frac{t}{T}}, \quad t \geq T$$

Theorem II.3.2 ***(A nonlinear integral inequality)*** *Let $E : [0, +\infty) \to [0, +\infty)$ be a non-increasing function and assume that there are two constants $\alpha > 0$ and $T > 0$ such that*

$$\int_t^{+\infty} E^{\alpha+1}(s)\,ds < T\,[E(0)]^{\alpha}\,E(t)\,, \quad t \geq 0$$

Then we have

$$E(t) < E(0)\left[\frac{T+\alpha t}{T+\alpha T}\right]^{-\frac{1}{\alpha}}, \quad t \geq T$$

II.3.2 Polynomial energy decay rate

Since system (II.1.12) is not uniformly stable, we will look for a polynomial energy decay rate for smooth solutions. As usual, we assume that $\Gamma = \Gamma_0 \cup \Gamma_1$ such that $\Gamma_0 \neq \emptyset$, $\Gamma_1 \neq \emptyset$ and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. Moreover, we assume that there exists a constant $\delta > 0$ and a point $x_0 \in \mathbb{R}^N$ such that, putting $m(x) = x - x_0$, we have

$$(m \cdot \nu) \geq \delta^{-1}, \quad \forall x \in \Gamma_1 \quad \text{and} \quad (m \cdot \nu) \leq 0, \quad \forall x \in \Gamma_0, \tag{II.3.1}$$

where $(\cdot)$ designates the scalar product in $\mathbb{R}^N$.

Theorem II.3.3 *Assume that (II.3.1) holds. Then for all $U_0 \in D(A)$, there exists a constant $M > 0$ continuously depending on the graph norm $\|U_0\|_{D(\mathcal{A})}$ such that the energy $E(t)$ of system (II.1.12) satisfies the following decay rate*

$$E(t) \leq E(0)\frac{2M}{M+t}, \quad \forall t \geq 0. \tag{II.3.2}$$

We will divide the proof into several steps.

Lemma II.3.1 *Let $U_0 \in D(A)$ and U be the corresponding solution of system (II.1.12). Then, for all $0 \leq S \leq T < \infty$, the following estimate holds*

$$\int_S^T \int_{\Gamma_1} |\eta|^2 d\Gamma dt \leq E(S). \tag{II.3.3}$$

Proof: Using (II.1.7), we get

$$\int_S^T \int_{\Gamma_1} |\eta|^2 d\Gamma dt = -\int_S^T E'(t)dt = E(S) - E(T) \leq E(S). \tag{II.3.4}$$

The proof is thus complete. ■

Lemma II.3.2 *Assume that (II.3.1) holds. Let* $U_0 \in D(A)$ *and* U *be the corresponding solution of system (II.1.12). Then there exist positive constants* $M_1 > 0, M_2 > 0$ *such that for all* $0 \leq S \leq T < \infty$*, the following estimate holds*

$$N \int_S^T \int_\Omega u_t^2 E(t)dxdt + (2-N) \int_S^T \int_\Omega |\nabla u|^2 E(t)dxdt \leq M_1 E^2(S) + M_2 E(S) E_1(S) \tag{II.3.5}$$

where the high order energy $E_1(t)$ *is defined by*

$$E_1(t) = \frac{1}{2}\|U_t(t)\|^2 = \frac{1}{2}\|(\mathcal{A}+\mathcal{B})U(t)\|^2. \tag{II.3.6}$$

Proof: Multiplying equation (II.1.1) by $2m \cdot \nabla u E(t)$ and integrating by parts, we get

$$\begin{aligned}
&-2\int_S^T \int_\Omega u_t(m\cdot\nabla u)E_t(t)dxdt - 2\int_S^T \int_\Omega u_t(m\cdot\nabla u_t)E(t)dxdt \\
&+ \left[2\int_\Omega u_t(m\cdot\nabla u)E(t)dx\right]_S^T - 2\int_S^T\int_\Omega \Delta u(m\cdot\nabla u)E(t)dxdt \\
&= 0.
\end{aligned} \tag{II.3.7}$$

Recall that for all $u \in H^2(\Omega)$, we have the following Rellich's identity (see [50]):

$$2\int_\Omega \Delta u(m\cdot\nabla u)dx = (N-2)\int_\Omega |\nabla u|^2 dx + 2\int_\Gamma \partial_\nu u(m\cdot\nabla u)d\Gamma - \int_\Gamma (m\cdot\nu)|\nabla u|^2 d\Gamma \tag{II.3.8}$$

This implies that

$$\begin{aligned}
&-2\int_S^T\int_\Omega \Delta u(m\cdot\nabla u)E(t)dxdt \\
&= (2-N)\int_S^T\int_\Omega |\nabla u|^2 E(t)dxdt - 2\int_S^T\int_\Gamma \partial_\nu u(m\cdot\nabla u)E(t)d\Gamma dt \\
&+ \int_S^T\int_\Gamma (m\cdot\nu)|\nabla u|^2 E(t)d\Gamma dt.
\end{aligned} \tag{II.3.9}$$

On the other hand, using Green's formula, we have

$$\begin{aligned}
-2\int_S^T\int_{\Gamma_1} |u_t(m\cdot\nabla u_t)E(t)dxdt = N\int_S^T\int_\Omega |u_t|^2 E(t)dxdt. \\
-\int_S^T\int_{\Gamma_1}(m\cdot\nu)|u_t|^2 E(t)d\Gamma dt.
\end{aligned} \tag{II.3.10}$$

Then, inserting (II.3.9) and (II.3.10) into (II.3.7), we get

$$\begin{aligned}
&N\int_S^T\int_\Omega |u_t|^2E(t)dxdt+(2-N)\int_S^T\int_\Omega |\nabla u|^2E(t)dxdt\\
&=2\int_S^T\int_\Omega u_t(m\cdot\nabla u)E_t(t)dxdt-\left[2\int_\Omega u_t(m\cdot\nabla u)E(t)dx\right]_S^T+\\
&\int_S^T\int_{\Gamma_1}(m\cdot\nu)|u_t|^2E(t)d\Gamma dt+2\int_S^T\int_\Gamma \partial_\nu u(m\cdot\nabla u)E(t)d\Gamma dt+\\
&-\int_S^T\int_\Gamma (m\cdot\nu)|\nabla u|^2E(t)d\Gamma dt. \qquad \text{(II.3.11)}
\end{aligned}$$

Now, thanks to Cauchy-Schwartz's inequality, we get

$$\int_\Omega u_t(m\cdot\nabla u)dx\le RE(t),\quad \forall t>0, \qquad \text{(II.3.12)}$$

where $R=\|m\|_\infty$. It follows that

$$\begin{aligned}
&-\left[2\int_\Omega u_t(m\cdot\nabla u)E(t)dx\right]_S^T+2\int_S^T\int_\Omega u_t(m\cdot\nabla u)E_t(t)dxdt\\
&\le 4RE^2(S)+2\int_S^T RE(t)E_1(t)dt\\
&\le 6RE^2(S). \qquad \text{(II.3.13)}
\end{aligned}$$

Next, for any $\epsilon>0$, using the geometric condition (II.3.1), we deduce that

$$\begin{aligned}
&2\int_S^T\int_{\Gamma_1}\partial_v u(m\cdot\nabla u)E(t)d\Gamma dt-\int_S^T\int_{\Gamma_1}(m\cdot v)|\nabla u|^2E(t)d\Gamma dt\\
&\le (R^2\epsilon-\delta^{-1})\int_S^T\int_{\Gamma_1}|\nabla u|^2E(t)d\Gamma dt+\frac{1}{\epsilon}\int_S^T\int_{\Gamma_1}|\partial_\nu u|^2E(t)d\Gamma dt. \qquad \text{(II.3.14)}
\end{aligned}$$

Choosing $\epsilon = \frac{\delta^{-1}}{2R^2}$ in (II.3.14), we get

$$\begin{aligned}
&2\int_S^T \int_{\Gamma_1} \partial_\nu u(m\cdot\nabla u)E(t)d\Gamma dt - \int_S^T \int_{\Gamma_1} (m\cdot v)|\nabla u|^2 E(t)d\Gamma dt \\
&\leq -\frac{\delta^{-1}}{2}\int_S^T \int_{\Gamma_1} |\nabla u|^2 E(t)d\Gamma dt + 2R^2\delta \int_S^T \int_{\Gamma_1} |\partial_\nu u|^2 E(t)d\Gamma dt \\
&\leq 2R^2\delta \int_S^T \int_{\Gamma_1} |\partial_\nu u|^2 E(t)d\Gamma dt \\
&= 2R^2\delta \int_S^T \int_{\Gamma_1} |\eta|^2 E(t)d\Gamma dt \leq 2R^2\delta E^2(S).
\end{aligned} \tag{II.3.15}$$

Using the condition $u=0$ on Γ_0 and the geometric condition (II.3.1), we get

$$\begin{aligned}
&2\int_S^T \int_{\Gamma_0} \partial_\nu u(m\cdot\nabla u)E(t)d\Gamma dt - \int_S^T \int_{\Gamma_0} (m\cdot \nu)|\nabla u|^2 E(t)d\Gamma dt \\
&= \int_S^T \int_{\Gamma_0} (m\cdot\nu)|\partial_\nu u|^2 E(t)d\Gamma dt \leq 0.
\end{aligned} \tag{II.3.16}$$

Since $U_0 \in D(\mathcal{A})$, by differentiating system (II.1.12) with respect to the variable t and using (II.3.3), we get

$$\int_S^T \int_{\Gamma_1} |\eta_t|^2 d\Gamma dt \leq E_1(S). \tag{II.3.17}$$

Then, using (II.1.4), (II.3.3) and (II.3.17), we deduce that

$$\begin{aligned}
\int_S^T \int_{\Gamma_1} (m\cdot\nu)|u_t|^2 E(t)d\Gamma dt &= \int_S^T \int_{\Gamma_1} (m\cdot\nu)(\eta(t)+\eta_t(t))^2 E(t)d\Gamma dt \\
&\leq 2RE(S)[E(S)+E_1(S)]
\end{aligned} \tag{II.3.18}$$

Finally, noting that $(m\cdot\nu)\leq 0$ on Γ_0 and by inserting (II.3.13), (II.3.15), (II.3.16) and (II.3.18) into (II.3.11) we get (II.3.5). The constant M_1 is given by

$$M_1 = 8R + 2R^2\delta, \quad M_2 = 2R.$$

The proof is thus complete. ■

Lemma II.3.3 *Let $U_0 \in D(A)$ and U be the corresponding solution of system (II.1.12). Then, for all $0 \leq S \leq T < \infty$, the following estimate holds*

$$-\int_S^T \int_\Omega |u_t|^2 E(t)dxdt + c_N \int_S^T \int_\Omega |\nabla u|^2 E(t)dxdt \leq M_3 E^2(S), \tag{II.3.19}$$

where $c_N = 1 - \frac{1}{2(N-1)}$ *and* M_3 *is a positive constant independent of initial data* U_0.

Proof: By multiplying equation (II.1.1) by $uE(t)$ and by integrating by parts, we obtain

$$\begin{aligned}
&-\int_S^T \int_\Omega |u_t|^2 E(t)dxdt + \int_S^T \int_\Omega |\nabla u|^2 E(t)dxdt \\
&= -\left[\int_\Omega u_t u E(t)dx\right]_S^T + \int_S^T \int_\Omega u_t u E_t(t)dxdt + \int_S^T \int_{\Gamma_1} u\partial_\nu u E(t)d\Gamma dt \qquad \text{(II.3.20)}
\end{aligned}$$

Thanks to Poincaré's inequality, there exists a constant c_0 such that

$$\int_\Omega u_t u dx \le c_0 E(t), \quad \forall t > 0. \qquad \text{(II.3.21)}$$

It follows that

$$-\left[\int_\Omega u_t u E(t)dx\right]_S^T + \int_S^T \int_\Omega u_t u E_t(t)dxdt \le 3c_0 E^2(S). \qquad \text{(II.3.22)}$$

On the other hand, thanks to Young's inequality and the traces theorem, there exists a constant $c_1 > 0$ independent of U_0 such that, for all $\varepsilon > 0$, the following estimate holds

$$\int_{\Gamma_1} u\partial_\nu u d\Gamma \le \frac{1}{2\varepsilon}\int_{\Gamma_1} |\partial_\nu u|^2 d\Gamma + \frac{\varepsilon c_1}{2}\int_\Omega |\nabla u|^2 dx. \qquad \text{(II.3.23)}$$

Then, using (II.1.3) and (II.3.4) we obtain

$$\begin{aligned}
\int_S^T \int_{\Gamma_1} u\partial_\nu u E(t)d\Gamma dt &\le \frac{1}{2\varepsilon}\int_S^T \int_{\Gamma_1} |\eta|^2 E(t)d\Gamma dt + \frac{\varepsilon c_1}{2}\int_S^T \int_\Omega |\nabla u|^2 E(t)dxdt \\
&\le \frac{1}{2\varepsilon}E^2(S) + \frac{\varepsilon c_1}{2}\int_S^T \int_\Omega |\nabla u|^2 E(t)dxdt. \qquad \text{(II.3.24)}
\end{aligned}$$

Choosing $\varepsilon = \frac{1}{c_1(N-1)}$ and inserting (II.3.22) and (II.3.24) into (II.3.20), we obtain (II.3.19). The constant M_3 is given by

$$M_3 = 3c_0 + \frac{c_1(N-1)}{2}.$$

The proof is thus complete. ■

Now, we *prove Theorem II.3.3*

Proof: By multiplying (II.3.19) by $(N-1)$ and by combining the resulting estimate with (II.3.3) and (II.3.5), we get

$$\int_S^{+\infty} E^2(t)dt \le ME(S)E(0), \qquad \text{(II.3.25)}$$

where

$$M = M_1 + (N-1)M_3 + \frac{1}{2} + M_2\frac{E_1(0)}{E(0)}. \tag{II.3.26}$$

Thanks to a classical integral inequality (see [50] Theorem 9.1), the integral inequality (II.3.25) leads to

$$E(t) \leq E(0)\frac{2M}{M+t}, \quad \forall t \geq 0. \tag{II.3.27}$$

■

Remark II.3.1 *In the one dimensional case, the energy decay rate (II.3.27) of type $\frac{1}{t}$ is in fact optimal (see [3], [5]). Moreover, using an approach in [10] for the Kirchhoff plates, we give a short proof of Theorem II.1.1 under the additional geometric condition (II.3.1).*

Corollary II.3.1 *Assume that (II.3.1) holds. For any initial data $U_0 \in H$, the energy $E(t)$ of system (II.1.12) asymptotically decreases to zero.*

Proof: Let $U_0 \in \mathcal{H}$. For any $\varepsilon > 0$, by density of $D(\mathcal{A})$ in $\mathcal{H}$, there exists $U_\varepsilon \in D(\mathcal{A})$ such that

$$\|U_0 - U_\varepsilon\|_{\mathcal{H}} < \frac{\varepsilon}{2}. \tag{II.3.28}$$

On the other hand, using Theorem II.3.3, there exists $T_\varepsilon > 0$ such that

$$\|S_{\mathcal{A}+\mathcal{B}}(t)U_\varepsilon\|_{\mathcal{H}} < \frac{\varepsilon}{2}, \quad \forall t > T_\varepsilon. \tag{II.3.29}$$

Since the semigroup $S_{\mathcal{A}+\mathcal{B}}$ is of contractions, then from (II.3.28) and (II.3.29) we deduce

$$\|S_{\mathcal{A}+\mathcal{B}}(t)U_0\|_{\mathcal{H}} \leq \|S_{\mathcal{A}+\mathcal{B}}(t)U_\varepsilon\|_{\mathcal{H}} + \|U - U_\varepsilon\|_{\mathcal{H}} < \varepsilon, \quad \forall t > T_\varepsilon. \tag{II.3.30}$$

The proof is thus complete. ■

II.4 Observability and exact controllability

In this section, we will study the observability and the exact controllability of the wave equation with a dynamical control:

$$u_{tt} - \Delta u = 0 \qquad \text{in } \Omega \times [0,T], \tag{II.4.1}$$

$$u = 0 \qquad \text{on } \Gamma_0 \times [0,T], \tag{II.4.2}$$

$$\frac{\partial u}{\partial \nu} + \eta = 0 \qquad \text{on } \Gamma_1 \times [0,T], \tag{II.4.3}$$

$$\eta_t - u_t = v \qquad \text{on } \Gamma_1 \times [0,T] \tag{II.4.4}$$

with the following initial condition:

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\ x\in\Omega,\quad \eta(x,0)=\eta_0(x),\ x\in\Gamma_1, \tag{II.4.5}$$

where v designates the dynamical boundary control. Denoting by $U=(u,u_t,\eta)$ the state of the system (II.4.1)-(II.4.5) and by $V=(0,0,v)$ the boundary control. We can formulate the system (II.4.1)-(II.4.5) into the following abstract problem:

$$U_t=\mathcal{A}U+V,\quad U(0)=U_0, \tag{II.4.6}$$

where $\mathcal{A}$ is the linear unbounded operator defined in (II.1.9)-(II.1.10).

i) We first establish an inequality of observability for the adjoint system :

$$\begin{aligned}
\varphi_{tt}-\Delta\varphi&=0 &&\text{in } \Omega\times[0,T], && \text{(II.4.7)}\\
\varphi&=0 &&\text{on } \Gamma_0\times[0,T], && \text{(II.4.8)}\\
\frac{\partial\varphi}{\partial\nu}+\xi&=0 &&\text{on } \Gamma_1\times[0,T], && \text{(II.4.9)}\\
\xi_t-\varphi_t&=0 &&\text{on } \Gamma_1\times[0,T] && \text{(II.4.10)}
\end{aligned}$$

with the following initial condition

$$\varphi(x,0)=\varphi_0(x),\quad \varphi_t(x,0)=\varphi_1(x),\ x\in\Omega,\quad \xi(x,0)=\xi_0(x),\ x\in\Gamma_1. \tag{II.4.11}$$

Setting $\Phi=(\varphi,\varphi_t,\xi)$ as the state of system (II.4.7)-(II.4.11), we formally transform system (II.4.7)-(II.4.11) into an evolution equation :

$$\Phi_t=\mathcal{A}\Phi,\quad \Phi(0)=\Phi_0. \tag{II.4.12}$$

Since $\mathcal{A}$ is skew adjoint and m-dissipative on $\mathcal{H}$, it generates a strongly continuous group of isometry $S_{\mathcal{A}}(t)$ on the energy space $\mathcal{H}$ (see Pazy [1]).

Theorem II.4.1 *Assume that condition (II.3.1) holds. There exist positive constants $T>0$ and $c>0$ independent of initial data U_0 such that the following inequality of observability*

$$\|\Phi_0\|^2_{\mathcal{H}}\le c\int_0^T\int_{\Gamma_1}(|\xi|^2+|\xi_t|^2)d\Gamma dt \tag{II.4.13}$$

holds for all $\Phi_0\in D(A)$.

Proof: Let $\psi=\varphi_t$. Then we have

$$\begin{aligned}
\psi_{tt}-\Delta\psi&=0 &&\text{in } \Omega\times[0,T], && \text{(II.4.14)}\\
\psi&=0 &&\text{on } \Gamma_0\times[0,T], && \text{(II.4.15)}\\
\frac{\partial\psi}{\partial\nu}+\psi&=0 &&\text{on } \Gamma_1\times[0,T]. && \text{(II.4.16)}
\end{aligned}$$

Using a classical result on the observability of wave equation with the Robin condition (see [50], [23]), there exist positive constants $T > 0$ and $c > 0$ such that

$$\|\psi_0\|^2_{L^2(\Omega)} + \|\psi_1\|^2_{V'} \leq c \int_0^T \int_{\Gamma_1} |\psi|^2 d\Gamma dt. \tag{II.4.17}$$

Noting that

$$\|\psi_0\|_{L^2(\Omega)} = \|\varphi_1\|_{L^2(\Omega)}, \quad \|\psi_1\|_{V'} = \|\Delta\varphi_0\|_{V'} = \|\varphi_0\|_V,$$

then it follows that

$$\|\varphi_0\|^2_V + \|\varphi_1\|^2_{L^2(\Omega)} \leq c \int_0^T \int_{\Gamma_1} |\xi_t|^2 d\Gamma dt. \tag{II.4.18}$$

On the other hand, the trace theorem implies that

$$\int_{\Gamma_1} |\xi_0|^2 d\Gamma \leq c' \int_0^T \int_{\Gamma_1} (|\xi|^2 + |\xi_t|^2) d\Gamma dt. \tag{II.4.19}$$

By combining (II.4.18)-(II.4.19), we get (II.4.13). The proof is complete. ■

Now we choose the control $v(t)$ of the following form

$$v(t) = v_0(t) - \frac{d}{dt} v_1(t) \quad \text{with} \quad v_0, v_1 \in L^2(0, T; L^2(\Gamma_1)), \tag{II.4.20}$$

where the derivation $\frac{d}{dt}$ is defined in the sense :

$$\int_0^T \int_{\Gamma_1} \frac{d}{dt} v_1(t)\mu(t) d\Gamma dt = -\int_0^T \int_{\Gamma_1} v_1(t)\mu_t(t) d\Gamma dt \tag{II.4.21}$$

for all $\mu \in H^1(0, T; L^2(\Gamma_1))$ (see [23], [?]).

Next let $\Phi = (\varphi, \varphi_t, \xi)$ be the solution of the homogeneous problem (II.4.12). Then by multiplying equation (II.4.6) by Φ and by integrating by parts, we obtain formally

$$\begin{aligned} &\langle U, \Phi \rangle_{D(\mathcal{A})' \times D(\mathcal{A})} \\ &= \langle U_0, \Phi_0 \rangle_{D(\mathcal{A})' \times D(\mathcal{A})} + \int_0^T \int_{\Gamma_1} (v_0(s)\xi(s) + v_1(s)\xi_t(s)) d\Gamma dt. \end{aligned} \tag{II.4.22}$$

We define the linear form L by

$$L(\Phi_0) = \langle U_0, \Phi_0 \rangle_{D(\mathcal{A})' \times D(\mathcal{A})} + \int_0^T \int_{\Gamma_1} (v_0(s)\xi(s) + v_1(s)\xi_t(s)) d\Gamma dt. \tag{II.4.23}$$

Then, we obtain a weak formulation of problem (II.4.6)

$$\langle U, \Phi \rangle_{D(\mathcal{A})' \times D(\mathcal{A})} = L(\Phi_0), \ \forall \Phi_0 \in D(\mathcal{A}). \tag{II.4.24}$$

Theorem II.4.2 *For all $U_0 \in D(A)'$ and all $v_0, v_1 \in L^2(0,T; L^2(\Gamma_1))$, system (II.4.6) admits a unique weak solution :*

$$U \in C^0([0,T]; D(\mathcal{A})') \tag{II.4.25}$$

in the sense that the variational equation (II.4.24) is satisfied for all $\Phi_0 \in D(A)$ on the interval $[0,T]$. Moreover, the linear application

$$(U_0, v_0, v_1) \to U \tag{II.4.26}$$

is continuous from $D(A)'L^2(0,T; L^2(\Gamma_1)) \times L^2(0,T; L^2(\Gamma_1))$ into $D(A)'$ with the corresponding strong topologies.

Proof: First, by Cauchy-Schwartz's inequality, it is easy to get

$$\begin{aligned} &\left| \int_0^t \int_{\Gamma_1} (v_0(s)\xi(s) + v_1(s)\xi_t(s)) d\Gamma dt \right| \\ &\leq \|(v_0, v_1)\|_{L^2(0,T;L^2(\Gamma_1))} \|\|\xi, \xi_t\|_{L^2(0,T;L^2(\Gamma_1))} \end{aligned} \tag{II.4.27}$$

On the other hand, we have

$$\|(\xi, \xi_t)\|^2_{L^2(0,T;L^2(\Gamma_1))} \leq \int_0^T (\|\Phi(t)\|^2_{\mathcal{H}} + \|\mathcal{A}\Phi(t)\|^2_{\mathcal{H}}) dt = T\|\Phi_0\|^2_{D(\mathcal{A})}.$$

It follows that

$$\|L\|_{\mathcal{L}(D(\mathcal{A}),\mathbb{R})} \leq c(\|(v_0, v_1)\|_{L^2(0,T;L^2(\Gamma_1))} + \|U_0\|_{D(\mathcal{A})'}). \tag{II.4.28}$$

By virtue of Riesz-Fréchet's representation theorem, for each $0 \leq t \leq T$, there exists a unique element $Z(t) \in D(\mathcal{A})'$ such that

$$L(\Phi_0) = \langle Z(t), \Phi_0 \rangle_{D(\mathcal{A})' \times D(\mathcal{A})}, \quad \forall \Phi_0 \in D(\mathcal{A}). \tag{II.4.29}$$

Then, setting $Z(t) = S_{\mathcal{A}}(-t)U(t)$ in (II.4.29), we get (II.4.24). Moreover, we have

$$\|U(t)\|_{D(\mathcal{A})'} \leq c(\|(v_0, v_1)\|_{L^2(0,T;L^2(\Gamma_1))} + \|U_0\|_{D(\mathcal{A})'}), \quad \forall t \in [0,T]. \tag{II.4.30}$$

Let $\{U_n\}_n$ be a sequence of approximate smooth solutions of (II.4.6). Since the estimation (II.4.30) is uniform on the interval $[0,T]$, we can prove that $U_n \to U$ uniformly on the interval $[0,T]$, therefore $U \in C^0([0,T]; D(\mathcal{A})')$. The proof is thus complete. ■

Theorem II.4.3 *Assume that the condition (II.3.1) holds. There exists $T > 0$ such that system (II.4.6) is exactly controllable at the moment T. More precisely, for any given initial data $U_0 \in H$, there exists a singular control $v(t)$ of the form (II.4.20) such that the weak solution U of the controlled problem (II.4.6) satisfies the final condition*

$$U(T) = 0. \tag{II.4.31}$$

Proof: *Proof.* Let $\Phi_0 = (\varphi_0, \psi_0, \xi_0) \in D(\mathcal{A})$ and $\Phi = (\varphi, \psi, \xi)$ be the corresponding solution of problem (II.4.12). Thanks to the inequality of observability (II.4.13), the semi-norm

$$\|\Phi_0\|_{\mathcal{F}} = \left(\int_0^T \int_{\Gamma_1} (|\xi|^2 + |\xi_t|^2) d\Gamma dt\right)^{1/2} \tag{II.4.32}$$

is in fact a norm on $D(\mathcal{A})$. We denote by $\mathcal{F}$ the closure of $D(\mathcal{A})$ with respect to this norm. On the other hand, it is easy to see that

$$\int_0^T \int_{\Gamma_1} (|\xi|^2 + |\xi_t|^2) d\Gamma dt \leq T(\|\Phi_0\|^2_{\mathcal{H}} + \|\mathcal{A}\Phi_0\|^2_{\mathcal{H}}).$$

Thus, we have the following inclusions:

$$D(\mathcal{A}) \subset \mathcal{F} \subset \mathcal{H}. \tag{II.4.33}$$

Then by identifying the Hilbert space $\mathcal{H}$ with its dual space, we get the following inclusions

$$\mathcal{H} \subset \mathcal{F}' \subset D(\mathcal{A})'. \tag{II.4.34}$$

Now, by choosing the control $v(t)$ such that

$$v(t) := \xi(t) - \frac{d}{dt}\xi_t(t), \tag{II.4.35}$$

where the derivation $\frac{d}{dt}$ is defined in the sense of (II.4.21).

Next we solve the backward problem

$$\Psi_t = \mathcal{A}\Psi + V, \quad \Psi(T) = 0. \tag{II.4.36}$$

Following Theorem II.4.2, the backward problem (II.4.36) admits a unique weak solution $\Psi(t) \in C^0([0, T]; D(\mathcal{A})')$. Then, we define the operator $\Lambda : D(\mathcal{A}) \to D(\mathcal{A})'$ by

$$\Lambda\Phi_0 = -\Psi(0), \quad \forall \Phi_0 \in D(\mathcal{A}). \tag{II.4.37}$$

From (II.4.22) and (II.4.36), it follows that

$$\langle \Lambda\Phi_0, \tilde{\Phi}_0 \rangle_{D(\mathcal{A})' \times D(\mathcal{A})} = \int_0^T \int_{\Gamma_1} (\xi\tilde{\xi} + \xi_t\tilde{\xi}_t) d\Gamma dt, \quad \forall \Phi_0, \tilde{\Phi}_0 \in D(\mathcal{A}) \tag{II.4.38}$$

Therefore, we have

$$|\langle \Lambda\Phi_0, \tilde{\Phi}_0\rangle_{D(\mathcal{A})'\times D(\mathcal{A})}| \leq \|\Phi_0\|_{\mathcal{F}}\|\tilde{\Phi}_0\|_{\mathcal{F}}, \quad \forall \Phi_0, \tilde{\Phi}_0 \in D(\mathcal{A}). \tag{II.4.39}$$

By definition, $D(\mathcal{A})$ is dense in $\mathcal{F}$. Then the above bilinear form can be continuously extended to $\mathcal{F}\times\mathcal{F}$. Therefore, (II.4.38) remains true on the space $\mathcal{F}\times\mathcal{F}$. Moreover we have

$$|\langle \Lambda\Phi_0, \Phi_0\rangle| = \|\Phi_0\|^2_{\mathcal{F}}, \quad \forall \Phi_0 \in \mathcal{F}. \tag{II.4.40}$$

By Lax-Milgram's Theorem, Λ is an isomorphism from $\mathcal{F}$ onto $\mathcal{F}'$. In particular, for any given $-U_0 \in \mathcal{H} \subset \mathcal{F}'$, there exists a unique $\Phi_0 \in \mathcal{F}$ such that

$$\Lambda\Phi_0 = -U_0. \tag{II.4.41}$$

This equality implies the exact controllability of (II.4.6). The proof is complete. ■

Remark II.4.1 *For the implication from (II.4.33) to (II.4.34), we have to identify the Hilbert space $\mathcal{H}$ with its dual space. Since $\mathcal{H} = V \times L^2(\Omega)\times L^2(\Gamma_1)$, we simultaneously identify $H^1(\Omega)$, respectively $L^2(\Omega)$ with their dual spaces. Fortunately, these identifications are independent and do not cause any confusions.*

Chapter III

Stabilization of the wave equation with variable coefficients and a delay in dissipative internal feedback

Contents

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ. It is assumed that Γ consists of two parts Γ_1 and Γ_2 ($\Gamma = \Gamma_1 \cup \Gamma_2$) with $\Gamma_2 \neq \emptyset$, $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$. Define

$$\mathcal{A}u = -\operatorname{div} A(x)\nabla u \qquad \text{for } u \in H^1(\Omega) \tag{III.0.1}$$

where $A(x) = (a_{ij}(x))$ is symmetric, positively definite matrices for each $x \in \mathbb{R}^n$ and $a_{ij}(x)$ are smooth functions on $\mathbb{R}^n$. We consider the stabilization of the wave equation with variable coefficients and a delay in the dissipative internal feedback:

$$\begin{cases} u_{tt} + \mathcal{A}u + a(x)[\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)] = 0 & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t)|_{\Gamma_2} = 0 & t \in (0,+\infty), \\ \dfrac{\partial u(x,t)}{\partial \nu_A}|_{\Gamma_1} = 0 & t \in (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & (x,t) \in \Omega \times (0,\tau) \end{cases} \tag{III.0.2}$$

where $a(x) \in C^1(\overline{\Omega})$ such that

$$\begin{cases} a(x) \geq 0 & x \in \Omega \\ a(x) > a > 0 & x \in \omega, \end{cases} \tag{III.0.3}$$

where $\omega \subset \Omega$ is a bounded domain with smooth boundary, $\partial\omega = \Gamma_1 \cup \Gamma_3$, $\overline{\Gamma}_1 \cap \overline{\Gamma}_3 = \emptyset$. $\frac{\partial u}{\partial \nu_A}$ is the co-normal derivative

$$\frac{\partial u}{\partial \nu_A} = \langle A(x)u, \nu \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard metric of the Euclidean space $\mathbb{R}^n$ and $\nu(x)$ is the outside unit normal vector for each $x \in \Gamma$. Moreover, $\tau > 0$ is a time delay, μ_1, μ_2 are real numbers with $\mu_1 > 0, \mu_2 \neq 0$, and the initial data (u_0, u_1, f_0) belong to a suitable space.

If $A(x)$ is a constant matrix on $\overline{\Omega}$, that is to say the system III.0.2 is a constant coefficient problem and many authors [[17]–[41], [?], [?]] have studied the problem. For a general $A(x)$, the main tools here to cope with the system III.0.2 are the differential geometrical methods which were introduced by [31] and were extended in [[29]–[43], [44], [8], [21], [22], [46], [53], [47], [32]–[54]], and many others. For a survey on the differential geometric methods, see [[40], [36]].

It is well known that if there is no delay (the case: $\mu_2 = 0$ in III.0.2), the energy of problem III.0.2 is exponentially decaying to zero, which was obtained by [[27], [14]] under the condition of constant coefficient and shown in [45] under the condition of variable coefficients.

If $\mu_1 \leq \mu_2$, an explicit sequence of arbitrary small delays that destabilize the system have been shown in [?]. On the contrary, if $|\mu_2| < \sqrt{1-d}\mu_1$, where d is dependent on τ, stabilization result of the energy of problem [42] has been proved under the conditions that the delay $\tau = \tau(t)$ is a function of time and $a(x)$ is a positive constant function, see [48]. To obtain our stabilization result, we assume that

$$|\mu_2| < \mu_1. \tag{III.0.4}$$

Define the energy of the system III.0.2 by

$$E(t) = \frac{1}{2}\int_\Omega \left(u_t^2 + \sum_{i,j=1}^n a_{ij}u_{x_i}u_{x_j} \right) dx + \xi \int_0^1 \int_\Omega a(x)u_t^2(x, t-\rho\tau)dxd\rho, \tag{III.0.5}$$

where ξ is a positive constant satisfying

$$\tau|\mu_2| < 2\xi < \tau(2\mu_1 - |\mu_2|). \tag{III.0.6}$$

Let

$$H^1_{\Gamma_2}(\Omega) = \{u \in H^1(\Omega) | u|_{\Gamma_2} = 0\}, \tag{III.0.7}$$

and

$$L^2_a(\Omega \times (0,1)) = \left\{ u \ \setminus \ \int_0^1 \int_\Omega a(x)u^2(x, -\rho\tau)dxd\rho < +\infty \right\}. \tag{III.0.8}$$

We assume the initial value $(u_0, u_1, f_0) \in H^1_{\Gamma_2}(\Omega) \times L^2(\Omega) \times L^2_a(\Omega \times (0,1))$, then we have $E(0) < +\infty$. From the below inequality III.2.8, we have $E(t)$ decreasing. We define

$$g = A^{-1}(x) \quad \text{for } x \in \mathbb{R}^n \tag{III.0.9}$$

as a Riemannian metric on $\mathbb{R}^n$ and consider the couple $(\mathbb{R}^n, g)$ as a Riemannian manifold. For each $x \in \mathbb{R}^n$, the metric g introduces an inner product and the norm on the tangent space on $\mathbb{R}^n_x = \mathbb{R}^n$ by

$$\langle X, Y\rangle_g = \langle A^{-1}(x)X, Y\rangle, \quad |X|^2_g = \langle X, X\rangle_g \quad X, Y \in \mathbb{R}^n_x.$$

Let D_g denote the Levi-Civita connection of the metric g and H be a vector field. Then the covariant differential D_gH of the vector field H is an order 2 tensor field. For the variable coefficients, the main assumption is:
Assumption A. There exist a vector field H on $\overline{\Omega}$ and a constant $\rho_0 > 0$ such that

$$D_gH(X, X) \geq \rho_0 |X|^2_g \quad \text{for } X \in \mathbb{R}^n_x, x \in \overline{\Omega}. \tag{III.0.10}$$

The main result of this paper is the following:

Theorem III.0.4 *Let the Assumption **A** hold true such that*

$$\langle H, \nu\rangle \leq 0 \quad \textit{for } x \in \Gamma_2 \tag{III.0.11}$$

where ν is the outside normal of the boundary Γ. Then there exist constants $C_1, C_2 > 0$, such that

$$E(t) \leq C_1 e^{-C_2 t} E(0), \quad t \geq 0 \tag{III.0.12}$$

The **Assumption A** was firstly introduced by [31] as a checkable assumption for the exact controllability of the wave equation with variable coefficients. Based on the Strongly convex function $D^2\rho^2$ (ρ is the distance function of $(\mathbb{R}^n, g)$), which is dependent on the Riemannian curvature of the metric g, many examples of the Assumption A have been obtained, see [[31], [36]].

III.1 Well-posedness of the system

In this section, we consider the existence and regularity of solutions to the system III.0.2. Setting

$$z(x, \rho, t) = u_t(x, t - \rho\tau) \quad x \in \overline{\Omega}, \rho \in (0,1), t > 0, \tag{III.1.1}$$

the system III.0.2 is equivalent to

$$\begin{cases} u_{tt} + \mathcal{A}u + a(x)[\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)] = 0 \quad (x,t) \in \Omega \times (0,+\infty), \\ u(x,t)|_{\Gamma_2} = 0 \quad t \in (0,+\infty), \\ \dfrac{\partial u(x,t)}{\partial \nu_A}|_{\Gamma_1} = 0 \quad t \in (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad x \in \Omega, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \quad (x,\rho,t) \in \overline{\Omega} \times (0,1) \times (0,+\infty), \\ z(x,0,t) = 0 \quad (x,t) \in \Gamma_2 \times (0,+\infty), \\ z(x,0,t) = u_t(x,t) \quad (x,t) \in \Gamma_1 \times (0,+\infty), \\ z(x,\rho,0) = f_0(x,-\rho\tau) \quad (x,t) \in \Omega \times (0,1). \end{cases} \tag{III.1.2}$$

We define

$$U := (u, u_t, z)^T,$$

then

$$U' := (u_t, u_{tt}, z_t)^T = \left(u_t, -\mathcal{A}u - a[\mu_1 u_t + \mu_2 z(\cdot,1,\cdot)], -\frac{1}{\tau} z_\rho \right)^T$$

Therefore, the system III.1.2 can be rewritten as

$$\begin{cases} U' = \mathcal{A}_0 U, \\ U(0) = (u_0, u_1, f_0(\cdot, -\cdot\,\tau))^T, \end{cases} \tag{III.1.3}$$

where

$$\mathcal{A}_0(u,v,z)^T = \left(v, -\mathcal{A}u - a[\mu_1 v + \mu_2 z(\cdot,1)], -\frac{1}{\tau} z_\rho \right)^T,$$

with domain

$$\begin{aligned} D(\mathcal{A}_0) = \Big\{ &(u,v,z) \in (H^2(\Omega) \cap H^1_{\Gamma_2}(\Omega)) \times H^1_{\Gamma_2}(\Omega) \times L^2_a(\Omega; H^1(0,1)) \\ &: \frac{\partial u}{\partial \nu_A}|_{\Gamma_1} = 0, z(\cdot,0) = v \text{ on } \overline{\Omega} \Big\} \end{aligned} \tag{III.1.4}$$

Denote by $\mathcal{H}^0$ the Hilbert space

$$\mathcal{H}^0 = H^1_{\Gamma_2}(\Omega) \times L^2(\Omega) \times L^2_a(\Omega \times (0,1)) \tag{III.1.5}$$

equipped with the inner product

$$\langle (u,v,z)^T, (\tilde{u},\tilde{v},\tilde{z})^T\rangle_{\mathcal{H}^0} = \int_\Omega (\langle \nabla u(x), A(x)\nabla \tilde{u}(x)\rangle + v(x)\tilde{v}(x))dx$$
$$+ 2\xi \int_0^1 \int_\Omega a(x) z(x,\rho)\tilde{z}(x,\rho)dxd\rho, \qquad \text{(III.1.6)}$$

where ξ is defined by III.0.6.

The methods to deal with the well-posedness result of the system III.0.2 were given by [?]. There are two major steps. The first step is to prove the operator $\mathcal{A}_0$ is dissipative and the second step is to prove $\lambda I - \mathcal{A}_0$ is surjective for a fixed $\lambda > 0$.

Here, we omit the process of the proof of the well-posedness result. By a similar proof with [18], we obtain the following well-posedness result.

Theorem III.1.1 *For any initial datum $U_0 \in H^0$, there exists a unique solution $U \in C([0,+\infty), H^0)$ of the system III.1.3. Moreover, if $U_0 \in D(A_0)$, then*

$$U \in C([0,+\infty), D(\mathcal{A}_0)) \cap C^1([0,+\infty), \mathcal{H}^0).$$

III.2 Proofs of Theorem III.0.4

In this section we work on Ω with two metrics at the same time, the standard dot metric $\langle\cdot,\cdot\rangle$ and the Riemannian metric $g = \langle\cdot,\cdot\rangle_g$ given by III.0.9.
If $f \in C^1(\mathbb{R}^n)$, we define the gradient $\nabla_g f$ of f in the Riemannian metric g, via the Riesz representation theorem, by

$$X(f) = \langle \nabla_g f, X\rangle_g, \qquad \text{(III.2.1)}$$

where X is any vector field on $(\mathbb{R}^n, g)$. The following lemma provides further relations between the two metrics, see [[5], Lemma 2.1].

Lemma III.2.1 *Let $x = (x_1,\ldots,x_n)$ be the natural coordinate system in R^n. Let f,h be functions and let H,X be vector fields. Then*
(a)

$$\langle H(x), A(x)X(x)\rangle_g = \langle H(x), X(x)\rangle, \quad x \in \mathbb{R}^n; \qquad \text{(III.2.2)}$$

(b)

$$\nabla_g f = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}(x) f_{x_j}\right)\frac{\partial}{\partial x_i} = A(x)\nabla f, \quad x \in \mathbb{R}^n, \qquad \text{(III.2.3)}$$

where ∇f is the gradient of f in the standard metric;
(c)

$$\nabla_g f(h) = \langle \nabla_g f, \nabla_g h\rangle_g = \langle \nabla f, A(x)\nabla h\rangle, \quad x \in \mathbb{R}^n, \qquad \text{(III.2.4)}$$

where the matrix $A(x)$ is given in the formula III.0.1;
(d)

$$\langle \nabla_g f, \nabla_g(\mathcal{H}(f))\rangle_g = D_g\mathcal{H}(\nabla_g f, \nabla_g f) + \frac{1}{2} div(|\nabla_g f|_g^2 \mathcal{H}) - \frac{1}{2}|\nabla_g f|_g^2 div\, \mathcal{H}, \quad \text{(III.2.5)}$$

where $div\ H$ is the divergence of H of the standard metric of R^n.

To prove Theorem III.0.4, we still need several lemmas further. Let

$$E_0(t) = \frac{1}{2}\int_\Omega (u_t^2 + |\nabla_g u|_g^2)dx, \quad \text{(III.2.6)}$$

with III.0.5, we have

$$E(t) = E_0(t) + \xi \int_0^1 \int_\Omega a(x)u_t^2(x, t-\rho\tau)dxd\rho. \quad \text{(III.2.7)}$$

Lemma III.2.2 *Suppose that the condition III.0.6 holds true. Let $u(x,t)$ be the solution of system III.0.2. Then there exist constants $C_1, C_2 > 0$ such that*

$$E(S) - E(T) \geq C_1 \int_S^T \int_\Omega a(x)(u_t^2(x,t) + u_t^2(x, t-\tau))dxdt \quad \text{(III.2.8)}$$

$$E(S) - E(T) \leq C_2 \int_S^T \int_\Omega a(x)(u_t^2(x,t) + u_t^2(x, t-\tau))dxdt \quad \text{(III.2.9)}$$

where $T \geq S \geq 0$ and C_1, C_2 are not dependent on u. The assertion III.2.8 implies that $E(t)$ is decreasing.

Proof: Differentiating III.0.5, we obtain

$$E'(t) = \int_\Omega (u_t u_{tt} + \nabla_g u \cdot \nabla u_t)dx + 2\xi \int_0^1 \int_\Omega a(x)u_t(x, t-\rho\tau)u_{tt}(x, t-\rho\tau)dxd\rho. \quad \text{(III.2.10)}$$

Applying Green's formula, referring to the fact that

$$u_t(x, t-\rho\tau) = -\frac{1}{\tau}u_\rho(x, t-\rho\tau), \quad u_{tt}(x, t-\rho\tau) = \frac{1}{\tau^2}u_{\rho\rho}(x, t-\rho\tau)$$

and by integrating by parts, we arrive at

$$E'(t) = \int_\Omega a(x)\left[(-\mu_1 u_t^2(x,t) - \mu_2 u_t(x,t)u_t(x,t-\tau)) + \frac{\xi}{\tau}(u_t^2(x,t) - u_t^2(x,t-\tau))\right]dx. \quad \text{(III.2.11)}$$

It follows from III.2.11 that

$$E'(t) \leq -C_1 \int_\Omega a(x)\left(u_t^2(x,t) + u_t^2(x,t-\tau)\right)dx \tag{III.2.12}$$

$$E'(t) \geq -C_2 \int_\Omega a(x)\left(u_t^2(x,t) + u_t^2(x,t-\tau)\right)dx \tag{III.2.13}$$

where $C_1, C_2 > 0$ and C_1 satisfies

$$C_1 = \min\left\{\frac{\xi}{\tau} - \frac{|\mu_2|}{2}, \mu_1 - \frac{|\mu_2|}{2} - \frac{\xi}{\tau}\right\}. \tag{III.2.14}$$

Then the inequality III.2.8 /III.2.9 follows directly from III.2.12/ III.2.13 integrating from S to T. ■

Lemma III.2.3 *Suppose that $u(x,t)$ solves equation*

$$u_{tt} + \mathcal{A}u + a(x)[\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)] = 0 \quad (x,t) \in \Omega \times (0,+\infty) \tag{III.2.15}$$

and that $\mathcal{H}$ is a vector field defined on $\overline{\Omega}$. Then

$$\begin{aligned}\int_0^T dt \int_\Gamma \frac{\partial u}{\partial \nu_A}\mathcal{H}(u)d\Gamma + \frac{1}{2}\int_0^T dt \int_\Gamma (u_t^2 - |\nabla_g u|_g^2)\mathcal{H}\cdot\nu d\Gamma \\ = (u_t, \mathcal{H}(u))|_0^T + \int_0^T dt \int_\Omega D_g\mathcal{H}(\nabla_g u, \nabla_g u)dx + \frac{1}{2}\int_0^T dt \int_\Omega (u_t^2 - |\nabla_g u|_g^2)\ div\ \mathcal{H}dx \\ + \int_0^T dt \int_\Omega a(x)\mathcal{H}(u)[\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)]dx.\end{aligned} \tag{III.2.16}$$

Moreover, assume that $P \in C^1(\overline{\Omega})$. Then

$$\begin{aligned}\int_0^T dt \int_\Omega (u_t^2 - |\nabla_g u|_g^2)Pdx = (u_t, uP)|_0^T + \frac{1}{2}\int_0^T dt \int_\Omega \nabla_g P(u^2)dx - \int_0^T dt \int_\Gamma Pu\frac{\partial u}{\partial \nu_A}d\Gamma \\ + \int_0^T dt \int_\Omega a(x)Pu[\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)]dx,\end{aligned} \tag{III.2.17}$$

Proof: Multiply III.2.15 by $H(u)$, with III.2.5 we have

$$\begin{aligned}
0 &= (u_{tt} + \mathcal{A}u + a(x)[\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)])\mathcal{H}(u) \\
&= (u_t\mathcal{H}(u))_t - \frac{1}{2}\mathcal{H}(u_t^2) - \operatorname{div}(\mathcal{H}(u)\nabla_g u) + \langle \nabla_g u, \nabla_g(\mathcal{H}(u))\rangle_g \\
&+ a(x)\mathcal{H}(u)[\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)] \\
&= (u_t\mathcal{H}(u))_t - \frac{1}{2}\operatorname{div}(u_t^2\mathcal{H}) + \frac{1}{2}u_t^2 \ \operatorname{div}\mathcal{H} - \operatorname{div}(\mathcal{H}(u)\nabla_g u) + D_g\mathcal{H}(\nabla_g f, \nabla_g f) \\
&+ \frac{1}{2}\operatorname{div}(|\nabla_g f|_g^2\mathcal{H}) - \frac{1}{2}|\nabla_g f|_g^2 \ \operatorname{div}\mathcal{H} + a(x)\mathcal{H}(u)[\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)].
\end{aligned} \tag{III.2.18}$$

The equality III.2.16 follows from the equality III.2.18 integrating by $\Omega \times (0,T)$. Multiplying III.2.15 by Pu and integrating by $\Omega \times (0,T)$, we obtain the equality III.2.17.
■

The following is the observability inequality for our stabilization problem.

Lemma III.2.4 *Suppose that the conditions III.0.10 and III.0.11 hold true. Let $u(x,t)$ be the solution of the system III.0.2. Then there exists a time $\overline{T} > 0$ such that, for $T > \overline{T}$, there exists a positive constant C_T for which*

$$E(0) \le C_T \int_0^T \int_\Omega a(x)(u_t^2(x,t) + u_t^2(x,t-\tau))dxdt, \tag{III.2.19}$$

where C_T is not dependent on u.

Proof: Set

$$\mathcal{H} = H, \quad P = \frac{1}{2}(\operatorname{div}\, H - \rho_0).$$

Substituting the identity III.2.17 into the identity III.2.16, we have

$$\begin{aligned}
\Pi_\Gamma &= (u_t, H(u) + Pu)|_0^T + \frac{\rho_0}{2}\int_0^T E_0(t)dt + \int_0^T dt \int_\Omega \left(D_g H(\nabla_g u, \nabla_g u) - \rho_0|\nabla_g u|_g^2 + \frac{1}{2}\nabla_g P(u^2)\right) dx \\
&+ \int_0^T dt \int_\Omega a(x)H(u)[\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)]dx \\
&+ \int_0^T dt \int_\Omega a(x)Pu[\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)]dx,
\end{aligned} \tag{III.2.20}$$

where

$$\Pi_\Gamma = \int_0^T dt \int_\Gamma \frac{\partial u}{\partial \nu_A}(H(u) + uP)d\Gamma + \frac{1}{2}\int_0^T dt \int_\Gamma (u_t^2 - |\nabla_g u|_g^2)H \cdot \nu d\Gamma. \tag{III.2.21}$$

We decompose Π_Γ as

$$\Pi_\Gamma = \Pi_{\Gamma_1} + \Pi_{\Gamma_2}.$$

Since $u|_{\Gamma_2} = 0$, we obtain $\nabla_\Gamma u|_{\Gamma_2} = 0$, that is,

$$\nabla_g u = \frac{\partial u}{\partial \nu_A} \frac{\nu_A}{|\nu_A|_g^2} \quad \text{for } x \in \Gamma_2. \tag{III.2.22}$$

Similarly, we have

$$H(u) = \langle H, \nabla_g u\rangle_g = \frac{\partial u}{\partial \nu_A} \frac{H \cdot \nu}{|\nu_A|_g^2} \quad \text{for } x \in \Gamma_2. \tag{III.2.23}$$

Using the formulas III.2.22 and III.2.23 in the formula III.2.21 on the portion Γ_2, we obtain

$$\Pi_{\Gamma_2} = \frac{1}{2}\int_0^T dt \int_{\Gamma_2} \left(\frac{\partial u}{\partial \nu_A}\right)^2 \frac{H \cdot \nu}{|\nu_A|_g^2} d\Gamma \le 0. \tag{III.2.24}$$

Since $\frac{\partial u(x,t)}{\partial \nu_A}|_{\Gamma_1} = 0$, with the formula III.2.21 on the portion Γ_1, we have

$$\Pi_{\Gamma_1} = \frac{1}{2}\int_0^T dt \int_{\Gamma_1} (u_t^2 - |\nabla_g u|_g^2) H \cdot \nu d\Gamma. \tag{III.2.25}$$

Let H_1 be a vector field on $\overline{\Omega}$ such that

$$\begin{cases} H_1 = H & x \in \Gamma_1 \\ H_1 = 0 & x \in \Gamma_2 \cup \Omega \setminus \overline{\omega}. \end{cases} \tag{III.2.26}$$

Set $\mathcal{H} = H_1$, from the identity III.2.16, we have

$$\Pi'_{\Gamma_1} = G \tag{III.2.27}$$

where

$$\Pi'_{\Gamma_1} = \frac{1}{2}\int_0^T dt \int_{\Gamma_1} (u_t^2 - |\nabla_g u|_g^2) H_1 \cdot \nu d\Gamma = \Pi_{\Gamma_1}. \tag{III.2.28}$$

$$\begin{aligned} G = (u_t, H_1(u))|_0^T + \int_0^T dt \int_\omega D_g H_1(\nabla_g u, \nabla_g u) dx + \frac{1}{2}\int_0^T dt \int_\omega (u_t^2 - |\nabla_g u|_g^2) \operatorname{div} H_1 dx \\ + \int_0^T dt \int_\omega a(x) H_1(u)[\mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau)] dx. \end{aligned} \tag{III.2.29}$$

From III.2.27-III.2.29, we obtain

$$\mathrm{II}_{\Gamma_1} = G \leq C(E(0) + E(T)) + C\int_0^T dt \int_\omega a(x)[u_t^2(x,t) + u_t^2(x,t-\tau) + |\nabla_g u|_g^2]dx. \tag{III.2.30}$$

Substituting III.2.24 and III.2.30 into III.2.20 by III.0.10, we obtain

$$\begin{aligned}\int_0^T E_0(t)dt \leq C(E(0) + E(T)) + \int_0^T dt \int_\Omega (C_\epsilon u^2 + \epsilon|\nabla_g u|_g^2)dx \\ + C\int_0^T dt \int_\Omega a(x)[u_t^2(x,t) + u_t^2(x,t-\tau) + |\nabla_g u|_g^2]dx.\end{aligned} \tag{III.2.31}$$

Set $P = a(x)$ and substituting the identity III.2.17 into the identity III.2.31, we have

$$\int_0^T E_0(t)dt \leq C(E(0)+E(T))+C\int_0^T dt \int_\Omega u^2 dx+C\int_0^T dt \int_\Omega a(x)[u_t^2(x,t)+u_t^2(x,t-\tau)]dx. \tag{III.2.32}$$

From III.2.7, we have

$$\begin{aligned}\int_0^T E_0(t)dt &= \int_0^T E(t)dt - \xi\int_0^T\int_0^1\int_\Omega a(x)u_t^2(x,t-\rho\tau)dxd\rho dt \\ &\geq \int_0^T E(t)dt - \xi\int_0^{T+\tau}\int_\Omega a(x)u_t^2(x,t-\tau)dxdt.\end{aligned} \tag{III.2.33}$$

From III.2.8 and III.2.9, we obtain

$$\int_0^T E(t)dt \geq TE(T). \tag{III.2.34}$$

$$\begin{aligned}E(0) &\leq E(T+\tau) + C_2\int_0^{T+\tau}\int_\Omega a(x)(u_t^2(x,t) + u_t^2(x,t-\tau))dxdt \\ &\leq E(T) + C_2\int_0^{T+\tau}\int_\Omega a(x)(u_t^2(x,t) + u_t^2(x,t-\tau))dxdt.\end{aligned} \tag{III.2.35}$$

Substituting III.2.33-III.2.35 into III.2.32, for sufficiently large T we obtain

$$E(0) \leq C\int_0^{T+\tau}\int_\Omega a(x)(u_t^2(x,t) + u_t^2(x,t-\tau))dxdt + C\int_0^{T+\tau}\int_\Omega u^2 dxdt. \tag{III.2.36}$$

The estimate III.2.19 follows from the inequality III.2.36 by a compactness-uniqueness argument as in [3]. ■

Proof: of Theorem III.0.4 Let $\overline{T} > 0$ be given by Lemma III.2.4. Then it follows from

III.2.8 and III.2.19 that, for $T > \overline{T}$,

$$E(0) \leq C_T \int_0^T \int_\Omega a(x)(u_t^2(x,t) + u_t^2(x,t-\tau))dxdt \leq C_T C_1^{-1}(E(0) - E(T)). \quad \text{(III.2.37)}$$

Then

$$E(T) \leq \frac{C_T C_1^{-1} - 1}{C_T C_1^{-1}} E(0). \quad \text{(III.2.38)}$$

The estimate III.0.12 follows from the inequality III.2.38. ■

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