People's Democratic Republic of Algeria
Ministry of Higher Learning and Scientific Research

El-oued University
Faculty of exact sciences
Department of Mathematics

Course Handout

Infinite series and generalized integrals

(Detailed lessons supported by various exercises)
Level: Second year university
Specialty: Mathematics

Prepared by

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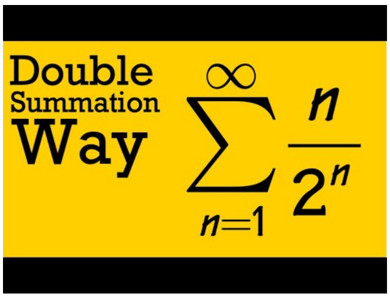

University year 2023 2024

Abstract : This manuscript includes a collection of lessons that fully cover the prescribed program for Analysis 03 Module and it is aimed primarily at second year undergraduate mathematics students. It is divided into 4 parts linked in their entirety to numerical series, sequences and series of functions, such as integer series and Fourier series, as well as to the generalized integral and to the various convergence tests. The lessons are supported by various solved and unsolved examples and exercises.

Keywords : Numerical series, partial sums, series of functions, sequence of functions, Fourier series, Power series, generalized integral, simple convergence, normal and uniform convergence.

الملخص: تتضمن هذه المطبوعة مجموعة من الدروس تغطي بالكامل المنهج المقرر لمقياس التحليل 03 وتستهدف في المقام الأول طلاب السنة الثانية ليسانس رياضيات. وهي مقسمة إلى 4 أجزاء تتعلق في مجملها بالسلاسل العددية والمتتاليات وسلاسل الدوال، مثل سلاسل القوى سلاسل فورييه، بالإضافة إلى التكامل المعمم ومعايير التقارب المختلفة. الدروس مدعمة بالعديد من الأمثلة والتمارين المحلولة وغير المحلولة.

الكلمات المفتاحية: سلسلة عددية ،مجاميع جزئية ، متتالية توابع، سلسلة توابع، سلسلة فوريية، سلسلة صحيحة، تكامل معمم، التقارب البسيط، التقارب النظيمي والمنتظم.

Contents

Preface

This course represents the content of the module of analysis 3. It is intended mainly to students of the 2nd year license mathematics (semester 3) or any other person wishing to know the techniques for calculating infinite sums in the continuous case and in the discrete case. Before being able to approach this course, it is essential to know what a numerical sequence and how to define it by a general term or a recurrent relation and to know everything concerning its convergence either by calculation or by convergence criteria and how to express the sum of n consecutive terms of a sequence and the calculation of its limit. The student must also master the concepts studied through the module of analysis 2, in particular the Taylor expansions and the equivalence of 2 functions in a neighborhood of a given point and the R-integrable functions. This course covers notions of analysis essentially concerning the calculation of infinite sums and generalized integrals. It is divided into 04 parts: the first part presents the different convergence tests of a numerical series while the second deals with sequences and series of functions and the different types of convergence (pointwise-uniform - normal-absolute), the 3rd part is devoted to the study of two interesting examples of series of functions (power series and the Fourier series), the last part concerns the generalized integrals and the functions defined by an improper integral depending on a parameter. This is a mathematics course, with rigorous and complete demonstrations of almost all the theorems presented. I have also inserted many of the examples and solved exercises in order to help comprehension and to make the text more explicit, as well as a series exercises are provided to students at the end of each chapter. Here are some methodological remarks. This course should be read as follows: for each concept covered, you must first to get a general idea of the phenomenon, and of the type of properties that are true, without going too much into the proofs but trying to understand the statements. Then you have to create a few examples or counter-examples yourself, to assimilate the concepts. It is at this stage that one must examine and really understand the technical details and the good hypotheses, which make a theorem really correct, and not only apparently correct but wrong. Finally, you have to try, without spending too much time, to prove yourself the different statements, then check in the text in case of failure or to compare. One way to test whether you have assimilated things well is to try to explain a chapter yourself without using the text, as follows: - give a general idea of the concepts covered and their usefulness, - state the definitions (exactly, without error), - state the main theoems with precise hypotheses and conclusions, - give good examples to illustrate these theorems or the way in which they are used, - give good counter-examples to abusive generalizations of theorems. Finally, you have to do many exercises, which make all this penetrate, among other things by going to the tutorial sessions.

Mathematicians played a major role in the topic of integrals and series This is a list of mathematicians rendered great services to humanity and innovated at a time when means were scarce

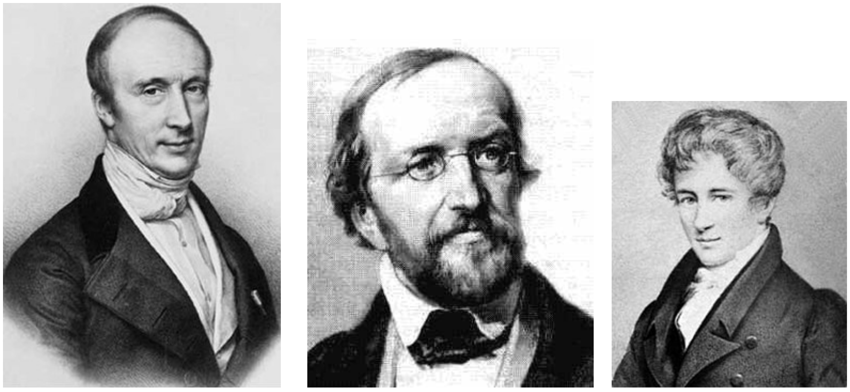

Figure 0.1: *From left to right: Cauchy(1789-1857) - Dirichlet(18051859) and Abel(1802 - 1829)*

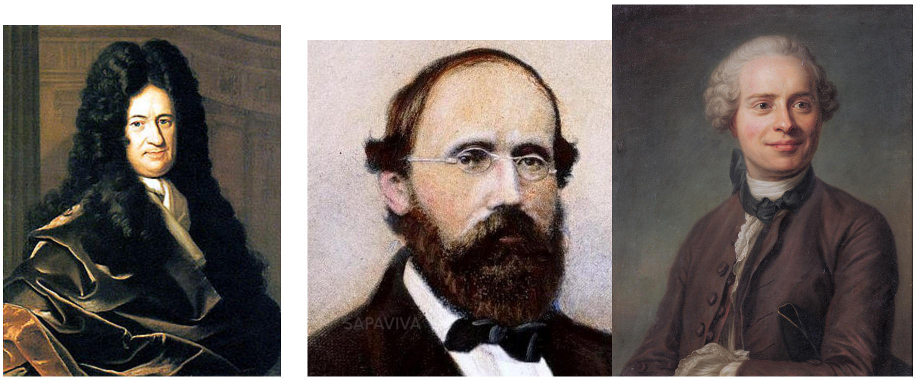

Figure 0.2: *From left to right: Leibnitz (1646-1716)- Riemann(1826-1866) and d'Alembert(1717 - 1783)*

Part A

Infinite numerical Series

Chapter 1

Infinite numerical series and sequence of partial sums

1.1 Introduction

Let $(u_n)_{n\geq 0}$ be a sequence of real (or complex) numbers. We know that the sum of a finite number of its successive terms is a real (or complex) number, but this does not remain true in the case of infinity of terms. To study the value of the sum $\sum u_n$, we form the partial sums

$$S_n = u_0 + u_1 + u_2 + ... + u_n = \sum_{p=0}^{p=n} u_p.$$

Then, we calculate the limit of the sequence (S_n). Take $\lim_{n\to\infty} S_n = S$, if this limit is real number, we call it sum of the series $\sum_{n\geq 0} u_n$ and we write $S = \sum_{n=0}^{+\infty} u_p$.

In general, the theory of series aims to give if possible, a meaning to the sum of an infinity of numbers (real or complex). For example, suppose we have an apple and a perfect knife to cut without ever losing any thing. If we start by eating half of the apple then, half of what is left and so on indefinitely, no one doubts that we will end up eating the whole apple. And if we note $u_0 = \frac{1}{2}$ the fraction of apple eaten the first time, $u_1 = \frac{1}{2}u_0 = \frac{1}{4}$ the fraction of apple eaten the second time and in general u_n the fraction of apple eaten the nth time . To say that we have eaten everything means that if we add up $u_0, u_1, u_2, .., u_n$, ... we will find 1, that we will note

$$u_0 + u_1 + u_2 + .. + u_n \ + ... = \sum_{p=0}^{+\infty} u_p = 1.$$

See fig (**??**)

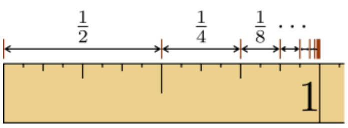

Figure 1.1: Sum

Zeno's paradox: An arrow is fired 20 meters from a target. he takes a certain amount of time to cover half the distance, namely ten meters. Then he still needs time to cover half the remaining distance, and again some time for half the distance still remaining. We thus add an infinity of non-zero durations, and Zeno concludes that the arrow never reaches his target, Zeno did not conceive that an infinity of finite distances could be traveled in a finite time. And yet, in the example of the apple that the sum of an infinity of terms can be a finite value.

1.2 Definitions and properties

Definition 1.2.1 *Let $(u_n)_{n\geq 0}$ be a sequence of (real or complex). Set*

$$S_n = u_0 + u_1 + u_2 + ... + u_n = \sum_{p=0}^{p=n} u_p.$$

The numerical sequence $(S_n(u))_{n\in\mathbb{N}}$ is called numerical series of general term u_n and denoted by one of the following symbols $\sum_{n\geq 0} u_n$, $\sum_{p=0}^{+\infty} u_p$,$\sum_{n\in\mathbb{N}} u_n$, $\sum u_n$.

$(S_n(u))_{n\in\mathbb{N}}$ is also called the sequence of ***partial sums****.*

Definition 1.2.2 *The numerical series $\sum_{n\geq 0} u_n$ is said to be* ***convergent*** *if its sequence of partial sums $(S_n(u))_{n\in\mathbb{N}}$converges to a limit S called* ***sum*** *of the series and we write briefly*

$$S = \lim_{n\to\infty} S_n(u) = \sum_{p=0}^{\infty} u_p$$

In this case, it will be useful to note $(R_n(u))$ the sequence of the ***remainders*** *associated to the convergent series $\sum_{n\geq 0} u_n$ defined by*

$$R_n(u) = S - S_n(u) = \sum_{p=n+1}^{\infty} u_p$$

$R_n(u)$ is called the remainder of the convergent series of order n.

In general, for all convergent series $\lim_{n\to\infty} R_n = S - \lim_{n\to\infty} S_n(u) = S - S = 0$.

The series $\sum_{n\geq 0} u_n$ is said to be ***divergent*** *if the sequence of its partial sums $(S_n(u))_{n\in\mathbb{N}}$ diverges.*

Remark 1.2.3 *1) To study the nature of a numerical series is to know whether it is convergent or divergent.*

2)A series and the sequence of its partial sums have the same sens of variations.

3) For our part, we will distinguish between any series $\sum_{n\geq 0} u_n$ and its sum, we will reserve the notation $\sum_{p=0}^{+\infty} u_p$ to a series convergent or to its sum.

4)Beware of the summation symbol $\sum_{p=0}^{+\infty} u_p$ it is not a sum in the usual sense, because it is not commutative in general.

5)The partial sums of a series are always definite, but the remainders of order n are only definite when the series converges.

Proposition 1.2.4 *If two numerical series $\sum_{n\geq 0} u_n$, $\sum_{n\geq 0} v_n$ do not differ except by just a finite number of terms (that is , $\exists n_0 \in \mathbb{N}, \forall n \geq n_0$; $u_n = v_n$) then, they are both convergent or both divergent.*

Proof. For fixed $n \geq n_0$, one can express the partial sums $S_n(u), S_n(v)$ of the sequences $(u_n), (v_n)$ respectively as follow

$$S_n(u) = \sum_{p=0}^{p=n_0-1} u_p + \sum_{p=n_0}^{p=n} u_p; S_n(v) = \sum_{p=0}^{p=n_0-1} v_p + \sum_{p=n_0}^{p=n} v_p.$$

Since $\sum_{p=n_0}^{p=n} u_p = \sum_{p=n_0}^{p=n} v_p$, for each $n \geq n_0$, then $S_n(u) - S_n(v) = a$ (constant). Consequently,

$\sum_{n\geq 0} u_n$ is convergent $\iff$ $(S_n(u))_{n\in\mathbb{N}}$ is convergent $\iff$ $(S_n(v))_{n\in\mathbb{N}}$ is convergent $\iff$ $\sum_{n\geq 0} v_n$ is convergent. ■

Remark 1.2.5 *1)The previous proposition allows us to say that two series may have the same nature, but they have different sums (in the case of convergence).*

2)The nature of a numerical series does not change if a finite number of terms are added to it or removed from it.

1.3 Nature of some usual numerical series

1.3.1 Geometric series

Let be the geometric sequence (q^n) where q is a real or complex number. The geometric series $\sum_{n\geq 0} q^n$ by definition, is the sequence of partial sums : $S_0 = q^0 = 1, S_1 = q^0 + q^1, S_2 = q^0 + q^1 + q^2, ...$

In the case $q = 1$, $S_n = 1^0+1^1+1^2+...+1^n = n+1$,which give $\lim S_n = +\infty$ therefore, $\sum_{n\geq 0} 1^n$ is divergent.

For $q \neq 1$,using the formula for the sum of consecutive terms of a geometric progression, S_n can be expressed as a function of n as follows:

$$S_n = 1 + q^1 + q^2 + ... + q^n = \frac{1 - q^{n+1}}{1 - q},$$

here, we distinguish two cases:

1) $|q| \geq 1$; $\lim S_n = \begin{cases} +\infty \text{ if } q > 1 \\ \text{does not exist if } q \leq -1 \end{cases}$, which prove that $\sum_{n\geq 0} q^n$ is divergent again.

2)$|q| < 1$; $\lim S_n = \frac{1}{1-q}$ because $\lim q^n = 0$.

Conclusion 1.3.1 *The geometric series $\sum_{n\geq 0} q^n$ converges if and only if $|q| < 1$. In this case its sum is $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$.*

Example 1.3.2 *1) $\sum_{n\geq 2} e^{-n}$ is geometric series with $q = e^{-1} \in \left]-1, 1\right[$, then it is convergent whose sum is equal*

$$\begin{aligned} \sum_{n=2}^{\infty} e^{-n} &= \sum_{n=0}^{\infty} e^{-n} - 1 - e^{-1} = \frac{1}{1 - e^{-1}} - 1 - e^{-1} \\ &= \frac{e^{-2}}{1 - e^{-1}} = \frac{1}{e^2 - e} \end{aligned}$$

2) $\sum_{n\geq 0}\left(-\frac{2}{3}\right)^n$ *is also a convergent geometric series because* $q=-\frac{2}{3}\in\,]-1,1[$, *then its sum is* $\sum_{p=0}^{+\infty}\left(-\frac{2}{3}\right)^n=\frac{1}{1+\frac{2}{3}}=\frac{3}{5}$.

1.3.2 Telescoping series

If $(a_n)_{n\in\mathbb{N}}$ is a given numerical sequence, $\sum_{n\geq 0}(a_{n+1}-a_n)$ is called a **telescoping series.**

This series represents the sequence of partial sums $(S_n)_{n\geq 0}$ where

$S_n=(a_1-a_0)+(a_2-a_1)+(a_3-a_2)+.....+(a_n-a_{n-1})+(a_{n+1}-a_n)$
$=a_{n+1}-a_0$.

Then, $\lim S_n=\lim(a_{n+1}-a_0)$,

Conclusion 1.3.3 *a sequence* $(a_n)_{n\in\mathbb{N}}$ *and the telescoping series* $\sum_{n\geq 0}(a_{n+1}-a_n)$ *have the same behaviour.*

If the sequence $(a_n)_{n\in\mathbb{N}}$ *converges to* l , *the series* $\sum_{n\geq 0}(a_{n+1}-a_n)$ *converges to* $l-a_0$.

Example 1.3.4 *1) In order to study the nature of the series* $\sum_{n\geq 1}\frac{1}{n(n+1)}$, *notice that* $\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$ *then by taking* $a_n=\frac{1}{n}$, *one can calculate the general term of its sequence of partial sums*

$S_n=\sum_{p=1}^{p=n}\left(\frac{1}{p}-\frac{1}{p+1}\right)=\sum_{p=1}^{p=n}(a_p-a_{p+1})=a_1-a_{n+1}=1-\frac{1}{n+1}$.

As $\lim S_n=\lim\left(1-\frac{1}{n+1}\right)=1$, *therefore,* $\sum_{n\geq 1}\frac{1}{n(n+1)}$ *is convergent with the sum* $\sum_{p=1}^{+\infty}\frac{1}{n(n+1)}=1$.

2) $\sum_{n\geq 1}\ln\left(\frac{n+1}{n}\right)$ *is also a telescoping series because we know that* $\ln\left(\frac{n+1}{n}\right)=\ln(n+1)-\ln n=a_{n+1}-a_n$, *where* $a_n=\ln n$. *So the series of partial sums is defined by:*

$S_n=\sum_{p=1}^{p=n}(\ln(p+1)-\ln p)=\ln(n+1)-\ln 1=\ln(n+1)$.

As $\lim S_n=\lim\ln(n+1)=+\infty$ *then,* $\sum_{n\geq 1}\frac{1}{n(n+1)}$ *is divergent.*

1.3.3 Harmonic series

It is the numerical series of general term $u_n=\frac{1}{n}$, i.e the series $\sum_{n\geq 1}\frac{1}{n}$.

The sequence of partial sums that it represents is defined by

$$S_n=\sum_{p=1}^{p=n}\frac{1}{p}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{n}.$$

We will try to study the nature of this series. By applying the mean value theorem for the function ln on the interval $I_k=[k,k+1]\,,k\in\mathbb{N}^*$ we obtain

$$\ln(k+1)-\ln k\leq\frac{1}{k};$$

and by writing this inequality for the values$k=1,2,3,\ldots,n$ then by adding it member to member we find that

$$\ln n<S_{n-1}<S_{n-1}+\frac{1}{n}=S_n.$$

Since $\lim\ln n=+\infty$ then, $\lim S_n=+\infty$ which prove that the harmonic series ($\sum_{n\geq 1}\frac{1}{n}$) is divergent.

Remark 1.3.5 *As can be clearly seen, the test of the sequence of partial sums is far from being easy to apply in most cases because it requires the exact explicit expression of this sequence.*

Indeed, it suffices to consider for example the harmonic series. To deal with this problem, it was necessary to design other criteria that allow us to judge the nature of a numerical series without going through the explicit expression of the sequence of partial sums. These criteria can constitute either a necessary condition of convergence, either a sufficient condition or a condition that is both necessary and sufficient.

1.4 Necessary condition of convergence

Theorem 1.4.1 *A necessary condition for the series $\sum_{n\geq 0} u_n$ to converge is the convergence of u_n to 0 but the reverse is not true.*

i.e. $\sum_{n\geq 0} u_n$ *is convergent* $\underset{\not\Leftarrow}{\Longrightarrow}$ $\lim u_n = 0.$

$\not\Leftarrow$ *means that the condition* $\lim u_n = 0$ *is not sufficient.*

Proof. Assume that $\sum_{n\geq 0} u_n$ is convergent, S is its sum and $(S_n(u))$ is the sequence of partial sums. Then,

$\lim u_n = \lim (S_n(u) - S_{n-1}(u)) = S - S = 0.$

On the other hand, we know from the above that the harmonic series is divergent while $\lim \frac{1}{n} = 0$, hence the invalidity of the sufficiency condition. ∎

Another counterexample: the telescopic series $\sum_{n\geq 0} \left(\sqrt{n+1} - \sqrt{n}\right)$ is divergent whereas $\lim \left(\sqrt{n+1} - \sqrt{n}\right) = \lim \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0.$

Remark 1.4.2 *To prove that the numerical series with the general term u_n is divergent, it is sufficient to prove that* $\lim u_n \neq 0.$

In fact, the contrapositive of the above implication $\lim u_n \neq 0 \Longrightarrow \sum_{n\geq 0} u_n$ is divergent is true.

Therefore, to prove that the series of general term u_n is divergent, it suffices to make sure that $\lim u_n \neq 0.$

Example 1.4.3 *The series* $\sum_{n\geq 1} \frac{3n+1}{2n-1}$, $\sum_{n\geq 1} \left(1+\frac{1}{n}\right)^n$ *are divergent because* $\lim \frac{3n+1}{2n-1} = \frac{3}{2} \neq 0$, $\lim \left(1+\frac{1}{n}\right)^n = e \neq 0$,

since they do not satisfy the necessary condition of convergence.

1.5 Cauchy's condition

We already know that the study of the series $\sum_{n\geq 0} u_n$ is equivalent to that of the sequence of partial sums associated with it. Sometimes it is not possible to calculate the limit of the partial sums sequence (S_n) because the expression of its general term is not known explicitly. As we also know that a numerical sequence is convergent if and only if it is of Cauchy then, in such cases, to know the nature of the numerical series, it is sufficient to prove that (S_n) is a Cauchy sequence. We can now state the Cauchy condition on the numerical series given by the following theorem.

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Theorem 1.5.1 *(Cauchy condition) For the series $\sum_{n\geq 0} u_n$ to converge, it is necessary and sufficient for $(S_n(u))$ to be a Cauchy sequence , i.e.*

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall m \geq n \geq N_\epsilon; \left| \sum_{p=n+1}^{m} u_p \right| < \epsilon$$

or in another form

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall p \in \mathbb{N}, \forall n \geq N_\epsilon; |S_{n+p}(u) - S_n(u)| < \epsilon$$

Proof. ($\Longrightarrow$) By definition if $\sum_{n\geq 0} u_n$ converges then, the sequence $(S_n(u))$ converges too and therefore it is a Cauchy sequence.

($\Longleftarrow$) If $(S_n(u))$ is a Cauchy sequence, it converges which imply that $\sum_{n\geq 0} u_n$ converges also. ■

Example 1.5.2 *We will use this test to prove that the harmonic series is divergent. Taking in the first formula $m = 2n$, we get*

$$\begin{aligned} |S_{2n}(u) - S_n(u)| &= \left| \sum_{p=1}^{p=2n} \frac{1}{p} - \sum_{p=1}^{p=n} \frac{1}{p} \right| \left| = \sum_{p=n+1}^{p=2n} \frac{1}{p} \right| \\ &= \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + ... + \frac{1}{2n} \\ &\geq \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n} + ... + \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}, \end{aligned}$$

thus, there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exist $m \geq n \geq N$; $|S_{2n}(u) - S_n(u)| \geq \epsilon$.

It suffices to take $\epsilon = \frac{1}{2}$and $m = 2n$. Which prove that the series $\sum_{n\geq 1} \frac{1}{n}$ is divergent.

1.6 Operations on the series - linearity

Definition 1.6.1 *The sum of the two series $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ is the series of general term $u_n + v_n$ i.e.*

$$\sum_{n\geq 0} u_n + \sum_{n\geq 0} v_n = \sum_{n\geq 0} (u_n + v_n)$$

Similarly, the product of the series $\sum_{n\geq 0} u_n$ by the real or complex number λ is the series of general term λu_n and one can write

$$\lambda \sum_{n\geq 0} u_n = \sum_{n\geq 0} \lambda u_n.$$

Proposition 1.6.2 *Let $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ be two numerical series and λ a non-zero real or complex.*

1) If $\sum_{n\geq 0} u_n$ converges to S and $\sum_{n\geq 0} v_n$ converges to L then the series $\sum_{n\geq 0} (u_n + v_n)$ converges to $S + L$.

2) If $\sum_{n\geq 0} u_n$ converges to S then the series $\sum_{n\geq 0} \lambda u_n$ converges to λS.

Proof. Denote by $(S_n(u)), (S_n(v)), (S_n(u+v))$ the partial sums sequences of the series of term general $u_n, v_n, u_n + v_n$ respectively.

1) It is clear that
$S_n(u+v) = \sum_{p=0}^{p=n}(u_p + v_p)$
$= \sum_{p=0}^{p=n} u_p + \sum_{p=0}^{p=n} v_p = S_n(u) + S_n(v)$.
passing to the limit in both members , we obtain

$$\begin{aligned}\lim S_n(u+v) &= \lim(S_n(u) + S_n(v)) \\ &= \lim S_n(u) + \lim S_n(v) = S + L\end{aligned}$$

then $\sum_{n\geq 0}(u_n + v_n)$ converges and whose sum is

$$\sum_{p=0}^{+\infty}(u_n + v_n) = \sum_{p=0}^{+\infty} u_n + \sum_{p=0}^{+\infty} v_n$$

2) Similarly, we have for an arbitrary $\lambda \in \mathbb{R}$ (or $\mathbb{C}$)
$S_n(\lambda u) = \sum_{p=0}^{p=n}(\lambda u_p) = \lambda \sum_{p=0}^{p=n} u_p$
hence

$$\lim S_n(\lambda u) = \lim \lambda \sum_{p=0}^{p=n} u_p = \lambda.S,$$

therefore, $\sum_{n\geq 0} \lambda u_n$ converges to λS. ■

Remark 1.6.3 *1) If $\sum_{n\geq 0} u_n$ converges and $\sum_{n\geq 0} v_n$ diverges then the series $\sum_{n\geq 0}(u_n + v_n)$ diverges.*

2) If the two series $\sum_{n\geq 0} u_n$, $\sum_{n\geq 0} v_n$ diverge, nothing can be said about the nature of their sum $\sum_{n\geq 0}(u_n + v_n)$.

Example 1.6.4 *1) The series $\sum_{n\geq 1} \frac{1}{n}, \sum_{n\geq 1} \frac{1}{n+1}$ diverge, while $\sum_{n\geq 1} \frac{1}{n(n+1)} = \sum_{n\geq 1}\left(\frac{1}{n} + (-1)\frac{1}{n+1}\right)$ converges as we have seen.*

2) Let $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ such that for all $n \in \mathbb{N}$; $u_n = 2$, $v_n = -2$..

These two series diverge because they do not satisfy the necessary condition of convergence. However $\sum_{n\geq 0}(u_n + v_n)$ converges.

1.6.1 Convergence of a series with complex terms

Proposition 1.6.5 *Let $\sum_{n\geq 0} u_n$ be a series such that for all $n \in \mathbb{N}$; $u_n \in \mathbb{C}$ (set of complex numbers). The series $\sum_{n\geq 0} u_n$ is convergent if and only if the series $\sum_{n\geq 0} Re(u_n)$ and $\sum_{n\geq 0} Im(u_n)$ are convergent.*

In the case of convergence, the sum of $\sum_{n\geq 0} u_n$ is

$$\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} Re(u_n) + i\sum_{n=0}^{\infty} Im(u_n)$$

Chapter 2

Numerical series with positive terms

Notation 2.0.1 *We will note often* $(\forall n \gg 0)$ *instead of* $(\exists n_0 \in \mathbb{N}; \forall n \geq n_0)$.

Definition 2.0.2 *We call* $\sum_{n\geq 0} u_n$ *a series with positive real terms any series satisfying* $u_n \geq 0$ *for any* $n \in \mathbb{N}$ *. Series verifying* $\forall n \gg 0; u_n \geq 0$ *can be considered as well with positive terms because the nature of those series does not change if we subtract a finite number of terms from it.*

2.1 Some convergence tests

The convergence criteria that we have seen so far rely on the explicit expression of the general term of the sequence of partial sums, which is not always easy to determine. It is for this reason that it was necessary to design other criteria that allow us to judge the nature of a numerical series $\sum_{n\geq 0} u_n$, without going through the determination of the general term of the sequence of the partial sums associated with it. In other words, criteria taking into account only the general term u_n.

2.1.1 Series with bounded above partial sums

Theorem 2.1.1 *Let* $\sum_{n\geq 0} u_n$ *be a series with positive real terms and* $(S_n(u))$ *its partial sums sequence.*

The series $\sum_{n\geq 0} u_n$ *is convergent if and only if* $(S_n(u))$ *is bounded above (i.e.* $\exists M > 0, \forall n \gg 0; 0 < S_n(u) \leq M$*).*

Proof. ($\Longleftarrow$) Notice that the 2nd implication is the contrapositive of the 1st one, $(S_n(u))$ is bounded above and $\forall n \gg 0; S_n(u) - S_{n-1}(u) = u_n \geq 0$ then $(S_n(u))_{n\geq n_0}$ is also an increasing sequence, which prove that $(S_n(u))$ is a convergent sequence.

($\Longrightarrow$) It well known that all convergent sequence is bounded. ■

Remark 2.1.2 *In the case of convergence,* $\sum_{n=n_0}^{+\infty} u_n = \sup_{n\geq n_0} S_n(u)$ *otherwise* $\sup S_n(u) = +\infty$.

Example 2.1.3 $\sum_{n\geq 0}\frac{2^n+1}{3^n+1}$ *is a convergent series , in fact we have for all $p\in\mathbb{N}$*

$$0\leq \frac{2^p+1}{3^p+1}\leq \left(\frac{2}{3}\right)^p+\frac{1}{3^p}.$$

Denoting by S_n the sequence of partial sums of this series, then

$$\begin{aligned} 0 &\leq \sum_{p=0}^{p=n}\frac{2^p+1}{3^p+1}\leq \sum_{p=0}^{p=n}\left(\frac{2}{3}\right)^p+\sum_{p=0}^{p=n}\frac{1}{3^p} \\ &= \frac{1-\left(\frac{2}{3}\right)^{n+1}}{1-\frac{2}{3}}+\frac{1-\left(\frac{1}{3}\right)^{n+1}}{1-\frac{1}{3}} \\ &\leq \frac{1}{1-\frac{2}{3}}+\frac{1}{1-\frac{1}{3}}=\frac{9}{2}, \end{aligned}$$

because, $\frac{2}{3}$, $\frac{1}{3}\in\,]0,1[$. Thus, the sequence (S_n) is bounded by $\frac{9}{2}$, so the corresponding series is convergent and has for sum a value S, such that $0<S<\frac{9}{2}$.

2.1.2 Comparison tests

Notation 2.1.4 *(Lando notation) Let (u_n), (v_n) be two real sequences.*

1) We note $u_n=O(v_n)$ if there exist $M>0$ such that $|u_n|\leq M|v_n|$ for all $n\gg 0$.

2)We note $u_n=\mathbf{o}(v_n)$ if for all $\epsilon>0$ and all $n\gg 0$; $|u_n|\leq\epsilon|v_n|$.

3) If $u_n-v_n=\mathbf{o}(u_n)$, we say that (u_n) , (v_n) are equialent sequences and one writes $u_n\sim v_n$.

Remark 2.1.5 *Suppose that $\forall n\gg 0$; $v_n>0$. then*

1)$u_n=O(v_n)$ if and only if $\frac{u_n}{v_n}$ is bounded.

2) $u_n=\mathbf{o}(v_n)$ if and only if $\lim\frac{u_n}{v_n}=0$.

3) $u_n\sim v_n$ if and only if $\lim\frac{u_n}{v_n}=1$.

Theorem 2.1.6 *Let (u_n) and (v_n) be two real sequences such that $\forall n\gg 0$; $0\leq u_n\leq v_n$*

then,

1) $\sum v_n$ is a convergent series $\Longrightarrow\sum u_n$ is a convergent series,

2) $\sum u_n$ is a divergent series $\Longrightarrow\sum v_n$ is a divergent series.

Proof. 1) Taking $S_n(u)=\sum_{p=n_0}^{p=n}u_p$, $S_n(v)=\sum_{p=n_0}^{p=n}v_p$. If the series $\sum v_n$ converge then the sequence of its partial sums converge also.Therefore, it is increased by a real $m>0$. Hence,

$0\leq S_n(u)\leq S_n(v)\leq m$. In view of the theorem **??** and the proposition **??**, we conclude that $\sum u_n$ is a convergent series.

2) It suffices to notice that the second implication is the contrapositive of the first one. ■

Example 2.1.7 *1) Consider the numerical series $\sum_{n\geq 2}\frac{1}{n^2}$, $\sum_{n\geq 2}\frac{1}{n(n-1)}$. It is clear that $0\leq\frac{1}{n^2}\leq\frac{1}{n(n-1)}$.*

As $\sum_{n\geq 2}\frac{1}{n(n-1)}=\sum_{n\geq 2}\left(\frac{1}{n-1}-\frac{1}{n}\right)$ is a covergent telescopic series then, the series $\sum_{n\geq 2}\frac{1}{n^2}$ conerge also.

2) Similarly, knowing that the harmonic seris $\sum_{n\geq 1} \frac{1}{n}$ *is divergent. Morover, we have* $0 \leq \frac{1}{n} \leq \frac{1}{\sqrt{n}}$ *for all* $n \geq 1$ *then the series* $\sum_{n\geq 1} \frac{1}{\sqrt{n}}$ *diverges* .

Exercise 2.1.8 *A student hopes to investigate the convergence of the series* $\sum_{n\geq 1} \frac{e^{-n}}{n}$ *by compary it with* $\sum_{n\geq 1} \frac{1}{n}$*. Is this student on the right track?*

Solution 2.1.9 *No, because for* $n \geq 1$ *we have* $0 < \frac{e^{-n}}{n} \leq \frac{1}{n}$*, however* $\sum_{n\geq 1} \frac{1}{n}$ *is a divergent series the comparison test therefore does not allow us to drow a conclusion about the convergence or divergence for the series* $\sum_{n\geq 1} \frac{e^{-n}}{n}$.

Corollary 2.1.10 *Let* $\sum u_n$ *and* $\sum v_n$ *be two series with positive real terms such that* $u_n = O(v_n)$ *, then*
1) $\sum v_n$ *converges* $\implies \sum u_n$ *converges.*
2) $\sum u_n$ *diverges* $\implies \sum v_n$ *diverges.*

Proof. As we know that
$u_n = O(v_n) \iff \exists M > 0, \exists n_0 \in \mathbb{N}; n \geq n_0 \implies u_n \leq Mv_n$,
then, by linearity $\sum v_n$ converges $\implies \sum Mv_n$ converges and by the comparison test the series $\sum u_n$ converges. ■

Corollary 2.1.11 *Let* (u_n) *and* (v_n) *be two real sequences with positive terms. If there exist two strictly positive real constants* a *and* b *such that*
$\forall n \gg 0;\ a \leq \frac{u_n}{v_n} \leq b$ *then, the numerical series* $\sum u_n$ *and* $\sum v_n$ *are of both covergent or divergent.*

Proof. It's clear that $u_n = O(v_n)$ and $v_n = O(u_n)$ because $\forall n \gg 0; u_n \leq bv_n$ and $v_n \leq \frac{1}{a}u_n$ then it suffices to apply **??**. ■

2.1.3 Equivalence tests

Corollary 2.1.12 *Let* (u_n) *and* (v_n) *be two real sequences such that* $\forall n \gg 0$; $u_n \geq 0$, $v_n > 0$*. Suppose that* $\lim \frac{u_n}{v_n} = l \in \mathbb{R}_+$*,then*
1) If $l \neq 0$ *then, both series* $\sum u_n$ *and* $\sum v_n$ *are of the same nature. In particular for* $l = 1$ *(i.e. the case* $u_n \sim v_n$*)*
2) If $l = 0$*, we distinguish two cases*
a) $\sum v_n$ *converges* $\implies \sum u_n$ *converges.*
b)$\sum u_n$ *diverges* $\implies \sum v_n$ *diverges.*

Proof. 1) By definition of limit, one can write,
$\lim \frac{u_n}{v_n} = l \neq 0 \iff \forall \epsilon > 0, \exists n_0 \in \mathbb{N}; n \geq n_0 \implies \left|\frac{u_n}{v_n} - l\right| < \epsilon$.
Choose $\epsilon < l$ (fixed) ,$\exists n_0 \in \mathbb{N}; n \geq n_0 \implies l - \epsilon < \frac{u_n}{v_n} \leq l + \epsilon$. Thus $\exists n_0 \in \mathbb{N}; n \geq n_0 \implies (l-\epsilon)v_n \leq u_n \leq (l+\epsilon)v_n$.
i.e. $u_n = O(v_n)$ and $v_n = O(u_n)$. In view of corollary **??**, $\sum u_n$ and $\sum v_n$ are both convegent ordivergent.
2) $\lim \frac{u_n}{v_n} = 0 \iff \forall \epsilon > 0, \exists n_0 \in \mathbb{N}; n \geq n_0 \implies 0 \leq u_n \leq \epsilon v_n$ then, it suffices to apply the comparison criteria to deduce a) and b). ■

Example 2.1.13 *1)* $\sum_{n\geq 0}\frac{2}{3n+5}$ *is a divergent series, in fact by using equivalence test, we have* $\forall\ n\geq 1, \frac{2}{3n+5}\geq 0, \frac{1}{n}>0$ *,*

$\lim\left(\frac{2}{3n+5}\div\frac{1}{n}\right)=\lim\frac{2n}{3n+5}=\frac{2}{3}\neq 0$ *and we also know that the harmonic series* $\sum_{n\geq 1}\frac{1}{n}$ *is divergent.*

2) $\sum_{n\geq 0}\frac{1}{e^{n-1}-n}$ *is a convergent series because by the equivalent test we have*

$$\lim\left(\frac{1}{e^{n-1}-n}\div e^{-n}\right)=\lim\left(\frac{1}{e^{-1}-ne^{-n}}\right)=e\neq 0$$

and as $0<e^{-1}<1$, *the geometric series with general term* e^{-n} *is convergent.*

3) Let $\sum_{n\geq 3}\frac{1}{\sqrt{n}(1+\ln n)}$ *be a numerical series. we have*

$$\lim\left(\frac{1}{n}\div\frac{1}{\sqrt{n}\,(1+\ln n)}\right)=\lim\frac{1+\ln n}{\sqrt{n}}=0.$$

Since the harmonic series is divergent, then $\sum_{n\geq 3}\frac{1}{\sqrt{n}(1+\ln n)}$ *is also divergent.*

2.1.4 D'Alembert's ratio test

Theorem 2.1.14 *Let* $\sum u_n$ *be a series with* ***strictly positive*** *real terms.*

1) If we have $\forall n\gg 0; \frac{u_{n+1}}{u_n}\geq 1$ *then,* $\sum u_n$ *diverges.*

2) If there exists $0<\alpha<1$ *such that* $\forall n\gg 0; \frac{u_{n+1}}{u_n}\leq\alpha$ *then* $\sum u_n$ *converges.*

Proof. 1) We have $u_{n+1}\geq u_n$ for all $n\geq n_0$, which mean that (u_n) is an increasing sequence starting from n_0.Then

$$\forall n\geq n_0; 0<u_{n_0}\leq u_n,$$

by passing to the limit, we obtain $\lim u_n\geq u_{n_0}>0$ i.e.$\lim u_n\neq 0$, which prove that necessary condition of convergence does not hold.

2) Suppose there exists $0<\alpha<1$ and $n_0\in\mathbb{N}$, such that $\forall p\geq n_0; \frac{u_{p+1}}{u_p}\leq\alpha$.

Let's write this inequality for the values of $p=n_0, n_0+1, ..., n-1$ and then multiply all the inequalities member by member, we find

$\frac{u_{n_0+1}}{u_{n_0}}.\frac{u_{n_0+2}}{u_{n_0+1}}.\frac{u_{n_0+3}}{u_{n_0+2}}...\frac{u_n}{u_{n-1}}\leq\alpha^{n-n_0}$. By simplification, we obtain $0<u_n\leq\frac{u_{n_0}}{\alpha^{n_0}}\alpha^n$. As $\sum\alpha^n$ is a convergent geometric series because $0<\alpha<1$, it suffices then to apply the comparison test. ■

Usual D'Alembert ratio test

Corollary 2.1.15 *Let* $\sum u_n$ *be a series with* ***strictly positive*** *real terms. Suppose that* $\lim\frac{u_{n+1}}{u_n}=l$ *then,*

1) If $l\in[0,1[$, *the series* $\sum u_n$ *converges .*

2) If $l\in\,]1,+\infty[$ *,the series* $\sum u_n$ *diverges.*

3) In the case $l=1$*,we cannot conclude about the nature of the series.*

Proof. By definition of the limite one can write

$$\forall\epsilon>0, \exists n_0\in\mathbb{N}, \forall n\in\mathbb{N}; n\geq n_0 \implies l-\epsilon\leq\frac{u_{n+1}}{u_n}\leq l+\epsilon.$$

1) In the case $l \in [0,1[$, we can choose $\epsilon = \frac{1-l}{2}$ and then $\frac{u_{n+1}}{u_n} \leq l + \epsilon = \frac{l+1}{2} < 1$.Thus, by D'Alembert ratio test, the series $\sum u_n$ is convergent.

2) For $l > 1$, taking $\epsilon = \frac{l-1}{2}$, we get $\frac{u_{n+1}}{u_n} \geq l - \epsilon = \frac{l+1}{2} > 1$.Again, by D'Alembert test the series $\sum u_n$ is divergent. ■

Example 2.1.16 *1) Let's study the nature of the series* $\sum_{n\geq 0} \frac{(2n)!}{(n!)^2}$. *Putting* $u_n = \frac{(2n)!}{(n!)^2}$.

Notice that $\forall n \in \mathbb{N}; u_n > 0$ *and*

$$\begin{aligned} \frac{u_{n+1}}{u_n} &= \frac{(2n+2)!}{((n+1)!)^2} \frac{(n!)^2}{(2n)!} = \frac{(2n+2)(2n+1)(2n)!}{(n+1)^2 .(n!)^2} \frac{(n!)^2}{(2n)!} \\ &= \frac{(2n+2)(2n+1)}{(n+1)^2 .} \end{aligned}$$

$\lim \frac{u_{n+1}}{u_n} = \lim \frac{(2n+2)(2n+1)}{(n+1)^2 .} = \lim \frac{4n^2}{n^2} = 4$. *As* $4 > 1$ *then, the series* $\sum_{n\geq 0} \frac{(2n)!}{(n!)^2}$ *diverges.*

2) Similarly for the series $\sum_{n\geq 1} \frac{n!}{n^n}$. *Set* $u_n = \frac{n!}{n^n}$. *We have*

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \frac{(n+1)\,n!}{(n+1)(n+1)^n} \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n},$$

then, $\lim \frac{u_{n+1}}{u_n} = \frac{1}{e} = e^{-1} \in [0,1[$, *which show that* $\sum_{n\geq 0} \frac{(2n)!}{(n!)^2}$ *is a convergent series.*

Remark 2.1.17 *The conditions of d'Alembert's test are sufficient but not necessary, i.e. the series* $\sum u_n$ *can be convergent without fulfilling the condition* $l < 1$ *or divergent and the condition* $l > 1$ *is not validated.*

For instance : $\sum_{n\geq 1} \frac{1}{\sqrt{n}}$ diverges, $\sum_{n\geq 1} \frac{1}{n^2}$ converges while in both cases $l = 1$.

2.1.5 Cauchy's root test

Theorem 2.1.18 *Let* $\sum u_n$ *be a series with positive real terms. .*

1)If we have : $\forall n \gg 0;\ \sqrt[n]{u_n} \geq 1$ *then, the series* $\sum u_n$ *diverges.*

2) If there exists $0 < \alpha < 1$ *such that* $\forall n \gg 0;\ \sqrt[n]{u_n} \leq \alpha$ *then,* $\sum u_n$ *is a convergent series.*

Proof. 1) Suppose that there exists $n_0 \in \mathbb{N}$, such that : $\forall n \geq n_0;\ \sqrt[n]{u_n} \geq 1$. We know that $f(t) = t^n$ is non-decreasing function on $\mathbb{R}_+$ so we conclude that $\forall n \geq n_0; u_n \geq 1$,

and then $\lim u_n \geq 1$.Therefore $\sum u_n$ is a divergent series.

2) Suppose that there exist $0 < \alpha < 1$ and $n_0 \in \mathbb{N}$ such that : $\forall n \geq n_0;\ \sqrt[n]{u_n} \leq \alpha$ then

$$\forall n \geq n_0; u_n \leq \alpha^n.$$

As $\sum \alpha^n$ is a convergent geometric series because $0 < \alpha < 1$ and by applying the comparison test, we deduce that $\sum u_n$ converges. ■

Corollary 2.1.19 *Let $\sum u_n$ be a series with positive real terms. Suppose* $\lim \sqrt[n]{u_n} = l$.

1) $l < 1 \implies \sum \alpha^n$ *converges*

2) $l > 1 \implies \sum \alpha^n$ *diverges.*

3)In the case $l = 1$,we cannot conclude about its nature .

Proof. We will apply the previous theorem. By definition of the limit, we have

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_0 \implies l - \epsilon \leq \sqrt[n]{u_n} \leq l + \epsilon.$$

1) For $l < 1$, we choose $\epsilon = \frac{1-l}{2}$ then, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ we get $\sqrt[n]{u_n} \leq l + \epsilon = \frac{1+l}{2} \in [0, 1[$. Thus, $\sum u_n$ is a convergent series.

2) If $l > 1$, taking $\epsilon = \frac{l-1}{2}$ (for example). Then, there exists $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$ we get $\sqrt[n]{u_n} \geq l - \epsilon = \frac{1+l}{2} > 1$. Therefore, $\sum u_n$ is a divergent series. ∎

Example 2.1.20 *To study the nature of the numerical series* $\sum_{n\geq 1} \left(\frac{an}{3n-2}\right)^n$ *(a is a positive real parameter). Taking* $u_n = \left(\frac{an}{3n-2}\right)^n \geq 0$. *It 's clear that* $\sum_{n\geq 1} u_n$ *is with positive real terms. we have*

$\lim \sqrt[n]{u_n} = \lim \frac{an}{3n-2} = \frac{a}{3}$. *Therefore, the series is convergent if* $0 \leq a < 3$ *, divergent if* $a > 3$.

For $a = 3$, by the cauchy test we can not conclude . Let's check the necessary condition

$$\lim \left(\frac{3n}{3n-2}\right)^n = \lim \frac{1}{\left(1 - \frac{2}{3}\frac{1}{n}\right)^n} = e^{\frac{2}{3}} \neq 0,$$

thus, the series diverges.

Comparison between the test of cauchy and that of D'Alembert

Proposition 2.1.21 *Let (u_n) be a sequence with strictly positive terms. If* $\lim \frac{u_{n+1}}{u_n} = l$ *exists then,* $l = \lim \sqrt[n]{u_n}$.

The reverse implication is false.

Proof. Suppose that, $l \neq 0$ and $0 < \epsilon < l$. There exists $N \in \mathbb{N}$ such that for all $p \geq N$ we have

$$l - \epsilon \leq \frac{u_{p+1}}{u_p} \leq l + \epsilon.$$

By multiplying these inequalities for the values $N \leq p \leq n-1$ member-to-member, we get

$$\forall n \geq N; (l-\epsilon)^{n-N} u_N \leq u_n \leq (l+\epsilon)^{n-N} u_N,$$

let's apply the nth root on all the members, we obtain

$$\sqrt[n]{(l-\epsilon)^{-N} u_N}\,(l-\epsilon) \leq \sqrt[n]{u_n} \leq \sqrt[n]{(l+\epsilon)^{-N} u_N}\,(l+\epsilon),$$

by passing to the limit, we deduce

$$(l-\epsilon) \leq \lim \sqrt[n]{u_n} \leq (l+\epsilon),$$

therefore, $\lim \sqrt[n]{u_n} = l$.

By a similar methode, one can prove that $\lim \sqrt[n]{u_n} = 0$ if $l = 0$ and $\lim \sqrt[n]{u_n} = +\infty$ if $l = +\infty$. ■

Counter-example : Set the numerical series $\sum_{n\geq 1} u_n$ defined by $u_{2p-1} = \left(\frac{2}{3}\right)^{p-1}$, $u_{2p} = \frac{2^{p-1}}{3^p}$ where $p = 1, 2, 3, ...$

In other words $u_n = \begin{cases} \left(\frac{2}{3}\right)^{\frac{n-1}{2}} \text{ ; for odd number } n \\ \frac{1}{2}\left(\frac{2}{3}\right)^{\frac{n}{2}} \text{ ; for even number } n. \end{cases}$

Let's calculate $\frac{u_{n+1}}{u_n}$ and $\sqrt[n]{u_n}$

$\frac{u_{n+1}}{u_n} = \begin{cases} \frac{1}{2}\left(\frac{2}{3}\right)^{\frac{n+1}{2}}\left(\frac{3}{2}\right)^{\frac{n-1}{2}} = \frac{1}{3}; \text{for odd number } n \\ \left(\frac{2}{3}\right)^{\frac{n}{2}}.2\left(\frac{3}{2}\right)^{\frac{n}{2}} = 2; \text{for an even number } n \end{cases}$ and

$\sqrt[n]{u_n} = \begin{cases} \left(\frac{2}{3}\right)^{\frac{n-1}{2n}} \text{; for an odd number } n \\ \sqrt{\frac{2}{3}}.\frac{1}{\sqrt[n]{2}}; \text{for an even number } n \end{cases}$

On the one hand, we have $\lim \frac{u_{n+1}}{u_n}$ does not exist, therefore we cannot know the nature of this series.

On the other hand $\lim \sqrt[n]{u_n} = \sqrt{\frac{2}{3}} \in [0, 1[$ then by Cauchy test, $\sum_{n\geq 1} u_n$ is a convergent series.

Remark 2.1.22 *Cauchy's root test is valid only if* $\lim \sqrt[n]{u_n}$ *exists. By contrast,* $l = \overline{\lim}\ \sqrt[n]{u_n}$ *can always be calculated. Then, the corollary* **??** *remains valid if we replace* $l = \lim \sqrt[n]{u_n}$ *by* $l = \overline{\lim}\ \sqrt[n]{u_n}$.

2.1.6 Other tests

2.1.6.1 Kummer's Test

Proposition 2.1.23 *Given a series* $\sum u_n$ *with strictly positive terms and a sequence* (b_n) *of finite positive constants let*

$$\mu = \lim \left(b_n \frac{u_n}{u_{n+1}} - b_{n+1}\right)$$

1) If $\mu > 0$*, the series* $\sum u_n$ *converges.*
2) If $\mu < 0$ *and* $\sum \frac{1}{b_n}$ *diverges then, the series* $\sum u_n$ *diverges.*
3) If $\mu = 0$*, the series may converge or diverge.*

Proof. By definition of the limit, we have for arbitrary $\epsilon > 0$ there exists $N \in \mathbb{N}$, such that for each $n \in \mathbb{N}$

$$n \geq N \implies \mu - \epsilon \leq b_n \frac{u_n}{u_{n+1}} - b_{n+1} \leq \mu + \epsilon.$$

1) For $\mu > 0$, choose $\epsilon = \frac{\mu}{2}$, take $l \geq N$ so

$$b_l \frac{u_l}{u_{l+1}} - b_{l+1} \geq \mu - \epsilon = \frac{\mu}{2} > 0 \iff u_{l+1} \leq \frac{2}{\mu}\left(b_l u_l - b_{l+1} u_{l+1}\right).$$

Consequently ,

$$\begin{aligned} \sum_{l=N}^{l=n} u_{l+1} &\leq \frac{2}{\mu} \sum_{l=N}^{l=n} \left(b_l u_l - b_{l+1} u_{l+1}\right) \\ &= \frac{2}{\mu}\left(b_N u_N - b_{n+1} u_{n+1}\right) \leq \frac{2}{\mu} b_N u_N = \text{constant}. \end{aligned}$$

The right-hand side of this inequality is a constant, independent of n. Therefore, the positive sequence of increasing terms $S_n(u)$ is bounded above, and consequently possesses a limit. Thus the series $\sum u_n$ is convergent.

2) In this case, choose $\epsilon = -\frac{\mu}{2}$ then, for all $n \geq N$, have

$$b_n \frac{u_n}{u_{n+1}} - b_{n+1} \leq \mu + \epsilon = \frac{\mu}{2} < 0 \Longleftrightarrow b_n u_n \leq b_{n+1} u_{n+1},$$

which mean that $(b_n u_n)_{n \geq N}$ is an increasing sequene, i.e for all $n \geq N$; $b_n u_n \geq b_N u_N$, There fore $u_n \geq \frac{b_N u_N}{b_n}$. By hypothesis, the series $\sum \frac{b_N u_N}{b_n}$ diverges then, the comparison test impliy that $\sum u_n$ diverges also. ■

2.1.6.2 Raabe-Duhamel's test (1839)

For the continuation we give rules allowing to explore the case where $\lim \frac{u_{n+1}}{u_n} = 1$ in the D'Alembert's test.

Proposition 2.1.24 *Let $\sum u_n$ be a series with strictly positive terms, such as*

$$\lim \frac{u_{n+1}}{u_n} = 1 \text{ and } \lim n \left(\frac{u_n}{u_{n+1}} - 1 \right) = m,$$

then, the series $\sum u_n$ will be convergent if $m > 1$ and divergent if $m < 1$. But for $m = 1$, this test fails.

Example 2.1.25 *1) Consider a real m and the series $\sum_{n \geq 1} \frac{1}{n^m}$. We have* $.\lim \frac{u_{n+1}}{u_n} = \lim \left(\frac{n}{n+1} \right)^m = 1$, *therefore the D'Alembert's test is inconclusive. However for n around $+\infty$*

$$\frac{u_n}{u_{n+1}} = \left(1 + \frac{1}{n} \right)^m = 1 + \frac{m}{n} + o\left(\frac{1}{n} \right),$$

then, $\lim n \left(\frac{u_n}{u_{n+1}} - 1 \right) = m$.

Raabe-Duhamel's test allows to conclude that the series$\sum_{n \geq 1} \frac{1}{n^m}$ diverges for $m \leq 1$ and converges for $m > 1$ as well known.

2)Consider the series $\sum_{n \geq 2} \frac{1}{(\ln n)^\alpha}$. Let's use Raabe's test. For large n, we have

$$n \left(\frac{u_n}{u_{n+1}} - 1 \right) = n \left(\left(\frac{\ln(n+1)}{\ln n} \right)^\alpha - 1 \right) = n \left(\left(1 + \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n} \right)^\alpha - 1 \right)$$
$$\approx n \left(\left(1 + \frac{1}{n \ln n}\right)^\alpha - 1 \right) \approx n \left(1 + \frac{\alpha}{n \ln n} - 1\right) \approx \frac{\alpha}{\ln n}.$$

we obtain, $\lim n \left(\frac{u_n}{u_{n+1}} - 1 \right) = \lim \frac{\alpha}{\ln n} = 0 < 1$, *then $\sum_{n \geq 2} \frac{1}{(\ln n)^\alpha}$ diverges.*

Proof. It suffices to put $b_n = n$ in Kummer's test because

$$\lim n \left(\frac{u_n}{u_{n+1}} - 1 \right) = m \Longleftrightarrow \lim n \frac{u_n}{u_{n+1}} - (n+1) = m - 1 = \mu.$$

Then,

1) $m > 1. \Longleftrightarrow \mu > 0$.

2) $m < 1 \Longleftrightarrow \mu < 0$, with $\sum \frac{1}{b_n} = \sum_{n \geq 1} \frac{1}{n}$ is the harmonic series so it's divergent. ■

2.1.6.3 Gauss's test (1812)

Proposition 2.1.26 *Let $\sum u_n$ be a series with strictly positive terms such that*

$$\frac{u_n}{u_{n+1}} = 1 + \frac{m}{n} + \frac{\beta_n}{n^2}$$

where, m is real constant, (β_n) is a bounded sequence as n tends to $+\infty$. Then, the series $\sum u_n$ converges for $m > 1$, diverges for $m \leq 1$.

Proof. For $m \neq 1$, by Raabe's test, we have

$$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim n\left(\frac{m}{n} + \frac{\beta_n}{n^2}\right) = \lim\left(m + \frac{\beta_n}{n}\right) = m,$$

in fact the boundedness of β_n when n tends to $+\infty$.

For $m = 1$, Raabe's test is inapplicable. In this case use Kummer's test by choosing $b_n = n \ln n$: for large n,

$$\begin{aligned}\left(b_n \tfrac{u_n}{u_{n+1}} - b_{n+1}\right) &= \left(n \ln n\left(1 + \tfrac{m}{n} + \tfrac{\beta_n}{n^2}\right) - (n+1)\ln(n+1)\right)\\ &= (n+m)\ln n + \tfrac{\beta_n}{n}\ln n - (n+1)\left(\ln n + \ln\left(1 + \tfrac{1}{n}\right)\right)\\ &\approx (n+m)\ln n + \beta_n \tfrac{\ln n}{n} - (n+1)\left(\ln n + \tfrac{1}{n}\right)\\ &= (m-1)\ln n - \tfrac{n+1}{n} + \beta_n \tfrac{\ln n}{n}\\ &\approx (m-1)\ln n - 1. < 0, \quad \text{for } m \leq 1.\end{aligned}$$

As $\sum \frac{1}{b_n} = \sum_{n\geq 2} \frac{1}{n \ln n}$ diverges then, $\sum u_n$ diverges in view of Kummer's test. ■

Example 2.1.27 *Let's study the nature of the series $\sum_{n\geq 1} \frac{1}{n^\alpha}$ (α is a real constant). We can test for convergence using Gauss' test, we have for large n*

$$\frac{u_n}{u_{n+1}} = \frac{1}{n^\alpha} \cdot \frac{(n+1)^\alpha}{1} = \left(\frac{n+1}{n}\right)^\alpha = \left(1 + \frac{1}{n}\right)^\alpha \approx 1 + \frac{\alpha}{n} + o\left(\frac{1}{n^2}\right),$$

.

In view of Gaus's test :$\sum_{n\geq 1} \frac{1}{n^\alpha}$ converges if $\alpha > 1$, diverges if $\alpha \leq 1$.

The function $\zeta(\alpha) = \sum_{n\geq 1} \frac{1}{n^\alpha}$ where $\alpha > 1$ is called Riemann zeta function.

2.1.7 Integral test

The concept of improper integrals is an extension to the concept of definite integrals. The reason for the term **improper** is because those integrals either include integration over infinite limits or the integrand may become infinite within the limits of integration. Here we will give a very short overview of this concept as far as we need it in the integral test for convergence of a numerical series and we will cover the topic of improper integrals in detail in the last chapter of this manuscript

Definition 2.1.28 *Let f be a continuous real value function on the non-compact interval $[a, b[$ (it's possible that $b = +\infty$). If $\lim_{x\to b^-} \int_a^x f(t)\,dt = l \in \mathbb{R}$, we say that the improper integral $\int_a^b f(t)\,dt$ converges and one can write $\int_a^b f(t)\,dt = l$. If $\lim_{x\to b^-} \int_a^x f(t)\,dt$ does not exist or it is infinite, we say that $\int_a^b f(t)\,dt$ diverges .*

Proposition 2.1.29 *Let f be a continuous, monotonically decreasing positive real function on $[a,+\infty[$ $(a \geq 0)$. Put $u_n = f(n)$. Then* $\sum_{n\in[a,+\infty[} u_n$ *converges if and only if* $\int_a^{+\infty} f(t)\,dt$ *converges.*

Proof. Let n_0 be the integer number defined by $n_0 = \begin{cases} a, \text{ if } a \in \mathbb{N}. \\ [a]+1, \text{if } a \notin \mathbb{N} \end{cases}$.

For all $p \geq n_0$, as f is monotonically decreasing on $[p,p+1] \subset [a,+\infty[$ we deduce

$\forall x \in [p,p+1]\,; f(p+1) < f(x) < f(p)$,

That's

$$\forall x \in [p,p+1]\,; u_{p+1} < f(x) < u_p,$$

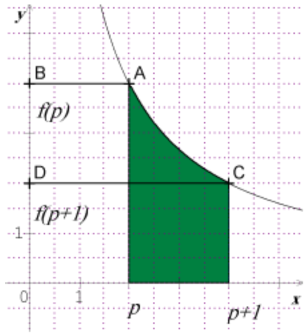

Figure 2.1:

By applying integration to all sides of double inequality, we get

$$u_{p+1} < \int_p^{p+1} f(x)\,dx < u_p,$$

Write this inequality for all values $p = n_0, n_0+1, n_0+2, ..., n_0+n-1$ and by addition member to member, we get

$\sum_{p=n_0}^{p=n_0+n-1} u_{p+1} < \int_{n_0}^{n_0+n} f(x)\,dx < \sum_{p=n_0}^{p=n_0+n-1} u_p$. Take $S_n = \sum_{p=n_0}^{p=n_0+n} u_p$ then $S_n - f(n_0) < \int_{n_0}^{n_0+n} f(x)\,dx < S_{n-1}$.

Assume that $\sum u_n$ is a convergent serirs i.e. $\lim S_n = \lim S_{n-1} = S \in \mathbb{R}$ so $S - f(n_0) \leq \int_{n_0}^{\infty} f(x)\,dx \leq S$ which mean that $\int_{n_0}^{\infty} f(x)\,dx$ is convergent.

Suppose thtat $\int_{n_0}^{\infty} f(x)\,dx$ is convergent then $0 \leq S_n \leq f(n_0) + \int_{n_0}^{\infty} f(x)\,dx$ and thus $\sum u_n$ converges. ■

Example 2.1.30 *Let $\sum_{n\geq 2} u_n$ be a series such that $u_n = \frac{1}{n(\ln n)^\beta}$ (β is a real parameter).*

1) $\beta \leq 0$,

since we know that the harmonic series is divergent and we have for each $n \geq 3$; $u_n = \frac{(\ln n)^{-\beta}}{n} \geq \frac{1}{n}$ then, $\sum_{n\geq 2} \frac{1}{n(\ln n)^\beta}$ is a divergent series.

2) $\beta > 0$,

it's clear that $f(t) = \frac{1}{t(\ln t)^\beta}$ is continious positive monotonically decreasing function on $[2,+\infty[$ because for all $t \in [2,+\infty[$

$f(t) > 0$ and $f'(t) = -\frac{\beta+\ln t}{\left(t(\ln t)^\beta\right)^2}(\ln t)^{\beta-1} < 0$.

Then, let's use integral test,

for $\beta = 1$;we have

$\int_2^{+\infty} \frac{dt}{t(\ln t)} = \lim_{x\to+\infty} [\ln(\ln t)]_2^x = \lim_{x\to+\infty} (\ln(\ln t) - \ln(\ln 2)) = +\infty$.

If $\beta \neq 1$, $\int_2^{+\infty} \frac{dt}{t(\ln t)^\beta} = \int_2^{+\infty} (\ln t)^{-\beta} d(\ln t) = \lim_{x\to+\infty} \left[\frac{(\ln t)^{1-\beta}}{1-\beta}\right]_2^x = \begin{cases} \frac{1}{(\beta-1)(\ln 2)^{\beta-1}}; \textit{ if } \beta > 1 \\ +\infty; \quad \textit{if } 0 < \beta < 1 \end{cases}$.

Conclusion:The series $\sum_{n\geq 2} \frac{1}{t(\ln t)^\beta}$ *converges if* $\beta > 1$, *diverges otherwise i.e. for.*$\beta \leq 1$.

2.1.7.1 Application : Series of Riemann and the Integral Test

Definition 2.1.31 *All numerical series* $\sum_{n\geq 1} \frac{1}{n^\alpha}$,*where* $\alpha \in \mathbb{R}$, *is called series of Riemann (also known as* $\alpha-$*series)*

1) If $\alpha \leq 0$, the necessary condition of convergence does not hold because $\lim \frac{1}{n^\alpha} = +\infty$. So $\sum_{n\geq 1} \frac{1}{n^\alpha}$ diverges.

2) In the $\alpha > 0$, to study its nature, we will use the test integral. Take $u_n = \frac{1}{n^\alpha} = f(n)$ notice that the function f where $f(x) = \frac{1}{x^\alpha}$ is continuous decreasing and positive on the domain $[1, +\infty[$. As we have

$\int_1^x \frac{dt}{t^\alpha} = \begin{cases} \frac{x^{1-\alpha}-1}{1-\alpha}, & \text{for } \alpha \neq 1 \\ \ln x, & \text{for } \alpha = 1, \end{cases}$ then, $\int_1^{+\infty} \frac{dt}{t^\alpha} = \begin{cases} +\infty, & \text{for } 0 < \alpha \leq 1 \\ \frac{1}{\alpha-1}, & \text{for } \alpha > 1. \end{cases}$

Hence the following result

Proposition 2.1.32 *(α-Series Test) The series of Riemann converges if* $\alpha > 1$ *and diverges if* $\alpha \leq 1$.

Another method to study the case is $\alpha > 0$

- For $0 < \alpha \leq 1$, note that

  $n = n^\alpha . n^{1-\alpha} \geq n^\alpha . 1 = n^\alpha$ so $\frac{1}{n^\alpha} \geq \frac{1}{n}$.As the harmonic series diverges then, according to the comparison test , $\sum_{n\geq 1} \frac{1}{n^\alpha}$ is divergent.

- For $\alpha > 1$, consider the function f defined by $f(t) = -\frac{1}{t^{\alpha-1}}$.on the domain $[2, +\infty[$. By applyinng the mean value theorem we get

  $\forall n \geq 2, \exists c \in]n-1, n[;\ f(n) - f(n-1) = f'(c) = \frac{\alpha-1}{c^\alpha} \geq \frac{\alpha-1}{n^\alpha}$.

  As $f(n)_{n\geq 2}$ is a convergent sequence then the telescopic series

  $\sum_{n\geq 2} (f(n) - f(n-1))$ is also convergent. In view of the comparison test, the series with general term $\frac{\alpha-1}{n^\alpha}$ converges. Thus by linearity, the series Riemann converges converges.

2.1.7.2 Riemann's rule

Corollary 2.1.33 *Let* $\sum u_n$ *be a series with positve terms,* $\alpha \in \mathbb{R}$.*Suppose* $\lim n^\alpha u_n = l \in \mathbb{R}_+ \cup \{+\infty\}$.*Then*

1) $\alpha > 1$ *and* $l \in \mathbb{R} \implies \sum u_n$ *converges,*

2) $\alpha \leq 1$ *and* $l \in \mathbb{R}^* \implies \sum u_n$ *diverges.*

Proof. It suffices to use the previous result and the equivalence test noting that $n^\alpha u_n = \frac{u_n}{\frac{1}{n^\alpha}}$. ■

Example 2.1.34 *We study the nature of the series with general term* $u_n = \frac{1}{(\ln n)^{\ln n}}$.

Notice that $n^3 u_n = \frac{n^3}{(\ln n)^{\ln n}} = \frac{e^{3\ln n}}{e^{(\ln n)\ln(\ln n)}} = e^{\ln n(3-\ln(\ln n))}$ *and*
$\ln n\,(3 - \ln(\ln n)) = -\infty$ *then,* $\lim n^3 u_n = 0 \in \mathbb{R}$*, thus* $\sum u_n$ *converges.*

2.1.7.3 Application to the series of Bertrand

Definition 2.1.35 *All numerical series* $\sum_{n\geq 2} \frac{1}{n^{\alpha}(\ln n)^{\beta}}$ *where* $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ *is called Series of Bertrand.*

Proposition 2.1.36 *the Bertrand series converges if* $\alpha > 1$*, diverges if* $\alpha < 1$*.*
In the case $\alpha = 1$*,it converges if and only if* $\beta > 1$*.*

Proof. Putting $u_n = \frac{1}{n^{\alpha}(\ln n)^{\beta}}$. notice that $\forall n \geq 3; 1 < \ln n < n$.

1) I f $\alpha > 1$, one can write $\lim n^{\frac{\alpha+1}{2}} u_n = \lim \frac{1}{n^{\frac{\alpha-1}{2}}(\ln n)^{\beta}} = 0 \in \mathbb{R}$ and as $\frac{\alpha+1}{2} > 1$ then according to Riemann's rule, we conclude that Bertrand's series converges.

2) I f $\alpha < 1$,we have $\lim n^1 \frac{1}{n^{\alpha}(\ln n)^{\beta}} = \lim \frac{n^{1-\alpha}}{(\ln n)^{\beta}} = +\infty \neq 0$, so Bertrand's series diverges.

3) $\alpha = 1$ (refer to example **??**). ■

Chapter 3

Numerical series with arbitrary terms

The previously studied tests of convergence are applied only to study the nature of a series with positive terms (or with a fixed sign) and cannot be applied to a numerical series $\sum u_n$ with arbitrary terms ((the terms may be positive or negative). Therefore, in what will come, we will be interested in studying the nature of the series $\sum |u_n|$.

3.1 Absolute convergence

Definition 3.1.1 *A series $\sum u_n$ is called* ***absolutely convergent*** *if the series of absolute values $\sum |u_n|$ is convergent.*

Example 3.1.2 *Are the following series absolutely convergent?*
$\sum_{n\geq 1} \frac{\cos n}{n\sqrt{n}}$, $\sum_{n\geq 1} \frac{(-1)^n e^{in}}{3n^2+n}$, $\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}}$

It is clear that $\left|\frac{\cos n}{n\sqrt{n}}\right| \leq \frac{1}{n^{\frac{3}{2}}}$.Since $\sum_{n\geq 1} \frac{1}{n^{\frac{3}{2}}}$ is a Rieamann's series with $\alpha = \frac{3}{2} > 1$, it converges and therefore, by using test comparison, $\sum_{n\geq 1} \left|\frac{\cos n}{n\sqrt{n}}\right|$ converges also. Then, $\sum_{n\geq 1} \frac{\cos n}{n\sqrt{n}}$ is absolutely convergent.

By the same method, we have $\left|\frac{(-1)^n e^{in}}{3n^2+n}\right| = \frac{1}{3n^2+n}$ and $\lim \left(\frac{1}{3n^2+n} \div \frac{1}{n^2}\right) = \lim \frac{n^2}{3n^2+n} = \frac{1}{3} \in \mathbb{R}^*$, the Rieamann's series $\sum_{n\geq 1} \frac{1}{n^2}$ with $\alpha = 2 > 1$is convergent then, in view of equivalence test the series $\sum_{n\geq 1} \left|\frac{(-1)^n e^{in}}{3n^2+n}\right|$ is convergent also. Therefore, $\sum_{n\geq 1} \frac{(-1)^n e^{in}}{3n^2+n}$ is absolutely convergent.

Example 3.1.3 *Since $\left|\frac{(-1)^n}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}}$ and $\sum_{n\geq 1} \frac{1}{\sqrt{n}}$ is a Rieamann's series with $\alpha = \frac{1}{2}$ so it's divergent . Then $\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}}$ is not absolutely convergent.*

Theorem 3.1.4 *If the series $\sum u_n$ is absolutely convergent, then it is converges, that is*
$\sum |u_n|$ is convergent $\Longrightarrow \sum u_n$ is convergent.
(The reverse implication is false)

Proof. Assuming (u_n) is a sequence of real numbers. we've got $-|u_n| \leq u_n \leq |u_n|$ then, $0 \leq u_n + |u_n| \leq 2|u_n|$.

Therefore, by linearity if $\sum |u_n|$ converges then $2\sum |u_n|$ and n view of comparison test, $\sum u_n + \sum |u_n|$ converges, which yields that $\sum u_n$ $(= (\sum u_n + \sum |u_n|) - \sum |u_n|)$ converges. ■

Remark 3.1.5 *If (u_n) is a sequence of complex numbers, in order to prove that the series$\sum \Re(u_n)$ and $\sum Im(u_n)$ are convergent, we use the same method by noting that*
$-|u_n| \leq \Re(u_n) \leq |u_n|$ *and* $-|u_n| \leq \Re(u_n) \leq |u_n|$.

Example 3.1.6 *Let $\sum_{n\geq 1} \frac{\sin n}{n^3}$ be a series with non fixed sign terms.*
Since $|\sin n| \leq 1$, we have $0 \leq \left|\frac{\sin n}{n^3}\right| \leq \frac{1}{n^3}$, Therefore the series $\sum_{n\geq 1} \left|\frac{\sin n}{n^3}\right|$ converges converges by comparison with the converging Riemann's series $\sum_{n\geq 1} \frac{1}{n^3}$ $(\alpha = 3 > 1)$.
Thus, the series $\sum_{n\geq 1} \frac{\sin n}{n^3}$ is convergent since it is absolutely convergent.

3.1.1 Conditional convergence

Definition 3.1.7 *We say that a series $\sum u_n$ is **conditionally convergent,** if it's convergent but not absolutely convergent.*

Example 3.1.8 *(**Alternating Harmonic series**) Let $\sum_{n\geq 1} \frac{(-1)^n}{n}$ be a series. We set $u_n = \frac{(-1)^n}{n}$. As well known that $\sum_{n\geq 1} |u_n| = \sum_{n\geq 1} \frac{1}{n}$ is divergent (the harmonic series) then, the series $\sum_{n\geq 1} \frac{(-1)^n}{n}$is not absolutely convergent.*
Let's prove tht's convergent. Let (a_n), (b_n) be two sequences defined for all $n \in \mathbb{N}^*$ as follow
$a_n = S_{2n} = -1 + \frac{1}{2} - \frac{1}{3} + ... - \frac{1}{2n-1} + \frac{1}{2n}$,
$b_n = S_{2n+1} = S_{2n} - \frac{1}{2n+1}$.
Note that for every integer number $n \geq 1$, we have
$a_{n+1} - a_n = S_{2(n+1)} - S_{2n} = -\frac{1}{2n+1} + \frac{1}{2n+2} = \frac{-1}{(2n+1)(2n+2)} < 0$,
$b_{n+1} - b_n = S_{2(n+1)+1} - S_{2n+1} = \frac{1}{2n+2} - \frac{1}{2n+3} = \frac{1}{(2n+2)(2n+3)} > 0$,
From which the sequence (a_n) is decreasing and the sequence (b_n) is increasing, and we also have
$\lim(a_n - b_n) = \lim \frac{1}{2n+1} = 0$. *So the two sequences are adjacent and therefore converge towards the same limit S. Therefore the sequence of partial sums (S_n) where $S_n = \sum_{p=1}^{p=n} \frac{(-1)^p}{p}$ is convergent, then, the series $\sum_{n\geq 1} \frac{(-1)^n}{n}$ converges.*

Conclusion 3.1.9 *The $\sum_{n\geq 1} funcseries \frac{(-1)^n}{n}$ is conditionally convergent.*

Remark 3.1.10 *If we rearrange the terms in a finite sum, the sum remains the same. This is not always the case for the sum infinite series.*

Proposition 3.1.11 *The sum of an absolutely convergent series does not change if we group its terms or rearrange them.*

In this proposition, the condition of absolute convergence is essential.

Example 3.1.12 *The series $\sum_{n\geq 0} \left(-\frac{1}{2}\right)^n$ is absolutely convergent with sum $S = \frac{2}{3}$ and hence any rearrangement of the terms has sum $\frac{2}{3}$.*

Counter example In view of the last example, $\sum_{n\geq 1}\frac{(-1)^n}{n}$ is convergent but not absolutely convergent. Denote by S to the sum of this series. We have

$-S=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+..$
$=\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\left(\frac{1}{7}-\frac{1}{14}\right)-\frac{1}{16}+..$
$=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}-\frac{1}{16}....=-\frac{1}{2}S$

Then, $S=0$, but we know that $S=-\ln 2\neq 0$.

Remark 3.1.13 *All convergence tests for a numerical series with positive terms remain usable in the study of absolute convergence of a series with arbitrary terms.*

To apply a convergence test for an arbitrary series, it suffices to replace u_n by $|u_n|$.

3.1.2 Abel's transformation and Dirichlet's test (Abel's rule)

For two given sequences (a_n) and (b_n) with $n\in\mathbb{N}$ one wants to study the partial sum $S_N=\sum_{n=0}^{n=N}a_nb_n$ of the series $\sum_{n\geq 0}a_nb_n$. If we define $B_n=\sum_{p=0}^{p=n}b_n$ then,

for every $n\geq 1, b_n=B_n-B_{n-1}$ and

$S_N=a_0b_0+\sum_{n=1}^{n=N}a_n\left(B_n-B_{n-1}\right)=a_0b_0+\sum_{n=1}^{n=N}a_nB_n-\sum_{n=1}^{n=N}a_nB_{n-1}$
$=a_0b_0+\sum_{n=1}^{n=N}a_nB_n-\sum_{n=0}^{n=N-1}a_{n+1}B_n$
$=a_0b_0+\left[a_NB_N+\sum_{n=1}^{n=N-1}a_nB_n\right]-\left[a_1B_0+\sum_{n=1}^{n=N-1}a_{n+1}B_n\right]$
$=a_0B_0-a_1B_0+a_NB_N+\sum_{n=1}^{n=N-1}a_nB_n-\sum_{n=1}^{n=N-1}a_{n+1}B_n$
$=a_NB_N+B_0\left(a_0-a_1\right)+\sum_{n=1}^{n=N-1}B_n\left(a_n-a_{n+1}\right)$
$=a_NB_N+\sum_{n=0}^{n=N-1}B_n\left(a_n-a_{n+1}\right)$
$=a_NB_N-\sum_{n=0}^{n=N-1}B_n\left(a_{n+1}-a_n\right).$

Therefore;

$$S_N=\sum_{n=0}^{n=N}a_nb_n=a_NB_N-\sum_{n=0}^{n=N-1}B_n\left(a_{n+1}-a_n\right)$$

Finally this process, called an **Abel's transformation**, can be used to prove several tests of convergence .

In general , we can prove that

Proposition 3.1.14 *For two given sequences (a_n) and (b_n) with $n\in\mathbb{N}$. For all $r,s\in\mathbb{N}$ with $s\geq r+1$we have,*

$$\sum_{n=r+1}^{n=s}a_n\left(b_n-b_{n-1}\right)=a_{s+1}b_s-a_{r+1}b_r-\sum_{n=r+1}^{n=s}b_n\left(a_{n+1}-a_n\right)$$

Proof. $\sum_{n=r+1}^{n=s}a_n\left(b_n-b_{n-1}\right)=\sum_{n=r+1}^{n=s}a_nb_n-\sum_{n=r+1}^{n=s}a_nb_{n-1}$
$=\sum_{n=r+1}^{n=s}a_nb_n-\sum_{n=r}^{n=s-1}a_{n+1}b_n$
$=\sum_{n=r+1}^{n=s}a_nb_n-\left[\sum_{n=r+1}^{n=s}a_{n+1}b_n-a_{s+1}b_s+a_{r+1}b_r\right]$
$=a_{s+1}b_s-a_{r+1}b_r-\sum_{n=r+1}^{n=s}b_n\left(a_{n+1}-a_n\right).$

Which complete the proof. ■

Definition 3.1.15 ***(Sequence of bounded variation)*** *A sequence (a_n) of real (or complex) numbers is said to be of bounded variation if $\sum_{n=0}^{\infty} |a_{n+1} - a_n| < \infty$.*

A typical example of such sequences are bounded monotone sequences of real numbers.

Proposition 3.1.16 *Any sequence of bounded variation converges. Moreover, any bounded monotone sequence of real numbers is of bounded variation.*

Proof. Let (a_n) be a sequence of bounded variation. Given $p < q$, notice that

$$a_q - a_p = \sum_{k=p}^{k=q} (a_{k+1} - a_k).$$

Hence $|a_q - a_p| \leq \sum_{k=p}^{k=q} |a_{k+1} - a_k|$.

By assumption, the sum $\sum_{k=0}^{\infty} |a_{k+1} - a_k|$ is bounded, so by Cauchy's criterion for series, the sum on the right hand side of this inequality can be made arbitrarily small as $p \to \infty$. Thus, (a_n) is Cauchy and hence converges.

Now let (a_n) be a non-increasing and bounded sequence. We shall prove that this sequence is of bounded variation; the proof for a non-decreasing sequence is similar. In this case, we have $a_{n+1} \leq a_n$ for each n, so each $n \in \mathbb{N}$, we have

$$\sum_{k=0}^{k=n} |a_{k+1} - a_k| = \sum_{k=0}^{k=n} (a_k - a_{k+1}) = a_0 - a_{n+1},$$

Since the sequence (a_n) is by assumption bounded, it follows that the partial sums of the series $\sum_{k \geq 0} |a_{n+1} - a_n|$ are bounded, hence (a_n) is of bounded variation. ■

Theorem 3.1.17 *(Dirichlet's test) if (a_n) is a sequence of real numbers and (v_n) a sequence of real (or complex) numbers satisfying the following conditions*

- *(a_n) is monotone with $\lim a_n = 0$,*
- *the sequence (V_n) where $V_n = \sum_{p=0}^{p=n} v_p$ is uniformly bounded i.e. $\exists M > 0, \forall n \in \mathbb{N}; \left|\sum_{p=0}^{n=n} v_p\right| \leq M$.*

Then the series $\sum_{n \geq 0} a_n v_n$ is convergent.

Proof. Let $S_n = \sum_{p=0}^{p=n} a_p v_p$ and $V_n = \sum_{p=0}^{n=n} v_p$.

From Abel's transformation, we have

$$S_n = a_n V_n + \sum_{p=0}^{p=n-1} V_p \left(a_p - a_{p+1}\right)$$

Since V_n is bounded by M and $\lim a_n = 0$, then $\lim a_n V_n = 0$. On the other hand we have, for each $p \in \mathbb{N}$,

$$\left|V_p \left(a_p - a_{p+1}\right)\right| \leq M \left|a_p - a_{p+1}\right|,$$

Since the sequence (a_n) is monotone, it is either decreasing or increasing, one can deduce

1) if (a_n) is decreasing;

$\sum_{p=0}^{p=n} M\,|a_p - a_{p+1}| = M\sum_{p=0}^{p=n-1}(a_p - a_{p+1}) = M(a_0 - a_{n+1})$ because it's a telescoping sum. Then $\lim \sum_{p=0}^{p=n} M\,|a_p - a_{p+1}| = Ma_0$.

2) if (a_n) is increasing, similarly, we get

$\sum_{p=0}^{p=n} M\,|a_p - a_{p+1}| = -M\sum_{p=0}^{p=n-1}(a_p - a_{p+1}) = M(a_{n+1} - a_0)$, which show that $\lim \sum_{p=0}^{p=n} M\,|a_p - a_{p+1}| = -Ma_0$.

So, in both cases, the series $\sum_{n\geq 0} V_n(a_n - a_{n+1})$ converges absolutely. Hence (S_n) converges. ■

Remark 3.1.18 *1) This theorem can be demonstrated otherwise by using Abel's transformation to show that (S_n) satisfies the Cauchy condition.*

2)The first condition of the previous theorem means that (a_n) is increasing with negative terms or else decreasing with positive terms.

3)The second condition does not mean that $\sum_{n\geq 0} v_n$ converges.

Proposition 3.1.19 *if (a_n) is a sequence of real numbers and (v_n) a sequence of complex numbers satisfying the following conditions*

- *(a_n) is of bounded variation and* $\lim a_n = 0$,
- *the sequence (V_n) where $V_n = \sum_{p=0}^{p=n} v_p$ is uniformly bounded .*

Then the series $\sum_{n\geq 0} a_n v_n$ is convergent

3.1.2.1 Applications on the series $\sum_{n\geq 1}\frac{\sin nx}{n^\alpha}, \sum_{n\geq 1}\frac{\cos nx}{n^\alpha}$

where $\alpha > 0$, $x \neq 2k\pi, (k \in \mathbb{Z})$,

- For $\alpha > 1$, we have $\left|\frac{\sin nx}{n^\alpha}\right| \leq \frac{1}{n^\alpha}, \left|\frac{\cos nx}{n^\alpha}\right| \leq \frac{1}{n^\alpha}$ then by comparison with the convergent series of Riemann $\sum_{n\geq 1}\frac{1}{n^\alpha}$, the two series $\sum_{n\geq 1}\frac{\sin nx}{n^\alpha}$ and $\sum_{n\geq 1}\frac{\cos nx}{n^\alpha}$ are absolutely convergent and thus convergent.

- In the case $0 < \alpha \leq 1$, we know that $\cos nx + i\sin nx = e^{inx}$, then $\sum_{p=0}^{p=n} e^{ipx} = \frac{e^{i(n+1)x}-1}{e^{ix}-1} = \frac{\cos(n+1)x - 1 + i\sin(n+1)x}{\cos x - 1 + i\sin x}$.

  Multiplying the numerator and denominator by the conjugate of the denominator, and after simplifying , we find

  $\sum_{p=0}^{p=n}\cos px = Re\left(\sum_{p=0}^{p=n} e^{ipx}\right) = \frac{\sin\left(\frac{n+1}{2}x\right)\cos\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}$

  $\sum_{p=0}^{p=n}\sin px = Im\left(\sum_{p=0}^{p=n} e^{ipx}\right) = \frac{\sin\left(\frac{n+1}{2}x\right)\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}$.

  Thus, $\left|\sum_{p=0}^{p=n}\cos px\right| \leq \frac{1}{\left|\sin\left(\frac{x}{2}\right)\right|}$ and $\left|\sum_{p=0}^{p=n}\sin px\right| \leq \frac{1}{\left|\sin\left(\frac{x}{2}\right)\right|}$.

  On the other hand the sequence $\left(\frac{1}{n^\alpha}\right)_{n\geq 1}$is decreasing with positive terms and $\lim \frac{1}{n^\alpha} = 0$.

  Which prove that all conditions of the theorem hold, then the series $\sum_{n\geq 1}\frac{\sin nx}{n^\alpha}, \sum_{n\geq 1}\frac{\cos nx}{n^\alpha}$ converge for $\alpha > 0$

3.1.3 Alternating series and Leibnitz's test

Definition 3.1.20 *Let $\sum v_n$ be a series with arbitrary terms. This series is said to be* ***alternating (or alternate)*** *if for all $n \in \mathbb{N}$; $v_n.v_{n+1} < 0$.*

Any alternating series can be written in the form $\sum (-1)^n u_n$ or $\sum (-1)^{n-1} u_n$ where (u_n) is a sequence with positive terms.

Corollary 3.1.21 *(**Leibnitz's test** also known as **alternating series test**) Let $\sum_{n\geq 0} (-1)^n u_n$ be an alternating series. If we have*

- *(u_n) is a decreasing sequence with positive terms*
- $\lim u_n = 0$.

Then, the alternating series $\sum_{n\geq 0} (-1)^n u_n$ converges.

In other words, if the absolute values of the terms of an alternating series are non-increasing and converge to zero, the series converges.
Proof. We take in Dirichlet's test $a_n = (-1)^n$, $v_n = u_n$ and apply Abel's rule, noting that $\left|\sum_{p=0}^{p=n} (-1)^p\right| \leq 2$. ∎

Example 3.1.22 *The alternating harmonic series $\sum_{n\geq 1} \frac{(-1)^n}{n}$ converges in view of Leibnitz's test because*
$\forall n \geq 1; \frac{1}{n} > 0$, $\frac{1}{n+1} \leq \frac{1}{n}$ *and* $\lim \frac{1}{n} = 0$.

Exercise 3.1.23 *Consider the series $\sum_{n\geq 2} \frac{(-1)^n}{n+(-1)^n}$.*

1) Show that this series is alternating and explain why the Leibnitz's test does not apply.

2)Calculate $\frac{(-1)^n}{n+(-1)^n} - \frac{(-1)^n}{n}$ and deduce the nature of proposed series.

Solution 3.1.24 *1) As $\frac{(-1)^n}{n+(-1)^n} = (-1)^n .u_n$ with $u_n = \frac{1}{n+(-1)^n} > 0$ for all $n \geq 2$ then, the series is alternating.*

The sequence $(u_n)_{n\geq 2}$ tends to 0, but does not decrease: the Leibnitz's test does not apply, and nothing can be said for the moment as to the nature of the series.

2) We have $\frac{(-1)^n}{n+(-1)^n} - \frac{(-1)^n}{n} = \frac{-1}{n(n+(-1)^n)}$ so $\frac{(-1)^n}{n+(-1)^n} = \frac{(-1)^n}{n} - \frac{1}{n(n+(-1)^n)}$,

the series$\sum_{n\geq 2} \frac{(-1)^n}{n}$ converges (alternating harmonic series) and the series $\sum_{n\geq 2} \frac{1}{n(n+(-1)^n)}$ is with positive terms convergent , in fact
$\frac{1}{n(n+(-1)^n)} \sim \frac{1}{n^2}$ at infinity and $\sum_{n\geq 2} \frac{1}{n^2}$ is convergent series of Riemann with $\alpha = 2 > 1$.

Then, the proposed series as the difference of two convergent series, is convergent.

Remark 3.1.25 *The conditions of the Leibnitz's test for the convergence of an alternating series are sufficient but not necessary. For example the alternating series*

Remark 3.1.26

$$\frac{1}{2} - 1 + \frac{1}{9} - \frac{1}{4} + \frac{1}{25} - \frac{1}{16} + \frac{1}{49} - \frac{1}{36} + ...$$

is absolutely convergent and thus convergent but does not satisfy the conditions of the Leibnitz's test since $u_{n+1} \leq u_n$ does not hold for all n.

3.1.3.1 Alternating series estimation theorem

Theorem 3.1.27 *If* $S = \sum_{n=0}^{\infty} (-1)^{n-1} u_n$ *is the sum of the alternating series* $\sum (-1)^{n-1} u_n$ *which satisfies the conditions of Leibnitz's test. Then,*

$$|R_n| = |S - S_n| \leq u_{n+1},$$

here R_n *stands for the remainder when we subtract the nth partial sum from the sum of the series.*

Proof. Notice that for all $n \in \mathbb{N}$; $S_{2n-1} \leq S \leq S_{2n}$. Then,

$$|R_n| = |S - S_n| \leq |S_{n+1} - S_n| = u_{n+1}.$$

■

Exercise 3.1.28 *Show that the following series converge:*
$\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}}$, $\sum_{n\geq 4} \frac{n(-1)^{n+1}}{n^3+1}$, $\sum_{n\geq 2} \frac{3}{e^n(-1)^{n+2}}$.

3.1.4 Cauchy product of two series

Definition 3.1.29 *Let* $\sum_{n\geq 0} u_n$ *and* $\sum_{n\geq 0} v_n$ *be two series. We call product series or Cauchy product of these two series the series* $\sum_{n\geq 0} w_n$ *where .*

$$w_n = \sum_{p=0}^{p=n} u_p v_{n-p} = u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + ... + u_n v_0,$$

for all $n \in \mathbb{N}$.

Example 3.1.30
$$\begin{aligned} \left(\sum_{n\geq 0} \frac{3^n}{n!}\right)\left(\sum_{n\geq 0} \frac{2^n}{n!}\right) &= \sum_{n\geq 0} \left(\sum_{p=0}^{p=n} \frac{3^p}{p!} \frac{2^{n-p}}{(n-p)!}\right) \\ &= \sum_{n\geq 0} \frac{1}{n!} \left(\sum_{p=0}^{p=n} \frac{n!}{(n-p)!p!} 2^{n-p} 3^p\right) \\ &= \sum_{n\geq 0} \frac{1}{n!} \left(\sum_{p=0}^{p=n} \begin{pmatrix} n \\ p \end{pmatrix} 2^{n-p} 3^p\right) \\ &= \sum_{n\geq 0} \frac{(3+2)^n}{n!} = \sum_{n\geq 0} \frac{5^n}{n!}. \end{aligned}$$

Theorem 3.1.31 *(Cauchy's theorem 1821) If the series* $\sum_{n\geq 0} u_n$ *and* $\sum_{n\geq 0} v_n$ *are absolutely convergent then, their Cauchy product series is absolutely convergent and its sum is* $P = \left(\sum_{p=0}^{\infty} u_p\right)\left(\sum_{p=0}^{\infty} v_p\right)$.

In the example **??**, the two series $\sum_{n\geq 0} \frac{3^n}{n!}$, $\sum_{n\geq 0} \frac{2^n}{n!}$ are absolutely convergent, their sums are respectively e^3 and e^2, so their product is an absolutely convergent series, and its sum is $e^5 = e^3.e^2$.
Proof. Take $U_n = \sum_{p=0}^{p=n} u_p$, $V_n = \sum_{p=0}^{p=n} v_p$, $P_n = \sum_{p=0}^{n} w_p$ with $w_n = \sum_{p=0}^{p=n} u_p v_{n-p}$. ■

1) If the series are with positive terms, we have $P_n \leq U_n V_n \leq P_{2n}$
in fact, $P_n = \sum_{(i,j)\in T_n} u_i v_j$ where T_n is the triangle $\{(i,j)\,; 0 \leq i \leq n, 0 \leq j \leq n, i+j \leq n\}$ while $U_n V_n = \sum_{(i,j)\in C_n} u_i v_j$ where C_n is the square $[0,n] \times [0,n]$ and $P_n = \sum_{(i,j)\in T_{2n}} u_i v_j$, noting that $T_n \subset C_n \subset T_{2n}$.

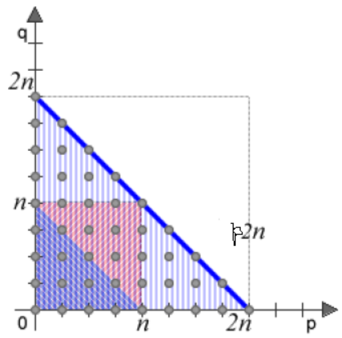

Figure 3.1: nested triangles

Moreover, all the terms of the series being positive or null then, the sequences of partial sums are increasing, so

$$P_n \leq U_n V_n \leq \left(\sum_{n=0}^{\infty} u_n\right)\left(\sum_{n=0}^{\infty} v_n\right),$$

The sequence (P_n) is therefore increasing and bounded above by $\left(\sum_{n=0}^{\infty} u_n\right)\left(\sum_{n=0}^{\infty} v_n\right)$ so converges. We obtain the required equality by passing to limit in all sides of inequality $P_n \leq U_n V_n \leq P_{2n}$.

2) If the series $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ are absolutely convergent then, let $\sum_{n\geq 0} h_n$ be the series Cauchy product of $\sum_{n\geq 0} |u_n|$ and $\sum_{n\geq 0} |v_n|$, according to the above, this series converges with $h_n = \sum_{p=0}^{p=n} |u_p|\,|v_{n-p}|$.

As $|w_n| = \left|\sum_{p=0}^{p=n} u_p v_{n-p}\right| \leq \sum_{p=0}^{p=n} |u_p|\,|v_{n-p}| = h_n$, then by comparison test, we deduce that $\sum_{n\geq 0} w_n$ is absolutely convergent. On the other hand, we have

$|U_n V_n - P_n| = \left|\sum_{(i,j)\in C_n - T_n} u_i v_j\right| \leq \sum_{(i,j)\in C_n - T_n} |u_i v_j|$

$= \left(\sum_{p=0}^{p=n} |u_p|\right)\left(\sum_{p=0}^{p=n} |v_p|\right) - \sum_{p=0}^{p=n} h_p,$

and this last expression tends to 0 as n tends to infinity.

Remark 3.1.32 *This result justifies the designation "product series".*

Proposition 3.1.33 *(Mertens's Theorem 1875) If at least one of the two convergent series $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ converges absolutely then, their Cauchy product converges with sum equal $\left(\sum_{n=0}^{\infty} u_n\right)\left(\sum_{n=0}^{\infty} v_n\right)$.*

Remark 3.1.34 *The Cauchy product of two convergent series is not necessarily convergent.*

Example 3.1.35 *1) By vertu of Leibnitz's test it's easy to show that the alternating series $\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}}$ is convergent. We will calculate the Cauchy product of this series by itself*

we have by definition $\left(\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}}\right)\left(\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}}\right) = \sum_{n\geq 1} w_n$, *where* $w_n = \sum_{p=1}^{p=n} \frac{(-1)^p}{\sqrt{p}} \frac{(-1)^{n++1-p}}{\sqrt{n+1-p}} = \sum_{p=1}^{p=n} \frac{1}{\sqrt{p}} \frac{(-1)^{n+1}}{\sqrt{n+1-p}} = (-1)^{n+1} \sum_{p=1}^{p=n} \frac{1}{\sqrt{p}\sqrt{n+1-p}}$.
Since for all $1 \leq p \leq n$ *we have* $\frac{1}{\sqrt{p}\sqrt{n+1-p}} \geq \frac{1}{\sqrt{n}}\frac{1}{\sqrt{n}} = \frac{1}{n}$ *so* $\sum_{p=1}^{p=n} \frac{1}{\sqrt{p}\sqrt{n+1-p}} \geq n.\frac{1}{n} = 1$ *i.e* $\lim w_n \geq 1$ *which prove that* $\sum_{n\geq 1} w_n$ *diverges.*

2) Similarly, the alternating Harmonic series $\sum_{n\geq 1}\frac{(-1)^n}{n}$ *is also convergent((by Leibnitz's test). The product series of this series by itself is* $\left(\sum_{n\geq 1}\frac{(-1)^n}{n}\right)\left(\sum_{n\geq 1}\frac{(-1)^n}{n}\right)=\sum_{n\geq 1}w_n$*, where*

$$w_n=\sum_{p=1}^{p=n}\frac{(-1)^p}{p}\frac{(-1)^{n++1-p}}{n+1-p}=(-1)^{n+1}\sum_{p=1}^{p=n}\frac{1}{p(n+1-p)}.$$
$$=\frac{(-1)^{n+1}}{n+1}\sum_{p=1}^{p=n}\left(\frac{1}{p}+\frac{1}{(n+1-p)}\right)$$
$$=\frac{(-1)^{n+1}}{n+1}\left(\sum_{p=1}^{p=n}\frac{1}{p}+\sum_{p=1}^{p=n}\frac{1}{(n+1-p)}\right)$$
$$=\frac{(-1)^{n+1}}{n+1}\left(\sum_{p=1}^{p=n}\frac{1}{p}+\sum_{k=n}^{k=1}\frac{1}{k}\right)$$
$$=2\frac{(-1)^{n+1}}{n+1}\sum_{p=1}^{p=n}\frac{1}{p}=(-1)^{n+1}a_n$$

Notice that for all $n\geq 1$*, we have* $a_n=\frac{2}{n+1}\sum_{p=1}^{p=n}\frac{1}{p}>0$ *and*

$$a_{n+1}-a_n=\frac{2}{n+2}\sum_{p=1}^{p=n+1}\frac{1}{p}-\frac{2}{n+1}\sum_{p=1}^{p=n}\frac{1}{p}$$
$$=2\left(\frac{1}{n+2}-\frac{1}{n+1}\right)\sum_{p=1}^{p=n}\frac{1}{p}+\frac{2}{n+2}\frac{1}{n+1}$$
$$=\frac{2}{(n+1)(n+2)}\left[1-\sum_{p=1}^{p=n}\frac{1}{p}\right]=\frac{-2}{(n+1)(n+2)}\sum_{p=2}^{p=n}\frac{1}{p}<0.$$

On the other hand, $a_n=\frac{2}{n+1}\left(\sum_{p=1}^{p=n}\frac{1}{p}-\ln(n+1)\right)+\frac{2}{n+1}\ln(n+1)$ *then* $\lim a_n=0+0=0$ *because we know that the sequence* $\left(\sum_{p=1}^{p=n}\frac{1}{p}-\ln(n+1)\right)_{n\geq 1}$ *converges. Consequently, in view Leibnitz's test , the series product* $\sum_{n\geq 1}w_n$ *converges.*

Exercise 3.1.36 *Show that the series* $\sum_{n\geq 0}\frac{n+1}{2^n}$ *converges, then as the Cauchy product of two other series calculate its sum.*

Solution 3.1.37 *By the test of D'Alembert, we have for any* $n\in\mathbb{N}$; $u_n=\frac{n+1}{2^n}>0$ *and* $\frac{u_{n+1}}{u_n}=\frac{1}{2}\frac{n+2}{n+1}$ *tends to* $\frac{1}{2}<1$ *then, the series converges.*

Note that $\frac{1}{2^n}=\frac{1}{2^p}\cdot\frac{1}{2^{n-p}}$, $p=0,1,2,...,n$ *then we can write* $\frac{n+1}{2^n}=\sum_{p=0}^{p=n}\frac{1}{2^n}=\sum_{p=0}^{p=n}\frac{1}{2^p}\cdot\frac{1}{2^{n-p}}$ *so*

$$\sum_{n\geq 0}\frac{n+1}{2^n}=\sum_{n\geq 0}\left(\sum_{p=0}^{p=n}\frac{1}{2^n}\right)=\sum_{n\geq 0}\left(\sum_{p=0}^{p=n}\frac{1}{2^p}.\frac{1}{2^{n-p}}\right)=\left(\sum_{n\geq 0}\frac{1}{2^n}\right)^2,$$

according to the previous theorem, the sum of this series is
$\sum_{n=0}^{\infty}\frac{n+1}{2^n}=\left(\sum_{n=0}^{\infty}\frac{1}{2^n}\right)\left(\sum_{n=0}^{\infty}\frac{1}{2^n}\right)=\left(\frac{1}{1-\frac{1}{2}}\right)^2=4.$

3.2 Double series

Definition 3.2.1 ***Double sequence*** *is a sequence of certain elements numbered by two indices* u_{nm} *with* $n,m=1,2,3.....$

For example $\frac{nm}{n+m},n.e^m..$

Definition 3.2.2 *We say that* $\alpha\in\mathbb{R}$ *i s a* ***double limit*** *of the double sequence of real numbers* $(u_{nm})_{n,m=1.2.3....}$ *if For any* $\epsilon>0$ *there exists an* $N_\epsilon\in\mathbb{N}$ *such that for all* $n,m\geq N_\epsilon$ *we have*

$$|u_{nm}-\alpha|<\epsilon,$$

we write $\alpha = \lim_{n,m\to\infty} u_{nm}$.

If for any $A > 0$ *there exists* $N_\epsilon \in \mathbb{N}$, *such that for all* $n, m \geq N_\epsilon$ *the inequality* $|u_{nm}| > A$ *is fulfilled, then the sequence* $(u_{nm})_{n,m=1.2.3....}$ *has infinity as its limit:* $\lim_{n,m\to\infty} u_{nm} = \infty$

The double limit of a double sequence is a special case of the double limit of a function of two variables over a set.

For example : $\lim_{n,m\to\infty} \frac{2}{n+m} = 0$; $\lim_{n,m\to\infty} n.e^m = +\infty$.

Definition 3.2.3 *If* $(u_{nm})_{n,m=1.2.3....}$ *is a double sequence of real numbers then,* $\sum_{n,m\geq 1} u_{nm}$ *is called a numerical double series.*

The finite sums $S_{nm} = \sum_{i=1}^{i=n} \sum_{j=1}^{j=m} u_{ij}$ *are said to be the partial sums of the double series* $\sum_{n,m\geq 1} u_{nm}$ *or its rectangular partial sums.*

If this sequence $(S_{nm})_{n,m=1.2.3....}$ *has a finite double limit* $S = \lim_{n,m\to\infty} S_{nm}$ *the series* $\sum_{n,m\geq 1} u_{nm}$ *is said to be convergent and the number* S *is said to be its sum:*

$$S = \sum_{n,m=1}^{\infty} u_{nm}.$$

If there is no finite limit for the double sequence $(S_{nm})_{n,m=1.2.3....}$ *the series* $\sum_{n,m\geq 1} u_{nm}$ *is said to be divergent.*

Properties: Double series have many of the properties of ordinary series. For instance,

1) if the double series $\sum_{n,m\geq 1} u_{nm}$ and $\sum_{n,m\geq 1} v_{nm}$ converge then, for any numbers α, β the double series

$$\alpha \sum_{n,m\geq 1} u_{nm} + \beta \sum_{n,m\geq 1} v_{nm} = \left(\sum_{n,m\geq 1} \alpha u_{nm} + \beta v_{nm} \right)$$

converges also.

2) Necessary condition for convergence of the series : if $\sum_{n,m\geq 1} u_{nm}$ converges then, $\lim_{n,m\to\infty} u_{nm} = 0$.

3)If all terms of the series $\sum_{n,m\geq 1} u_{nm}$ are non-negative, the sequence of its partial sums $(S_{nm})_{n,m=1.2.3....}$ always has a finite or an infinite limit, and $\lim_{n,m\to\infty} S_{nm} = \sup_{n,m\geq 1} S_{nm}$.

4)The specific properties of a double series are due to the presence of double-indexed terms. If the double series $\sum_{n,m\geq 1} u_{nm}$ converges and if for all $n = 1, 2.3$ the series $\sum_{m,\geq 1} u_{nm}$ converges as well then the repeated series $\sum_{n\geq 1} \left(\sum_{m,\geq 1} u_{nm} \right)$ also converges, and its sum is equal to the sum of the given series i.e.

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{nm} = \sum_{n,m=1}^{\infty} u_{nm}.$$

Moreover, any series obtained by rearrangement of its terms also converges, and the sum of any such arbitrary series is the same as the sum of the initial series.

Definition 3.2.4 *If there exists a bijection* $\varphi : \mathbb{N}^* \to \mathbb{N}^* \times \mathbb{N}^*$ *such that the (single) series* $\sum_{k\geq 1} u_{\varphi(k)}$ *converges absolutely then, the double series* $\sum_{n,m\geq 1} u_{nm}$ *is said to be absolutely convergent .*

If a double series is absolutely convergent, it is also convergent; and its sum $\sum_{n,m=1}^{\infty} u_{nm} = \sum_{k=1}^{\infty} u_{\varphi(k)}$.

Lemma 3.2.5 *The double series* $\sum_{n,m\geq 1} u_{nm}$ *is absolutely convergent if and only if there exits* $K \in \mathbb{R}_+$ *such that for all* $M, N \in \mathbb{N}^*$, *we have*

$$\sum_{i=1}^{i=N} \sum_{j=1}^{j=M} |u_{ij}| \leq K.$$

Proof. Suppose that the series $\sum_{n,m\geq 1} u_{nm}$ is absolutely convergent and that, in the definition, the bijection φ is given such that the series

$\sum_{k\geq 1} u_{\varphi(k)}$ is absolutely convergent. We can take $K = \sum_{k=1}^{\infty} \left|u_{\varphi(k)}\right|$. Indeed, if N, M are given, $I = [1, N] \times [1, M] \cap \mathbb{N}^* \times \mathbb{N}^*$ being finite set, there exists L such that $I \subset \varphi\left([1, L] \cap \mathbb{N}\right)$ so

$$\sum_{i=1}^{i=n} \sum_{j=1}^{j=m} |u_{ij}| \leq K.$$

The converse is not more difficult. For each $L \in \mathbb{N}$, there exists $N = M$ such that all $\varphi(k)$ are in the square $[0, N] \times [0, N]$ for $k = 0, 1, ..., L$. Thus,

$\sum_{k=1}^{L} \left|u_{\varphi(k)}\right| \leq \sum_{i=1}^{i=N} \sum_{j=1}^{j=N} |u_{ij}| \leq K$ ■

Example 3.2.6 *For the double series* $\sum_{n,m\geq 1} \frac{1}{2^n 3^m}$ *we have* $\sum_{i=1}^{i=M} \frac{1}{2^j 3^i} = \frac{3}{2^j} \frac{1-\frac{1}{3^M}}{2} \leq \frac{3}{2^{j+1}}$ *then,*

$$\sum_{i=1}^{i=N} \sum_{j=1}^{j=M} \frac{1}{2^j 3^i} \leq \sum_{i=1}^{i=N} \frac{3}{2^{j+1}} = 3\left(1 - \left(\frac{1}{2}\right)^N\right) \leq 3.$$

In view of the above lemma, the series is absolutely convergent .

Theorem 3.2.7 *(Fubini theorem about the double series) If the double series* $\sum_{n,m\geq 1} u_{nm}$ *converges absolutely then, for each fixed* $n \in \mathbb{N}^*$,*the single series* $\sum_{m\geq 1} u_{nm}$ *is convergent and for any fixed* $m \in \mathbb{N}^*$,*the single series* $\sum_{n\geq 1} u_{nm}$ *is convergent . Moreover, we have*

$$\sum_{i,j=1}^{\infty} u_{ij} = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} u_{ij}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} u_{ij}\right)$$

Proof. Given K, as in the Lemma, we have

$$\forall N \in \mathbb{N}^*, \ \sum_{j=1}^{i=N} |u_{ij}| \leq \sum_{i=1}^{i=M} \sum_{j=1}^{j=M} |u_{ij}| \leq K,$$

where $M = \max\left(i, N\right)$.

This proves that the terms of the same line are the terms of an absolutely convergent series. Similar reasoning shows that the same is true for columns.

Denote by $l_i = \sum_{j=1}^{\infty} u_{ij}$, $c_j = \sum_{i=1}^{\infty} u_{ij}$ the sums whose existence we have justified, and denote by $L_i = \sum_{j=1}^{\infty} |u_{ij}|$, $C_j = \sum_{i=1}^{\infty} |u_{ij}|$; it's clear that $l_i \leq L_i$, $c_j \leq C_j$..

In the inequality

$$\left| \sum_{i=1}^{i=N} \sum_{j=1}^{j=M} u_{ij} \right| \leq \sum_{i=1}^{i=N} \left| \sum_{j=1}^{j=M} u_{ij} \right| \leq \sum_{i=1}^{i=N} \sum_{j=1}^{j=M} |u_{ij}|$$

we can pass to the limit in M,

$$\lim_{M\to\infty} \left| \sum_{i=1}^{i=N} \sum_{j=1}^{j=M} u_{ij} \right| \leq \sum_{i=1}^{i=N} \left| \lim_{M\to\infty} \sum_{i=1}^{i=M} u_{ij} \right| = \sum_{i=1}^{i=N} |l_i| \leq K$$

This proves that $\sum l_n$ is absolutely convergent. Similarly, we have that $\sum c_n$ is also absolutely convergent. Each of the sides of the Fubini identity is thus well defined.

Take $\epsilon > 0$ and consider N', M' such that

$$\left| \sum_{i,j=1}^{\infty} u_{ij} - \sum_{i=1}^{i=N} \sum_{j=1}^{j=M} u_{ij} \right| < \frac{\epsilon}{2}$$

as soon as $M \geq M'$ and $N \geq N'$. Fixing $N \geq N'$. Since each of the series l_i is absolutely convergent, we also have estimates of the remainders, there exists for each $i = 1, 2, ..N$, a number M_i such that for all $S \geq M_i$ we have $\left| l_i - \sum_{j=1}^{j=S} u_{ij} \right| \leq \frac{\epsilon}{2N}$.

Thus, considering $M^* = \max\{1, 2, ..N, M'\}$. For all $M \geq M^*$ and $i = 1, 2, ..N$, we have

$$\left| l_i - \sum_{j=1}^{j=M} u_{ij} \right| \leq \frac{\epsilon}{2N} \quad \text{and} \quad \left| \sum_{i=1}^{i=N} l_i - \sum_{i=1}^{i=N} \sum_{j=1}^{j=M} u_{ij} \right| \leq \frac{\epsilon}{2}.$$

So, for all $\epsilon > 0$, there exists N', M^* such that if $N \geq N', M \geq M^*$ then,

$$\begin{aligned} \left| \sum_{i,j=1}^{\infty} u_{ij} - \sum_{i=1}^{i=N} l_i \right| &\leq \left| \sum_{i,j=1}^{\infty} u_{ij} - \sum_{i=1}^{i=N} \sum_{j=1}^{j=M} u_{ij} \right| \\ + \left| \sum_{i=1}^{i=N} l_i - \sum_{i=1}^{i=N} \sum_{j=1}^{j=M} u_{ij} \right| &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This proves the first Fubini equality, the second is similar. ■

3.3 Exercises

Exercise 1 For each from the following series, use the sequence of partial sums to determine whether the series converges or diverge

$\sum_{n\geq 1} \frac{n}{n+2}$; $\sum_{n\geq 1} (1 - (-1)^n)$; $\sum_{n\geq 1} \frac{1}{(n+1)(n+2)}$; $\sum_{n\geq 0} \frac{n+1}{\pi^n}$; $\sum_{n\geq 1} \frac{1}{4n^2-1}$; $\sum_{n\geq 1} \frac{1+2+3+...+n}{1^3+2^3+3^3+...+n^3}$ (Hint: use the equality $\sum_{i=1}^{i=n} i^3 = \left(\sum_{i=1}^{i=n} i\right)^2$);

$\sum_{n\geq 1}\frac{1}{n(n+1)(n+2)}$; $\sum_{n\geq 1}\frac{n^4-(n+1)^3}{n!}$; $\sum_{n\geq 0}\frac{n}{n^4+n^2+1}$; $\sum_{n\geq 0}\frac{1}{\sqrt{n+2}+\sqrt{n+1}}$; $\sum_{n\geq 2}\ln\left(1-\frac{1}{n^2}\right)$.

Exercise 2 Let (a_n) be a sequence with positive terms. Take $u_n = \frac{a_n}{(1+a_1)(1+a_2)(1+a_3)...(1+a_n)}$.

Show that the series $\sum u_n$ and then calculate its sum in the case $a_n = \frac{1}{\sqrt{n}}$.

Exercise 3 Calculate the following sums

$\sum_{n=2}^{\infty}\frac{1}{n^2-1}$; $\sum_{n=1}^{\infty}\ln\left(1+\frac{2}{n(n+3)}\right)$; $\sum_{n=1}^{\infty}\frac{1}{n^2(n+1)^2}$ (help: use the information $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$).

Exercise 4 Use the appropriate test to determine whether the series with positive terms is convergent or divergent in each case .

$\sum_{n\geq 3}\frac{3n}{n-2}$; $\sum_{n\geq 1}\frac{1}{n^3+8n}$; $\sum_{n\geq 2}\frac{1}{\sqrt{n^2-3}}$; $\sum_{n\geq 1}\frac{1}{n^2+\sqrt{n}}$; $\sum_{n\geq 1}\frac{n^3}{n^5+4n+1}$;

$\sum_{n\geq 1}\frac{4}{n!+4^n}$; $\sum_{n\geq 0}n^2e^{-n}$; $\sum_{n\geq 0}\frac{n+\sqrt{n}}{3n^2+1}$; $\sum_{n\geq 3}\sin^3\left(\frac{1}{n}\right)$;

$\sum_{n\geq 1}\frac{2+\cos n}{n\sqrt{n}}$; $\sum_{n\geq 2}\frac{1}{n(n++\ln n)}$; $\sum_{n\geq 1}\frac{4^n-1}{3^n-1}$; $\sum_{n\geq 0}\frac{n^3}{2^n+1}$;

$\sum_{n\geq 1}\frac{n^n\sqrt{n}}{n!e^n}$; $\sum_{n\geq 1}\left(\frac{n}{n+2}\right)^n$; $\sum_{n\geq 2}\frac{1+\ln n}{n\sqrt{n}}$;

$\sum_{n\geq 1}\frac{2^n}{n^2}(\sin\alpha)^2$,with $\alpha\in\left[0,\frac{\pi}{2}\right]$; $\sum_{n\geq 1}\frac{1}{n^2+\sin n}$.

Exercise 5 Use the integral test to determine whether the following series converges or diverges

$\sum_{n\geq 5}\frac{1}{\sqrt{n-4}}$; $\sum_{n\geq 25}\frac{n^2}{(n^3+9)^{\frac{5}{2}}}$; $\sum_{n\geq 1}\frac{1}{n^2+1}$; $\sum_{n\geq 4}\frac{1}{n^2+1}$; $\sum_{n\geq 1}ne^{-n^2}$.

Exercise 6 Use the suitable test to determine whether the following series with arbitrary terms is absolutely convergent, conditionally convergent or divergent in each case

$\sum_{n\geq 1}\frac{(-1)^n}{\sqrt{n^2+n}}$; $\sum_{n\geq 1}\frac{\cos(\sqrt{n})}{n\sqrt{2}n}$; $\sum_{n\geq 1}(-1)^n\frac{n+\sqrt{n}}{2n^3+1}\sin\left(\frac{\sqrt{n+}}{n}\right)$; $\sum_{n\geq 2}\frac{(-1)^n}{n+(-1)^n\sqrt{n}}$.

Exercise 7 1) If k,n and $p\in\mathbb{N}$ with $k\leq n$;

prove that $\sum_{n=k}^{n=p}\frac{C_n^k-C_n^{k+1}}{2^n}=\frac{C_{p+1}^{k+1}}{2^p}$.

2)Let $\sum u_n$ be a convergent series. prove that the series $\sum v_n$ where $v_n=\frac{1}{2^n}\sum_{p=0}^{p=n}C_n^p u_p$ is convergent.

Part B

Sequences and series of Functions

Chapter 4

Sequences of Functions

In this chapter, we are going to study the sequences of real (or complex) valued functions $(f_n)_{\in \mathbb{N}}$, i.e. sequences whose terms are functions defined on the same subset $I \subset \mathbb{R}$. For this, it is clear that we will combine all the fundamental notions known on the numerical sequences with those established on functions with a single variable. We will be interested in the properties of the function limit in terms of continuity, differentiability, integrability, etc . . ., being known those of the functions $(f_n)_{\in \mathbb{N}}$. The notion of the simple passage to the known limit in the context of numerical sequences makes it possible to define a criterion called pointwise convergence of sequences of functions.

In what follow, let I be a subset from $\mathbb{R}$, denote by $F(I;\ \mathbb{k})$ the set of all functions $f : I \to \mathbb{k}$ where $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$. Often, I denotes an interval of $\mathbb{R}$ or an interval except a number countable of values.

Definition 4.0.1 *Suppose $f_n : I \to \mathbb{k}$ be a real (or complex)-valued function defined on I for each $n \in \mathbb{N}$ (i.e. $f_n \in F(I;\ \mathbb{k})$) , then the set $\{f_n; n = 0, 1, 2, ...\}$ is called a* ***sequence of functions*** *on I. It is denoted by $(f_n)_{n\in\mathbb{N}}$.*

4.1 Pointwise Convergence

Definition 4.1.1 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions on I .We say that $(f_n)_{n\in\mathbb{N}}$ is* ***pointwise convergent,*** *if the real(or complex)-valued sequence of functions $(f_n(x))_{n\in\mathbb{N}}$ converges for every real number $x \in I$. It's to say that :*

$\forall x \in I,\ \lim_{n\to\infty} f_n(x) = f(x) \in \mathbb{k}$ *(here the writing $f(x)$ says that the limit depends on x) .*

In other word ,

$(f_n)_{n\in\mathbb{N}}$ is pointwise convergent if and only if

$$\forall x \in I, \forall \epsilon > 0, \exists N(\epsilon, x); \forall n \in \mathbb{N}, n \geq N(\epsilon, x) \implies |f_n(x) - f(x)| < \epsilon.$$

The function $f : I \to \mathbb{k}$ defined by $f(x) = \lim_{n\to\infty} f_n(x)$ is called the ***pointwise limit*** *of the sequence $(f_n)_{\in\mathbb{N}}$. And we express it briefly as follows*

$f_n \to f$ (pointwisely) or $\lim f_n = f$.

Example 4.1.2 *1) Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of functions defined on $I = [0, 1]$ by its general term $f_n(x) = x^n$.*

It's clear that

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} x^n = \begin{cases} 1; & \textit{if } x = 1 \\ 0; & \textit{if } x \in [0,1[\end{cases} = f(x),$$

Then, the sequence $(f_n)_{n\in\mathbb{N}}$*is pointwise convergent on* $[0,1]$ *and its pointwise limit is the function* f.

2) Suppose the sequence of functions $(g_n)_{n\in\mathbb{N}}$ *such that*

$$g_n(x) = \cos\left(\frac{n\pi x + 3}{2n+1}\right); x \in \mathbb{R}.$$

$(g_n)_{n\in\mathbb{N}}$ *is pointwise convergent on* $\mathbb{R}$ *to a function* g *because for all* $x \in \mathbb{R}$, $\lim_{n\to+\infty} g_n(x) = \lim_{n\to+\infty} \cos\left(\frac{n\pi x+3}{2n+1}\right) = \cos\left(\frac{\pi x}{2}\right) = g(x)$.

3) Let $(\varphi_n)_{n\in\mathbb{N}}$ *be a sequence of function defined by :* $\varphi_n(x) = ne^{-nx}, x \in \mathbb{R}$.

We have $\lim_{n\to\infty} \varphi_n(x) = \begin{cases} 0; & \textit{if } x > 0 \\ +\infty; & \textit{if } x \leq 0 \end{cases}$, *then* $(\varphi_n)_{n\in\mathbb{N}}$ *does not pointwise convergent on* $\mathbb{R}$. *But on* $\mathbb{R}_+^*$ *it converges pointwisely to the function* $\varphi \equiv 0$.

4) The sequence of function defined by $f_n(x) = \left(1 + \frac{x}{n}\right)^n$, $x \in \mathbb{R}_+$, $n \geq 1$.

For a fixed $x \in \mathbb{R}_+$, *we have* $\lim_{n\to\infty} f_n(0) = \lim 1 = 1 = e^0$. *For* $x > 0$; $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n\to\infty} e^{x\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{x}{n}}} = e^{x.1} = e^x$ *.Then,* $f_n \to f$ *(pointwisely) where* $f\ (x) = e^x$.

Remark 4.1.3 *Pointwise convergence is often easy to establish, but it does not preserve certain properties of functions (continuity, bounded aspect, integrability), as shown in the following example.*

Example 4.1.4 *1) In the previous example 1) all the functions* $f_n(x) = x^n$ *;* $n = 0, 1, 2, ...$ *are continuous on* $[0,1]$ *while the pointwise limit* f *where*

$$f(x) = \begin{cases} 1; & \textit{if } x = 1 \\ 0; & \textit{if } x \in [0,1[\end{cases}$$

isn't continuous on this interval.

2) It is easy to verify that the sequence of function $(f_n)_{n\in\mathbb{N}}$ *where* $f_n(x) = n^2x^n(1-x)$ *is pointwise convergent towards the null function on* $[0,1]$ *whose integral on this segment is* 0. *However,*

$$\lim_{n\to\infty} \int_0^1 n^2x^n(1-x)\,dx = \lim_{n\to\infty} n^2\left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$$
$$= \lim_{n\to\infty} \frac{n^2}{(n+1)(n+2)} = 1 \neq 0.$$

Thus, pointwise convergence does not preserve the properties of functions: this justifies the definition of a new form of convergence, more difficult to verify, but valid for passages to the limit.

4.2 Uniform convergence

Definition 4.2.1 *A sequence of functions $(f_n)_{n\in\mathbb{N}}$ is said to be* ***uniformly convergent*** *to a function f on the set I, if for each $\epsilon > 0$,*

there exists $N(\epsilon) \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N(\epsilon)$ and any $x \in I$.

Alternatively, we can define the uniform convergence of a sequence of functions, as follows.

A sequence of functions $(f_n)_{n\in\mathbb{N}}$ is said to be uniformly convergent to a function f on the set I if ;

$$\lim_{n\to\infty}\left(\sup_{x\in I}|f_n(x) - f(x)|\right) = 0.$$

Briefly , we express it as : $f_n \rightrightarrows f$ or $f_n \to f$ uniformly

Example 4.2.2 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined by :*

$$f_n(x) = \frac{2nx^3}{nx^2+1}, x \in \mathbb{R}.$$

Let's first study the pointwise convergence. We have

$\lim_{n\to\infty}\frac{2nx^3}{nx^2+1} = \begin{cases} 0; \text{ if } x = 0 \\ 2x; \text{ if } x \neq 0 \end{cases} = 2x$. *Then $f_n \to f$ (pointwise) such that for all $x \in \mathbb{R}$, $f(x) = 2x$.*

Now calculate $\sup_{x\in\mathbb{R}}|f_n(x) - f(x)|$

$g(x) = |f_n(x) - f(x)| = \left|\frac{2nx^3}{nx^2+1} - 2x\right| = \frac{2|x|}{nx^2+1}$,*noting that g is an even function then, $\sup_{x\in\mathbb{R}} g(x) = \sup_{x\geq 0} g(x)$. we have for all $x \geq 0$*

$g'(x) = \frac{2(1+x\sqrt{n})}{(nx^2+1)^2}(1 - x\sqrt{n}) = 0 \implies x = \frac{1}{\sqrt{n}}$

$g'(x) < 0 \implies x > \frac{1}{\sqrt{n}}$ *and $g'(x) > 0 \implies x < \frac{1}{\sqrt{n}}$. which yields the following variation table*

x	0		$1/\sqrt{n}$		$+\infty$
$g'(x)$		$+$	0	$-$	
$g(x)$	0	$\nearrow$	$1/\sqrt{n}$	$\searrow$	0

Figure 4.1: Variation table

So, $\sup_{x\in\mathbb{R}}|f_n(x) - f(x)| = g\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{\sqrt{n}}$ *therefore,* $\lim_{n\to\infty}\sup_{x\in\mathbb{R}}|f_n(x) - f(x)| = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$ *i.e. $f_n \to f$ uniformly see (***fig ??***) .*

2) Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of functions where $f_n(x) = x^n(1-x)$, $x \in [0,1]$. We have

$\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} n.1^n.0 = 0$ *and if $0 \leq x < 1$,* $\lim_{n\to\infty} f_n(x) = (1-x)\lim_{n\to\infty} x^n = 0$ *. Then, $f_n \to f \equiv 0$ (pointwisely).*

To determine $H_n = \sup_{x\in[0,1]}|f_n(x) - f(x)| = \sup_{x\in[0,1]} f_n(x)$ we study the variations of f_n:

$$f_n'(x) = nx^{n-1}(n - (n+1)x)$$

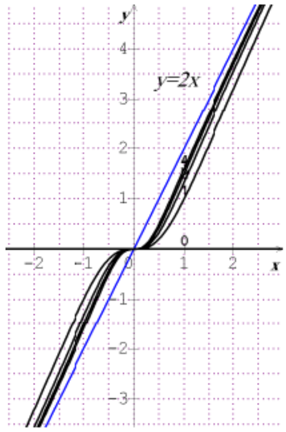

Figure 4.2: uniform convergence of fn

so f_n is increasing in the interval $\left[0, \frac{n}{n+1}\right]$ and decreasing thereafter, then it takes its maximum at $x_0 = \frac{n}{n+1}$,that's $0 \leq H_n = f_n\left(\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \frac{1}{n+1} \to \frac{1}{e}.0 = 0$ as $n \to \infty$.

Then, $(f_n)_{n\in\mathbb{N}}$ converges uniformly to the zero function on $[0,1]$.

4.2.1 Uniform Cauchy criterion

Proposition 4.2.3 *A sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function f on the set I, if and only if, it satisfies the uniform Cauchy criterion with respect to x:*

$$\forall\epsilon > 0, \exists N(\epsilon) \in \mathbb{N}, \forall n \geq N(\epsilon), \forall p \in \mathbb{N}, \forall x \in I; \quad |f_{n+p}(x) - f_n(x)| < \epsilon,$$

or in another form

$$\forall\epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \forall p \in \mathbb{N}; \quad \sup_{x\in I} |f_{n+p}(x) - f_n(x)| < \epsilon \tag{4.1}$$

Proof. Assuming that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ satisfies (**??**), let's prove that it converges uniformly towards a function f on the domain I. For every (fixed) $x \in I$ and according to condition **??** $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy's sequence of real value and therefore converges towards a real number $f(x)$ (when we change x in I the limit $f(x)$ changes). Then, $(f_n)_{n\in\mathbb{N}}$ converges pointwisely to the function f. To prove that the convergence is uniform, it is sufficient to take $p \to \infty$ in (**??**).

Suppose that $f_n \to f$ (uniformly), let's prove that condition **??** is true. we've got

$\forall\epsilon > 0, \exists N(\epsilon) \in \mathbb{N}, \forall n \geq N(\epsilon), \forall p \in \mathbb{N}, \forall x \in I; |f_n(x) - f(x)| < \frac{\epsilon}{2}$,

Let n and p two integer numbers such that $n \geq N(\epsilon), p \in \mathbb{N}$ then, $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ and $|f_{n+p}(x) - f(x)| < \frac{\epsilon}{2}$. Therefore, by triangular inequality, we get for all $x \in I$

$|f_{n+p}(x) - f_n(x)| \leq |f_{n+p}(x) - f(x)| + |f_n(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$

So, $(f_n)_{n\in\mathbb{N}}$ satisfies (**??**) . ■

Remark 4.2.4 *The Cauchy criterion allows the study of uniform convergence without knowing the function limit.*

4.2.2 Sufficient condition for uniform convergence

Proposition 4.2.5 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of function. Then, $(f_n)_{n\in\mathbb{N}}$converge uniformly on I to a function f, if and only if there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of non-negative real numbers converging to 0 such that*

$$\forall x \in I; |f_n(x) - f(x)| \leq a_n \quad \text{for } n \text{ sufficiently large}$$

Proof. Suppose that $(f_n)_{n\in\mathbb{N}}$ converges uniformly on I to f . There exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < 1$ for all $n \geq N$ and $x \in I$. Hence, $a_n = \sup_{x\in I} |f_n(x) - f(x)| \geq 0$ is well-defined for all $n \geq N$. Define $a_n \geq 0$ arbitrarily for $1 \leq n \leq N-1$. Given an arbitrary $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that if $n \geq K$ then $|f_n(x) - f(x)| < \epsilon$ for all $x \in I$. We can assume that $K \geq N$. Therefore, if $n \geq K$ then $a_n = \sup_{x\in I} |f_n(x) - f(x)| < \epsilon$. This prove that $\lim a_n = 0$. Conversely, suppose that there exists $a_n \geq 0$ such that $\lim a_n = 0$ and $\sup_{x\in I} |f_n(x) - f(x)| \leq a_n$ for all $n \geq N$. Let $\epsilon > 0$ be arbitrary. Then there exists $K \in \mathbb{N}$ such that if $n \geq K$ then $a_n < \epsilon$. Hence, if $n \geq K \geq N$ then $\sup_{x\in I} |f_n(x) - f(x)| \leq a_n < \epsilon$. This implies that if $n \geq K$ then $|f_n(x) - f(x)| < \epsilon$ for all $x \in I$ and thus $(f_n)_{n\in\mathbb{N}}$ converges uniformly on I to f . ■

Example 4.2.6 *In the previous example (**??**), it's clear that for all $x \in I$, $n \in \mathbb{N}^*$*
$|f_n(x) - f(x)| \leq \frac{1}{\sqrt{n}} = a_n$ and $\lim \frac{1}{\sqrt{n}} = 0$. Then , $f_n \to f$ uniformly.

4.2.3 Necessary condition for uniform convergence

Proposition 4.2.7 *If the sequence of function $(f_n)_{n\in\mathbb{N}}$converges uniformly on I to a function f then, $(f_n)_{n\in\mathbb{N}}$ converges pointwisely on I to the same function. But the converse is false.*

We can write briefly:

$$f_n \to f \ (\text{uniformly}) \begin{bmatrix} \Longrightarrow \\ \not\Leftarrow \end{bmatrix} f_n \to f \ (\text{pointwisely}) \ .$$

Proof. Suppose that $f_n \to f$ (uniformly). In order to prove the pointwise convergence, it suffices to take in its definition $N(\epsilon, x) = N(\epsilon)$. ■

As counter example showing that the converse is false, we can take

$$f_n(x) = x^n, x \in [0,1] .$$

It clear that $f_n \to f$ (pointwisely) where $f(x) = \begin{cases} 1; \text{if } x = 1 \\ 0; \text{ if } x \in [0,1[\end{cases}$.

But it does not converge uniformly, in fact $\lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| = \lim_{n\to\infty} \max\left(0, \sup_{x\in[0,1[} x^n\right) = 1 \neq 0$.

On the interval $[0,1[$, this sequence is still not uniformly convergent also because $\sup_{x\in[0,1[} |f_n(x) - f(x)| = 1$ (i.e does not change). But it converges uniformly on each domain of the form $[0,\alpha[$ where $0 \leq \alpha < 1$ where we remark that: $\sup_{x\in[0,\alpha[} |f_n(x) - f(x)| = \alpha^n \to 0$ as $n \to \infty$.

4.2.4 Properties of uniformly convergent sequence of functions

4.2.4.1 Continuous sequence of functions

Theorem 4.2.8 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from $F(I,R)$ and $x_0 \in I$. If*

- *$(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function f on I, and*
- *f_n is continuous at x_0 for all $n \in \mathbb{N}$,*

then, f is continuous at x_0.

Proof. Let ϵ be a strictly positive real number. Since $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent then,

$$\exists N(\epsilon) \in \mathbb{N}, \forall n \geq N(\epsilon), \forall x \in I; |f(x) - f_n(x)| < \frac{\epsilon}{3}$$

and by continuity of $f_{N(\epsilon)}$ at x_0

$$\exists \delta > 0; \forall x \in I, |x - x_0| < \delta \implies \left|f_{N(\epsilon)}(x) - f_{N(\epsilon)}(x_0)\right| < \frac{\epsilon}{3}.$$

Then, by triangular inequality we have for all $x \in I$

$$\begin{aligned} |x - x_0| &< \delta \implies |f(x) - f(x_0)| \leq \left|f(x) - f_{N(\epsilon)}(x)\right| + \left|f_{N(\epsilon)}(x) - f_{N(\epsilon)}(x_0)\right| \\ + \left|f_{N(\epsilon)}(x_0) - f(x_0)\right| &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

That is, f is continuous at x_0. ■

Corollary 4.2.9 *In the previous theorem, if we replace the statement (f_n is continuous at x_0) by (f_n is continuous on I), the result is the function f is continuous on I.*

Remark 4.2.10 *1) This result can be seen as an inversion of limits, since we have shown that:* $\lim_{x\to x_0} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to x_0} f_n(x) = f(x_0)$.

2) This result is often useful in ***contrapositive form****: if a sequence of continuous functions converges pointwisely to a non-continuous function so, this convergence is not uniform.*

That's, if

- *$(f_n)_{n\in\mathbb{N}}$ converges pointwisely to a non -continuous function f on I,*
- *f_n is continuous on I for all $n \in \mathbb{N}$.*

Then, the convergence is not uniform.

As example the sequence of functions $(f_n)_{n\in\mathbb{N}}$ with $f_n(x) = x^n, x \in [0,1]$ (example studied previously)

$f_n \to f$ (pointwisely) where $f(x) = \begin{cases} 0; \text{ if } x \in [0,1[\\ 1; \text{ if } x = 1 \end{cases}$

Note That f_n is continuous on I for all $n \in \mathbb{N}$, but f is not continuous because $\lim_{x\to 1^-} f(x) = 0 \neq f(1) = 1$.

Which suffices to conclude that $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly.

Example 4.2.11 *Another example: let be the sequence* $(f_n)_{n\in\mathbb{N}}$ *where*

$$f_n(x) = \frac{1-nx^n}{1+2nx^n}, x \in \mathbb{R}_+.$$

For $0 \leq x < 1, \lim_{n\to\infty} f_n(x) = 1$ *because* $\lim_{n\to\infty} nx^n = \lim_{n\to\infty} ne^{n\ln x} = 0$ *and for* $x \geq 1, \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\frac{1}{nx^n}-1}{\frac{1}{nx^n}+2} = -\frac{1}{2}$ *because* $\lim_{n\to\infty} nx^n = +\infty$.

Then, $f_n \to f$ *(pointwise) where* $f(x) = \begin{cases} 1; & \text{if } x \in [0,1[\\ -\frac{1}{2}; & \text{if } x \geq 1 \end{cases}$. *As* f_n *is continuous on* $\mathbb{R}_+$ *for all* $n \in \mathbb{N}$ *but* f *is not continuous on* $\mathbb{R}_+$ *because* $\lim_{x\to1^-} f(x) = 1 \neq f(1) = -\frac{1}{2}$ *so, this convergence is not uniform.*

Theorem 4.2.12 *(Dini's theorem)If the sequence of functions.* $(f_n)_{n\in\mathbb{N}}$

- *converges pointwisely on the interval* $I = [a,b]$ *to a function* f *continuous on* I*,*
- f_n *is monotone on* I *for all* $n \in \mathbb{N}$ *(not necessarily continuous),*

Then, $(f_n)_{n\in\mathbb{N}}$ *converges uniformly on* I.

4.2.4.2 Intgrable sequence of functions

Theorem 4.2.13 *If* $(f_n)_{n\in\mathbb{N}}$ *is a sequence of real valued functions, where*

- f_n *is integrable on* $I = [a,b]$ *for every positive integer number* n.
- $(f_n)_{n\in\mathbb{N}}$ *converges uniformly towards a function* f *over the interval* I.

Then, f *is integrable on* I *and*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof. For all $x \in I$ and all $n \in \mathbb{N}$ one can write

$$|f(x)| = |f_n(x) + f(x) - f_n(x)| \leq |f_n(x) - f(x)| + |f_n(x)|$$

Thus,

$$\begin{aligned}\int_a^b |f(x)|\,dx &\leq \int_a^b |f_n(x) - f(x)|\,dx + \int_a^b |f_n(x)|\,dx \\ &\leq \int_a^b |f_n(x)|\,dx + (b-a)\sup_{x\in[a,b]} |f_n(x) - f(x)|.\end{aligned}$$

The uniform convergence of $(f_n)_{n\in\mathbb{N}}$ means that $\left(\sup_{x\in[a,b]} |f_n(x) - f(x)|\right)_{n\in\mathbb{N}}$ is a real valued sequence that converges to 0 so , it's bounded and as f_n is integrable on the interval I so $\int_a^b |f_n(x)|\,dx$ is also bounded. Then,

$$\int_a^b |f(x)|\,dx \leq M + (b-a)M',$$

where, M, M' are positive real constants. Which proves that f is integrable on I . On the other hand

$$\begin{aligned}\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| &= \left|\int_a^b (f_n(x) - f(x))\,dx\right| \\ &\leq \int_a^b |f_n(x) - f(x)|\,dx \leq (b-a)\sup_{x\in[a,b]} |f_n(x) - f(x)|,\end{aligned}$$

as $\lim_{n\to\infty} \sup_{x\in[a,b]} |f_n(x) - f(x)| = 0$ then, $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$. Which complete the proof. ■

Remark 4.2.14 *In the previous theorem, if all functions* f_n, $n = 0, 1, 2, \ldots$ *are continuous on* $[a, b]$ *then, they are integrable on this interval and*

$$\lim_{n\to\infty} \int_a^b f_n dx = \int_a^b \lim_{n\to\infty} f_n dx.$$

Corollary 4.2.15 *If* $(f_n)_{n\in\mathbb{N}}$ *is a sequence of real valued functions*

- *integrable for every positive integer number* n *on the interval* $[a; b]$
- *uniformly convergent to a function* f *on* $[a; b]$.

Then, $(F_n)_{n\in\mathbb{N}}$ *converges uniformly to the function* F *on* $[a; b]$ *where* $F_n(x) = \int_a^x f_n(t)\,dt$ *and* $F(x) = \int_a^x f(t)\,dt$.

Proof. In view the above theorem, $F_n \to F$ (pointwisely). Let's prove the uniform convergence.

$\forall x \in [a, b]\,; |F_n(x) - F(x)| = \left|\int_a^x (f_n(t) - f(t))\,dt\right|$
$\leq \int_a^x |f_n(t) - f(t)|\,dt$
$\leq (b - a)\sup_{t\in[a,b]} |f_n(t) - f(t)| \to 0$ as $n \to \infty$.

Then, by the proposition **??** $F_n \to F$ (uniformly). ■

Remark 4.2.16 *The uniform convergence of* $(f_n)_{n\in\mathbb{N}}$ *in the previous theorem is a sufficient condition but not necessary.*

Example 4.2.17 *It's well known that the sequence of functions* $(f_n)_{n\in\mathbb{N}}$ *where* $f_n(x) = x^n$, $x \in [0, 1]$ *is pointwise convergent to the function* f *where* $f(x) = \begin{cases} 1 \text{ if } x = 1 \\ 0 \text{ if } x \in [0, 1[\end{cases}$ *and*

$\lim_{n\to\infty} \int_0^1 x^n dx = \lim_{n\to\infty} \frac{1}{n+1} = 0 = \int_0^1 f(x)\,dx$. *While the convegence is not uniform.*

4.2.4.3 Continuously differentiable sequence of functions

Theorem 4.2.18 *Let* $(f_n)_{n\in\mathbb{N}}$ *be a continuously differentiable sequence of functions on the domain* $[a; b]$, *i.e. of class* $C^1([a, b])$ *and check:*

- *There exists* $x_0 \in [a; b]$ *such that the sequence of real number* $(f_n(x_0))_{n\in\mathbb{N}}$ *is convergent.*
- *The sequence of derivative functions* $(f_n')_{n\in\mathbb{N}}$ *converges uniformly on the domain* $[a; b]$ *to a function* g.

Then the sequence $(f_n)_{n\in\mathbb{N}}$ *converges uniformly on the domain* $[a; b]$ *to a function* $f \in C^1([a, b])$ *and* $f' = g$ *i.e.*

$$\forall x \in [a, b]\,; \left(\lim_{n\to\infty n} f_n(x)\right)' = \lim_{n\to\infty} f_n'(x)$$

Proof. The function f_n' is continuous on $[a;b]$ because f_n is of class $C^1([a,b])$. As above, the function g is continuous on $[a;b]$ because $f_n' \to g$ (uniformly). We have for each $x \in [a;b]$;

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(t)\,dt$$

By calculating the limit of both sides and taking into account the convergence of the sequence $(f_n(x_0))_{n\in\mathbb{N}}$ and the uniform convergence of $(f_n')_{n\in\mathbb{N}}$, we get

$$\lim_{n\to\infty n} f_n(x) = \lim_{n\to\infty n} f_n(x_0) + \lim_{n\to\infty} \int_{x_0}^{x} f_n'(t)\,dt = f(x_0) + \int_{x_0}^{x} g(t)\,dt,$$

then, $(\lim_{n\to\infty n} f_n(x))' = g(x) = \lim_{n\to\infty n} f_n'(x)$. Now, let's prove that $f_n \to f$ (uniformly). We have for all $x \in [a,b]$,

$$\begin{aligned} |f_n(x) - f(x)| &= \left| f_n(x_0) + \textstyle\int_{x_0}^{x} f_n'(t)\,dt - f(x_0) - \int_{x_0}^{x} g(t)\,dt \right| \\ &\leq |f_n(x_0) - f(x_0)| + \textstyle\int_{x_0}^{x} |f_n'(t) - g(t)|\,dt \\ &\leq |f_n(x_0) - f(x_0)| + (b-a)\sup_{t\in[a,b]} |f_n'(t) - g(t)|. \end{aligned}$$

As $\lim_{n\to\infty} f_n(x_0) = f(x_0)$ and $f_n' \to g$ (uniformly) then, for $\epsilon > 0$ arbitrary there exist two non-negative integers N_1 and N_2 such that

$$\forall n \in \mathbb{N}; n \geq N_1 \implies |f_n(x_0) - f(x_0)| < \frac{\epsilon}{2}$$

and

$$\forall n \in \mathbb{N}; n \geq N_2 \implies \sup_{t\in[a,b]} \left| f_n'(t) - g(t) \right| < \frac{\epsilon}{2(b-a)}$$

.

Then,

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq N \implies |f_n(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

where $N = \max(N_1, N_2)$. Therefore $f_n \to f$ (uniformly). ∎

Example 4.2.19 *Consider the sequence $(f_n)_{n\in\mathbb{N}}$ defined on $I = [-1,1]$ by $f_n(x) = \frac{2nx+(-1)^n x^2}{n}$. We compute that $f_n'(x) = \frac{2(n+(-1)^n x)}{n}$ and clearly f_n' is continuous on I for each $n \in \mathbb{N}$. Now $\lim_{n\to\infty} f_n'(x) = 2$ for all $x \in I$ and thus (f_n') converges pointwise to g where $g(x) = 2$. To prove that the convergence is uniform we note that $|f_n'(x) - g(x)| = \left|\frac{2((-1)^n x)}{n}\right| = \frac{|2x|}{n} \leq \frac{2}{n}$ and as $\lim \frac{2}{n} = 0$, Therefore, $f_n' \to g$ (uniformly) on I. As $f_n(0) = 0$ by the above theorem $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function f with $f(0) = 0$ and $f' = g$. Then, $f(x) = \int_0^x g(t)\,dt = \int_0^x 2dt = 2x$.*

4.2.4.4 Uniform boundedness

Definition 4.2.20 *A sequence of function $(f_n)_{n\in\mathbb{N}}$ on I is said to be uniformly bounded on I if there exists a constant $M > 0$ such that*

$$|f_n(x)| \leq M,$$

for all $n \in \mathbb{N}$ and all $x \in I$.(i.e. M is independent of n and x).

Proposition 4.2.21 *Suppose that* $f_n \to f$ *uniformly on* I. *If for each* $n \in \mathbb{N}$, f_n *is bounded on* I *then, the sequence* $(f_n)_{n\in\mathbb{N}}$ *is uniformly bounded on* I *and* f *is bounded on* I.

Proof. In the definition of the uniform convergence, choose $\epsilon = 1$. Then, there exists $K \in \mathbb{N}$, such that

$$|f(x)| \leq |f_n(x) - f(x)| + |f_n(x)| \leq 1 + |f_n(x)|,$$

for all $x \in I$. and all $n \geq K$. Since f_K is bounded, then $|f(x)| \leq 1 + \max_{x\in I} |f_K(x)| = M'$ for all $x \in I$. and thus f is bounded on I . Therefore, $|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| \leq 1 + M'$ for all $x \in I$. and all $n \geq K$. Let M_n be an upper bound of f_n for any $n \in \mathbb{N}$. Then,

$$|f_n(x)| \leq \max\left(M_0, M_1, ..., M_{K-1}, M' + 1\right) = M,$$

for all $n \in \mathbb{N}$ and all $x \in I$. ■

Example 4.2.22 *The sequence of functions* $f_n \in F(]0,1[, \mathbb{R})$ *defined by* $f_n(x) = \frac{n}{nx+1}$.

It's clear that (f_n) *is pointwise convergent on* $]0,1[$ *and it's pointwise limit is* $f(x) = \frac{1}{x}$*,but this convergence is not uniform, indeed*

$$\sup_{0<x<1} |f_n(x) - f(x)| = \sup_{0<x<1} \frac{1}{x(nx+1)} = +\infty.$$

Since each f_n *is bounded on* $]0,1[$ *but* $f(x) = \frac{1}{x}$ *is not. The sequence* (f_n) *does, however converge uniformly to* f *on every interval* $[\alpha, 1[$ *with* $0 < \alpha < 1$. *To prove this,we estimate for all* $\alpha \leq x < 1$ *that* $|f_n(x) - f(x)| = \frac{1}{x(nx+1)} \leq \frac{1}{\alpha(n\alpha+1)}$ *because the function* $x \longmapsto \frac{1}{x(nx+1)}$ *is decreasing in* $[\alpha, 1[$. *As* $\lim_{n\to\infty} \frac{1}{\alpha(n\alpha+1)} = 0$ *which is a sufficient condition of uniform convergence. Note that* $|f(x)| = \frac{1}{x} \leq \frac{1}{\alpha}$ *for all* $x \in [\alpha, 1[$. *so the uniform limit f is bounded on* $[\alpha, 1[$.

Chapter 5

Infinite series of functions

In this chapter, we consider series whose terms are functions.

Definition 5.0.1 *Let I be a non-empty subset of $\mathbb{R}$. An infinite series of real (or complex) valued functions on I is a series of the form $\sum_{n\geq 0} f_n(x)$ for any $x \in I$, where $(f_n)_{n\geq 0}$ is a sequence of real (or complex) valued functions on I.*

The sequence of functions $(S_n)_{n\geq 0}$ defined by $S_n(x) = \sum_{p=0}^{p=n} f_p(x)$ for all $x \in I$ is called ***sequence of partial sums*** *generated by the series $\sum_{n\geq 0} f_n(x)$.*

Remark 5.0.2 *For each x fixed in I, $\sum_{n\geq 0} f_n(x)$ is a series of real (or complex) numbers.*

Example 5.0.3 $\sum_{n\geq 0} n!(x-1)^n$ *is an infinite series of functions on $\mathbb{R}$ where $f_n(x) = n!(x-1)^n$.*

Very importante note

The convergence of an infinite series of functions can be treated by considering the convergence of the sequence of partial sums (which are functions). It is now clear that our previous work the sequence of function translate essentially directly to infinite series of functions.

5.1 Pointwise convergence

Definition 5.1.1 *We say that the series of functions $\sum_{n\geq 0} f_n(x)$ is* ***pointwise convergent*** *on I if its sequence of partial sums $(S_n)_{n\geq 0}$ converges pointwisely on I to a function S. This function is called sum of the series and we write: for all $x \in I$*

$$\lim_{n\to\infty} S_n(x) = S(x) = \sum_{n=0}^{\infty} f_n(x)$$

We can express this definition in short as follow
$\sum_{n\geq 0} f_n(x) \to S(x)$ *(pointwise)* $\Longleftrightarrow S_n \to S$ *(pointwisely).*

Remark 5.1.2 *1) To study the pointwise convergence of a series of functions $\sum_{n\geq 0} f_n(x)$ on I, we study the nature of the infinite numerical series $\sum_{n\geq 0} f_n(a)$ for each constant $a \in I$.*

2)The criteria established on the convergence of numerical series are all valid for studying the pointwise convergence of a series of functions (considering x as real parameter).

3) If a series $\sum_{n\geq 0} f_n(x)$ converges pointwisely on I to a function S then, its nth remainder is defined for all $x \in I$ by

$$R_n(x) = S(x) - S_n(x) = \sum_{p=n++1}^{\infty} f_p(x).$$

In this case we have, $R_n \to 0$ (pointwisely.

5.2 Absolute convergence

Definition 5.2.1 *The series of functions $\sum_{n\geq 0} f_n(x)$ is said to be* ***absolutely convergent*** *on I, if the series of functions $\sum_{n\geq 0} |f_n(x)|$ converges pointwisely on I.*

Example 5.2.2 *Let's study the pointwise convergence of the series*

$$\sum_{n\geq 0} x^2 (3-x)^n$$

.

It's obvious that this series converges for $x = 0$, otherwise, we have

$$S_n(x) = \sum_{p=0}^{p=n} x^2 (3-x)^p = x^2 \frac{(3-x)^{n+1} - 1}{2-x}$$

The geometric series of general term $(3-x)^n$ converges if and only if $|3-x| < 1$, that's $2 < x < 4$. In this case , we have

$$\forall x \in \left]2, 4\right[, \lim_{n\to\infty} S_n(x) = \frac{x^2}{x-2} = S(x),$$

then, the series of functions $\sum_{n\geq 0} x^2 (3-x)^n$ is poinwise convergent to the function S on $\left]2,4\right[\cup \{0\}$, where $S(x) = \frac{x^2}{x-2}$.

Now, let's check the absolute convergence of this series . Applying the root test of Cauchy, put $f_n(x) = x^2 (3-x)^n$,

$\lim_{n\to\infty} \sqrt[n]{|f_n(x)|} = \lim_{n\to\infty} \sqrt[n]{x^2 |3-x|^n}$

$$= \lim_{n\to\infty} |3-x| |x|^{\frac{2}{n}} = \begin{cases} 0 & \text{if } x = 0 \\ |3-x| & \text{if } x \neq 0. \end{cases}$$

Then, $\sum_{n\geq 0} x^2 (3-x)^n$ converges absolutely if and only if $|3-x| < 1$ or $x = 0$.

Proposition 5.2.3 *(Necessary condition of pointwise convergence) If a series of functions $\sum_{n\geq 0} f_n(x)$ converges pointwisely on I then, the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges pointwisely on I to the zero-function . But the converse is false. In short, one can write :*

$$\sum_{n\geq 0} f_n(x) \to S \text{ (pointwise on } I\text{)} \begin{bmatrix} \Longrightarrow \\ \not\Leftarrow \end{bmatrix} f_n \to 0 \text{ (pointwise on } I\text{)}.$$

5.3 Uniform convergence

Definition 5.3.1 *We say that a series of functions* $\sum_{n\geq 0} f_n(x)$ ***converges uniformly*** *to a function* S *on* I *if its sequence of partial sums* $(S_n)_{n\in\mathbb{N}}$ *converges uniformly to the function* $S=\sum_{n=0}^{\infty} f_n(x)$ *on* I.

That is to say the sequence of real (or complex) numbers of general term $\sup_{x\in I}\left|\sum_{p=0}^{p=n} f_p(x) - S(x)\right|$ *converges to* 0.

Example 5.3.2 *In the last example, we have*

$$\forall x\in\,]2,4[\,\cup\{0\}\,, |S_n(x)-S(x)| = \frac{x^2}{x-2}\,|3-x|^{n+1}$$

Let r *be a real number such that* $0<r<1$, *then for all* $x\in[3-r,3+r]$ $(\subset\,]2,4[)$ *we get* $|S_n(x)-S(x)|\leq \frac{(3-r)^2}{1-r}r^{n+1}$. *So*

$$0\leq \sup_{x\in[3-r,3+r]} |S_n(x)-S(x)| \leq \frac{(3-r)^2}{1-r}r^{n+1}$$

.

Thus, $\lim_{n\to\infty}\sup_{x\in[3-r,3+r]}|S_n(x)-S(x)|=0$ *because* $\frac{(3-r)^2}{1-r}r^{n+1}\to 0$ *as* $n\to\infty$.

Conclusion : $\sum_{n\geq 0} x^2(3-x)^n$ *converges uniformly to* $S(x)=\frac{x^2}{x-2}$ *on* $[3-r,3+r]\cup\{0\}$ *with* $0<r<1$.

5.3.1 Necessary condition of the uniform convergence

Proposition 5.3.3 *1) If a series of functions* $\sum_{n\geq 0} f_n(x)$ *converges uniformly on* I *then, it converges pointwisely on* I *i.e.*

$\sum_{n\geq 0} f_n(x)\to S$ *(uniformly on* I *)* $\implies$ $.\sum_{n\geq 0} f_n(x)\to S$ *(pointwisely on* I *)*

2) If a series of functions $\sum_{n\geq 0} f_n(x)$ *converges uniformly on* I *then, the sequence of functions* $(f_n)_{n\in\mathbb{N}}$ *converges uniformly on* I *to the zero function. i.e*

$\sum_{n\geq 0} f_n(x)\to S$ *(uniformly on* I *)* $\implies$ $f_n\to f\equiv 0$ *(uniformly on* I *).*

Proof. 2) It is sufficient to note that, for all $x\in I$ and all $n\geq 1$

$$|f_n(x)| = |S_n(x)-S_{n-1}(x)| \leq |S_n(x)-S(x)| + |S_{n-1}(x)-S(x)|\,,$$

As for all $x\in I$, $\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} S_{n-1}(x) = S(x)$ then, by definition of limit, for all arbitrary strictly positive ϵ, there exists $N\in\mathbb{N}$ such that for each $n\geq N$ we have

$|S_n(x)-S(x)|<\frac{\epsilon}{2}$ and $|S_{n-1}(x)-S(x)|<\frac{\epsilon}{2}$. This prove that

$$\forall\epsilon>0,\exists N\in\mathbb{N},\forall n\in\mathbb{N};\forall x\in I; n\geq N \implies |f_n(x)-0|\leq\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

Therefore, $f_n\to f\equiv 0$ (uniformly on I). ■

Example 5.3.4 *Let* $\sum_{n\geq 0} f_n(x)$ *be the series of functions defined for all* $x\in\mathbb{R}_+$ *by* $f_n(x) = n\sqrt{n}x^3e^{-x\sqrt{n}}$.

Proof. It's clear that $\lim_{n\to\infty} f_n(x) = 0$ so $f_n \to 0$ (pointwisely on $\mathbb{R}_+$). Let's study the uniform convergence of the sequence $(f_n)_{n\geq 1}$ .We have

$$f_n'(x) = n\sqrt{n}x^2\left(3 - x\sqrt{n}\right)e^{-x\sqrt{n}},$$

$f_n'(x) = 0 \Longleftrightarrow x = 0$ or $x = \frac{3}{\sqrt{n}}$, $f_n'(x) > 0 \Longleftrightarrow 0 < x < \frac{3}{\sqrt{n}}$,

$f_n'(x) < 0 \Longleftrightarrow x > \frac{3}{\sqrt{n}}$ hence, $\sup_{x\geq 0}|f_n(x)| = f_n\left(\frac{3}{\sqrt{n}}\right) = \frac{27}{e^3} \nrightarrow 0$ as $n \to \infty$, which show that the sequence of functions $(f_n)_{n\geq 1}$ does not converge uniformly to the zero function on $\mathbb{R}_+$. Thus, the series of functions $\sum_{n\geq 0} f_n(x)$ does not converge uniformly on $\mathbb{R}_+$. ■

Sometimes it is not easy to calculate explicitly the pointwise limit $S(x)$ of a series of functions, then to study the uniform convergence of such series we can use the following Theorem.

Theorem 5.3.5 ***(Cauchy criterion)*** *The series of functions* $\sum_{n\geq 0} f_n(x)$ *converges uniformly on* I *if and only if*

$$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}, \forall m, n \in \mathbb{N}; m \geq n \geq N(\epsilon) \implies \sup_{x\in I}\left|\sum_{i=n+1}^{i=m} f_i(s)\right| < \epsilon.$$

Theorem 5.3.6 ***(Abel's theorem for uniform convergence)*** *Let* $(v_n)_{n\in\mathbb{N}}$ $(a_n)_{n\in\mathbb{N}}$ *be two sequences of real (or complex) valued functions on* I *checking*

- $\exists M > 0, \forall n \in \mathbb{N}, \forall x \in I; \left|\sum_{i=0}^{i=n} v_i(s)\right| \leq M$ *(that's the partial sums of the series* $\sum_{n\geq 0} v_n(x)$ *are uniformly bounded).*
- *the series of non-negative real numbers* $\sum_{n\geq 0}\sup_{t\in I}|a_{n+1}(t) - a_n(t)|$ *converges,*
- *the sequence of functions* $(a_n)_{n\in\mathbb{N}}$ *converges uniformly to zero function on* I

Then, the series of functions $\sum_{n\geq 0} a_n(x)\, v_n(x)$ *converges uniformly on* I.

Example 5.3.7 *Let* $\sum_{n\geq 1} f_n(x)$ *be the series of functions defined for all* $x \geq 1$ *by* $f_n(x) = \frac{e^{-nx}}{n}$. *We may write* $f_n(x) = a_n(x)\, v_n(x)$ *where* $a_n(x) = \frac{1}{n}$, $v_n(x) = e^{-nx}$. *For all* $x \geq 1$ *and all* $n \in \mathbb{N}^*$, *we have*

$$\left|\sum_{p=0}^{p=n} v_p(s)\right| = \left|\sum_{p=0}^{p=n} e^{-px}\right| = \left|\frac{1 - e^{-(n+1)x}}{1 - e^{-x}}\right| \leq \frac{1}{1 - e^{-x}} \leq \frac{1}{1 - e^{-1}} \simeq 1.582\,0,$$

because the function $x \longmapsto \frac{1}{1-e^{-x}}$ *is decreasing on* $[1, +\infty[$.

On the other hand, $\sup_{t\in I}|a_{n+1}(t) - a_n(t)| = \sup_{t\in I}\left|\frac{1}{n+1} - \frac{1}{n}\right| = \left(\frac{1}{n} - \frac{1}{n+1}\right)$ *then,* $\sum_{n\geq 1}\sup_{t\in I}|a_{n+1}(t) - a_n(t)|$ *is a convergent telecopic real numbers series whose sum is* 1.

Finally, we have $a_n \to 0$ *(uniformly) because* a_n *does not depend to* x. *Thus, we conclude that* $\sum_{n\geq 1} f_n(x)$ *converges uniformly on* $[1, +\infty[$.

Theorem 5.3.8 *Let* $(v_n)_{n\in\mathbb{N}}$ $(a_n)_{n\in\mathbb{N}}$ *be two sequences of real (or complex) valued functions on* I *satisfying*

- $\exists M > 0, \forall n \in \mathbb{N}, \forall x \in I; \left|\sum_{i=0}^{i=n} v_i(s)\right| \leq M$ *(that's the partial sums of the series* $\sum_{n\geq 0} v_n(x)$ *are uniformly bounded).*
- *for all* $x \in I$*, the sequence* $(a_n(x))_{n\in\mathbb{N}}$ *is monotone*
- *the sequence of functions* $(a_n)_{n\in\mathbb{N}}$ *converges uniformly to zero function on* I

Then, the series of functions $\sum_{n\geq 0} a_n(x) v_n(x)$ converges uniformly on I.

(This criterion is used especially to prove uniform convergence when the series is not absolute convergent

Example 5.3.9 *Consider the same series of function* $\sum_{n\geq 1} \frac{e^{-nx}}{n}$.

It's clear that $(a_n)_{n\in\mathbb{N}}$ *is a monotone decreasing sequence because for all* $n \geq 1, a_{n+1} - a_n = -\frac{1}{n(n+1)} < 0$.

Then, by this theorem the series $\sum_{n\geq 1} \frac{e^{-nx}}{n}$ *converges uniformly.*

Theorem 5.3.10 ***(Weiertrass's M-test)*** *If there exists a convergent series* $\sum_{n\geq 0} M_n$ *of positive numbers such that*

$$\forall x \in I, \forall n \in \mathbb{N}; |f_n(x)| \leq M_n$$

then, the series of functions $\sum_{n\geq 0} f_n(x)$*will converge uniformly (and absolutely) on* I

Proof. Since $\sum_{n\geq 0} M_n$ is convergent, therefore by Cauchy criterion for convergence of series, for all $\epsilon > 0$, there exists and integer positive N such that for each $p \in \mathbb{N}$ and $n \geq N$; $\left|\sum_{i=n+1}^{i=n+p} M_i\right| < \epsilon$. As $M_i \geq 0$ for all $i \in \mathbb{N}$ hence, for all $x \in I$ and all $n \geq N, p \in \mathbb{N}$ we have

$$\left|\sum_{i=n+1}^{i=n+p} f_i(x)\right| \leq \sum_{i=n+1}^{i=n+p} |f_i(x)| \leq \sum_{i=n+1}^{i=n+p} M_i \leq \left|\sum_{i=n+1}^{i=n+p} M_i\right| < \epsilon. \tag{5.1}$$

This gives that$\sum_{n\geq 0} f_n(x)$ is uniformly convergent on I. Also from (**??**), $\sum_{n\geq 0} |f_n(x)|$ is uniformly thus pointwisely convergent which allow us to conclude that $\sum_{n\geq 0} f_n(x)$ is absolutely convergent. ■

5.4 Normal convergence

According to the above, to prove the uniform convergence of a series of functions, we need the explicit expressions of the general term of the associated sequence of partial sums $S_n(x)$ as well as of its sum $S(x)$, which is not available in many cases. In order to overcome this difficulty, we will study the convergence of a numerical series extracted from the given series of functions, which allows us to study the uniform convergence via the general term of the new series.

Definition 5.4.1 *Let $f \in F(I, \mathbb{k})$ be a real (or complex) valued function.* $\sup_{x\in I} |f(x)|$ *is called norme of f and denoted by $\|f\|$.*

Definition 5.4.2 *We say that a series of functions $\sum_{n\geq 0} f_n(x)$ is* ***normally convergent*** *on I if the series of non- negative real numbers $\sum_{n\geq 0} \|f_n\|$ converges.*

Example 5.4.3 *The series $\sum_{n\geq 1} \frac{\cos(n\sqrt{x})}{n\sqrt{n}}$ is normally convergent on $\mathbb{R}_+$ because:*

$\forall x \in \mathbb{R}_+, \forall n \in \mathbb{N}^*$; $|f_n(x)| = \left|\frac{\cos(n\sqrt{x})}{n\sqrt{n}}\right| \leq \frac{1}{n^{\frac{3}{2}}}$ and since $f_n(0) = \frac{1}{n^{\frac{3}{2}}}$ then, $\|f_n\| = \sup_{x\in\mathbb{R}_+} |f_n(x)| = \frac{1}{n^{\frac{3}{2}}}$.

The numerical series $\sum_{n\geq 1} \|f_n\| = \sum_{n\geq 1} \frac{1}{n^{\frac{3}{2}}}$ is a series of Riemann ($\alpha-$series) with $\alpha = \frac{3}{2} > 1$, thus it's convergent.

5.4.1 Sufficient condition for normal convergence

Proposition 5.4.4 *If there exists a convergent series of positive real numbers $\sum u_n$ such that for all $x \in I$ and for all $n \in \mathbb{N}$*

$$|f_n(x)| \leq u_n, \tag{5.2}$$

then, the series of functions $\sum_{n\geq 0} f_n(x)$ is normally convergent on I.

Proof. According to the assumption (**??**), we deduce

$$\forall n \in \mathbb{N}, 0 \leq \|f_n\| = \sup_{x\in I} |f_n(x)| \leq u_n,$$

and as $\sum_{n\geq 0} u_n$ is a convergent series with positive terms, thus by the comparison test the series $\sum_{n\geq 1} \|f_n\|$ converges. Therefore, $\sum_{n\geq 0} f_n(x)$ is normally convergent on I. ■

It can be proved in another way by observing that $\sum_{n\geq 0} u_n$ fulfills the Cauchy's criterion because it is convergent and that $\sum_{i=n+1}^{i=n+p} \|f_i\| \leq \sum_{i=n+1}^{i=n+p} u_i$.

Example 5.4.5 *In the previous example, noting that $\sum_{n\geq 1} \frac{1}{n^{\frac{3}{2}}}$ converges and since*

$$\forall n \in \mathbb{N}, \forall x \in I; \left|\frac{\cos(n\sqrt{x})}{n\sqrt{n}}\right| \leq \frac{1}{n^{\frac{3}{2}}},$$

then, the series of functions $\sum_{n\geq 0} f_n(x)$ converges normally on $\mathbb{R}_+$.

5.4.2 The relationship between different types of convergence of a series of functions

see (Diagram **??**)

Proposition 5.4.6 *Let $\sum_{n\geq 0} f_n(x)$ be a series of functions on I.*

1) If the series $\sum_{n\geq 0} f_n(x)$ converges normally on I then, it converges uniformly on I.

2) If the series $\sum_{n\geq 0} f_n(x)$ converges normally on I then, it converges absolutely on I.

3) If the series $\sum_{n\geq 0} f_n(x)$ converges uniformly or absolutely on I then, it converges pointwisely on I.

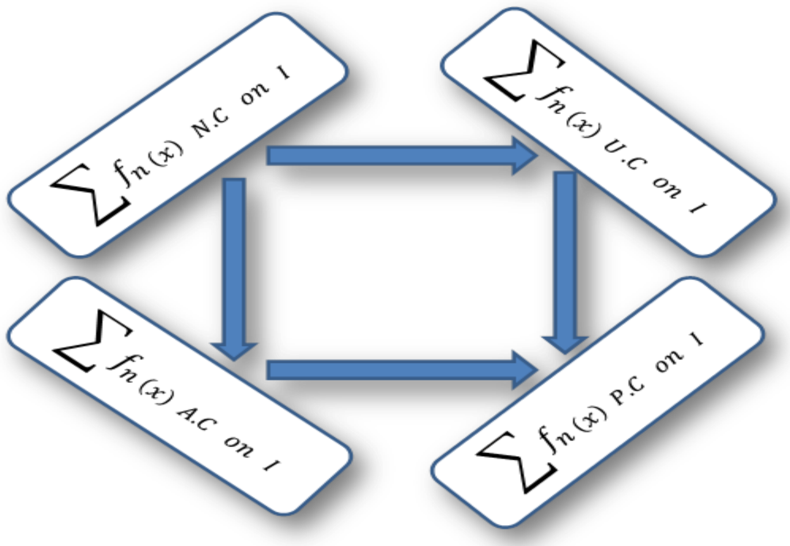

Figure 5.1: **Types of convergences**

Proof. 1) 2) For all $x \in I$ and all $n \in \mathbb{N}$, we have

$$|f_n(x)| \leq \sup_{x \in I} |f_n(x)| = M_n.$$

Suppose that $\sum_{n \geq 0} f_n(x)$ converges normally on I, hence $\sum_{n \geq 0} M_n$ is a convergent series of positive real terms. Then by Weierstrass's M-test, the series $\sum_{n \geq 0} f_n(x)$ converges uniformly and absolutely on I.

3)It is obvious. ■

Remark 5.4.7 *The implication: $\left(\sum_{n \geq 0} f_n(x)\right.$ U.C. on I. $\Longrightarrow \sum_{n \geq 0} f_n(x)$ N.C on $\left.I\right)$ is false.*

Counter example The series of functions $\sum_{n \geq 1} f_n(x)$ where $f_n(x) = \frac{(-1)^n}{x^2+n}$, converges uniformly because according to Abel's theorem, we have

$$\forall n \geq 1; \left| \sum_{i=1}^{i=n} (-1)^i \right| \leq 2.$$

For all $x \in \mathbb{R}$, the sequence $\left(\frac{1}{x^2+n}\right)_{n \geq 1}$ is decreasing and as $\sup_{x \in \mathbb{R}} \left|\frac{1}{x^2+n}\right| = \frac{1}{n} \to 0$ as $n \to \infty$, this show that $\frac{1}{x^2+n} \to 0$ uniformly. But $\sum_{n \geq 1} f_n(x)$ does not converge normally because $\sum_{n\geq 1}\sup_{x\in\mathbb{R}}\left|\frac{1}{x^2+n}\right| = \sum_{n\geq 1}\frac{1}{n}$ is the harmonic series.

5.4.3 Uniform convergence and sum of a series of functions

Let $\sum_{n\geq 0} f_n(x)$ be a uniformly convergent series of functions on an interval I of $\mathbb{R}$ and $S(x) = \sum_{i=0}^{\infty} f_n(x)$ its sum.

5.4.3.1 Series of continuous functions

Theorem 5.4.8 *If the functions f_n $n = 0, 1, 2, ...$ are continuous on I .*

Then, the sum $S(x) = \sum_{i=0}^{\infty} f_i(x)$ is a continuous function on I i.e. for all $a \in I$,

$$\lim_{x\to a} S(x) = \lim_{x\to a}\sum_{i=0}^{\infty} f_n(x) = \sum_{i=0}^{\infty}\lim_{x\to a} f_n(x) = \sum_{i=0}^{\infty} f_n(a).$$

Proof. Since $\sum_{n\geq 0} f_n(x)$ is uniformly convergent series of functions on I, the sequence of its partial sums (S_n) where $S_n(x) = \sum_{i=0}^{i=n} f_i(x)$ is thus uniformly convergent to the function S on the same domain. On the other hand, the function S_n is continuous on I for every $n \in \mathbb{N}$ as a sum of continuous functions. Then, in view of theorem **??**, S is continuous on I. ■

Example 5.4.9 *Let $\sum_{n\geq 0}\left(x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}}\right)$ be a series of functions. Take $a_i(x) = x^{\frac{1}{2i-1}}$ we can find the associated sequence of partial sums (S_n) as follow*

$S_n(x) = \sum_{i=0}^{i=n}\left(x^{\frac{1}{2i+1}} - x^{\frac{1}{2i-1}}\right) = \sum_{i=0}^{i=p}\left(a_{i+1}(x) - a_i(x)\right)$

$= a_{n+1}(x) - a_0(x) = x^{\frac{1}{2n+1}} - x.$

Then, the series converge pointwisely on $\mathbb{R}$ to the function sum

$S(x) = \lim_{n\to\infty}\left(x^{\frac{1}{2n+1}} - x\right) = \begin{cases} 1 - x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 - x & \text{if } x < 0, \end{cases}$

notice that all functions $f_n(x) = x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}}, n = 0, 1, 2, ..$ are continuous at 0 but S does not which show that the convergence of $\sum_{n\geq 0} f_n(x)$ is not uniform.

5.4.3.2 Series of integrable functions

Theorem 5.4.10 *If the functions f_n $n = 0, 1, 2, ...$ are integrable on $I = [a, b]$.*

Then, the sum $S(x) = \sum_{i=0}^{\infty} f_i(x)$ is an integrable function on I , and

$$\int_a^b \sum_{i=0}^{\infty} f_i(x)\,dx = \sum_{i=0}^{\infty}\int_a^b f_i(x)\,dx.$$

Proof. Since, the sequence (S_n) where $S_n(x) = \sum_{i=0}^{i=n} f_i(x)$ is uniformly convergent to the function S on $[a, b]$ then, the function sum S is integrable on $[a,b]$ therefore, by theorem **??** and the linearity of integral we have

$$\begin{aligned}\int_a^b \lim_{n\to\infty}\sum_{i=0}^{i=n} f_i(x)\,dx &= \lim_{n\to\infty}\int_a^b \sum_{i=0}^{i=n} f_i(x)\,dx \\ &= \lim_{n\to\infty}\sum_{i=0}^{i=n}\int_a^b f_i(x)\,dx = \sum_{i=0}^{\infty}\int_a^b f_i(x)\,dx.\end{aligned}$$

which complete the proof.

■

Example 5.4.11 *The geometric series* $\sum_{n\geq 0}(-1)^n x^n$ *is pointwise convergent to the function* $S(x) = \frac{1}{1+x}$ *on the interval* $]-1,1[$ *and uniformly convergent on all interval* $[a,b] \subset]-1,1[$. *As all functions* $x \longmapsto (-1)^n x^n, n = 0,1,2,,..$ *are continuous thus integrable on* $[a,b](\subset]-1,1[$ *). Then,* S *is integrable and*

$\int_a^b \frac{dx}{1+x} = \sum_{n=0}^{\infty}\int_a^b (-x)^n\,dx$ *i.e.* $\ln\left|\frac{1+b}{1+a}\right| = \sum_{n=0}^{\infty}(-1)^n \frac{b^{n+1}-a^{n+1}}{n+1}$.

5.4.4 Series of continuously differentiable functions

Theorem 5.4.12 *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of continuously differentiable functions on the interval* $[a,b]$ *and verify:*

- *There exists* x_0 *such that the numerical series* $\sum_{n\geq 0} f_n(x_0)$ *converges.*
- *The series of derivative functions* $\sum_{n\geq 0} f_n'(x)$ *converges uniformly on* $[a,b]$ *to a function* g.

Then, the series $\sum_{n\geq 0} f_n(x)$ *converges uniformly on* $[a,b]$ *to a the function* $S \in C^1([a,b])$ *and*

$\forall x \in [a,b], \left(\sum_{i=0}^{\infty} f_i(x)\right)' = \sum_{i=0}^{\infty} f_i'(x)$.

Proof. It suffices to notice that all the conditions of the theorem **??** are satisfied for the sequence of partial sums. ■

5.5 Exercises

Exercise 01: Test the following sequences of functions for pointwise then uniform convergence

1) $f_n(x) = \ln\left(x + \frac{1}{n}\right),\ x \in \mathbb{R}_+, n > 0$
2) $f_n(x) = \frac{x}{n(1+x^n)}, x \in [1,+\infty[, n \geq 1$
3) $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}, x \in [0,2\pi[, n \geq 1$
4) $f_n(x) = \frac{x}{n+x}, x \in [0,+\infty[,\ n \geq 2,$
5) $f_n(x) = \frac{nx}{1+nx^2}, x \in I$ (arbitrary interval) $n \in \mathbb{N}$
6) $f_n(x) = \begin{cases} \left(1-\frac{x}{n}\right)^n & \text{if } x \in [0,n[\\ 0 & \text{if x} \geq n \end{cases}$

Exercise 02 : Let $f_n : \mathbb{R} \to \mathbb{R}$ be the sequence of functions defined by $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$.

Show that each f_n, $n = 1,2,3,...$ is of class $C^1(\mathbb{R})$ and that the sequence $(f_n)_{n\geq 1}$ converges uniformly on $\mathbb{R}$ to a function f which is not of class $C^1(\mathbb{R})$.

Exercise 03

1) Show that the sequence of functions $f_n(x) = x(1 + n^\alpha e^{-nx})$ defined on $\mathbb{R}_+$ with $\alpha \in \mathbb{R}, n \in \mathbb{N}$ converges pointwisely to a function f to be determined.

2) Determine the values of α for which there is uniform convergence.

3) Calculate $\lim_{n\to\infty} \int_0^1 x(1 + \sqrt{n}e^{-nx})\,dx$.

Exercise 04 Test the following series of functions for pointwise, absolute, normal and uniform convergence

1) $\sum_{n\geq 1} \frac{\sin(nx)}{x^2+n^2}$ on $\mathbb{R}$ 2) $\sum_{n\geq 1} \frac{(-1)^n}{nx+(-1)^n}$ on $[2, +\infty[$

3) $\sum_{n\geq 1} \frac{(-1)^n e^{-nx^2}}{n^2+1}$ on $\mathbb{R}$

Exercise 05 For all $x \in [0, 1]$ and all $n \in \mathbb{N}$, put

$$u_0(x) = 1, u_{n+1}(x) = \int_0^x u_n\left(t - t^2\right) dt.$$

Test the pointwise and the uniform convergence of the series of functions $\sum u_n(x)$, $\sum u_n'(x)$.

Exercise 06 Let be the sequence of function (f_n) defined on $[0, 1]$ by

$$f_n(x) = \frac{n\left(x^3 + x\right)e^{-x}}{nx + 1}.$$

1) Show that (f_n) converges pointwisely on $[0, 1]$ to a function f to be determined.

2) Prove that (f_n) converges uniformly on all interval $[a, 1]$ where $0 < a < 1$.

2) Does (f_n) uniformly convergent on $[0, 1]$?.

Exercise 07 Let be the sequence of function (f_n) such that

$$f_n(x) = \frac{e^{-nx}}{n^2 + x^2}$$

.

1) Determine D the domain of convergence of the series $\sum f_n(x)$.

2) Prove that this series of functions is uniformly convergent on D.

3) Conclude that its sum $S(x) = \sum_{i=0}^{\infty} f_i(x)$ is a continuous on D.

4) Prove that S is differentiable on $[\alpha, +\infty[$ with $\alpha > 0$.

Part C

Power series and Fourier series

Chapter 6

Power series

Definition 6.0.1 *All series of functions of the form*

$$\sum_{n\geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + ...$$

is called a ***power series*** *centred at* $x = 0$*, where* $a_0, a_1, a_2, ...a_n, ..$ *are real (or complex) constants called coefficients of the series and* x *is a real or complex variable.*

In general, A series of the form

$$\sum_{n\geq 0} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + ...$$

is a power series centred at $x = x_0$*, where* x_0 *is given.*

Remark 6.0.2 *To make this definition precise, we provide that* $x^0 = 1$ *and* $(x - x_0)^0 = 1$ *even when* $x = 0$ *and* $x = x_0$ *respectively.*

Example 6.0.3 *The series*
$\sum_{n\geq 0} nx^n = x + 2x^2 + 3x^3 + ...$ *and* $\sum_{n\geq 0} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$
are both power series centred at $x = 0$*. the series*

$$\sum_{n\geq 1} \frac{(x-1)^n}{n^2+1} = 1 + \frac{1}{2}(x-1) + \frac{1}{5}(x-1)^2 + ...$$

is a power series centred at $x = 1$*.*

6.1 Convergence of a power series

Since the terms in a power series involve a variable x the series may converge for certain values of x and diverge for other values . For a power series centered at $x = x_0$, the value of the series at x_0 is given by a_0. Therefore, a power series always converges at its center. Some power series converge only at that value. Most power series, however, converge for more than one value of x . In that case, the power series either converges for all value x from $\Bbbk\,(= \mathbb{R}$ or $\mathbb{C})$. or converges for all x in the disk $B_R(x_0) = \{x \in \Bbbk; |x - x_0| < R\}$ where R is a positive real number. For example, the geometric series $\sum_{n\geq 0} x^n$, $x \in \mathbb{R}$ converges for all $x \in]-1, 1[$, but diverges for all x outside this interval.

Remark 6.1.1 *In what follows, we will concentrate our study on the power series centered at $x = 0$, because for a series centered at a value x_0 other than zero, all results follow by letting $z = x - x_0$ and considering $\sum_{n\geq 0} a_n z^n$ (series power centered at $z = 0$).*

Definition 6.1.2 *The set D of all values of x for which the power series $\sum_{n\geq 0} a_n x^n$ converges is called the domain of convergence of this series i.e.*

$$D = \left\{ x \in \mathbb{k};\ \sum_{n\geq 0} a_n x^n \text{ converges} \right\},$$

D is non- empty set because $0 \in D$.

Example 6.1.3 *1) $\sum_{n\geq 0} x^n$ is a power series. Its domain of convergence is $D = B_1(0) = \{x \in \mathbb{k};\ |x| < 1\}$ (open disk with center 0 and radius 1). If $\mathbb{k} = \mathbb{R}$, $D =]-1, 1[$.*

This series converges to the function $x \longmapsto \frac{1}{1-x}$ on D.

2) $\sum_{n\geq 0} \frac{x^n}{n!}$ is a power series with coefficients $a_n = \frac{1}{n!}, n = 0, 1, 2, ...$ By using D'alembert's racio test of convergence, we get for $x \neq 0$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \right| \div \left| \frac{x^n}{n!} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} \left| \frac{x^{n+1}}{x^n} \right| = \lim_{n\to\infty} \frac{1}{n+1} |x| = 0 < 1,$$

so this series is absolutely convergent therefore, it's convergent pointwisely on $\mathbb{k}$ and its sum is

$$\forall x \in \mathbb{k}; S(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

For the sequel we only consider the real power series, that is to say the series of the form $\sum_{n\geq 0} a_n x^n$ where (a_n) is a sequence of real numbers and x is a real variable.

Theorem 6.1.4 ***(Abel's theorem)*** *Let $\sum_{n\geq 0} a_n x^n$ be a real power series.*

If there exists a real number $\alpha \neq 0$ such that the numerical sequence $(a_n \alpha^n)_n$ is bounded, then the power series $\sum_{n\geq 0} a_n x^n$ converges absolutely for every real number x satisfying $|x| < |\alpha|$.

Proof. By hypothesis, the sequence $(a_n \alpha^n)_n$ is bounded means that

$$\exists M > 0, \forall n \in \mathbb{N}; |a_n \alpha^n| \leq M,$$

so, $|a_n| \leq \frac{M}{|\alpha|^n}$. Then , for all $x \in \mathbb{R}$ such that $|x| < |\alpha|$ we have

$$0 \leq |\ a_n x^n| = |a_n| |x|^n \leq \frac{M}{|\alpha|^n} |x|^n = M \left| \frac{x}{\alpha} \right|^n.$$

As $r = \left|\frac{x}{\alpha}\right| < 1$, so the geometric series $\sum_{n\geq 0} \left|\frac{x}{\alpha}\right|^n$ converges hence, by comparison test $\sum_{n\geq 0} a_n x^n$ is absolutely convergent. ■

Corollary 6.1.5 *If there exists a real number $\alpha \neq 0$ such that the numerical series $\sum_{n\geq 0} a_n \alpha^n$ is convergent then, the power series $\sum_{n\geq 0} a_n x^n$ converges absolutely for every real number x satisfying $|x| < |\alpha|$.*

Proof. By the necessary condition of convergence we have

$\sum_{n\geq 0} a_n\alpha^n$ converges $\Longrightarrow$ $(a_n\alpha^n)_n$ converges to 0 $\Longrightarrow$ $(a_n\alpha^n)_n$ is bounded,

then, it is sufficient to apply Abel's theorem. ■

Corollary 6.1.6 *If there exists a real number $\alpha \neq 0$ such that the numerical series $\sum_{n\geq 0} a_n\alpha^n$ is divergent then, the power series $\sum_{n\geq 0} a_n x^n$ diverges for every real number x satisfying $|x| > |\alpha|$.*

Proof. Let x_1 be a real number such that $|x_1| > |\alpha|$. Suppose that $\sum_{n\geq 0} a_n x_1^n$ is convergent so, according to the previous corollary, the power series $\sum_{n\geq 0} a_n\alpha^n$ is convergent as $|\alpha| < |x_1|$, which contradict the fact that this series diverges. ■

6.1.1 Radius of convergence

Definition 6.1.7 *Let $\sum_{n\geq 0} a_n x^n$ be a power series. If there exists a real number $R > 0$ such that the series converges for all x satisfying $|x| < R$ and diverges for all x satisfying $|x| > R$ then, R is called the radius of convergence. of this series. In paticular,*

If the series converges only at $x = 0$, we say the radius of convergence is $R = 0$. If the series converges for all real numbers x, we say the radius of convergence is $R = \infty$.

In general , the radius of convergence of the power series $\sum_{n\geq 0} a_n x^n$ is:

$$R = \sup\left\{\alpha \in \mathbb{R}_+ ; \sum_{n\geq 0} a_n\alpha^n \text{ is convergent}\right\}.$$

Example 6.1.8 *1) The radius of convergence ot the series $\sum_{\geq n0} x^n$ is $R = 1$ because it converges when $|x| < 1$ and diverges otherwise. But for the series $\sum_{n\geq 0} \frac{x^n}{n!}$ its radius of convergence is $R = +\infty$ because it converges anywhere.*

2) To check the convergence of the power series $\sum_{\geq n1} \frac{3^n}{n}(x-2)^n$ centred at $x = 2$, apply Cauchy's root test, we have

$$\lim_{n\to\to\infty} \sqrt[n]{\frac{1}{n}|x-2|^n} = \lim_{n\to\to\infty} 3|x-2|\sqrt[n]{\frac{1}{n}} = 3|x-2|$$

then,

$\sum_{\geq n1} \frac{1}{n}(x-2)^n$ converges absolutely if $3|x-2| < 1$ i.e. $|x-2| < \frac{1}{3}$ and diverges if $|x-2| > \frac{1}{3}$ so its radius of convergence is $R = \frac{1}{3}$.

Remark 6.1.9 *Let $\sum_{\geq n0} a_n x^n$ be a power series and R is its radius of convergence then,*

1) it is absolutely convergent thus pointwisely on $D =]-R, R[$ (domain of convergence).

2) $R = 0 \Longleftrightarrow D = \{0\}$

3) $R = +\infty \Longleftrightarrow D = \mathbb{R}$.

Solved exercise: Find the interval and the radius of convergence for the power series $\sum_{n\geq 0} \frac{(x-2)^n}{(n+1)3^n}$.

In order to apply the ratio test of D'Alembert, we have for all $x \neq 2$
$\rho = \lim_{n\to\infty} \left(\left| \frac{(x-2)^{n+1}}{(n+2)3^{n+1}} \right| \div \left| \frac{(x-2)^n}{(n+1)3^n} \right| \right) = \lim_{n\to\infty} \frac{|x-2|^{n+1}}{(n+2)3^{n+1}} \times \frac{(n+1)3^n}{|x-2|^n}$
$= \lim_{n\to\infty} \frac{n+1}{3(n+2)} |x-2| = \frac{1}{3} |x-2|$.

If $\rho < 1$ then $|x-2| < 3$ since $|x-2| < 3$ implies that $-1 < x < 5$, the series converges absolutely on the interval $]-1, 5[$ .The ratio $\rho > 1$ if x$\notin]-1,5[$. Therefore, the series diverges if $x < -1$ or $x > 5$.

The ratio test is inconclusive if $\rho = 1$. Since $\rho = 1$ is equivalent to $x = -1$ or $x = 5$. We will test theese values of x separately.

For $x = -1$, the series given by $\sum_{n\geq 0} \frac{(-1)^n}{n+1}$ is the alternating harmonic series, then converges (leibnitz's test). For $x = 5$, the series $\sum_{n\geq 0} \frac{1}{n+1}$ is the harmonic series, then it diverges. We conclude that the domain of convergence is $[-1, 5[$ and the radius of convergence is $R = 3$.

Remark 6.1.10 *If a power series converges on a finite interval, the series may or may not converge at the endpoints.*

Proposition 6.1.11 *If there exist two real numbers m and M such that*

$$\forall n \in \mathbb{N}; 0 < m \leq |a_n| \leq M$$

Then, the radius of convergence ot the power series $\sum_{n\geq 0} a_n x^n$ is $R = 1$.

Proof. By multiplying all sides of the previous inequality by α^n, we obtain: $m\alpha^n \leq a_n\alpha^n \leq M\alpha^n$, let us denote by :

$E_m = \{\alpha \in \mathbb{R}_+; (m\alpha^n)_n \text{ is bounded}\}, E = \{\alpha \in \mathbb{R}_+; (a_n\alpha^n)_n \text{ is bounded}\}$and $E_M = \{\alpha \in \mathbb{R}_+; (M\alpha^n)_n \text{ is bounded}\}$.

So, we have the following inclusions $E_M \subset E \subset E_m$ then, $\sup E_M = 1 \leq \sup E \leq \sup E_m = 1$. ■

Example 6.1.12 *By this proposition, $R = 1$ is the radius of convergence ot the series $\sum_{\geq n0} (-1)^n \frac{e^n}{e^n+1} x^n$, because $0 < \frac{1}{2} \leq |a_n| = \frac{e^n}{e^n+1} \leq 1$.*

Theorem 6.1.13 *If R is the radius of convergence of a power series $\sum_{n\geq 0} a_n x^n$ then, this series is normally convergent on each interval $[-q, q]$ where $0 \leq q < R$.*

Proof. For each $x \in [-q, q]$ we have

$$|a_n x^n| = |a_n| |x|^n \leq |a_n| q^n.$$

According to Abel's theorem $\sum_{n\geq 0} |a_n| q^n$ is a convergent because $q < R$. Then, $\sum_{n\geq 0} a_n x^n$ converges normally on $[-q, q]$. ■

Theorem 6.1.14 (*Hadamard's rule*) *If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \in \mathbb{R}_+ \cup \{+\infty\}$ then, the radius of convergence of the power series $\sum_{\geq n0} a_n x^n$ is $R = \frac{1}{L}$.*

Proof. Suppose that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, by using the ratio test of D'Alembert, we have for $x \neq 0$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x| = L|x|.$$

The series converges absolutely for each real $x \in \mathbb{R}$ satisfying $L|x| < 1$ that's $|x| < \frac{1}{L}$ and diverges for all $x \in \mathbb{R}$ such that $|x| > \frac{1}{L}$ then, $R = \frac{1}{L}$. ■

Proposition 6.1.15 *If* $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L \in \mathbb{R}_+ \cup \{+\infty\}$ *then, the radius of convergence of the power series* $\sum_{\geq n0} a_n x^n$ *is* $R = \frac{1}{L}$.

Proof. We have by root test of Cauchy,

$$\lim_{n\to\infty}\sqrt[n]{|a_n x^n|} = \lim_{n\to\infty}\sqrt[n]{|a_n||x|^n} = \lim_{n\to\infty}|x|\sqrt[n]{|a_n|} = L|x|.$$

We deduce for the power series in this case also , it's absolutely convergent if $|x| < \frac{1}{L}$, divergent if $|x| > \frac{1}{L}$, which prove that $R = \frac{1}{L}$. ■

Remark 6.1.16 *1) In the case where the two limits* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$*and* $\lim_{n\to\infty}\sqrt[n]{|a_n|}$ *do not exist, the radius of convergence of the power series* $\sum_{\geq n0} a_n x^n$ *is* $R = \frac{1}{L}$ *where*

$$L = \limsup \sqrt[n]{|a_n|}$$

2) $L = 0 \implies R = +\infty$ *and* $L = +\infty \implies R = 0.$

3)Let $\sum_{n\geq 0} a_n x^n$ *be a power series of radius of convergence R. Then its convergence domain is one of the four intervals :* $]-R,R[\,,[-R,R[\,,]-R,R]\,,[-R,R]\,.$

Example 6.1.17 *1) Consider the power series* $\sum_{n\geq 0}\frac{n!}{2^n+1}x^n$. *We have*

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+1)!}{2^{n+1}+1}\times\frac{2^n+1}{n!}\\ &= \lim_{n\to\infty}(n+1)\frac{2^n+1}{2^{n+1}+1}\\ &= \lim_{n\to\infty}(n+1)\frac{1+2^{-n}}{2+2^{-n}} = +\infty,\end{aligned}$$

by Hadamard's rule, the radius of convergence is $R = 0$, *that's to say the given series converges only for* $x = 0$ *and diverges on* $\mathbb{R}^*$.

2) $\sum_{n\geq 0}\sqrt{n}x^{3n+2}$ *is a power series because it's of the form* $\sum_{n\geq 0} a_n x^n$ *where*

$$a_n = \begin{cases} 0 & \text{if } n = 3p \text{ or } n = 3p+1 \\ \sqrt{p} & \text{if } n = 3p+2. \end{cases}$$

Let's calculate the radius of convergence . We have

$$\lim_{n\to\infty}\sqrt[n]{\left|\sqrt{n}x^{3n+2}\right|} = \lim_{n\to\infty} n^{\frac{1}{2n}}|x|^{\frac{3n+2}{n}} = \lim_{n\to\infty} e^{\frac{\ln(n)}{2n}}|x|^{3+\frac{2}{n}} = |x|^3$$

then, this series converges absolutely and thus pointwisely if $|x|^3 < 1$ *and diverges if* $|x|^3 > 1$.*Since,*

$|x|^3 < 1 \iff |x| < 1 \iff x \in]-1,1[$ *and*

$|x|^3 > 1 \iff |x| > 1 \iff x > 1$ *or* $x < 1$.

Therefore, the radius of convergence is $R = 1$. now, we need to study the cases $x = -1$ and $x = 1$.Substituting x by these values we remark that $\sum_{n\geq 0}\sqrt{n}$; $\sum_{n\geq 0}(-1)^n\sqrt{n}$ are divergent series because $\lim_{n\to\infty}\sqrt{n} \neq 0$. *Consequently, the domain of convergence is* $D = \left]-1, 1\right[$.

6.1.1.1 Radius of convergence of a sum or a product of two power series

Proposition 6.1.18 *Let $\sum_{n\geq 0} a_n x^n$ and $\sum_{n\geq 0} b_n x^n$ be two power series with radius of convergence R_a, R_b respectively, denote by R_{a+b} the radius of convergence of the power series $\sum_{n\geq 0}(a_n + b_n)x^n$.Then*
$R_{a+b} = \min(R_a, R_b)$ if $R_a \neq R_b$ and $R_{a+b} \geq R_a$ if $R_a = R_b$.

Proof. 1) Assume that $R_b < R_a$.
If $|x| < R_b$ then, $|x| < R_a$ hence, the two series $\sum_{n\geq 0} a_n x^n$ and $\sum_{n\geq 0} b_n x^n$ are absolutely convergent. As $|(a_n + b_n)x^n| \leq |a_n x^n| + |b_n x^n|$ then, the series $\sum_{n\geq 0}(a_n + b_n)x^n$ converges absolutely for $|x| < \min(R_a, R_b)$.
If $|x| > R_b$; two cases arise:
i) $R_b < |x| < R_a$, $\sum_{n\geq 0} a_n x^n$ converges absolutely and $\sum_{n\geq 0} b_n x^n$ diverges. So $\sum_{n\geq 0}(a_n + b_n)x^n$ diverges.
ii) $R_b < R_a < |x|$, the two series diverge. let's show $\sum_{n\geq 0}(a_n + b_n)x^n$ diverges also. Let's reason by the absurd. Assume that the series $\sum_{n\geq 0}(a_n + b_n)x^n$ is convergent, so by Abel's theorem, it converges absolutely for all $x_0 \in \mathbb{R}$ such that $|x_0| < |x|$ and in particular for all x_0 verifying $R_b < |x_0| < R_a$. Hence the contradiction.
2) Now, assume that $R_b = R_a$. It is clear that the series $\sum_{n\geq 0}(a_n + b_n)x^n$ converges absolutely if $|x| < R_b = R_a$; so the radius of convergence $R_{a+b} \geq R_a$. ■

Example 6.1.19 *Consider the two power series $\sum_{n\geq 0} x^n$ and $\sum_{n\geq 0}\frac{2-3^n}{3^n}x^n$.It is easy to prove that both series have a radius of convergence $R = 1$, on the other hand, the series deduced from the sum of the two series (i.e.$\sum_{n\geq 0}\frac{2}{3^n}x^n$) has 3 as radius of convergence.*

Proposition 6.1.20 *Let $\sum_{n\geq 0} a_n x^n$ and $\sum_{n\geq 0} b_n x^n$ be two power series having respectively R_a and R_b for radius of convergence . Then, their series product is a power series of general term $w_n(x) = \left(\sum_{i=0}^{\hat{}=n} a_i b_{n-i}\right)x^n$ and its radius of convergence R_{ab} checked $R_{ab} \geq \min(R_a, R_b)$.*

Example 6.1.21 *Find a power series whose sum is the function $f(x) = \frac{1}{(1-x)(1-2x)}$.*
We see that $\frac{1}{(1-x)(1-2x)} = \frac{1}{1-x}\cdot\frac{1}{1-2x}$ and each factor is the sum following power series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with radius of convergence 1and interval of convergence $\left]-1, 1\right[$, $\frac{1}{1-2x} = \sum_{n=0}^{\infty}(2x)^n$ with radius of convergence $\frac{1}{2}$ and interval of convergence $\left]-\frac{1}{2}, \frac{1}{2}\right[$ then, the radius of convergence of the series product is $R = \min\left(1, \frac{1}{2}\right) = \frac{1}{2}$. So for $x \in \left]-\frac{1}{2}, \frac{1}{2}\right[$,

$$\begin{aligned}\frac{1}{1-x}\cdot\frac{1}{1-2x} &= \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty}(2x)^n\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{\hat{}=n} 2^{n-i}\right)x^n\\ &= \sum_{n=0}^{\infty}\left(2^{n+1} - 1\right)x^n.\end{aligned}$$

Remark 6.1.22 *Let* $\sum_{n\geq 0} a_n (x - x_0)^n$ *be a series power centred at* $x = x_0$ *and* R *be its radius of convergence . Then*

1) R *is itself the radius of convergence of the power series* $\sum_{n\geq 0} a_n x^n$.
2) If $R \in \mathbb{R}_+^*$ *the series* $\sum_{n\geq 0} a_n (x - x_0)^n$ *is*

- *absolutely convergent on* $]x_0 - R, x_0 + R[$ *, divergent on* $]-\infty, -R[\cup]R, +\infty[$ *,*
- *normally convergent on all interval* $[a, b] \subset]x_0 - R, x_0 + R[$ *.*
- *The particular cases* $x = x_0 - R$*,* $x = x_0 + R$ *should be studied separately.*

6.1.1.2 Continuity and integrability of a power series sum

Theorem 6.1.23 *Let* $\sum_{n\geq 0} a_n (x - x_0)^n$ *be a power series,* R *is its radius of convergence and* $S(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ *is its sum. Then, the function* S *is continuous on* $]x_0 - R, x_0 + R[$ *and for all intervall* $[a, b] \subset]x_0 - R, x_0 + R[$ *we have,*

$$\begin{aligned} \int_a^b S(x)\,dx &= \int_a^b \sum_{n=0}^{\infty} a_n (x - x_0)^n\,dx = \sum_{n=0}^{\infty} \int_a^b a_n (x - x_0)^n\,dx \\ &= \sum_{n=0}^{\infty} a_n \left(\frac{(b - x_0)^{n+1} - (a - x_0)^{n+1}}{n + 1} \right) \end{aligned}$$

Proof. For this series the general term $f_n(x) = a_n (x - x_0)^n$ is a polynomial function for every $n \in \mathbb{N}$, that is continuous on $\mathbb{R}$Also, this series is normally and thus uniformly convergent on each interval $[a, b] \subset]x_0 - R, x_0 + R[$. Hence, it is sufficient to apply theorems **??**, **??** about continuity and integrability of the sum of uniformly convergent series of functions on a domain I to prove our mentioned. results ■

Remark 6.1.24 *The power series* $\sum_{n\geq 0} a_n (x - x_0)^n$ *and* $\sum_{n\geq 0} \frac{a_n}{n+1} (x - x_0)^{n+1}$ *have the same radius of convergence* R*.*

6.2 Differentiation of power series

In general, one cannot differentiate a uniformly convergent sequence or series. We can, however, differentiate power series, and they behaves as nicely as one can imagine in this respect. The sum of a power series

$$S(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

is infinitely differentiable inside its interval of convergence, and its derivative is given by term-by-term differentiation as follow

$$S'(x) = a_1 + 2a_2 x + 3a_3 x^2 + ...$$

To prove this, we first show that the term-by-term derivative of a power series has the same radius of convergence as the original power series. The idea is that the geometrical decay of the terms of the power series inside its radius of convergence dominates the algebraic growth of the factor n.

Theorem 6.2.1 *Suppose that R is the radius of convergence of the power series $\sum_{n\geq 0} a_n (x-x_0)^n$. Then, the power series $\sum_{n\geq 0} na_n (x-x_0)^{n-1}$ has the same radius of convergence .*

Proof. Assume without loss of generality that $x_0 = 0$, and suppose $|x| < R$. Choose κ such that $|x| < \kappa < R$, and let $r = \frac{|x|}{\kappa}$, notice that $0 < r < 1$. To estimate the terms in the differentiated power series by the terms in the original series, we rewrite their absolute values as follows

$$\left|na_n x^{n-1}\right| = \frac{n}{\kappa}\left(\frac{|x|}{\kappa}\right)^{n-1} |a_n| \kappa^n = \frac{n}{\kappa} r^{n-1} |a_n| \kappa^n,$$

The ratio test of D'Alembert shows that the series $\sum_{n\geq 1} nr^{n-1}$ converges, So the sequence $\left(nr^{n-1}\right)_{n\geq 1}$ is bounded by a positive real number M. Thus

$$\left|na_n x^{n-1}\right| \leq \frac{M}{\kappa} |a_n| \kappa^n.$$

By Abel's theorem, the series $\sum_{n\geq 0} |a_n| \kappa^n$ converges, since $\kappa < R$, so the comparison test implies that $\sum_{n\geq 0} na_n x^{n-1}$ converges absolutely.

Conversely, suppose $|x| > R$. Then $\sum_{n\geq 0} |a_n x^n|$ diverges because the series $\sum_{n\geq 0} a_n x^n$ diverges and as

$$\left|na_n x^{n-1}\right| \geq \frac{1}{|x|} |a_n x^n|, \text{ for } n \geq 1,$$

so the comparison test implies that $\sum_{n\geq 0} na_n x^{n-1}$ diverges. Thus the power series $\sum_{n\geq 0} a_n x^n$, $\sum_{n\geq 0} na_n x^{n-1}$ have the same radius of convergence R. ■

Theorem 6.2.2 *Suppose that $R > 0$ is the radius of convergence of the power series $\sum_{n\geq 0} a_n (x-x_0)^n$ and $S(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ is its sum. Then, the function S is differentiable in $]x_0 - R, x_0 + R[$ and*

$$S'(x) = \left(\sum_{n=0}^{\infty} a_n (x-x_0)^n\right)' = \sum_{n=0}^{\infty} na_n (x-x_0)^{n-1}.$$

Proof. By the previous theorem, The term-by-term differentiated power series $\sum_{n\geq 0} na_n (x-x_0)^{n-1}$converges in $]x_0 - R, x_0 + R[$. We denote its sum by

$$g(x) = \sum_{n=0}^{\infty} na_n (x-x_0)^{n-1}.$$

Let $0 < \kappa < R$, the two power series $\sum_{n\geq 0} a_n (x-x_0)^n$ and $\sum_{n\geq 0} na_n (x-x_0)^{n-1}$ converges normally an then uniformly on $[x_0 - \kappa, x_0 + \kappa]$. Applying Theorem **??** to their partial sums, we conclude that S is differentiable in $[x_0 - \kappa, x_0 + \kappa]$ and $S' = g$. Since this holds for every $0 \leq \kappa < R$, it follows that S is differentiable$]x_0 - R, x_0 + R[$ in and $S' = g$. ■

Corollary 6.2.3 *If the power series* $\sum_{n\geq 0} a_n (x - x_0)^n$ *has radius of convergence* $R > 0$ *then the function* S *defined by*

$$\forall x \in]x_0 - R, x_0 + R[\,; S(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is infinitely differentiable on $]x_0 - R, x_0 + R[$ *and*

$$\forall n \in \mathbb{N}; a_n = \frac{S^{(n)}(x_0)}{n!}$$

Proof. By applying the previous theorem to the power series

$$S(x) = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + a_3 (x - x_0)^3 + ... + a_n (x - x_0)^n + ..$$

p times, we find
$S'(x) = a_1 + 2a_2 (x - x_0) + 3a_3 (x - x_0)^2 + ...na_n (x - x_0)^{n-1} + ..$
$S''(x) = 2a_2 + 2.3a_3 (x - x_0) + ... (n-1) na_n (x - x_0)^{n-2} + ...$
$S'''(x) = 1.2.3a_3 + ... (n-2)(n-1) na_n (x - x_0)^{n-3} + ...$
.
.
$S^{(p)}(x) = 1.2.3...p.a_p + ... + (n - p + 1) .... (n-1) na_n (x - x_0)^{n-p} + ...$
$= p!a_p + ... + \frac{n!}{(n-p)!} a_n (x - x_0)^{n-p} + ...$
where all of these power series have radius of convergence R. Setting $x = x_0$ in these series, we get
$a_0 = S(x_0)$, $a_1 = S'(x_0)$, $a_2 = \frac{S''(x_0)}{2}, ..., a_p = \frac{S^{(p)}(x_0)}{p!}, ..$
which proves the result . ∎

One consequence of this result is that convergent power series with different coefficients cannot converge to the same sum.

Remark 6.2.4 *If two power series* $\sum_{n\geq 0} a_n (x - x_0)^n$, $\sum_{n\geq 0} b_n (x - x_0)^n$ *have non-zero radius of convergence and have the same sum on some neighbourhood of* x_0, *then* $a_n = b_n$ *for* $n = 0, 1, 2, 3, ...$

In fact, if the common sum in $]x_0 - \delta, x_0 + \delta[$ where $0 < \delta < R$ is S then, $a_n = \frac{S^{(n)}(x_0)}{n!} = b_n$.

Definition 6.2.5 [*Real analytic function*] *Given a real constant* x_0. *We say that a real valued function* f *is real* ***analytic*** *at* x_0 *if* f *is the sum of a power series centred at* x_0 *on some neighbourhood of* x_0 *with non-zero radius of convergence. i.e.*

there exists a sequence of real numbers $(a_n)_{n\geq 0}$ *and* $r > 0$ *such that for all* $x \in]x_0 - r, x_0 + r[$ *we have*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

6.2.1 Taylor's series

We have already proven that all function f can be represented as sum of a power series, this series must be of the form $\sum_{n\geq 0} a_n (x-x_0)^n$ .It therefore seems worthwhile to study power series having this form.

Definition 6.2.6 *If f is a real valued function whose nth derivative at $x = x_0$ exists for every $n \in \mathbb{N}$ then, the **Taylor's series** for $f(x)$ at $x = x_0$ is*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$

It is most common to deal with Taylor series at $x = x_0 = 0$, since the terms in the power series are simplest to write in this case.Such series are often called **Maclaurin series.**

Remark 6.2.7 *Since Taylor's series are special cases of power series, all of our results about power series automatically apply to them thus, for example the Taylor series for $f'(x)$ is obtained by differentiating the Taylor's series for $f(x)$ term-by-term.*

Example 6.2.8 *The Taylor's series of the function exp at $x = 0$ (Maclorin series) is*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Definition 6.2.9 *(Smooth function) A function f is **smooth** (or C^∞ or infinitely differentiable) on an intervali $I \subset \mathbb{R}$, written $f \in C^\infty(I)$ if has continuous derivatives of all order.*

Theorem 6.2.10 *Let f be a function of real variable..If f is real analytic in some neighbourhood of x_0 then, there exists $r > 0$ such that $f \in C^\infty$ $(]x_0 - r, x_0 + r[)$ i.e f is smooth on $]x_0 - r, x_0 + r[$. In this case one can write*

$$\forall x \in]x_0 - r, x_0 + r[\,; f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$

Remark 6.2.11 *The condition $f \in C^\infty$ $(]x_0 - r, x_0 + r[)$ in the previous theorem is necessary for the analycity of f at x_0 but not sufficient.*

Counter example: Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x>0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

The infinite differentiability of $f(x)$ at $x = 0$ follows from the chain rule (derivative of the composition of two differentiable functions: $\left(h\left(-\frac{1}{x}\right)\right)' = \frac{1}{x^2}.h'\left(-\frac{1}{x}\right)$). Moreover, we can easily prove by induction that its nth derivative has the form

$$f^{(n)}(x) = \begin{cases} p_n\left(\frac{1}{x}\right) e^{-\frac{1}{x}} & \text{if } x>0 \\ 0 & \text{if } x < 0 \end{cases}$$

where $p_n(\frac{1}{x})$ is a polynomial in $\frac{1}{x}$. Thus, we just have to show that f has derivatives of all orders at 0 and that these derivatives are equal to zero. It's clear that $f'(0^-) = 0$ because $f(0) = 0$ and $f(x) = 0$ for all $x < 0$.

For the right derivative we have

$$f'\left(0^+\right) = \lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x} = \lim_{h\to +\infty} h.e^{-h} = 0.$$

Since both the left and right derivatives equal zero, we have $f'(0) = 0$.

Using a proof by induction. to show that all the derivatives of f at 0 exist and are zero. Suppose that $f^{(n)}(0) = 0$, which we have verified for $n = 1$. The left derivative $f^{(n+1)}(0^-)$ is clearly zero, so we just need to prove that the right derivative is zero.

$$f^{(n+1)}\left(0^+\right) = \lim_{x\to 0^+} \frac{f^{(n)}(x)}{x} = \lim_{x\to 0^+} \frac{p_n\left(\frac{1}{x}\right) e^{-\frac{1}{x}}}{x} = \lim_{h\to +\infty} hp_n(h)\, e^{-h} \qquad = 0$$

Which prove that $f \in C^\infty(\mathbb{R})$. The Taylor's series of f at $x_0 = 0$ is $\sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^\infty 0.x^n$ which converges to 0, so its sum is different to f in any neighbourhood of 0, which mean that f is not analytic at 0.

Theorem 6.2.12 *(Sufficient condition for the analyticity) Let $f : I \to \mathbb{R}$ be a function $x_0 \in I$ (fixed) such that*

- *$f \in C^\infty(]x_0 - r, x_0 + r[)$ (i.e. f is smooth on a neighbourhood of x_0),*
- *there exists $M > 0$ such that*

$$\forall k \in \mathbb{N}, \forall x \in\,]x_0 - r, x_0 + r[\,; \left|f^{(k)}(x)\right| \le M.$$

Then, f is analytic at x_0 that's

$$\forall x \in\,]x_0 - r, x_0 + r[\,; f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

Proof. The Taylor formula of order n for the function f on a neighbourhood of x_0 is given by

$$\forall x \in\,]x_0 - r, x_0 + r[\,; f(x) = \sum_{k=0}^{k=n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + R_n(x),$$

where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ is the nth remainder. We have for all $x \in\,]x_0 - r, x_0 + r[$

$$|R_n(x)| \le M \frac{r^{n+1}}{(n+1)!}.$$

Taking $a_n(r) = M\frac{r^{n+1}}{(n+1)!}$. As $\lim_{n\to\infty} \frac{a_{n+1}(r)}{a_n(r)} = \lim_{n\to\infty} \frac{r}{n+2} = 0 < 1$ then, by ratio test of D'Alembert the series of positive real numbers $\sum a_n(r)$ converges. Therefore, $\lim_{n\to\infty} M\frac{r^{n+1}}{(n+1)!} = 0$.

Which show that $f(x)$ is the sum of Taylor's series on $]x_0 - r, x_0 + r[$. ■

Example 6.2.13 *In all examples we take* $x_0 = 0$ *and let* $]-r, r[$ *be a neighbourhood of zero where* $r > 0$.

1) $f(x) = \sin x$. *By induction , we can prove that the nth derivative of* f *is* $f^{(n)}(x) = \sin\left(x + n\frac{\pi}{2}\right)$. *It's clear that* $f \in C^{\infty}(\mathbb{R})$ *and for all* $x \in \ ]-r, r[$ *we get* $\left|f^{(n)}(x)\right| \leq 1$. *According to the above theorem,, one can write*

$$\forall x \in \mathbb{R}; \sin x = \sum_{n=0}^{\infty} \frac{\sin\left(n\frac{\pi}{2}\right)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

2) $f(x) = \frac{1}{1-x}$. *It's well known that* $f(x)$ *is the sum of the geometric power series* $\sum_{n=0}^{\infty} x^n$ *on its domain of convergence* $]-1, 1[$ *i.e*

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + .$$

Solved exercise: Find a function f represented the sum of the following infinite power series and state the domain in every case

$\sum_{n\geq 0} 7x^{3n}$, $\sum_{n\geq 0} 2\left(\frac{1}{3}(x-1)\right)^n$.

The first series $\sum_{n\geq 0} 7x^{3n} = \sum_{n\geq 0} 7\left(x^3\right)^n$ is a geometric series. It converges if $|x|^3 < 1$, which means that its interval of convergence is $D =]-1, 1[$. In this interval, the sum $\sum_{n=0}^{\infty} 7x^{3n} = \frac{7}{1-x^3}$.

Therefore, $f(x) = \frac{7}{1-x^3}$ in the domain $D =]-1, 1[$. The series does not represent this function outside of this domain because it diverges.

The second series $\sum_{n\geq 0} 2\left(\frac{1}{3}(x-1)\right)^n$ is also a geometric series which converges if $\frac{1}{3}|x-1| < 1$ then, its domain of convergence is $D' =]-2, 4[$ .Within $]-2, 4[$ the sum

$\sum_{n=0}^{\infty} 2\left(\frac{1}{3}(x-1)\right)^n . = \frac{2}{1-\frac{1}{3}(x-1)} = \frac{6}{4-x} = g(x)$. Consequently, $g(x) = \frac{6}{4-x}$ in $]-2, 4[$.

Listed below are some common functions, their power series forms, and convergence interval.

Power Series	Interval of Convergence
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$	$]-1, +1[$
$\sin x. = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$	$]-\infty, +\infty[$
$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$	$]-\infty, +\infty[$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$	$]-\infty, +\infty[$
$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$	$]-1, +1]$
$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)! x^{2n+1}}{2^{2n}(n!)^2(2n+1)} = x + \frac{x^3}{6} + \frac{3}{40}x^5 + ...$	$[-1, +1]$
$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$	$]-1, +1[$
$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + ...$	$]-\infty, +\infty[$
$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...$	$]-\infty, +\infty[$

Proposition 6.2.14 *Suppose that* f *and* g *are two analytic functions at* x_0 *i.e there exist two sequences of real numbers* (a_n) , (b_n) *and* $r > 0$ *such that for all* $x \in]x_0 - r, x_0 + r[$,

$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n$. *Then,*

1) the function $f + g$ *is analytic at* x_0 *and for all* $x \in]x_0 - r, x_0 + r[$

$$(f + g)(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n ,$$

2) he function fg *is analytic at* x_0 *and for all* $x \in]x_0 - r, x_0 + r[$ *,we have*

$$(fg)(x) = \sum_{n=0}^{\infty} \sum_{p=0}^{p=n} a_p b_{n-p} (x - x_0)^n .$$

6.3 Exercises

Exercise 1 For each of the following power series, find the radius of convergence and the interval of convergence

1) $\sum_{n\geq 0} \frac{n^2+n}{5^n} x^n$ 2) $\sum_{n\geq 1} \frac{\ln n}{n^2} x^n$ 3) $\sum_{n\geq 1} \sin\left(\frac{1}{n}\right) x^n$

4) $\sum_{n\geq 0} \cosh(n) x^{2n}$ 5) $\sum_{n\geq 1} (-1)^n n^2 x^n$ 6) $\sum_{n\geq 1} \frac{2^n}{n^2} (x - 3)^n$

7) $\sum_{n\geq 1} (-1)^n \frac{10^n}{n!} (x - 10)^n$ 8) $\sum_{n\geq 1} \frac{n}{4^n (n+1)} x^{2n}$

Exercise 2 Calculate the sums of the following power series each in their open interval of convergence after determining their convergence radius

1) $\sum_{n\geq 2} \frac{x^n}{n(n-1)}$ 2) $\sum_{n\geq 0} \frac{n^2+4n-1}{(n+2)n!} x^{3n}$ (Hint: $\forall x \in \mathbb{R}, \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$)

Exercise 3 Use a known series to find a power series in x that has the given function as its sum:

1) $f(x) = x.\sin\left(x^2\right)$ 2) $f(x) = \frac{\ln\left(1+x^3\right)}{x}$ 3) $f(x) = \frac{x-\arctan(x)}{x^3}$.

Exercise 4 In every case from the following, determine the Taylor series for the function f around the point $x = x_0$ and find its radius of convergence

1) $f(x) = \sin x$, $x_0 = \pi$ 2) $f(x) = \ln x$, $x_0 = 1$ 3) $f(x) = \frac{1}{1+x}$, $x_0 = 0$

4) $f(x) = \frac{x}{4-x^2}$, $x_0 = 0$ 5) $f(x) = \frac{1+2x}{1+x+x^2}$, $x_0 = 0$

6) $f(x) = \ln\sqrt{1 + x + x^2} =$, $x_0 = 0$.

Exercise 5 **1)** determine the interval of convergence of the series

$$\sum_{n\geq 0} \frac{x^{4n+1}}{(4n+1)!}.$$

2) Write the function f such that $f(x) = \sin x + \sinh x$ as a sum of a power series on $\mathbb{R}$.

3) Deduce the sum $\sum_{n=0}^{\infty} \frac{x^{4n+1}}{(4n+1)!}$.

Exercise 6 Consider the power series $\sum_{n\geq 0} \frac{x^{4n}}{(4n)!}$.

1) Determine the radius of convergence R and the domain of convergence D of this series.

2) For all $x \in D$ denote by $S(x) = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!}$ the sum of the power series.Show S is a solution of the differential equations

$y - y'' = \cos x$ and $y + y'' = \cosh x$. Deduce the expression of $S(x)$ without the sign $\sum$.

3) Calculate the sums of the following numerical series : $\sum_{n\geq 0} \frac{1}{(4n)!}$, $\sum_{n\geq 1} \frac{4^n}{(4n-1)!}$ and $\sum_{n\geq 1} \frac{n-1}{(4n-3)!}$.

Chapter 7

Fourier series

7.1 Some important definitions

Definition 7.1.1 *(Periodic function) Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function.*
f *is said to be* ***periodic*** *if there exists* $T > 0$ *such that*

$$f(x+T) = f(x), \quad \text{for all } x \in \mathbb{R}. \tag{7.1}$$

i.e. $f(x) = f(x+T) = f(x+2T) = f(x+3T) = ...$
T *and any other multiple of* T *are called periods of* f. *The smallest positive one satisfied* **??** *is called the period of* f.

Example 7.1.2 *1)* $\sin$ *is a periodic function with period* $T = 2\pi$ *because* $\sin(x+2\pi) = \sin x$, *for all* $x \in \mathbb{R}$.
Similarly, the functions : $\cos$, $\sec$, $\cos ec$, *are periodic of period* $T = 2\pi$.
2) $\tan$, $\cot$, *are also periodic with* $T = \pi$.

Consequences
Suppose that f is periodic function with period T and a is non-zero real constant then,
1) the function f_a defined by $f_a(x) = f(ax)$, for all $x \in \mathbb{R}$ is also periodic and its period is: $\frac{T}{a}$.
2) If further f is integrable we get $\forall c \in \mathbb{R}; \int_c^{c+T} f(t)\,dt = \int_0^T f(t)\,dt$.

Proof. 1) We have for any $x \in \mathbb{R}$
$f_a\left(x + \frac{T}{a}\right) = f\left(a\left(x + \frac{T}{a}\right)\right) = f(ax+T) = f(ax) = f_a(x)$
because T is the period of f.
2) For an arbitrary $c \in \mathbb{R}$, we have $\int_c^{c+T} f(t)\,dt = \int_c^0 f(t)\,dt + \int_0^T f(t)\,dt + \int_T^{c+T} f(t)\,dt$
Take the auxilary variable $z = t - T$ in the integral $\int_T^{c+T} f(t)\,dt$, we get

$$\int_T^{c+T} f(t)\,dt = \int_0^c f(z+T)\,dz = \int_0^c f(z)\,dz$$

Then, $\int_c^{c+T} f(t)\,dt = \int_c^0 f(t)\,dt + \int_0^T f(t)\,dt + \int_0^c f(t)\,dt = \int_0^T f(t)\,dt$. ■

Definition 7.1.3 *(Odd and even function) Let* $f : D \subseteq \mathbb{R} \to \mathbb{R}$ *be a function where* D *is a symmetric subset of* $\mathbb{R}$ *(i.e. if* $x \in D$ *then* $-x \in \mathbb{R}$*).*
1) f *is said to be* ***odd*** *function if* $f(-x) = -f(x)$ *for all* $x \in D$.
2) f *is said to be* ***even*** *function if* $f(-x) = f(x)$ *for all* $x \in D$.

The graph of an odd function is symmetric with respect to the origin , while the graph of an even function is symmetric with respect to $y-$ axis.

Note: Let $f, g; D \subset \mathbb{R} \to \mathbb{R}$ be two functions.

1) f, g are odd $\Longrightarrow$ $f.g$ is an even function,

2) f, g are even $\Longrightarrow$ $f.g$ is an even function,

3) f is even and g is odd $\Longrightarrow$ $f.g$ is an odd function.

If f is an integrable function on $[-a, a]$ where $0 < a \leq +\infty$ then,

$$\int_{-a}^{a} f(x)\, dx = \begin{cases} 0; & \text{if } f \text{ is an odd function} \\ 2\int_0^a f(x)\, dx; & \text{if } f \text{ is an even function} \end{cases}$$

7.2 Trigonometric series -Fourier's series

Definition 7.2.1 *Let $(a_n)_{n\in\mathbb{N}}$ $(b_n)_{n\in\mathbb{N}}$ be two series of real numbers and $l > 0$. Any series of function of the form*

$$\frac{a_0}{2} + \sum_{n\geq 1} \left[a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right) \right], \tag{7.2}$$

is called trigonometric series or Fourier's series.

Remark 7.2.2 *If the two numerical series $\sum_{n\geq 0} a_n$ and $\sum_{n\geq 0} b_n$ are absolutely convergent, then the trigonometric series $\frac{a_0}{2} + \sum_{n\geq 1} \left[a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right) \right]$ is normally convergent.*

Because, in this case the series $\sum_{n\geq 0} |a_n| + |b_n|$ converges and

$$\sup_{x\in\mathbb{R}} \left| a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right) \right| \leq |a_n| + |b_n|.$$

Another form of a trigonometric series:

One can write the trigonometric series **??** in the following new form

$$\sum_{n\in\mathbb{Z}} c_n e^{i\frac{n\pi}{l}x},$$

where $c_0 = \frac{a_0}{2}$ and for all $n \geq 1$; $c_{-n} = \frac{a_n + ib_n}{2}$, $c_n = \frac{a_n - ib_n}{2}$

Proof. We know that , $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$ thus, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ by substitution in **??**, we obtain for $n \geq 1$.

$$\begin{aligned} a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right) &= a_n \left(\frac{e^{i\frac{n\pi}{l}x} + e^{-i\frac{n\pi}{l}x}}{2} \right) + b_n \left(\frac{e^{i\frac{n\pi}{l}x} - e^{-i\frac{n\pi}{l}x}}{2i} \right) \\ &= a_n \left(\frac{e^{i\frac{n\pi}{l}x} + e^{-i\frac{n\pi}{l}x}}{2} \right) + b_n \left(\frac{-ie^{i\frac{n\pi}{l}x} + ie^{-i\frac{n\pi}{l}x}}{2} \right) \\ &= \left(\frac{a_n - ib_n}{2} \right) e^{i\frac{n\pi}{l}x} + \left(\frac{a_n + ib_n}{2} \right) e^{-i\frac{n\pi}{l}x} \\ &= c_n e^{i\frac{n\pi}{l}x} + c_{-n} e^{-i\frac{n\pi}{l}x}. \end{aligned}$$

Which complete the proof. ■

Definition 7.2.3 *The sequence of functions* $1, \cos\left(\frac{\pi}{l}x\right), \sin\left(\frac{\pi}{l}x\right), \cos\left(\frac{2\pi}{l}x\right)$, $\sin\left(\frac{2\pi}{l}x\right), \cos\left(\frac{3\pi}{l}x\right), \sin\left(\frac{3\pi}{l}x\right), ..., \cos\left(\frac{n\pi}{l}x\right), \sin\left(\frac{n\pi}{l}x\right), ...$*is called a* ***principal trigonometric system.***

Property : The principal trigonometric system defined above is orthogonal i.e. for each $n, m \in \mathbb{N}$ we have

$$\int_{-l}^{l}\cos\left(\frac{n\pi}{l}x\right)\cos\left(\frac{m\pi}{l}x\right)dx = \int_{-l}^{l}\sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{m\pi}{l}x\right)dx = \begin{cases} l & \text{if } n = m \\ 0 & \text{if } \ n \neq m \end{cases}$$

$$\int_{-l}^{l}\cos\left(\frac{n\pi}{l}x\right)\sin\left(\frac{m\pi}{l}x\right)dx = 0$$

Proof. To calculate these integrals, it is sufficient to use the following well-known results

$\cos\alpha.\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$,
$\sin\alpha.\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$,
$\cos\alpha.\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$. ■

Theorem 7.2.4 *If the trigonometric series* $\frac{a_0}{2}+\sum_{n\geq1}\left[a_n\cos\left(\frac{n\pi}{l}x\right)+b_n\sin\left(\frac{n\pi}{l}x\right)\right]$ *converges pointwisely on the interval* $[-l, l]$ *then, its sum* $S(x)$ *is a* $2l-$*periodic function. Further if the convergence is uniform, then, the coefficients* a_n *and* b_n *with* $n = 0, 1, 2, ...$ *can be expressed as a function of* $S(x)$ *as follows:*

$$\begin{cases} a_0 = \frac{1}{l}\int_{-l}^{l}S(x)\,dx \\ \forall n\geq 1,\ a_n = \frac{1}{l}\int_{-l}^{l}S(x)\cos n\left(\frac{\pi}{l}x\right)dx \\ \forall n\geq 1, b_n = \frac{1}{l}\int_{-l}^{l}S(x)\sin n\left(\frac{\pi}{l}x\right)dx. \end{cases}$$

Proof. Since the functions cos and sin are $2\pi-$periodic then,
$\cos\left(\frac{n\pi}{l}(x+2l)\right) = \cos\left(\frac{n\pi}{l}x+2n\pi\right) = \cos\left(\frac{n\pi}{l}x\right)$ and
$\sin\left(\frac{n\pi}{l}(x+2l)\right) = \sin\left(\frac{n\pi}{l}x+2n\pi\right) = \sin\left(\frac{n\pi}{l}x\right)$,
therefore,

$$\begin{aligned} S(x+2l) &= \frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi}{l}(x+2l)\right)+b_n\sin\left(\frac{n\pi}{l}(x+2l)\right)\right] \\ &= \frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi}{l}x\right)+b_n\sin\left(\frac{n\pi}{l}x\right)\right] = S(x)\ . \end{aligned}$$

So, $S(x)$ is a $2l-$periodic function.

Now, suppose the series converges uniformly to $S(x)$ on $[-l, l]$. Since the general term $u_n(x) = a_n\cos\left(\frac{n\pi}{l}x\right)+b_n\sin\left(\frac{n\pi}{l}x\right)$ of the trigonometric series is a continuous function on $\mathbb{R}$ then, $S(x)$ is continuous and thus integrable on $[-l, l]$.

By multiplying both sides of the equality $S(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi}{l}x\right)+b_n\sin\left(\frac{n\pi}{l}x\right)\right]$ by each term of the principal trigonometric system and by integrating over the domain $[-l, l]$, taking into account the property of orthogonality we find for all $k \in \mathbb{N}$

$$\int_{-l}^{l}S(x)\cos\left(\frac{k\pi}{l}x\right)dx = \int_{-l}^{l}\frac{a_0}{2}\cos\left(\frac{k\pi}{l}x\right)dx+\sum_{n=1}^{\infty}\left[\begin{array}{c} a_n\int_{-l}^{l}\cos\left(\frac{n\pi}{l}x\right)\cos\left(\frac{k\pi}{l}x\right)dx \\ +b_n\int_{-l}^{l}\sin\left(\frac{n\pi}{l}x\right)\cos\left(\frac{k\pi}{l}x\right)dx, \end{array}\right] \tag{7.3}$$

and

$$\int_{-l}^{l}S(x)\sin\left(\frac{k\pi}{l}x\right)dx = \int_{-l}^{l}\frac{a_0}{2}\sin\left(\frac{k\pi}{l}x\right)dx+\sum_{n=1}^{\infty}\left[\begin{array}{c} a_n\int_{-l}^{l}\cos\left(\frac{n\pi}{l}x\right)\sin\left(\frac{k\pi}{l}x\right)dx \\ +b_n\int_{-l}^{l}\sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{k\pi}{l}x\right)dx. \end{array}\right] \tag{7.4}$$

Taking $k=0$ in (**??**) (i.e. $\cos\left(\frac{k\pi}{l}x\right)=1$), we get $\int_{-l}^{l} S(x)dx=\int_{-l}^{l}\frac{a_0}{2}dx+0=la_0$ because $n\neq k$ in the sum $\sum_{n=1}^{\infty}[..]$. Thus, $a_0=\frac{1}{l}\int_{-l}^{l} S(x)\,dx$.

Suppose $k\geq 1$, then by the property of orthogonality one can deduce

$$\int_{-l}^{l} S(x)\cos\left(\frac{k\pi}{l}x\right)dx=0+a_n\int_{-l}^{l}\cos\left(\frac{n\pi}{l}x\right)\cos\left(\frac{n\pi}{l}x\right)dx=a_nl,$$

and

$$\int_{-l}^{l} S(x)\sin\left(\frac{k\pi}{l}x\right)dx=0+b_n\int_{-l}^{l}\sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{n\pi}{l}x\right)dx=b_nl,$$

which prove the result. ■

7.3 Fourier series of a function

Definition 7.3.1 *Let $f:\mathbb{R}\to\mathbb{R}$ be $2l-$periodic function. Then, the trigonometric series*

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi}{l}x\right)+b_n\sin\left(\frac{n\pi}{l}x\right)\right]$$

defined on the interval $[-l,l]$ where
$a_0=\frac{1}{l}\int_{-l}^{l} f(x)\,dx$ *and for all* $n\geq 1$
$a_n=\frac{1}{l}\int_{-l}^{l} f(x)\cos n\left(\frac{\pi}{l}x\right)dx,\ \ b_n=\frac{1}{l}\int_{-l}^{l} f(x)\sin n\left(\frac{\pi}{l}x\right)dx,$
is called ***Fourier series of the function*** f *and if these integrals are defined, the constants a_n,b_n $(n=0,1,2,3,...)$ are called* ***Fourier coefficients*** *of the function f (or also Euler-Fourier formulas).*

The partial sums of this series $(S_nf)(x)=\frac{a_0}{2}+\sum_{p=1}^{p=n}\left[a_p\cos\left(\frac{p\pi}{l}x\right)+b_p\sin\left(\frac{p\pi}{l}x\right)\right]$ *are called* ***Fourier sums*** *of f.*

Particular case: If f is $2\pi-$periodic (i.e.$l=\pi$) function then its Fourier series is

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos(nx)+b_n\sin(nx)\right].$$

Remark 7.3.2 *1) As f is $2l-$periodic, the interval $[-l,l]$ can be replaced by any other interval of the form $[c,c+2l]$.*

2) If f is an odd function then, $a_n=0$ and $b_n=\frac{2}{l}\int_0^l f(x)\sin n\left(\frac{\pi}{l}x\right)dx$,

3) If f is an even function then, $a_n=\frac{2}{l}\int_0^l f(x)\cos n\left(\frac{\pi}{l}x\right)dx$ and $b_n=0$.

Definition 7.3.3 ***(Half range Fourier series)*** *If a $2l-$periodic function f is defined over half the range, say 0 to l, instead of the full range from $-l$ to l, it may be expanded in a series of sine terms only or of cosine terms only. The series produced is then called a* ***half range Fourier series.***

For example the Fourier series of the half range even function is given by:
$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx)$ where $a_0=\frac{2}{l}\int_0^l f(x)\,dx$ and $a_n=\frac{2}{l}\int_0^l f(x)\cos n\left(\frac{\pi}{l}x\right)dx$, $n=1,2,3,...$

Example 7.3.4 *Find half range sine series for* $f(x)=e^x$ *on* $(0,1)$.

So, required half range sine series of this function

$$f(x)=e^x=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right), l=1$$

where for all $n\geq 1$,

$b_n=\frac{2}{l}\int_0^l f(x)\sin n\left(\frac{\pi}{l}x\right)dx=2\int_0^1 e^x\sin(n\pi x)\,dx=2\Im\left(\int_0^1 e^{(1+in\pi)x}dx\right).$

Since

$\int_0^1 e^{(1+in\pi)x}dx=\left[\frac{e^{(1+in\pi)x}}{1+in\pi}\right]_0^1=\frac{e^{1+in\pi}-1}{1+in\pi}$

$=\frac{e(-1)^n-1}{1+in\pi}=\frac{(1-in\pi)(e(-1)^n-1)}{1+(n\pi)^2}$

$=\frac{(e(-1)^n-1)}{1+(n\pi)^2}+i\frac{n\pi(1-e(-1)^n)}{1+(n\pi)^2}.$

Then, $b_n=\frac{2n\pi(1-e(-1)^n)}{1+(n\pi)^2}$, *therefore* $e^x=\sum_{n=1}^{\infty}\frac{2n\pi(1-e(-1)^n)}{1+(n\pi)^2}\sin(n\pi x)$.

This series is the Fourier series of an odd 2*-periodic function* f *defined by* $f(x)=e^x$ *on* $(0,1)$.

Some properties Let us denote $a_n(f), b_n(f)$ the Fourier coefficients of f.

1) Fourier coefficients are bounded. Because $|a_n(f)|\leq\frac{1}{l}\int_{-l}^{l}|f(x)|\,dx$ and $|b_n(f)|\leq\frac{1}{l}\int_{-l}^{l}|f(x)|\,dx$.

2) Let f,g two function be integrable on $[-l,l]$ and $\alpha,\beta\in\mathbb{R}$. Then,
$a_n(\alpha f+\beta g)=\alpha a_n(f)+\beta a_n(g)$ and $b_n(\alpha f+\beta g)=\alpha b_n(f)+\beta b_n(g)$.

3) Let $f\in C^1([-l,l])$. Then, $a_n(f)=-\frac{b_n(f')}{n}$, $b_n(f)=\frac{a_n(f')}{n}$.

Bessel's inequality

Let $L^2([-l,l])$ be the set of all functions $f:\mathbb{R}\to\mathbb{R}$ such that $\int_{-l}^{l}|f(t)|^2dt<\infty$. Denote by $\|f\|_2=\sqrt{\int_{-l}^{l}|f(t)|^2dt}$

Proposition 7.3.5 *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a function* $2l-$*periodic and* $f\in L^2([-l,l])$. *Then, the trigonometric polynomial of degree n which best approximate* f *in the norm* $\|.\|_2$ *is the nth Fourier sum of the function* f.

i.e $\inf\|f-p_n\|_2=\|f-S_nf\|_2$.

Proof. For an arbitrary trigonometric polynomial of degree n,

$$p_n(x)=\frac{c_0}{2}+\sum_{p=1}^{p=n}\left[c_p\cos\left(\frac{p\pi}{l}x\right)+d_p\sin\left(\frac{p\pi}{l}x\right)\right],$$

we have $\int_{-l}^{l}|f(x)-p_n(x)|^2dx=\int_{-l}^{l}\left(f^2-2f.p_n+p_nx^2\right)dx$.

By using orthogonality property we get

$\int_{-l}^{l}p_n^2(x)\,dx=\int_{-l}^{l}\left(\frac{c_0}{2}+\sum_{p=1}^{p=n}\left[c_p\cos\left(\frac{p\pi}{l}x\right)+d_p\sin\left(\frac{p\pi}{l}x\right)\right]\right)^2dx$

$=\int_{-l}^{l}\left(\left(\frac{c_0}{2}\right)^2+\sum_{p=1}^{p=n}\left[c_p^2\cos^2\left(\frac{p\pi}{l}x\right)+d_p^2\sin^2\left(\frac{p\pi}{l}x\right)\right]\right)dx$

$=2l\left(\frac{c_0}{2}\right)^2+\sum_{p=1}^{p=n}\left[c_p^2\int_{-l}^{l}\cos^2\left(\frac{p\pi}{l}x\right)dx+d_p^2\int_{-l}^{l}\sin^2\left(\frac{p\pi}{l}x\right)dx\right]$

$=\frac{c_0^2}{2}l+\sum_{p=1}^{p=n}\left(lc_p^2+ld_p^2\right).=l\left[\frac{c_0^2}{2}+\sum_{p=1}^{p=n}\left(c_p^2+d_p^2\right)\right],$

which is known as Plancherel-Parseval's identity for trigonometric polynomials. In the second place, using the definition of Fourier coefficients:

$\int_{-l}^{l} f(x).p_n(x)\,dx. = \int_{-l}^{l} f(x)\left(\frac{c_0}{2} + \sum_{p=1}^{p=n}\left[c_p \cos\left(\frac{p\pi}{l}x\right) + d_p \sin\left(\frac{p\pi}{l}x\right)\right]\right) dx$
$= \int_{-l}^{l} f(x)\frac{c_0}{2}dx + \sum_{p=1}^{p=n}\left[c_p \int_{-l}^{l} f(x)\cos\left(\frac{p\pi}{l}x\right) dx + d_p \int_{-l}^{l} f(x)\sin\left(\frac{p\pi}{l}x\right) dx\right]$
$= .la_0\frac{c_0}{2} + \sum_{p=1}^{p=n}\left[lc_pa_p + ld_pb_p\right] = l\left(\frac{a_0c_0}{2} + \sum_{p=1}^{p=n}\left[c_pa_p + d_pb_p\right]\right).$

Then,
$\int_{-l}^{l} |f(x) - p_n(x)|^2 dx = \|f\|_2^2 + l\left[\frac{c_0^2}{2} - a_0c_0 + \sum_{p=1}^{p=n}\left(c_p^2 + d_p^2 - 2c_pa_p - 2d_pb_p\right)\right]$
$= \|f\|_2^2 + l\left[\frac{(c_0-a_0)^2}{2} + \sum_{p=1}^{p=n}(c_p - a_p)^2 + \sum_{p=1}^{p=n}(d_p - b_p)^2\right]$
$-l\left[\frac{a_0^2}{2} + \sum_{p=1}^{p=n}\left(a_p^2 + b_p^2\right)\right].$

The first and the last terms of the last expression do not depend on p_n. Then, the only choice to minimise is $c_p = a_p$ and $d_p = b_p$ for all $0 \le p \le n$. i.e. $p_n = S_nf$. Which was to be shown. ■

Corollary 7.3.6 ***(Bessel's inequality)***. *Let $f \in L^2([-l,l])$. Then,* $\frac{a_0^2}{2} + \sum_{p=1}^{p=n}\left(a_p^2 + b_p^2\right) \le \frac{1}{l}\|f\|_2^2$.

Notice that the right term in the equality is finite because f is square-integrable by hypothesis.

Proof. Let $p_n = S_nf$ in the last exression of the above proof we get

$0 \le \int_{-l}^{l} |f(x) - p_n(x)|^2 dx = \|f\|_2^2 - l\left[\frac{a_0^2}{2} + \sum_{p=1}^{p=n}\left(a_p^2 + b_p^2\right)\right]$. then, $\|f\|_2^2 \ge l\left[\frac{a_0^2}{2} + \sum_{p=1}^{p=n}\left(a_p^2 + b_p^2\right)\right]$. ■

An essential consequence of this result is that the Fourier coefficients of any square integrable function tend to 0, otherwise the numerical series $\sum_{p=1}^{p=n}\left(a_p^2 + b_p^2\right)$ diverges and could not bounded .

Corollary 7.3.7 ***(Plancherel-Parseval's identity)***. *Let $f \in L^2([-l,l])$ bounded. Then,* $\frac{1}{2l}\|f\|_2^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{p=1}^{\infty}\left(a_p^2 + b_p^2\right)$.

Proof. We in the proof above that

$$\int_{-l}^{l} |f(x) - S_nf(x)|^2 dx = \|f\|_2^2 - l\left[\frac{a_0^2}{2} + \sum_{p=1}^{\infty}\left(a_p^2 + b_p^2\right)\right],$$

and taking the limit when $n \to \infty$. ■

7.3.1 Dirichlet's conditions for Fourier series

Theorem 7.3.8 *A function $f : \mathbb{R} \to \mathbb{R}$ $2l$-periodic defined on $[-l,l]$ can be expanded in Fourier series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right)\right]$$

provided that f

- *is single valued and bounded in $[-l,l]$,*
- *has almost a finite number of minima and maxima in $[-l,l]$,*

- *is piecewise continuous with finite number of discontinuities in* $[-l, l]$.

(These conditions are sufficient but not necessary).

Theorem 7.3.9 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a* $2l$*-periodic function of class* C^1 *piecewise in* $[-l, l]$ *(called also piecewise smooth). Then, the Fourier series of this function converges pointwisely on* $[-l, l]$ *to the function* $\widetilde{f}$ *defined by* $\widetilde{f}(x) = \frac{f(x^+)+f(x^-)}{2}$, *where* $f(x^+) = \lim_{t\to x^+} f(t)$ *and* $f(x^-) = \lim_{t\to x^-} f(t)$.

Remark 7.3.10 *1)* f *is continuous at* $x_0 \in \,]-l, l[\iff \widetilde{f}(x_0) = f(x_0)$, *because* $f\left(x_0^+\right) = f\left(x_0^-\right) = f(x_0)$

2) $\widetilde{f}(x_0) = \frac{f(-l^+)+f(l^-)}{2}$ *if* f *has a discontinuity* x_0 *at end point of* $[-l, l]$ *i.e.*$x_0 = -l$ *or* l.

Solved exercise: Consider $f : \mathbb{R} \to \mathbb{R}$ $2\pi-$periodic function defined on $[-\pi, \pi]$ by

$$f(x) = \begin{cases} x \sin x & \text{if } x \in [0, \pi[\\ 0 & \text{if } x \in [-\pi, 0[\end{cases}.$$

1) Calculate the Fourier coefficients $a_0(f)$, $a_1(f)$, $b_1(f)$.

2) For $n \geq 2$, determine the other Fourier coefficients $a_n(f)$, $b_n(f)$ and deduce the Fourier series of the function f.

3) Deduce the sums of the following numerical series $\sum_{n\geq 0} \frac{(-1)^n}{n^2-1}$,

Solution 7.3.11 *using integration by part taking into account* $l = \pi$ *we get in general for* $p \in \mathbb{N}^*$

$\int_0^\pi x \sin(px)\, dx = \frac{1}{p}\left[-x\cos px + \frac{\sin px}{p}\right]_0^\pi = \frac{(-1)^{p+1}}{p}\pi$ *and*

$\int_0^\pi x \cos(px)\, dx = \frac{1}{p}\left[x\sin px + \frac{\cos px}{p}\right]_0^\pi = \frac{(-1)^p-1}{p^2}$

1) Then, one can easily calulate the coefficients

$$\begin{aligned}
a_0(f) &= \frac{1}{l}\int_{-l}^{l} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0dx + \int_0^\pi x\sin x dx\right) \\
&= \frac{1}{\pi}\int_0^\pi x \sin x dx = \frac{1}{\pi}\frac{(-1)^{1+1}}{1}\pi = 1, \\
a_1(f) &= \frac{1}{l}\int_{-l}^{l} f(x)\cos x dx = \frac{1}{\pi}\int_0^\pi x\sin x.\cos x dx = \frac{1}{2\pi}\int_0^\pi x\sin 2x dx \\
&= \frac{1}{2\pi}\frac{(-1)^3}{2}\pi = -\frac{1}{4}, \\
b_1(f) &= \frac{1}{l}\int_{-l}^{l} f(x)\sin x dx = \frac{1}{\pi}\int_0^\pi x\sin x.\sin x dx = \frac{1}{\pi}\int_0^\pi x\left(\frac{1-\cos 2x}{2}\right)dx \\
&= \frac{1}{2\pi}\left[\int_0^\pi x dx - \int_0^\pi x\cos 2x dx\right] = \frac{1}{2\pi}\left[\frac{\pi^2}{2} - 0\right] = \frac{\pi}{4}.
\end{aligned}$$

.

2) For $n \geq 2$

$$\begin{aligned} a_n(f) &= \frac{1}{l}\int_{-l}^{l} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{\pi} x\sin x.\cos nx dx = \frac{1}{2\pi}\int_0^{\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx \\ &= \frac{1}{2\pi}\left[\int_0^{\pi} x\sin(n+1)x dx - \int_0^{\pi} x\sin(n-1)x dx\right] = \frac{1}{2\pi}\left[\frac{(-1)^{n+2}}{n+1}\pi - \frac{(-1)^n}{n-1}\pi\right] \\ &= \frac{(-1)^{n+1}}{n^2-1} \end{aligned}$$

$$\begin{aligned} b_n(f) &= \frac{1}{l}\int_{-l}^{l} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{\pi} x\sin x.\sin nx dx = \frac{1}{2\pi}\int_0^{\pi} x\left[\cos(n-1)x - \cos(n+1)x\right]dx \\ &= \frac{1}{2\pi}\left[\int_0^{\pi} x\cos(n-1)x dx - \int_0^{\pi} x\cos(n+1)x dx\right] = \frac{1}{2\pi}\left[\frac{(-1)^{n-1}-1}{(n-1)^2} - \frac{(-1)^{n+1}-1}{(n+1)^2}\right] \\ &= \frac{\left[(-1)^n+1\right]}{2\pi}\frac{-4n}{(n^2-1)^2} = \frac{-2n}{\pi}\frac{\left[(-1)^n+1\right]}{(n^2-1)^2} \end{aligned}$$

Thus, the Fourier series of the periodic function f is

$$\frac{1}{2} - \frac{1}{4}\cos x + \frac{\pi}{4}\sin nx + \sum_{n\geq 2}\left[\frac{(-1)^{n+1}}{n^2-1}\cos nx - \frac{2n}{\pi}\frac{\left[(-1)^n+1\right]}{(n^2-1)^2}\sin nx\right]$$

3) According to Didichlet's theorem, the Fourier series for $x = 0$

$$\frac{1}{4} - \sum_{n\geq 2}\frac{(-1)^n}{n^2-1}$$

converges to $\widetilde{f}(0) = f(0) = 0 = \frac{1}{4} - \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}$. *Then,*

$$\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1} = \frac{1}{4}.$$

7.4 Exercises

Exercise 1 Calculate the Fourier series of the $2\pi-$ periodic function

$$f(x) = x^2$$

where $0 < x < 2\pi$.

Exercise 2 A function f has the properties that $f(x+\pi) = -f(x)$ and $f(-x) = f(x)$ for all x. Show that all its Fourier coefficients satisfy

$$\forall n \in \mathbb{N}; a_{2n} = 0 = b_n$$

Exercise 3 Let f be a function of period 2π such that

$$f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0 \\ 0 & \text{if } \ 0 < x < \pi. \end{cases}$$

1) Sketch a graph of f in the interval $-2\pi < x < 2\pi$.
2) Show that the Fourier series for $f$in the interval $-\pi < x < \pi$is

$$\frac{1}{2} - \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right)$$

3)By giving an appropriate value to x, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

Exercise 4 Let f be a function of period 2π such that

$$f(x) = \begin{cases} \pi - x & \text{if } 0 < x < \pi \\ 0 & \text{if } \quad \pi < x < 2\pi. \end{cases}$$

1) Sketch a graph of f in the interval $[-2\pi, 2\pi]$
2) Show that the Fourier series for f in the interval $0 < x < 2\pi$ is

$$\frac{\pi}{4} + \frac{2}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + ...\right)$$

3) By giving an appropriate value to x, show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + ...$$

Exercise 5 We consider the function $f : \mathbb{R} \to \mathbb{R}$ of period 2π and such that:

$$f(x) = \pi^2 - x^2 \; ; -\pi < x < \pi$$

1) Draw the graph of f (we will take at least $x \in [-2\pi, 2\pi]$).
2) Calculate the Fourier series associated with f , study its convergence and give its limit. Does it converge uniformly on $\mathbb{R}$?
3) Calculate the sums: $S_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, $S_2 = \sum_{n=1}^{\infty} \frac{1}{n^2}$ and $S_3 = \sum_{n=1}^{\infty} \frac{1}{n^4}$, $S_4 = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^4}$
4) Find a particular solution of class C^2 and period 2π of the differential equation $(E) : y'' - 3y = f(x)$. Give the general solution of (E).

Part D

Improper integrals and integrals depending on a parameter

Chapter 8

Improper Integrals

8.1 Introduction

Recall that the fundamental theorem of calculus says that if f is a continuous function on the closed interval $[a, b]$ then,

$$\int_a^b f(x)\, dx = F(b) - F(a),$$

where F is any anti-derivative of f. Both the continuity condition and closed interval must hold to use the fundamental theorem of calculus, and in this case if $f(x) \geq 0$ in the interval $[a, b]$, $\int_a^b f(x)\, dx$ represents the area under the graph of f from a to b.

We begin with an example where blindly applying the fundamental fheorem of calculus can give an incorrect result. When we apply this theorem in order to calculate $\int_0^3 \frac{dx}{(x-1)^2}$ we find

$$\int_0^3 \frac{dx}{(x-1)^2} = \left[-\frac{1}{x-1}\right]_0^3 = -\frac{1}{2} - 1 = -\frac{3}{2}.$$

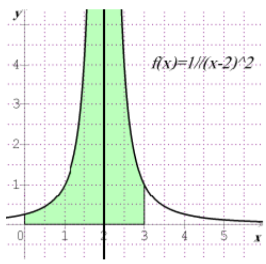

Figure 8.1: Divergent integral

However, this answer is wrong since f is not continuous on $[0, 3]$ we cannot directly apply the fundamental theorem of calculus. Intuitively, we can see why is not the correct answer by looking at the graph of f on $[0, 3]$. In general, the Riemann integrals already studied, relate to functions f defined and bounded on a bounded closed interval $[a; b]$. It is useful to generalize this notion to the case of arbitrary functions f defined on any interval, except perhaps at a finite number of points and not necessarily bounded. We distinguish two cases:

1) integral of an unbounded function on a bounded interval,

2) integral of any function on an unbounded interval.

In what will come, we consider I **is a non-compact interval from** $\mathbb{R}$ **and** $f : I \to \mathbb{R}$ **is any function.**

Definition 8.1.1 *We say that f is l**locally integrable** on I if f is Riemann integrable on any compact interval $[\alpha, \beta]$ included in I. We denote $R_{loc}(I)$ the set of functions locally integrable on I. In particular, Any continuous or piecewise continuous function on an interval I is locally integrable. on the same interval.*

Example 8.1.2 *The function* $\ln$ *is locally integrable on* $]0, +\infty[$ *because it's continuous in all interval compact* $[\alpha, \beta] \subset]0, +\infty[$.

Definition 8.1.3 *(Singular point) We say that $c \in \overline{I}$ (closure of I) is a **singular point** of the function f if* $\lim_{x \to c} f(x) = \infty$ *or does not exist.*

Example 8.1.4 *1) Since* $\lim_{x \to 1} \frac{1}{(x-1)^2} = +\infty$ *then, $c = 1$ is a singular point of the function* $x \longmapsto \frac{1}{(x-1)^2}$ *on* $]0, 3[$.

2) $c = 0$ is a singular point of the function $x \longmapsto \sin \frac{1}{x}$ *on* $]0, \pi[$ *because* $\lim_{x \to 0} \sin \frac{1}{x}$ *does not exist.*

In what follows, we consider :$f : I \to \mathbb{R}$ **is a locally integrable function on the interval** $I = (a; b)$ **. where** $(a; b)$ **is an open or half-open interval of** $\mathbb{R}$ **with** $-\infty \leq a < b \leq +\infty$.

Definition 8.1.5 *(Improper integral We say that* $\int_a^b f(x)\,dx$ *is an **improper integral** if at least one of integration bounds is a singular point or infinite ($-\infty$ or $+\infty$).*

1) If b is the singular point then, the improper integral of f over $I = [a; b[$ is $\int_a^b f(x)\,dx = \lim_{\lambda \to b^-} \int_a^\lambda f(x)\,dx$.

2) If a the singular point then, the improper integral of f over $I =]a; b]$ is $\int_a^b f(x)\,dx = \lim_{\lambda \to a^+} \int_\lambda^b f(x)\,dx$.

3) If $b = +\infty$ then, the improper integral of f over $I = [a; +\infty[$ is $\int_a^b f(x)\,dx = \lim_{\lambda \to +\infty} \int_a^\lambda f(x)\,dx$.

4) If $a = -\infty$ then, the improper integral of f over $I =]-\infty; b]$ is $\int_a^b f(x)\,dx = \lim_{\lambda \to -\infty} \int_\lambda^b f(x)\,dx$.

8.2 Convergence of an improper integral

Definition 8.2.1 *1) The improper integral* $\int_a^b f(x)\,dx$ *is said to be convergent at b if* $\lim_{\lambda \to b^-} \int_a^\lambda f(x)\,dx = l \in \mathbb{R}$ *(finite real number) and we write* $\int_a^b f(x)\,dx = l$.

2) The improper integral $\int_a^b f(x)\,dx$ *is said to be convergent at a if* $\lim_{\lambda \to a^+} \int_\lambda^b f(x)\,dx. = l \in \mathbb{R}$ *(finite real number) and we write* $\int_a^b f(x)\,dx = l$.

Similarly, we can define the convergence of an improper integral at infinity as follow

1) The improper integral $\int_a^{+\infty} f(x)\,dx$ *is said to be convergent at* $+\infty$ *if* $\lim_{\lambda\to+\infty}\int_a^{\lambda} f(x)\,dx = l \in \mathbb{R}$ *(finite real number) and we write* $\int_a^{+\infty} f(x)\,dx = l$.

2)The improper integral $\int_{-\infty}^{b} f(x)\,dx$ *is said to be convergent at* $-\infty$ *if* $\lim_{\lambda\to-\infty}\int_{\lambda}^{b} f(x)\,dx. = l \in \mathbb{R}$ *(finite real number) and we write* $\int_{-\infty}^{b} f(x)\,dx = l$.

Definition 8.2.2 *In any of the cases mentioned in the above definition, if the limit is infinite (*$=\infty$*) or does not exist, the corresponding integral is* ***divergent.***

Remark 8.2.3 *If there is no confusion, we will just say that the integral converges or is convergent rather than converges at* b *or is convergent at* b.

Example 8.2.4 *1)* $\int_0^{+\infty} xe^{-x}dx$ *is an improper integral because its above bound equal* $+\infty$. *The function* $x \longmapsto xe^{-x}$ *is locally integrable over* $[0,+\infty[$, *then for all* $\lambda > 0$ *an integration by part yields*

$$\begin{aligned}\int_0^{\lambda} xe^{-x}dx &= \left[-xe^{-x}\right]_0^{\lambda} - \int_0^{\lambda} -e^{-x}dx = \left[-xe^{-x}\right]_0^{\lambda} + \left[-e^{-x}\right]_0^{\lambda} \\ &= -\lambda e^{-\lambda} - e^{-\lambda} + 1.\end{aligned}$$

Then,

$$\int_0^{+\infty} xe^{-x}dx = \lim_{\lambda\to+\infty}\int_0^{\lambda} xe^{-x}dx = \lim_{\lambda\to+\infty}\left(-\lambda e^{-\lambda} - e^{-\lambda} + 1\right) = 1 \in \mathbb{R}.$$

Therefore, $\int_0^{+\infty} xe^{-x}dx$ *converges and it's equal to* 1.

2) $\int_1^2 \frac{x}{x^2-1}dx$ *is an improper integral because* $\lim_{x\to1^+}\frac{x}{x^2-1} = +\infty$ *i.e.* 1 *is a singular point. The function* $x \longmapsto \frac{x}{x^2-1}$ *is locally integrable over* $]1,2]$ *. Let* $\lambda \in]1,2]$ *be arbitrary*

$$\int_{\lambda}^{2} \frac{x}{x^2-1}dx = \left[\frac{1}{2}\ln\left|x^2-1\right|\right]_{\lambda}^{2} = \frac{1}{2}\ln\frac{3}{|\lambda^2-1|}.$$

.

Then,

$$\int_1^2 \frac{x}{x^2-1}dx = \lim_{\lambda\to1^+}\int_{\lambda}^{2}\frac{x}{x^2-1}dx = \lim_{\lambda\to1^+}\frac{1}{2}\ln\frac{3}{|\lambda^2-1|} = +\infty,$$

therefore $\int_1^2 \frac{x}{x^2-1}dx$ *diverges.*

8.2.1 Integals of Riemann (called also $\alpha-$Integrals)

Definition 8.2.5 *Consider any* $a > 0$ *and a fixed* $\alpha \in \mathbb{R}$. *The improper integrals* $\int_a^{+\infty}\frac{dx}{x^{\alpha}}$, $\int_0^{a}\frac{dx}{x^{\alpha}}$ *are called* ***integrals of Riemann*** *or again* $\alpha-$***Integrals.***

Theorem 8.2.6 *1)* $\int_a^{+\infty}\frac{dx}{x^{\alpha}}$ *converges if and only if* $\alpha > 1$

2) $\int_0^{a}\frac{dx}{x^{\alpha}}$ *converges if and only if* $\alpha < 1$

Proof. 1) For each $\lambda > a$, we have

if $\alpha = 1$, $\int_a^\lambda \frac{dx}{x} = [\ln x]_a^\lambda = \ln\lambda - \ln a = \ln\left(\frac{\lambda}{a}\right)$, then $\int_a^{+\infty} \frac{dx}{x} = \lim_{\lambda\to+\infty} \ln\left(\frac{\lambda}{a}\right) = +\infty$.

If $\alpha \neq 1$, $\int_a^\lambda \frac{dx}{x^\alpha} = \left[\frac{x^{1-\alpha}}{1-\alpha}\right]_a^\lambda = \frac{\lambda^{1-\alpha}}{1-\alpha} - \frac{a^{1-\alpha}}{1-\alpha}$, then

$\int_a^{+\infty} \frac{dx}{x^\alpha} = \lim_{\lambda\to+\infty}\left(\frac{\lambda^{1-\alpha}}{1-\alpha} - \frac{a^{1-\alpha}}{1-\alpha}\right) = \begin{cases} \frac{a^{1-\alpha}}{\alpha-1} & \text{if } \alpha > 1 \\ +\infty & \text{if } \alpha < 1, \end{cases}$

which prove that $\int_a^{+\infty} \frac{dx}{x^\alpha}$ converges for $\alpha > 1$ and diverges otherwise.

2)On the other hand, similarly we have for any $0 < \lambda < a$

$\int_\lambda^a \frac{dx}{x} = [\ln x]_\lambda^a = \ln a - \ln\lambda = \ln\left(\frac{a}{\lambda}\right)$, then $\int_\lambda^a \frac{dx}{x} = \lim_{\lambda\to0^+} \ln\left(\frac{a}{\lambda}\right) = +\infty$.

if $\alpha \neq 1$, $\int_a^\lambda \frac{dx}{x^\alpha} = \left[\frac{x^{1-\alpha}}{1-\alpha}\right]_\lambda^a = \frac{a^{1-\alpha}}{1-\alpha} - \frac{\lambda^{1-\alpha}}{1-\alpha}$, then

$\int_0^a \frac{dx}{x^\alpha} = \lim_{\lambda\to+\infty}\left(\frac{a^{1-\alpha}}{1-\alpha} - \frac{\lambda^{1-\alpha}}{1-\alpha}\right) = \begin{cases} \frac{a^{1-\alpha}}{1-\alpha} & \text{if } \alpha < 1 \\ +\infty & \text{if } \alpha > 1, \end{cases}$

which prove that $\int_0^a \frac{dx}{x^\alpha}$ converges if $\alpha < 1$ and diverges otherwise. ■

Example 8.2.7 *The improper integrals* $\int_2^{+\infty} \frac{dx}{x^2\sqrt{x}}$, $\int_0^3 \frac{dx}{\sqrt[3]{x}}$ *converge while* $\int_0^2 \frac{dx}{x^2\sqrt{x}}$, $\int_1^{+\infty} \frac{dx}{\sqrt[3]{x}}$ *diverge.*

Corollary 8.2.8 *Suppose that* $-\infty < a < b < +\infty$*. Then,*

The improper integrals $\int_a^b \frac{dx}{(x-a)^\alpha}$ *and* $\int_a^b \frac{dx}{(b-x)^\alpha}$ *converge if and only if* $\alpha < 1$.

Proof. It suffices to take the auxiliary variables $u = x - a$ then $u = b - x$ and applying the previous theorem. ■

Proposition 8.2.9 *For all* $c \in [a, b[$, *the improper integrals* $\int_c^b f(x)\,dx$ *and* $\int_a^b f(x)\,dx$ *are both convergent or both divergent .*

Proof. By Chasles relationship, we have for all $\lambda \in [a, b[$, $\int_a^\lambda f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\lambda f(x)\,dx$.

As $f \in R_{Loc}([a, b[)$ then f is integrable on $[a, c] \subset [a, b[$ and thus by taking the limit in both sides of the equality, we get

$\lim_{\lambda\to b^+} \int_a^\lambda f(x)\,dx = \int_a^c f(x)\,dx + \lim_{\lambda\to b^+} \int_c^\lambda f(x)\,dx$ which yields

$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx}_{\in\mathbb{R}} + \int_c^b f(x)\,dx.$$

That's prove the result. ■

Remark 8.2.10 *In the case of an open interval i.e.* $I =]a, b[$ $(-\infty \le a < b \le +\infty)$*.If for each* $c \in]a, b[$, *both* $\int_a^c f(x)\,dx$ *and* $\int_c^b f(x)\,dx$ *converge then, the improper integral* $\int_a^b f(x)\,dx$ *converges and*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example 8.2.11 *Let's study the convergence of the improper integral:* $\int_{-\infty}^{+\infty} \frac{dx}{x^2+1}$.
The function $x \longmapsto \frac{1}{x^2+1}$ *is continuous on* $\mathbb{R}$*, then it is integrable locally over* $\mathbb{R}$*. We have for all* $c, \lambda \in \mathbb{R}$

$\int_{\lambda}^{c} \frac{dx}{x^2+1} = [\arctan(x)]_{\lambda}^{c} = \arctan(c) - \arctan(\lambda)$ *then,*

$\int_{-\infty}^{c} \frac{dx}{x^2+1} = \lim_{\lambda \to -\infty} [\arctan(c) - \arctan(\lambda)] = \arctan(c) + \frac{\pi}{2} \in \mathbb{R}$ *and*

$\int_{c}^{\lambda} \frac{dx}{x^2+1} = [\arctan(x)]_{c}^{\lambda} = \arctan(\lambda) - \arctan(c)$ *then,*

$\int_{c}^{+\infty} \frac{dx}{x^2+1} = \lim_{\lambda \to +\infty} (\arctan(\lambda) - \arctan(c)) = \frac{\pi}{2} - \arctan(c) \in \mathbb{R}$.

Consequently, $\int_{-\infty}^{+\infty} \frac{dx}{x^2+1}$ *converges to its value*

$\int_{-\infty}^{+\infty} \frac{dx}{x^2+1} = \int_{-\infty}^{c} \frac{dx}{x^2+1} + \int_{c}^{+\infty} \frac{dx}{x^2+1} = \arctan(c) + \frac{\pi}{2} + \frac{\pi}{2} - \arctan(c) = \pi$.

Remark that the value of $\int_{-\infty}^{+\infty} \frac{dx}{x^2+1}$ *does not depend on the choice of* c.

Proposition 8.2.12 *Suppose that* $I = [a, b[$ *with* $b \neq +\infty$. *If* $\lim_{x \to b^-} f(x) = d \in \mathbb{R}$ *then, the improper integral* $\int_a^b f(x)\, dx$ *converges.*

Proof. Let g be the function defined on $[a, b]$ by $g(x) = f(x)$ if $x \in [a, b[$ and $g(b) = d$. It clear that g is integrable on the closed interval $[a, b]$ and $\int_a^b f(x)\, dx = \int_a^b g(x)\, dx$. Thus, $\int_a^b f(x)\, dx$.converges . ■

Generalization: Suppose that f is defined and locally integrable on $[a, b[- \{c_1, c_2, c_3, ...c_n\}$ where $a < c_1 < c_2 < c_3 < ... < c_n < b$.

If the improper integrals $\int_a^{c_1} f(x)\, dx$, $\int_{c_1}^{c_2} f(x)\, dx$, $\int_{c_2}^{c_3} f(x)\, dx$, ...,$\int_{c_n}^{b} f(x)\, dx$ converge, we say that $\int_a^b f(x)\, dx$ converges and

$$\int_a^b f(x)\, dx = \int_a^{c_1} f(x)\, dx + \int_{c_1}^{c_2} f(x)\, dx + \int_{c_2}^{c_3} f(x)\, dx + ... + \int_{c_n}^{b} f(x)\, dx.$$

Some remarks

1)Writing $\int_{-\infty}^{+\infty} f(x)\, dx = \lim_{\lambda \to +\infty} \int_{-\lambda}^{+\lambda} f(x)\, dx$ is usually wrong. To study the nature of the integral $\int_{-\infty}^{+\infty} f(x)\, dx$, we divide it into two integrals $\int_{-\infty}^{c} f(x)\, dx$ $\int_{c}^{+\infty} f(x)\, dx$ $(c \in \mathbb{R})$.

For example : we have $\lim_{\lambda \to +\infty} \int_{-\lambda}^{+\lambda} x dx = \lim_{\lambda \to +\infty} \left(\frac{\lambda^2}{2} - \frac{(-\lambda)^2}{2} \right) = 0$ but $\int_{-\infty}^{+\infty} x dx$ diverges.

2) If f is a complex valued function with real variable defined on $I = [a, b[$ .Then,

$\int_a^b f(x)\, dx$ converges at b if and only if $\int_a^b Ref(x)\, dx$ and $\int_a^b Imf(x)\, dx$ converge at b .

8.2.2 Properties

1) **Linearity**

Proposition 8.2.13 *Let f and g be two real valued functions locally integrable on $[a, b[$ and $\alpha, \beta \in \mathbb{R}$. So*

1) If the improper integrals $\int_a^b f(x)\, dx$ *and* $\int_a^b g(x)\, dx$ *converge at b then,* $\int_a^b (\alpha f + \beta g)(x)\, dx$ *converges also. Moreover, we have*

$$\int_a^b (\alpha f + \beta g)(x)\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx.$$

2) If $\int_a^b f(x)\,dx$ diverges and $\int_a^b g(x)\,dx$ converges, then the improper integrals $\int_a^b (f+g)(x)\,dx$ and $\int_a^b \alpha f(x)\,dx$ $(\alpha \neq 0)$ diverge.

3) If the improper integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ diverge so we cannot conclude anything about the convergence or divergence of $\int_a^b (f+g)(x)\,dx$.

For example $\int_1^{+\infty} \frac{2-x}{x\sqrt{x}}dx$, $\int_1^{+\infty} \frac{dx}{\sqrt{x}}$ are divergent integral while its sum $\int_1^{+\infty} \frac{2-x}{x\sqrt{x}}dx + \int_1^{+\infty} \frac{dx}{\sqrt{x}} = \int_1^{+\infty} \frac{2dx}{x\sqrt{x}} = 2\int_1^{+\infty} \frac{dx}{x^{\frac{3}{2}}}$ whict is $\alpha-$integral with $\alpha = \frac{3}{2} > 1$ thus it converges.

Proof. 1) It suffices to write for all $\lambda \in \mathbb{R}$ such that $a < \lambda < b$

$\int_a^\lambda (\alpha f + \beta g)(x)\,dx = \alpha \int_a^\lambda f(x)\,dx + \beta \int_a^\lambda g(x)\,dx$ because $f, g \in R_{Loc}([a,b[)$. Then, passing to the limit we find the result.

2) By the result 1) $\int_a^b -g(x)\,dx$ converges. Suppose $\int_a^b (f+g)(x)\,dx$ converges then, by the result 1) again $\int_a^b (f+g)(x)\,dx + \int_a^b -g(x) = \int_a^b f(x)\,dx$ converges which contradict the fact that $\int_a^b f(x)\,dx$ diverges. Therefore $\int_a^b (f+g)(x)\,dx$ diverges. ■

2) **Integration by change of variable**

Theorem 8.2.14 *Let $\varphi : (\alpha, \beta) \to (a,b)$ be a bijection of class C^1 on the interval (α, β). Then, the two improper integrals $\int_a^b f(x)\,dx$ and $\int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$ are both convergent or both divergent in the case of convergence, we have $\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$.*

Example 8.2.15 *To calculate the integral $\int_1^2 \frac{dx}{x\ln(x)}$ we will make the following change of variable $t = \ln(x)$ i.e. $x = e^t$ then, $\varphi :]0, \ln 2] \to]1,2]$ where $\varphi(t) = e^t$ is the bijection mentioned in the theorem. We have $\int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt = \int_0^{\ln 2} \frac{e^t}{e^t t}dt = \int_0^{\ln 2} \frac{dt}{t}$. Then, this integral is divergent because it's a $\alpha-$integral (integral of Riemann) with $\alpha = 1$. Thus $\int_1^2 \frac{dx}{x\ln(x)}$ diverges also.*

3) **Integration by part**

Theorem 8.2.16 *Let u and v be two functions of class C^1 on $[a;b[$ .If $u.v$ has a finite limit at b then, the improper integrals $\int_a^b u'v dx$ and $\int_a^b uv' dx$ are both convergent or both divergent . In the case of the convergence we have $\int_a^b u'v dx = \lim_{\lambda \to b^-} [uv]_a^\lambda - \int_a^b uv' dx$.*

Example 8.2.17 *To calculate $\int_0^3 x\ln x dx$ by part, take $u' = x$ and $v = \ln x$ i.e. $u = \frac{x^2}{2}$ and $v' = \frac{1}{x}$. Note that, $\lim_{x\to 0^+} uv = \lim_{x\to 0^+} \frac{x^2}{2}\ln x = 0.$ We have $\int_0^3 uv' dx = \int_0^3 \frac{x^2}{2}\frac{1}{x}dx = \int_0^3 \frac{x}{2}dx = \left[\frac{x^2}{4}\right]_0^3 = \frac{9}{4}$ which imply that this integral converges then, $\int_0^3 x\ln x dx$ converges also and*

$$\int_0^3 x\ln x dx = \lim_{\lambda\to 0^+}\left[\frac{x^2}{2}\ln x\right]_\lambda^3 - \int_0^3 \frac{x^2}{2}\frac{1}{x}dx = \frac{9}{2}\ln 3 - \frac{9}{4}.$$

In what follows, we will suffice with studying the convergence or divergence of a generalized integrals on $I = [a, b[$ with $a < b \leq +\infty$. In all the results where we use $\lim_{\lambda\to b^-}$ will be replaced by $\lim_{\lambda\to+\infty}$ in the case $b = +\infty$ (Similar results hold for the other improper integrals where $I =]a, b]$ with $-\infty \leq a < b$)

Theorem 8.2.18 ***(Cauchy criterion for the convergence of an improper integral)*** *The integral $\int_a^b f(x)\,dx$ converges at b if and only if*

$$\forall \epsilon > 0, \exists c \in I = [a, b[\,, \forall x', x'' \in I; \left(x' > c\right) \wedge \left(x'' > c\right) \implies \left| \int_{x'}^{x''} f(x)\,dx \right| < \epsilon.$$

8.3 Convergence tests for improper integrals of functions with constant sign

The following test allows us to determine convergence or divergence information about improper integrals that are hard to compute by comparing them to easier ones. If f is a function that only takes negative values on the interval I, then $-f$ only takes positive values on the same domain, thus studying the nature of $\int_a^b f(x)\,dx$ returns to the study of the nature of $\int_a^b -f(x)\,dx$. We will therefore limit ourselves to the case of positive functions.

Theorem 8.3.1 *Let f be a positive function such that $f \in R_{Loc}([a, b[)$. Then, the improper integral $\int_a^b f(x)\,dx$ converges at b if and only if there exists $M > 0$ such that*

$$\forall \lambda \in [a, b[\,; 0 \leq \int_a^\lambda f(x)\,dx \leq M.$$

And thus $\int_a^b f(x)\,dx$ diverges if and only if $\lim_{\lambda \to b^-} \int_a^\lambda f(x)\,dx = +\infty$.

Proof. The function $\lambda \longmapsto \int_a^\lambda f(x)\,dx$ is increasing on $[a, b[$ because its derivative f is positive and therefore in order for it to have a finite limit at b it is necessary and sufficient to be bounded. ■

Example 8.3.2 *Let's test the convergence of $\int_0^1 t \sin^2\left(\frac{1}{t}\right) dt$. It's known that $0 \leq \sin^2\left(\frac{1}{t}\right) \leq 1$, then $0 \leq t \sin^2\left(\frac{1}{t}\right) \leq t$ for all $t > 0$. Therefore, for evry $\lambda \in]0, 1]$ we have*
$0 \leq \int_\lambda^1 t \sin^2\left(\frac{1}{t}\right) dt \leq \int_\lambda^1 t dt = \frac{1}{2} - \frac{\lambda^2}{2} \leq \frac{1}{2}$. According to the last theorem, $\int_0^1 t \sin^2\left(\frac{1}{t}\right) dt$ converges at 0.

8.3.1 Comparison test for improper integrals

Theorem 8.3.3 *Let f and g be two locally integrable functions defined on $[a, b[$ such that $0 \leq f(x) \leq g(x)$ for all $x \in [a, b[$. Then, we have*
1) If $\int_a^b g(x)\,dx$ converges then, $\int_a^b f(x)\,dx$ converges
in the case of convergence we have $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$
2) If $\int_a^b f(x)\,dx$ diverges then, $\int_a^b g(x)\,dx$ diverges.

Proof. 1) Suppose that $\int_a^b g(x)\,dx$ converges. We have for all $\lambda \in [a, b[$,

$$0 \leq \int_a^\lambda f(x)\,dx \leq \int_a^\lambda g(x)\,dx \leq \int_a^b g(x)\,dx = M.$$

By the theorem **??**, the integral $\int_a^b f(x)\,dx$ converges. On the other hand, $\int_a^b f(x)\,dx = \lim_{\lambda \to b^+} \int_a^\lambda f(x)\,dx \leq \int_a^b g(x)\,dx$.

2) Suppose that $\int_a^b f(x)\,dx$ diverges, then $\lim_{\lambda\to b^+}\int_a^\lambda f(x)\,dx = +\infty$. Since $\int_a^\lambda f(x)\,dx \le \int_a^\lambda g(x)\,dx$ So, $\lim_{\lambda\to b^+}\int_a^\lambda g(x)\,dx = +\infty$. ∎

Example 8.3.4 *1) Let's study the convergence of* $\int_0^1 \frac{\ln t}{t^2+1}dt$ *by comparison test. We have* $\forall t \in]0,1]$, $\ln t < \frac{\ln t}{t^2+1} \le 0$ *i.e.* $0 \le \frac{-\ln t}{t^2+1} < -\ln t$ *and since* $\int_0^1 -\ln t dt = -\left[1\ln 1 - \lim_{\lambda\to 0^+}\lambda\ln\lambda - \int_0^1 dt\right] = 1$, *then* $-\int_0^1 \frac{\ln t}{t^2+1}dt$ *converges.*

2) Which of the following integrals converge? $\int_1^{+\infty} e^{-x^2}dx$, $\int_{-\infty}^{-2}\frac{\sin^2 x}{x^2}dx$.

Note that $0 < e^{-x^2} \le e^{-x}$ *for all* $x \ge 1$ *and it clear that*
$\int_1^{+\infty} e^{-x}dx = \lim_{\lambda\to+\infty}\left[-e^{-x}\right]_1^\lambda = \lim_{\lambda\to+\infty}\left(e^{-1}-e^{-\lambda}\right) = \frac{1}{e}$,
so $\int_1^{+\infty} e^{-x^2}dx$ *converges.*

Also $0 \le \sin^2 x \le 1$ *for all* $x \le -2$ *then* $0 < \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$. *Since* $\int_{-\infty}^{-2}\frac{1}{x^2}dx. = \left[-\frac{1}{x}\right]_{\lambda\to-\infty}^{-2} = \frac{1}{4}$ *so* $\int_{-\infty}^{-2}\frac{\sin^2 x}{x^2}dx$ *converges.*

8.3.2 Limit comparison test

Corollary 8.3.5 *If positive functions* f *and* g *are locally integrable on* $I = [a,b[$ *and* $\lim_{x\to b^-}\frac{f(x)}{g(x)} = L$, *then*

1) In the case $L \in \mathbb{R}_+^*$,*the integrals* $\int_a^b f(x)\,dx$ *and* $\int_a^b g(x)\,dx$ *are both convergent or both divergent In the case* $L \in \mathbb{R}$.

2) If $L = 0$; $\int_a^b g(x)\,dx$ *converges* $\Longrightarrow \int_a^b f(x)\,dx$ *converges.*

3) If $L = +\infty$, $\int_a^b g(x)\,dx$ *diverges* $\Longrightarrow \int_a^b f(x)\,dx$ *divverges.*

Proof. 1) Suppose $L \in \mathbb{R}_+^*$, by definition of the limit, we have
$\forall\epsilon > 0, \exists\delta > 0, \forall x \in [a,b[\,;\ b-\delta < x < b \Longrightarrow \left|\frac{f(x)}{g(x)} - L\right| < \epsilon$
then,

$$\forall\epsilon > 0, \exists\delta > 0, \forall x \in [a,b[\,;\ b-\delta < x < b \Longrightarrow (L-\epsilon)\,g(x) < f(x) < (L+\epsilon)\,g(x).$$

Choosing $\epsilon = \frac{L}{2}$ for example so for all $x \in [a,b[\,\cap\,]b-\delta,b[$ we have $0 < \frac{L}{2}g(x) < f(x) < \frac{3L}{2}g(x)$. For $c \in [a,b[\,\cap\,]b-\delta,b[$, by comparison test one can deduce that
$\int_a^b g(x)\,dx$ converges $\Longleftrightarrow \int_c^b g(x)\,dx$ converges $\Longleftrightarrow \int_c^b f(x)\,dx$ converges $\Longleftrightarrow \int_a^b f(x)\,dx$ converges .

2) 3) By a similar method we prove the remaining two cases. ∎

Example 8.3.6 *Does the integral* $\int_1^{+\infty}\frac{dx}{\sqrt{x}+2}$ *converge or diverge?*

As $\lim_{x\to+\infty}\frac{1}{\sqrt{x}+2} \div \frac{1}{\sqrt{x}} = \lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x}+2} = 1 \in \mathbb{R}_+^*$ *and* $\int_1^{+\infty}\frac{dx}{\sqrt{x}}$ *diverges because it's an* $\alpha-$*integral with* $\alpha = \frac{1}{2} < 1$. *Then,* $\int_1^{+\infty}\frac{dx}{\sqrt{x}+2}$ *diverges also.*

Corollary 8.3.7 *Let* f *be a locally integrable positive function on* $[a,+\infty[$ $(a > 0)$ *and let* α *be fixed real number. Suppose that* $\lim_{t\to+\infty} t^\alpha f(t) = L$. *Then,*

1) if $L \in \mathbb{R}_+$ *and* $\alpha > 1$,*the improper integral* $\int_a^{+\infty} f(t)\,dt$ *converges,*

2) if $L = +\infty$ *and* $\alpha \le 1$,*the improper integral* $\int_a^{+\infty} f(t)\,dt$ *diverges.*

Proof. Note that $t^\alpha f(t) = \frac{f(t)}{\frac{1}{t^\alpha}}$ then, it suffices to take $g(t) = \frac{1}{t^\alpha}$ and applying the limit comparison test. ∎

Corollary 8.3.8 *Let f be a locally integrable positive function on $[a,b[$ $(0 < a < b)$. If there exists $\alpha < 1$ such that $\lim_{t\to b^-} t^\alpha f(t) = L \in \mathbb{R}_+$, then $\int_a^b f(t)\,dt$ converges.*

8.3.3 Dirichlet's test

Theorem 8.3.9 *Let f and g be two locally integrable functions defined on $[a,b[$ such that*

- *f is monotone on $[a,b[$ and $\lim_{t\to b^-} f(t) = 0$, and*
- *$\exists M > 0, \forall x \in [a,b[\,; \left|\int_a^x g(t)\,dt\right| \leq M$,*

then, the improper integral $\int_c^b f(t)\,g(t)\,dt$ converges at b.

Proof. As $\lim_{t\to b^-} f(t) = 0$, then for an arbitrary $\epsilon > 0$ there exists $c \in [a,b[$ such that

$$\forall t \in\,]c,b[\,; |f(t)| < \frac{\epsilon}{4M}$$

By the second condition we have

$$\begin{aligned}\left|\int_{x'}^{x''} g(t)\,dt\right| &= \left|\int_a^{x''} g(t)\,dt - \int_a^{x'} g(t)\,dt\right| \\ &\leq \left|\int_a^{x''} g(t)\,dt\right| + \left|\int_a^{x'} g(t)\,dt\right| \leq 2M.\end{aligned}$$

Let $x\prime, x'' \in\,]c,b[$ such that $x\prime < x''$

In view of the mean theorem, we have

$$\begin{aligned}\left|\int_{x'}^{x''} f(t)\,g(t)\,dt\right| &= \left|f(x')\int_{x'}^{\xi} g(t)\,dt + f(x'')\int_{\xi'}^{x''} g(t)\,dt\right| \\ &\leq |f(x')|\left|\int_{x'}^{\xi} g(t)\,dt\right| + |f(x'')|\left|\int_{\xi'}^{x''} g(t)\,dt\right| \\ &\leq \tfrac{\epsilon}{4M}.2M + \tfrac{\epsilon}{4M}.2M = \epsilon.\end{aligned}$$

The convergence of $\int_c^b f(t)\,g(t)\,dt$ results from Cauchy criterion. ■

Corollary 8.3.10 ***(Abel's test)*** *Let f and g be two locally integrable functions on $[a,b[$. If*

- *the function f is monotone and bounded on $[a,b[$.*
- *$\int_c^b g(t)\,dt$ converges at b,*

then, $\int_c^b f(t)\,g(t)\,dt$ converges at b..

Proof. It suffices to take in the previous theorem $F = f - \lim_{t\to b^-} f(t)$. ■

8.3.4 Absolute and conditional convergence

Definition 8.3.11 *Let f be a locally integrable function on $[a,b[$.*

1) An improper integral $\int_a^b f(x)\,dx$ is said to be ***absolutely convergent*** *if the integral $\int_a^b |f(x)|\,dx$ converges.*

2)If the integral $\int_a^b f(x)\,dx$ converges but $\int_a^b |f(x)|\,dx$ diverges, we say that $\int_a^b f(x)\,dx$ is ***conditionally convergent.***

Example 8.3.12 *The integral $\int_0^{+\infty} e^{-t}\cos(\ln(1+t))\,dt$ is absolutely convergent because $0<\left|e^{-t}\cos(\ln(1+t))\,dt\right|\le e^{-t}$ and $\int_0^{+\infty} e^{-t}dt=1$.*

Theorem 8.3.13 *If the improper integral $\int_a^b f(x)\,dx$ converges absolutely, then it converges and*

$$\left|\int_a^b f(x)\,dx\right|\le\int_a^b |f(x)|\,dx.$$

The converse is false

Proof. Suppose that $\int_a^b |f(x)|\,dx$ converges. We consider the following functions for $x\in[a,b[$

$$f_-(x)=\frac{|f(x)|-f(x)}{2},\quad f_+(x)=\frac{|f(x)|+f(x)}{2}.$$

Note that $0\le f_-(x)\le|f(x)|$, $0\le f_+(x)\le|f(x)|$ and $f(x)=f_+(x)-f_-(x)$.

By comparison test, the integrals $\int_a^b f_-(x)\,dx$, $\int_a^b f_+(x)\,dx$ are convergent thus by linearity the integral $\int_a^b f(x)\,dx=\int_a^b f_+(x)\,dx-\int_a^b f_-(x)\,dx$ converges. ■

Example 8.3.14 *The integral $\int_1^{+\infty}\frac{\sin x}{x}dx$ is conditionally convergent. Indeed, according to the integration by parts formula, we have*

$\int_1^{\lambda}\frac{\sin x}{x}dx=\left[-\frac{\cos x}{x}\right]_1^{\lambda}-\int_1^{\lambda}\frac{\cos x}{x^2}dx$ *then,* $\int_1^{+\infty}\frac{\sin x}{x}dx=\cos 1-\int_1^{+\infty}\frac{\cos x}{x^2}dx.$

The integral $\int_1^{+\infty}\frac{\cos x}{x^2}dx$ converges absolutely because $0\le\left|\frac{\cos x}{x^2}\right|\le\frac{1}{x^2}$ and $\int_1^{+\infty}\frac{1}{x^2}dx$ converge ($\alpha-$ integral with $\alpha=2>1$). Thus $\int_1^{+\infty}\frac{\cos x}{x^2}dx$ converges which imply that $\int_1^{+\infty}\frac{\sin x}{x}dx$ converges. Next, we show that $\int_1^{+\infty}\frac{|\sin x|}{x}dx$ diverges. We estimate for all $n\ge 2$

$$\int_\pi^{n\pi}\frac{|\sin x|}{x}dx=\sum_{p=1}^{p=n-1}\int_{p\pi}^{(p+1)\pi}\frac{|\sin x|}{x}dx\ge\sum_{p=1}^{p=n-1}\frac{1}{(p+1)\pi}\int_{p\pi}^{(p+1)\pi}|\sin x|\,dx$$
$$=\sum_{p=1}^{p=n-1}\frac{1}{(p+1)\pi}\int_0^{\pi}\sin x\,dx=\frac{2}{\pi}\sum_{p=1}^{p=n-1}\frac{1}{(p+1)}=\frac{2}{\pi}\sum_{p=2}^{p=n}\frac{1}{p}.$$

As $\lim_{n\to\infty}\sum_{p=2}^{p=n}\frac{1}{p}=+\infty$, Then $\int_\pi^{+\infty}\frac{|\sin x|}{x}dx$ diverges, and consequently $\int_1^{+\infty}\frac{|\sin x|}{x}dx$ diverges also.

Example 8.3.15 *Now, we can use Dirichlet's test to prove that $\int_1^{+\infty}\frac{\sin x}{x}dx$ converges.*

- *The function $x\longmapsto\frac{1}{x}$ is monotonically decreasing on $[1,+\infty[$ because for all $x\ge 1$; $\left(\frac{1}{x}\right)'=-\frac{1}{x^2}<0$. Moreover, $\lim_{x\to\infty}\frac{1}{x}=0$.*

- *For all* $\lambda \geq 1$; $\left|\int_1^\lambda \sin x dx\right| = \left|[-\cos x]_1^\lambda\right| = |\cos\lambda - \cos 1| = 2\left|\sin\left(\frac{\lambda-1}{2}\right)\sin\left(\frac{\lambda+1}{2}\right)\right| \leq 2$.

Then , $\int_1^{+\infty} \frac{\sin x}{x} dx$ *converges* .

Chapter 9

Integrals depending on a parameter

Let J be an open interval of $\mathbb{R}$. We consider a half-open interval $I = [a, b[$ and a function of two variables

$$\begin{aligned} f\ &: J \times [a; b[\to \mathbb{k} \qquad (= \mathbb{R} \text{ or } \mathbb{C}) \\ (x,t) &\longmapsto f(x,t), \end{aligned}$$

We assume either that $b = \infty$ or that b is a singular point, so $\int_a^b f(x,t)\,dt$ is a generalized integral depending on the parameter x. So the first problem posed is that of the convergence of a family of generalized integrals, we are led to determine the set D of values of x for which $\int_a^b f(x,t)\,dt$ converges. D is the domain of definition of the function F where

$$F(x) = \int_a^b f(x,t)\,dt.$$

If for every $x \in J$, $\int_a^b f(x,t)\,dt.$ converges, we say that F is a function defined by a generalized integral. Our aim of this chapter is to compute limits and to study the continuity and derivability of such functions

9.1 Uniform convergence of a generalized integral depending on a parameter

Definition 9.1.1 *1) $I = [a, +\infty[$:we say that the improper integral $\int_a^{+\infty} f(x,t)\,dt.$ converges uniformly on J if.*

$$\forall \epsilon > 0, \exists \eta(\epsilon) > 0, \forall T \in I; T \geq \eta(\epsilon) \implies \sup_{x \in J} \left| \int_T^{+\infty} f(x,t)\,dt. \right| < \epsilon$$

2) $I = [a, +b[$:we say that the improper integral $\int_a^b f(x,t)\,dt.$ converges uniformly on J if

$$\forall \epsilon > 0, \exists \eta(\epsilon) > 0, \forall T \in I; \quad 0 < b - T < \eta(\epsilon) \implies \sup_{x \in J} \left| \int_{b-T}^{b} f(x,t)\,dt. \right| < \epsilon.$$

9.2 Normal convergence of a generalized integral depending on a parameter:

Definition 9.2.1 *We say that the integral $\int_a^b f(x,t)\,dt$ is normally convergent on J if there exists a function $\varphi : I \to \mathbb{R}_+$ checking*
1) $\forall (x,t) \in J \times I$, $|f(x,t)| \le \varphi(t)$ 2) $\int_a^b \varphi(t)\,dt$ converges.

Example 9.2.2 $F(x) = \int_0^{+\infty} \frac{e^{-tx}}{1+t^2} dt.$

The function $f(x,t) = \frac{e^{-tx}}{1+t^2}$ is defined and continuous on $J \times I = \mathbb{R} \times [0,+\infty[$, moreover it's positive .Then, $\int_0^{+\infty} \frac{e^{-tx}}{1+t^2} dt$ is an improper integral.

$\lim_{t\to+\infty} t^\alpha \frac{e^{-tx}}{1+t^2} = \begin{cases} 0 & \text{if } \alpha = 2 \text{ and } x > 0 \\ +\infty & \text{if } \alpha = 1 \text{and } x < 0. \end{cases}$

For $x = 0$, $F(0) = \int_0^{+\infty} \frac{dt}{1+t^2} = \frac{\pi}{2}$. Which. mean that this integral converges for all $x \in [0,+\infty[$, thus the domain of definition of F is $D = [0,+\infty[$.

Moreover , we have, for all $x \in [0,+\infty[$, $\frac{e^{-tx}}{1+t^2} \le \frac{1}{1+t^2} = \varphi(t)$ and since $\int_0^{+\infty} \varphi(t)\,dt = \frac{\pi}{2}$ then, $\int_0^{+\infty} \frac{e^{-tx}}{1+t^2} dt$ converges normally on $I = [0,+\infty[$.

Proposition 9.2.3 *If the integral $\int_a^b f(x,t)\,dt$ converges normally on I then, it converges uniformly on I .*

9.3 Continuity of a function defined by an integral depends on a parameter

Theorem 9.3.1 *We suppose that*

- *for each $x \in J$, the function $t \longmapsto f(x,t)$ is piecewise continuous on I,*
- *for each $t \in I$, the function $x \longmapsto f(x,t)$ is continuous on J,*
- *there exists a function $\varphi : I \to \mathbb{R}_+$ piecewise continuous and integrable over I such that, for each $(x,t) \in J \times I$*

$$|f(x,t)| \le \varphi(t) \quad \textit{(dominance hypothesis)}.$$

then, The function F is defined and continuous on J and one can write for all $x_0 \in J$

$$\lim_{x\to x_0} F(x) = \lim_{x\to x_0} \int_a^b f(x,t)\,dt = \int_a^b \lim_{x\to x_0} f(x,t)\,dt = \int_a^b f(x_0,t)\,dt = F(x_0).$$

(Note that the last condition of this theorem means that $\int_a^b f(x,t)\,dt$ is normally convergent and thus uniformly convergent).

Proof. For $x \in J$, the function $t \longmapsto f(x,t)$ is piecewise continuous on I and for all $t \in I$, $|f(x,t)| \le \varphi(t)$, then $\int_a^b |f(x,t)|\,dt \le \int_a^b \varphi(t)\,dt \in \mathbb{R}$ because φ is integrable over I .Therefore, $\int_a^b f(x,t)\,dt$ converges absolutely thus it converges . we deduce that F is define on J. Let's prove that F is continuous at x. Let (x_n) be a sequence of element of J such that $\lim x_n = x$. We know that $t \longmapsto f(x_n,t)$ is piecewise continuous on I . Since for every $t \in I; x \longmapsto f(x,t)$

is continuous on J and $x_n \to x$ then, for all $t \in I$, $f(x_n, t) \to f(x,t)$. On the other hand, for all $n \in \mathbb{N}$ and all $t \in I$, $|f(x_n,t)| \leq \varphi(t)$. According to dominated convergenc theorem, $F(x_n) = \int_a^b f(x_n,t)\,dt$ converges to $\int_a^b f(x,t)\,dt = F(x)$ which show that F is continuous at $x \in I$ (arbitrary), hence F is continuous on J. ■

Example 9.3.2 *Let's prove that the function $F(x) = \int_0^\pi \sqrt{x+\cos t}dt$ is defined and continuous on $J = [1, +\infty[$.*

1) It's clear that $t \longmapsto \sqrt{x+\cos t}$ is continuous on $[0,\pi]$ because for all $x \geq 1$, $x + \cos t \geq 1 + \cos t \geq 0$. Then, $\int_0^\pi \sqrt{x+\cos t}dt$ converges on $[0,\pi]$ for every $x \geq 1$ which means that F is defined on $[1,+\infty[$.

2) For any $A \geq 1$ (fixed) we have

- *The function $t \longmapsto \sqrt{x+\cos t}$ is continuous on $[0,\pi]$ (stronger than piecewise continuous).*
- *The function $x \longmapsto \sqrt{x+\cos t}$ is continuous on $[0, A]$.*
- *For all $(x,t) \in [0,A] \times [0,\pi]$, we have $\sqrt{x+\cos t} \leq \sqrt{A+\cos t} = \varphi(t)$ with $\varphi(t) \geq 0$ and continuous on $[0,\pi]$ thus integrable.*

In view of the above theorem F is continuous on $[0, A]$ for all $A \geq 1$ then, it's continuous on $[1,+\infty[$.

Theorem 9.3.3 *Within the conditions of theorem **??**, the function $F(x) = \int_a^b f(x,t)\,dt$ is locally integrable on J i.e*

$$\forall \alpha, \beta \in J; \int_\alpha^\beta F(x)\,dx = \int_\alpha^\beta \left(\int_a^b f(x,t)\,dt\right) dx = \int_a^b \left(\int_\alpha^\beta f(x,t)\,dx\right) dt$$

Proof. As F is continuous on J, then it's locally integrable on thi interval. On the other hand we have $t \longmapsto f(x,t)$ is integrable on $I = [a,b[$ for all $x \in J$ and according to the second condition the function $x \longmapsto f(x,t)$ is locally integrable on J for all $t \in I$. Our result follows immediately from Fubini's theorem. ■

9.4 Passing to the limit under the integral sign

Theorem 9.4.1 *Let d be real or infinite adherent to J. Suppose that*

- *for each $x \in J$, the function $t \longmapsto f(x,t)$ is piecewise continuous on I,*
- *for each $t \in I$, the function $x \longmapsto f(x,t)$ has a limit $l(t)$ when x tends to d . Moreover, the function $t \longmapsto l(t)$ is piecewise continuous on I*
- *there exists a function $\varphi : I \to \mathbb{R}_+$ piecewise continuous and integrable over I such that, for each $(x,t) \in J \times I$*

  $$|f(x,t)| \leq \varphi(t) \quad \text{(dominance hypothesis).}$$

  then, the function F has a limit at d and

  $$\lim_{x\to d} F(x) = \lim_{x\to d} \int_a^b f(x,t)\,dt = \int_a^b \lim_{x\to d} f(x,t)\,dt = \int_a^b l(t).$$

Proof. 1) If $d \in \mathbb{R}$, Pose for all $(x,t) \in (J \cup \{d\}) \times I$, $g(x,t) = \begin{cases} f(x,t) & \text{if } x \neq d \\ l(t) & \text{if } x = d. \end{cases}$

The function g checks the assumptions of theorem **??** on $(J \cup \{d\}) \times I$, so the function G:: $x \longmapsto \int_a^b g(x,t)\,dt$ is continuous on $J \cup \{d\}$ and in particular at d.

2) If $d = +\infty$, it suffices to apply the previous result to the function $(x,t) \longmapsto f\left(\frac{1}{x}, t\right)$ in 0^+. ∎

Example 9.4.2 *In order to calculate* $\lim_{x \to +\infty} \int_0^{+\infty} e^{-t^2\sqrt{x}}dt$, *we apply this theorem.*

- *For all* $x \geq 1$, *the function* $t \longmapsto e^{-t^2\sqrt{x}}$ *continuous on* $[0, +\infty[$.
- $\lim_{x \to +\infty} e^{-t^2\sqrt{x}} = l(t) \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{if } t > 0, \end{cases}$ *note that* $t \longmapsto l(t)$ *is piecewise continuous on* $[0, +\infty[$.
- *We have also,* $e^{-t^2\sqrt{x}} \leq e^{-t^2} = \varphi(t)$ *and it's clear that* φ *is continuous and integrable on* $I = [0, +\infty[$.

Then,
$\lim_{x \to +\infty} \int_0^{+\infty} e^{-t^2\sqrt{x}}dt = \int_0^{+\infty} \lim_{x \to +\infty} e^{-t^2\sqrt{x}}dt = \int_0^{+\infty} 0dt = 0.$

9.5 Differentiation of a function defined by an integral depends on a parameter

Theorem 9.5.1 ***(Leibniz's theorem)*** *Suppose that*

- *for each* $x \in J$, *the function* $t \longmapsto f(x,t)$ *is piecewise continuous and integrable on* $I = [a, b[$,

  We further assume that f *admits on* $J \times I$ *a partial derivative with respect to its first variable* x *satisfying;*
- *for each* $x \in J$, *the function* $t \longmapsto \frac{\partial f}{\partial x}(x,t)$ *is piecewise continuous on* I,
- *for each* $t \in I$, *the function* $x \longmapsto \frac{\partial f}{\partial x}(x,t)$ *is continuous on* J,
- *there exists a function* $\varphi_1 : I \to \mathbb{R}_+$ *piecewise continuous and integrable over* I *such that, for each* $(x,t) \in J \times I$

  $$\left|\frac{\partial f}{\partial x}(x,t)\right| \leq \varphi_1(t) \quad \textit{(dominance hypothesis).}$$

  then, The function F *is of class* C^1 *on* J *and one can write for all* $x_0 \in J$ *and its derivative is obtained by derivation under the sign integral*

  $$\forall x \in J, F'(x) = \int_a^b \frac{\partial f}{\partial x}(x,t)\,dt.$$

Proof. Since for each x of J, the function $t \longmapsto f(x,t)$ is piecewise continuous and integrable over I, the function F is defined on J. Let x_0 be an element of J. For $(x,t) \in J \times I$, we pose

$$g(x,t) = \begin{cases} \frac{f(x,t)-f(x_0,t)}{x-x_0} & \text{if} \quad x \neq x_0 \\ \frac{\partial f}{\partial x}(x_0,t) \quad \text{if} & x = x_0. \end{cases}$$

Note that for $x \neq x_0$,

$$\begin{aligned} \frac{F\ (x) - F\ (x_0)}{x - x_0} &= \frac{\int_a^b f(x,t)\,dt - \int_a^b f(x_0,t)\,dt}{x - x_0} \\ &= \int_a^b \frac{f(x,t) - f(x_0,t)}{x - x_0} dt = \int_a^b g(x,t)\,dt. \end{aligned}$$

We have then,

for each $x \in J$, the function $t \longmapsto |g(x,t)|$ is piecewise continuous on I,

for each $t \in I$, the function $x \longmapsto g(x,t)$ is continuous on J (including in $x = x_0$ because by definition $g(x_0,t) = \lim_{x \to x_0} g(x,t)$).

Let $(x,t) \in (J - \{x_0\}) \times I$, by applying the mean value theorem

$$|g(x,t)| = \left| \frac{f(x,t) - f(x_0,t)}{x - x_0} \right| \leq \sup_{u \in J} \frac{\partial f}{\partial x}(u,t) \leq \varphi_1(t),$$

which remains true when $x = x_0$ and $t \in I$. According to the theorem **??**, the function $G(x) = \int_a^b g(x,t)\,dt.$ is continuous on J and in particular at x_0. We deduce that $\frac{F\ (x)-F\ (x_0)}{x-x_0}$ has a limit when x tends to x_0 and therefore F is differentiable at x_0. Further,

$$F'(x_0) = \lim_{x \to x_0} \frac{F\ (x) - F\ (x_0)}{x - x_0} = \lim_{x \to x_0} \int_a^b g(x,t)\,dt = \int_a^b \lim_{x \to x_0} g(x,t)\,dt = \int_a^b \frac{\partial f}{\partial x}(x_0,t)\,dt.$$

Finally, as well according to the theorem **??** the function $F'(x) = \int_a^b \frac{\partial f}{\partial x}(x,t)\,dt.$ is continuous on J and therefore F is of class C^1on J. ∎

Solved exercise:

For any real x, we consider $F(x) = \int_0^{+\infty} \frac{e^{-t}\sin xt}{t} dt$

1) Prove that F is defined on $\mathbb{R}$

2) Test the differentiability of F, then calculate its derivative F'

3) Deduce the expression of $F(x)$ as a function of x.

Solution 9.5.2 *1) For all $x \in [-A,A]$ where $A > 0$*

$t \longmapsto \frac{e^{-t}\sin xt}{t}$ is continuous on $[0,+\infty[$ (at $t = 0$ the function can be prolonged by continuity because $\lim_{t \to 0^+} \frac{e^{-t}\sin xt}{t} = x$).

for each $(x,t) \in [-A,A] \times [0,+\infty[$, we have $\left|\frac{e^{-t}\sin xt}{t}\right| \leq \frac{e^{-t}}{t}|x|\,t = Ae^{-t} = \varphi(t)$. Since the function φ is continuous and integrable on $[0,+\infty[$ ($\int_0^{+\infty} \varphi(t)\,dt = A$), Then, the function F is defined on $[-A,A]$ for all $A > 0$, thus its domain of definition is $D = \mathbb{R}$.

2) Let $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{e^{-t}\sin xt}{t}\right) = \frac{e^{-t}}{t} t\cos xt = e^{-t}\cos xt$. Applying the last theorem, we get

- *for each $x \in \mathbb{R}$, the function $t \longmapsto e^{-t}\cos xt$ is continuous on $I = [0,+\infty[$.*

- *for each* $t \in I$, *the function* $x \longmapsto e^{-t}\cos xt$ *is continuous on* $\mathbb{R}$.
- *for each* $(x,t) \in \mathbb{R}\times[0,+\infty[$, $\left|e^{-t}\cos xt\cos xt\right| \le e^{-t} = \varphi_1(t)$, *with* φ_1 *is continuous and integrable on* $[0,+\infty[$.

Then, the functon F *is of class* C^1 *on* $\mathbb{R}$ *and* $F'(x) = \int_0^{+\infty}\frac{\partial}{\partial x}\left(\frac{e^{-t}\sin xt}{t}\right)dt = \int_0^{+\infty}e^{-t}\cos xt dt$, *We know that* $\cos xt = \frac{e^{ixt}+e^{-ixt}}{2}$, *then* ,

$$F'(x) = \int_0^{+\infty}e^{-t}\cos xt dt = \frac{1}{2}\int_0^{+\infty}\left(e^{t(-1+ix)}+e^{t(-1-ix)}\right)dt$$
$$= -\frac{1}{2}\left[\frac{1}{1-ix}e^{t(-1+ix)}+\frac{1}{1+ix}e^{t(-1-ix)}\right]_0^{\lambda\to\infty}$$
$$= -\frac{1}{2}\left[0-\frac{1}{1-ix}-\frac{1}{1+ix}\right] = \frac{1}{2}\left(\frac{1}{1-ix}+\frac{1}{1+ix}\right) = \frac{1}{1+x^2}.$$

3) We obtained for all $x\in\mathbb{R}$, $F'(x) = \frac{1}{1+x^2}$ *and we have* $F(0)=0$ *then,* $F(x) = \int_0^x \frac{dt}{1+t^2} = \arctan(x)$.

9.6 Exercises

Exercise 1: Test whether each of the following integrals converges or diverges and calculate its value in case of convergence

1) $\int_0^1 t\ln(t)\,dt$ 2) $\int_0^{+\infty}\frac{dt}{\sqrt{t}(t+1)}$ 3) $\int_0^1\frac{tdt}{(1-t)^2}$ 3) $\int_1^{+\infty}\frac{\arctan(t)dt}{1+t^2}$ 4) $\int_0^{+\infty}\frac{dt}{\sqrt{t(t+1)}}$ 5) $\int_0^{\frac{\pi}{2}}\tan(t)\,dt$ 6) $\int_0^1\frac{e^t}{t}dt$.

Exercise 2: Use the appropriate test to study the convergence of each of the following generalized integrals

1) $\int_1^{+\infty}\frac{t^3dt}{\ln t+t^4}$ 2) $\int_1^{+\infty}\frac{dt}{\sqrt{t^4+t^3-t^{22}}}$

Exercise 3 Show that the integral $\int_0^{+\infty}\frac{\ln(t)dt}{1+t^2}$ is convergent. Calculate its value.

Exercise 4 Show that the following integrals are absolutely convergent

1) $\int_0^1\sin\left(\frac{1}{t}\right).\ln(t)\,dt$ 2) $\int_0^{+\infty}e^{-t^2}\cos tdt$ 3) $\int_0^{+\infty}te^{-t^2}\left(t^2\sin t-\cos\left(\frac{1}{t}\right)\right)dt$ 4) $\int_1^{+\infty}\frac{\ln(t)\cos t}{t\sqrt{t}}dt.$

Exercise 5: 1) Determine according to the value of $\alpha\in\mathbb{R}$, whether the integral $\int_0^{+\infty}(t-1)^{\alpha}e^{-t^2}dt$ converges or diverges.

2) Do the integral $\int_0^{+\infty}\frac{\sin t}{t}e^{-t}dt$ converges or diverges.

Exercise 6 : We study in this exercise the generalized integral depending on a parameter $x\in\mathbb{R}$:

$$F(x) = \int_0^{+\infty}\frac{\cos(tx)}{1+t^2}dt$$

1) Show that the function F is defined continuous and bounded over $D=\mathbb{R}$.

2) Prove that F is an even function and calculate $F(0)$.

3) Show that for all $x>0$; $F(x) = xG(x)$ where

$$G(x) = \int_0^{+\infty}\frac{\cos(u)}{x^2+u^2}du.$$

4) Show that $G(x)$ is twice differentiable on $]0,+\infty[$ and express $G'(x)$ and $G''(x)$ in the form of a generalized integrals depending on the parameter $x\in]0,+\infty[$.

5) Deduce that F is twice differentiable on $]0,+\infty[$ and that its second derivative is given by

$$F''(x) = x\int_0^{+\infty} \frac{\left(2x^2-6u^2\right)\cos(u)}{\left(x^2+u^2\right)^2}du.$$

Bibliography

[1] K. Allab; éléments d'Analyse (fonction d'une variable réelle), Office des publications universitaires N^0 d'édition 600 (Janvier 1986).

[2] G.Baranenkov, R. Chostak, B.Demidovitch, V.Efimenko, S. Frolov, S.Kogan, G. Lountz, E.Porchnéva, ESytchéva et AYanpolski ; Recueil d'éxercises et problèmes d'analyse mathématiques, traduit du russe par H.Damadian, édition Mir,Moscou, (1997).

[3] T.J. Bromwich; An introduction to the theory of infinite series , Macmillan (1947).

[4] A. Dufetel; Analyse Séries de Fourier et Équations Différentielles, Exercices et problèmes résolus avec rappels de cours, Vuibert- paris aout 2010.

[5] F. Delmer; Mathématique, Les Séries, rappels de cours et exercices résolus, Dunod- paris 1995; ISBN : 2 10 002635 6.

[6] Livio Flaminio; Séries numériques et intégrales généralisées; notes de cours; Octobre 2008

[7] S. G. Delabrière; Suites, Séries, Intégrales, Cours et exercices, ellipses-paris 2009.

[8] W. J. Kaczor and M. T. Nowak; Problems in Mathematical Analysis III: Integration, Student Mathematical Library Volume: 21; 2003; 356 pp.

[9] G. Monna (avec la participation de R. Morvan); Problèmes de mathématiques, Séries numériques, Séries de fonctions, Séries entières, Cepadués-edition France 2008; ISBN :978.2.85428.832.2.

[10] N. Pisconov; Calcul diffrentiel et intégral (tome 2), Traduction française, édition Mir Moscou,1980.

[11] A.J. Riquet; Calcules Scientifiques Suites et Séries, Cours et exercices corrigés, ellipses- paris 2010; ISBN : 978-2-7298-5421-8.

[12] G.S. Salekhov; Computation of series , Moscow (1955) (In Russian).

[13] E.T. Whittaker, G.N. Watson, A course of modern analysis , Cambridge Univ. Press (1952) pp. Chapt. 6.