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**Thèse**

# Doctorat LMD

Domaine: Mathématique

Filière: Mathématiques

Spécialité**:** Analyse mathématique et application

## Thème

# Sur une classe de problèmes aux limites non locaux

Présenté par
Bellamouchi Chahinez

Soutenu publiquement devant le jury composé de

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EL-OUED UNIVERSITY
FACULTY OF EXACTES SCIENCES

Thesis
# Doctorat LMD

**Theme**

# On a class of nonlocal boundary value problems

**presented by**
Bellamouchi Chahinez

**Publicly defended before the jury composed**

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**University year : 2024/2025**

# Abstract

In this thesis, we use Krasnoselskii's fixed point theorem to prove the existence of a positive solution to a one-dimensional nonlocal elliptic problem and the existence of a positive radial solution to a multidimensional nonlocal elliptic problem under weak conditions on the reaction terms and the diffusion coefficients. Moreover, uniqueness results are obtained by imposing some additional conditions on the given data.

**Key words:** Nonlocal elliptic problem, Regular set, Positive solution, Fixed point, Krasnoselskii theorem, Existence, Uniqueness.

# ملخص

في هذه الأطروحة، نستخدم نظرية النقطة الثابتة لكراسنوسيلسكي لإثبات وجود حل موجب لمسألة ناقصية غير محلية ذات بعد واحد ووجود حل موجب على مجموعة منتظمة لمسألة ناقصية غير محلية متعددة الأبعاد تحت شروط ضعيفة على مكونات المسألتين. من جهة أخرى، تم الحصول على نتائج في الوحدانية بفرض شروط إضافية على المعطيات.

**الكلمات المفتاحية** : مسألة ناقصية غير محلية، مجموعة منتظمة ،الحل الموجب، النقطة الثابتة، نظرية كراسنوسيلسكي، الوجود، الوحدانية.

## Résumé

Dans cette thèse, nous utilisons le théorème du point fixe de Krasnoselskii pour démontrer l'existence d'une solution positive à un problème elliptique non local unidimensionnel et l'existence d'une solution radiale positive à un problème elliptique non local multidimensionnel sous des conditions faibles sur les termes de réaction et les coefficients de diffusion. De plus, des résultats d'unicité sont obtenus en imposant des conditions supplémentaires sur les données fournies.

**Key words :** Problème elliptique non local, Ensemble régulier, Solution positive, Point fixe, Théorème de Krasnoselskii, Existence, Unicité.

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Discussing this thesis.

# Dedication

Praise be to Allah, by whose grace good deeds are accomplishe Allah gave me the ability and success to complete my thesis no endeavor is completed except by allah.

Praise be to Allah through words, deeds, gratitude and contentment.

I dedicate this graduation to the one who taught me to give and to the one whose name I carry with all my might

Be proud and ask Allah to extend your life so that you can see the fruits that have come

Picking it up after a long wait, "Dear Father".

To the smile of life and the secret of my existence, and to the one whose prayers were the secret of my success

My dearest beloved, "My beloved mother"

And to those who are greatly credited with encouraging and motivating me, my brothers:

Mouna, Raouan, Sirin, Taki Eddin and Ahmed.

And to my dear fiance, who was a great, supporter to me at every moment.

To all my colleagues, friends, and everyone who helped me complete it.

I wish my Allah bless you and protect you for me.

# General notations

| | |
|---|---|
| $\mathbb{N}$ | Set of natural numbers. |
| $\mathbb{R}$ | Set of real numbers. |
| $\mathbb{R}^n$ | Real n-dimensional vector space constructed over the field of reals. |
| $\mathbb{R}_+$ | Set of positive real numbers. |
| $[a, b]$ | The closed interval $a \leq x \leq b$. |
| $(a, b)$ | The open interval $a < x < b$. |
| $(\cdot)' = \frac{d(\cdot)}{dt}$ | The ordinary derivative with respect to $t$. |
| $\Omega \subset \mathbb{R}^n$ | Open in $\mathbb{R}^n$. |
| $\partial\Omega$ | Border of $\Omega$. |
| $\lvert \cdot \rvert$ | The Euclidean norm on $\mathbb{R}^n$. |
| $\lVert \cdot \rVert_X$ | The norm define on space X. |
| $E = C([0, 1], \mathbb{R})$ | Banach space of continuous functions from $[0, 1]$ to $\mathbb{R}$. |
| $\rightarrow$ | Strong convergence. |
| *a.e.* | Almost everywhere. |
| *i.e.* | Stands for the Latin phrase "id est," which means "that is" or "in other words. |

# Contents

# General Introduction

This thesis is devoted to study the existence and uniqueness of a positive solution to a one-dimensional and positive radial solution to a multidimensional nonlocal elliptic problems, respectively. Under weak conditions on the reaction terms and the diffusion coefficients.

Boundary value problems (BVPs for short) have a lot of applications. Our interest is in the kind of nonlocal elliptic problems, which have applications across scientific and engineering domains. In physics, these problems are instrumental in describing anomalous diffusion, where particle spreading deviates from classical patterns. They find application in percolation theory, aiding the understanding of how substances permeate through interconnected networks in porous media, see [4, 6, 25, 28, 29, 49, 63]. In biology, nonlocal elliptic problems are employed to model the spread of diseases in populations, ecological dynamics, and species migration influenced by long-range interactions, as in [23, 35, 61]. Financial modeling benefits from these equations by capturing global influences on local market dynamics, enhancing risk assessment and predictions in complex financial systems, for more detail see [2, 10, 24, 58, 65]. Materials science utilizes nonlocal elliptic problems to study the mechanical properties of materials, especially those with nonlocal interactions. Image processing applications involve denoising and restoration, where nonlocal interactions help preserve crucial features. Additionally, these problems play a role in fractional calculus, providing a mathematical foundation for systems with fractional derivatives. They are also employed in solving inverse problems in medical imaging, geophysics, and environmental monitoring, for instance [1, 34, 51, 59, 67]. The broad range of applications underscores the significance of nonlocal elliptic problems in advancing our understanding of diverse natural and engineered systems.

$$\begin{cases} -\Big(\displaystyle\int_0^1 g(u)d\tau\Big)^{\beta} u'' = cg(u)^{\alpha} \quad t \in (0,1), \\ u'(0) = u(1) = 0, \end{cases} \tag{1}$$

and

$$\begin{cases} -\Big(\displaystyle\int_{\Omega} K(u)dy\Big)^{\beta} \Delta u = c(K(u))^{\alpha}, \quad \text{in}\, \Omega, \\ u = 0 \quad \text{on}\, \partial\Omega. \end{cases}$$

A similar problem as (1), JA. Carrillo in [32], used bifurcation argument, where

$$g(u) = e^{-u}.$$

In [20], F.Corrêa. et al, obtained the existence of positive solutions by two method. Used a result of fixed point index theory to sub and super solution and a comparison principle. Additionally, they proved the uniqueness and asymptotic behaviour of the solutions for the evolution case in a bounded smooth domain of $\mathbb{R}^n$ to

$$\begin{cases} -\mathcal{A}\Big(\displaystyle\int_{\Omega} u\Big)\Delta u = g(x,u) \quad \text{in}\, \Omega; \\ u = 0, \quad \text{on}\, \partial\Omega. \end{cases}$$

M. Chipot and B. Lovat in [40], studied analogue problem used index fixed point theorem. In [19], FJSA. Corrêa applied the Krasnoselskii fixed point theorem, Schaefer's fixed point theorem and the comparison principle to get positive solutions for

$$\begin{cases} -\mathcal{B}\Big(\displaystyle\int_0^1 |v|^p d\tau\Big) v'' = m(t)g(v), \quad t \in (0,1); \\ v'(0) = v(1) = 0. \end{cases}$$

An analogous problem was studied by CS. Goodrich, in [13], with second term is $\lambda f(t,v)$, used topological fixed point theory and a novel order cone. In addition, when $p \geq 1$, obtained radial positive solution in an annular domain for

$$\begin{cases} -A\Big(\displaystyle\int_{\Omega} |u|^p d\tau\Big)\Delta u = \lambda g(u(t)). \quad t \in \Omega \\ u = 0, \quad \text{on}\, \partial\Omega. \end{cases}$$

CO. Alves and DP. Covei proved in [11], when the second term is

$$h_1(x,u)f\Big(\int_{\Omega}|u|^q d\tau\Big)+h_2(x,u)g\Big(\int_{\Omega}|u|^r d\tau\Big)$$

where $p,q,r\in[1,+\infty)$. Addition $u(t)>0$, on $t\in\Omega$ with Dirichlet boundary value conditions, B. Yan and T. Ma in [8], with second term is $f_\lambda(x,u)$, proved the existence by sub-super solution. By bifurcation theory, obtained the existence and multiplicity of positive solutions with the changes of the parameter. Moreover, B. Yan and D. Wang in [7], when the second term is $\lambda f(t,u)$, used theory of fixed point index to obtained the multiplicity of positive solutions.

Many results are obtained on the existence and multiplicity of nontrivial solutions or positive solutions and radial positive solutions by using different techniques, as fixed point theorems see [39, 42, 43, 45, 50, 53, 56, 66], Bolzano theorem see [14], bifurcation arguments as in [62], Galerkin approach as [21], sub-supersolution method in [26], variational methods for instance [17, 22, 36, 44, 46, 47], comparison principle method see [41], monotone operator method likes [55], dynamical methods in [12], approximation methods see [18].

This thesis is organized as follows. The first chapter is devoted to recall some fundamental notions which are used for proving the results: Metric and Banach spaces, spaces of k-times continuously differentiable functions and integrable functions, function defined by an integral and derivation under the integral, the fixed point index, theorem of Krasnoselskii fixed point on cones in a Banach space, its proof and how to apply it with a simple example.

In the second chapter, we study the existence of positive solutions in a one-dimension and the uniqueness in a special case for the following problem

$$\begin{cases}-M\Big(t,\displaystyle\int_0^1 f(u)d\tau\Big)u''=g(t,u), & t\in(0,1);\\ u'(0)=u(1)=0,\end{cases}\tag{2}$$

where $f:[0,\infty)\to[0,\infty)$, $g:[0,1]\times[0,\infty)\to[0,\infty)$ and $M:[0,1]\times[0,\infty)\to(0,\infty)$ are continuous functions satisfy weak conditions (2.2) - (2.9). The good choice of cone on the Banach space in Krasnoselskii fixed point theorem gives us a positive fixed point, which is a positive solution to the problem (2). Also, we introduce an example illustrates our study.

In the third chapter, we prove the existence of radial positive solutions in an annular domain of $\mathbb{R}^n$, such that $n \geq 2$ and the uniqueness in a special case, under weak conditions (3.2) - (3.7) for nonlocal elliptic problem:

$$\begin{cases} -M\Big(|x|, \displaystyle\int_{\Omega} f(u)dy\Big)\Delta u = g(|x|, u), \quad \text{in}\,\Omega, \\ u = 0, \quad \text{on}\,\partial\Omega, \end{cases} \tag{3}$$

where $\Omega = \{x \in \mathbb{R}^N |\, a < |x| < b\}$, $N \geq 2$, $a, b \in (0, \infty)$, such that $\quad a < b$. The continuous functions $f : [0, \infty) \to [0, \infty)$, $g : [a, b] \times [0, \infty) \to [0, \infty)$ and $M : [a, b] \times [0, \infty) \to (0, \infty)$. We apply some changes on problem (3) to make the problem one-dimensional, then used Krasnoselskii fixed point theorem to obtain the radial positive solution. Our results can be found in [9].

# Chapter 1

# Preliminary

This first chapter will be devoted to the concepts, theories, and results used throughout the remainder of the dissertation.

## 1.1 Metric Spaces

**Definition 1.1.1.** *[27]*

*Let $X$ be a set. $d : X \times X \to \mathbb{R}$ is a function define distance, if satisfies the following conditions:*

1. *Non-negativity:* $d(x, y) \geq 0 \quad \forall x, y \in X$.
2. *Identity of Indiscernibles:* $d(x, y) = 0 \Leftrightarrow x = y$.
3. *Symmetry:* $d(x, y) = d(y, x) \quad \forall x, y \in X$.
4. *Triangle Inequality:* $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$.

**Example 1.1.1.**

- *On* $\mathbb{R}^n$, *we can consider the following distances:*

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|, \text{ (Manhattan distance)}$$

$$d_2(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}, \text{ (Euclidean distance)}$$

$d_\infty(x,y) = \max\limits_{1\leqslant i\leqslant n}|x_i - y_i|.$

- *On* $C([0,1],\mathbb{R})$, *we define the distances:*

$$d_1(f,g) = \int_0^1 |f(t)-g(t)|dt.$$

$$d_2(f,g) = \sqrt{\int_0^1 (f(t)-g(t))^2 dt}.$$

$$d_\infty(f,g) = \sup_{t\in[0,1]} |f(t)-g(t)|.$$

**Example 1.1.2.**

*The function* $d : \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}_+$, $(x,y) \mapsto |\frac{1}{x} - \frac{1}{y}|$ *define a distance, because :*

- $d(x,y) = 0 \Leftrightarrow |\frac{1}{x} - \frac{1}{y}| = 0 \Leftrightarrow \frac{1}{x} = \frac{1}{y} \Leftrightarrow x = y.$
- $d(x,y) = |\frac{1}{x} - \frac{1}{y}| = |(-1)(\frac{1}{y} - \frac{1}{x})| = |(-1)||\frac{1}{y} - \frac{1}{x}| = d(y,x).$
- $d(x,z) = |\frac{1}{x} - \frac{1}{z}| = |\frac{1}{x} - \frac{1}{y} + \frac{1}{y} - \frac{1}{z}| \leq |\frac{1}{x} - \frac{1}{y}| + |\frac{1}{y} - \frac{1}{z}| = d(x,y) + d(y,z).$

**Definition 1.1.2.** *[27](Metric Space)*

*We call metric space any non-empty set* $X$ *provided with a distance* $d$*, denoted by* $(X,d)$*.*

**Definition 1.1.3.** *[27] (Convergence of a Sequence)*

*Let* $(X,d)$ *be a metric space. A sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $X$ *is said to converge to a point* $x \in X$ *if:*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : \quad d(x_n, x) \leq \varepsilon.$$

**Definition 1.1.4.** *[27] ( Cauchy Sequence)*

*Let* $(X,d)$ *be a metric space. A sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $X$ *is called a Cauchy if*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p,q \geq N : \quad d(x_p, x_q) \leq \varepsilon.$$

**Example 1.1.3.**

*Consider the space* $C([0,1],\mathbb{R})$ *provided by* $d_1$*. Let us define sequence of function*

$$\forall n \in \mathbb{N}^* : \quad f_n(x) = \min(n, \frac{1}{\sqrt{x}}) = \begin{cases} n & si \quad 0 \leq x \leq \dfrac{1}{n^2}; \\ \dfrac{1}{\sqrt{x}} & si \quad \dfrac{1}{n^2} \leq x \leq 1. \end{cases}$$

*For all* $\varepsilon > 0$, *and* $m, n \in \mathbb{N}^*$ *with* $n \geq m$, *we have*

$$\begin{aligned}
d(f_m, f_n) &= \int_0^1 |f_m(x) - f_n(x)| dx \\
&= \int_0^{\frac{1}{n^2}} (n-m) dx + \int_{\frac{1}{n^2}}^{\frac{1}{m^2}} (\frac{1}{\sqrt{x}} - m) dx + \int_{\frac{1}{m^2}}^{1} (\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}}) dx \\
&= (\frac{1}{n}) - (\frac{m}{n^2}) + 2(\frac{1}{m} - \frac{1}{n}) - m(\frac{1}{m^2} - \frac{1}{n^2}) \\
&= \frac{1}{m} - \frac{1}{n} < \frac{1}{m}.
\end{aligned}$$

*We can take* $N_0 = [\frac{1}{\varepsilon}] + 1$. *Then,* $f_n$ *is Cauchy sequence.*

**Proposition 1.1.1.** *[27]*

1. *In a metric space, any convergent sequence is Cauchy.*
2. *Any Cauchy sequence is bounded.*

**Definition 1.1.5.** *[27] ( Complete Space)*

*The metric space* $(X, d)$ *is said to be complete if any Cauchy sequence in* $X$ *is convergent in* $X$.

**Example 1.1.4.**

*The space of real numbers* $\mathbb{R}$ *equipped with metric is a complete metric space.*
*Let* $(x_n)_{n\in\mathbb{N}}$ *be a Cauchy sequence in* $\mathbb{R}$, *we have*

$$\forall \varepsilon > 0, \exists N_1 \in \mathbb{N}, \forall n \geq m \geq N_1 : \quad |x_n - x_m| < \frac{\varepsilon}{2}. \tag{1.1}$$

*The sequence* $(x_n)_{n\in\mathbb{N}}$ *is bounded by Proposition 1.1.1, and by the Bolzano-Weierstrass theorem, every bounded sequence in* $\mathbb{R}$ *has a convergent subsequence. Let* $(x_{n_k})_{k\in\mathbb{N}}$ *be a subsequence of* $(x_n)_{n\in\mathbb{N}}$ *that converges to some limit* $x \in \mathbb{R}$. *Then,*

$$\forall \varepsilon > 0, \exists N_2 \in \mathbb{N}, \forall k \geq N_2 : \quad |x_{n_k} - x| < \frac{\varepsilon}{2}. \tag{1.2}$$

*Choose* $k \geq N_2$ *such that* $n_k \geq N_1$. *From* (1.1) *and* (1.2), *we get*

$$\forall n \geq N : \ |x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

*Therefore,* $(x_n)_{n\in\mathbb{N}}$ *converges to* $x$.

**Definition 1.1.6.** *[27] (Convex set)*
*Let $X$ be a set. We say that a subset $K$ of $X$ is convex if:*

$$\forall x, y \in K,\ \forall \lambda \in [0,1] : \quad \lambda x + (1-\lambda) y \in K.$$

**Theorem 1.1.1.** *[27] (Closed Set)*
*A set $C$ is closed if and only if the limit of any convergent sequence of elements of $C$ belongs to $C$.*

**Theorem 1.1.2.** *[27] (Compact Set)*
*A subset $K \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

**Definition 1.1.7.** *[27](Closure Set)*
*The closure of a set $A$ in a banach space $X$ is given by:*

$$\overline{A} = \bigcap \{F \subseteq X : F \text{ is closed and } A \subseteq F\}.$$

**Definition 1.1.8.** *[27] (Relatively Compact Set)*
*Let $X$ be a metric space and let $A \subseteq X$. The set $A$ is called relatively compact if the closure of $A$ is compact.*

## 1.2 Banach Spaces

**Definition 1.2.1.** *[27] ( Normated Vector Space)*
*Let $X$ be a vector space over $\mathbb{R}$. A norm on $X$ is an application $\| \cdot \| : X \to \mathbb{R}_+$ satisfied*

- $x \in X,\ \|x\| = 0 \Leftrightarrow x = 0 \quad (separation).$
- $\forall x \in X, \forall \lambda \in \mathbb{R},\ \|\lambda x\| = |\lambda| \|x\| \quad (homogeneity).$
- $\forall x, y \in X,\ \|x + y\| \leq \|x\| + \|y\| \quad (triangular\ inequality).$

**Definition 1.2.2.** *[27] (Norm Space)*
*The couple $(X, \| \cdot \|)$, where $X$ is a vector space and $\| \cdot \|$ is a norm on $X$, is called norm vector space.*

**Remark 1.2.1.**

*Set*

$$\forall x, y \in X : \quad d(x,y) = \|x - y\|.$$

*The function $d$ define a distance on $X$. So, everything that was mentioned in the metric space remains verified in Banach space.*

**Example 1.2.1.**

*The space $C([0,1],\mathbb{R})$ provided by $\|\cdot\|_\infty : X \to \mathbb{R}_+, f \mapsto \|f\|_\infty = \sup\limits_{t\in[0,1]} |f(t)|$. Indeed,*

- $\|f\|_\infty = 0 \Leftrightarrow \sup\limits_{t\in[0,1]} |f(t)| = 0 \Leftrightarrow |f(t)| = 0, \forall t \in [0,1] \Leftrightarrow f(t) = 0,\ \forall t \in [0,1]$.
- $\forall \alpha \in \mathbb{R}, \forall f \in C :$

  $\|\alpha f\|_\infty = \sup\limits_{t\in[0,1]} |(\alpha f)(t)| = \sup\limits_{t\in[0,1]} |\alpha f(t)| = |\alpha| \sup\limits_{t\in[0,1]} |f(t)| = |\alpha| \|f\|_\infty$.
- $\forall t \in [0,1], \forall f, g \in C :\ |(f+g)(t)| = |f(t) + g(t)| \le |f(t)| + |g(t)|$.

*Taking the supremum on both sides, we get*

$$\sup_{t\in[0,1]} |(f+g)(t)| \le \sup_{t\in[0,1]} \left(|f(t)| + |g(t)|\right).$$

*By properties of the supremum,*

$$\sup_{t\in[0,1]} \left(|f(t)| + |g(t)|\right) \le \sup_{t\in[0,1]} |f(t)| + \sup_{t\in[0,1]} |g(t)|.$$

*Therefore,*

$$\|f+g\|_\infty = \sup_{t\in[0,1]} |f(t) + g(t)| \le \sup_{t\in[0,1]} |f(t)| + \sup_{t\in[0,1]} |g(t)| = \|f\|_\infty + \|g\|_\infty.$$

**Definition 1.2.3.** *[27] (Banach Space)*

*We call Banach space a complete normed vector space, that is to say in which any Cauchy sequence is convergent.*

**Example 1.2.2.**

*Consider $X = C([a,b],\mathbb{R})$. We show that $(X, \|\cdot\|_\infty)$ is a Banach space. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $X$, this means*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \ge m \ge N : \quad \|f_n - f_m\|_\infty \le \frac{\varepsilon}{3}.$$

*which implies,*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq m \geq N, \forall x \in [0,1] : \quad |f_n(x) - f_m(x)| \leq \frac{\varepsilon}{3}. \tag{1.3}$$

*Since* $(f_n(x))_{n\in\mathbb{N}}$ *is a Cauchy sequence on* $\mathbb{R}$*, where* $(\mathbb{R}, |\cdot|)$ *is complete, we obtain*

$$\lim_{n\to\infty} f_n(x) = f(x), \text{ for each } x \in [0,1].$$

*Using* (1.3)*, fix* $n$ *and taking the limit as* $m \to \infty$*, we get*

$$\forall n \geq N, \forall x \in [0,1] : \quad |f_n(x) - f(x)| \leq \frac{\varepsilon}{3}. \tag{1.4}$$

*Therefore, for all* $n \geq N$*:*

$$\|f_n - f\|_\infty \leq \frac{\varepsilon}{3}.$$

*This shows that* $f_n \to f$ *uniformly on* $[0,1]$*. We will establish* $f \in X$*. For* $x, x_0 \in [0,1]$*, we have*

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|. \tag{1.5}$$

*From the continuity of* $f_n$ *at* $x_0$*,* $n \geq N$ *:*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in [0,1], |x - x_0| < \delta : \quad |f_n(x) - f_n(x_0)| \leq \frac{\varepsilon}{3}. \tag{1.6}$$

*According to* (1.4)*,* (1.5) *and* (1.6)*, it follows that*

$$\forall x \in [0,1], |x - x_0| < \delta : \quad |f(x) - f(x_0)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

*Hence,* $f$ *is continuous.*

**Definition 1.2.4.** *(Converge Weakly)*

*Let* $X$ *be a Banach space. A sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $X$ *is said to converge weakly to* $x \in X$ *if:*

$$\forall f \in X^*, \quad f(x_n) \to f(x) \text{ as } n \to \infty,$$

*where* $X^*$ *denotes the dual space of* $X$*, the space of continuous linear functionals on* $X$*.*

**Definition 1.2.5.** *[27]*

*Consider* $X = C([a,b], \mathbb{R})$ *with the norm* $\|\cdot\|_\infty$ *and* $A$ *is a subset of* $X$*.*

- *We say that* $A$ *is uniformly bounded, if there exists a constant* $C > 0$, *such that*

$$\|f\|_\infty \leq C, \forall f \in A.$$

- *We say that* $A$ *is uniformly equicontinuous, if for all* $\varepsilon > 0$ *there exists* $\delta > 0$,

$$\forall\, x, y \in [a, b] : \quad |x - y| < \delta \Rightarrow \Big| f(x) - f(y) \Big| \leq \varepsilon, \ \forall\, f \in A.$$

**Example 1.2.3.**

*Consider the family of functions*

$$\mathbb{G} = \{g_n \mid n \in \mathbb{N}^*\},$$

*where* $\forall n \in \mathbb{N}^*, \forall x \in [0,1] : \ g_n(x) = \sin(nx)$. *We prove that this family is equicontinuous. We have*

$$|\sin(nx) - \sin(ny)| = 2 \left| \cos\left(\frac{nx + ny}{2}\right) \sin\left(\frac{n(x - y)}{2}\right) \right|.$$

*Since* $\left|\cos\left(\frac{nx+ny}{2}\right)\right| \leq 1$, *we obtain*

$$|\sin(nx) - \sin(ny)| \leq 2 \left| \sin\left(\frac{n(x - y)}{2}\right) \right|. \tag{1.7}$$

*Apply the Mean Value Theorem on* $[0, \theta]$, *where* $\theta > 0$ :

$$\sin'(\xi) = \frac{\sin(\theta) - \sin(0)}{\theta - 0}, \quad \forall \xi \in (0, \theta).$$

*Since* $\sin'(\theta) = \cos(\theta)$ *such that* $\cos(\theta) \leq 1$, *it follows that*

$$\sin(\theta) \leq \theta, \quad \forall\, \theta > 0. \tag{1.8}$$

*Remplacing* $\frac{n(x-y)}{2} = \theta$ *at* (1.8), *we obtain*

$$\sin\left(\frac{n(x - y)}{2}\right) \leq \frac{n(x - y)}{2}, \ \forall n > 0. \tag{1.9}$$

*From* (1.9) *and* (1.7), *we have*

$$|\sin(nx) - \sin(ny)| \leq 2 \left| \frac{n(x - y)}{2} \right| = n|x - y|.$$

*To ensure that* $|\sin(nx) - \sin(ny)| < \varepsilon$*, we need*

$$n|x - y| \leq \varepsilon,$$

*which implies*

$$|x - y| \leq \frac{\varepsilon}{n} < \varepsilon.$$

*Hence, for all* $n \in \mathbb{N}, \delta = \varepsilon$*, we get*

$$\forall n \in \mathbb{N},\ |x - y| < \varepsilon \ \Rightarrow\ |g_n(x) - g_n(y)| \leq \varepsilon.$$

*This leads to, the family of functions* $(g_n)_{n\in\mathbb{N}}$ *is equicontinuous on* $[0, 1]$.

**Theorem 1.2.1.** *[27] (Ascoli-Arzela )*
*Let* $A$ *be a subset of the space* $C([a, b], \mathbb{R})$*. Then,* $A$ *is relatively compact if and only if the functions of* $A$ *are uniformly bounded and equicontinuous.*

**Example 1.2.4.**
*Consider the set* $A$ *of functions defined by:*

$$A = \{f_\alpha,\ \alpha \in [0, 1]\},$$

*where,*

$$f_\alpha(x) = e^{-\alpha x},\ \forall x \in [0, 1].$$

*We will show that* $A$ *is relatively compact.* $A$ *is uniformly Bounded. Indeed, for any* $\alpha \in [0, 1]$ *and* $x \in [0, 1]$,

$$|f_\alpha(x)| = \left|e^{-\alpha x}\right| \leq 1,$$

*Thus, the set* $A$ *is uniformly bounded by 1.*
$A$ *is equicontinuous, for* $f_\alpha \in A$*, we have*

$$|f_\alpha(x) - f_\alpha(y)| = \left|e^{-\alpha x} - e^{-\alpha y}\right|.$$

*Applying the Mean Value Theorem on the function* $e^{-\alpha t}$ *on the interval* $[x, y]$*, there exists* $c \in (x, y)$*, such that*

$$-\alpha e^{-\alpha c} = \frac{e^{-\alpha y} - e^{-\alpha x}}{y - x}.$$

*Thus,*

$$|e^{-\alpha y} - e^{-\alpha x}| = \alpha e^{-\alpha c}|y - x| \leq \alpha|y - x|.$$

*Now, for a given* $\varepsilon > 0$*, such that*

$$|e^{-\alpha y} - e^{-\alpha x}| < \varepsilon.$$

*For any* $\alpha \in [0, 1], \delta = \varepsilon : |x - y| < \delta$*, then*

$$|e^{-\alpha y} - e^{-\alpha x}| \leq \alpha|x - y| < \alpha\varepsilon \leq \varepsilon.$$

*Hence, the set* $A$ *is equicontinuous.*

**Definition 1.2.6.** *[16]( Cone)*
*A non-empty subset* $P$ *of a Banach space* $X$ *is said to be a cone on* $X$ *if it is convex, closed and verifies the following two conditions:*

1. *For all* $x \in P$ *and* $\lambda \geq 0$*, implies* $\lambda x \in P$*.*
2. *For all* $x \in P$ *and* $-x \in P$ *implies* $x = 0$*.*

**Example 1.2.5.**
*Consider* $P$ *a subset* $\mathbb{R}^2$ *define by:*

$$P = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}.$$

*We need to show that* $P$ *satisfies the conditions of the cone.*
*1. For all* $(x, y) \in P$ *and* $\lambda \geq 0$*, we have* $\lambda(x, y) = (\lambda x, \lambda y) \in P$*, since* $\lambda x \geq 0$ *and* $\lambda y \geq 0$ *(by the definition of* $P$*).*
*2. For all* $(x, y) \in P$ *and* $(-x, -y) \in P$*, we have* $x \geq 0, y \geq 0, -x \geq 0$ *and* $-y \geq 0$ *then* $x = 0, y = 0$*. Therefore,* $(x, y) = (0, 0)$*.*
*3. Convexity: For any* $(x_1, y_1), (x_2, y_2) \in P$ *and any* $\lambda \in [0, 1]$ *:*

$$\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) = (\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2).$$

*Since* $x_1, x_2, y_1, y_2 \geq 0$*, it follows that*

$$\lambda x_1 + (1 - \lambda)x_2 \geq 0,\ \lambda y_1 + (1 - \lambda)y_2 \geq 0.$$

*Therefore,* $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) \in P$, *then* $P$ *is convex.*

*4. Closedness: Let* $(x_n, y_n)_{n\in\mathbb{N}}$ *be a sequence in* $P$, *such that*

$$\lim_{n\to\infty}(x_n, y_n) = (x, y),$$

*which implies*

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y.$$

*From* $(x_n, y_n) \in P$, *we have*

$$x_n \geq 0 \quad \text{and} \quad y_n \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

*Since* $x_n \geq 0$ *for all* $n$, *and the limit of a non-negative sequence is also non-negative, it follows that* $x \geq 0$. *Similarly,* $y_n \geq 0$ *for all* $n$ *gives* $y \geq 0$. *Therefore,* $(x, y) \in P$. *Hence* $P$ *is closed.*

**Example 1.2.6.**

*Consider* $X = C([0,1], \mathbb{R})$. *We define the following set:*

$$P = \{f \in X \mid f \text{ is non-decreasing}, f(0) = 0\}.$$

*1. For any* $f \in P$, *if* $-f \in P$, *then* $f(0) = 0$, $-f(0) = 0$, *and* $f, -f$ *are both nondecreasing. This implies*

$$f(t_1) \leq f(t_2) \quad \text{and} \quad -f(t_1) \leq -f(t_2),$$

*for all* $t_1, t_2 \in [0,1]$ *with* $t_1 \leq t_2$. *Thus,* $f(t_1) = f(t_2)$ *for all* $t_1, t_2 \in [0,1]$, *which implies* $f(t) = f(0) = 0, \forall t \in [0,1]$. *Therefore,* $f = 0$.

*2. Let* $f \in P$ *and* $\lambda \geq 0$. *Then,*

$$(\lambda f)(0) = \lambda f(0) = \lambda 0 = 0.$$

*In addition, if* $f$ *is non-decreasing function, then* $\lambda f$ *is also non-decreasing because* $\lambda \geq 0$. *Thus,* $\lambda f \in P$.

*3. Convexity. Let* $f, g \in P$ *and* $\alpha \in [0,1]$. *We have*

$$f(0) = 0 \quad \text{and} \quad g(0) = 0 \implies (\alpha f + (1-\alpha)g)(0) = \alpha f(0) + (1-\alpha)g(0) = 0.$$

*Also, for any* $t_1, t_2 \in [0,1]$ *with* $t_1 \leq t_2$, $f(t_1) \leq f(t_2)$ *and* $g(t_1) \leq g(t_2)$, *it follows that*

$$\alpha f(t_1) + (1-\alpha)g(t_1) \leq \alpha f(t_2) + (1-\alpha)g(t_2).$$

*Therefore,* $\alpha f + (1-\alpha)g \in P$.

*4. Closedness. Suppose* $(f_n)_{n\in\mathbb{N}}$ *is a sequence in* $P$*, such that* $f_n \to f$ *uniformly.*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : \quad \|f_n - f\|_\infty < \varepsilon,$$

*which implies*

$$|f_n(0) - f(0)| < \varepsilon \quad \text{for all } n \geq N.$$

*Since* $f_n \in P$*, we have* $f_n(0) = 0$ *for all* $n$*, then*

$$|0 - f(0)| < \varepsilon \implies |f(0)| < \varepsilon, \forall \varepsilon > 0.$$

*Thus,* $f(0) = 0$.

*From the fact that* $f_n$ *is non-decreasing, for any* $t_1, t_2 \in [0,1]$ *with* $t_1 \leq t_2$,

$$f_n(t_1) \leq f_n(t_2).$$

*Passing to the limit as* $n \to \infty$*, we get*

$$f(t_1) \leq f(t_2).$$

*Hence,* $f$ *is non-decreasing. Therefore,* $f \in P$*. We concude that* $P$ *is cone.*

**Theorem 1.2.2.** *[16] (Continuous Operator)*

*Let* $X$ *be a Banach spaces. An operator* $T : X \to X$ *is called continuous at* $x \in X$*, if for any sequence* $(x_n)_{n\in\mathbb{N}} \subset X$ *which converges by the norm to* $x$*,* $T((x_n)_{n\in\mathbb{N}})$ *converges by the norm to* $T(x)$*.*

$T$ *is said to be continuous on a set* $A \subset X$ *if* $T$ *is continuous at every point* $x \in A$*.*

**Proposition 1.2.1.** *[16](Compact Operator)*

*For* $X, Y$ *are two Banach spaces. An operator* $T : X \to Y$ *is called compact if and only if for every bounded sequence* $(x_n)_{n\in\mathbb{N}}$ *of* $X$*, we can extract a subsequence* $(x_{n_k})_{k\in\mathbb{N}}$ *of* $X$ *such that the sequence* $(T(x_{n_k}))_{k\in\mathbb{N}}$ *converges in* $Y$*. In other words,* $T$ *transforms any bounded set of* $X$ *into a relatively compact set of* $Y$*.*

**Definition 1.2.7.** *[16](Completely Continuous Operator)*
*Let $X$ and $Y$ be two Banach spaces. An operator $T : X \to Y$ is said to be completely continuous if for every sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ that converges weakly to some $x \in X$, the image sequence $(T(x_n))_{n\in\mathbb{N}}$ converges strongly to $T(x)$ in $Y$. In other words, $T$ is continuous and compact.*

**Example 1.2.7.**
*Consider the space $X = C([0, 1], \mathbb{R})$ equipped with the $\|\cdot\|_\infty$ norm. Define the operator*

$$T : X \to X$$
$$f \mapsto Tf,$$

*where,*

$$\forall x \in [0, 1] : \ \ Tf(x) = \int_0^x f(t)dt.$$

*We aim to prove that $T$ is a compact operator. Consider a bounded set $B \subset X$. This means*

$$\exists C > 0, \forall f \in B : \quad \|f\|_\infty \leq C. \tag{1.10}$$

*Showing the set $\mathcal{M} = \{Tf : f \in B\}$ is uniform boundedness and equicontinuity. For any $f \in B, x \in [0, 1]$, we have*

$$|Tf(x)| = \left|\int_0^x f(t)\,dt\right| \leq \int_0^x |f(t)|\,dt \leq \int_0^1 \|f\|_\infty\,dt \leq \|f\|_\infty. \tag{1.11}$$

*From (1.10) and (1.11), we get*

$$\|Tf\|_\infty \leq C, \forall f \in B.$$

*Hence, $\mathcal{M}$ is uniformly bounded.*
*For $f \in B$ and $x_1, x_2 \in [0, 1]$ with $x_1 < x_2$:*

$$|Tf(x_2) - Tf(x_1)| = \left|\int_0^{x_2} f(t)\,dt - \int_0^{x_1} f(t)\,dt\right| = \left|\int_{x_1}^{x_2} f(t)\,dt\right| \leq \int_{x_1}^{x_2} |f(t)|. \tag{1.12}$$

*Using (1.10) and (1.12), it follows that*

$$\left|\int_{x_1}^{x_2} f(t)\,dt\right| \leq \int_{x_1}^{x_2} |f(t)|\,dt \leq \int_{x_1}^{x_2} C\,dt \leq C|x_2 - x_1|.$$

*Given any* $\varepsilon > 0$*, choose* $\delta = \frac{\varepsilon}{C}$*. Then for* $|x_2 - x_1| < \delta$*, we have:*

$$|Tf(x_2) - Tf(x_1)| \leq C|x_2 - x_1| < C\delta \leq \varepsilon.$$

*This shows that* $\mathcal{M}$ *is equicontinuous. By the Arzelà–Ascoli Theorem* $\mathcal{M}$ *is relatively compact. Thus,* $T$ *is completely continuous.*

**Definition 1.2.8.** *[27] ( Lipschitzian)*
*We say that a function* $f : (E; \|\cdot\|_E) \to (F; \|\cdot\|_F)$ *is Lipschitz with constant* $k > 0$ *(or k-Lipschitzian), if it satisfies:*

$$\forall x, y \in E : \quad \|f(x) - f(y)\|_F \leq k\|x - y\|_E.$$

**Example 1.2.8.**
*Let the function* $f : \mathbb{R}_+ \to \mathbb{R}$, $x \mapsto \frac{x}{x+1}$ *is Lipschitzian because:*

$$\begin{aligned} \forall x, y \in \mathbb{R}_+ : \quad |f(x) - f(y)| &= |\frac{x}{x+1} - \frac{y}{y+1}| \\ &\leq \frac{|x-y|}{(1+y)(x+1)} \\ &\leq |x - y|. \end{aligned}$$

*So,* $f$ *is* $1$ *- Lipschitzian on* $\mathbb{R}_+$*.*

## 1.3 Spaces of k-times Continuously Differentiable Functions and Integrable Functions

**Definition 1.3.1.** *[52](Space* $C^k([a,b],\mathbb{R})$*)*
*Let* $k \in \mathbb{N}^*$ *and* $[a, b] \subset \mathbb{R}$*. The space* $C^k([a,b],\mathbb{R})$ *is defined by the space of k-times continuously differentiable functions on* $[a, b]$ *with values in* $\mathbb{R}$*. It is a Banach space when we equip it with the norm*

$$\|f\|_{C^k([a,b],\mathbb{R})} = \sum_{n=0}^{k} \|f^{(n)}\|_\infty = \sum_{n=0}^{k} \sup_{x\in[a,b]} |f^{(n)}(x)|.$$

*In particular, for* $k = 0$, $C^0([a,b],\mathbb{R}) = C([a,b],\mathbb{R})$ *is the space of continuous functions on* $[a, b]$ *endowed with the norm*

$$\|f\|_{C([a,b],\mathbb{R})} = \sup_{x\in[a,b]} |f(x)|.$$

**Definition 1.3.2.** *[27] (The space $L^p(\Omega)$ )*
*Let $\Omega$ be an open set of $\mathbb{R}^n$ endowed with the Lebesgue measure and $p \in [1, +\infty[$. The space $L^p(\Omega)$ is defined by:*

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R};\ f measurable\ and\ \int_\Omega |f(x)|^p dx < +\infty\}.$$

*For all $f \in L^p(\Omega)$, we denote*

$$\|f\|_{L^p} = \Big(\int_\Omega |f(x)|^p dx\Big)^{\frac{1}{p}}.$$

**Proposition 1.3.1.** *[27] (Cauchy-Schwarz Inequality)*
*In $\mathbb{R}^n$, let $\Omega$ be an open set. If $f \in L^2(\Omega)$ and $g \in L^2(\Omega)$, then:*

$$\left|\int_\Omega f(x)g(x)\, dx\right| \leq \|f\|_2\|g\|_2.$$

**Theorem 1.3.1.** *[48](Convergence Dominated by Lebesgue)*
*Consider an open set $\Omega \subset \mathbb{R}^N$. Let $p \in [1, +\infty[$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions on $L^p(\Omega)$ converging almost everywhere to a measurable function $f$. We assume that there is a function $g \in L^p(\Omega)$, such that for all $n \in \mathbb{N}$,*

$$|f_n(x)| \leq g(x) \quad a.e.\ on\ \Omega.$$

*Then, $f \in L^p(\Omega)$ and we have*

$$\lim_{n\to+\infty} \|f_n - f\|_{L^p(\Omega)} = 0.$$

**Theorem 1.3.2.** *[54]( A Convergence Theorem for the Riemann Integral)*
*Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of Riemann integrable that converging uniformly to $f : [a, b] \to \mathbb{R}$. Then $f$ is also Riemann integrable and*

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

## 1.4 Function Defined by an Integral and Derivation Under the Integral

**Theorem 1.4.1.** *[38] (Continuity of an Integral with Parameter )*
*Either a measurable subset $\Omega \subset \mathbb{R}^n$, or an interval $I \subset \mathbb{R}$ with non-empty interior and a*

*function* $f : \Omega \times I \to \mathbb{R}$. *If the following three conditions are satisfied:*

*1. for all* $t \in I$, *the function* $x \mapsto f(x,t)$ *is measurable.*

*2. for almost everywhere* $x \in \Omega$, *the function* $t \mapsto f(x,t)$ *is continuous on* $I$.

*3. there is a function* $l : \Omega \to \mathbb{R}^+$ *Lebesgue-integrable such that*

$$\left|\frac{\partial f}{\partial t}(x,t)\right| \leq l(x) \quad a.e.\ x \in \Omega, \quad \forall t \in I.$$

*So, the function*

$$t \mapsto \int_\Omega f(x,t)dx,$$

*is continuous over the interval* $I$.

**Theorem 1.4.2.** *[38] (Derivation Under the Integral)*

*Either a measurable subset* $\Omega \subset \mathbb{R}^n$, *an interval* $I \subset \mathbb{R}$ *with non-empty interior and function* $f : \Omega \times I \to \mathbb{R}$. *If the following three conditions are satisfied:*

*1.for all* $t \in I$, *the function* $x \mapsto f(x,t)$ *is measurable.*

*2. for almost everywhere* $x \in \Omega$, *the function* $t \mapsto f(x,t)$ *is derivable from* $\frac{\partial f}{\partial t}(x,t)$.

*3. there is a function* $l : \Omega \to \mathbb{R}^+$ *Lebesgue-integrable such that*

$$\left|\frac{\partial f}{\partial t}(x,t)\right| \leq l(x) \quad a.e.\ x \in \Omega, \forall t \in I.$$

*So, the function:*

$$t \mapsto \int_\Omega f(x,t)dx,$$

*is differentiable on* $I$ *of derivative:*

$$\frac{d}{dt}\int_\Omega f(x,t)dx = \int_\Omega \frac{\partial f}{\partial t}(x,t)dx.$$

**Theorem 1.4.3.** *( Integration by Parts)*

*Let* $f, g : [a,b] \to \mathbb{R}$ *of class* $C^1$. *We have*

$$\int_a^b f'(x)g(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)dx.$$

**Theorem 1.4.4.** *[31](Theorem of Change Variables in a Multiple Integral )*

*Suppose that* $0 \leq a < b$, *and set* $A = \{x \in \mathbb{R}^n : a \leq |x| \leq b\}$. *Let* $g : [a,b] \to \mathbb{R}$ *be Riemann* *integrable on* $[a, b]$*, and let* $f(x) = g(|x|)$ *for* $x \in A$*. Then* $f$ *is Riemann integrable on* $A$*, and*

$$\int_A f = n\mathcal{V}_n \int_a^b g(r)r^{n-1}dr,$$

*where* $\mathcal{V}_n$ *is the volume of the unit ball in* $\mathbb{R}^n$*.*

***Proof.***

Let $n \geq 2$ and $g(r) \geq 0$, $r \in [a, b]$. Given $\epsilon > 0$, choose $a = r_0 < r_1 < r_2 < ... < r_N = b$ and $0 \leq m_k \leq M_K$ for $k = \overline{1, N}$ such that

$$\begin{aligned} r_k - r_{k-1} < \epsilon \quad &\text{for} \quad 1 \leq k \leq N; \\ m_k \leq g(r) \leq M_k \quad &\text{for} \quad r_{k-1} \leq r \leq r_k, \quad 1 \leq k \leq N; \end{aligned} \tag{1.13}$$

$$\sum_{k=1}^{N}(M_k - m_k)(r_k - r_{k-1}) < \epsilon. \tag{1.14}$$

For $1 \leq k \leq N$, let $A_k = \{x \in A : r_{k-1} \leq |x| \leq r_k\}$. Note that $m_k \leq f(x) \leq M_k$ for $x \in A_k$, and that the n-dimensional volume of $A_k$ is $\mathcal{V}_n(r_k^n - r_{k-1}^n)$. Hence the upper and lower Riemann integrals of $f$ over $A$ satisfy

$$\sum_{k=1}^{N} m_k\mathcal{V}_n(r_k^n - r_{k-1}^n) \leq \underline{\int}_A f \leq \overline{\int}_A f \leq \sum_{k=1}^{N} M_k\mathcal{V}_n(r_k^n - r_{k-1}^n).$$

But, by the mean value theorem, we have

$$nr_{k-1}^{n-1}(r_k - r_{k-1}) \leq r_k^n - r_{k-1}^n \leq nr_k^{n-1}(r_k - r_{k-1})$$

for $1 \leq k \leq N$. It follows that

$$L_\epsilon \leq \underline{\int}_A f \leq \overline{\int}_A f \leq U_\epsilon,$$

where $L_\epsilon = n\mathcal{V}_n \sum_{k=1}^{n} m_k r_{k-1}^{n-1}(r_k - r_{k-1})$ and $U_\epsilon = n\mathcal{V}_n \sum_{k=1}^{n} M_k r_k^{n-1}(r_k - r_{k-1})$. On the other hand,

$$\sum_{k=1}^{N} m_k r_{k-1}^{n-1}(r_k - r_{k-1}) \leq \int_a^b g(r)r^{n-1}dr \leq \sum_{k=1}^{N} M_k r_k^{n-1}(r_k - r_{k-1}).$$

We conclude that $\underline{\int}_A f$, $\overline{\int}_A f$, and $n\mathcal{V}_n \int_a^b g(r)r^{n-1}dr$ all belong to the interval $[L_\epsilon, U_\epsilon]$ and this is so for every $\epsilon > 0$.

To complete the proof it suffices to demonstrate that $U_\epsilon - L_\epsilon \to 0$ as $\epsilon \to 0^+$. To this end, first note that if $0 \le m \le M$, $0 \le r \le R \le b$, $R - r < \epsilon$, and $n \ge 2$. Then, by another appeal to the mean value theorem, we find

$$\begin{aligned} 0 \le MR^{n-1} - mr^{n-1} &= (M-m)R^{n-1} + m(R^{n-1} - r^{n-1}) \\ &\le (M-m)R^{n-1} + m(n-1)R^{n-2}(R-r) \\ &\le (M-m)b^{n-1} + M(n-1)b^{n-2}\epsilon. \end{aligned}$$

Thus, for $0 < \epsilon < 1$ and with $M = \max\{M_1, ..., M_N\}$, we have

$$\begin{aligned} 0 \le (n\mathcal{V}_n)^{-1}(U_\epsilon - L_\epsilon) &= \sum_{k=1}^{N} [M_k r_k^{n-1} - m_k r_{k-1}^{n-1}](r_k - r_{k-1}) \\ &\le \sum_{k=1}^{N} [(M_k - m_k)b^{n-1} + M_k(n-1)b^{n-2}\epsilon](r_k - r_{k-1}) \\ &\le b^{n-1}\sum_{k=1}^{N}(M_k - m_k)(r_k - r_{k-1}) + M(n-1)b^{n-2}\epsilon\sum_{k=1}^{N}(r_k - r_{k-1}) \\ &\le b^{n-1}\epsilon + M(n-1)b^{n-2}\epsilon(b-a) \end{aligned}$$

according to (1.13) and (1.14). Hence $U_\epsilon - L_\epsilon \to 0$ as $\epsilon \to 0^+$. □

## 1.5 Fixed Point Theory

**Definition 1.5.1.** *[52](Fixed Point)*

*Let $X$ be a Banach space and $T : X \to X$ be an operator. We call fixed point of $T$ any point $x \in X$ such that $Tx = x$.*

**Example 1.5.1.**

*Let $X = \mathbb{R}^2$ with the Euclidean norm, and define the operator*

$$\begin{aligned} &T : X \to X \\ &(x, y) \mapsto T(x, y) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right). \end{aligned}$$

*We have,*

$$T(1,1) = \left(\frac{1+1}{2}, \frac{1+1}{2}\right) = (1,1).$$

*Therefore, $(1,1)$ is a fixed point of $T$.*

**Example 1.5.2.**
*Let $T$ be a function defined from $\mathbb{R}$ to $\mathbb{R}$ by:*

$$T(x) = x + \frac{\pi}{2} - \arctan(x).$$

*We look for the fixed points of $T$*

$$T(x) = x \Leftrightarrow x + \frac{\pi}{2} - \arctan(x) = x \Leftrightarrow \frac{\pi}{2} = \arctan(x),$$

*which is impossible because the function tan is not defined at $\frac{\pi}{2}$. Then, $T$ has no fixed in $\mathbb{R}$.*

**Theorem 1.5.1.** *[27] (Theorem of Heine )*
*Let $X$ and $Y$ be two metric spaces, such that $X$ is compact. Any continuous mapping from $X$ to $Y$ is uniformly continuous.*

## 1.6 Reminder on Green Function

George Green introduced the mathematical function known as "Green's function" in 1828. It is employed in many dimensions to solve partial and ordinary differential equations. The kernel of the integral operator is the term used to describe Green's function.

We consider the ordinary differential equations:

$$(H) \quad (py')' + py = 0,$$

$$(NH) \quad (py')' + py = f,$$

where $p \in C^1([a,b],\mathbb{R})$, $p(t) \neq 0$ for all $t \in [a,b]$, $f \in C([a,b],\mathbb{R})$ as well as the conditions:

$$(CB)_h \qquad \begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) = 0, \\ \beta_1 y(b) + \beta_2 y'(b) = 0, \end{cases}$$

$$(CB)_{nh} \qquad \begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) = \gamma, \\ \beta_1 y(b) + \beta_2 y'(b) = \delta, \end{cases}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma$ and $\delta$ are constants such as $|\alpha_1| + |\alpha_2|, |\beta_1| + |\beta_2| \neq 0$.

**Definition 1.6.1.** *[57]*

*We call Green function associated with the homogeneous problem* $(H), (CB)_h$ *a function* $G : [a, b] \times [a, b] \to \mathbb{R}$ *satisfying the following properties:*

1. *$G$ is continuous on $[a, b] \times [a, b]$.*
2. *$G$ is symmetric: $G(t, s) = G(s, t)$ for all $(t, s) \in [a, b] \times [a, b]$.*
3. *for $s \in [a, b]$, the function $t \mapsto \frac{\partial G}{\partial t}(t, s)$ is continuous for all $t \neq s$.*
4. *$\lim\limits_{t \to s^+} \frac{\partial G}{\partial t}(t, s) - \lim\limits_{t \to s^-} \frac{\partial G}{\partial t}(t, s) = \frac{1}{p(s)}$ for all $s \in [a, b]$.*
5. *for $s \in [a, b]$, the partial function $t \mapsto G(t, s)$ is a solution of the equation $(H)$ for all $t \neq s$.*
6. *for $s \in [a, b]$, the partial function $t \mapsto G(t, s)$ satisfies the conditions $(CB)_h$.*

**Theorem 1.6.1.** *(Existence and Uniqueness of Green Function)*

*Suppose that the homogeneous problem $(H), (CB)_h$ does not admit a non-trivial solution. Then, there exists only function $G$ and is called the function of Green, such that, for any function $f$, the solution $y$ of the non-homogeneous problem $(NH) - (CB)_h$ can be written uniquely as:*

$$\forall t \in [a, b] : \quad y(t) = \int_0^1 G(t, s) f(s) ds.$$

**Example 1.6.1.**

*Consider the following problem:*

$$\begin{cases} -u''(t) = 0, \ 0 < t < 1, \\ u'(0) = u(1) = 0. \end{cases} \tag{1.15}$$

*Let us construct Green function of (1.15). Remembering that any solution of (1.15) is of the form $u(t) = at + b$. According to the associated boundary conditions, we obtain $a = b = 0$, which shows that (1.15) does not admit a non-trivial solution. We now consider Green function as follows :*

$$G(t, s) = \begin{cases} a_1(s)t + b_1(s), \ 0 \leqslant t \leqslant s \leqslant 1, \\ a_2(s)t + b_2(s), \ 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

*Using the properties of Green function, it follows*

*1. The function $G$ is continuous on $[0,1]\times[0,1]$, so*

$$a_1(s)s + b_1(s) = a_2(s)s + b_2(s). \tag{1.16}$$

*2. It results from $\lim\limits_{s\to 0^+}\frac{\partial G}{\partial t}(t,s) - \lim\limits_{t\to s^-}\frac{\partial G}{\partial t}(t,s) = -1$, $\forall s\in[0,1]$ such that*

$$a_2(s) - a_1(s) = -1. \tag{1.17}$$

*3. As $G$ satisfies the boundary conditions, it comes*

$$a_1(s) = 0,\ a_2(s) = -b_2(s). \tag{1.18}$$

*According to (1.17), (1.18), we have*

$$a_1(s) = 0,\ a_2(s) = -1,\ b_2(s) = 1. \tag{1.19}$$

*Combining (1.15), (1.16) and (1.19), we get*

$$G(t,s) = \begin{cases} 1-t, & 0\le s\le t\le 1,\\ 1-s, & 0\le t\le s\le 1. \end{cases}$$

## 1.7 Krasnoselskii Fixed Point Theorem

In recent years, the Krasnoselskii fixed point theorem for cone maps and its many generalizations have been successfully applied to establish the existence of multiple solutions of boundary value problems of various types. It is called " Fixed point theorem of cone expansion and compression" or " Guo-Krasnoselskii Theorem " or "Krasnoselskii Theorem". The majority of known proofs of Krasnoselskii Theorem using topological index degree theory. The first appearance of Krasnoselskii Theorem in 1960, where the positive cone has interior points. by Amann [30], on chapter $II$ has a discussion and proof it with the general boundary conditions. D. Guo and V. Lakshmikantham are introduce excellent proof see [16]. There are also researchers who have proven Krasnoselskii Theorem without using topological index degree theory as Potter see [5], Chaljub-Simon and Volkmann see [3].

In this section, We will introduce Krasnoselskii Theorem with proofs, and explain how to applied to obtain the positive solution and evaluate this with a simple exapmle.

**Theorem 1.7.1.** *[16] (Krasnoselskii Fixed Point Theorem)*
*Let $E$ be a Banach space, $P \subset E$ a cone in $E$, $\Omega_1$ and $\Omega_2$ two open bounded sets in $E$, such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let $S : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ be a completely continuous operator, such that one of the following conditions is satisfied:*

1. $\|Sx\| \leq \|x\|$ *for all* $x \in P \cap \partial\Omega_1$ *and* $\|Sx\| \geq \|x\|$ *for all* $x \in P \cap \partial\Omega_2$.
2. $\|Sx\| \geq \|x\|$ *for all* $x \in P \cap \partial\Omega_1$ *and* $\|Sx\| \leq \|x\|$ *for all* $x \in P \cap \partial\Omega_2$.

*Then, the operator $S$ admits at least one fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.*

We will need some definitions and lemmas to prove the above theorem.

**Definition 1.7.1.** *[16](Condensing Operator)*
*An operator $A : D \to E, D$ be a subset on Banach space $E$. Condensing operator $A$ is defined as a continuous and bounded operator such that for any bounded set $Z$ in the domain $D$ satisfies*

$$\mu(A(Z)) < \mu(Z) \quad with\, 0 < \mu(Z),$$

*where $\mu(Z)$ denotes the measure of noncompactness of $Z$. Readers will find the definitions and properties of measure of noncompactness in Lakshmikantham and Leela [64] and Guo [15].*

**Definition 1.7.2.** *[16]*
*The fixed point of the operator $A$ on the set $U$ with respect to $X$ is denoted as $i(A, U, X)$.*

**Lemma 1.7.1.** *[16]*
*Consider the condensing operator $A : P \cap \overline{\Omega} \to P$ where $0 \in \Omega$. Assuming that the inequality*

$$Ax \neq \mu x, \, \forall x \in P \cap \partial\Omega, \, \mu \geq 1. \tag{1.20}$$

*Then $i(A, P \cap \Omega, P) = 1$.*

**Lemma 1.7.2.** *[16]*
*Consider a completely continuous operator, define $A : P \cap \overline{\Omega} \to P$. Suppose that*

1. $\inf\limits_{x \in P \cap \partial\Omega} \|Ax\| > 0$.

2. $Ax \neq \mu x, \forall x \in P \cap \partial\Omega,\ 0 < \mu \leq 1.$

*Therefore* $i(A, P \cap \Omega, P) = 0$.

**Theorem 1.7.2.** *[33]( Theorem Dugundji Extension)*
*Let $E_1$ and $E_2$ be two real Banach spaces and $C$ be a closed subset of $E_1$. Suppose that operator $T : C \to E_2$ is completely continuous. Then, there exists a completely continuous operator $\tilde{T} : E_1 \to E_2$ such that*

1. $\tilde{T}x = Tx$ *for any* $x \in C$.
2. $\tilde{T}(E_1) \subset \overline{co}\,T(C)$, *where* $\overline{co}\,T(C)$ *denotes the closed convex hull of* $T(C)$.

## Proof of Theorem 1.7.1

We prove the first condition of theorem, since the proof is similar when second condition is satisfied.
By Theorem Dugundji 1.7.2, it is possible extended $S$ to a completely continuous operator from $P \cap \overline{\Omega_2}$ to $P$. Let us assume that $S$ has no fixed points on $P \cap \partial\Omega_1$ and $P \cap \partial\Omega_2$. Firstly, suppose that condition 1 is verified, i.e., cone expansion. That much is evident.

$$Sx \neq \mu x,\ \forall x \in P \cap \partial\Omega_1,\ \mu \geq 1. \tag{1.21}$$

On the other hand,

$$\exists\, x_0 \in P \cap \partial\Omega_1, \mu_0 > 1 : \quad Sx_0 = \mu_0 x_0 \Rightarrow \|Sx_0\| = \mu_0 \|x_0\| > \|x_0\|,$$

this contradicts with condition 1. Hence, from (1.21) and Lemma 1.7.1, we deduce

$$i\,(S, P \cap \Omega_1, P) = 1. \tag{1.22}$$

Similarly, we showing

$$Sx \neq \mu x,\ \forall x \in P \cap \partial\Omega_2,\ 0 < \mu \leq 1. \tag{1.23}$$

Indeed,

$$\exists\, x_1 \in P \cap \partial\Omega_2, 0 < \mu_1 < 1 : \quad Sx_1 = \mu_1 x_1 \Rightarrow \|Sx_1\| = \mu_1 \|x_1\| < \|x_1\|,$$

which leading to contradiction with condition 1. Furthermore, by condition 1 we get

$$\inf_{x_1 \in P \cap \partial \Omega_2} \|Sx_1\| \geq \inf_{x_1 \in P \cap \partial \Omega_2} \|x_1\| > 0. \tag{1.24}$$

From Lemma 1.7.2, (1.23) and (1.24), it follows

$$i\left(S, P \cap \Omega_2, P\right) = 0. \tag{1.25}$$

As a result, combining (1.22) and (1.25), we have

$$i\left(S, P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right), P\right) = i\left(S, P \cap \Omega_2, P\right) - i\left(S, P \cap \Omega_1, P\right) = -1 \neq 0. \tag{1.26}$$

We conclude $S$ has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Similarly, when condition 2 is satisfied, instead of (1.22), (1.25), we obtain

$$i\left(S, P \cap \Omega_1, P\right) = 0,\ i\left(S, P \cap \Omega_2, P\right) = 1.$$

This leads to (1.26). Then, $S$ has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$. □

**Remark 1.7.1.** *(Technique of Applying Krasnoselskii's Theorem)*

*The technique of applying the cone fixed point theorem to obtain the results of the existence from a boundary value problem is to rewrite the problem as an integral equation, usually via the use of Green's function. The Banach space is the space of continuous functions with an appropriate norm, and the positive cone is the set of continuous positive functions or some suitable subset of it. The integral operator is a completely continuous cone map and if one can find suitable constants a and b such that the hypotheses of the cone theorem are satisfied, then the annular region has a fixed point, which is a positive solution of the boundary value problem.*

## 1.8 Example

We study the following problem:

$$\begin{cases} -x''(t) = f(x(t)), \quad 0 \leq t \leq 1; \\ x(0) = x(1) = 0, \end{cases} \tag{1.27}$$

where $f$ is continuous and nonnegative on $[0, \infty)$ and $f(0) = 0$, which satisfies

$$0 \leq \overline{\lim_{x \to 0^+}} \frac{f(x)}{x} < 8. \tag{1.28}$$

$$24\sqrt{3} < \overline{\lim_{x \to +\infty}} \frac{f(x)}{x} \leq +\infty. \tag{1.29}$$

Under assumptions (1.28) and (1.29), the problem (1.27) has at least one non trivial positive solution.

## Solution.

The function $x(t) \equiv 0$ is the trivial solution of the problem (1.27). We prove that problem has nontrivial solution satisfying

$$x(t) = \int_0^1 k(t, s) f(x(s)) ds,$$

where $k$ is the Green function,

$$k(t, s) = \begin{cases} t(1-s), & t \leq s; \\ s(1-t), & t > s. \end{cases} \tag{1.30}$$

Define two cones with $P_\varepsilon \subset P$, for $0 < \varepsilon < \frac{1}{2}$, as

$$\begin{aligned} P &= \{x \in C([0,1], \mathbb{R}) \mid \quad x \geq 0\}, \\ P_\varepsilon &= \{x \in P \mid \quad \min_{\frac{1}{2}-\varepsilon < t < \frac{1}{2}+\varepsilon} x(t) \geq (\frac{1}{2} - \varepsilon) \|x\|_\infty\}. \end{aligned}$$

Claim that $P_\varepsilon$ is cone.

1. Suppose $x \in P_\varepsilon$ and $-x \in P_\varepsilon$, we have

$$\min_{\frac{1}{2}-\varepsilon < t < \frac{1}{2}+\varepsilon} x(t) \geq \left(\frac{1}{2} - \varepsilon\right) \|x\|_\infty,$$

$$\min_{\frac{1}{2}-\varepsilon < t < \frac{1}{2}+\varepsilon} (-x(t)) \geq \left(\frac{1}{2} - \varepsilon\right) \|-x\|_\infty.$$

Since $\|-x\|_\infty = \|x\|_\infty$, this implies

$$\begin{aligned} \min_{\frac{1}{2}-\varepsilon < t < \frac{1}{2}+\varepsilon} (-x(t)) = - \min_{\frac{1}{2}-\varepsilon < t < \frac{1}{2}+\varepsilon} x(t) \\ \geq \left(\frac{1}{2} - \varepsilon\right) \|x\|_\infty. \end{aligned}$$

Thus

$$\max_{\frac{1}{2}-\varepsilon<t<\frac{1}{2}+\varepsilon} x(t) \leq -\left(\frac{1}{2}-\varepsilon\right)\|x\|_\infty.$$

It follows that

$$\left(\frac{1}{2}-\varepsilon\right)\|x\|_\infty \leq \min_{\frac{1}{2}-\varepsilon<t<\frac{1}{2}+\varepsilon} x(t) \leq \max_{\frac{1}{2}-\varepsilon<t<\frac{1}{2}+\varepsilon} x(t) \leq -\left(\frac{1}{2}-\varepsilon\right)\|x\|_\infty.$$

This implies

$$\left(\frac{1}{2}-\varepsilon\right)\|x\|_\infty \leq -\left(\frac{1}{2}-\varepsilon\right)\|x\|_\infty,$$

which is only possible if $\|x\|_\infty = 0$. Therefore, $x = 0$.

2. Let $x \in P_\varepsilon$ and $\lambda \geq 0$, we get

$$\begin{aligned}\min_{\frac{1}{2}-\varepsilon<t<\frac{1}{2}+\varepsilon}(\lambda x(t)) &= \lambda \min_{\frac{1}{2}-\varepsilon<t<\frac{1}{2}+\varepsilon} x(t)\\ &\geq \lambda\left(\frac{1}{2}-\varepsilon\right)\|x\|_\infty\\ &= \left(\frac{1}{2}-\varepsilon\right)\|\lambda x\|_\infty.\end{aligned}$$

Thus

$$\min_{\frac{1}{2}-\varepsilon<t<\frac{1}{2}+\varepsilon}(\lambda x(t)) \geq \left(\frac{1}{2}-\varepsilon\right)\|\lambda x\|_\infty.$$

This shows that $\lambda x \in P_\varepsilon$.

3. Let $x, y \in P_\varepsilon$, we have

$$\min_{\frac{1}{2}-\varepsilon<t<\frac{1}{2}+\varepsilon} x(t) \geq \left(\frac{1}{2}-\varepsilon\right)\|x\|_\infty,$$

$$\min_{\frac{1}{2}-\varepsilon<t<\frac{1}{2}+\varepsilon} y(t) \geq \left(\frac{1}{2}-\varepsilon\right)\|y\|_\infty.$$

Using the properties of the minimum function:

$$\begin{aligned}\min_{\frac{1}{2}-\varepsilon<t<\frac{1}{2}+\varepsilon}(\alpha x(t)+(1-\alpha)y(t)) &\geq \alpha \min_{\frac{1}{2}-\varepsilon<t<\frac{1}{2}+\varepsilon} x(t) + (1-\alpha)\min_{\frac{1}{2}-\varepsilon<t<\frac{1}{2}+\varepsilon} y(t)\\ &\geq \alpha\left(\frac{1}{2}-\varepsilon\right)\|x\|_\infty + (1-\alpha)\left(\frac{1}{2}-\varepsilon\right)\|y\|_\infty\\ &= \left(\frac{1}{2}-\varepsilon\right)(\alpha\|x\|_\infty + (1-\alpha)\|y\|_\infty).\end{aligned}$$

Using $\|\alpha x + (1-\alpha)y\| \leq \alpha\|x\|_\infty + (1-\alpha)\|y\|_\infty$, we obtain

$$\left(\frac{1}{2}-\varepsilon\right)(\alpha\|x\|_\infty + (1-\alpha)\|y\|_\infty) \geq \left(\frac{1}{2}-\varepsilon\right)\|\alpha x + (1-\alpha)y\|_\infty.$$

Therefore

$$\min_{\frac{1}{2}-\varepsilon<t<\frac{1}{2}+\varepsilon} (\alpha x(t)+(1-\alpha)y(t)) \geq \left(\frac{1}{2}-\varepsilon\right) \|\alpha x+(1-\alpha)y\|_{\infty}.$$

This shows that $\alpha x+(1-\alpha)y \in P_{\varepsilon}$. Then $P_{\varepsilon}$ is convex.

3. Suppose that $x_n \in P_{\varepsilon}$ and $x_n \to x$. We need to show that $x \in P_{\varepsilon}$. Since $x_n \in P_{\varepsilon}$, we have

$$\min_{\frac{1}{2}-\varepsilon<t<\frac{1}{2}+\varepsilon} x_n(t) \geq \left(\frac{1}{2}-\varepsilon\right) \|x_n\|_{\infty}.$$

Taking the limit as $n \to \infty$ and using the continuity of the minimum function and the norm:

$$\min_{\frac{1}{2}-\varepsilon<t<\frac{1}{2}+\varepsilon} x(t) \geq \left(\frac{1}{2}-\varepsilon\right) \|x\|_{\infty}.$$

This shows that $x \in P_{\varepsilon}$. So $P_{\varepsilon}$ is closed.

Let us define the map $A : P \to P$, $x \mapsto Ax$ such that

$$\forall\, x \in P, \forall t \in [0,1] : \quad Ax(t) = \int_0^1 k(t,s) f(x(s)) ds.$$

Showing that $A$ is continuous. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence on $P$, where $(x_n)_{n\in\mathbb{N}}$ converges uniformly to $x$. This means

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : \quad \|x_n - x\|_{\infty} \leq \varepsilon,$$

which implies

$$x_n(s) \to x(s), \;\forall s \in [0,1]. \tag{1.31}$$

Denoting $y_n = Ax_n$ and $y = Ax$. By the continuity of $f, k$ on $[0,1]$, $[0,1]\times[0,1]$, respectively and (1.31), it follows from Theorem 1.3.2 :

$$\begin{aligned}\lim_{n\to\infty} y_n &= \lim_{n\to\infty} \int_0^1 k(t,s) f(x_n(s))\, ds \\ &= \int_0^1 k(t,s) f(x(s))\, ds = y.\end{aligned}$$

Hence, $A$ is continuous.

Now, we claim $A$ is compact. Let $B$ be a bounded set in $P$,

$$\exists\, M_1 > 0, \forall x \in B : \;\; \|x\|_{\infty} \leq M_1.$$

Setting $K(t) = \{Ax(t),\, x \in B\}$ with $t \in [0,1],\ K = \{Ax,\, x \in B\}$. We prove that $K(t)$ is equicontinuous and relatively compact, which means that $K$ is relatively compact. We have

$$\begin{aligned} \forall t \in [0,1]: \quad |Ax(t)| &= \left| \int_0^1 k(t,s) f(x(s)) ds \right| \\ &\leq \int_0^1 k(t,s) |f(x(s))| ds. \end{aligned} \tag{1.32}$$

From the continuity of $f$ on the compact $[0, M_1]$, we get

$$\exists M_2 > 0, \forall s \in [0, M_1]: \quad 0 \leq |f(s)| \leq M_2. \tag{1.33}$$

Using (1.32), (1.33) and the fact that $0 \leq k \leq 1$, we obtain

$$\begin{aligned} \forall t \in [0,1]: \quad 0 \leq |Ax(t)| &\leq \int_0^1 k(t,s)|f(x(s))| ds \\ &\leq M_2 \int_0^1 ds = M_2, \end{aligned} \tag{1.34}$$

which implies $K(t)$ is bounded. Therefore, $K(t)$ is relatively compact in $\mathbb{R}$. Next, for all $t_1, t_2 \in [0,1]$ and $x \in B$, we have

$$\begin{aligned} |Ax(t_1) - Ax(t_2)| &= \left| \int_0^1 k(t_1,s) f(x(s)) ds - \int_0^1 k(t_2,s) f(x(s)) ds \right| \\ &= \left| \int_0^1 \Big( k(t_1,s) - k(t_2,s) \Big) f(x(s)) ds \right| \\ &\leq \int_0^1 \left| \Big( k(t_1,s) - k(t_2,s) \Big) f(x(s)) \right| ds \\ &= \int_0^1 |k(t_1,s) - k(t_2,s)| |f(x(s))| ds \\ &\leq \|k(t_1,.) - k(t_2,.)\|_\infty \int_0^1 |f(x(s))| ds \end{aligned} \tag{1.35}$$

Since $k$ is continuous function on $[0,1] \times [0,1]$, then, $k$ is uniformly continuous. So,

$$\forall \varepsilon > 0, \exists \delta > 0, \forall s \in [0,1]: \quad |t_1 - t_2| \leq \delta \Rightarrow \|k(t_1,s) - k(t_2,s)\|_\infty \leq \varepsilon. \tag{1.36}$$

Combining (1.35), (1.33) and (1.36), it follows that

$$\begin{aligned} |Ax(t_1) - Ax(t_2)| &\leq \int_0^1 |k(t_1,s) - k(t_2,s)| |f(x(s))| ds \\ &\leq \varepsilon M_2 \int_0^1 ds \\ &= \varepsilon M_2, \end{aligned}$$

which implies that $K(t)$ is equicontinuous. From the Arzela-Ascoli Theorem, we deduce $K$ is a relatively compact set. Therefore, $A$ is compact.

We show $A(P_\varepsilon) \subset P_\varepsilon$ for any $\varepsilon \in (0, \frac{1}{2})$. Let $x \in P$. From (1.30), we have

$$\begin{aligned}\forall t \in [0,1]: \quad |Ax(t)| &= \left| \int_0^1 k(t,s) f(x(s)) ds \right| \\ &\leq \int_0^1 k(t,s) |f(x(s))| ds \\ &\leq \int_0^1 s(1-s) |f(x(s))| ds \\ &= \int_0^1 s(1-s) f(x(s)) ds,\end{aligned}$$

which gives

$$\|Ax\|_\infty \leq \int_0^1 s(1-s) f(x(s)) ds. \tag{1.37}$$

Moreover, for all $t \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right]$, $s \in [0,1]$, we obtain

$$k(t,s) = \begin{cases} t(1-s) \geq \left(\dfrac{1}{2} - \varepsilon\right)(1-s) & \text{for } t \leq s; \\ s(1-t) \geq \varepsilon \left[1 - \left(\dfrac{1}{2} + \varepsilon\right)\right] = \left(\dfrac{1}{2} - \varepsilon\right) & \text{for } t > s, \end{cases}$$

which implies

$$\forall t \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right], \forall s \in [0,1]: \quad k(t,s) \geq \left(\frac{1}{2} - \varepsilon\right) s(1-s). \tag{1.38}$$

According to (1.37), (1.38), we get

$$\begin{aligned}\int_0^1 k(t,s) f(x(s)) ds &\geq \left(\frac{1}{2} - \varepsilon\right) \int_0^1 s(1-s) f(x(s)) ds \\ &\geq \left(\frac{1}{2} - \varepsilon\right) \|Ax\|_\infty.\end{aligned}$$

This leads to

$$\begin{aligned}\min_{t \in [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]} Ax(t) &= \min_{t \in [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]} \left( \int_0^1 k(t,s) f(x(s)) ds \right) \\ &\geq \left(\frac{1}{2} - \varepsilon\right) \|Ax\|_\infty,\end{aligned}$$

which means

$$A(P_\varepsilon) \subset P_\varepsilon, \ \ \forall \varepsilon \in (0, \frac{1}{2}). \tag{1.39}$$

Moreover, from (1.28), we have

$$\forall \varepsilon > 0, \exists \eta > 0 : \quad \sup_{0<x\leq \eta} \Big( \frac{f(x)}{x} \Big) < 8 + \varepsilon,$$

which gives

$$\forall \varepsilon > 0, \exists \eta > 0, \forall x \in [0, \eta] : \quad 0 \leq f(x) < x(8 + \varepsilon). \tag{1.40}$$

Also, in view of the condition (1.29), we obtain

$$\forall \varepsilon > 0, \exists r > 0 : \quad 24\sqrt{3} + \varepsilon < \sup_{x \geq r} \Big( \frac{f(x)}{x} \Big),$$

which implies

$$\forall \varepsilon > 0, \exists r > 0, x \geq r : \quad f(x) > (24\sqrt{3} + \varepsilon)x. \tag{1.41}$$

For all $x \in P$ such that $\|x\| = r$ and (1.40), which yields

$$\begin{aligned} Ax(t) &= \int_0^1 k(t, s) f(x(s)) ds \\ &\leq 8 \int_0^1 k(t, x) x(s) ds \\ &\leq 8 \|x\| \int_0^1 k(t, s) ds \\ &= 8t(1 - t)\|x\|. \end{aligned}$$

We define the function $\varphi(t) = t(1 - t)$. Then, $\varphi'(t) = 1 - 2t$.

| $t$ | $0$ | | $\frac{1}{2}$ | | $1$ |
|---|---|---|---|---|---|
| $\varphi'(t)$ | | $-$ | $0$ | $+$ | |
| $\varphi(t)$ | $0$ | ↗ | $1$ | ↘ | $0$ |

This leads to

$$\begin{aligned} Ax(t) &= 8t(1 - t)\|x\|_\infty \\ &\leq \|x\|_\infty, \end{aligned}$$

this means

$$Ax \leq \|Ax\|_\infty \leq \|x\|_\infty.$$

From this set

$$\Omega_1 = \{x \in C([0,1],\mathbb{R}), \quad \|x\|_\infty < r\}. \tag{1.42}$$

We deduce the following estimate

$$\|Ax\|_\infty \leq \|x\|_\infty. \tag{1.43}$$

Otherwise, for all $0 < \varepsilon < \frac{1}{2}$, denoting

$$r_\varepsilon = \max\left\{2r, \frac{\eta}{\frac{1}{2}-\varepsilon}\right\}.$$

Since $r_\varepsilon > r$, choose $x \in P_\varepsilon$ such that $\|x\|_\infty = r_\varepsilon$, then

$$\begin{aligned} \min_{t\in[\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon]} x(t) &\geq \left(\frac{1}{2}-\varepsilon\right)\|x\|_\infty \\ &= \left(\frac{1}{2}-\varepsilon\right) r_\varepsilon. \end{aligned} \tag{1.44}$$

We have

$$\begin{aligned} Ax\left(\frac{1}{2}\right) &= \int_0^1 k(\frac{1}{2},s)f(x(s))ds \\ &\geq \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} k\left(\frac{1}{2},s\right) f(x(s))ds. \end{aligned} \tag{1.45}$$

From the fact that $s \mapsto k\left(\frac{1}{2},s\right) f(x(s))$ is nonnegative map and (1.41), we obtain

$$\begin{aligned} Ax\left(\frac{1}{2}\right) &\geq \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} k\left(\frac{1}{2},s\right) f(x(s))ds \\ &\geq 24\sqrt{3}\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} k\left(\frac{1}{2},s\right) x(s)ds. \end{aligned} \tag{1.46}$$

Applying (1.44), it follows

$$24\sqrt{3}\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} k\left(\frac{1}{2},s\right) x(s)ds \geq 24\sqrt{3}\left(\frac{1}{2}-\varepsilon\right)\|x\|_\infty \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} k\left(\frac{1}{2},s\right) ds. \tag{1.47}$$

Combining (1.46) and (1.47), gives

$$\begin{aligned} Ax\left(\frac{1}{2}\right) &\geq 24\sqrt{3}\left(\frac{1}{2}-\varepsilon\right)\|x\|_\infty \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} k\left(\frac{1}{2},s\right) ds \\ &\geq 12\sqrt{3}\left(\frac{1}{2}-\varepsilon\right)\varepsilon(1-\varepsilon)\|x\|_\infty. \end{aligned}$$

We define the function $\theta(\varepsilon)=\varepsilon(1-\varepsilon)(\frac{1}{2}-\varepsilon)$, $0<\varepsilon<\frac{1}{2}$. We have

$$\theta'(\varepsilon)=3\varepsilon^2-3\varepsilon+\frac{1}{2}.$$

Then,

$$\theta'(\varepsilon)=0\Leftrightarrow\varepsilon=\frac{3-\sqrt{3}}{6}\vee\varepsilon=\frac{3+\sqrt{3}}{6}.$$

| $\varepsilon$ | $0$ | | $\frac{3-\sqrt{3}}{6}$ | | $\frac{1}{2}$ |
|---|---|---|---|---|---|
| $\theta'(\varepsilon)$ | | $+$ | $0$ | $-$ | |
| $\theta(\varepsilon)$ | $0$ | ↗ | $\frac{\sqrt{3}}{36}$ | ↘ | $0$ |

This implies

$$\begin{aligned}\|Ax\|_\infty &\geq \|Ax\left(\frac{1}{2}\right)\|_\infty\\ &\geq 12\sqrt{3}\left(\frac{1}{2}-\varepsilon\right)\varepsilon(1-\varepsilon)\|x\|_\infty\\ &\geq 12\sqrt{3}\frac{\sqrt{3}}{36}\|x\|_\infty\\ &=\|x\|_\infty.\end{aligned}$$

Under this assumption

$$\Omega_2=\{x\in C([0,1],\mathbb{R}),\quad \|x\|_\infty<r_{\varepsilon'}\}. \tag{1.48}$$

We obtain

$$\|Ax\|_\infty\geq\|x\|_\infty. \tag{1.49}$$

According to (1.42), (1.43),(1.48) and (1.49), we deduce by Theorem (1.7.1), the operator $A$ has a positive fixed point in $P_{\varepsilon'}\cap\left(\overline{\Omega}_2\backslash\Omega_1\right)$, which is solution of problem (1.27) $\square$

# Chapter 2

# Existence of a Positive Solution in One-Dimensional

We study the existence of a positive solution to the problem

$$\begin{cases} -M\Big(t, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)u''(t) = g(t,u), \quad t \in (0,1); \\ u'(0) = u(1) = 0. \end{cases} \tag{2.1}$$

To establish the existence result, we make some assumptions on $f, g$ and $M$.

- $f : [0,\infty) \to [0,\infty)$ is continuous function satisfying for some two nondecreasing continuous functions $\alpha, \beta : [0,\infty) \to [0,\infty)$,

$$\alpha(s) \le f(s) \le \beta(s). \tag{2.2}$$

- $g : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous function,

$$\xi(t)h(s) \le g(t,s) \le \eta(t)k(s), \tag{2.3}$$

while $h, k : [0,\infty) \to [0,\infty)$ are increasing continuous functions, and the measurables functions $\xi, \eta : [0,1] \to [0,\infty)$. Also $g(\cdot, 0) \neq 0$.

- $M : [0,1] \times [0,\infty) \to (0,\infty)$ is continuous function,

$$\theta(t)v(s) \le M(t,s) \le \mu(t)w(s), \tag{2.4}$$

where $v, w : [0,\infty) \to (0,\infty)$ are increasing continuous functions, and $\theta, \mu : [0,1] \to (0,\infty)$ are measurables functions. In addition

$$0 < \int_0^1 \frac{\eta(s)}{\theta(s)} ds < \infty, \tag{2.5}$$

and

$$0 < \int_0^d G(0,s) \frac{\xi(s)}{\mu(s)} ds < \infty, \tag{2.6}$$

with $d \in (0,1)$ and $G$ denoting the Green function,

$$G(t,s) = \begin{cases} 1-t, & 0 \le s \le t \le 1; \\ 1-s, & 0 \le t \le s \le 1. \end{cases} \tag{2.7}$$

Moreover, suppose that

$$\exists\, R_1 > 0 : \quad \inf_{0<s\le R_1} \frac{h((1-d)s)}{sw(\beta(s))} \ge \frac{1}{\displaystyle\int_0^d G(0,s)\frac{\xi(s)}{\mu(s)} ds}. \tag{2.8}$$

$$\exists\, R_2 > R_1 : \quad \sup_{s \ge R_2} \frac{k(s)}{sv(d\alpha(1-d)s)} \le \frac{1}{\displaystyle\int_0^1 \frac{\eta(s)}{\theta(s)} ds}. \tag{2.9}$$

Assume that

$$\liminf_{s\to 0^+} \frac{h((1-d)s)}{sw(\beta(s))} = \infty.$$

It is equivalent to

$$\sup_{\delta>0} \inf_{s\le\delta} \{\frac{h((1-d)s)}{sw(\beta(s))}\} = \infty.$$

Using upper bound property, it follows

$$\forall \varepsilon > 0, \exists\, \delta_\varepsilon > 0 : \quad \inf_{s\le\delta_\varepsilon} \{\frac{h((1-d)s)}{sw(\beta(s))}\} > \underbrace{\sup_{s>0} \inf_{s\le\delta_\varepsilon} \{\frac{h((1-d)s)}{sw(\beta(s))}\}}_{\infty} - \varepsilon.$$

We choose $R_1 = \delta_\varepsilon$ to

$$\inf_{0<s\le R_1} \{\frac{h((1-d)s)}{sw(\beta(s))}\} > \infty > \frac{1}{\displaystyle\int_0^d G(0,s)\frac{\xi(s)}{\mu(s)} ds},$$

which implies (2.8). Now, suppose that

$$\limsup_{s\to\infty} \frac{k(s)}{sv(d\alpha(1-d)s)} = 0.$$

This limit, can be written as

$$\inf_{\sigma>R_1}\sup_{s\geq\sigma}\{\frac{k(s)}{sv(d\alpha(1-d)s)}\}=0.$$

From lower bound property,

$$\forall\varepsilon>0,\,\exists\,\sigma_\varepsilon>R_1:\quad \underbrace{\inf_{\sigma_\varepsilon>R_1}\sup_{s\geq\sigma_\varepsilon}\{\frac{k(s)}{sv(d\alpha(1-d)s)}\}}_{0}+\varepsilon>\sup_{s\geq\sigma_\varepsilon}\{\frac{k(s)}{sv(d\alpha(1-d)s)}\}.$$

If $\varepsilon=\dfrac{1}{\displaystyle\int_0^1\frac{\eta(s)}{\theta(s)}ds}>0$, $R_2=\sigma_\varepsilon$, we find

$$\sup_{s\geq R_2}\{\frac{k(s)}{sv(d\alpha(1-d)s)}\}<\frac{1}{\displaystyle\int_0^1\frac{\eta(s)}{\theta(s)}ds},$$

which gives (2.9).

## 2.1 Existence of a Positive Solution

**Theorem 2.1.1.**

*Assume that the conditions (2.2)-(2.9) are verified. Then, the problem (2.1) has a positive solution.*

### Proof.

Consider the following problem

$$\begin{cases}-u''(t)=\psi(t), & t\in(0,1),\\ u'(0)=u(1)=0,\end{cases}\tag{2.10}$$

which has a unique solution, given by ( see Theorem 1.6.1):

$$\forall t\in[0,1]:\quad u(t)=\int_0^1 G(t,s)\psi(s)ds,$$

where $\psi:[0,1]\to\mathbb{R}$ is a continuous function. Let $u$ be a solution of the problem (2.1). By replacing $\psi$ with the continuous function

$$s\mapsto\frac{g(s,u(s))}{M\Big(s,\displaystyle\int_0^1 f(u(\tau))d\tau\Big)},$$

we see that $u$ is solution of the integral equation

$$\forall t \in [0,1]: \quad u(t) = \int_0^1 G(t,s) \frac{g(s,u(s))}{M\Big(s, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)} ds. \tag{2.11}$$

On the other hand, assume that $u$ verifies (2.11). By using Theorem 1.4.2, we find

$$\forall t \in ]0,1[: \quad u'(t) = -\int_0^t \frac{g(s,u(s))}{M\Big(s, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)} ds.$$

Let $(t_j)_{j\in\mathbb{N}} \in ]0,1[$, such that $\lim\limits_{j\to\infty} t_j = 0$. We have

$$\begin{aligned}\lim_{j\to\infty} u'(t_j) &= \lim_{j\to\infty} -\int_0^{t_j} \frac{g(s,u(s))}{M\Big(s, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)} ds \\ &= -\int_0^0 \frac{g(s,u(s))}{M\Big(s, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)} ds = 0.\end{aligned}$$

This gives $u'(0) = 0$. If $(t_j)_{j\in\mathbb{N}} \in ]0,1[$, where $\lim\limits_{j\to\infty} t_j = 1$, it follows that

$$\begin{aligned}\lim_{j\to\infty} u'(t_j) &= \lim_{j\to\infty} -\int_0^{t_j} \frac{g(s,u(s))}{M\Big(s, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)} ds \\ &= -\int_0^1 \frac{g(s,u(s))}{M\Big(s, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)} ds := u'(1).\end{aligned}$$

According to the extension, we have

$$\forall t \in [0,1]: \quad u'(t) = -\int_0^t \frac{g(s,u(s))}{M\Big(s, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)} ds,$$

and

$$\forall t \in ]0,1[: \quad u''(t) = -\frac{g(t,u(t))}{M\Big(s, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)}.$$

For $(t_j)_{j\in\mathbb{N}} \in ]0,1[$ with $\lim\limits_{j\to\infty} t_j = 0$, we get

$$\begin{aligned}\lim_{j\to\infty} u''(t_j) &= \lim_{j\to\infty} -\frac{g(t_j,u(t_j))}{M\Big(t_j, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)} \\ &= -\frac{g(0,u(0))}{M\Big(0, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)} := u''(0).\end{aligned}$$

If $(t_j)_{j\in\mathbb{N}} \in ]0,1[$ such that $\lim\limits_{j\to\infty} t_j = 1$, we obtain

$$\begin{aligned}\lim_{j\to\infty} u''(t_j) &= \lim_{j\to\infty} -\frac{g(t_j, u(t_j))}{M\left(t_j, \displaystyle\int_0^1 f(u(\tau))d\tau\right)}\\ &= -\frac{g(1, u(1))}{M\left(1, \displaystyle\int_0^1 f(u(\tau))d\tau\right)} := u''(1).\end{aligned}$$

By the extension theorem, we deduce $u \in C^2([0,1],\mathbb{R})$. Since

$$\begin{aligned}u(1) &= \int_0^1 G(1,s)\frac{g(s,u(s))}{M\left(s, \displaystyle\int_0^1 f(u(\tau))d\tau\right)}ds\\ &= \int_0^1 0\frac{g(s,u(s))}{M\left(s, \displaystyle\int_0^1 f(u(\tau))d\tau\right)}ds = 0.\end{aligned}$$

We conclude $u$ is solution of (2.1). Define the following map:

$$S : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R}),$$

$$u \mapsto Su,$$

where

$$\forall t \in [0,1] : \quad Su(t) = \int_0^1 G(t,s)\frac{g(s,u(s))}{M\left(s, \displaystyle\int_0^1 f(u(\tau))d\tau\right)}ds.$$

From the continuity of $u$ and $g$ on $[0,1]$ and $[0,1]\times[0,+\infty[$, respectively, the function

$$s \mapsto g(s,u(s))$$

is continuous on $[0,1]$. Also, $G$ and $s \mapsto M\left(s, \displaystyle\int_0^1 f(u(\tau))d\tau\right)$ are continuous on $[0,1]\times[0,1]$ and $[0,1]$, respectively. Then,

$$(t,s) \mapsto G(t,s)\frac{g(s,u(s))}{M\left(s, \displaystyle\int_0^1 f(u(\tau))d\tau\right)}$$

is continuous on $[0,1]\times[0,1]$. The function $t \mapsto \dfrac{\partial}{\partial t}G(t,s)\dfrac{g(s,u(s))}{M\left(s, \displaystyle\int_0^1 f(u(\tau))d\tau\right)}$ is bounded by

$$s \mapsto \frac{g(s,u(s))}{M\left(s, \displaystyle\int_0^1 f(u(\tau))d\tau\right)}.$$

We deduce the above map $S$ is well define by Theorem 1.4.1.

We will research a positive fixed point of the map $S$ which is positive solution for the problem (2.1), by using Krasnoselskii Theorem. Firstly, we prove that $S$ is continuous and compact, as demonstrated by the following Lemma.

**Lemma 2.1.1.**
*Consider the map* $S : (F, \| \cdot \|_\infty) \to (F, \| \cdot \|_\infty)$, *where*

$$F = \{u \in E \,|\, u(t) \geq 0, \ \forall t \in [0,1]\}.$$

*Then,* $S$ *is completely continuous function.*

**Proof.**

We prove that $S$ is continuous. Let $(u_j)_{j \in \mathbb{N}} \subset F$ be a sequence, has a limit $u \in F$,

$$\lim_{j \to +\infty} \|u_j - u\|_\infty = 0, \tag{2.12}$$

which can be written as

$$\lim_{j \to +\infty} \sup_{t \in [0,1]} |u_j(t) - u(t)| = 0.$$

Then,

$$\forall \rho > 0, \exists j_0 \in \mathbb{N}, \forall j \in \mathbb{N}, j \geq j_0 : \quad \sup_{t \in [0,1]} |u_j(t) - u(t)| \leq \rho,$$

which gives

$$\forall j \geq j_0 : \quad u_j(t) - u(t) \leq \rho.$$

This leads to

$$\forall j \geq j_0 : \quad u_j(t) \leq \rho + u(t),$$

which implies

$$\forall t \in [0,1], \forall j \geq j_0 : \quad 0 \leq u_j(t) \leq \rho + \max_{t \in [0,1]} u(t). \tag{2.13}$$

Set

$$c_1 = \max_{t \in [0,1]} \left\{ \max_{t \in [0,1]} u(t) + \rho, \max_{t \in [0,1]} u_0(t), \max_{t \in [0,1]} u_1(t), \max_{t \in [0,1]} u_2(t), ..., \max_{t \in [0,1]} u_{j_0 - 1}(t) \right\} > 0. \tag{2.14}$$

Using (2.13) and (2.14) to obtain

$$\forall j \in \mathbb{N}: \quad u_j, u \in [0, c_1]. \tag{2.15}$$

Since $g$ is continuous on the compact $[0,1] \times [0, c_1]$, then $g$ is bounded,

$$\exists\, c_2 > 0, \forall x, y \in [0,1] \times [0, c_1]: \quad 0 \leq g(x,y) \leq c_2. \tag{2.16}$$

For $s = x, u_j(s) = y$ in (2.16), it follows that

$$\forall s \in [0,1], \forall j \in \mathbb{N}: \quad 0 \leq g(s, u_j(s)) \leq c_2. \tag{2.17}$$

Moreover, using the continuity of $g$ and (2.12), we find

$$\forall s \in [0,1]: \quad \lim_{j \to +\infty} g(s, u_j(s)) = g(s, u(s)). \tag{2.18}$$

From the continuity of $f$ on the compact $[0, c_1]$, $f$ is uniformly continuous. Then,

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in [0, c_1]: \quad |x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \varepsilon. \tag{2.19}$$

Take $\rho = \delta$ in (2.13), and $u_j(s) = x, u(s) = y$ in (2.19), to get

$$\forall \varepsilon > 0, \forall s \in [0,1], \forall j \geq j_0: \quad |f(u_j(s)) - f(u(s))| \leq \varepsilon,$$

which means that $f(u_j)$ converges uniformly to $f(u)$. Using Theorem 1.3.2, it follows that

$$\begin{aligned} \lim_{j \to +\infty} \int_0^1 f(u_j(\tau))d\tau &= \int_0^1 \lim_{j \to +\infty} f(u_j(\tau))d\tau \\ &= \int_0^1 f(\lim_{j \to +\infty} u_j(\tau))d\tau \\ &= \int_0^1 f(u(\tau))d\tau, \end{aligned} \tag{2.20}$$

which gives

$$\forall \varepsilon > 0, \exists j_0 \in \mathbb{N}, \forall j \geq j_0: \quad \left| \int_0^1 f(u_j(\tau))d\tau - \int_0^1 f(u(\tau))d\tau \right| \leq \varepsilon.$$

Then,

$$\forall j \geq j_0: \quad \int_0^1 f(u_j(\tau))d\tau \leq \int_0^1 f(u(\tau))d\tau + \varepsilon. \tag{2.21}$$

Let us set

$$c_3 = \max\Bigg\{ \int_0^1 f(u(\tau))d\tau + \varepsilon, \int_0^1 f(u_0(\tau))d\tau, \int_0^1 f(u_1(\tau))d\tau, \\ \int_0^1 f(u_2(\tau))d\tau, ..., \int_0^1 f(u_{j_0-1}(\tau))d\tau \Bigg\} > 0. \tag{2.22}$$

From (2.21) and (2.22), we obtain

$$\forall j \in \mathbb{N}: \quad \int_0^1 f(u_j(\tau))d\tau, \int_0^1 f(u(\tau))d\tau \in [0, c_3].$$

Since $\frac{1}{M}$ is continuous on the compact $[0,1]\times[0,c_3]$, then $\frac{1}{M}$ is bounded,

$$\exists\, c_4 > 0, \forall (x,y) \in [0,1]\times[0,c_3]: \quad 0 < \frac{1}{M(x,y)} \leq c_4. \tag{2.23}$$

Furthermore, by (2.20), we have

$$\lim_{j\to\infty} \frac{1}{M\left(s, \int_0^1 f(u_j(\tau))d\tau\right)} = \frac{1}{M\left(s, \int_0^1 f(u(\tau))d\tau\right)}. \tag{2.24}$$

Combining (2.18) and (2.23), we find

$$\exists c_5 > 0,\ \forall j \in \mathbb{N},\ \forall s \in [0,1]: \quad \frac{g(s,u_j(s))}{M\left(s, \int_0^1 f(u_j(\tau))d\tau\right)} \leq c_2 c_4 = c_5, \tag{2.25}$$

which leads, by the Lebesgue dominated convergence theorem, to

$$\forall s \in [0,1]: \quad \lim_{j\to+\infty} \frac{g(s,u_j(s))}{M\left(s, \int_0^1 f(u_j(\tau))d\tau\right)} = \frac{g(s,u(s))}{M\left(s, \int_0^1 f(u(\tau))d\tau\right)}. \tag{2.26}$$

For all $j \in \mathbb{N}$, it follows that

$$\begin{aligned} &\Big|Su_j(t) - Su(t)\Big| \\ &= \left| \int_0^1 G(t,s) \frac{g(s,u_j(s))}{M\Big(s, \int_0^1 f(u_j(\tau))d\tau\Big)} ds - \int_0^1 G(t,s) \frac{g(s,u(s))}{M\Big(s, \int_0^1 f(u(\tau))d\tau\Big)} ds \right| \\ &= \left| \int_0^1 G(t,s) \Bigg( \frac{g(s,u_j(s))}{M\Big(s, \int_0^1 f(u_j(\tau))d\tau\Big)} - \frac{g(s,u(s))}{M\Big(s, \int_0^1 f(u(\tau))d\tau\Big)} \Bigg) ds \right| \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^1 \left| G(t,s)\Big( \frac{g(s,u_j(s))}{M\Big(s, \int_0^1 f(u_j(\tau))d\tau\Big)} - \frac{g(s,u(s))}{M\Big(s, \int_0^1 f(u(\tau))d\tau\Big)} \Big) \right| ds \\
&= \int_0^1 G(t,s) \left| \frac{g(s,u_j(s))}{M\Big(s, \int_0^1 f(u_j(\tau))d\tau\Big)} - \frac{g(s,u(s))}{M\Big(s, \int_0^1 f(u(\tau))d\tau\Big)} \right| ds.
\end{aligned}$$

As $0 \leq G \leq 1$, we get

$$\Big| Su_j(t) - Su(t) \Big| \leq \int_0^1 \left| \frac{g(s,u_j(s))}{M\Big(s, \int_0^1 f(u(\tau))d\tau\Big)} - \frac{g(s,u(s))}{M\Big(s, \int_0^1 f(u(\tau))d\tau\Big)} \right| ds.$$

Passing through the upper bound, we find

$$\begin{aligned}
0 &\leq \|Su_j(t) - Su(t)\|_\infty \\
&\leq \sup_{t\in[0,1]} \left( \int_0^1 \left| \frac{g(s,u_j(s))}{M\Big(s, \int_0^1 f(u(\tau))d\tau\Big)} - \frac{g(s,u(s))}{M\Big(s, \int_0^1 f(u(\tau))d\tau\Big)} \right| ds \right) \\
&= \int_0^1 \left| \frac{g(s,u_j(s))}{M\Big(s, \int_0^1 f(u(\tau))d\tau\Big)} - \frac{g(s,u(s))}{M\Big(s, \int_0^1 f(u(\tau))d\tau\Big)} \right| ds.
\end{aligned} \tag{2.27}$$

Passing to the limit in (2.27), and using (2.26), we obtain

$$\lim_{j\to+\infty} \|Su_j - Su\|_\infty = 0.$$

Thus, $S$ is continuous.

Now, we prove that $S$ is compact. Assume that $A \subset F$ is a bounded subset and $t \in [0,1]$. Set

$$B = \{Su; u \in A\}, \quad B(t) = \{Su(t); u \in A\}.$$

We show that $B$ is relatively compact. In other words $B(t)$ is equicontinuous and uniformly bounded. For all $t_1, t_2 \in [0,1]$ and $u \in A$, we have

$$\begin{aligned}
&\left|Su\left(t_1\right) - Su\left(t_2\right)\right| \\
&= \left|\int_0^1 G(t_1,s)\frac{g(s,u(s))}{M\Big(s,\displaystyle\int_0^1 f(u(\tau))d\tau\Big)}ds - \int_0^1 G(t_2,s)\frac{g(s,u(s))}{M\Big(s,\displaystyle\int_0^1 f(u(\tau))d\tau\Big)}ds\right| \\
&= \left|\int_0^1 \Big(G(t_1,s)-G(t_2,s)\Big)\frac{g(s,u(s))}{M\Big(s,\displaystyle\int_0^1 f(u(\tau))d\tau\Big)}ds\right| \\
&\leq \int_0^1 \left|\Big(G(t_1,s)-G(t_2,s)\Big)\frac{g(s,u(s))}{M\Big(s,\displaystyle\int_0^1 f(u(\tau))d\tau\Big)}\right|ds \\
&\leq \int_0^1 \left|G(t_1,s)-G\left(t_2,s\right)\right|\frac{g(s,u(s))}{M\Big(s,\displaystyle\int_0^1 f(u(\tau))d\tau\Big)}ds.
\end{aligned} \tag{2.28}$$

Since $G$ is continuous function on the compact $[0,1]\times[0,1]$, we deduce $G$ is uniformly continuous,

$$\forall\varepsilon > 0, \exists\delta > 0, \forall s \in [0,1] : |t_1 - t_2| \leq \delta \Rightarrow |G(t_1,s) - G(t_2,s)| \leq \varepsilon. \tag{2.29}$$

From the boundedness of $A$, we have

$$\exists\, c_6 > 0, \forall u \in A : \quad 0 \leq \|u\|_\infty \leq c_6.$$

By (2.4),

$$\alpha(u) \leq f(u) \leq \beta(u).$$

Since $\beta$ is increasing function, we find

$$\alpha(0) \leq f(u) \leq \beta(c_6).$$

Integrating over $[0,1]$ the above inequation, we get

$$\alpha(0) \leq \int_0^1 f(u(\tau))d\tau \leq \beta(c_6),$$

which gives

$$\int_0^1 f(u(\tau))d\tau \in [\alpha(0), \beta(c_6)].$$

From the continuity of $\frac{1}{M}$ on the compact $[0,1] \times [\alpha(0), \beta(c_6)]$, it follows $\frac{1}{M}$ is bounded,

$$\exists\, c_7 > 0, \forall u \in A: \quad 0 < \frac{1}{M\Big(s, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)} \leq c_7. \tag{2.30}$$

Also, $g$ is continuous on the compact $[0,1] \times [0, c_6]$, which implies $g$ is bounded,

$$\exists\, c_8 > 0, \forall u \in A: \quad 0 \leq g(s, u(s)) \leq c_8. \tag{2.31}$$

According to (2.30) and (2.31), the following map

$$s \mapsto \frac{g(s, u(s))}{M\Big(s, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)}$$

is bounded,

$$\exists\, c_9 > 0, \forall s \in [0,1]: \quad \frac{g(s, u(s))}{M\Big(s, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)} \leq c_7 c_8 = c_9. \tag{2.32}$$

Combining (2.28), (2.29) and (2.32), for all $t_1, t_2 \in [0,1]$, we obtain

$$\begin{aligned}\Big|Su\,(t_1) - Su\,(t_2)\Big| &\leq \int_0^1 \Big|G(t_1, s) - G\,(t_2, s)\Big| \frac{g(s, u(s))}{M\Big(s, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)} ds \\ &\leq \varepsilon c_9 \int_0^1 ds = \varepsilon c_9.\end{aligned}$$

For all $t_1, t_2 \in [0,1]$, such that

$$|t_1 - t_2| \leq \delta \Rightarrow \Big|Su\,(t_1) - Su\,(t_2)\Big| \leq \varepsilon c_9.$$

Then, $B(t)$ is equicontinuous.

On the other hand, we claim that $B(t)$ is bounded. Using (2.7), we find

$$\begin{aligned}Su(t) &= \int_0^1 G\,(t, s) \frac{g(s, u(s))}{M\Big(s, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)} ds \\ &\leq \int_0^1 \frac{g(s, u(s))}{M\Big(s, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)} ds \\ &= c_9 \int_0^1 ds = c_9,\end{aligned}$$

which implies

$$\exists\, c_9 > 0, \forall u \in A : \quad \|Su\|_\infty \leq c_9.$$

Hence $B(t)$ is uniformly bounded. According to the Arzela-Ascoli Theorem, $B$ is a relatively compact. Therefore, $S$ is compact.

Let us define the following set

$$P = \{u \in F : \quad \min_{t\in[0,d]} u(t) \geq (1-d)\|u\|_\infty\}.$$

We would like point out that $P$ is a cone on $E$.

- $P$ is a nonempty subset on $E$. It is clear that $P \subset E$, and $P \neq \emptyset$ because $0 \in P$.
- $P$ is convex. For $\alpha \in [0,1]$ and $\zeta, \psi \in P$, we get

$$\begin{aligned} \zeta, \psi \geq 0, \quad &\min_{t\in[0,d]} \zeta(t) \geq (1-d)\|\zeta\|_\infty, \\ &\min_{t\in[0,d]} \psi(t) \geq (1-d)\|\psi\|_\infty. \end{aligned}$$

Since $(1-\alpha)\zeta + \alpha\psi \geq 0$, it follows

$$\begin{aligned} (1-d)\|(1-\alpha)\zeta + \alpha\psi\|_\infty &\leq (1-d)\{\|(1-\alpha)\zeta\|_\infty + \|\alpha\psi\|_\infty\} \\ &= (1-d)\left\{(1-\alpha)\|\zeta\|_\infty + \alpha\|\psi\|_\infty\right\}. \end{aligned}$$

For a fixed $d \in (0,1)$ we have $(1-d)(1-\alpha) \leq (1-\alpha)$ and $(1-d)\alpha \leq \alpha$. Then,

$$\begin{aligned} (1-d)\|(1-\alpha)\zeta + \alpha\psi\|_\infty &\leq (1-d)\left\{(1-\alpha)\|\zeta\|_\infty + \alpha\|\psi\|_\infty\right\} \\ &\leq (1-\alpha)\|\zeta\|_\infty + \alpha\|\psi\|_\infty \\ &\leq \min_{t\in[0,d]}(1-\alpha)\zeta(t) + \min_{t\in[0,d]} \alpha\psi(t) \\ &\leq \min_{t\in[0,d]}((1-\alpha)\zeta + \alpha\psi)(t). \end{aligned}$$

Hence, $P$ is convex.

- $P$ is a closed. Let $(u_j)_{j\in\mathbb{N}} \subset P$ be a sequence, has a limit $u \in E$. We have

$$\forall j \in \mathbb{N} : \quad u_j \geq 0, \tag{2.33}$$

and

$$\forall j \in \mathbb{N}, s \in [0,d] : \quad (1-d)\left\|u_j\right\|_\infty \leq \min_{t\in[0,d]} u_j(t). \tag{2.34}$$

Recalling that

$$\forall j \in \mathbb{N}, \forall t \in [0,1] : \quad 0 \leq \min_{t\in[0,d]} u_j(t) \leq u_j. \tag{2.35}$$

According to (2.34), (2.35), we find

$$\forall j \in \mathbb{N}, t \in [0,d] : \quad (1-d)\left\|u_j\right\|_\infty \leq u_j(t). \tag{2.36}$$

From

$$\lim_{j\to+\infty} \|u_j - u\|_\infty = 0,$$

we have

$$\lim_{j\to+\infty} u_j(t) = u(t). \tag{2.37}$$

Passing to the limit as $j \to \infty$ in (2.33) and (2.36), we get

$$u \geq 0, \tag{2.38}$$

and

$$(1-d)\|u\|_\infty \leq u. \tag{2.39}$$

Combining (2.38) and (2.39), it follows $u \in P$. We deduce $P$ is a closed.

- if $\lambda \geq 0$ is a real number and $u \in P$, we have $\lambda u \geq 0$ and

$$\begin{aligned}(1-d)\|\lambda u\|_\infty &= (1-d)\lambda\|u\|_\infty \\ &\leq (1-d)\lambda \min_{t\in[0,d]} u(t).\end{aligned}$$

Since $(1-d)\lambda \leq \lambda$, we find

$$\begin{aligned}(1-d)\|\lambda u\|_\infty &\leq (1-d)\lambda \min_{t\in[0,d]} u(t) \\ &\leq \lambda \min_{t\in[0,d]} u(t) \\ &= \min_{t\in[0,d]} (\lambda u)(t).\end{aligned}$$

Then $\lambda u \in P$.

- It is clear that if $u \in P, -u \in P$, also $u \geq 0, -u \leq 0$. We get $u = 0$.

Showing the desired result $S(P) \subset P$. Let $u \in P$ and $G, g, M$ are positives functions, then $Su \geq 0$. Set

$$j(s) = \frac{g(s,u(s))}{M\Big(s, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)}.$$

We have

$$\begin{aligned}\min_{t\in[0,d]} Su(t) &= \min_{t\in[0,d]} \int_0^1 G(t,s)\frac{g(s,u(s))}{M\Big(s,\displaystyle\int_0^1 f(u(\tau))d\tau\Big)}ds\\ &= \min_{t\in[0,d]} \int_0^1 G(t,s)j(s)ds.\end{aligned}$$

Applying (2.7), we get

$$\int_0^1 G(t,s)j(s)ds = \int_0^t (1-t)j(s)ds + \int_t^1 (1-s)j(s)ds,$$

which gives

$$\begin{aligned}\min_{t\in[0,d]} Su(t) &= \min_{t\in[0,d]} \int_0^1 G(t,s)j(s)ds\\ &= \min_{t\in[0,d]} \left(\int_0^t (1-t)j(s)ds + \int_t^1 (1-s)j(s)ds\right)\\ &\geq (1-d) \min_{t\in[0,d]} \left(\int_0^t j(s)ds + \int_t^1 (1-s)j(s)ds\right)\\ &\geq (1-d) \min_{t\in[0,d]} \left(\int_0^t (1-s)j(s)ds + \int_t^1 (1-s)j(s)ds\right)\\ &= (1-d) \min_{t\in[0,d]} \int_0^1 (1-s)j(s)ds.\end{aligned}$$

We know that

$$\begin{aligned}\min_{t\in[0,d]} \int_0^1 (1-s)j(s)ds &= \int_0^1 (1-s)j(s)ds\\ &= \max_{t\in[0,1]} \int_0^1 (1-s)j(s)ds,\end{aligned}$$

which leads to

$$\begin{aligned}\min_{t\in[0,d]} Su(t) &\geq (1-d) \min_{t\in[0,d]} \int_0^1 (1-s)j(s)ds\\ &= (1-d) \max_{t\in[0,1]} \int_0^1 (1-s)j(s)ds\\ &= (1-d) \max_{t\in[0,1]} \left(\int_0^t (1-s)j(s)ds + \int_t^1 (1-s)j(s)ds\right).\end{aligned}$$

Since $0 \leq s \leq t$, then $1-s \geq 1-t$. So

$$\int_0^1 (1-s)j(s)ds \geq \int_0^1 (1-t)j(s)ds.$$

It follows that

$$\begin{aligned}\min_{t\in[0,d]} Su(t) &= (1-d)\max_{t\in[0,1]}\left(\int_0^t (1-s)j(s)ds + \int_t^1 (1-s)j(s)ds\right)\\ &\geq (1-d)\max_{t\in[0,1]}\left(\int_0^t (1-t)j(s)ds + \int_t^1 (1-s)j(s)ds\right)\\ &= (1-d)\max_{t\in[0,1]}\int_0^1 G(t,s)j(s)ds\\ &= (1-d)\|Su\|_\infty.\end{aligned}$$

Combining these inequalities, we find

$$\min_{s\in[0,d]} Su(s) \geq (1-d)\|Su\|_\infty.$$

Thus, $S(P)\subset P$.

Next, we will show that the first condition of Theorem 1.7.1 is satisfied. We have

$$Su(0) = \int_0^1 G(0,s)\frac{g(s,u(s))}{M\left(s,\int_0^1 f(u(\tau))d\tau\right)}ds. \tag{2.40}$$

Since the map

$$s\mapsto G(0,s)\frac{g(s,u(s))}{M\left(s,\int_0^1 f(u(\tau))d\tau\right)},$$

is non-negative on $[0,1]$, $[0,d]\subset[0,1]$ and (2.40), it follows

$$\begin{aligned}Su(0) &= \int_0^1 G(0,s)\frac{g(s,u(s))}{M\left(s,\int_0^1 f(u(\tau))d\tau\right)}ds\\ &\geq \int_0^d G(0,s)\frac{g(s,u(s))}{M\left(s,\int_0^1 f(u(\tau))d\tau\right)}ds.\end{aligned}\tag{2.41}$$

Furthermore, from (2.3), we find

$$\xi(s)h(u(s)) \leq g(s,u(s)),\quad \text{a.e. } s\in[0,1]. \tag{2.42}$$

Also, using (2.4), we get

$$M\left(s,\int_0^1 f(u(\tau))d\tau\right) \leq \mu(s)w\Big(\int_0^1 f(u(\tau))d\tau\Big),\quad \text{a.e. } s\in[0,1]. \tag{2.43}$$

According to (2.41)-(2.43), it yields

$$\begin{aligned} Su(0) &\geq \int_0^d G(0,s)\frac{g(s,u(s))}{M\left(s, \int_0^1 f(u(\tau))d\tau\right)}ds \\ &\geq \frac{1}{w\Big(\int_0^1 f(u(\tau))d\tau\Big)}\int_0^d G(0,s)\frac{\xi(s)h(u(s))}{\mu(s)}ds. \end{aligned} \tag{2.44}$$

Using (2.39) and the fact that $h$ is non-decreasing function, we derive

$$h(u) \geq h((1-d)\|u\|_\infty). \tag{2.45}$$

In view of (2.2), it follows

$$f(u) \leq \beta(u). \tag{2.46}$$

Recalling that

$$u \leq \|u\|_\infty. \tag{2.47}$$

Since $\beta$ is increasing function, we obtain from (2.47),

$$\beta(u) \leq \beta(\|u\|_\infty). \tag{2.48}$$

Using (2.45) and (2.48), we get

$$f(u) \leq \beta(\|u\|_\infty). \tag{2.49}$$

Integrating over $[0,1]$ the above inequality (2.49), we find

$$\begin{aligned} \int_0^1 f(u(\tau))d\tau &\leq \int_0^1 \beta(\|u\|_\infty)d\tau \\ &= \beta(\|u\|_\infty)\int_0^1 d\tau \\ &= \beta(\|u\|_\infty). \end{aligned} \tag{2.50}$$

Using the equation (2.50), we have

$$w\Big(\int_0^1 f(u)d\tau\Big) \leq w\Big(\beta(\|u\|_\infty)\Big). \tag{2.51}$$

According to (2.44), (2.45) and (2.51), we derive that

$$\begin{aligned} Su(0) &\geq \frac{1}{w\Big(\int_0^1 f(u(\tau))d\tau\Big)}\int_0^d G(0,s)\frac{\xi(s)h(u(s))}{\mu(s)}ds \\ &\geq \frac{h\left((1-d)\|u\|_\infty\right)}{w\Big(\beta(\|u\|_\infty)\Big)}\int_0^d G(0,s)\frac{\xi(s)}{\mu(s)}ds. \end{aligned} \tag{2.52}$$

From the condition (2.8), we get

$$\forall s \in ]0, R_1] : \quad \frac{h\left((1-d)s\right)}{sw\Big(\beta(s)\Big)} \geq \inf_{0<s\leq R_1} \frac{h\left((1-d)s\right)}{sw\Big(\beta(s)\Big)} \geq \frac{1}{\displaystyle\int_0^d G(0,s)\frac{\xi(s)}{\mu(s)}ds},$$

which gives

$$\forall s \in ]0, R_1] : \quad \frac{h((1-d)s)}{w\Big(\beta(s)\Big)} \geq \frac{s}{\displaystyle\int_0^d G(0,s)\frac{\xi(s)}{\mu(s)}ds}.$$

It follows that

$$\forall s \in [0, R_1] : \quad \frac{h\left((1-d)s\right)}{w\Big(\beta(s)\Big)} \geq \frac{s}{\displaystyle\int_0^d G(0,s)\frac{\xi(s)}{\mu(s)}ds}.$$

Since $u \in E$, $\|u\|_\infty \leq R_1$, we have

$$\frac{h\left((1-d)\|u\|_\infty\right)}{w\Big(\beta(\|u\|_\infty)\Big)} \geq \frac{\|u\|_\infty}{\displaystyle\int_0^d G(0,s)\frac{\xi(s)}{\mu(s)}ds}. \tag{2.53}$$

From (2.52), (2.53), we obtain

$$\begin{aligned} Su(0) &\geq \frac{h\left((1-d)\|u\|_\infty\right)}{w\Big(\beta(\|u\|_\infty)\Big)} \int_0^d G(0,s)\frac{\xi(s)}{\mu(s)}ds \\ &\geq \frac{\|u\|_\infty}{\displaystyle\int_0^d G(0,s)\frac{\xi(s)}{\mu(s)}ds} \int_0^d G(0,s)\frac{\xi(s)}{\mu(s)}ds \\ &= \|u\|_\infty. \end{aligned}$$

Arguing as above, for this assumption

$$\Omega_1 = \{u \in E;\ \|u\|_\infty \leq R_1\}. \tag{2.54}$$

We obtain the following estimate

$$\forall u \in P \cap \partial\Omega_1 : \quad \|Su\|_\infty \geq \|u\|_\infty. \tag{2.55}$$

On the other hand, in view of condition (2.2), for all $u \in P$,

$$\alpha(u) \leq f(u). \tag{2.56}$$

Using (2.39) and the fact that $\alpha$ is non-decreasing function, we have

$$\alpha((1-d)\|u\|_\infty) \leq \alpha(u). \tag{2.57}$$

From (2.56) and (2.57), we get

$$\alpha((1-d)\|u\|_\infty) \leq f(u). \tag{2.58}$$

Integrating from 0 to $d$ the inequation (2.58), we obtain

$$\begin{aligned}\int_0^d f(u(s))ds &\geq \int_0^d \alpha((1-d)\|u\|_\infty)ds \\ &= \alpha((1-d)\|u\|_\infty)\int_0^d ds \\ &= d\alpha((1-d)\|u\|_\infty).\end{aligned} \tag{2.59}$$

From conditions (2.4) and (2.59), it follows that

$$\begin{aligned}M\Big(s, \int_0^1 f(u(\tau))d\tau\Big) &\geq \theta(s)v\Big(\int_0^1 f(u(\tau))d\tau\Big) \\ &\geq \theta(s)v\Big(\int_0^d f(u(\tau))d\tau\Big) \\ &\geq \theta(s)v\Big(d\alpha((1-d)\|u\|_\infty)\Big).\end{aligned} \tag{2.60}$$

From (2.4), we have

$$g(s,u(s)) \leq \eta(s)K(u(s)), \quad \text{a.e. } s \in [0,1]. \tag{2.61}$$

Using (2.47) and the fact that $K$ is increasing function, we obtain

$$K(u) \leq K(\|u\|_\infty). \tag{2.62}$$

By (2.62) and (2.61), gives

$$\eta(s)K(u(s)) \leq \eta(s)K(\|u\|_\infty), \quad \text{a.e. } s \in [0,1]. \tag{2.63}$$

Furthermore, according to (2.60) and (2.63), we derive that

$$\begin{aligned}\forall t\in[0,1]:\quad Su(t)&=\int_0^1 G(t,s)\frac{g(s,u(s))}{M\left(s,\displaystyle\int_0^1 f(u(\tau))d\tau\right)}ds\\&\leq\int_0^1 G(t,s)\frac{K(u(s))\eta(s)}{v\Big(\displaystyle\int_0^1 f(u(\tau))d\tau\Big)\theta(s)}ds\\&\leq\frac{K(\|u\|_\infty)}{v\Big(d\alpha((1-d)\|u\|_\infty\Big)}\int_0^1 G(t,s)\frac{\eta(s)}{\theta(s)}ds\\&\leq\frac{K(\|u\|_\infty)}{v\Big(d\alpha((1-d)\|u\|_\infty\Big)}\int_0^1\frac{\eta(s)}{\theta(s)}ds.\end{aligned}\tag{2.64}$$

From condition (2.9), we obtain

$$\begin{aligned}\exists\, R_2>R_1:\quad\frac{K(s)}{sv\Big(d\alpha((1-d)s)\Big)}&\leq\sup_{s\geq R_2}\frac{K(s)}{sv\Big(d\alpha((1-d)s)\Big)}\\&\leq\frac{1}{\displaystyle\int_0^1 G(t,s)\frac{\eta(s)}{\theta(s)}ds}.\end{aligned}$$

If $u\in E$ such that $\|u\|_\infty\geq R_2$, we find

$$\frac{K(\|u\|_\infty)}{v\Big(d\alpha((1-d)\|u\|_\infty\Big)}\leq\frac{\|u\|_\infty}{\displaystyle\int_0^1 G(t,s)\frac{\eta(s)}{\theta(s)}ds}.\tag{2.65}$$

Combining (2.64) and (2.65), it follows that

$$\begin{aligned}Su(t)&\leq\frac{k(\|u\|_\infty)}{v\Big(d\alpha((1-d)\|u\|_\infty\Big)}\int_0^1\frac{\eta(s)}{\theta(s)}ds\\&\leq\frac{\|u\|_\infty}{\displaystyle\int_0^1\frac{\eta(s)}{\theta(s)}ds}\int_0^1\frac{\eta(s)}{\theta(s)}ds\\&=\|u\|_\infty.\end{aligned}$$

Let us set

$$\Omega_2=\{u\in E:\quad\|u\|_\infty\leq R_2\}.\tag{2.66}$$

Then, we have

$$\forall u\in P\cap\partial\Omega_2:\quad\|Su\|_\infty\leq\|u\|_\infty.\tag{2.67}$$

Before concluding this section, we would like point out that $P$ is a cone in $E$ with $\overline{\Omega_1} \subset \Omega_2$. Also, $S$ is completly continuous. Combining (2.54), (2.55), (2.66) and (2.67), we conclude from Krasnoselskii Theorem 1.7.1 that $S$ has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$,

$$\forall t \in [0,1]: \quad u(t) = \int_0^1 G(t,s) \frac{g(s,u(s))}{M\left(s, \displaystyle\int_0^1 f(u(\tau))d\tau\right)} ds.$$

□

## 2.2 Example

**Example 2.2.1.**

*We study the existence of a positive solution of following problem*

$$\begin{cases} -\dfrac{e^{t\left(\int_0^1 \frac{e^u}{u^2+u+1} d\tau\right)^2}}{\displaystyle\int_0^1 \frac{e^u}{u^2+u+1} d\tau + 1} u'' = t\ln((u^2-u+1)t+1) \qquad t \in (0,1), \\ u'(0) = u(1) = 0. \end{cases} \tag{2.68}$$

*Define* $g : [0,1] \times [0,\infty) \to [0,\infty)$, *such that*

$$g(t,s) = t\ln((s^2-s+1)t+1).$$

*To get* (2.3), *we are trying to find the functions* $\xi, h, k$ *and* $\eta$. *Notice that*

$$\begin{aligned}(s^2-s+1)t+1 &= s^2t - st + t + 1 \\ &\leq s^2t + t + 1,\end{aligned}$$

*for any* $t \in [0,1]$, *then*

$$(s^2-s+1)t+1 \leq s^2+2.$$

*Since* ln *is increasing function, we get*

$$\ln((s^2-s+1)t+1) \leq \ln(s^2+2).$$

*Multiplying by* $t$,

$$t\ln((s^2-s+1)t+1) \leq t\ln(s^2+2). \tag{2.69}$$

*On the other hand, the function* $s \mapsto s^2 - s + 1$, *that derivative is* $s \mapsto 2s - 1$.

| $s$ | 0 | | $\frac{1}{2}$ | | $+\infty$ |
|---|---|---|---|---|---|
| $2s-1$ | | $-$ | 0 | $+$ | |
| $s^2-s+1$ | 1 | ↘ | $\frac{-1}{4}$ | ↗ | $+\infty$ |

*We find*

$$g(t, \frac{-1}{4}) = t\ln(\frac{3}{4}t + 1). \tag{2.70}$$

*From* (2.69) *and* (2.70), *we can choose*

$$\xi : t \mapsto t\ln(\frac{3}{4}t + 1), \; h : s \mapsto 1,$$

*and*

$$k : s \mapsto \ln(s^2 + 2), \; \eta : t \mapsto t.$$

*Furthermore,* $M : [0,1] \times [0,\infty) \to [0,\infty)$, *where* $M(t,s) = \frac{e^{ts^2}}{s+1}$. *We have*

$$\forall t \in [0,1] : \quad ts^2 \le s^2.$$

*Since exp is non-decreasing function, then*

$$\forall t \in [0,1] : \quad e^{ts^2} \le e^{s^2}. \tag{2.71}$$

*Also,*

$$\forall s \in [0,\infty) : \quad \frac{1}{s+1} \le 1. \tag{2.72}$$

*From* (2.71), (2.72), *it follows that*

$$\frac{e^{ts^2}}{s+1} \le e^{ts^2} \le e^{s^2}.$$

*Moreover, will derivative* $M$ *by* $s$, *we obtain*

$$\frac{\partial M}{\partial s}(t,s) := \frac{(2ts^2 + 2ts - 1)e^{ts^2}}{(s+1)^2}.$$

*This leads to*

$$2ts^2+2ts-1=0 \Rightarrow s=\frac{-t+\sqrt{t^2+2t}}{2t} \vee s=\frac{-t-\sqrt{t^2+2t}}{2t}.$$

| $s$ | 0 | | $\frac{-t+\sqrt{t^2+2t}}{2t}$ | | $+\infty$ |
|---|---|---|---|---|---|
| $\frac{\partial M}{\partial s}$ | | $-$ | 0 | $+$ | |
| $M(t,s)$ | $+\infty$ | ↘ | $\frac{e^{t(\frac{-t+\sqrt{t^2+2t}}{2t})^2}}{\frac{\sqrt{t^2+2t}-t}{2t}+1}$ | ↗ | $+\infty$ |

*We can choose*

$$\theta : t \mapsto \frac{e^{t(\frac{\sqrt{t^2+2t}-t}{2t})^2}}{\frac{\sqrt{t^2+2t}-t}{2t}+1},\ v : s \mapsto 1,$$

*and*

$$w : s \mapsto e^{s^2},\ \mu : t \mapsto 1,$$

*Thus, the function $M$ satisfies (2.4).*

*Let $f : [0,\infty) \to [0,\infty)$, such that $f(s)=\frac{e^s}{s^2+s+1}$. We have*

$$s^2+s+1 \geq 1 \Rightarrow \frac{1}{s^2+s+1} \leq 1.$$

*Multiplying the above inequation by $e^s$, we obtain*

$$\frac{e^s}{s^2+s+1} \leq e^s. \tag{2.73}$$

*On the other hand, the derivative of $f$ is*

$$s \mapsto \frac{(s^2-s)e^s}{(s^2+s+1)^2}.$$

| $s$ | 0 | | 1 | | $+\infty$ |
|---|---|---|---|---|---|
| $f'(s)$ | | $-$ | 0 | $+$ | |
| $f(s)$ | 1 | ↘ | $\frac{e}{3}$ | ↗ | $+\infty$ |

*To get* (2.2)*, it suffices to choose*

$$\beta : s \mapsto e^s, \ \alpha : s \mapsto \frac{e}{3}.$$

*We can see* $g, f$ *and* $M$ *are non-monotonic functions. We have*

$$0 < \int_0^d G(0,s)\frac{\xi(s)}{\mu(s)}ds = \int_0^d (1-s)s\ln(\frac{3}{4}s+1)ds < \infty,$$

*and*

$$0 < \int_0^1 \frac{\eta(s)}{\theta(s)}ds = \int_0^1 \frac{\sqrt{s^2+2s}+s}{2e^{s(\frac{\sqrt{s^2+2s}-s}{2s})^2}}ds < \infty,$$

*since the functions*

$$s \mapsto (1-s)s\ln(\frac{3}{4}s+1)$$

*and*

$$s \mapsto \frac{\sqrt{s^2+2s}+s}{2e^{s(\frac{\sqrt{s^2+2s}-s}{2s})^2}},$$

*are positives and continuous on* $[0,1]$*. In addition,*

$$\lim_{s\to 0^+} \frac{h((1-d)s)}{sw(\beta(s))} = \lim_{s\to 0^+} \frac{1}{se^{e^{2s}}} = \infty,$$

*and*

$$\lim_{s\to\infty} \frac{k(s)}{sv(d\alpha(1-d)s)} = \lim_{s\to\infty} \frac{\ln(s^2+2)}{s} = 0.$$

*Thus, there exists at least one positive solution of problem* (2.68)*, satisfying*

$$\forall t \in [0,1]: \quad u(t) = -\int_0^1 G(t,s)\frac{t\ln((u^2-u+1)t+1)e^u}{(u^2+u+1)e^{\left(\int_0^1 \frac{e^u}{u^2+u+1}d\tau\right)^2}}ds.$$

$\square$

## 2.3 Uniqueness of a Positive Solution

Suppose the diffusion coefficient function $M$ is dependent of the second variable

$$\exists\, m_0 > 0, \forall s \in [0,\infty): \quad M(s) \geq m_0, \tag{2.74}$$

with

$$\begin{aligned}&\exists\, c_8 > 0,\ \forall t \in [0,1], \forall\, s_1, s_2 \in [0,\infty) : \\ &\qquad (g(t,s_1) - g(t,s_2))(s_1 - s_2) \leq c_8 |s_1 - s_2|^2, \text{ with } \quad m_0 > c_8.\end{aligned} \tag{2.75}$$

In addition, both $M$ and $f$ are Lipschitz functions, such that

$$\exists\, c_9 > 0, \forall x, y \in [0,\infty) : \quad |f(x) - f(y)| \leq c_9 |x - y|. \tag{2.76}$$

$$\exists\, c_{10} > 0, \forall s_1, s_2 \in [0,\infty) : \quad |M(s_1) - M(s_2)| \leq c_{10} |s_1 - s_2|. \tag{2.77}$$

Moreover

$$c_9 c_{10} \|g(\cdot,0)\|_2 < (m_0 - c_8)^2. \tag{2.78}$$

We study the uniqueness for the problem

$$\begin{cases} -M\Big(\displaystyle\int_0^1 f(u(\tau))d\tau\Big)u''(t) = g(t,u), \quad t \in (0,1); \\ u'(0) = u(1) = 0. \end{cases} \tag{2.79}$$

**Theorem 2.3.1.**

*Assume that the conditions (2.2)-(2.9) and (2.74)- (2.78) are verified. Then, the problem (2.79) has a unique solution.*

## Proof.

Let $u$ be a positive solution to (2.79). Then, $u$ is solution to

$$-M\Big(\int_0^1 f(u(\tau))d\tau\Big)u'' = g(t,u), \quad t \in (0,1). \tag{2.80}$$

Multiplying by $u$ the equation (2.80),

$$-M\Big(\int_0^1 f(u(\tau))d\tau\Big)u''u = g(t,u)u. \tag{2.81}$$

Integrating by parts (2.81) over $[0,1]$, we get

$$-M\Big(\int_0^1 f(u(\tau))d\tau\Big)\Big[[u'u]_0^1 - \int_0^1 u'^2 dt\Big] = \int_0^1 g(t,u)u\, dt.$$

Since $u'(0) = u(1) = 0$, it follows that

$$M\Big(\int_0^1 f(u(\tau))d\tau\Big)\int_0^1 u'^2 dt = \int_0^1 g(t,u)u\, dt. \tag{2.82}$$

Moreover, in view of the equation (2.74), we deduce

$$m_0 \int_0^1 u'^2 dt \leq M\Big(\int_0^1 f(u(\tau))d\tau\Big)\int_0^1 u'^2 dt. \tag{2.83}$$

From (2.75), we have

$$(g(t,s_1)-g(t,s_2))(s_1-s_2) \leq c_8|s_1-s_2|^2.$$

We choose $u=s_1$, $0=s_2$ in the above inequality to obtain

$$g(t,u)u \leq c_8|u|^2 + g(t,0)u,$$

which implies

$$\begin{aligned}\int_0^1 g(t,u)u\,dt &\leq \int_0^1 \Big(c_8|u|^2 + g(t,0)u\Big)dt\\ &= c_8\int_0^1 |u|^2 dt + \int_0^1 g(t,0)u\,dt \\ &\leq c_8\|u\|_2^2 + \int_0^1 |g(t,0)u|\,dt.\end{aligned} \tag{2.84}$$

Using Cauchy-Schwarz inequality, we have

$$c_8\int_0^1 |u|^2 dt + \int_0^1 g(t,0)u\,dt \leq c_8\|u\|_2^2 + \|g(\cdot,0)\|_2\|u\|_2. \tag{2.85}$$

Notice that $\|u\|_2 \leq \|u'\|_2$ because

$$\forall x \in [0,1]: \quad u(x) = \int_0^x u'(t)\,dt + u(0).$$

Since $u(0)=0$, we get

$$\forall x \in [0,1]: \quad u(x) = \int_0^x u'(t)\,dt.$$

Then,

$$\begin{aligned}\forall x \in [0,1]: \quad |u(x)| &= \left|\int_0^x u'(t)\,dt\right| \\ &\leq \int_0^x |u'(t)|dt \\ &\leq \sup_{t\in[0,1]} |u'(t)| \int_0^x dt \\ &= \|u'\|_\infty \int_0^x dt \\ &= x\|u'\|_\infty \\ &\leq \|u'\|_\infty.\end{aligned}$$

From $\|u\|_\infty \leq \|u'\|_\infty$ and (2.85), it follows that

$$c_8\|u\|_2^2 + \|g(\cdot,0)\|_2\|u\|_2 \leq c_8\|u'\|_2^2 + \|g(\cdot,0)\|_2\|u'\|_2. \tag{2.86}$$

According to (2.82) - (2.86), we arrive at

$$m_0\|u'\|_2^2 \leq (c_8\|u'\|_2 + \|g(\cdot,u)\|_2)\|u'\|_2.$$

Then

$$(m_0 - c_8)\|u'\|_2^2 \leq \|g(\cdot,u)\|_2\|u'\|_2,$$

Since $\|u'\|_2 > 0$, we find

$$\|u'\|_2 \leq \frac{\|g(\cdot,u)\|_2}{m_0 - c_8}, \quad \text{with} \quad m_0 > c_8. \tag{2.87}$$

Let $u_1$ and $u_2$ be two positives solutions of (2.79). Then,

$$-M\Big(\int_0^1 f(u_1(\tau))d\tau\Big)u_1'' = g(t,u_1). \tag{2.88}$$

$$-M\Big(\int_0^1 f(u_2(\tau))d\tau\Big)u_2'' = g(t,u_2). \tag{2.89}$$

Subtracting (2.88) from (2.89), we get

$$-M\Big(\int_0^1 f(u_1(\tau))d\tau\Big)u_1'' + M\Big(\int_0^1 f(u_2(\tau))d\tau\Big)u_2'' = g(t,u_1) - g(t,u_2).$$

It follows that

$$\begin{aligned}-M\Big(\int_0^1 f(u_1(\tau))d\tau\Big)u_1'' + M\Big(\int_0^1 f(u_2(\tau))d\tau\Big)u_2'' - M\Big(\int_0^1 f(u_1(\tau))d\tau\Big)u_2''&\\ + M\Big(\int_0^1 f(u_1(\tau))d\tau\Big)u_2'' = g(t,u_1) - g(t,u_2).&\end{aligned}$$

Setting $u_3 = u_1 - u_2$, we obtain

$$\begin{aligned}-M\Big(\int_0^1 f(u_1(\tau))d\tau\Big)u_3'' = \Big\{M\Big(\int_0^1 f(u_1(\tau))d\tau\Big) - M\Big(\int_0^1 f(u_2(\tau))d\tau\Big)\Big\}u_2''&\\ + \Big(g(t,u_1) - g(t,u_2)\Big).&\end{aligned} \tag{2.90}$$

Multiplying (2.90) by $u_3$, we get

$$\begin{aligned}&-M\Big(\int_0^1 f(u_1(\tau))d\tau\Big)u_3''u_3 =\\ &\Big\{M\Big(\int_0^1 f(u_1(\tau))d\tau\Big) - M\Big(\int_0^1 f(u_2(\tau))d\tau\Big)\Big\}u_2''u_3 + \Big(g(t,u_1) - g(t,u_2)\Big)u_3.\end{aligned} \tag{2.91}$$

Integrating by parts (2.91) over $[0,1]$, we obtain

$$\begin{aligned}
&- M\Big(\int_0^1 f(u_1(\tau))d\tau\Big)\Big[[u_3' u_3]_0^1 - \int_0^1 u_3'^2 dt\Big]\\
&= -\Big\{M\Big(\int_0^1 f(u_1(\tau))d\tau\Big) - M\Big(\int_0^1 f(u_2(\tau))d\tau\Big)\Big\}\Big[[u_2' u_3]_0^1 - \int_0^1 u_2' u_3' dt\Big]\\
&+ \int_0^1 \Big(g(t,u_1) - g(t,u_2)\Big)u_3\, dt.
\end{aligned}$$

Since $u_3'(0) = u_1'(0) - u_2'(0) = 0$, $u_3(1) = u_1(1) - u_2(1) = 0$, it follows that

$$\begin{aligned}
&M\Big(\int_0^1 f(u_1(\tau))d\tau\Big)\|u_3'\|_2^2\\
&= -\Big\{M\Big(\int_0^1 f(u_1(\tau))d\tau\Big) - M\Big(\int_0^1 f(u_2(\tau))d\tau\Big)\Big\}\int_0^1 u_2' u_3'\, dt\\
&+ \int_0^1 \Big(g(t,u_1) - g(t,u_2)\Big)u_3\, dt.
\end{aligned} \tag{2.92}$$

Let us set

$$I = -\Big\{M\Big(\int_0^1 f(u_1(\tau))d\tau\Big) - M\Big(\int_0^1 f(u_2(\tau))d\tau\Big)\Big\}\int_0^1 u_2' u_3'\, dt, \tag{2.93}$$

and

$$J = \int_0^1 \Big(g(t,u_1) - g(t,u_2)\Big)u_3\, dt. \tag{2.94}$$

In view of equation (2.77), we have

$$\begin{aligned}
M\Big(\int_0^1 f(u_1(\tau))d\tau\Big) - M\Big(\int_0^1 f(u_2(\tau))d\tau\Big) &\leq \Big|M\Big(\int_0^1 f(u_1(\tau))d\tau\Big) - M\Big(\int_0^1 f(u_2(\tau))d\tau\Big)\Big|\\
&\leq c_{10}\Big(\int_0^1 f(u_1(\tau))d\tau - \int_0^1 f(u_2(\tau))d\tau\Big)\\
&= c_{10}\int_0^1 \Big(f(u_1(\tau)) - f(u_2(\tau))\Big)d\tau\\
&\leq c_{10}\int_0^1 \Big|f(u_1(\tau)) - f(u_2(\tau))\Big|d\tau.
\end{aligned} \tag{2.95}$$

From (2.76), it follows

$$\begin{aligned}
|f(u_1) - f(u_2)| &\leq c_9|u_1 - u_2|\\
&= c_9|u_3|.
\end{aligned} \tag{2.96}$$

Then,

$$\int_0^1 |f(u_1(\tau)) - f(u_2(\tau))| d\tau \leq c_9 \int_0^1 |u_3| d\tau.$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int_0^1 |f(u_1(\tau)) - f(u_2(\tau))| d\tau &\leq c_9 \int_0^1 |u_3| d\tau \\ &\leq c_9 \Big( \int_0^1 1 d\tau \Big)^{\frac{1}{2}} \Big( \int_0^1 |u_3|^2 d\tau \Big)^{\frac{1}{2}} \\ &= c_9 \|u_3\|_2. \end{aligned} \tag{2.97}$$

From (2.95) and (2.97), we find

$$\begin{aligned} M\Big(\int_0^1 f(u_1(\tau)) d\tau\Big) - M\Big(\int_0^1 f(u_2(\tau)) d\tau\Big) &\leq c_{10} \int_0^1 \Big| f(u_1(\tau)) - f(u_2(\tau)) \Big| d\tau \\ &= c_{10} c_9 \|u_3\|_2. \end{aligned} \tag{2.98}$$

In addition, using (2.93) and (2.98), we get

$$\begin{aligned} I \leq |I| &= \Bigg| - \Bigg\{ M\Big(\int_0^1 f(u_1(\tau)) d\tau\Big) - M\Big(\int_0^1 f(u_2(\tau)) d\tau\Big) \Bigg\} \int_0^1 u_2' u_3' dt \Bigg| \\ &\leq c_9 c_{10} \|u_3\|_2 \Bigg| \int_0^1 u_2' u_3' dt \Bigg| \\ &\leq c_9 c_{10} \|u_3\|_2 \int_0^1 |u_2' u_3'| dt. \end{aligned}$$

By Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} I &\leq c_9 c_{10} \|u_3\|_2 \int_0^1 |u_2' u_3'| dt \\ &\leq c_9 c_{10} \|u_3\|_2 \Big( \int_0^1 |u_2'|^2 d\tau \Big)^{\frac{1}{2}} \Big( \int_0^1 |u_3'|^2 d\tau \Big)^{\frac{1}{2}} \\ &= c_9 c_{10} \|u_3\|_2 \|u_2'\|_2 \|u_3'\|_2 \\ &= c_9 c_{10} \|u_3'\|_2^2 \|u_2'\|_2. \end{aligned} \tag{2.99}$$

From (2.87) and (2.99), it follows

$$\begin{aligned} I &\leq c_9 c_{10} \|u_3'\|_2^2 \|u_2'\|_2 \\ &\leq c_9 c_{10} \|u_3'\|_2^2 \frac{\|g(\cdot, 0)\|_2}{m_0 - c_8}. \end{aligned} \tag{2.100}$$

Now, by (2.94), we arrive at

$$J \leq \int_0^1 \Big(g(t,u_1) - g(t,u_2)\Big) u_3 \, dt. \tag{2.101}$$

From (2.74),

$$\begin{aligned} \Big(g(t,u_1) - g(t,u_2)\Big) u_3 &\leq c_8 |u_1 - u_2|^2 \\ &= c_8 |u_3|^2. \end{aligned} \tag{2.102}$$

Integrating over $[0,1]$ the inequation (2.102), it follows that

$$\int_0^1 \Big(g(t,u_1) - g(t,u_2)\Big) u_3 dt \leq c_8 \int_0^1 |u_3|^2 dt.$$

Using Cauchy-Schwarz inequality and $\|u_3\|_2 \leq \|u_3'\|_2$, we get

$$\begin{aligned} \int_0^1 \Big(g(t,u_1) - g(t,u_2)\Big) u_3 dt &\leq c_8 \int_0^1 |u_3|^2 dt \\ &\leq c_8 \Big(\int_0^1 1 d\tau\Big)^{\frac{1}{2}} \Big(\int_0^1 |u_3|^2 d\tau\Big)^{\frac{1}{2}} \\ &= c_8 \|u_3\|_2^2 \\ &\leq c_8 \|u_3'\|_2^2. \end{aligned} \tag{2.103}$$

According to (2.101) - (2.103), we obtain

$$J \leq c_8 \|u_3'\|_2^2. \tag{2.104}$$

Moreover, using (2.74), we find

$$m_0 \|u_3^{'}\|_2^2 \leq M\Big(\int_0^1 f(u_1(\tau)) d\tau\Big) \|u_3^{'}\|_2^2. \tag{2.105}$$

Combining (2.100), (2.104) and (2.105), we derive the following estimate

$$\begin{aligned} m_0 \|u_3^{'}\|_2^2 &\leq \frac{c_2 c_3}{m_0 - c_8} \|g(\cdot,0)\|_2 \|u_3^{'}\|_2^2 + c_8 \|u_3^{'}\|_2^2 \\ &= \Big(\frac{c_2 c_3}{m_0 - c_8} \|g(\cdot,0)\|_2 + c_8\Big) \|u_3^{'}\|_2^2, \end{aligned}$$

which gives

$$\Big(m_0 - c_8 - \frac{c_9 c_{10}}{m_0 - c_8} \|g(.,0)\|_2\Big) \|u_3^{'}\|_2^2 \leq 0.$$

Since

$$\Big(m_0 - c_8 - \frac{c_9 c_{10}}{m_0 - c_8} \|g(\cdot,0)\|_2\Big) > 0,$$

it follows that

$$\|u_3'\|_2^2 = 0.$$

Then,

$$u_3 = 0 \Leftrightarrow u_1 - u_2 = 0.$$

This leads to $u_1 = u_2$. Thus, the uniqueness of the solution of the problem (2.79). $\square$

# Chapter 3

# Existence of Positive Radial Solution in the Multidimensional Case

We claim the existence of a positive radial solution in an annular domain of $\mathbb{R}^N$ with $N \geq 2$, for following problem

$$\begin{cases} -M\Big(|x|, \displaystyle\int_\Omega f(u(\tau))d\tau\Big)\Delta u = g(|x|, u) \quad \text{in}\, \Omega; \\ u = 0 \quad \text{on}\, \partial\Omega. \end{cases} \tag{3.1}$$

Assume that

- $\Omega = \{x \in \mathbb{R}^N |\, a < |x| < b\},\ a, b \in (0, \infty)$, such that $\quad a < b$.
- $f : [0, \infty) \to [0, \infty)$ is a continuous function such that

$$\alpha(s) \leq f(s) \leq \beta(s), \quad \forall s \in [0, \infty), \tag{3.2}$$

where $\alpha, \beta : [0, \infty) \to [0, \infty)$ are two nondecreasing continuous functions.

- $g : [a, b] \times [0, \infty) \to [0, \infty)$ is a continuous function satisfying

$$\xi(t)h(s) \leq g(t, s) \leq \eta(t)k(s), \tag{3.3}$$

for some measurables functions $\xi, \eta : [a, b] \to [0, \infty)$ and nondecreasing continuous functions $h, k : [0, \infty) \to [0, \infty)$. Also, $g(\cdot, 0) \neq 0$.

- $M : [a, b] \times [0, \infty) \to (0, \infty)$ is a continuous function such that

$$\theta(t)v(s) \leq M(t, s) \leq \mu(t)w(s), \tag{3.4}$$

where $v, w : [0, \infty) \to (0, \infty)$ are increasing continuous functions and $\theta, \mu : [a, b] \to (0, \infty)$ are measurables functions. Set

$$A = \frac{(ab)^{N-2}}{b^{N-2} - a^{N-2}}, \quad B = \frac{b^{N-2}}{b^{N-2} - a^{N-2}},$$

and consider the functions on $[0, 1]$,

$$q_N(s) = \Big(b(\frac{a}{b})^s \log(\frac{b}{a})\Big)^2, \quad N = 2,$$

and

$$q_N(s) = \frac{A^{\frac{2}{N-2}}}{(N-2)^2 (B-s)^{2\frac{N-1}{N-2}}}, \quad N \geq 3.$$

For $d_1, d_2 \in (0, 1)$ such that $d_1 < d_2$, set $d_0 = \min(1 - d_2, d_1)$. There exists positive constant $R_1$, depending on $N$, satisfying

$$\begin{cases} \underset{0<s\leq R_1}{inf} \dfrac{h(d_0 s)}{sw\Big(\omega b^2 \log(\frac{b}{a})\beta(s)\Big)} \geq \dfrac{1}{\displaystyle\int_{d_1}^{d_2} \mathbf{G}(\frac{1}{2}, s) q_2(s) \frac{\xi\Big(b(\frac{a}{b})^s\Big)}{\mu\Big(b(\frac{a}{b})^s\Big)} ds}, & if N = 2; \\ \underset{0<s\leq R_1}{inf} \dfrac{h(d_0 s)}{sw\Big(\frac{\omega}{A(N-2)}(\frac{A}{B-s})^{2\frac{N-1}{N-2}}\beta(s)\Big)} \geq \dfrac{1}{\displaystyle\int_{d_1}^{d_2} \mathbf{G}(\frac{1}{2}, s) q_N(s) \frac{\xi\Big((\frac{A}{B-1})^{\frac{1}{N-2}}\Big)}{\mu\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)} ds}, & if N \geq 3, \end{cases} \tag{3.5}$$

and there exists $R_2 > R_1$ depending on $N$, such that

$$\begin{cases} \underset{s\geq R_2}{sup} \dfrac{k(s)}{sv\Big(\omega a^2 (d_2 - d_1) \log(\frac{b}{a})\alpha(d_0 s)\Big)} \leq \dfrac{1}{\displaystyle\int_0^1 q_2(s) \frac{\eta\Big(b(\frac{a}{b})^s\Big)}{\theta\Big(b(\frac{a}{b})^s\Big)} ds}, & if\, N = 2; \\ \underset{s\geq R_2}{sup} \dfrac{k(s)}{sv\Big((d_2 - d_1)\frac{\omega}{A(N-2)}(\frac{A}{B})^{2\frac{N-1}{N-2}}\alpha(d_0 s)\Big)} \leq \dfrac{1}{\displaystyle\int_0^1 q_N(s) \frac{\eta\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)}{\theta\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)} ds}, & if N \geq 3; \end{cases} \tag{3.6}$$

where $\omega$ is the measure of the unit sphere in $\mathbb{R}^N$, and $\mathbf{G}$ is green function

$$\mathbf{G}(t, s) = \begin{cases} (1-t)s, & 0 \leq s \leq t \leq 1; \\ (1-s)t, & 0 \leq t \leq s \leq 1. \end{cases} \tag{3.7}$$

## 3.1 Existence of Positive Radial Slution

**Theorem 3.1.1.**

*Assume that the hypotheses (3.2)-(3.7) are verified. The problem (3.1) has a positive radial solution.*

## Proof.

We are able to change as in [37, 60]. Problem (3.1) can be reformulated as the following problem

$$\begin{cases} \mathcal{M}\Big(t, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)U''(t) = q_N(t)\mathcal{G}(t,U) \quad t \in (0,1); \\ U(0) = U(1) = 0. \end{cases} \tag{3.8}$$

Indeed, if $N = 2$, we establish

$$\begin{aligned} r = b(\frac{a}{b})^t \Leftrightarrow \quad dr &= b\frac{d}{dt}\Big(\frac{a}{b}\Big)^t \\ &= b\frac{d}{dt}e^{t\log(\frac{a}{b})} \\ &= be^{t\log\left(\frac{a}{b}\right)}\log\Big(\frac{a}{b}\Big)dt \\ &= b\Big(\frac{a}{b}\Big)^t\log\Big(\frac{a}{b}\Big)dt \\ &= r\log\Big(\frac{a}{b}\Big)dt, \end{aligned} \tag{3.9}$$

and

$$U(t) = u(r), \quad \mathcal{G}(t,U) = g\Big(b\Big(\frac{a}{b}\Big)^t, U\Big). \tag{3.10}$$

The measure of unit ball is $\pi$. Using Theorem 1.4.4 for $n = 2$, gives

$$\int_0^1 \mathcal{F}(U(\tau))d\tau = 2\pi\int_a^b f(u(r))rdr.$$

If $r = a \Rightarrow s = 1$, $r = b \Rightarrow s = 0$, we have

$$\begin{aligned}\int_0^1 \mathcal{F}(U(\tau))d\tau &= 2\pi \int_1^0 b(\frac{a}{b})^s f(U(s)) b(\frac{a}{b})^s \log\left(\frac{a}{b}\right) ds\\ &= 2\pi b^2 \log\left(\frac{a}{b}\right) \int_1^0 (\frac{a}{b})^{2s} f(U(s))ds\\ &= -2\pi b^2 \log\left(\frac{a}{b}\right) \int_0^1 (\frac{a}{b})^{2s} f(U(s))ds\\ &= 2\pi b^2 \log\left(\frac{b}{a}\right) \int_0^1 (\frac{a}{b})^{2s} f(U(s))ds.\end{aligned}$$

This leads to

$$\begin{aligned}\mathcal{M}\Big(t, \int_0^1 \mathcal{F}(U(\tau))d\tau\Big) =& M\Big(r,\ 2\pi \int_a^b r f(u(r))dr\Big)\\ =& M\Big(b\Big(\frac{b}{a}\Big)^t,\ 2\pi b^2 \log\left(\frac{b}{a}\right) \int_0^1 (\frac{a}{b})^{2s} f(U(s))ds\Big).\end{aligned}$$

If $N \geq 3$, we have

$$\begin{aligned}t = B - \frac{A}{r^{N-2}} \Leftrightarrow r = (\frac{A}{B-t})^{\frac{1}{N-2}}, \quad dt &= -(-\frac{A(N-2)}{r^{N-1}})dr\\ &= \frac{A(N-2)}{r^{N-1}}dr,\end{aligned}$$

and

$$U(t) = u(r), \quad \mathcal{G}(t, U) = g\Big((\frac{A}{B-t})^{\frac{1}{N-2}},\ U\Big). \tag{3.11}$$

The measure of unit ball for $N = 3$ is $\frac{4\pi}{3}$. Using Theorem 1.4.4, we obtain

$$\int_0^1 \mathcal{F}(U(\tau))d\tau = 3\frac{4\pi}{3} \int_a^b f(u(r))r^2 dr = 4\pi \int_a^b f(u(r))r^2 dr.$$

If $r = a \Rightarrow s = 0$, $r = b \Rightarrow s = 1$, we get

$$\int_0^1 \mathcal{F}(U(\tau))d\tau = \frac{4\pi}{A} \int_0^1 \Big(\frac{A}{B-s}\Big)^4 f(U(s))ds.$$

Then,

$$\begin{aligned}\mathcal{M}\Big(t, \int_0^1 \mathcal{F}(U(\tau))d\tau\Big) =& M\Big(r,\ 4\pi \int_a^b r f(u(r))dr\Big)\\ =& M\Big(\frac{A}{B-t},\ \frac{4\pi}{A} \int_0^1 \Big(\frac{A}{B-s}\Big)^4 f(U(s))ds\Big).\end{aligned}$$

If $N > 3$, it follows that

$$\begin{aligned}\mathcal{M}\Big(t, \int_0^1 \mathcal{F}(U(\tau))d\tau\Big) &= M\Big(r,\, \omega \int_a^b r^{N-1} f(u(r))dr\Big)\\ &= M\Big((\frac{A}{B-t})^{\frac{1}{N-2}},\, \frac{\omega}{A(N-2)}\int_0^1 (\frac{A}{B-s})^{2\frac{N-1}{N-2}} f(U(s))ds\Big),\end{aligned}$$

where $\omega = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$, $\Gamma(k) = (k-1)! = \displaystyle\int_0^\infty e^{-s}s^{t-1}ds$, $t > 0$. As

$$n = 4: \quad \omega = \frac{\pi^{4/2}}{\Gamma\left(\frac{4}{2}+1\right)} = \frac{\pi^2}{\Gamma(3)} = \frac{\pi^2}{2!} = \frac{\pi^2}{2}$$

and

$$n = 5: \quad \omega = \frac{\pi^{5/2}}{\Gamma\left(\frac{5}{2}+1\right)} = \frac{\pi^{5/2}}{\Gamma\left(\frac{7}{2}\right)} = ... = \frac{4\pi^2}{15}.$$

Now, we prove that the problem (3.8) has a positive radial solution such that

$$U(t) = \int_0^1 \mathbf{G}(t,s)\frac{q_N(s)\mathcal{G}(s,U(s))}{\mathcal{M}\Big(s, \int_0^1 \mathcal{F}(U(\tau))d\tau\Big)}ds.$$

Let us define the operator

$$\begin{aligned} T:\ C([0,1],\mathbb{R}) &\to C([0,1],\mathbb{R})\\ U &\mapsto TU,\end{aligned}$$

where

$$\forall t \in [0;1]: \quad TU(t) = \int_0^1 \mathbf{G}(t,s)\frac{q_N(s)\mathcal{G}(s,U(s))}{\mathcal{M}\Big(s, \int_0^1 \mathcal{F}(U(\tau))d\tau\Big)}ds.$$

The above operator is well define by Theorem 1.4.1. Indeed, by the continuity of functions $q, \mathbf{G}, s \mapsto \mathcal{G}(s,U(s))$ and $s \mapsto \dfrac{1}{\mathcal{M}\Big(s, \int_0^1 \mathcal{F}(U(\tau))d\tau\Big)}$ on $[0,1]$, the function

$$(t,s) \mapsto \mathbf{G}(t,s)\frac{q_N(s)\mathcal{G}(s,U(s))}{\mathcal{M}\Big(s, \int_0^1 \mathcal{F}(U(\tau))d\tau\Big)}$$

is continuous on $[0,1]$ and bounded by

$$s \mapsto \frac{q_N(s)\mathcal{G}(s,U(s))}{\mathcal{M}\Big(s, \int_0^1 \mathcal{F}(U(\tau))d\tau\Big)}.$$

We claim the following map

$$T : (F, \|\cdot\|_\infty) \to (F, \|\cdot\|_\infty),$$

where $F = \{U \in E |\ U(t) \geq 0,\ \forall t \in [a,b]\}$ is a completly continuous. Firstly, $T$ is continuous. Let $(U_j)_{j\in\mathbb{N}} \subset F$ be a sequence, which converges to $U \in F$, such that

$$\lim_{j\to+\infty} \|U_j - U\|_\infty = 0, \tag{3.12}$$

which can be written as

$$\lim_{j\to+\infty} \sup_{t\in[0,1]} |U_j(t) - U(t)| = 0.$$

Using the limit definition to write

$$\forall \gamma > 0, \exists j_1 \in \mathbb{N}, \forall j \in \mathbb{N}, j \geq j_1 : \quad |U_j(t) - U(t)| \leq \gamma,$$

which gives

$$\forall j \geq j_1 : \quad U_j(t) \leq \gamma + U(t). \tag{3.13}$$

Let us set

$$C_1 = \max_{t\in[0,1]} \left\{ \max_{t\in[0,1]} U(t) + \gamma, \max_{t\in[0,1]} U_0(t), \max_{t\in[0,1]} U_1(t), \max_{t\in[0,1]} U_2(t), ..., \max_{t\in[0,1]} U_{j_1-1}(t) \right\} > 0. \tag{3.14}$$

By (3.13) and (3.14), we find

$$\forall j \in \mathbb{N}, \forall t \in [0,1] : \quad U_j(t), U(t) \in [0, C_1].$$

From the continuity of $\mathcal{G}$ on the compact $[0,1] \times [0, C_1]$, $\mathcal{G}$ is dounded,

$$\exists\, C_2 > 0, \forall x, y \in [0,1] \times [0, C_1] : \quad 0 \leq \mathcal{G}(x,y) \leq C_2. \tag{3.15}$$

By taking $s = x$, $U_j(s) = y$ in (3.15), it follows that

$$\forall s \in [0,1], \forall j \in \mathbb{N} : \quad 0 \leq \mathcal{G}(s, U_j(s)) \leq C_2. \tag{3.16}$$

Moreover, using (3.12) and the continuity of $\mathcal{G}$, we find

$$\forall s \in [0,1] : \quad \lim_{j\to+\infty} \mathcal{G}(s, U_j(s)) = \mathcal{G}(s, U(s)). \tag{3.17}$$

Since $\mathcal{F}$ is continuous on the compact $[0, C_1]$, then $\mathcal{F}$ is uniformly continuous,

$$\forall \varepsilon > 0,\ \exists\, \delta > 0, \forall x, y \in [0, C_1] : \quad |x - y| \leq \delta \Rightarrow |\mathcal{F}(x) - \mathcal{F}(y)| \leq \varepsilon. \tag{3.18}$$

Choosing $U_j(s) = x, U(s) = y, \gamma = \delta$ in (3.18), and (3.12), we obtain

$$\forall \varepsilon > 0, \forall j \in \mathbb{N}, \forall s \in [0, 1] : \quad |\mathcal{F}(U_j(s)) - \mathcal{F}(U(s))| \leq \varepsilon.$$

Hence, $(\mathcal{F}(U_j))_{j\in\mathbb{N}}$ converge uniformly to $\mathcal{F}(U(\tau))$. From Theorem 1.3.2, we find

$$\begin{aligned} \lim_{j\to+\infty} \int_0^1 \mathcal{F}(U_j(\tau))d\tau &= \int_0^1 \lim_{j\to+\infty} \mathcal{F}(U_j(\tau))d\tau \\ &= \int_0^1 \mathcal{F}(\lim_{j\to+\infty} U_j(\tau))d\tau \\ &= \int_0^1 \mathcal{F}(U(\tau))d\tau, \end{aligned} \tag{3.19}$$

which implies

$$\forall \varepsilon > 0,\ \exists\, j_0 \in \mathbb{N}, \forall j \geq j_0 : \quad \left| \int_0^1 \mathcal{F}(U_j(\tau))d\tau - \int_0^1 \mathcal{F}(U(\tau))d\tau \right| \leq \varepsilon,$$

then,

$$\forall j \geq j_0 : \quad \int_0^1 \mathcal{F}(U_j(\tau))d\tau \leq \int_0^1 \mathcal{F}(U(\tau))d\tau + \varepsilon.$$

Set

$$\begin{aligned} C_3 = \max\bigg\{ &\int_0^1 \mathcal{F}(U(\tau))d\tau + \varepsilon, \int_0^1 \mathcal{F}(U_0(\tau))d\tau, \int_0^1 \mathcal{F}(U_1(\tau))d\tau, \\ &\int_0^1 \mathcal{F}(U_2(\tau))d\tau, ..., \int_0^1 \mathcal{F}(U_{j_0-1}(\tau))d\tau \bigg\} > 0, \end{aligned}$$

which gives

$$\forall j \in \mathbb{N} : \quad \int_0^1 \mathcal{F}(U_j(\tau))d\tau, \int_0^1 \mathcal{F}(U(\tau))d\tau \in [0, C_3].$$

By (3.19), we find

$$\lim_{j\to+\infty} \frac{1}{\mathcal{M}\left(s, \int_0^1 \mathcal{F}(U_j(\tau))d\tau\right)} = \frac{1}{\mathcal{M}\left(s, \int_0^1 \mathcal{F}(U(\tau))d\tau\right)}. \tag{3.20}$$

Since $(x, y) \mapsto \dfrac{1}{\mathcal{M}(x, y)}$ is continuous on the compact $[0, 1] \times [0, C_3]$. Then, it is bounded

$$\exists\, C_4 > 0, \forall s \in [0, 1], \forall j \in \mathbb{N} : \quad 0 < \frac{1}{\mathcal{M}\left(s, \int_0^1 \mathcal{F}(U_j(\tau))d\tau\right)} \leq C_4. \tag{3.21}$$

Combining (3.16) and (3.21), yields

$$\exists C_5 > 0, \forall j \in \mathbb{N}, \forall s \in [0,1] : \quad \frac{\mathcal{G}(s, U_j(s))}{M\left(s, \displaystyle\int_0^1 \mathcal{F}(U_j(\tau))d\tau\right)} \leq C_2 C_4 = C_5. \tag{3.22}$$

From (3.17) and (3.20), and the Lebesgue dominated convergence theorem, we have

$$\forall s \in [0,1] : \quad \lim_{j \to +\infty} \frac{\mathcal{G}(s, U_j(s))}{\mathcal{M}\left(s, \displaystyle\int_0^1 \mathcal{F}(U_j)d\tau\right)} = \frac{\mathcal{G}(s, U(s))}{\mathcal{M}\left(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\right)}. \tag{3.23}$$

Furthermore, for all $j \in \mathbb{N}$, we get

$$\begin{aligned}
&\Big|TU_j(t) - TU(t)\Big| \\
&= \left| \int_0^1 \mathbf{G}(t,s) q_N(s) \frac{\mathcal{G}(s, U_j(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U_j(\tau))d\tau\Big)} ds - \int_0^1 \mathbf{G}(t,s) q_N(s) \frac{\mathcal{G}(s, U(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)} ds \right| \\
&= \left| \int_0^1 \mathbf{G}(t,s) q_N(s) \Big( \frac{\mathcal{G}(s, U_j(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U_j(\tau))d\tau\Big)} - \frac{\mathcal{G}(s, U(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)} \Big) ds \right| \\
&\leq \int_0^1 \Big| \mathbf{G}(t,s) q_N(s) \Big| \left| \frac{\mathcal{G}(s, U_j(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U_j(\tau))d\tau\Big)} - \frac{\mathcal{G}(s, U(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)} \right| ds \\
&\leq \int_0^1 q_N(s) \left| \frac{\mathcal{G}(s, U_j(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U_j(\tau))d\tau\Big)} - \frac{\mathcal{G}(s, U(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)} \right| ds.
\end{aligned} \tag{3.24}$$

Since $q_N$ is continuous function on the compact $[0,1]$, then

$$\exists C_6 > 0, \forall s \in [0,1] : \quad q_N(s) \leq C_6. \tag{3.25}$$

By passing the norm infinity, we get

$$\begin{aligned}
0 \leq \|TU_j(t) - TU(t)\|_\infty &\leq \sup_{t \in [0,1]} \left( \int_0^1 C_6 \left| \frac{\mathcal{G}(s, U_j(s))}{\mathcal{M}(s, \int_0^1 \mathcal{F}(U_j(\tau))d\tau)} - \frac{\mathcal{G}(s, U(s))}{\mathcal{M}(s, \int_0^1 \mathcal{F}(U(\tau))d\tau)} \right| ds \right) \\
&= C_6 \int_0^1 \left| \frac{\mathcal{G}(s, U_j(s))}{\mathcal{M}(s, \int_0^1 \mathcal{F}(U_j(\tau))d\tau)} - \frac{\mathcal{G}(s, U(s))}{\mathcal{M}(s, \int_0^1 \mathcal{F}(U(\tau))d\tau)} \right| ds.
\end{aligned} \tag{3.26}$$

Passing to the limit in (3.26), as $j$ tends to $\infty$ and using (3.23), we obtain

$$\lim_{j\to\infty} \|TU_j(t) - TU(t)\|_\infty = 0.$$

This means that $T$ is continuous. We show $T$ is compact. Let $A \subset F$ be a bounded subset. For any element $t \in [0,1]$, we define

$$B = \{TU; U \in A\}, \quad B(t) = \{TU(t); U \in A\}.$$

The set $B$ is relatively compact. Let us prove that $B(t)$ is equicontinuous and uniformly bounded. For all $t_1, t_2 \in [0,1]$ and $U \in A$, we have

$$\begin{aligned}
&\Big|TU\left(t_1\right) - TU\left(t_2\right)\Big| \\
&= \left|\int_0^1 \mathbf{G}(t_1,s) q_N(s) \frac{\mathcal{G}(s,U(s))}{\mathcal{F}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)} ds - \int_0^1 \mathbf{G}(t_2,s) q_N(s) \frac{\mathcal{G}(s,U(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)} ds\right| \\
&= \left|\int_0^1 \Big(\mathbf{G}(t_1,s) - \mathbf{G}(t_2,s)\Big) q_N(s) \frac{\mathcal{G}(s,U(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)} ds\right| \\
&\leq \int_0^1 \left|\Big(\mathbf{G}(t_1,s) - \mathbf{G}(t_2,s)\Big) q_N(s) \frac{\mathcal{G}(s,U(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)}\right| ds \\
&\leq \int_0^1 \Big|\mathbf{G}(t_1,s) - \mathbf{G}\left(t_2,s\right)\Big| q_N(s) \frac{\mathcal{G}(s,U(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)} ds.
\end{aligned} \tag{3.27}$$

Using the continuity of $\mathbf{G}$ on the compact $[0,1] \times [0,1]$. Then, $\mathbf{G}$ is uniformly continuous,

$$\forall \varepsilon > 0, \exists\, \delta > 0, \forall s \in [0,1]: \ |t_1 - t_2| \leq \delta \Rightarrow |\mathbf{G}(t_1,s) - \mathbf{G}(t_2,s)| \leq \varepsilon. \tag{3.28}$$

Since $A$ is bounded, we find

$$\exists\, C_7 > 0, \forall\, U \in A: \quad 0 \leq \|U\|_\infty \leq C_7.$$

By (3.2),

$$\alpha(U) \leq f(U(\tau)) \leq \beta(U),$$

Since $\beta$ is increasing function,

$$\beta(U) \leq \beta(\|U\|_\infty).$$

Then,

$$\alpha(0) \leq f(U(\tau)) \leq \beta(C_7),$$

which implies, by integrating over $[0,1]$ the above inequation,

$$\alpha(0) \leq \int_0^1 f(U(\tau))d\tau \leq \beta(C_7).$$

From the continuity of $\frac{1}{\mathcal{M}}$ on the compact $[0,1] \times [\alpha(0), \beta(C_7)]$, it follows $\frac{1}{\mathcal{M}}$ is bounded,

$$\exists\, C_8 > 0, \forall U \in A : \quad 0 < \frac{1}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)} \leq C_8. \tag{3.29}$$

Also, $\mathcal{G}$ is continuous on the compact $[0,1] \times [0, C_7]$, which implies $\mathcal{G}$ is bounded,

$$\exists\, C_9 > 0, \forall u \in A : \quad 0 \leq \mathcal{G}(s, U(s)) \leq C_9. \tag{3.30}$$

According to (3.29) and (3.30), we find

$$s \mapsto \frac{\mathcal{G}(s, U(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)}$$

is bounded,

$$\exists\, C_{10} > 0, \forall s \in [0,1] : \quad \frac{g(s, u(s))}{M\Big(s, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)} \leq C_8 C_9 = C_{10}. \tag{3.31}$$

Combining (3.27), (3.28) and (3.31), for all $t_1, t_2 \in [0,1]$, we obtain

$$\begin{aligned}\Big|Su\left(t_1\right) - Su\left(t_2\right)\Big| &\leq \int_0^1 \Big|\mathbf{G}(t_1, s) - \mathbf{G}\left(t_2, s\right)\Big| \frac{g(s, u(s))}{M\Big(s, \displaystyle\int_0^1 f(u(\tau))d\tau\Big)} ds \\ &\leq \varepsilon C_{10} \int_0^1 ds = \varepsilon C_{10} = \varepsilon',\end{aligned}$$

which means

$$\forall \varepsilon' > 0, \exists\, \delta > 0, \forall s \in [0,1] : \quad |t_1 - t_2| \leq \delta \Rightarrow |TU(t_1) - TU(t_2)| \leq \varepsilon'.$$

Then, $B(t)$ is equicontinuous. On the other hand, we claim that $B(t)$ is bounded. We have

$$\begin{aligned}\|TU\|_\infty &\leq \sup_{t\in[0,1]} \int_0^1 \mathbf{G}\,(t,s)\, q_N(s) \frac{\mathcal{G}(s,U(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)} ds \\ &= \int_0^1 q_N(s) \frac{\mathcal{G}(s,U(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)} ds.\end{aligned}$$

From (3.22) and (3.25), we have

$$\begin{aligned}\|TU\|_\infty &\leq q_N(s) \int_0^1 \frac{\mathcal{G}(s,U(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)} ds \\ &\leq C_6 C_{10} \int_0^1 ds \\ &= C_6 C_{10} = C_{11},\end{aligned}$$

which implies

$$\exists\, C_{11} > 0, \forall U \in A : \quad \|TU\|_\infty \leq C_{11}.$$

So, $B(t)$ is uniformly bounded. Therefore, $B$ is relatively compact. According to the Arzela-Ascoli Theorem, $T$ is compact.

We prove that the set

$$P = \{U \in F, \min_{t\in[d_1,d_2]} U(t) \geq d_0 \|U\|_\infty\}$$

is a cone on $E$.

- $P$ is a nonempty subset on $E$. We have $P \subset E$ and since $0 \in P$, the set $P$ is non-empty.
- $P$ is convex. For $\alpha \in [0,1]$ and $\zeta, \psi \in E$, we have

$$\zeta, \psi \geq 0, \min_{t\in[d_1,d_2]} \zeta(t) \geq d_0 \|\zeta\|_\infty, \min_{t\in[d_1,d_2]} \psi(t) \geq d_0 \|\psi\|_\infty.$$

Since $(1-\alpha)\zeta + \alpha\psi \geq 0$, then

$$\begin{aligned}d_0 \|(1-\alpha)\zeta + \alpha\psi\|_\infty &\leq d_0 \left\{\|(1-\alpha)\zeta\|_\infty + \|\alpha\psi\|_\infty\right\} \\ &= d_0 \left\{(1-\alpha)\|\zeta\|_\infty + \alpha\|\psi\|_\infty\right\}.\end{aligned}$$

From $d_0(1-\alpha) \leq 1-\alpha$ and $d_0\alpha \leq \alpha$, we get

$$\begin{aligned} d_0\|(1-\alpha)\zeta + \alpha\psi\|_\infty &\leq d_0 \left\{(1-\alpha)\|\zeta\|_\infty + \alpha\|\psi\|_\infty\right\} \\ &\leq (1-\alpha)\|\zeta\|_\infty + \alpha\|\psi\|_\infty \\ &\leq \min_{t\in[d_1,d_2]}((1-\alpha)\zeta)(t) + \min_{t\in[d_1,d_2]}(\alpha\psi)(t) \\ &\leq \min_{t\in[d_1,d_2]}((1-\alpha)\zeta + \alpha\psi)(t). \end{aligned}$$

Hence, $P$ is convex.

- $P$ is a closed. Let $(U_j)_{j\in\mathbb{N}} \subset P$ be a sequence, has a limit $U \in E$. Therefore,

$$\forall j \in \mathbb{N}: \quad U_j \geq 0, \tag{3.32}$$

and

$$\forall j \in \mathbb{N}: \quad d_0 \left\|U_j\right\|_\infty \leq \min_{t\in[d_1,d_2]} U_j(t). \tag{3.33}$$

Recalling to

$$\min_{t\in[d_1,d_2]} U_j(t) \leq U_j(s). \tag{3.34}$$

According (3.33) and (3.34), we have

$$\forall j \in \mathbb{N}, s \in [d_1,d_2]: \quad d_0 \left\|U_j\right\|_\infty \leq U_j(s). \tag{3.35}$$

With

$$\lim_{j\to+\infty} \|U_j - U\|_\infty = 0,$$

which implies

$$\lim_{j\to+\infty} U_j(t) = U(t). \tag{3.36}$$

As $j \to \infty$ in (2.35) and (3.35), we find

$$U \geq 0, \tag{3.37}$$

and

$$d_0\|U\|_\infty \leq U. \tag{3.38}$$

Combining (3.37) and (3.38), it follows $U \in P$. We deduce $P$ is a closed on $E$.

- For $\lambda \geq 0$ is a real number and $U \in P$, we have $\lambda U \geq 0$ and

$$\begin{aligned} d_0\|\lambda U\|_\infty &= d_0\lambda\|U\|_\infty \\ &\leq d_0\lambda \min_{t\in[d_1,d_2]} U(t). \end{aligned}$$

Since $d_0\lambda \leq \lambda$, we find

$$\begin{aligned} d_0\|\lambda U\|_\infty &\leq d_0\lambda \min_{t\in[d_1,d_2]} U(t) \\ &\leq \lambda \min_{t\in[d_1,d_2]} U(t) \\ &= \min_{t\in[d_1,d_2]} (\lambda U)(t). \end{aligned}$$

Then, $\lambda U \in P$.

- It is clear that if $U \in P, -U \in P$, also $U \geq 0, -U \leq 0$. We obtain $U = 0$.

We claim $T(P) \subset P$. Indeed, denoting

$$\phi : [0,1] \to \mathbb{R}, s \mapsto \frac{q_N(s)\mathcal{G}(s,U(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)},$$

it follows that

$$\begin{aligned} \min_{t\in[d_1,d_2]} TU(t) &= \min_{t\in[d_1,d_2]} \int_0^1 \mathbf{G}(t,s)\frac{q_N(s)\mathcal{G}(s,U(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)}ds \\ &= \min_{t\in[d_1,d_2]} \int_0^1 \mathbf{G}(t,s)\phi(s)ds. \end{aligned}$$

Using (2.7), we have

$$\min_{t\in[d_1,d_2]} TU(t) = \min_{t\in[d_1,d_2]} \{\int_0^t (1-t)s\phi(s)ds + \int_t^1 (1-s)t\phi(s)ds.$$

For $t \in [d_1, d_2]$, gives $1 - t \geq 1 - d_2$, then

$$\begin{aligned} \min_{t\in[d_1,d_2]} TU(t) &= \min_{t\in[d_1,d_2]} \{\int_0^t (1-t)s\phi(s)ds + \int_t^1 (1-s)t\phi(s)ds\} \\ &\geq \min_{t\in[d_1,d_2]} \{(1-d_2)\int_0^t s\phi(s)ds + d_1\int_t^1 (1-s)\nu(s)ds\} \\ &\geq d_0 \min_{t\in[d_1,d_2]} \{\int_0^t (1-s)s\phi(s)ds + \int_t^1 (1-s)s\phi(s)ds\} \\ &= d_0\int_0^1 (1-s)s\phi(s)ds. \end{aligned} \tag{3.39}$$

In addition,

$$\begin{aligned} \max_{t\in[0,1]} TU(t) &= \max_{t\in[0,1]} \int_0^1 \mathbf{G}(t,s)\phi(s)ds \\ &= \max_{t\in[0,1]} \{\int_0^t (1-t)s\phi(s)ds + \int_t^1 (1-s)t\phi(s)ds\}. \end{aligned}$$

Since $0 \leq s \leq t \leq 1$, then $1-s \geq 1-t$. So

$$\begin{aligned}\int_0^1 (1-s)s\phi(s)ds &\geq \int_0^t (1-s)s\phi(s)ds\\ &\geq \int_0^t (1-t)s\phi(s)ds\\ &\geq \int_0^1 (1-t)s\phi(s)ds.\end{aligned}$$

It follows that

$$\max_{t\in[0,1]} \int_0^1 (1-s)s\phi(s)ds = \min_{t\in[0,1]} \int_0^1 (1-s)s\phi(s)ds.$$

This leads to

$$\begin{aligned}\max_{t\in[0,1]} TU(t) &= \max_{t\in[0,1]}\{\int_0^t (1-t)s\phi(s)ds + \int_t^1 (1-s)t\phi(s)ds\}\\ &\leq \min_{t\in[0,1]}\{\int_0^t (1-s)s\phi(s)ds + \int_t^1 (1-s)\phi(s)ds\}\\ &\leq \min_{t\in[0,1]}\{\int_0^t (1-s)s\phi(s)ds + \int_t^1 (1-s)s\phi(s)ds\}\\ &= \int_0^1 (1-s)s\phi(s)ds.\end{aligned}\tag{3.40}$$

From (3.39), (3.40) we get

$$\min_{t\in[d_1,d_2]} TU(t) \geq d_0 \max_{t\in[0,1]} TU(t).$$

Therefore, $T(P) \subset P$.

On the other hand, if $N=2$, for all $U \in P$, we have

$$\begin{aligned}TU(\frac{1}{2}) &= \int_0^1 \mathbf{G}\Big(\frac{1}{2},s\Big)\frac{q_2(s)\mathcal{G}(s,U(s))}{\mathcal{M}\Big(s,\displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)}ds\\ &= \int_0^1 \mathbf{G}\Big(\frac{1}{2},s\Big)\frac{q_2(s)g\Big(b(\frac{a}{b})^s,U(s)\Big)}{M\Big(b(\frac{a}{b})^s,\omega b^2\log(\frac{b}{a})\displaystyle\int_0^1(\frac{a}{b})^{2\tau}f(U(\tau))d\tau\Big)}ds.\end{aligned}\tag{3.41}$$

Since the function

$$s \mapsto \mathbf{G}\Big(\frac{1}{2},s\Big)\frac{q_2(s)g\Big(b(\frac{a}{b})^s,U(s)\Big)}{M\Big(b(\frac{a}{b})^s,\omega b^2\log(\frac{b}{a})\int_0^1(\frac{a}{b})^{2\tau}f(U(\tau))d\tau\Big)},$$

is non-negative on $[0,1]$, $[d_1, d_2] \subset [0,1]$, and (3.41), we get

$$\begin{aligned} TU(\frac{1}{2}) &= \int_0^1 \mathbf{G}\Big(\frac{1}{2}, s\Big) \frac{q_2(s) g\Big(b(\frac{a}{b})^s, U(s)\Big)}{M\Big(b(\frac{a}{b})^s, \omega b^2 \log(\frac{b}{a}) \displaystyle\int_0^1 (\frac{a}{b})^{2\tau} f(U(\tau)) d\tau\Big)} ds \\ &\geq \int_{d_1}^{d_2} \mathbf{G}\Big(\frac{1}{2}, s\Big) \frac{q_2(s) g\Big(b(\frac{a}{b})^s, U(s)\Big)}{M\Big(b(\frac{a}{b})^s, \omega b^2 \log(\frac{b}{a}) \displaystyle\int_0^1 (\frac{a}{b})^{2\tau} f(U(\tau)) d\tau\Big)} ds. \end{aligned} \tag{3.42}$$

From (3.3),

$$\xi\Big(b(\frac{a}{b})^s\Big) h(U(s) \leq g\Big(b(\frac{a}{b})^s, U(s)\Big), \quad \text{a.e. } s \in [0,1]. \tag{3.43}$$

Using (3.4), we obtain

$$\begin{aligned} & M\Big(b(\frac{a}{b})^s, \omega b^2 \log(\frac{b}{a}) \int_0^1 (\frac{a}{b})^{2\tau} f(U(\tau)) d\tau\Big) \\ & \qquad \leq \mu\Big(b(\frac{a}{b})^s\Big) w\Big(\omega b^2 \log(\frac{b}{a}) \int_0^1 (\frac{a}{b})^{2\tau} f(U(\tau)) d\tau\Big), \quad \text{a.e. } s \in [0,1]. \end{aligned} \tag{3.44}$$

According to (3.42) - (3.44), we find

$$\begin{aligned} TU(\frac{1}{2}) &\geq \int_{d_1}^{d_2} \mathbf{G}\Big(\frac{1}{2}, s\Big) \frac{q_2(s) g\Big(b(\frac{a}{b})^s, U(s)\Big)}{M\Big(b(\frac{a}{b})^s, \omega b^2 \log(\frac{b}{a}) \displaystyle\int_{d_1}^{d_2} (\frac{a}{b})^{2\tau} f(U(\tau)) d\tau\Big)} ds \\ &\geq \frac{1}{w\Big(\omega b^2 \log(\frac{b}{a}) \displaystyle\int_0^1 (\frac{a}{b})^{2\tau} f(U(\tau)) d\tau\Big)} \int_{d_1}^{d_2} \mathbf{G}\Big(\frac{1}{2}, s\Big) q_2(s) \frac{\xi\Big(b(\frac{a}{b})^s\Big) h(U(s))}{\mu\Big(b(\frac{a}{b})^s\Big)} ds. \end{aligned} \tag{3.45}$$

From (3.38) and the fact that $h$ is non-decreasing function, we derive

$$h(d_0 \|U\|_\infty) \leq h(U). \tag{3.46}$$

In view of (3.2), it follows

$$f(U) \leq \beta(U). \tag{3.47}$$

Recalling that

$$U \leq \|U\|_\infty. \tag{3.48}$$

Since $\beta$ is increasing function, we obtain from (3.48),

$$\beta(U) \leq \beta(\|U\|_\infty). \tag{3.49}$$

From (3.47), (3.49), gives

$$f(U) \leq \beta(\|U\|_\infty). \tag{3.50}$$

Integrating over $[0,1]$ the above inequation (3.50), and using $(\frac{a}{b})^{2\tau} \leq 1$, we obtain

$$\begin{aligned}\int_0^1 (\frac{a}{b})^{2\tau} f(U(\tau))d\tau &\leq \int_0^1 (\frac{a}{b})^{2\tau} \beta(\|U\|_\infty)d\tau \\ &= \beta(\|U\|_\infty) \int_0^1 d\tau \\ &= \beta(\|U\|_\infty).\end{aligned}$$

This gives

$$w\Big(\omega b^2 \log(\frac{b}{a}) \int_0^1 (\frac{a}{b})^{2\tau} f(U(\tau))d\tau\Big) \leq w\Big(\omega b^2 \log(\frac{b}{a})\beta(\|U\|_\infty)\Big). \tag{3.51}$$

Furthermore, according to (3.45), (3.46) and (3.51), we get

$$\begin{aligned} TU(\frac{1}{2}) &\geq \frac{1}{w\Big(\omega b^2 \log(\frac{b}{a}) \int_0^1 (\frac{a}{b})^{2\tau} f(U(\tau))d\tau\Big)} \int_{d_1}^{d_2} \mathbf{G}\Big(\frac{1}{2}, s\Big) q_2(s) \frac{\xi\Big(b(\frac{a}{b})^s\Big) h(U(s))}{\mu\Big(b(\frac{a}{b})^s\Big)} ds \\ &\geq \frac{h\left(d_0\|U\|_\infty\right)}{w\Big(\omega b^2 \log(\frac{b}{a})\beta(\|U\|_\infty)\Big)} \int_{d_1}^{d_2} \mathbf{G}\Big(\frac{1}{2}, s\Big) q_2(s) \frac{\xi\Big(b(\frac{a}{b})^s\Big)}{\mu\Big(b(\frac{a}{b})^s\Big)} ds. \end{aligned} \tag{3.52}$$

From the condition (3.5), we have

$$\begin{aligned} \forall s \in ]0, R_1] : \quad \frac{h\left(d_0 s\right)}{w\Big(\omega b^2 \log(\frac{b}{a})\beta(s)\Big)} &\geq \inf_{0<s\leq R_1} \frac{h\left(d_0 s\right)}{w\Big(\omega b^2 \log(\frac{b}{a})\beta(s)\Big)} \\ &\geq \frac{s}{\displaystyle\int_{d_1}^{d_2} \mathbf{G}\Big(\frac{1}{2}, s\Big) q_2(s) \frac{\xi\Big(b(\frac{a}{b})^s\Big)}{\mu\Big(b(\frac{a}{b})^s\Big)} ds}. \end{aligned}$$

This implies

$$\forall s \in [0, R_1] : \quad \frac{h\left(d_0 s\right)}{w\Big(\omega b^2 \log(\frac{b}{a})\beta(s)\Big)} \geq \frac{s}{\displaystyle\int_{d_1}^{d_2} \mathbf{G}\Big(\frac{1}{2}, s\Big) q_2(s) \frac{\xi\Big(b(\frac{a}{b})^s\Big)}{\mu\Big(b(\frac{a}{b})^s\Big)} ds}.$$

Since $u \in E$ such that $\|U\|_\infty \leq R_1$, we get

$$\frac{h\left(d_0\|U\|_\infty\right)}{w\Big(\omega b^2 \log(\frac{b}{a})\beta(\|U\|_\infty)\Big)} \geq \frac{\|U\|_\infty}{\displaystyle\int_{d_1}^{d_2} \mathbf{G}\Big(\frac{1}{2}, s\Big) q_2(s) \frac{\xi\Big(b(\frac{a}{b})^s\Big)}{\mu\Big(b(\frac{a}{b})^s\Big)} ds}. \tag{3.53}$$

Using (3.52) and (3.53), we obtain

$$
\begin{aligned}
TU(\frac{1}{2}) &\geq \frac{h\left(d_0\|U\|_\infty\right)}{w\Big(\omega b^2\log(\frac{b}{a})\beta(\|U\|_\infty)\Big)}\int_{d_1}^{d_2}\mathbf{G}\Big(\frac{1}{2},s\Big)q_2(s)\frac{\xi\Big(b(\frac{a}{b})^s\Big)}{\mu\Big(b(\frac{a}{b})^s\Big)}ds\\
&\geq \frac{\|U\|_\infty}{\displaystyle\int_{d_1}^{d_2}\mathbf{G}\Big(\frac{1}{2},s\Big)q_2(s)\frac{\xi\Big(b(\frac{a}{b})^s\Big)}{\mu\Big(b(\frac{a}{b})^s\Big)}ds}\int_{d_1}^{d_2}\mathbf{G}\Big(\frac{1}{2},s\Big)q_2(s)\frac{\xi\Big(b(\frac{a}{b})^s\Big)}{\mu\Big(b(\frac{a}{b})^s\Big)}ds\\
&= \|U\|_\infty.
\end{aligned}
$$

If we set

$$\Omega_1=\{U\in E;\ \|U\|_\infty\leq R_1\}, \tag{3.54}$$

we find the following inequality

$$\forall U\in P\cap\partial\Omega_1:\quad \|TU\|_\infty\geq\|U\|_\infty. \tag{3.55}$$

On the other hand, in view of the condition (3.2), for all $U\in P$, we have

$$\alpha(U)\leq f(U). \tag{3.56}$$

Using the fact that $\alpha$ is increasing function and (3.38), we derive

$$\alpha(d_0\|U\|_\infty)\leq\alpha(U). \tag{3.57}$$

According to (3.56) and (3.57), we find

$$\alpha(d_0\|U\|_\infty)\leq f(U). \tag{3.58}$$

Integrating over $[d_1,d_2]$ the inequation (3.58), and using $(\frac{a}{b})^{2\tau}\geq(\frac{a}{b})^2$, we obtain

$$
\begin{aligned}
\int_{d_1}^{d_2}(\frac{a}{b})^{2\tau}f(U(\tau))ds &\geq \int_{d_1}^{d_2}(\frac{a}{b})^{2\tau}\alpha(d_0\|U\|_\infty)ds\\
&= \alpha(d_0\|U\|_\infty)(\frac{a}{b})^{2\tau}\int_{d_1}^{d_2}ds\\
&= (d_2-d_1)(\frac{a}{b})^{2\tau}\alpha(d_0\|U\|_\infty)\\
&\geq (d_2-d_1)(\frac{a}{b})^{2}\alpha(d_0\|U\|_\infty).
\end{aligned}
\tag{3.59}
$$

By (3.4), (3.59), we get

$$\begin{aligned}\theta\Big(b(\frac{a}{b})^s\Big)v\Big(\omega b^2\log(\frac{b}{a})\int_0^1(\frac{a}{b})^{2\tau}f(U(\tau))d\tau\Big)\\ \leq M\Big(b(\frac{a}{b})^s,\omega b^2\log(\frac{b}{a})\int_0^1(\frac{a}{b})^{2\tau}f(U(\tau))d\tau\Big),\quad \text{a.e. } s\in[0,1].\end{aligned}\tag{3.60}$$

Since the following map

$$\tau\mapsto\int_0^1(\frac{a}{b})^{2\tau}f(U(\tau))d\tau,$$

is nonnegative function, and $[d_1,d_2]\subset[0,1]$, it follows that

$$\int_0^1(\frac{a}{b})^{2\tau}f(U(\tau))d\tau\geq\int_{d_1}^{d_2}(\frac{a}{b})^{2\tau}f(U(\tau))d\tau.$$

Also,

$$\begin{aligned}v\Big(\omega b^2\log(\frac{b}{a})\int_0^1(\frac{a}{b})^{2\tau}f(U(\tau))d\tau\Big)&\geq v\Big(\omega b^2\log(\frac{b}{a})\int_{d_1}^{d_2}(\frac{a}{b})^{2\tau}f(U(\tau))d\tau\Big)\\ &\geq v\Big(\omega a^2\log(\frac{b}{a})(d_2-d_1)\alpha(d_0\|U\|_\infty)\Big).\end{aligned}\tag{3.61}$$

From (3.60) and (3.61), we have

$$\begin{aligned}M\Big(b(\frac{a}{b})^s,\omega b^2\log(\frac{b}{a})\int_0^1(\frac{a}{b})^{2\tau}f(U(\tau))d\tau\Big)&\geq\theta\Big(b(\frac{a}{b})^s\Big)v\Big(\omega b^2\log(\frac{b}{a})\int_0^1(\frac{a}{b})^{2\tau}f(U(\tau))d\tau\Big)\\ &\geq\theta\Big(b(\frac{a}{b})^s\Big)v\Big(\omega a^2\log(\frac{b}{a})(d_2-d_1)\alpha(d_0\|U\|_\infty)\Big).\end{aligned}\tag{3.62}$$

By (3.3), it yields

$$g\Big(b(\frac{a}{b})^s,U(s)\Big)\leq\eta\Big(b(\frac{a}{b})^s\Big)k(U(s)),\quad \text{a.e. } s\in[0,1].\tag{3.63}$$

Using (3.48) and the fact that $K$ be a increasing function, then

$$k(U)\leq K(\|u\|_\infty).\tag{3.64}$$

From (3.63) and (3.64), we find

$$g\Big(b(\frac{a}{b})^s,U(s)\Big)\leq\eta\Big(b(\frac{a}{b})^s\Big)K(\|u\|_\infty),\quad \text{a.e. } s\in[0,1].\tag{3.65}$$

According to (3.62)- (3.65), we get

$$\begin{aligned}\forall t\in[0,1]:\quad TU(t) &= \int_0^1 \mathbf{G}(t,s)q_2(s)\frac{\mathcal{G}(s,U(s))}{\mathcal{M}\Big(s,\displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)}ds\\ &= \int_0^1 \mathbf{G}(t,s)\frac{q_2(s)g(b(\frac{a}{b})^s,U(s))}{M\Big(b(\frac{a}{b})^s,\omega b^2\log(\frac{b}{a})\displaystyle\int_0^1 (\frac{a}{b})^{2\tau}f(U(\tau))d\tau\Big)}ds\\ &\le \frac{K(\|u\|_\infty)}{v\Big(\omega a^2\log(\frac{b}{a})(d_2-d_1)\alpha(d_0\|U\|_\infty)\Big)}\int_0^1 \mathbf{G}(t,s)q_2(s)\frac{\eta\Big(b(\frac{a}{b})^s\Big)}{\theta\Big(b(\frac{a}{b})^s\Big)}ds\\ &\le \frac{K(\|u\|_\infty)}{v\Big(\omega a^2\log(\frac{b}{a})(d_2-d_1)\alpha(d_0\|U\|_\infty)\Big)}\int_0^1 q_2(s)\frac{\eta\Big(b(\frac{a}{b})^s\Big)}{\theta\Big(b(\frac{a}{b})^s\Big)}ds.\end{aligned}\tag{3.66}$$

From the condition (3.6), we have

$$\begin{aligned}\exists\ R_2>R_1:\quad \frac{K(s)}{v\Big(\omega a^2\log(\frac{b}{a})(d_2-d_1)\alpha(d_0 s)\Big)} &\le \sup_{s\ge R_2}\frac{K(s)}{v\Big(\omega a^2\log(\frac{b}{a})(d_2-d_1)\alpha(d_0 s)\Big)}\\ &\le \frac{s}{\displaystyle\int_0^1 q_2(s)\frac{\eta\Big(b(\frac{a}{b})^s\Big)}{\theta\Big(b(\frac{a}{b})^s\Big)}ds}.\end{aligned}$$

For $U\in E, \|U\|_\infty\ge R_2$, we obtain

$$\frac{K(\|u\|_\infty)}{v\Big(\omega a^2\log(\frac{b}{a})(d_2-d_1)\alpha(d_0\|U\|_\infty)\Big)}\le\frac{\|U\|_\infty}{\displaystyle\int_0^1 q_2(s)\frac{\eta(b(\frac{a}{b})^s)}{\theta(b(\frac{a}{b})^s)}ds}.\tag{3.67}$$

Combining (3.66) and (3.67), we get

$$\begin{aligned}TU(t) \le&\frac{K(\|u\|_\infty)}{v\Big(\omega a^2\log(\frac{b}{a})(d_2-d_1)\alpha(d_0\|U\|_\infty)\Big)}\int_0^1 q_2(s)\frac{\eta\Big(b(\frac{a}{b})^s\Big)}{\theta\Big(b(\frac{a}{b})^s\Big)}ds\\ \le&\frac{\|U\|_\infty}{\displaystyle\int_0^1 q_2(s)\frac{\eta(b(\frac{a}{b})^s)}{\theta(b(\frac{a}{b})^s)}ds}\int_0^1 q_2(s)\frac{\eta\Big(b(\frac{a}{b})^s\Big)}{\theta\Big(b(\frac{a}{b})^s\Big)}ds\\ =&\|U\|_\infty.\end{aligned}$$

Setting

$$\Omega_2=\{U\in E;\ \|U\|_\infty\le R_2\},\tag{3.68}$$

it follows that

$$\forall U \in P \cap \partial\Omega_2: \quad \|TU\|_\infty \le \|U\|_\infty. \tag{3.69}$$

In addition, if $N \ge 3$, for all $U \in P$, we have

$$\begin{aligned} TU(\frac{1}{2}) &= \int_0^1 \mathbf{G}\Big(\frac{1}{2}, s\Big) q_N(s) \frac{\mathcal{G}(s, U(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)} ds \\ &= \int_0^1 \mathbf{G}\Big(\frac{1}{2}, s\Big) q_N(s) \frac{g\Big((\frac{A}{B-s})^{\frac{1}{N-2}}, U(s)\Big)}{M\Big((\frac{A}{B-s})^{\frac{1}{N-2}}, \frac{\omega}{A(N-2)} \displaystyle\int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}} f(U(\tau))d\tau\Big)} ds. \end{aligned} \tag{3.70}$$

Since the map

$$s \mapsto \mathbf{G}\Big(\frac{1}{2}, s\Big) q_N(s) \frac{g\Big((\frac{A}{B-s})^{\frac{1}{N-2}}, U(s)\Big)}{M\Big((\frac{A}{B-s})^{\frac{1}{N-2}}, \frac{\omega}{A(N-2)} \displaystyle\int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}} f(U(\tau))d\tau\Big)}$$

is non-negative on $[0, 1]$, $[d_1, d_2] \subset [0, 1]$, we obtain from (3.70),

$$\begin{aligned} TU(\frac{1}{2}) &= \int_0^1 \mathbf{G}\Big(\frac{1}{2}, s\Big) q_N(s) \frac{g\Big((\frac{A}{B-s})^{\frac{1}{N-2}}, U(s)\Big)}{M\Big((\frac{A}{B-s})^{\frac{1}{N-2}}, \frac{\omega}{A(N-2)} \displaystyle\int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}} f(U(\tau))d\tau\Big)} ds \\ &\ge \int_{d_1}^{d_2} \mathbf{G}\Big(\frac{1}{2}, s\Big) q_N(s) \frac{g\Big((\frac{A}{B-s})^{\frac{1}{N-2}}, U(s)\Big)}{M\Big((\frac{A}{B-s})^{\frac{1}{N-2}}, \frac{\omega}{A(N-2)} \displaystyle\int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}} f(U(\tau))d\tau\Big)} ds. \end{aligned} \tag{3.71}$$

Therefore, by (3.3), we get

$$\xi\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big) h(U(s)) \le g\Big((\frac{A}{B-s})^{\frac{1}{N-2}}, U(s)\Big), \quad \text{a.e. } s \in [0, 1]. \tag{3.72}$$

Using (3.4), we deduce

$$\begin{aligned} &M\Big((\frac{A}{B-s})^{\frac{1}{N-2}}, \frac{\omega}{A(N-2)} \int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}} f(U(\tau))d\tau\Big) \\ &\le \mu\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big) w\Big(\frac{\omega}{A(N-2)} \int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}} f(U(\tau))d\tau\Big), \quad \text{a.e. } s \in [0, 1]. \end{aligned} \tag{3.73}$$

According to (3.71)-(3.73), it follows

$$\begin{aligned} TU(\frac{1}{2}) &\geq \int_{d_1}^{d_2} \frac{q_N(s)g\Big((\frac{A}{B-s})^{\frac{1}{N-2}}, U(s)\Big)}{M\Big((\frac{A}{B-s})^{\frac{1}{N-2}}, \frac{\omega}{A(N-2)}\int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}} f(U(\tau))d\tau\Big)} ds \\ &\geq \frac{1}{w\Big(\frac{\omega}{A(N-2)}\int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}} f(U(\tau))d\tau\Big)} \int_{d_1}^{d_2} G\Big(\frac{1}{2}, s\Big) q_N(s) \frac{\xi\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)h(U(s))}{\mu\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)} ds. \end{aligned} \tag{3.74}$$

For all $\tau \in [0,1]$, we have

$$(\frac{A}{B-\tau})^{2\frac{N-1}{N-2}} \leq (\frac{A}{B-1})^{2\frac{N-1}{N-2}}. \tag{3.75}$$

From (3.50) and (3.75), it follows that

$$\begin{aligned} \int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}} f(U(\tau))d\tau &\leq \int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}} \beta(\|U\|_\infty)d\tau \\ &\leq (\frac{A}{B-1})^{2\frac{N-1}{N-2}} \beta(\|U\|_\infty)\int_0^1 ds \\ &= (\frac{A}{B-1})^{2\frac{N-1}{N-2}} \beta(\|U\|_\infty). \end{aligned}$$

Hence,

$$w\Big(\frac{\omega}{A(N-2)}\int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}} f(U(\tau))d\tau\Big) \leq w\Big(\frac{\omega}{A(N-2)}(\frac{A}{B-1})^{2\frac{N-1}{N-2}}\beta(\|U\|_\infty)\Big). \tag{3.76}$$

According (3.46), (3.74) and (3.76), we get

$$\begin{aligned} &TU(\frac{1}{2}) \\ &\geq \frac{1}{w\Big(\frac{\omega}{A(N-2)}\int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}} f(U(\tau))d\tau\Big)} \int_{d_1}^{d_2} \mathbf{G}\Big(\frac{1}{2}, s\Big) q_N(s) \frac{\xi\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)h(U(s))}{\mu\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)} ds \\ &\geq \frac{h\left(d_0\|U\|_\infty\right)}{w\Big(\frac{\omega}{A(N-2)}(\frac{A}{B-1})^{2\frac{N-1}{N-2}}\beta(\|U\|_\infty)\Big)} \int_{d_1}^{d_2} \mathbf{G}\Big(\frac{1}{2}, s\Big) q_N(s) \frac{\xi\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)}{\mu\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)} ds. \end{aligned} \tag{3.77}$$

From the condition (3.5), we have

$$\begin{aligned}\forall s \in ]0, R_1] : \quad \frac{h\left(d_0 s\right)}{w\Big(\frac{\omega}{A(N-2)}(\frac{A}{B-1})^{2\frac{N-1}{N-2}}\beta(s)\Big)} &\geq \inf_{0<s\leq 1} \frac{h\left(d_0 s\right)}{w\Big(\frac{\omega}{A(N-2)}(\frac{A}{B-1})^{2\frac{N-1}{N-2}}\beta(s)\Big)} \\ &\geq \frac{s}{\displaystyle\int_{d_1}^{d_2} \mathbf{G}\Big(\frac{1}{2}, s\Big) q_N(s) \frac{\xi\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)}{\mu\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)} ds}.\end{aligned}$$

This leads to

$$\forall s \in [0, R_1] : \quad \frac{h\left(d_0 s\right)}{w\Big(\frac{\omega}{A(N-2)}(\frac{A}{B-1})^{2\frac{N-1}{N-2}}\beta(s)\Big)} \geq \frac{s}{\displaystyle\int_{d_1}^{d_2} \mathbf{G}\Big(\frac{1}{2}, s\Big) q_N(s) \frac{\xi\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)}{\mu\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)} ds}.$$

Let $u \in E$ such that $\|U\|_\infty \leq R_1$. Then,

$$\frac{h\left(d_0 \|U\|_\infty\right)}{w\Big(\frac{\omega}{A(N-2)}(\frac{A}{B-1})^{2\frac{N-1}{N-2}}\beta(\|U\|_\infty)\Big)} \geq \frac{\|u\|_\infty}{\displaystyle\int_{d_1}^{d_2} \mathbf{G}\Big(\frac{1}{2}, s\Big) q_N(s) \frac{\xi\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)}{\mu\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)} ds}. \tag{3.78}$$

Combining (3.77) and (3.78), we obtain

$$\begin{aligned}TU(\frac{1}{2}) &\geq \frac{h\left(d_0 \|U\|_\infty\right)}{w\Big(\frac{\omega}{A(N-2)}(\frac{A}{B-1})^{2\frac{N-1}{N-2}}\beta(\|U\|_\infty\Big)} \int_{d_1}^{d_2} \mathbf{G}\Big(\frac{1}{2}, s\Big) q_N(s) \frac{\xi\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)}{\mu\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)} ds \\ &\geq \frac{\|u\|_\infty}{\displaystyle\int_{d_1}^{d_2} \mathbf{G}\Big(\frac{1}{2}, s\Big) q_N(s) \frac{\xi\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)}{\mu\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)} ds} \int_{d_1}^{d_2} \mathbf{G}\Big(\frac{1}{2}, s\Big) q_N(s) \frac{\xi\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)}{\mu\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)} ds \\ &= \|u\|_\infty.\end{aligned}$$

For

$$\Omega_1 = \{U \in E;\ \|U\|_\infty \leq R_1\}, \tag{3.79}$$

we get

$$\forall U \in P \cap \partial\Omega_1 : \quad \|TU\|_\infty \geq \|U\|_\infty. \tag{3.80}$$

In addition, using (3.4), we find

$$\begin{aligned}&\theta\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big) v\Big(\frac{\omega}{A(N-2)} \int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}} f(U(\tau)) d\tau\Big) \\ &\leq M\Big((\frac{A}{B-s})^{\frac{1}{N-2}}, \frac{\omega}{A(N-2)} \int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}} f(U(\tau)) d\tau\Big), \quad \text{a.e. } s \in [0, 1].\end{aligned} \tag{3.81}$$

For all $\tau \in [0,1]$, we have

$$(\frac{A}{B-\tau})^{2\frac{N-1}{N-2}} \geq (\frac{A}{B})^{2\frac{N-1}{N-2}}. \tag{3.82}$$

The use of (3.58) and (3.82), gives

$$\begin{aligned}\int_{d_1}^{d_2}(\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}f(U(\tau))d\tau &\geq \int_{d_1}^{d_2}(\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}\alpha(\|U\|_\infty)d\tau\\ &\geq (\frac{A}{B})^{2\frac{N-1}{N-2}}\alpha(\|U\|_\infty)\int_{d_1}^{d_2}d\tau\\ &\geq (d_2-d_1)(\frac{A}{B})^{2\frac{N-1}{N-2}}\alpha(d_0\|U\|_\infty).\end{aligned} \tag{3.83}$$

The following map

$$\tau \mapsto \int_0^1(\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}f(U(\tau))d\tau,$$

is nonnegative function and $[d_1,d_2] \subset [0,1]$, then, from (3.83), it follows that

$$\begin{aligned}\int_0^1(\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}f(U(\tau))d\tau &\geq \int_{d_1}^{d_2}(\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}f(U(\tau))d\tau\\ &\geq (d_2-d_1)(\frac{A}{B})^{2\frac{N-1}{N-2}}\alpha(d_0\|U\|_\infty).\end{aligned} \tag{3.84}$$

Using (3.81) and (3.84), we obtain

$$\begin{aligned}&M\Big((\frac{A}{B-s})^{\frac{1}{N-2}}, \frac{\omega}{A(N-2)}\int_0^1(\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}f(U(\tau))d\tau\Big)\\ &\geq \theta\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)v\Big(\frac{\omega}{A(N-2)}\int_0^1(\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}f(U(\tau))d\tau\Big)\\ &\geq \theta\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)w\Big(\frac{\omega}{A(N-2)}(\frac{A}{B})^{2\frac{N-1}{N-2}}(d_2-d_1)\alpha(d_0\|U\|_\infty)\Big).\end{aligned} \tag{3.85}$$

Moreover, from (3.4) and (3.48), we have

$$\begin{aligned}g\Big((\frac{A}{B-s})^{\frac{1}{N-2}}, U(s)\Big) &\leq \eta\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)k(U(s))\\ &\leq \eta\Big((\frac{A}{B-s})^{\frac{1}{N-2}}\Big)k(\|u\|_\infty).\end{aligned} \tag{3.86}$$

According to (3.85) and (3.86), for all $t \in [0,1]$, we get

$$\begin{aligned} TU(t) &= \int_0^1 \mathbf{G}(t,s) q_N(s) \frac{\mathcal{G}(s,U(s))}{\mathcal{M}\left(s, \int_0^1 \mathcal{F}(U(\tau))d\tau\right)} ds \\ &= \int_0^1 \mathbf{G}(t,s) q_N(s) \frac{g\left(\left(\frac{A}{B-s}\right)^{\frac{1}{N-2}}, U(s)\right)}{M\left(\left(\frac{A}{B-s}\right)^{\frac{1}{N-2}}, \frac{\omega}{A(N-2)} \int_0^1 \left(\frac{A}{B-\tau}\right)^{2\frac{N-1}{N-2}} f(U(\tau))d\tau\right)} ds \\ &\leq \frac{K(\|u\|_\infty)}{v\left(\frac{\omega}{A(N-2)}\left(\frac{A}{B}\right)^{2\frac{N-1}{N-2}}(d_2-d_1)\alpha(d_0\|U\|_\infty)\right)} \int_0^1 \mathbf{G}(t,s) q_N(s) \frac{\eta\left(\left(\frac{A}{B-s}\right)^{\frac{1}{N-2}}\right)}{\theta\left(\left(\frac{A}{B-s}\right)^{\frac{1}{N-2}}\right)} ds \\ &\leq \frac{K(\|u\|_\infty)}{v\left(\frac{\omega}{A(N-2)}\left(\frac{A}{B}\right)^{2\frac{N-1}{N-2}}(d_2-d_1)\alpha(d_0\|U\|_\infty)\right)} \int_0^1 q_N(s) \frac{\eta\left(\left(\frac{A}{B-s}\right)^{\frac{1}{N-2}}\right)}{\theta\left(\left(\frac{A}{B-s}\right)^{\frac{1}{N-2}}\right)} ds. \end{aligned} \tag{3.87}$$

From the condition (3.6),

$$\begin{aligned} \exists R_2 \geq R_1 : \quad \frac{K(s)}{v\left(\frac{\omega}{A(N-2)}\left(\frac{A}{B}\right)^{2\frac{N-1}{N-2}}(d_2-d_1)\alpha(d_0 s)\right)} &\leq \sup_{s\in[0,1]} \frac{K(s)}{v\left(\frac{\omega}{A(N-2)}\left(\frac{A}{B}\right)^{2\frac{N-1}{N-2}}(d_2-d_1)\alpha(d_0 s)\right)}) \\ &\leq \frac{s}{\int_0^1 q_N(s) \frac{\eta\left(\left(\frac{A}{B-s}\right)^{\frac{1}{N-2}}\right)}{\theta\left(\left(\frac{A}{B-s}\right)^{\frac{1}{N-2}}\right)} ds}. \end{aligned}$$

For $U \in E, \|U\|_\infty \geq R_2$, we obtain

$$\frac{K(\|u\|_\infty)}{v\left(\frac{\omega}{A(N-2)}\left(\frac{A}{B}\right)^{2\frac{N-1}{N-2}}(d_2-d_1)\alpha(d_0\|u\|_\infty)\right)} \leq \frac{\|u\|_\infty}{\int_0^1 q_N(s) \frac{\eta\left(\left(\frac{A}{B-s}\right)^{\frac{1}{N-2}}\right)}{\theta\left(\left(\frac{A}{B-s}\right)^{\frac{1}{N-2}}\right)} ds}. \tag{3.88}$$

Combining (3.87) and (3.88), we have

$$\begin{aligned} TU(t) &\leq \frac{K(\|u\|_\infty)}{v\left(\frac{\omega}{A(N-2)}\left(\frac{A}{B}\right)^{2\frac{N-1}{N-2}}(d_2-d_1)\alpha(d_0\|U\|_\infty)\right)} \int_0^1 q_N(s) \frac{\eta\left(\left(\frac{A}{B-s}\right)^{\frac{1}{N-2}}\right)}{\theta\left(\left(\frac{A}{B-s}\right)^{\frac{1}{N-2}}\right)} ds \\ &\leq \frac{\|U\|_\infty}{\int_0^1 q_N(s) \frac{\eta\left(\left(\frac{A}{B-s}\right)^{\frac{1}{N-2}}\right)}{\theta\left(\left(\frac{A}{B-s}\right)^{\frac{1}{N-2}}\right)} ds} \int_0^1 q_N(s) \frac{\eta\left(\left(\frac{A}{B-s}\right)^{\frac{1}{N-2}}\right)}{\theta\left(\left(\frac{A}{B-s}\right)^{\frac{1}{N-2}}\right)} ds. \\ &= \|u\|_\infty. \end{aligned}$$

Hereafter, if assume that

$$\Omega_2 = \{U \in E;\ \|U\|_\infty \leq R_2\}, \tag{3.89}$$

we deduce

$$\forall U \in P \cap \partial\Omega_2 : \quad \|TU\|_\infty \leq \|U\|_\infty. \tag{3.90}$$

Finally, $P$ is a cone in $E, 0 \in \Omega_1$, with $\overline{\Omega_1} \subset \Omega_2$, and $T(P) \subset P$. As well, $T$ is complety continuous. Combining (3.54), (3.55), (3.68), (3.69), (3.79), (3.80), (3.89) and (3.90). Using Krasnoselskii Theorem 1.7.1 $T$ has a positive fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$, which is solution to (3.1),

$$\forall t \in [0,1] : \quad U(t) = \int_0^1 \mathbf{G}(t,s) \frac{q_N(s)\mathcal{G}(s,U(s))}{\mathcal{M}\Big(s, \displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)} ds.$$

□

## 3.2 Uniqueness of Positive Radial Solution

Assume the diffusion coefficient function $M$ is independent of the first variable, satisfying

$$\exists\, m_1 > 0,\ \forall s \in [0,\infty) : \quad M(s) \geq m_1, \tag{3.91}$$

and

$$\begin{aligned} &\exists\, C_{12} > 0, \forall t \in [0,1],\ \forall s_1, s_2 \in [0,\infty) : \\ &\qquad (g(t,s_1) - g(t,s_2))(s_1 - s_2) \leq C_{12}|s_1 - s_2|^2, \quad \text{with} \quad m_1 > C_6 C_{12}. \end{aligned} \tag{3.92}$$

Furthermore, suppose that $M, f$ are Lipschitz functions, such that

$$\exists\, C_{13} > 0, \forall x, y \in [0,\infty) : \quad |f(x) - f(y)| \leq C_{13}|x - y|. \tag{3.93}$$

$$\exists\, C_{14} > 0, \forall s_1, s_2 \in [0,\infty) : \quad |M(s_1) - M(s_2)| \leq C_{14}|s_1 - s_2|, \tag{3.94}$$

with

$$\begin{aligned} &\text{if } N = 2 : \quad (m_1 - C_6 C_{12})^2 > C_6 C_{15}\|g(t,0)\|_2, \\ &\text{if } N \geq 3 : \quad (m_1 - C_6 C_{12})^2 > C_6 C_{16}\|g(t,0)\|_2, \end{aligned} \tag{3.95}$$

where

$$\begin{aligned} C_{15} &= C_{14} 2\pi b^2 \log\left(\frac{b}{a}\right)\sqrt{\frac{\left(\frac{a}{b}\right)^4 - 1}{4\ln\left(\frac{a}{b}\right)}}, \\ C_{16} &= C_{13}C_{14}\frac{\omega}{A(N-2)}\left(A^{4\frac{N-1}{N-2}}\frac{(B)^{\frac{2-3N}{N-2}} - (B-1)^{\frac{2-3N}{N-2}}}{\frac{2-3N}{N-2}}\right)^{\frac{1}{2}}. \end{aligned} \tag{3.96}$$

We prove the uniqueness of the solution of the problem

$$\begin{cases} \mathcal{M}\Big(\displaystyle\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)U''(t) = q_N(t)\mathcal{G}(t,U) \quad t \in (0,1), \\ U'(0) = U(1) = 0. \end{cases} \tag{3.97}$$

**Theorem 3.2.1.**

*Suppose the conditions* (3.2)*-*(3.8) *and* (3.91)*-*(3.95) *are satisfied. Then, the solution of the problem* (3.97) *is unique.*

## Proof.

Let $U$ be a positive solution for (3.97). Then, $U$ is solution of

$$-\mathcal{M}\Big(\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)U'' = q_N(t)\mathcal{G}(t,U). \tag{3.98}$$

Multiplying by $U$ the equation (3.98), it follows that

$$-\mathcal{M}\Big(\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)U''U = q_N(t)\mathcal{G}(t,U)U. \tag{3.99}$$

Integrating by parts over $[0,1]$ the equation (3.99), we obtain

$$-\mathcal{M}\Big(\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)\Big[[U'U]_0^1 - \int_0^1 U'^2 dt\Big] = \int_0^1 q_N(t)\mathcal{G}(t,U)U dt.$$

Taking into account $U'(0) = U(1) = 0$, we get

$$\mathcal{M}\Big(\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)\int_0^1 U'^2 dt = \int_0^1 q_N(t)\mathcal{G}(t,U)U dt. \tag{3.100}$$

Using Cauchy-Schwarz inequality and (3.100), we find

$$\begin{aligned} \mathcal{M}\Big(\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)\int_0^1 U'^2 dt &\geq m_1 \int_0^1 U'^2 dt \\ &\geq m_1\Big(\int_0^1 1dt\Big)^{\frac{1}{2}}\Big(\int_0^1 |U'|^2 dt\Big)^{\frac{1}{2}} \\ &= m_1 \|U'\|_2. \end{aligned} \tag{3.101}$$

Choosing $U = s_1$, $0 = s_2$ in (3.92), then

$$\mathcal{G}(t,U)U \leq C_{12}|U|^2 + \mathcal{G}(t,0)U,$$

integrating over $[0,1]$, and using (3.25), we have

$$\begin{aligned}
\int_0^1 q_N(t)\mathcal{G}(t,U)U dt &\leq \int_0^1 C_6\mathcal{G}(t,U)U dt \\
&\leq C_6\int_0^1 \Big(C_{12}|U|^2 + \mathcal{G}(t,0)U\Big) dt \\
&\leq C_6\Big[C_{12}\int_0^1 |U|^2 dt + \int_0^1 \mathcal{G}(t,0)U dt\Big].
\end{aligned}$$

Applying Cauchy-Schwarz inequality,

$$\begin{aligned}
&\int_0^1 q_N(t)\mathcal{G}(t,U)U dt \\
&\leq C_6\Big[C_{12}\int_0^1 |U|^2 dt + \int_0^1 \mathcal{G}(t,0)U dt\Big] \\
&\leq C_6\Big[C_{12}\int_0^1 |U|^2 dt + \int_0^1 |\mathcal{G}(t,0)U| dt\Big] \\
&\leq C_6\Big[C_{12}\Big(\int_0^1 1 dt\Big)^{\frac{1}{2}}\Big(\int_0^1 |U|^2 dt\Big)^{\frac{1}{2}} + \Big(\int_0^1 |\mathcal{G}(t,0)|^2 dt\Big)^{\frac{1}{2}}\Big(\int_0^1 |U|^2 dt\Big)^{\frac{1}{2}}\Big] \\
&= C_6\Big(C_{12}\|U\|_2 + \|\mathcal{G}(t,0)\|_2\|U\|_2\Big).
\end{aligned}$$

From $\|U\|_2 \leq \|U'\|_2$, it follows

$$C_6\Big(C_{12}\|U\|_2 + \|\mathcal{G}(t,0)\|_2\|U\|_2\Big) \leq C_6\Big(C_{12}\|U'\|_2 + \|\mathcal{G}(t,0)\|_2\|U'\|_2\Big). \tag{3.102}$$

According to (3.100) and (3.102), it yields

$$m_1\|U'\|_2^2 \leq C_6C_{12}\|U'\|_2^2 + C_6\|\mathcal{G}(t,0)\|_2\|U'\|_2,$$

then,

$$(m_1 - C_6C_{12})\|U'\|_2^2 \leq C_6\|\mathcal{G}(t,0)\|_2\|U'\|_2.$$

Since $\|U'\|_2 > 0$, we find

$$\|U'\|_2 \leq \frac{C_6\|\mathcal{G}(t,0)\|_2}{m_1 - C_6C_{12}}, \quad \text{with} \quad m_1 > C_6C_{12}. \tag{3.103}$$

On the other hand, in the case $N=2$. Assume $U_1, U_2$ are two positives solutions of (3.97). So,

$$-M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_1)ds\Big)U_1''=q_2(t)g\Big(b\Big(\frac{a}{b}\Big)^t,U_1\Big), \tag{3.104}$$

$$-M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_2)ds\Big)U_2''=q_2(t)g\Big(b\Big(\frac{a}{b}\Big)^t,U_2\Big). \tag{3.105}$$

Subtracting (3.104) from (3.105), we derive

$$\begin{aligned}&-M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_1)ds\Big)U_1''+M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_2)ds\Big)U_2''\\&=q_2(t)g\Big(b\Big(\frac{a}{b}\Big)^t,U_1\Big)-q_2(t)g\Big(b\Big(\frac{a}{b}\Big)^t,U_2\Big),\end{aligned}$$

which can be written as

$$\begin{aligned}&-M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_1)ds\Big)U_1''+M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_2)ds\Big)U_2''\\&+M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_2)ds\Big)U_2''-M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_2)ds\Big)U_2''\\&=q_2(t)\Big(g\Big(b\Big(\frac{a}{b}\Big)^t,U_1\Big)-g\Big(b\Big(\frac{a}{b}\Big)^t,U_2\Big)\Big).\end{aligned}$$

Let $U_3=U_1-U_2$, then

$$\begin{aligned}&-M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_1)ds\Big)U_3''\\&\quad=\Big\{M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_1)ds\Big)-M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_2)ds\Big)\Big\}U_2''\\&\quad+q_2(t)\Big(g\Big(b\Big(\frac{a}{b}\Big)^t,U_1\Big)-g\Big(b\Big(\frac{a}{b}\Big)^t,U_2\Big)\Big).\end{aligned} \tag{3.106}$$

Multiplying the equation (3.106) by $U_3$, it follows that

$$\begin{aligned}&-M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_1)ds\Big)U_3''U_3\\&\quad=\Big\{M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_1)ds\Big)-M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_2)ds\Big)\Big\}U_2''U_3\\&\quad+\Big(q_2(t)g\Big(b\Big(\frac{a}{b}\Big)^t,U_1\Big)-q_2(t)g\Big(b\Big(\frac{a}{b}\Big)^t,U_2\Big)\Big)U_3.\end{aligned} \tag{3.107}$$

Integrating by parts over $[0,1]$ the equation (3.107), we obtain

$$\begin{aligned}
&-M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_1)ds\Big)\Big[[U_3'U_3]_0^1-\int_0^1 U_3'^2dt\Big]\\
&=\Big\{M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_1)ds\Big)-M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_2)ds\Big)\Big\}\Big[[U_2'U_3]_0^1\\
&-\int_0^1U_3'U_2'dt\Big]+\int_0^1q_2(t)\Big(g\Big(b\Big(\frac{a}{b}\Big)^t,U_1\Big)-g\Big(b\Big(\frac{a}{b}\Big)^t,U_2\Big)\Big)U_3dt.
\end{aligned}$$

Since $U_3'(0)=U_1'(0)-U_2'(0)=0$, $U_3(1)=U_1(0)-U_2(0)=0$, we get

$$\begin{aligned}
&-M\Big(b\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_1)ds\Big)\Big[-\int_0^1U_3'^2dt\Big]=\\
&\Big\{M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_1)ds\Big)-M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_2)ds\Big)\Big\}\Big[-\int_0^1U_3'U_2'dt\Big]\\
&+\int_0^1q_2(t)\Big(g\Big(b\Big(\frac{a}{b}\Big)^t,U_1\Big)-g\Big(b\Big(\frac{a}{b}\Big)^t,U_2\Big)\Big)U_3dt.
\end{aligned}\tag{3.108}$$

Denote

$$I=\Big\{M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_1)ds\Big)-M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_2)ds\Big)\Big\}\Big[-\int_0^1U_3'U_2'dt\Big],\tag{3.109}$$

and

$$J=\int_0^1q_2(t)\Big(g\Big(b\Big(\frac{a}{b}\Big)^t,U_1\Big)-g\Big(b\Big(\frac{a}{b}\Big)^t,U_2\Big)\Big)U_3dt.\tag{3.110}$$

By (3.94), we obtain

$$\begin{aligned}
&M\Big(b\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_1)ds\Big)-M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_2)ds\Big)\\
&\leq\Big|M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_1)ds\Big)-M\Big(2\pi b^2\log\Big(\frac{b}{a}\Big)\int_0^1(\frac{a}{b})^{2s}f(U_2)ds\Big)\Big|\\
&\leq C_{14}2\pi b^2\log\Big(\frac{b}{a}\Big)\Big|\int_0^1(\frac{a}{b})^{2s}\Big(f(U_1)-f(U_2)\Big)ds\Big|.
\end{aligned}\tag{3.111}$$

Using Cauchy-Schwarz inequality, we get

$$\Big|\int_0^1\Big(\frac{a}{b}\Big)^{2s}\Big(f(U_1)-f(U_2)\Big)ds\Big|\leq\sqrt{\int_0^1\Big(\Big(\frac{a}{b}\Big)^{2s}\Big)^2ds}\sqrt{\int_0^1\Big(f(U_1)-f(U_2)\Big)^2ds}.$$

Let $x=\left(\frac{a}{b}\right)^4$:

$$\int_0^1\Big(\Big(\frac{a}{b}\Big)^{2s}\Big)^2ds=\int_0^1\Big(\frac{a}{b}\Big)^{4s}ds=\int_0^1x^s\,ds=\Big[\frac{x^s}{\ln x}\Big]_0^1=\frac{x-1}{\ln x}=\frac{\left(\frac{a}{b}\right)^4-1}{4\ln\left(\frac{a}{b}\right)}.$$

Then,

$$\sqrt{\int_0^1 \left(\left(\frac{a}{b}\right)^{2s}\right)^2 ds} = \sqrt{\frac{\left(\frac{a}{b}\right)^4 - 1}{4\ln\left(\frac{a}{b}\right)}}.$$

By (3.93), it follows that

$$\begin{aligned} f(U_1) - f(U_2) &\leq |f(U_1) - f(U_2)| \\ &\leq C_{13}|U_1 - U_2| \\ &= C_{13}|U_3|, \end{aligned}$$

which implies

$$\begin{aligned} \left(\int_0^1 \left(f(U_1) - f(U_2)\right)^2 ds\right)^{\frac{1}{2}} &\leq \left(C_{13}\int_0^1 |U_3|^2 ds\right)^{\frac{1}{2}} \\ &= C_{13}\|U_3\|_2. \end{aligned} \tag{3.112}$$

From (3.111) and (3.112), we have

$$\begin{aligned} &\left|M\left(2\pi b^2 \log\left(\frac{b}{a}\right)\int_0^1 (\frac{a}{b})^{2s} f(U_1)ds\right) - M\left(2\pi b^2 \log\left(\frac{b}{a}\right)\int_0^1 (\frac{a}{b})^{2s} f(U_2)ds\right)\right| \\ &\leq C_{14} 2\pi b^2 \log\left(\frac{b}{a}\right)\sqrt{\frac{\left(\frac{a}{b}\right)^4 - 1}{4\ln\left(\frac{a}{b}\right)}}\left|\int_0^1 (\frac{a}{b})^{2s}\left(f(U_1(s)) - f(U_2(s))\right)ds\right| \\ &= C_{14} 2\pi b^2 \log\left(\frac{b}{a}\right)\sqrt{\frac{\left(\frac{a}{b}\right)^4 - 1}{4\ln\left(\frac{a}{b}\right)}}\|U_3\|_2. \end{aligned} \tag{3.113}$$

According to (3.96), (3.109) and (3.113), we get

$$\begin{aligned} I &\leq C_{15}\|U_3\|_2\left| - \int_0^1 U_2' U_3' ds\right| \\ &\leq C_{15}\|U_3\|_2\int_0^1 |U_2' U_3'| ds. \end{aligned}$$

Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} I &\leq C_{15}\|U_3\|_2\left(\int_0^1 |U_2'|^2 ds\right)^{\frac{1}{2}}\left(\int_0^1 |U_3'|^2 ds\right)^{\frac{1}{2}} \\ &= C_{15}\|U_3\|_2\|U_2'\|_2\|U_3'\|_2. \end{aligned}$$

Since $\|U_3\|_2 \leq \|U_3'\|_2$, then

$$I \leq C_{15}\|U_3'\|_2^2\|U_2'\|_2. \tag{3.114}$$

From (3.103) and (3.114), it follows

$$I \le C_{15}\|U_3'\|_2^2 \frac{C_6\|\mathcal{G}(t,0)\|_2}{m_1 - C_6C_{12}}. \tag{3.115}$$

According to (3.25), (3.110), we have

$$\begin{aligned} J &\le \int_0^1 q_2(t)\Big(g\Big(b\Big(\frac{a}{b}\Big)^t, U_1\Big) - g\Big(b\Big(\frac{a}{b}\Big)^t, U_2\Big)\Big)U_3 dt \\ &\le C_6\int_0^1 \Big|\Big(g\Big(b\Big(\frac{a}{b}\Big)^t, U_1\Big) - g\Big(b\Big(\frac{a}{b}\Big)^t, U_2\Big)\Big)U_3\Big| dt. \end{aligned} \tag{3.116}$$

By (3.92), we derive

$$\begin{aligned} \Big(g\Big(b\Big(\frac{a}{b}\Big)^t, U_1\Big) - g\Big(b\Big(\frac{a}{b}\Big)^t, U_2\Big)\Big)U_3 &\le C_{12}|U_1 - U_2|^2 \\ &= C_{12}|U_3|^2. \end{aligned} \tag{3.117}$$

Integrating over $[0,1]$ the equation (3.117), we obtain

$$\int_0^1 \Big(g\Big(b\Big(\frac{a}{b}\Big)^t, U_1\Big) - g\Big(b\Big(\frac{a}{b}\Big)^t, U_2\Big)\Big)U_3 dt \le C_{12}\int_0^1 |U_3|^2 dt.$$

Applying Cauchy-Schwarz inequality and $\|U_3\|_2 \le \|U_3'\|_2$, we deduce

$$\begin{aligned} \int_0^1 \Big(g\Big(b\Big(\frac{a}{b}\Big)^t, U_1\Big) - g\Big(b\Big(\frac{a}{b}\Big)^t, U_2\Big)\Big)U_3 dt &\le C_{12}\int_0^1 |U_3|^2 dt \\ &\le C_{12}\Big(\int_0^1 1 d\tau\Big)^{\frac{1}{2}}\Big(\int_0^1 |U_3|^2 d\tau\Big)^{\frac{1}{2}} \\ &= C_{12}\|U_3\|_2 \\ &\le C_{12}\|U_3'\|_2^2. \end{aligned} \tag{3.118}$$

According (3.116),(3.118), we obtain

$$J \le C_6C_{12}\|U_3'\|_2^2. \tag{3.119}$$

Applying Cauchy-Schwarz inequality

$$\begin{aligned} -\mathcal{M}\Big(\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)\Big[-\int_0^1 U_3'^2 dt\Big] &\le \mathcal{M}\Big(\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)\Big(\int_0^1 1 d\tau\Big)^{\frac{1}{2}}\Big(\int_0^1 |U_3'|^2 d\tau\Big)^{\frac{1}{2}} \\ &= \mathcal{M}\Big(\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)\|U_3'\|_2 \\ &\le \mathcal{M}\Big(\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)\|U_3'\|_2^2. \end{aligned}$$

Moreover, using (3.91), we find

$$m_1\|U_3'\|_2^2 \leq \mathcal{M}\Big(\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)\|U_3'\|_2^2. \tag{3.120}$$

Combining (3.108), (3.115), (3.119) and (3.120), it follows that

$$\begin{aligned} m_1\|U_3'\|_2^2 &\leq C_{15}\|U_3'\|_2^2\frac{C_6\|\mathcal{G}(t,0)\|_2}{m_1-C_6C_{12}} + C_6C_{12}\|U_3'\|_2^2 \\ &= \Big(\frac{C_6C_{15}\|\mathcal{G}(t,0)\|_2}{m_1-C_6C_{12}} + C_6C_{12}\Big)\|u_3'\|_2^2, \end{aligned}$$

which gives

$$\Big(m_1 - \frac{C_6C_{15}\|\mathcal{G}(t,0)\|_2}{m_1-C_6C_{12}} - C_6C_{12}\Big)\|U_3'\|_2^2 \leq 0.$$

Recalling that

$$m_1 - \frac{C_6C_{15}\|\mathcal{G}(t,0)\|_2}{m_1-C_6C_{12}} - C_6C_{12} > 0,$$

it follows that

$$\|U_3'\|_2^2 = 0.$$

Then

$$U_3 = 0 \Rightarrow U_1 - U_2 = 0.$$

Hence $U_1 = U_2$. We conclude the problem (3.97) has unique positive solution for $N = 2$.

With similar steps, if $N \geq 3$, we have

$$\begin{aligned} &M\Big(\frac{\omega}{A(N-2)}\int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}f(U_1)d\tau\Big)\Big[-\int_0^1 U_3'^2 dt\Big] = \\ &\Big\{M\Big(\frac{\omega}{A(N-2)}\int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}f(U_1)ds\Big) - \\ &M\Big(\frac{\omega}{A(N-2)}\int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}f(U_2)ds\Big)\Big\}\Big[-\int_0^1 U_3'U_2'dt\Big] \\ &+\int_0^1 q_N(t)\Big(g\Big((\frac{A}{B-t})^{\frac{1}{N-2}}, U_1\Big) - g\Big((\frac{A}{B-t})^{\frac{1}{N-2}}, U_2\Big)\Big)U_3 dt. \end{aligned} \tag{3.121}$$

Setting

$$\begin{aligned} I = &\Big\{M\Big(\frac{\omega}{A(N-2)}\int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}f(U_1)d\tau\Big) - \\ &M\Big(\frac{\omega}{A(N-2)}\int_0^1 (\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}f(U_2)d\tau\Big)\Big\}\Big[-\int_0^1 U_3'U_2'dt\Big], \end{aligned} \tag{3.122}$$

and

$$J=\int_0^1 q_N(t)\Big(g\Big((\frac{A}{B-t})^{\frac{1}{N-2}},U_1\Big)-g\Big((\frac{A}{B-t})^{\frac{1}{N-2}},U_2\Big)\Big)U_3dt. \tag{3.123}$$

By (3.94), we obtain

$$\begin{aligned}&M\Big(\frac{\omega}{A(N-2)}\int_0^1(\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}f(U_1)d\tau\Big)-M\Big(\frac{\omega}{A(N-2)}\int_0^1(\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}f(U_2)d\tau\Big)\\&\leq\left|M\Big(\frac{\omega}{A(N-2)}\int_0^1(\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}f(U_1)d\tau\Big)-M\Big(\frac{\omega}{A(N-2)}\int_0^1(\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}f(U_2)d\tau\Big)\right|\\&\leq C_{14}\frac{\omega}{A(N-2)}\left|\int_0^1(\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}\Big(f(U_1(\tau))-f(U_2(\tau))\Big)d\tau\right|.\end{aligned} \tag{3.124}$$

Using Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\left(\int_0^1\left(\frac{A}{B-\tau}\right)^{\frac{2(N-1)}{N-2}}\left(f(U_1(\tau))-f(U_2(\tau))\right)\,d\tau\right)^2\\&\leq\left(\int_0^1\left(\frac{A}{B-\tau}\right)^{4\frac{N-1}{N-2}}\,d\tau\right)\left(\int_0^1\left(f(U_1(\tau))-f(U_2(\tau))\right)^2\,d\tau\right).\end{aligned} \tag{3.125}$$

For simplify the first integral, let $u=B-\tau\Rightarrow du=-d\tau$ and $\tau=0\Rightarrow u=B,\tau=1\Rightarrow u=B-1$. The integral becomes

$$\begin{aligned}\int_0^1\left(\frac{A}{B-\tau}\right)^{4\frac{N-1}{N-2}}\,d\tau&=A^{4\frac{N-1}{N-2}}\int_{B-1}^B u^{-4\frac{N-1}{N-2}}\,du\\&=A^{4\frac{N-1}{N-2}}\left[\frac{u^{\frac{2-3N}{N-2}}}{\frac{2-3N}{N-2}}\right]_{B-1}^B\\&=A^{4\frac{N-1}{N-2}}\frac{(B)^{\frac{2-3N}{N-2}}-(B-1)^{\frac{2-3N}{N-2}}}{\frac{2-3N}{N-2}}.\end{aligned} \tag{3.126}$$

From (3.124) - (3.126), we have

$$\begin{aligned}&M\Big(\frac{\omega}{A(N-2)}\int_0^1(\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}f(U_1(\tau))d\tau\Big)-M\Big(\frac{\omega}{A(N-2)}\int_0^1(\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}f(U_2(\tau))d\tau\Big)\\&\leq C_{14}\frac{\omega}{A(N-2)}\left|\int_0^1(\frac{A}{B-\tau})^{2\frac{N-1}{N-2}}\Big(f(U_1(s))-f(U_2(s))\Big)ds\right|\\&=C_{13}C_{14}\frac{\omega}{A(N-2)}\Big(A^{4\frac{N-1}{N-2}}\frac{(B)^{\frac{2-3N}{N-2}}-(B-1)^{\frac{2-3N}{N-2}}}{\frac{2-3N}{N-2}}\Big)^{\frac{1}{2}}\|U_3\|_2.\end{aligned} \tag{3.127}$$

Combining (3.96), (3.122) and (3.127), we get

$$\begin{aligned}I &\leq C_{16}\|U_3\|_2\left|-\int_0^1 U_2'U_3'ds\right|\\ &\leq C_{16}\|U_3\|_2\int_0^1 |U_2'U_3'|ds.\end{aligned}$$

Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}I &\leq C_{16}\|U_3\|_2\Big(\int_0^1 |U_2'|^2ds\Big)^{\frac{1}{2}}\Big(\int_0^1 |U_3'|^2ds\Big)^{\frac{1}{2}}\\ &= C_{16}\|U_3\|_2\|U_2'\|_2\|U_3'\|_2.\end{aligned}$$

Since $\|U_3\|_2 \leq \|U_3'\|_2$, then

$$I \leq C_{16}\|U_3'\|_2^2\|U_2'\|_2. \tag{3.128}$$

From (3.103) and (3.128), it follows

$$I \leq C_{16}\|U_3'\|_2^2\frac{C_6\|\mathcal{G}(t,0)\|_2}{m_1 - C_6C_{12}}. \tag{3.129}$$

According to (3.25), (3.123), we have

$$\begin{aligned}J &\leq \int_0^1 q_N(t)\Big(g\Big(\big(\frac{A}{B-t}\big)^{\frac{1}{N-2}}, U_1\Big) - g\Big(\big(\frac{A}{B-t}\big)^{\frac{1}{N-2}}, U_2\Big)\Big)U_3dt\\ &\leq C_6\int_0^1 \Big|\Big(g\Big(\big(\frac{A}{B-t}\big)^{\frac{1}{N-2}}, U_1\Big) - g\Big(\big(\frac{A}{B-t}\big)^{\frac{1}{N-2}}, U_2\Big)\Big)U_3\Big|dt.\end{aligned} \tag{3.130}$$

By (3.92):

$$\begin{aligned}\int_0^1 \Big|\Big(g\Big(\big(\frac{A}{B-t}\big)^{\frac{1}{N-2}}, U_1\Big) - g\Big(\big(\frac{A}{B-t}\big)^{\frac{1}{N-2}}, U_2\Big)\Big)U_3\Big|ds &\leq C_{12}\int_0^1 \Big|U_1 - U_2\Big|^2dt\\ &\leq C_{12}\|U_3\|_2^2\\ &\leq C_{12}\|U_3'\|_2^2.\end{aligned} \tag{3.131}$$

Using (3.130), (3.131), we obtain

$$J \leq C_6C_{12}\|U_3'\|_2^2. \tag{3.132}$$

Also, by (3.91):

$$m_1\|U_3'\|_2^2 \leq \mathcal{M}\Big(\int_0^1 \mathcal{F}(U(\tau))d\tau\Big)\|U_3'\|_2^2. \tag{3.133}$$

Combining (3.121), (3.129), (3.132) and (3.133), it follows that

$$\begin{aligned} m_1\|U_3'\|_2^2 &\leq C_{16}\|U_3'\|_2^2\frac{C_6\|\mathcal{G}(t,0)\|_2}{m_1-C_6C_{12}}+C_6C_{12}\|U_3'\|_2^2 \\ &= \Big(\frac{C_6C_{16}\|\mathcal{G}(t,0)\|_2}{m_1-C_6C_{12}}+C_6C_{12}\Big)\|U_3'\|_2^2, \end{aligned}$$

then

$$\Big(m_1-\frac{C_6C_{16}\|\mathcal{G}(t,0)\|_2}{m_1-C_6C_{12}}-C_6C_{12}\Big)\|U_3'\|_2^2\leq 0.$$

We know that

$$m_1-\frac{C_6C_{16}\|\mathcal{G}(t,0)\|_2}{m_1-C_6C_{12}}-C_6C_{12}>0,$$

which implies

$$\|U_3'\|_2^2=0.$$

Therefore

$$U_3=0\Rightarrow U_1-U_2=0.$$

Hence, the problem (3.97) has unique positive solution for $N\geq 3$. □

# Conclusion

In conclusion, this thesis presents results that contribute to the study of nonlocal boundary problems, offering new pathways for scientific exploration in this developing field. After introducing preliminary concepts crucial to understanding this work, we demonstrated results on the existence and the uniqueness of positive solutions for a one-dimensional nonlocal boundary problem. Moreover, we obtained the existence and the uniqueness of positive radial solutions in multidimensional nonlocal elliptic problems with simple example.

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