Democratic and Popular Republic of Algeria Ministry of Higher Education and Scientific Research

---

Hamma Lakhdar University of El Oued

Faculty of Exact Sciences

Final study memory

Academic Master

Field: Mathematics and Informatique
Major: Mathematics
Specialization: Fundamental and Applied Mathematics

Theme

On the study of some singular integral equations

Presented by: Bekkari Maroua and Ben Ali Dhikra
Under the supervision of: Guesba Messaoud

Defended before the jury:

M:Dr. Abdel Wahab Mansour	University of El-Oued	President
M: Dr. Guesba Messaoud	University of El-Oued	Supervisor
M :Dr. Ben Ali Ibrahim	University of El-Oued	Examiner

University year : 2023 – 2024

ABSTRACT

In this thesis, we have studied some singular integral equations. This thesis consists of three main chapters. In the first chapter, we have defined the integral equation, its types, and solutions. In the second chapter, we discussed the types of singular integral equations, the Abel problem, and weak integral equations. The final chapter is dedicated to the solutions of integral equations using complex analysis.

In conclusion, we hope that we have succeeded in presenting and writing this thesis.

keywords : Fredholm integral equation, Volterra integral equation,Singular integral equation, Kernel ,Singular point .

Résumé :

Dans cette thèse, nous avons étudié quelques équations intégrales singulières. Cette thèse se compose de trois chapitres principaux. Dans le premier chapitre, nous avons défini l'équation intégrale, ses types et ses solutions. Dans le deuxième chapitre, nous avons discuté des types d'équations intégrales singulières, du problème d'Abel et des équations intégrales faibles. Le dernier chapitre est consacré aux solutions d'équations intégrales à l'aide d'une analyse complexe.

En conclusion, nous espérons avoir réussi à présenter et rédiger cette thèse.

mots-clés : Équation intégrale de Fredholm, équation intégrale de Volterra, équation intégrale singulière, noyau, point singulier.

الملخص:

في هذه المذكرة قمنا بدراسة بعض المعادلات التكاملية الشاذة. تتكون هذه الأطروحة من ثلاثة فصول رئيسية. وقد عرفنا في الفصل الأول المعادلة التكاملية وأنواعها وحلولها. وفي الفصل الثاني تناولنا أنواع المعادلات التكاملية الشاذة ومسألة هابيل والمعادلات التكاملية الضعيفة. أما الفصل الأخير فقد خصص لحلول المعادلات التكاملية باستخدام التحليل المركب.

وفي الختام نأمل أن نكون قد وفقنا في تقديم وكتابة هذه الأطروحة.

الكلمات الرئيسية: معادلة فريدهولم التكاملية، معادلة فولتيرا التكاملية، معادلة التكامل الشاذة، النواة، النقطة الشاذة.

ACKNOWLEDGEMENT

“In the name of Allah, the Most Gracious, the Most Merciful ”

Praise be to Allah who has granted us the success to complete this humble work. We would like to thank everyone who has helped and supported us.
We extend our heartfelt thanks to our supervisor, Professor ***Guesba Messaoud****, for his guidance, directions, and for overseeing this thesis.*
We also express our gratitude and appreciation to the members of the thesis committee. We must not forget our esteemed professors in the Department of Mathematics at the University of El Oued, who have provided us with a great deal of knowledge and have exerted significant efforts in building the next generation.
Finally, we would like to thank our families for their encouragement, patience, and for being a pillar of support throughout this academic journey.

Thank you.

الاهداء

اهدي تخرجي الى النور الذي انار دربي والسراج الذي لا يختفي نوره ابدا **والدي** العزيز.

الى من اخص الله الجنة تحت قدميها وغمرتني بالحب والحنان واشعرتني بالسعادة والامان **والدتي** العزيزة .

الى **اخوتي واخواتي** .

ذكرى بن علي ، مروة بكاري

Contents

List of Figures

NOTATIONS

- K *being* $\mathbb{C};\mathbb{R}$
- $\mathbb{C};\mathbb{R}$ *: complex field and real field .*
- $\mathcal{L}(f)$*:Laplace transformation*
- $(A\phi)(n)$:*The able integral operator .*
- Z : *Number complexe.*
- C :*Simple closed contour .*
- *P.V: Cauchy principle value .*
- $C_{\mathbb{R}}$*: The circle* $|Z| = \mathbb{R}$
- $\Gamma(x)$: *The gamma function.*

GENERAL INTRODUCTION

The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics.

It has enormous applications in many physics problems. Integral equations were first encountered in the theory of Fourier integral. In 1826, Abel obtained other integral equations. The actual development of the theory of integral equations began with the work of the Italian mathematician Volterra in 1896 and the Swedish mathematician Fredholm 1900.

Therefore, the aim of the current work is to study singular integral equation, therefore, in the first chapter, we discuss generality of integral equation, its classification, and solutions to the Volterra and Fredholm equations with examples. We also touched on their solutions using the Laplace transform. As for the second chapter, we defined the singular integral equation and the Cauchy value, the main Cauchy value for integration, the eigenvalues and the eigenrays, and we touched on the linear and non-linear Apple problem and the Cauchy singularity.

Finally, in the third chapter, we present the solution of integral equations using analytical methods, especially complex analysis methods, which can be considered one of the most promising ways to solve singular integral equations. These methods are distinguished by their usefulness, either by subtracting the *singularity from the kernel of the integral equations or by using Primary conversion methods. Such methods also work when singularity occurs within the limits of integration.*

CHAPTER I

GENERALITY IN THE INTEGRAL EQUATIONS

I.1 Concept of the integral equations

Definition I.1.1

An integral equation is defined as an equation in which the unknown function u(x) to be determined appear under the integral sign.

The standard type of integral equation is of the form

$$u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt, \tag{I.1}$$

where:

$f(x), u(x)$ *and* $K(x,t)u(x)$ *three functions are given* $K(x,t)u(x)$*is called the kernel.*

$\alpha(x)$ *and* $\beta(x)$ *are the limits of integration.*

$u(x)$ *it the unknown function appears under the integral sign.*

λ *a no null constant parameter in to* $\mathbb{R}$ *or* $\mathbb{C}$*.*

Remark I.1.1

- *If* $f(x)$ *in integral equation then I.1 the resulting equation is called a homogeneous integral equation, otherwise it is called non-homogeneous integral equation*
- *If the kernel*$K(x,t,u(x)) = K(x,t)u(x)$*then the integral equation I.1 are classified as linear integral equation, otherwise it is called non-linear integral equation.*
- *The function* $\phi(x)$ *will determine type kind of the integral equation*
- $\alpha(x),\beta(x)$ *will determine type of the integral equation.*

Example I.1.1

$$\begin{aligned} u(x) &= \lambda \int_5^2 K(x,t)u^2(t)dt \quad \textit{homogenous integral equation} \\ u(x) &= x - \frac{1}{6}x^3 + \int_0^x (x,t)u(t)dt \quad \textit{non homogenous integral equation} \\ u(x) &= f(x) + a\int_a^b K(n,t)u(t)dt \quad \textit{linear integral equation} \\ u(x) &= \frac{1}{2} + x - \int_0^2 (x-t)u^2(t)dt \quad \textit{,nonlinear integral equation} \end{aligned} \tag{I.2}$$

I.2 Classifications of integral equations

I.2.1 Linear Integral Equations

Volterra integral equations

Difinition I.2.1.1

The most standard form of linear Volterra integral equation is given by

$$\phi(x)u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt, \tag{I.3}$$

where the limits of integration a (constants) and x and unknown function $u(x)$ appears linearly under the integral sign.

1. *If* $\phi(x) = 0$ *then equation I.3 simply becomes*

$$f(x) = \lambda \int_a^x K(x,t)u(t)dt, \tag{I.4}$$

   and this equation is known as the linear Volterra integral equation of the first kind.

2. *If*$\phi(x) = \phi$ *constant* $\neq 0$ *then equation I.3 becomes*

$$\phi u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt, \tag{I.5}$$

   which is known as the linear Volterra integral equation of the second kind.

3. *If the function* $\phi(x) \neq 0$ *then equation I.3 is known as the Volterra linear integral equation of the third kind*

Example I.2.1.1

$$xe^{2x-1} = \int_1^x e^{xt}u(t)\,dt \qquad \textit{first kind}$$

$$x^2 - u(x) = \int_0^x x\cos u(t)\,dt \qquad \textit{second kind}$$

$$(x^2 - x)u(x) = \int_1^x x\cos u(t)\,dt \qquad \textit{third kind}$$

Fredholm integral equations

Definition I.2.1.2

The most standard form of linear Fredholm integral equation is given by

$$\phi u(x)u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \tag{I.6}$$

where the limits of integration a and b are constants and the unknown function u(x) appears linearly under the integral sign.

1. *if* $\phi(x) = 0$ *then equation I.3 simply becomes*

   $$f(x) = \lambda \int_a^b K(x,t)u(t)dt, \tag{I.7}$$

   and this equation is known as the linear Fredholm integral equation of the first kind.

2. $\phi(x) = \phi$ *constant* $\neq 0$ *then equation I.3 becomes*

   $$\phi u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, \tag{I.8}$$

   Which is known as the linear Fredholm integral equation of the second kind.

3. *if the function* $\phi(x) \neq 0$ *,then equation I.3 is known as the Fredholm linear integral equation of the third kind*

Example I.2.1.2

$$\cos(x) = \int_0^1 (x-t)u(t)\,dt \qquad \textit{first kind}, \tag{I.9}$$

$$u(x) = x^2 + x + \int_0^1 xtu(t)\,dt \qquad \textit{second kind}, \tag{I.10}$$

$$xu(x) = \int_0^1 x^2tu(t)\,dt \qquad \textit{third kind}, \tag{I.11}$$

Wiener-Hoph integral equations

Difinition I.2.1.3

We call wiener-Hoph integral equation the equation of form

$$\phi(x)u(x) = f(x) + \lambda \int_a^{\infty} k(x-t)u(t)dt, \tag{I.12}$$

Example I.2.1.4

$$e^x u(x) = 2 + 2\int_0^{\infty} (t-x)^2 u(t)dt, \tag{I.13}$$

Renwal integral equations

Definition I.2.1.4

We call Renwal integral equation the equation of form

$$\phi u(x)u(x) = f(x) + \lambda \int_a^x k(x-t)u(t)dt, \tag{I.14}$$

Abel integral equations

Definition I.2.1.5

We call linear Abel integral equation the equation of form

$$f(x) = \int_a^x \frac{u(t)}{(x-t)^\alpha}dt \qquad 0 < \alpha = constant < 1, \tag{I.15}$$

Example I.2.1.5

$$x^3 = \int_2^x \frac{u(t)}{\sqrt{x-t}}dt, \tag{I.16}$$

I.2.2 non-Linear Integral Equations

Volterra Integral Equations

Definition I.2.2.1

We called nonlinear Volterra integral equation of the first kind given by form

$$f(x) = \lambda \int_a^x k(x,t,u(t))dt, \tag{I.17}$$

and we called nonlinear Volterra integral equation of the second kind given by form

$$\phi(x) = f(x) + \lambda \int_a^x k(x,t,u(t))dt \qquad \phi = \ constant\ \neq 0, \tag{I.18}$$

We called nonlinear Volterra integral equation of the third kind given by form

$$\phi u(x) = f(x) + \lambda \int_a^x k(x,t,u(t))dt \quad \phi(x) \neq 0, \tag{I.19}$$

Example I.2.2.1

$$\frac{1}{x}\int_1^x x\ln(u(t))\,dt, \tag{I.20}$$

$$cos x = \int_0^1 (x-t)u(t)dt, \tag{I.21}$$

$$u(x) = x^2 + x + \int_0^1 x + u(t)dt, \tag{I.22}$$

$$xu(x) = \int_0^1 x^2 + x + u(t)dt, \tag{I.23}$$

Fredholm Integral Equations

Definition I.2.2.2

We called non linear Fredholm integral equation of the first kind given by form

$$f(x) = \lambda \int_a^b k(x,t,u(t))dt, \tag{I.24}$$

and we called nonlinear Fredholm integral equation of the second kind given by form

$$\phi(x)u(x) = f(x) + \lambda \int_a^b k(x,t,u(t))dt \qquad \phi = constant \neq 0, \tag{I.25}$$

We called non linear Fredholm integral equation of the third kind given by form

$$\phi(x)u(x) = f(x) + \lambda \int_a^x k(x,t,u(t))dt \qquad \phi(x) \neq 0, \tag{I.26}$$

Remarque I.2.2.1

- *if $f(x) = 0$ in integral equation , then the resulting equation is called a homogeneous integral equation, otherwise it is called non-homogeneous integral equation.*

Integro-differential equations

Definition I.2.2.3

The non linear integro-differential equation appears in the form:

$$\phi(x)u(x) = f(x) + \lambda \int_a^b k(x,tu(t))dx \qquad \phi \neq 0, \tag{I.27}$$

and the standard form of the non linear integro-differential equation of the first kind is given by

$$\int_G K_1(t.x.u(x))dx + \int_G K_2(t.x.u^{(n)}(x))dx = f(t) \qquad K_2(t.x.u^{(n)}(x)) \neq 0$$

Where $u^{(n)}$ indicates the nth derivative of $u(x)$,the kernels K, K_1, K_2 and the function $f(x)$aregiven real valued functions.

The Volterra-Fredholm integro-differential equation arise in the same manner as Volterra-Fredholm integral equation with one or more of ordinary derivatives in addition to the integral operators

Example I.2.2.2

$$u^{(n)}(t) = f(x) + \lambda \int_G k(x,t,u(t))dx \quad x,y \in G, \tag{I.28}$$

Abel integral equations

Definition I.2.2.4

We call Abel integral equation the equation of form

$$u(x) = \int_{-\infty}^{x} (x-t)^{\alpha-1} g(u(t))dt, \tag{I.29}$$

$0 < \alpha < 1$ *and* $g : [0.\infty) \longrightarrow [0.\infty) g(0) = 0$ *and* $g(x) > 0$

Example I.2.2.3

$$\pi = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt, \tag{I.30}$$

Lalesco integral equations

Definition I.2.2.5

We call Lalesco integral equation the equation of form

$$u(x) = f(x) + \int_0^x [K_1(x,t)u(t) + K_2(x,t)u^2(t) + K_3(x,t)u^3(t)....... + K_n(x,t)u^n(t)]dt, \tag{I.31}$$

I.2.3 Mixed Integral Equations

Volterra-Fredholm Integral Equations

Definition I.2.3.6

The Volterra-Fredholm integral equation second kind appear in two forms, namely

$$u(x) = f(x) + 1 \int_0^x K_1(x,t)u(t)dt + 2 \int_a^b K_2(x,t)u(t)dt, \tag{I.32}$$

and the mixed form

$$u(x) = f(x) + 1 \int_0^x \int_a^b K(r,t)u(t)dtdr, \tag{I.33}$$

where $f(x)$ and $K(x,t)$ are analytic functions.For example

$$u(x) = sin(x) + \int_0^x \int_a^b r^2 u(t)dtdr, \tag{I.34}$$

In the next section, the concern will be paid to the approximation methods for solving Fredholm and Volterra integral equations, respectively. Also, two numerical methods will be given so that to solve the integral equations.

I.3 Solution of Integral Equation

I.3.1 Fredholm Integral Equation:

The methods for solving non-homogenous Fredholm integral equations are depending on the degeneracy case of the kernel. A degenerate kernel is a kernel in the form of a finite sum of the products of a function of x only, say, $a_k(x)$, and a function of t only, say, $b_k(t)$, for all $k = 1,2,\dots,n$, in other words:

$$k(x,t) = \sum_{k=1}^{n} a_k(x)b_k(t), \tag{I.35}$$

where n is a positive integer.

To solve the non-homogenous Fredholm integral equation of the second kind with degenerate kernel, one has:

$$\begin{aligned} y(x) &= f(x) + \lambda \int_a^b k(x,t)a_k(x)b_k(t)y(t)dt \\ &= f(x) + \lambda \int_a^b \sum_{k=1}^{n} \{a_k(x)b_k(t)\}y(t)dt \\ &= f(x) + \lambda \sum_{k=1}^{n} \{a_k(x) \int_a^b b_k(t)y(t)dt\}, \end{aligned} \tag{I.36}$$

Let

$$c_k = \int_a^b b_k(t)y(t)dt, \tag{I.37}$$

so equation (I.36) becomes:

$$y(x) = f(x) + \lambda \sum_{k=1}^{n} c_k a_k(x), \tag{I.38}$$

This gives $y(x)$, our limited solution, once the constants c_k have been determined. We may findc_k by multiplying eq.(I.38) by $b_m(x)$ and integrating both sides from a to b, to eliminate the *x-dependence. Use of eq.*(I.37) *, will be made to get:*

$$\begin{aligned} c_m &= \int_a^b b_m(x)y(x)dx \\ &= \int_a^b b_m(x)f(x)dx + \lambda \sum_{k=1}^{n} \{c_k \int_b^a b_m(x)a_k(x)dx\}, \end{aligned} \tag{I.39}$$

Let

$$a_{mk}(x) = \int_a^b b_m(x)a_k(x)dx, \tag{I.40}$$

$$f_m = \int_a^b b_m(x)f(x)dx, \tag{I.41}$$

This implies that eq.(I.39) *becomes:*

$$c_m = f_m + \lambda \sum_{k=1}^{n} a_{mk}C_k, \tag{I.42}$$

where $m = 1, 2, .., n$.

It's clear that equation (I.42) *is a set of linear system algebraic equations in* $c_{n-1}, c_1 \ldots, , c_n$. *Here* f_m *and* a_{mk} *are considered known since we are given* $b_m(x)$, $f(x)$, *and* $a_k(x)$.

So, the solution of the Fredholm integral equation of the second kind with degenerate kernel reduces to solving for c_m *from the system of n linear equations* (I.42).

If we use matrix notation, the system (I.42) *can be written in matrix form:*

$$c = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} + \lambda \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = F + \lambda AC, \tag{I.43}$$

or in the matrix form:

$$(I - \lambda A)C = F, \tag{I.44}$$

From the theory of linear systems of equations, it is well known that equation (I.44) *has a unique solution if* $|I - \lambda A| \neq 0$ *and has either an infinite number of solutions or no solution when* $|I - \lambda A| = 0$.

Example I.3.1.1

solve the following fredholm integral equation:

$$u(x)-\lambda\int_{-\pi}^{\pi}\left(x\cos(t)+t^2\sin(x)+\cos(x)\sin(t)\right)u(t)\,dt=x$$

Solution

$$u(x)=x+\lambda\left(x\int_{-\pi}^{\pi}\cos(t)u(t)dt+\sin(x)\int_{-\pi}^{\pi}t^2u(t)dt+\cos(x)\int_{-\pi}^{\pi}t^2u(t)dt\right)$$

$$u(x)=x+\lambda\left(xc_1+c_2\sin x+c_3 cosx\right)$$

$$c_1=\int_{-\pi}^{\pi}\cos t u(t)dt=\int_{-\pi}^{\pi}\cos t[t+\lambda(tc_1+c_2\sin(t)+c_3\cos t]dt$$

$$\int_{-\pi}^{\pi}t\cos t dt+\lambda c_1\int_{-\pi}^{\pi}t\cos(t)dt+\lambda c_2\int_{-\pi}^{\pi}t\cos(t)\sin(t)dt+\lambda c_3\int_{-\pi}^{\pi}t\cos^2(t)dt$$

$$\begin{cases} c_1-\lambda\pi c_3=0\\ c_2+4\lambda\pi c_3=0\\ -2\lambda\pi c_1-\lambda\pi c_2+c_3=2\pi\end{cases}$$

$$\begin{cases} c_1=\dfrac{2\lambda\pi^2}{1+2\lambda^2\pi^2}\\ c_2=-\dfrac{8\lambda\pi^2}{1+2\lambda^2\pi^2}\\ c_3=-\dfrac{2\pi}{1+2\lambda^2\pi^2}\end{cases}$$

$$u(x)=\frac{2\lambda\pi}{1+2\lambda^2\pi^2}(\lambda\pi x-4\lambda\pi\sin(x)+\cos(x))+x$$

Now, if the Fredholm integral equ is homogenous and with degenerate kernel, the problem is then known as "Eigenvalue problem" [9]and could be solved similarly as in the above approach.

Another approximate method can be implemented by approximating a non degenerate kernel by a degenerate one by using Taylor or other series expansion. The error involved in this method is according to the approximation method used for the kernel [12].

I.3.2 Volterra Integral Equations

Numann, Liouville, and Volterra developed a method to find the unknown function $y(x)$ as a power series in λ. To illustrate this, consider the second kind Volterra integral equation:

$$y(x) = f(x) + \lambda \int_a^x k(x,t)y(t)dt, \tag{I.45}$$

The solution of (I.45) *may be interpreted and found using the following integral equation:*

$$y(x) = f(x) + \lambda \int_a^x S(x,t,\lambda)y(t)dt, \tag{I.46}$$

where $S(x,t,\lambda)$ is a resolvent kernel of the integral equation. The resolvent kernel $S(x,t,\lambda)$ can be approximated by the Numann series [19]:

$$S(x,t,\lambda) = \sum_{i=0}^{\infty} \lambda^i + k_{i+1}(x,t), \tag{I.47}$$

where:

$$\begin{cases} k_{i+1}(x,t) = \int_a^x k(x,z)k_i(z,t)dz \\ k_1(x,t) = k(x,t), \end{cases} \tag{I.48}$$

Example I.3.2.1

solve the following volterra integral equation:

$$u(x) = e^{x^2} - \int_0^x e^{x^2-t^2} u(t)\,dt$$

Solution.

$$k(x,t) = k_1(x,t) = e^{x^2-t^2}$$

$$k_2(x,t) = \int_t^x e^{x^2-s^2}.e^{s^2-t^2}ds$$

$$= \int_t^x e^{x^2-t^2}ds = e^{x^2-t^2}\int_t^x ds = e^{x^2-t^2}(x-t)$$

$$k_3(x,t) = \int_t^x e^{x^2-s^2}e^{s^2-t^2}(s-t) = e^{x^2-t^2}\int_t^x (s-t)ds$$

$$e^{x^2-s^2}[\frac{1}{3}(s-t)^2]_t^x$$

$$= \frac{1}{2}e^{x^2-s^2}(x-t)^2$$

$$k_4(x,t) = \int_t^x e^{x^2-s^2}.\frac{1}{2}e^{s^2-t^2}.(s-t)^2 ds$$

$$= \int_t^x \frac{1}{2}e^{x^2-t^2}(s-t)^2 ds$$

$$= \frac{1}{6}e^{x^2-t^2}(x-t)^3$$

$$k_n(x,t) = \frac{(x-t)^{n-1}}{(n-1)!}e^{x^2-t^2}$$

$$R(x,t,\lambda) = \sum_{n=0}^{+\infty} \lambda^n k_{n+1}(x,t)$$

$$= \sum_{n=0}^{+\infty} \lambda^n \frac{(x-t)^n}{n!}e^{x^2-t^2}$$

$$= e^{\lambda(x-t)}e^{x^2-t^2}$$

$$R(x,t,-1) = e^{-(x-t)}e^{x^2-t^2}$$

$$u(x) = e^{x^2} - \int_0^x e^{-(x-t)}e^{x^2-t^2}.e^{t^2} dt$$

$$= e^{x^2} - \int_0^x e^{-(x-t)}.e^{x^2} dt$$

$$= e^{x^2} - \int_0^x e^{-x+t+x^2} dt$$

$$= e^{x^2} - \int_0^x e^{-x+x^2}.e^{x^2} dt$$

$$= e^{x^2} - \int_0^x e^{-x+x^2}.e^{t} dt$$

$$= e^{x^2} - e^{x^2-x}\int_0^x e^t dt$$

$$= e^{x^2} - e^{x^2-x}[e^x - 1]$$

$$= e^{x^2} - e^{x^2} + e^{x^2-x}$$

$$u(x) = e^{x^2-x}$$

the convergence of Numann series and the solution can be proved by assuming the following series for the solution $y(x)$:

$$y(x) = y_0(x) + \lambda y_1(x) + \lambda^2 y_2(x) + \ldots, \tag{I.49}$$

and substituting this series in (I.47) to obtain:

$$y(x) = f(x) + \lambda \int_a^x k(x,t) y_0(t) dt + \lambda^2 \int_a^x k(x,t) y_1(t) dt + ..., \tag{I.50}$$

Equating the coefficient of both sides for λ of the same power, one obtains:

$$\begin{cases} y_0(x) = f(x) \\ y_1(x) = \int_a^x k(x,t) y_0(t) dt \\ \vdots \\ y_n(x) = \int_a^x k(x,t) y_{n-1}(t) dt, \end{cases} \tag{I.51}$$

thus:

$$\begin{aligned} y_1(x) &= \int_a^x k(x,t) f(t) dt, \\ y_2(x) &= \int_a^x k(x,z) \left\{ \int_a^z k(x,z) f(t) dt \right\} dz, \\ &= \int_a^x f(t) \{ \int_t^x k(x,z) k(z,t) dz \} dt, \\ &= \int_a^x f(t) k_2(x,t) dt, \end{aligned}$$

where:

$$k_2(x,t) = \int_x^t k(x,z) k_1(z,t) dz$$

and so on, the general term for the iterated kernel can be obtained as:

$$k_{n+1}(x,y) = \int_x^t k_n(x,t)\, k(x,y)\, k_n(y,t)\, dt$$

Example I.3.2.2

find the solvent of the integral equation of volterra with kernel:

Solution.

$$k(x,t) = e^{x-t}$$

we have

$$k_1(x,t) = k(x,t) = e^{x-t}$$

$$k_2(x,t) = \int_t^x k(x,s)k(s,t)ds$$

$$= \int_t^x e^{x-t}ds = e^{x-t}\int_t^x 1ds$$

$$k_2 = e^{x-t}(x-t)$$

$$k_3(x,t) = \int_t^x k(x,s)k_2(s,t)ds$$

$$= \int_t^x e^{x-s}e^{s-t}(s-t)ds$$

$$= \int_t^x e^{x-t}(s-t)ds$$

$$= e^{x-t}\int_t^x (s-t)ds$$

$$= e^{x-t}[\frac{s^2}{2} - ts]_t^x$$

$$= e^{x-t}(\frac{x^2}{2} - xt - \frac{t^2}{2} + t^2)$$

$$= e^{x-t}(\frac{x^2}{2} - xt + \frac{t^2}{2})$$

$$k_3(x,t) = \frac{1}{2}e^{x-t}(x-t)^2$$

$$k_4(x,t) = \frac{1}{2}\int_t^x e^{x-s}e^{s-t}(s-t)^2ds$$

$$= \frac{1}{2}\int_t^x e^{x-t}(s-t)^2ds$$

$$= \frac{1}{2}e^{x-t}\int_t^x (s-t)^2ds$$

$$= \frac{1}{2}e^{x-t}\frac{(s-t)^3}{3}|_t^x$$

$$= \frac{1}{2}e^{x-t}\frac{(x-t)^3}{3}$$

There is a method for solving the Volterra integral equation of the second kind, starting by sub-

stituting initially a zero for $y(x)$ in the integral (with λ as a constant number, suppose $\lambda = 1$) to obtain a first approximation $y_1(x)$.

$$y_1(x) = f(x) + \int_0^x k(x,t)y(0)dt = f(x), \tag{I.52}$$

Then, we substitute $y_1(x)$ for $y(x)$ again into the integral of the right member of the equation to obtain a second approximation $y_2(x)$:

$$y_2(x) = f(x) \int_0^x k(x,t)y_1(t)dt, \tag{I.53}$$

Continuing this process, we obtain the n^{th} approximation:

$$y_n(x) = f(x) \int_0^x k(x,t)y_{n-1}(t)dt \quad n = 1,2,\dots, \tag{I.54}$$

This method can also be used for Fredholm integral equation of the second kind, and it turns out that if $f(x)$ is a continuous function for $0 \leq x \leq a$ and if $k(x,t)$ is also continuous for $0 \leq x \leq a$ and $0 \leq t \leq x$, then the sequence $y_n(x)$ will converge to the solution $y(x)$ as $n \to \infty$ [8].

Example I.3.2.3

Use the successive approximation method to solve the following Fredholm integral equation

$$u_{n+1}(x) = x + e^x - \int_0^1 xtu_n(t)dt \quad x \in \mathrm{N}$$

Solution.*on pose, $u_0(x) = 0$*

$$u_1(x) = x + e^x - \int_0^1 xtu_0(t)dt = x + e^x$$

$$u_2(x) = x + e^x - \int_0^1 xtu_1(t)dt$$

$$u_2(x) = x + e^x - x\int_0^1 t(e^t)dt$$

$$= x + e^x - x(\frac{1}{3}t^3|_0^1 + 1)$$

$$= x + e^x\frac{1}{3}x - x$$

$$u_2(x) = e^x\frac{1}{3}x$$

$$u_3(x) = x + e^x - \int_0^1 xt(e^t - \frac{1}{3}t)dt$$

$$= x + e^x - x\int_0^1 te^t - \frac{1}{3}t^2 dt$$

$$= x + e^x - x(1 - \frac{1}{9}) = x + e^x - \frac{8}{9}x$$

$$u_3(x) = e^x + \frac{x}{9}$$

$$u_4(x) = x + e^x - \int_0^1 xt(e^t - \frac{t}{9})dt$$

$$= x + e^x - x\int_0^1 t(e^t) + \frac{1}{9}t^2 dt$$

$$= x + e^x - x\left[te^t - e^t + \frac{1}{27}t^3\right]_0^1$$

$$= x + e^x - x\left[\frac{1}{27} + 1\right]$$

$$u_4(x) = e^x - \frac{x}{27}$$

on :

$$u_n(x) = e^x + \frac{(-1)^{n-1}}{3^{n-1}}x, \quad n \in \mathbb{N}*$$

$$u_n(x) = \lim_{n\to\infty}(\frac{(-1)^{n-1}}{3^{n-1}}x)$$

$$= e^x + \lim \frac{(-1)^{n-1}}{3^{n-1}}x$$

$$u(x) = e^x$$

I.3.3 Laplace Transform

The Laplace transform $\mathcal{L}(f) = \mathcal{L}\{f(s);p\}$ of a function $f(s)$ is defined as

$$\tilde{f}(s) = \int_0^\infty e^{-ps}f(s)ds \quad , \tag{I.55}$$

The inverse $\mathcal{L}^{-1}(F) = \mathcal{L}\{F(p);s\}$ is

$$\mathcal{L}^{-1}(F) = f(s) = \frac{1}{2\pi i}\int_{y-i\infty}^{y+i\infty} F(p)e^{ps}dp, \tag{I.56}$$

This transformation is also linear because

$$\mathcal{L}(af+bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$$

for any two constants a and b.

Following are some of the basic properties of the Laplace transform [6]:

1. *If a is a constant,*
$$\mathcal{L}\{e^{ps} f(s) : p\} = F(p-a), \tag{I.57}$$
2. *If a is a constant,*
$$\mathcal{L}\{f(as) : p\} = \frac{1}{a}\mathcal{L}\{f(s); \frac{p}{a}\}, \tag{I.58}$$
3. *The transforms of the derivatives are obtained as follows:*
$$\mathcal{L}\{f'(s)\} = p\mathcal{L}\{f(s) : p\} - f(0), \tag{I.59}$$
and more generally
$$\mathcal{L}\{f^{(k)}(s)\} = p^k \mathcal{L}\{f(s); p\} - p^{k-1} f(0) - \ldots - f^{(k-1)(0)}, \tag{I.60}$$
Also,
$$\frac{d}{dp}F(p) = -\mathcal{L}\{sf(s); p\}, \tag{I.61}$$
4. *If*
$$h(s) = \int_0^x f(x)dx, \tag{I.62}$$
then
$$\mathcal{L}\{h(s); p\} = H(p) = \frac{F(p)}{p}, \tag{I.63}$$
5. *For the convolution integral*
$$h(s) = \int_0^s f(x)g(s-x)dx = \int_0^s g(x)f(s-x)dx, \tag{I.64}$$
we have
$$\mathcal{L}\{h(s); p\} = \mathcal{L}\{f(s); p\}\mathcal{L}\{g(s); p\}, \tag{I.65}$$

or

$$H(p) = F(p)G(p), \tag{I.66}$$

These transforms are especially effective in solving the integral equations with difference kernels, that is, equations with kernels of the form

$$k(s,t) = k(s-t), \tag{I.67}$$

We demonstrate it for the Volterra integral equation of the second kind,

$$g(s) = f(s) + \int_0^s k(s-t)g(t)dt, \tag{I.68}$$

Applying the Laplace transform to both sides of this equation and using (I.66)*, we obtain*

$$G(p) = F(p) + K(p)G(p),$$

or

$$G = \frac{F(p)}{1-K(p)}, \tag{I.69}$$

Inverting this equation, we find the solution $g(s)$*.*

We can also evaluate the resolvent kernel $\gamma(s,t)$ *as defined in* (I.38) *. It is easily proved that if the kernel* $k(s,t)$ *is a difference kernel, then so is the resolvent kernel. Accordingly, the solution of the integral equation* (I.68) *is*

$$g(s) = f(s) + \int_0^b \Gamma(s-t)f(t)dt, \tag{I.70}$$

Taking the Laplace transform of both sides of (I.70)*, we have*

$$G(p) = F(p) + \Omega F(p), \tag{I.71}$$

where

$$\Omega(p) = \mathcal{L}[\Gamma(s);p].$$

From (I.69)and (I.71), it follows that

$$\frac{F(p)}{1-K(p)} = F(p)\left[1+\Omega(p)\right],$$

or

$$\Omega(p) = \frac{K(p)}{1-K(p)},$$

which when inverted yields $\Gamma(s-t)$. *Finally, we substitute this value in* (I.70) *and obtain the solution* $g(s)$.

Example I.3.3.1

Solve the integral equation using the Laplace transform:

$$\int_0^s e^{x-t}u(t)dt = xe^x$$

Solution.

$$\int_0^s e^{x-t}u(t)dt = xe^x$$

$$\mathcal{L}(\int_0^s e^{x-t}u(t)dt) = \mathcal{L}(xe^x)$$

$$\mathcal{L}(e^x)\ \mathcal{L}(u(t) = \mathcal{L}(xe^x)$$

we put

$$\mathcal{L}(u(t) = \phi(p)$$

and we know that

$$\mathcal{L}(e^x) = \frac{1}{(p-1)^2}, \mathcal{L}(xe^x) = \frac{1}{p-1}$$

$$\frac{1}{p-1}\phi(p) = \frac{1}{(p-1)^2}$$

$$\Rightarrow \phi(p) = \frac{1}{p-1}$$

$$u(x) = \mathcal{L}^{-1}\phi(p) = \mathcal{L}^{-1}\left(\frac{1}{p-1}\right)$$

$$u(x) = e^x$$

CHAPTER II

SINGULAR INTEGRAL EQUATIONS

II.1 Singular integral equations

In this chapter we shall discuss the integral equations [6]

$$f(s)=\int_a^b K(s,t)g(t)\,dt, \tag{II.1}$$

$$g(s)=f(s)+\int_a^b K(s,t)g(t)\,dt, \tag{II.2}$$

$$f(s)=\int_a^b K(s,t)g(t)\,dt, \tag{II.3}$$

$$g(s)=f(s)+\int_a^b K(s,t)g(t)\,dt, \tag{II.4}$$

The kernel $K(s,t)$ and the function $f(s)$ are given while $g(s)$ is to be evaluated. Equations (II.1)and (II.2) are Fredholm integral equations of the first and second kind, respectively. Equations (II.3) and (II.4) are called Volterra integral equations of the first and second kind, respectively. If the domain of definition of the kernel is infinite, or if the kernel has a singularity within its domain of definition, the integral equation is said to be singular.

Examples II.1.1

The first type of singular integral equation :

$$u(x)=2x+6\int_0^{\infty}\sin(x-t)u(t)dt$$

$$u(x)=x+\frac{1}{3}\int_{-\infty}^{0}\cos(x+t)u(t)dt$$

$$u(x)=1+x^2+\frac{1}{6}\int_{-\infty}^{+\infty}(x+t)u(t)dt$$

example of the seconde kind of singular integral equation :

$$x^2=\int_0^x\frac{1}{\sqrt{x-t}}u(t)dt$$

$$x=\int_0^x\frac{1}{(x-t)^{\alpha}}u(t)dt \quad 0<\alpha<1$$

$$u(x)=1-2\sqrt{x}-\int_0^x\frac{1}{\sqrt{x-t}}u(t)dt$$

where thee singular behavior in this kind of equation has resulted from the kernel $k(x.t)$ becoming *infinite ast* $\to x$

II.2 Cauchy Principal Value for Integrals

We found that the limit [6] :

$$\int_a^b g(s)\,ds = \lim_{\substack{\varepsilon\to 0\\ \eta\to 0}} \left[\int_a^{c-\varepsilon} g(s)\,ds + \int_{c+\eta}^{b} g(s)\,dS\right], \tag{II.5}$$

if it exists, is called the improper integral of the function $g(s)$ *in the range* (a,b)*. Here it is implied that* η *and* ε *tend to zero independently. But it may happen that the limit* (II.5)*does not exist when* η *and* ε *tend to zero independently of each other, but it exists if* η *and* ε *are related. The classic example is the function* $f(s) = \frac{1}{s-c}$, $a < c < b$*; the limit* (II.5) *in this case is*

$$\int_a^{c-\varepsilon} \frac{ds}{s-c} + \int_{c+\eta}^{b} \frac{ds}{s-c} = \ln\frac{b-c}{c-a} + \ln\left(\frac{\varepsilon}{\eta}\right).$$

If η *and* ε *tend to zero independently of each other, then the quantity* $\log(\varepsilon/\eta)$ *will vary arbitrarily. However, if* η *and* ε *are related, then the above limit exists. In the special case* $\eta = \varepsilon$*, this limit is*

$$p.v. \int_a^b \frac{ds}{s-c} = \ln\left(\frac{b-c}{a-c}\right), \tag{II.6}$$

and is called the Cauchy principal value or Cauchy principal integral.

The same definition applies to a general function $g(s)$*. The Cauchy principal value of the integral of a function* $g(s)$ *that becomes infinite at an interior point* $z = c$ *of the range of integration* (a,b) *is the limit*

$$\lim_{\varepsilon\to 0}\left(\int_a^{c-\varepsilon} g(s)\,ds + \int_{c+\varepsilon}^{b} g(s)\,dS\right), \tag{II.7}$$

where $0 < e \leq \min(c-a, b-c)$*. As we have explained, we denote this limit as*

$$p.v. \int_a^b g(s)\,ds,$$

A similar definition for the Cauchy principal value is given for integrals with infinite ranges of integration. For instance, the limit

$$\int_{-\infty}^{\infty} g(s)\,ds = \lim_{\substack{A\to\infty\\ B\to\infty}} \int_{-A}^{B} g(s)\,ds, \tag{II.8}$$

may not exist when A and B tend to infinity independently of each other, but the limit may exist when $A = B$. This limit,

$$\lim_{A\to\infty}\int_{-A}^{A} g(s)\,ds, \tag{II.9}$$

is called the Cauchy principal value. The limits (II.7) and (II.9) are also called singular integrals.

II.3 Expansion of the Kernel In $|x-y|$

The functions $T_n(\cos\theta) = \cos n\theta$ are called Chebyshev polynomials of the first kind. They form a complete orthogonal set in the interval $[-1,1]$ with respect to the weight $(1-x^2)^{-1/2}$, namely [6],

$$\int_{-1}^{1}\frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = \begin{cases}\pi & n=m=0,\\ \pi/2 & n=m\neq 0,\\ 0 & n\neq m.\end{cases} \tag{II.10}$$

while every function $f(x)$ defined in $[-1,1]$ and satisfying

$$\int_{-1}^{1}\frac{|f(x)|^2}{\sqrt{1-x^2}}\,dx < \infty, \tag{II.11}$$

admits an expansion of the form

$$f(x) = c_0 + c_1T_1(x) + c_2T_2(x) + \cdots, \tag{II.12}$$

where

$$C_0 = \frac{2}{\pi}\int_{-1}^{1}\frac{f(x)\,dx,}{\sqrt{1-x^2}} \quad C_n = \frac{2}{\pi}\int_{-1}^{1}\frac{f(x)T_n(x)\,dx}{\sqrt{1-x^2}}, \quad n\geq 1, \tag{II.13}$$

The expansion(II.13) converges in the mean square with respect to the weight $(1-x^2)^{-1/2}$. The kernel $\ln|x-y|$, $-1\leq x,y\leq 1$, admits the expansion

$$\ln|x-y| = -\ln 2 - \sum_{n=1}^{\infty}\frac{2}{n}T_n(x)T_n(y), \tag{II.14}$$

in terms of the Chebyshev polynomials.

Next, we substitute expansion (II.14) in the integral

$$\int_{-1}^{1}\ln|x-y|T_n(y)(1-y^2)^{-1/2}\,dy,$$

and use the orthogonality conditions (II.10) *. This yields the identities*

$$\int_{-1}^{1} \frac{\ln|x-y|\,dy}{\sqrt{1-y^2}} = -\pi \ln 2,, \quad -1<x<1, \tag{II.15}$$

$$\int_{-1}^{1} \frac{\ln|x-y| T_n(y)\,dy}{\sqrt{1-y^2}} = -\frac{\pi}{n} - T_n(x), \quad -1<x<1, \quad n \geq 1, \tag{II.16}$$

An immediate corollary that follows from (II.15) *is that the solution of the integral equation*

$$\int_{-1}^{1} \ln|x-y| g_0(y)\,dy = 1, \quad -1<x<1, \tag{II.17}$$

is

$$g_0(y) = \frac{-1}{\pi \ln 2 \sqrt{1-y^2}}, \quad -1<y<1, \tag{II.18}$$

Relations (II.15)*and* (II.16)*state that the operator*

$$L_0\{f(y);x\} = \int_{-1}^{1} \frac{\ln|x-y| f(y)}{\sqrt{1-y^2}}\,dy, \tag{II.19}$$

has the eigenvalues

$$\lambda_n = \begin{cases} -\pi \ln 2 & \textit{for } n=0, \\ -\pi/n & \textit{for } n \geq 1, \end{cases} \tag{II.20}$$

and the associated eigenfunctions are $T_n(x)$, $n \geq 0$. *Observe that* $|\lambda_1| \geq |\lambda_0| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$ *and thus sometimes it will be convenient to interchange the first two eigenvalues.*

Alternatively, we can use the symmetric operator

$$L_1\{f(y);x\} = \int_{-1}^{1} (1-x^2)^{-1/4}(1-y^2)^{-1/4} \ln|x-y| f(y)\,dy, \tag{II.21}$$

whose eigenvalues are still given by (II.20) *but whose eigenfunctions are now the functions* $(1-x^2)^{-1/4} T_n(x)$, $n \geq 0$, *that form a complete orthogonal set in* $[-1,1]$ *with the weight* $w(x)=1$.

The problem of finding the eigenvalues of the operator $\int_{-1}^{1} \ln|x-y| f(y)\,dy$ *is more difficult. The asymptotic behavior of the eigenvalues of this operator is considered in Section 7.10.*

The integral equation of the second kind

$$(1-\lambda L_0)(g) = g(x) - \lambda \int_{-1}^{1} \frac{\ln|x-y| g(y)\,dy}{\sqrt{1-y^2}} = f(x), \tag{II.22}$$

$-1 < x < 1$*can be readily solved by expanding* $f(x)$ *and* $g(x)$ *in a series of Chebyshev polynomials,*

$$f(x) = c_0 + c_1 T_1(x) + c_2 T_2(x) + \cdots , \tag{II.23}$$

$$g(x) = d_0 + d_1 T_1(x) + d_2 T_2(x) + \cdots , \tag{II.24}$$

which when substituted in(II.22) *the integral equation immediately give the relations*

$$(1 + \lambda \pi \ln 2) d_0 = c_0, \tag{II.25}$$

$$(1 + \frac{\lambda \pi}{n}) d_n = c_n, \quad n \geq 1, \tag{II.26}$$

Therefore, if $\lambda \lambda_n \neq 1$*, for all* $n \in \mathbb{N}$*, the solution is unique and is given by*

$$g(x) = \int_{-1}^{1} R_\lambda(x,y) f(y)\, dy, \tag{II.27}$$

where

$$\sqrt{1-y^2} R_\lambda(x,y) = \frac{1}{\pi(1+\lambda\pi\ln 2)} + \sum_{n=1}^{\infty} \frac{2T_n(x)T_n(y)}{\pi(1+\frac{\lambda\pi}{n})}, \tag{II.28}$$

When $\lambda = 1/\lambda_n$*, the solution exists if and only if*

$$\int_{-1}^{1} \frac{f(x)T_n(x)\, dx}{\sqrt{1-y^2}} = 0, \tag{II.29}$$

and if that is the case, the solution is

$$g(x) = \int_{-1}^{1} R_{\lambda_n}(x,y) f(y)\, dy + cT_n(x), \quad -1 < x < 1, \tag{II.30}$$

where c is an arbitrary constant and where $R_{\lambda_n}(x,y)$ *is given by a series like that in* (II.28) *except that the n-th term is omitted.*

II.4 Eigenvalues and eigenfunctions

We consider the homogeneous Fredholm integral equation [15]

$$u(x) = \lambda \int_a^b K(x,t) u(t)\, dt, \tag{II.31}$$

for which $u(x) = 0$ is an obvious solution. This solution is taken as zero or trivial solution.

The values of the parameter λ for which Eq. (II.31) possesses nonzero solutions $[u(x) \neq 0]$ are the eigenvalues of Eq. (II.31) or of the kernel $K(x,t)$. Further, corresponding to such eigenvalues of λ, every non-zero solution of Eq.(II.31) is called an eigenfunction.

Remarks:

1. *Eigenvalues are also termed as characteristic values or characteristic numbers. Similarly, eigenfunctions are also called characteristic functions or fundamental functions.*
2. *$\lambda = 0$ is not considered as an eigenvalue, since it corresponds to $u(x) = 0$.*
3. *Corresponding to eigenvalue λ, $u(x)$ and $Cu(x)$ are eigenfunctions for arbitrary constant C.*
4. *The set of all characteristic numbers of an integral equation with a kernel $K(x,t)$ is called the spectrum of the kernel (or of the integral equation).*
5. *A homogeneous Fredholm integral equation may not possess any real eigenvalue or eigenfunction.*

II.5 Initial value problem and boundary value problem

Initial value problem

When an ordinary differential equation is solved under conditions which involve dependent variable and its derivative at the same value of its independent variable, then the problem u nder consideration is said to be an initial value problem [6].

For instance, consider the differential equation

$$y''' + xy'' + 2(x^2 - x)y = e^x - x,$$

with the conditions $y(0) = 1$, $y'(0) = -1$, $y''(0) = k$.

Boundary value problem

When an ordinary differential equation is solved under conditions which involve dependent variable and its derivative at two different values of the independent variable, then the problem under consideration is said to be a boundary value problem.

For example,

$$y'' + xy' + 2y = e^{-x}$$

with the conditions $y(0) = 0$, $y'(1) = -1$.

II.6 ABEL integral equation and its generalization

In this section, we present some Abel type integral equations and their solutions [11].

First, we consider the Abel integral equation given by

$$(\tilde{A}_u)phi(x) = \frac{1}{\pi}\frac{d}{dx}\left[\int_0^x \frac{f(t)}{(x-t)^{1/2}}dt\right], \tag{II.32}$$

where $f(0) = 0$*, and the operator* $\tilde{A}$ *may be regarded as the Abel integral operator.*

We can solve the integral equation (II.32)*for the class of functions whose Laplace transforms exist. The Laplace transforms of the functions* $u(t)$ *and* $f(x)$ *are defined by [16] and [5]:*

$$(u(p),F(p)) = \int_0^\infty (\breve{(}x),f(x))e^{-px}dx, Re\ p > \delta > 0, \tag{II.33}$$

where δ *is some positive number.*

The integral equation (II.32)*is sometimes called Abel integral equation of the first type. The second type Abel integral equation is*

$$\int_x^b \frac{u(x)}{(t-x)^{1/2}}dt = f(x), \quad 0 < x < b, \tag{II.34}$$

where $f(b) = 0$. *Its solution can easily be obtained as*

$$u(x) = -\frac{1}{\pi}\frac{d}{dx}\int_x^b \frac{f(t)}{(t-x)^{1/2}}dt, \tag{II.35}$$

A slight generalization of first type Abel integral equation is

$$\int_0^x \frac{u(x)}{(x-t)^\alpha}dt = f(x), \quad x > 0, \tag{II.36}$$

where $f(0) = 0$ *and* $0 < \alpha < 1$. *Its solution can be obtained, using the Laplace transform method* *or an obvious very elementary method, as*

$$u(x) = \frac{\sin(\pi\alpha)}{\pi}\frac{d}{dx}\left[\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt,\right] \quad x>0, \tag{II.37}$$

The solution of the second type Abel integral equation

$$\int_x^b \frac{u(x)}{(t-x)^{\alpha}}\,dt = f(x) \quad 0<x<b,$$

where $f(b)=0$ *and* $0<\alpha<1$*, is*

$$u(x) = -\frac{\sin\pi\alpha}{\pi}\frac{d}{dx}\left[\int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}\,dt,\right] \quad 0<x<b, \tag{II.38}$$

The most general form of first type Abel integral equation is

$$\int_a^x \frac{u(x)}{\{h(x)-h(t)\}^{\alpha}}\,dt = f(x) \quad a<x<b, \tag{II.39}$$

where $f(a)=0,\ 0<\alpha<1,$ *and* $h(x)$ *is a strictly monotonically increasing and differentiable function on* $[a,b]$ *with* $h'(x)\neq 0$ *on* $[a,b]$*. Its solution is*

$$u(x) = \frac{\sin\pi\alpha}{\pi}\frac{d}{dx}\left[\int_a^x \frac{f(t)h'(t)}{\{h(x)-h(t)\}^{1-\alpha}}\,dt\right], \quad a<x<b, \tag{II.40}$$

The most general form of second type Abel integral equation is

$$\int_x^b \frac{u(x)}{\{h(t)-h(x)\}^{1/2}}\,dt = f(x), \quad a<x<b, \tag{II.41}$$

where $f(b)=0,\ 0<\alpha<1,$ *and* $h(x)$ *is as in (e) above. Its solution is*

$$u(x) = -\frac{\sin(\pi\alpha)}{\pi}\frac{d}{dx}\int_x^b \frac{f(t)}{(h(t)-h(x))^{1-\alpha}}\,dt, \quad a<x<b, \tag{II.42}$$

A special case:

For the special case $h(x)=x^2,\ a=0,\ b=1,$ *and* $\alpha=\frac{1}{2}$*, the Abel integral equation of first kind* (II.39) *has the form*

$$(\mathcal{A}u)(x) = \int_0^x \frac{u(x)}{(x^2-t^2)^{1/2}}\,dt = f(x), \quad 0<x<1,(f(0)=0), \tag{II.43}$$

having the solution

$$u(x) = (\mathcal{A}^{-1}f)(x) = \frac{2}{\pi}\frac{d}{dx}\left[\int_0^x \frac{f(t)}{(x^2-t^2)^{1/2}}\,dt\right], \quad 0 < x < 1, \tag{II.44}$$

while the Abel integral equation of the second kind (II.41) *has the form*

$$\int_x^1 \frac{u(x)}{(t^2-x^2)^{1/2}}\,dt = f(x), \quad 0 < x < 1, (f(1) = 0), \tag{II.45}$$

having the solution

$$u(x) = -\frac{2}{\pi}\frac{d}{dx}\left[\int_x^1 \frac{f(t)}{(t^2-x^2)^{1/2}}\,dt\right], \quad 0 < x < 1, \tag{II.46}$$

II.6.1 Systems of Generalized Abel Integral Equations

The system of generalized Abel integral equations in two unknowns is of the form [17]:

$$\begin{aligned} f_1(x) &= \int_0^x (K_{11}(x,t)u(t) + K_{12}(x,t)v(t))\,dt, \\ f_2(x) &= \int_0^x (K_{21}(x,t)u(t) + K_{22}(x,t)v(t))\,dt, \end{aligned} \tag{II.47}$$

The kernels $K_{ij}(x,t)$, $1 \le i,j \le 2$, *and the functions* $f_i(x)$, $i = 1,2$, *are given real-valued functions. Recall that the kernels* K_{ij} *are singular kernels given by:*

$$K_{ij} = \frac{1}{(x-t)^{\alpha_{ij}}}, \quad 1 \le i,j \le 2, \tag{II.48}$$

For $\alpha_{ij} = \frac{1}{2}$, $1 \le i,j \le 2$, *the system is called a system of Abel integral equations. Abel's systems of three equations in three unknowns will be examined in detail in the next section.*

The system of Abel's generalized singular integral equations gives a solution if

$$\det\left(\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\right) \neq 0, \tag{II.49}$$

In what follows, we will apply the Laplace transform method to handle the system (II.47).

Taking the Laplace transform of both sides of the system (II.47) *gives the linear system in* $U(s)$

and $V(s)$*:*

$$\begin{cases} F_1(s) = K_{11}(s)U(s) + K_{12}(s)V(s), \\ F_2(s) = K_{21}(s)U(s) + K_{22}(s)V(s), \end{cases} \tag{II.50}$$

where

$$\begin{cases} U(s) = \mathcal{L}\{u(x)\}, \quad V(s) = \mathcal{L}\{v(x)\}, \\ F_i(s) = \mathcal{L}\{f_i(x)\}, \quad 1 \leq i \leq 2, \\ K_{ij}(s) = \mathcal{L}\{K_{ij}(x)\}, \quad 1 \leq i, j \leq 2, \end{cases} \tag{II.51}$$

Solving the system (II.50) *for* $U(s)$ *and* $V(s)$ *by using Cramer's rule gives:*

$$U(s) = \frac{\det \begin{vmatrix} F_1(s) & K_{12}(s) \\ F_2(s) & K_{22}(s) \end{vmatrix}}{\det \begin{vmatrix} K_{11}(s) & K_{12}(s) \\ K_{12}(s) & K_{22}(s) \end{vmatrix}}, V(s) = \frac{\det \begin{vmatrix} K_{11}(s) & F_1(s) \\ K_{21}(s) & F_2(s) \end{vmatrix}}{\det \begin{vmatrix} K_{11}(s) & K_{12}(s) \\ K_{21}(s) & K_{22}(s) \end{vmatrix}}, \tag{II.52}$$

Having determined $U(s)$ *and* $V(s)$*, the unique solution for* $u(x)$ *and* $v(x)$ *can be determined by using the inverse Laplace transform method.*
We will focus our study only on the specific case where

$$\alpha_{11} = \alpha_{22}, \quad \alpha_{12} = \alpha_{21}, \tag{II.53}$$

The other cases of α_{ij} *can be handled in a similar manner.*

Example II.6.1.1

Solve the system of singular integral equations by using the Laplace transform method:

$$\begin{cases} 2\sqrt{x}\left(1 + \frac{2}{3}x\right) + \frac{3}{2}x^{2/3}\left(1 - \frac{3}{5}x\right) = \int_0^x \left(\frac{1}{(x-t)^{1/2}}u(t) + \frac{1}{(x-t)^{1/3}}v(t)\right) dt, \\ 2\sqrt{x}\left(1 - \frac{2}{3}x\right) + \frac{3}{2}x^{2/3}\left(1 + \frac{3}{5}x\right) = \int_0^x \left(\frac{1}{(x-t)^{1/3}}u(t) + \frac{1}{(x-t)^{1/2}}v(t)\right) dt, \end{cases} \tag{II.54}$$

Taking the Laplace transform of both sides of each equation in (II.54) *gives:*

$$\begin{cases} \sqrt{\pi}s^{-5/2}(1+s) - \Gamma\left(\frac{2}{3}\right)s^{-8/3}(1-s) = \sqrt{\pi}s^{-1/2}U(s) + \Gamma\left(\frac{2}{3}\right)s^{-2/3}V(s), \\ -\sqrt{\pi}s^{-5/2}(1-s) - \Gamma\left(\frac{2}{3}\right)s^{-8/3}(1+s) = \Gamma\left(\frac{2}{3}\right)s^{-2/3}U(s) + \sqrt{\pi}s^{-1/2}V(s), \end{cases} \tag{II.55}$$

Solving this system of equations for $U(s)$ and $V(s)$ by using any computer symbolic system, such as Maple or Mathematica, gives:

$$U(s) = \frac{1}{s} + \frac{1}{s^2}, V(s) \quad = \frac{1}{s} - \frac{1}{s^2}, \tag{II.56}$$

By taking the inverse Laplace transform of both sides of each equation in (II.56), the exact solutions are given by:

$$(u(x), v(x)) = (1+x, 1-x), \tag{II.57}$$

II.6.2 Nonlinear Abel's Integral Equation

The standard form of the nonlinear Abel's integral equation is given by [17] :

$$f(x) = \int_0^x \frac{1}{\sqrt{(x-t)}} F(u(t))dt, \tag{II.58}$$

where the function $f(x)$ is a given real-valued function, and $F(u(x))$ is a nonlinear function of $u(x)$. Recall that the unknown function $u(x)$ occurs only inside the integral sign for the Abel's integral equation (II.58).

To determine a solution for the nonlinear Abel's integral equation (II.58), we first convert it to a linear Abel's integral equation of the form:

$$f(x) = \int_0^x \frac{1}{\sqrt{(x-t)}} v(t)dt, \tag{II.59}$$

by using the transformation:

$$v(x) = F(u(x)), \tag{II.60}$$

where $F(u(x))$ is invertible, i.e., $F^{-1}(u(x))$ exists. This, in turn, means that:

$$u(x) = F^{-1}(v(x)), \tag{II.61}$$

In this section, we will handle Equation (II.59) by using the Laplace transform method.

Example II.6.2.1

Solve the nonlinear Abel integral equation:

$$2\pi\sqrt{x} = \int_0^x \frac{1}{\sqrt{x-t}} u^2(t)dt, \tag{II.62}$$

Assume $u^2(x)$ is invertible. The transformation:

$$v(x) = u^2(x), \quad u(x) = \pm\sqrt{v(x)}, \tag{II.63}$$

carries (II.62) *into:*

$$2\pi\sqrt{x} = \int_0^x \frac{1}{\sqrt{x-t}} v(t)dt, \tag{II.64}$$

Substituting $f(x) = 2\pi\sqrt{x}$ into (**??**) *gives:*

$$v(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{2\pi\sqrt{t}}{\sqrt{x-t}}dt = \pi, \tag{II.65}$$

This in turn gives the solutions:

$$u(x) = \pm\sqrt{\pi}, \tag{II.66}$$

obtained upon using (II.63).

II.7 The Weakly Singular Volterra Equations

The weakly-singular Volterra-type integral equations of the second kind are given by [17] :

$$u(x) = f(x) + \int_0^x \frac{\beta}{\sqrt{x-t}} u(t)\,dt, \quad x \in [0,T], \tag{II.67}$$

and

$$u(x) = f(x) + \int_0^x \frac{\beta}{[g(x)-g(t)]^\alpha} u(t)\,dt, \quad 0 < \alpha < 1, \quad x \in [0,T], \tag{II.68}$$

where β is a constant. Equation (II.68) *is known as the generalized weakly singular Volterra equation.*

The weakly-singular and the generalized weakly-singular equations (II.67)*and* (II.68) *fall under the category of singular equations with singular kernels:*

$$\begin{gathered} K(x,t) = \frac{1}{\sqrt{x-t}}, \\ K(x,t) = \frac{1}{[g(x)-g(t)]^\alpha}, \end{gathered} \tag{II.69}$$

respectively. Notice that the kernel is called weakly singular.

Definition II.7.1

The weakly-singular Volterra-type integral equations of the second kind are given by:

$$u(x) = f(x) + \lambda \int_0^x \frac{\beta}{\sqrt{(x-t)}} u(t)\,dt, \quad x \in [0,T], \tag{II.70}$$

and

$$u(x) = f(x) + \lambda \int_0^x \frac{\beta}{[g(x)-g(t)]^\alpha} u(t)\,dt, \quad 0 < \alpha < 1, \quad x \in [0,T], \tag{II.71}$$

Example II.7.1

We consider the weakly-singular second-kind Volterra integral equation.

$$u(x) = x + \frac{4}{3}x^{3/2} \int_0^x \frac{1}{\sqrt{(x-t)}} u(t)\,dt, \quad I \in [0,2], \tag{II.72}$$

Proceeding as before, we se:

$$\begin{aligned} u_0(x) &= x + \frac{4}{3}x^{3/2}, \\ u_1(x) &= -\int_0^x \frac{t+\frac{4}{3}t^{3/2}}{\sqrt{(x-t)}} dt, \\ &= -\frac{4}{3}x^{3/2} - \frac{1}{2}\pi x^2, \end{aligned} \tag{II.73}$$

Cancelling the noise terms between the components $u_0(x)$ and $u_1(x)$, and verifying that the remaining term in $u_0(x)$ satisfies the equation (II.7)gives the exact solution

$$u(x) = x, \tag{II.74}$$

II.8 Nonlinear Weakly-Singular Volterra Integral Equations

The nonlinear weakly-singular Volterra integral equation of the second kind reads [18]

$$u(x) = f(x) + \int_0^x \frac{1}{\sqrt{x-t}} F(u(t))\,dt, \quad x \in [0,T], \tag{II.75}$$

and can be generalized to

$$u(x) = f(x) + \int_0^x \frac{1}{[g(x)-g(t)]^\alpha} F(u(t))\,dt, \quad 0 < \alpha < 1, \quad x \in [0,T], \tag{II.76}$$

where $F(u(t))$ is a nonlinear function of $u(t)$ such as $u^2(x)$ and $u^3(x)$, and the data function $f(x)$ is a given real-valued function. The unknown function $u(x)$ appears inside and outside the integral signs, a characteristic feature of a second-kind integral equation.

Definition II.8.1 *The weakly-singular Fredholm-type integral equation of the second space are given by*

$$u(x) = f(x) + \lambda \int_a^b M(x,t)K(x,t)u(t)\,dt, \quad x \in [0,T], \tag{II.77}$$

where $K(x,t)$ is the weakly singular kernel defined as

$$K(x;t) = \begin{cases} (x-t)^{-\alpha}, & 0 < \alpha < 1, \\ \log(x-t), \end{cases}$$

For example,

$$u(x) = e^{-x} + \int_0^1 \sqrt{x-t}\log(x-t)u(t)\,dt, \tag{II.78}$$

II.9 Nonlinear Weakly-Singular Fredholm Integral Equations

The nonlinear weakly-singular Fredholm integral equations are of the form [18]:

$$u(x) = f(x) + \int_0^1 \frac{1}{\sqrt{|x-t|}} F(u(t))\,dt, \quad x \in [0,1], \tag{II.79}$$

and its generalized form is given by

$$u(x) = f(x) + \int_0^1 \frac{1}{[|g(x)-g(t)|]^{\alpha}} F(u(t))\,dt, \quad 0 < \alpha < 1, \quad x \in [0,1], \tag{II.80}$$

where $F(u(t))$ is a nonlinear function of $u(t)$, such as $u^2(x)$, $u^3(x)$, $e^{u(x)}$, etc.

Example II.9.1

Consider the nonlinear weakly-singular Fredholm integral equation :

$$u(x) = x - \frac{2}{3(x\sqrt{x} + \sqrt{|x^3-1|})} + \int_0^1 \frac{u^2(t)}{\sqrt{|x^3-t^3|}}\,dt, \quad 0 < x < 1, \tag{II.81}$$

Remark: *Where the limit of integration $b = x$ then we call (II.77) Weakly Singular Volterra equation.*

II.10 Strong and Super Strong Singularities

1. *The integral equation will be regarded as being of singular type when the kernel under consideration takes the form [18]:*

$$k(x,y) = \frac{A(x,y)}{x-y}, \tag{II.82}$$

Where the numerator, $A(x,y)$ is a differentiable function of x and y. In this case, the following integral diverges:

$$\int_a^b k(x,y)\varphi(y)\,dy = \int_a^b \frac{A(x,y)}{x-y}\varphi(y)\,dy.$$

2. *A kernel is said to have a strong singularity if we write it in the form :*

$$k(x,y) = \frac{B(x,y)}{(x-y)^2}, \tag{II.83}$$

where $B(x,y)$ is a differentiable function of x and y.

II.11 Cauchy Singularity

Let D be a bounded and connected domain in the complex plane, then the Cauchy integral is given by the formula :

$$\frac{1}{\pi i}\int_{\partial D}\frac{u(t)}{(t-x)}\,dt = f(x), \quad t \in \mathbb{C}, \tag{II.84}$$

For example,

$$\frac{1}{i\pi}\int_{\partial(D(0;1))}\frac{u(t)}{t-x}\,dt = x^2, \tag{II.85}$$

CHAPTER III

SOLUTION OF SINGULAR INTEGRAL EQUATIONS USING COMPLEX ANALYSIS METHODS

III.1 Introductin

Solution of integral equations using analytical methods, especially, complex analysis methods could be considered as one of the most promising methods. These methods have its utility of either subtraction of the singularity from the integral equations from using elementary transformations. Such methods work also when the singularity occurs in the limits of integration.

III.2 Mathematical background

III.2.1 Basic concepts in complex analysis:

a. Isolated singular point:

A singular point z_0 is said to be isolated if, in addition, there is some neighborhood of z_0 through which u is analytic except at the point [3].

b. Residues:

Before that; when z_0 is an isolated singular point of u, then there is a positive number r_1, such that the function is analytic at each point z for which $0 < |z - z_0| < r_1$. In that domain, the function is represented by the Laurent series:

$$u(z) = \sum_{n=0}^{\infty} \left(a_n (z - z_0)^n + \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} \right) + ..., \tag{III.1}$$

In particular,

$$b_1 = \frac{1}{2\pi i} \int_c u(z) dz, \tag{III.2}$$

The complex number b_m, which is the coefficient of $\frac{1}{z - z_0}$ in the expansion (III.1), is called the residue of u at the isolated singular point z_0.

c. Poles:

We have seen that if a function u has an isolated singular point z_0, the function is represented by a Laurent series:

$$u(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \tag{III.3}$$

in a domain $0 < |z - z_0| < r_1$ centered at z_0. The portion of the series involving negative powers of $z - z_0$ is called the principal part of u at z_0.

If the principal part of u at z_0 contains at least one nonzero term but the number of such terms is finite, there exists a positive integer m such that $b_m \neq 0$ and $b_{m+1} = b_{m+2} = \ldots = 0$. That is, *the expansion* (III.3) *takes the form:*

$$u(z) = \sum_{n=0}^{\infty} \left(a_n (z - z_0)^n + \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \ldots\ldots + \frac{b_m}{(z - z_0)^m} \right), \tag{III.4}$$

in a domain $0 < |z - z_0| < r_1$ *centered at* z_0*. In this case, the isolated singular point* z_0 *is called "a pole of order m". A pole of order* $m = 1$ *is called a simple pole.*

III.2.2 The residue theorem:

The following theorem is a precise statement for evaluating the value of a given integral around contour C containing n singularities in terms of the residue of each of these points.

Theorem III.2.1.

Let C be a simple closed contour within and on which a function u is analytic except for a finite number of singular points $z_1, z_2, \ldots, z_n$ *lying interior to C. If* $B_1, B_2, \ldots, B_n$ *denote the residues of u at these points, then:*

$$\int_C u(z)\, dz = 2\pi i \,(B_1 + B_2 + \ldots + B_n)\,, \tag{III.5}$$

where C is described in the positive sense.

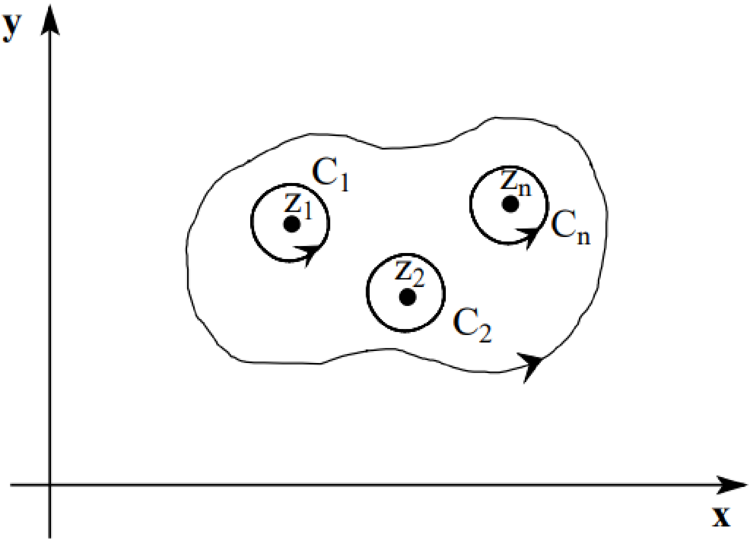

Figure III.1: III.2.2.1

The major difficulty in practice is the evaluation of the residues related to each singular point. Therefore, the next theorem helps in evaluating such residues.

Theorem III.2.2.

Given a function u and suppose that for some positive integer m the function $\phi(z) = (z - z_0)^m u(z)$ *can be defined at* z_0 *so that it is analytic there and* $\phi'(z_0) \neq 0$. *Then u has a pole of order m at* z_0. *Its residue there is given by:*

$$b_m = \frac{\phi^{m-1}(z_0)}{(m-1)!} \qquad if \quad m > 1, \tag{III.6}$$

and by formula

$$b_m = \phi(z_0) = \lim_{z \to z_0} (z - z_0) u(z) \qquad if \quad m = 1, \tag{III.7}$$

III.3 Singular integral equations

In this section, we are dealing with singular integrals, in which the singularity may occur either in the integrand function or in the limits of integration (i.e., one of the integration limits approaches ∞ *or* $-\infty$*). For illustration purposes, we will consider the most singular integrals that may occur as a quotient of two polynomials.*

III.3.1 Quotients of analytic functions

The most encountered type of singular integrals related to integral equations occurs when the integral is a quotient function and the denominator vanishes at some points. Such types of singular integral equations may take the form:

$$u(x) = f(x) + \int_C k(x,z) u(z)\, dz, \tag{III.8}$$

where:

$$k(x,z) = \frac{p(z)}{q(z)} S(x)$$

and p and q are both analytic at z_0 *and* $p(z_0) \neq 0$. *We first note that* z_0 *is an isolated point of k if and only if* $q(z_0) = 0$. *For if* $q(z_0) = 0$, *then* $q(z) \neq 0$ *at any other point in some neighborhood of* z_0; *this is because the zeros of an analytic function which is not identically zero are isolated.*

It follows that k *is analytic everywhere in that neighborhood of* z_0 *except at the point* z_0 *itself, and* z_0 *is therefore an isolated singular point of* k*. Conversely, if* z_0 *is an isolated singular point of* k*, then* $q(z_0) = 0$*. For, when* $q(z_0) \neq 0$*, it follows by continuity that* $q(z) \neq 0$ *throughout some neighborhood of* z_0*. It then follows that* k *is analytic at* z_0*, and this contradicts the fact that* z_0 *is an isolated singular point.*

The function $k(x,z)$ *given in equation* (III.8) *has a simple pole at* z_0 *if, in addition to the conditions given there,* $q(z_0) = 0$ *and* $q'(z_0) \neq 0$*. The residue of* k *at the simple pole* z_0 *is given by the formula:*

$$b_1 = \frac{p(z_0)}{q'(z_0)}, \tag{III.9}$$

Example III.3.1.

In order to solve the singular integral equation:

$$u(x) = x^2 \int_C \frac{z^2 x}{1+z} u(z)dz, \tag{III.10}$$

where

$$C = \{|z| \leq 1\}$$

We notice first that the kernel is given by:

$$k(x,z) = \frac{p(z)}{q(z)} x = \frac{z^2}{1+z} x, \quad x \in [-1,1]$$

and we can find the integrals by using the quotients method, by letting:

$$u(x) = a_0 + a_1 x + a_2 x^2, \tag{III.11}$$

hence:

$$u(z) = a_0 + a_1 z + a_2 z^2, \tag{III.12}$$

Substituting $u(x)$ *and* $u(z)$ *into the original equation, we get:*

$$a_0 + a_1 x + a_2 x^2 = x^2 + \int_c \left(\frac{z^2 x}{1+z} \right) (a_0 + a_1 z + a_2 z^2) dz$$

Now, we must find the integral:

$$\begin{aligned}\int_C \frac{a_0z^2+a_1z^3+a_2z^4}{1+z}xdz &\approx a_0\int_C \frac{z^2x}{1+z}dz + a_1\int_C \frac{z^3x}{1+z}dz + a_2\int_C \frac{z^4x}{1+z}dz \\ &= x\left[a_0\int_C \frac{z^2}{1+z}dz + a_1\int_C \frac{z^3}{1+z}dz + a_2\int_C \frac{z^4}{1+z}dz\right]\end{aligned}$$

Letting:

$$J_1 = \int_C \frac{z^2}{1+z}dz \approx \int_C \frac{p(z)}{q(z)}$$

Then the singular point is equals to $z_0=-1$*. Since* $p(z)=z^2$*, then* $p(z_0)=(-1)^2=1\neq 0$*, and hence* $q'(z)=1$ *and* $q'(z_0)\neq 0$*. The singularity is a simple pole with residue:*

$$b_1 = \frac{p(z_0)}{q'(z_0)} = 1$$

$$J_1 = 2\pi i b_1 = 2\pi i$$

$$J_2 = \int_C \frac{z^3}{1+z}dz$$

It is clear that $p(z)=z^3$*,* $p(z_0)=(-1)^3\neq 0$ *,* $q(z_0)=0$*,* $q'(z_0)=1\neq 1$*, then* $b_2=-1$ *and hence* $J_2=2\pi i b_2=-2\pi i$*.*

Similarly,

$$J_3 = \int_C \frac{z^4}{1+z}dz \approx 2\pi,$$

Now,

$$a_0+a_1x+a_2x^2 = x^2+a_0x2\pi i+a_1x2\pi i+a_2x2\pi i,$$

and hence:

$$a_0+[(1+2\pi i)a_1-a_02\pi i-a_22\pi i]x+a_2x^2 = x^2, \tag{III.13}$$

Therefore, we get:

$$a_0=0, a_2=1, \tag{III.14}$$

Substituting (III.14) *in* (III.13) *, we get:*

$$[(1+2\pi i)a_1-2\pi i]=0 \implies a_1 = \frac{2\pi i}{1+2\pi i},$$

Therefore,

$$a_1 = \frac{-4\pi^2}{1+4\pi^2}+\frac{2\pi^2}{1+4\pi^2}i,$$

and hence as a result, the solution of the singular integral equation is given by:

$$u(x) = \left(\frac{-4\pi^2 + 2\pi i}{1 + 4\pi^2}\right) x + x^2,$$

III.3.2 Subtraction of the Singularity:

Dealing with singular kernels $k(x,z)$ *is in general complicated by the fact that the position of the singularity may be a function both of x and z. We shall introduce a method in which the singularity can be weakened, by a suitable subtraction technique of the singularity [4].*

In this method, the kernel has a singularity at a fixed value of z for all x, it arises in the theory of quantum scattering problems, for example. We suppose that the singularity can be factored out of $k(x,z)$*; that is taking for definiteness the case of a pole of order* α*, we write:*

$$k(x,z) = \frac{k_0(x,z)}{(z-q)^{\alpha}}, \tag{III.15}$$

Where k_o *is assumed to be regular function and rewrite the integral operator as:*

$$\int_a^b k(x,z)u(z)dz = \int_a^b \frac{[k_0(x,z)u(z) - k_0(x,q)u(q)]}{(z-q)^{\alpha}} dz + k_0(x,q)u(q) \int_a^b \frac{dz}{(z-q)^{\alpha}}, \tag{III.16}$$

Then the following two cases will be arise:

- *If* $\alpha \leq 1$*, the first term is now regular at* $z = q$*, and the second can be evaluated exactly;*
- *If* $\alpha > 1$*, then neither integral exists in the Riemann sense and nor does the original integral operator. In that case, the integral operator will normally have been defined in some extended sense (such as a principal value integral); then* (III.16) *remains valid and weakens the form of the singularity. Note that the revised equation* (III.16) *includes the function value* $u(q)$.

If we consider a solution of (III.16) *with a rule containing points* $\xi_1, \xi_2, \ldots, \xi_n$*, we would normally write down the approximation equations at these N points.*

It is now necessary to write the equation also at the point $x = q$*,* $u(q)$ *appearing as the* $(N+1)$*th unknown in the equations (unless q happens to be a quadrature point).*

Example III.3.2.

To solve the singular integral equation:

$$u(x) = x^2 \int_0^1 \frac{xz}{(z-1)^{\frac{1}{2}}} u(z)dz,$$

using the subtraction method. Then according to this method, in terms of the singular point $z = 1$*, and the kernel* $k(x,z)$ *is given by:*

$$k(x,z) = \frac{xz}{(z-1)^{\frac{1}{2}}} = \frac{k_0(x,z)}{(z-q)^{\alpha}},$$

Since the integration $\int_0^1 \left[\frac{xz}{(z-1)^{\frac{1}{2}}}\right] u(z)dz$*, can be written as:*

$$\int_0^1 \frac{xz}{(z-1)^{\frac{1}{2}}} u(z)dz = \int_0^1 \frac{[(xz)u(z) - xqu(q)]}{(z-1)^{\frac{1}{2}}} dz + xqu(q) \int_0^1 \frac{dz}{(z-1)^{\frac{1}{2}}},$$

The first term is now regular at $z = q = 1$*, and the second can be evaluated exactly as:*

$$u(x) = x^2 - xf(1)(-2i) = x^2 - 2ixf(1),$$

where $f(1)$ *is a value boundary problem.*

III.3.3 Integral equations with an infinite range

An important application of the theory of residues is the evaluation of certain types of real definite integrals. The examples treated here and in the remainder of this chapter illustrate this application of the theory [3]. Recall from elementary calculus that an improper integral of the form:

$$\int_{-\infty}^{\infty} u(x)dx, \tag{III.17}$$

where u is continuous for all x, is said to converge and have the value:

$$\lim_{R\to\infty} \int_{-R}^{0} u(x)dx + \lim_{R\to\infty} \int_{0}^{R} u(x)dx, \tag{III.18}$$

When the individual limits exist. Another number associated with the integration (III.17) is also *useful. Namely, the "Cauchy principal value" of integral (III.17) is defined by the equation:*

$$P.V. \int_{-\infty}^{\infty} u(x)dx = \lim_{R\to\infty} \int_{-R}^{R} u(x)dx, \tag{III.19}$$

where P.V. refers to the principle value of the integral. Provided that the limit on the right hand side exists.

If the integral(III.17) converges, the value obtained is also the Cauchy principal value. On the other hand, when $u(x) = x$, for example; the Cauchy principal value of integral (III.17) is zero, whereas that integral does not converge according to definition (III.18). Suppose that u is an even function; that is, $u(-x) = u(x)$ for all real numbers x. Then if the Cauchy principal value of integral (III.17) exists, integral (III.17) actually converges.

For when u is even,

$$\int_{-R}^{0} u(x)dx = \int_{0}^{R} u(x)dx = \frac{1}{2}\int_{-R}^{R} u(x)dx,$$

and the existence of the limit in expression (3.19) thus implies the existence of both the limits in expression (III.18).

Suppose now that the integrand $u(x)$ function in integral (III.17) can be rewritten as $u(x) = p(x)/q(x)$ where $p(x)$ and $q(x)$ are real polynomials with no factors in common and $q(x)$ has no real zeros. If the degree of $q(x)$ is at least two greater than the degree of $p(x)$, then the integral converges. The value to which that integral converges can often be found quite easily by determining its Cauchy principal value by means of the theory of residues. To illustrate the method, let us evaluate the convergent integral:

$$\int_{0}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}dx, \tag{III.20}$$

Note that, the integral on the right-hand side represents an integration of the function:

$$u(z) = \frac{2z^2-1}{z^4+5z^2+4} = \frac{2z^2-i}{(z^2+1)(z^2+4)},$$

along the entire real axis.

This function has simple poles at the points $z = \pm i, \pm 2i$ and is analytic everywhere else. When $R > 2$, the singular points of u in the upper half-plane lie in the interior of the semicircular region bounded by the segment $-R \le x \le R, y = 0$ of the x-axis and the upper half C_R of the circle $|z| = R$ (See Fig III.3.3).

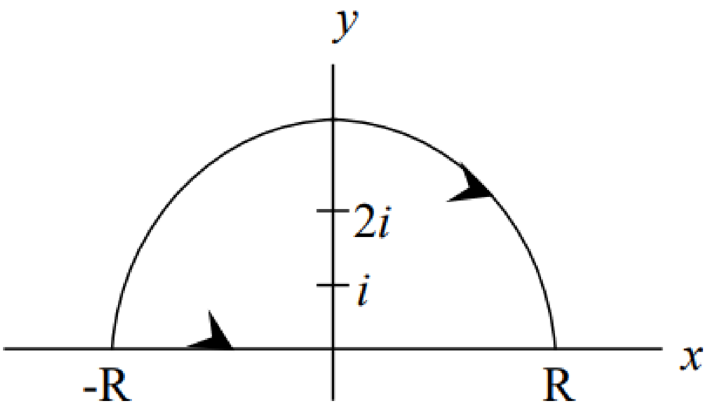

Figure III.2: III.3.3.1

Integrating u counterclockwise direction around the boundary of this semicircular region, we obtain that:

$$\int_{-R}^{R} u(x)dx + \int_{C_R} u(z)dz = 2\pi i(B_1 + B_2), \tag{III.21}$$

where B_1 is the residue of u at the point $z = i$ and B_2 is the residue of u at the point $z = 2i$. According to formula (III.20) *and using theorem (III.3.2), we have:*

$$B_1 = \lim_{z \to i}(z-i)u(z) = \frac{i}{2},$$

$$B_2 = \lim_{z \to 2i}(z-2i)u(z) = -\frac{3}{4}i,$$

Therefore, equation (III.21) *takes the form:*

$$\int_{-R}^{R} u(x)dx = \frac{\pi}{2} - \int_{C_R} u(z)dz, \tag{III.22}$$

The last equation is valid for all values of $R > 2$. We now show that the integral on the right in equation (III.22) *approaches zero as R tends to ∞. To do this, we observe that:*

$$|z^4 + 5z^2 + 4| = |z^2 + 1||z^2 + 4| \geq (|z|^2 - 1)(|z|^2 - 4),$$

Hence, when z is a point on C_R, we have:

$$|z^4 + 5z^2 + 4| \geq (R^2 - 1)(R^2 - 4),$$

Also, on C_R, we have $|2z^2-1| \leq 2|z|^2+1 = 2R^2+1$. Consequently:

$$\left|\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}dz\right| \leq \frac{2R^2+1}{(R^2-1)(R^2-4)}\pi R,$$

where πR is the length of C_R. The desired limit is now evident; that is:

$$\lim_{R\to\infty}\int_{C_R} u(z)dz = 0,$$

It thus follows from equation (III.22) that:

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{2x^2-1}{x^4+5x^2+4}dx = \frac{\pi}{2} \text{ or } P.V\int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}dx = \frac{\pi}{2},$$

and, since the integral here actually converges, we arrive at the result:

$$\int_{0}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}dx = \frac{\pi}{4},$$

We shall generalize the above idea (procedure) to treat the infinite range (singular) integral equation of the general form:

$$u(x) = f(x) + \int_{-\infty}^{\infty} k(x,t)f(t)dt,$$

We will illustrate this approach by the following illustrative example:

Example III.3.3.

Consider the singular integral equation:

$$f(s) = s^2 + \int_{0}^{\infty} \frac{x^2 S}{(x^2+1)(x^2+4)}u(x)dx, \tag{III.23}$$

Then using the collocation method, let:

$$f(s) = a_0 + a_1 s + a_2 s^2, \tag{III.24}$$

Substituting (III.24) in (III.23), we get:

$$a_0 + a_1 s + a_2 s^2 = s^2 + s\int_{0}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}(a_0 + a_1 x + a_2 x^2)dx,$$

For the integral operator, we have:

$$\begin{aligned}\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}(a_0+a_1x+a_2x^2)dx &= \int_0^\infty \frac{a_0x^2+a_1x^3+a_2x^4}{(x^2+1)(x^2+4)}dx \\ = a_0\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx + a_1\int_0^\infty \frac{x^3}{(x^2+1)(x^2+4)}dx &+ a_2\int_0^\infty \frac{x^4}{(x^2+1)(x^2+4)}dx,\end{aligned} \tag{III.25}$$

for simplicity, let:

$$J_1 = \int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx,$$

$$J_2 = \int_0^\infty \frac{x^3}{(x^2+1)(x^2+4)}dx,$$

$$J_3 = \int_0^\infty \frac{x^4}{(x^2+1)(x^2+4)}dx,$$

We can find the integrals J_1, J_2, and J_3, using the method of improper integrals:

Since

$$J_1 = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx$$

$$J_2 = \frac{1}{2}\int_{-\infty}^\infty \frac{x^3}{(x^2+1)(x^2+4)}dx,$$

$$J_3 = \frac{1}{2}\int_{-\infty}^\infty \frac{x^4}{(x^2+1)(x^2+4)}dx,$$

then, the above integrals are real integrals and we can reformulate these integrals as complex integrals, as follows:

$$J_1 = \frac{1}{2}\int_{-\infty}^\infty \frac{z^2}{(z^2+1)(z^2+4)}dz,$$

Similarly for J_2, J_3. The singular points related to J_1 are $z_0 = i$, $z_1 = -i$, $z_2 = 2i$, $z_3 = -2i$. And hence for the upper semicircle with line segment from $-R$ to R, we have:

$$\oint_C u(z) = \int_{-R}^R u(x)dx + \int_{C_R} u(z)dz = 2\pi i(B_1+B_2),$$

where B_1 is the residue of (i) and B_2 is the residue of $(2i)$. Then we can find the residues by the quotients method, as follows: Since

$$u(z) = \frac{z^2}{(z^2+1)(z^2+4)},$$

then $p(z) = z^2$ implies $p(z_0) = i^2 = -1$, *and* $q(z) = (z^2+1)(z^2+4)$ *implies* $q(z_0) = 0$*,*

$q(z)$ *has the first derivative:*

$$q'(z) = 2z(z^2+1) + 2z(z^2+4) = 2z^3+2z+2z^3+8z = 4z^3+10z = z(4z^2+10),$$

Hence,

$$q'(z_0) = i(-4+10) = 6i,$$

and therefore

$$b_1 = -\frac{1}{6i} = \frac{i}{6},$$

Now, similarly b_2 *at* $z_2 = 2i$*, equals to* $b_2 = \frac{i}{3}$ *Hence the first complex integration* J_1 *is given by:*

$$\oint_C u(z) = 2\pi i(b_1+b_2) = 2\pi i\left(\frac{i}{6} - \frac{i}{6}\right) = \frac{\pi}{3},$$

We now show that the integral:

$$\int_{-R}^{R} u(x)dx,$$

Approaches zero as R tends to ∞*. To do this, we observe that:*

$$|(z^2+1)(z^2+4)| = |(z^2+1)||(z^2+4)| \geq (|z|^2-1)(|z|^2-4),$$

Hence, when z is a point on C_R*, we have:*

$$|(z^2+1)||(z^2+4)| \geq (R^2-1)(R^2-4)$$

also on C_R*:*

$$|z^2| \leq |z|^2 = R^2,$$

Consequently:

$$\left|\int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}dz\right| \leq \frac{2R^2-1}{(R^2-1)(R^2-4)}\pi R,$$

where πR *is the length of* C_R*, then the desired limit is now evident; that is:*

$$\lim_{R\to\infty}\int_{C_R} u(z)dz = 0, \lim_{R\to\infty}\int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{\pi}{3}$$

or

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx = \frac{\pi}{3},$$

hence:

$$J_1 = \int_{0}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx = \frac{\pi}{6},$$

Similarly carrying out the integrals for J_2 and J_3, Since we have the same singular points for J_1 ($z_0 = i, z_2 = 2i$) and hence for b_1, b_2 of J_2 equals $\frac{-1}{6}$ and $\frac{2}{3}$. Since

$$\oint_C u(z) = 2\pi i - (\frac{-1}{6} + \frac{2}{3}) = \pi i,$$

Similarly, b_1, b_2 of J_3 equals to $\frac{-i}{6}$ and $\frac{4}{3}i$, Since

$$\oint_C u(z) = 2\pi i(\frac{-1}{6} + \frac{4i}{3}) = \frac{-7}{3}\pi,$$

and, in the above integral equation, the following integral$\int_{-R}^{R} u(x)dx$approaches zero as R tends to ∞. Then:

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x^3}{(x^2+1)(x^2+4)} dx = \pi i \quad or \int_{-\infty}^{\infty} \frac{x^3}{(x^2+1)(x^2+4)} dx = \pi i,$$

and hence:

$$J_2 = \int_{0}^{\infty} \frac{x^3}{(x^2+1)(x^2+4)} dx = \frac{\pi i}{2},$$

Similarly, $J_3 = \frac{-7}{6}\pi$. Now, substituting J_1, J_2, and J_3 in (III.25), we get:

$$S\left(a_0 \int_{0}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx + a_1 \int_{0}^{\infty} \frac{x^3}{(x^2+1)(x^2+4)} dx + a_2 \int_{0}^{\infty} \frac{x^4}{(x^2+1)(x^2+4)} dx\right)$$

$$= s\left(a_0 \frac{\pi}{6} + a_1 \frac{\pi i}{6} - a_2 \frac{7}{6}\pi\right)$$

Now, $a_0 + a_1 s + a_2 s^2 = s^2 + s\left(a_0 \frac{\pi}{6} + a_1 \frac{\pi i}{6} - a_2 \frac{7}{6}\pi\right), a_0 = 0, a_2 = 1$ then we get $a_1 = \frac{-7\pi}{6-3\pi i}$
Therefore the solution of integral equation (III.23) is given by:

$$u(x) = \frac{-7\pi}{(6-3\pi i)} x + x^2$$

GENERAL CONCLUSION

This thesis has three main objectives:

- ***The first goal:*** *studying integral equations .*
- ***The second goal:*** *Study integral equations individuality with some characteristics and propertie Main.*
- ***The third goal:*** *is to employ and use numerical analysis methods to find the analytical solution for singular integral equations whenever the kernel function is singular or the limits of integration are no specific.*

REFERENCES

[1] *C. A. Brebbia, J. H. Seinfeld, A. S. Cakmak, P. Silvester, C. S. Desai, G. Pinder, Fusion-Plasma Laser, R. McCrory, S. Orszag, S. Yip, et al. Lecture Notes in Engineering. Springer, 1989.*

[2] *J. W. Brown and R. V. Churchill. Complex variables and applications. McGraw-Hill, 2009.*

[3] *R. V. Churchill, J. W. Brown, R. F. Verhey. Complex Variables and Applications. McGraw-Hill, Inc., 1974.*

[4] *L. M. Delves and J. L. Mohamed. Computational methods for integral equations. CUP Archive, 1985.*

[5] *G. Doetsch. Introduction to the Theory and Application of the Laplace Transformation. Springer Science & Business Media, 2012.*

[6] *R. Estrada and R. P. Kanwal. Singular integral equations. Springer Science & Business Media, 2012.*

[7] *M. Guesba. Sur quelques équations intégrales non linéaires.Mémoire de Magister, UNIVERSITE KASDI MERBAH OUARGLA, 2012.*

[8] *F. B. Hildebrand. Methods of applied mathematics prentice-hall. New Jersey, 1965.*

[9] *A. J. Jerri. Introduction to integral equations with applications. John Wiley & Sons, 1985.*

[10] *B. Malgrange and R. Narasimhan. Lectures on the theory of functions of several complex variables. Vol. 13. Springer Berlin, 1984.*

[11] *B. N. Mandal and A. Chakrabarti. Applied singular integral equations. CRC Press, 2016.*

[12] *I. G. Petrovskii. Lectures on the theory of integral equations. Courier Corporation, 1996.*

[13] *W. Pogorzelski. Integral equations and their applications, 1966.*

[14] *M. D. Raisinghania. Integral equations and boundary value problems. S. Chand Publishing, 2007.*

[15] *D. C. Sharma and M. C. Goyal. Integral equations. PHI Learning Pvt. Ltd., 2017.*

[16] *I. N. Sneddon. The use of integral transforms, 1972.*

[17] *A.-M. Wazwaz. Linear and nonlinear integral equations. Vol. 639. Springer, 2011.*

[18] *A.-M. Wazwaz. First course in integral equations, A. World Scientific Publishing Company, 2015.*

[19] *H. J. Weber and G. B. Arfken. Elementary methods to solve linear integral equation , ISE. Elsevier, 1978.*