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On the numerical range and the numerical radius of operators in C*-algebras

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Dedication

We dedicate this humble work to:

Our dear parents, our first source of strength in life, who have never failed to support us and have made countless sacrifices throughout our journey. Mom and Dad, your love is beyond words, and our greatest wish is to make you proud, just as you have always lifted us up with your kindness and belief in us.

To our loyal friends and relatives who have always stood by our side with advice and support we will never forget your presence through every stage. Especially our university friends, with whom we shared the most beautiful moments during these five wonderful years.

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Our final prayer is that all praise belongs to Allah, the Lord of all worlds.

Contents

Notations

1. $\mathcal{H}$: Complex Hilbert space.
2. $\mathfrak{A}$: be a C^*-algebra .
3. $\|a\|$: The norm of a.
4. a^{-1} : The inverse operator of a.
5. a^* : The adjoint operator of a.
6. $\Re(a)$: The real part of a.
7. $\Im(a)$: The imaginary part of a.
8. $\sigma(a)$: The spectrum of a.
9. $\rho(a)$: The resolvent of a.
10. $r(a)$: Spectral radius of a.
11. $W(a)$: The numerical range of a.
12. $\omega(a)$: The numerical radius of a.
13. $\langle ., . \rangle$: The inner product of $\mathcal{H}$.
14. $\bar{A}$: The closure of A.
15. ∂M : The boundary of M.
16. $\oplus$: The sign of direct sum.
17. $\mathcal{B}(\mathcal{H})$: Banach algebra of all bounded linear operators on Hilbert space $\mathcal{H}$.
18. $\mathcal{I}(\mathcal{H})$: The set of invertible operators in $\mathcal{B}(\mathcal{H})$.
19. $a = T$: A bounded linear operator defined on $\mathcal{H}(T \in \mathcal{B}(\mathcal{H}) = \mathfrak{A})$.

Chapter 1

Preliminaries

1.1 Norms and Banach Spaces :

Definition 1.1.1 *Let E be a vector space over $\mathbb{F}$. A norm on E is a function $\|\cdot\| : E \to \mathbb{R}$ such that for all $x, y, \in E$ and $\alpha \in \mathbb{F}$,*

1. $\|x\| \geq 0$.
2. $\|x\| = 0$ *if and only if* $x = 0$.
3. $\|\alpha x\| = |\alpha|\|x\|$.
4. $\|x + y\| \leq \|x\| + \|y\|$.

And we denote $(E, \|.\|)$ is a normed space.

Theorem 1.1.1 *Let E be a vector space over $\mathbb{F}$ with norm $\|\cdot\|$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in E which converge to x, y in E respectively and let $\{\alpha_n\}$ be a sequence in $\mathbb{F}$ which converges to α in $\mathbb{F}$. Then:*

1. $|\|x\| - \|y\|| \leq \|x - y\|$.
2. $\lim_{n\to\infty} \|x_n\| = \|x\|$.
3. $\lim_{n\to\infty} (x_n + y_n) = x + y$.
4. $\lim_{n\to\infty} \alpha_n x_n = \alpha x$.

☛ **Proof**

1. By the triangle inequality, $\|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\|$ and so

   $$\|x\| - \|y\| \leq \|x - y\|$$

   Interchanging x and y we obtain $\|y\| - \|x\| \leq \|y - x\|$. However, as $\|x - y\| = \|(-1)(y - x)\| = \|y - x\|$ we have

   $$-\|x - y\| \leq \|x\| - \|y\| \leq \|x - y\|$$

   Hence $|\|x\| - \|y\|| \leq \|x - y\|$.

2. Since $\lim_{n\to\infty} x_n = x$ and $|\|x\| - \|x_n\|| \leq \|x - x_n\|$ for all $n \in \mathbb{N}$, it follows that $\lim_{n\to\infty} \|x_n\| = \|x\|$.

3. Since $\lim_{n\to\infty} x_n = x$, $\quad \lim_{n\to\infty} y_n = y$ and

   $$\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \leq \|x_n - x\| + \|y_n - y\|$$

   for all $n \in \mathbb{N}$, it follows that $\lim_{n\to\infty} (x_n + y_n) = x + y$.

4. Since $\{\alpha_n\}$ is convergent it is bounded, so there exists $K > 0$ such that $|\alpha_n| \leq K$ for all $n \in \mathbb{N}$. Also,

$$\begin{aligned} \|\alpha_n x_n - \alpha x\| &= \|\alpha_n (x_n - x) + (\alpha_n - \alpha) x\| \\ &\leq |\alpha_n| \|x_n - x\| + |\alpha_n - \alpha| \|x\| \\ &\leq K \|x_n - x\| + |\alpha_n - \alpha| \|x\| \end{aligned}$$

   for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} \alpha_n x_n = \alpha x$.

Definition 1.1.2 *Let E be a vector space and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on E. The norm $\|\cdot\|_2$ is equivalent to the norm $\|\cdot\|_1$ if there exists $M, m > 0$ such that for all $x \in X$*

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1.$$

Definition 1.1.3 (Banach Spaces) *Let $(E, \|.\|)$ be a normed space. Then $(E, \|.\|)$ is said to be a Banach space if every Cauchy sequence has a limit in E **i.e.** if $\|x_n - x_m\| \longrightarrow 0$ as $n \longrightarrow \infty$ and $m \longrightarrow \infty$, there exists $x \in X$ such that $\|x_n - x\| \longrightarrow 0$ as $n \longrightarrow \infty$.*

1.2 Inner Product Spaces, Hilbert Spaces :

Definition 1.2.1 *Let E be a vector space over $\mathbb{K}$. E is said to be an inner product space if there exists a map $\langle .,. \rangle : E \times E \longrightarrow \mathbb{K}$ holds the following :*

1. *$\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for all $x, y, z \in X$.*
2. *$\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ for all $x, y \in X$ and $\lambda \in \mathbb{K}$.*
3. *$\langle x, y\rangle = \overline{\langle y, x\rangle}$ for any $x, y \in X$.*
4. *$\langle x, x\rangle > 0$ for all $x \in X\backslash\{0\}$.*

And we denote $(X, \langle .,. \rangle)$ is an inner product space, and the map $\langle .,. \rangle$ is called an inner product.

Remark 1.2.1 *1. $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$ for all $x, y, z \in E$.*

2. $\langle x, \lambda y\rangle = \bar{\lambda}\langle x, y\rangle$ for all $x, y \in E$ and $\lambda \in \mathbb{K}$.

Theorem 1.2.1 (Cauchy-Schwarz inequality) *Let $(X, \langle .,. \rangle)$ be an inner product space. Then we have :*

$$|\langle x, y\rangle| \leq |\langle x, x\rangle|^{1/2} |\langle y, y\rangle|^{1/2} \quad \textit{for all } x, y \in X$$

Moreover if $|\langle x, y\rangle| = |\langle x, x\rangle|^{1/2}|\langle y, y\rangle|^{1/2}$, then x and y are linearly dependent.

Theorem 1.2.2 *Any inner product space $(X, \langle .,. \rangle)$ is a normed space where $\|x\| = |\langle x, x\rangle|^{1/2}$, and this norm is called the associated norm of the inner product.*

☛ **Proof**

Using the above properties of inner products we now verify that the given definition of $\|x\|$ satisfies all the defining properties of a norm :

1. $\|x\| = (x, x)^{1/2} \geq 0$.
2. $\|x\| = 0 \Longleftrightarrow (x, x)^{1/2} = 0 \Longleftrightarrow x = 0$.
3. $\|\alpha x\| = (\alpha x, \alpha x)^{1/2} = (\alpha\bar{\alpha})^{1/2}(x, x)^{1/2} = |\alpha|\|x\|$.
4. $\|x + y\|^2 = \|x\|^2 + 2\Re\mathrm{e}(x, y) + \|y\|^2$
   $\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$ (by Cauchy-Schwarz inequality)
   $= (\|x\| + \|y\|)^2$.

☛ **Remark**
We can write Cauchy-Schwarz inequality that way

$$|\langle x, y\rangle| \leq \|x\|\|y\| \quad \text{for all } x, y \in X$$

Proposition 1.2.1 *Let $(X, \langle ., .\rangle)$ be an inner product space and suppose that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are convergent sequences in X, with $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then $\lim_{n\to\infty}\langle x_n, y_n\rangle = \langle x, y\rangle$*

Theorem 1.2.3 *Let $(X, \langle ., .\rangle)$ be an inner product space. Then we have :*

1. $\forall x, y \in X : \|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$ *(Parallelogram law)*
2. *if X is a real vector space, then $\forall x, y \in X : \langle x, y\rangle = \frac{1}{4}\left\{\|x+y\|^2 - \|x-y\|^2\right\}$*
3. *if X is a complex vector space, then*

$$\forall x, y \in X : \langle x, y\rangle = \frac{1}{4}\left\{\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right\} \qquad \textbf{\textit{(Polarization identity)}}$$

Where $\|x\| = |\langle x, x\rangle|^{1/2}$ and i is the imaginary unit of complex numbers $(i^2 = -1)$.

Definition 1.2.2 *Let $\mathcal{H}$ be a vector space over $\mathbb{C}$. $\mathcal{H}$ is said to be a Hilbert space if it is an inner product space and $\mathcal{H}$ with associated norm is a Banach space, and we denote $(\mathcal{H}, \langle ., .\rangle)$ is a complex Hilbert space.*

Exempel 1.2.1

- ☛ The euclidean inner product in $\mathbb{C}^n (n \in \mathbb{N})$ is defined as follows :

$$\forall x, y \in \mathbb{C}^n : \quad \langle x, y\rangle = \sum_{k=1}^{k=n} x_k \overline{y_k}$$

  $\mathbb{C}^n$ with its euclidean inner product $(\mathbb{C}^n, \langle ., \rangle.)$ is a Hilbert space.

- ☛ Consider the following vector space over $\mathbb{C}$, $\quad \ell^2(\mathbb{C}) = \left\{x = (x_n)_{n\in\mathbb{N}} \subset \mathbb{C}; \quad \sum_{n=1}^{\infty} |x_n|^2 < \infty\right\}$.
  This space is a Hilbert space when it is endowed with the following inner product:

$$\forall x, y \in \ell^2 : \quad \langle x, y\rangle_{\ell^2} = \sum_{n=1}^{+\infty} x_n \overline{y_n}$$

- ☛ If $f, g \in L^2(X)$ then $f\bar{g} \in L^1(X)$ and the function $(\cdot, \cdot) : L^2(X) \times L^2(X) \to \mathbb{F}$ defined by $(f, g) = \int_X f\bar{g}d\mu$ is an inner product on $L^2(X)$. This inner product will be called the standard inner product on $L^2(X)$.

1.2.1 Riesz representation theorem :

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$, and $\varphi \in \mathcal{H}'$ be bounded linear functional on $\mathcal{H}$.

Theorem 1.2.4 *For every $\varphi \in \mathcal{H}'$, there exists a unique $f \in \mathcal{H}$, such that for every $v \in \mathcal{H}$ we hav*

$$\langle \varphi, v\rangle = \langle v \mid f\rangle \qquad \forall v \in \mathcal{H}$$

Moreover,

$$\|\varphi\|_{\mathcal{H}'} = \|f\|_{\mathcal{H}}$$

☛ **Proof**
Denote by

$$N(\varphi) = \{v \in \mathcal{H}; \langle \varphi, v\rangle = 0\}$$

the nul subspace of $\varphi.N(\varphi)$ is a closed subspace of $\mathcal{H}$. If $\langle\varphi, v\rangle = 0$ for every $v \in \mathcal{H}$, then it suffices to choose $f = 0$. Assume that $\varphi \neq 0$, then, $N(\varphi) \neq \mathcal{H}$, consequently, $(N(\varphi))^{\perp} \neq \{0\}$ and $\mathcal{H} = N(\varphi) \oplus (N(\varphi))^{\perp}$ and there exists $z \in (N(\varphi))^{\perp}$ such that $\langle\varphi, z\rangle \neq 0$, Clearly $z \neq 0$ and one can take

$$\langle\varphi, z\rangle = 1.$$

For every $v \in \mathcal{H}$ we have

$$\langle\varphi, v - \langle\varphi, v\rangle z\rangle = \langle\varphi, v\rangle - \langle\varphi, v\rangle\langle\varphi, z\rangle = 0,$$

Thus, $v - \langle\varphi, v\rangle z \in N(\varphi)$ and since $z \in (N(\varphi))^{\perp}$ one gets

$$\langle v - \langle\varphi, v\rangle z \mid \bar{z}\rangle = 0$$

Set $f = \frac{\bar{z}}{\|z\|^2}$, then

$$\langle v \mid \bar{z}\rangle = \langle\varphi, v\rangle\langle z \mid \bar{z}\rangle = \langle\varphi, v\rangle\|z\|^2.$$

Thus,

$$\langle\varphi, v\rangle = \frac{\langle v \mid \bar{z}\rangle}{\|z\|^2} = \left\langle v \,\middle|\, \frac{\bar{z}}{\|z\|^2}\right\rangle = \langle v \mid f\rangle.$$

This completes the proof of the first statement. On the other hand, let $\|v\| \leq 1$, then,

$$\|\varphi\|_{\mathcal{H}'} = \sup_{\|v\|\leq 1} |\langle\varphi, v\rangle| = \sup_{\|v\|\leq 1} |\langle f \mid v\rangle| \leq \sup_{\|v\|\leq 1} \|f\|\|v\| \leq \|f\|_{\mathcal{H}}.$$

Set $v = \frac{f}{\|f\|}$, then $\|v\| = 1$, we have

$$\|\varphi\| \geq |\langle\varphi, v\rangle| = \frac{|\langle\varphi, f\rangle|}{\|f\|} = \frac{\langle f \mid f\rangle}{\|f\|} = \|f\|,$$

Thus, $\|\varphi\|_{\mathcal{H}'} = \|f\|_{\mathcal{H}}$.

1.3 Definitions and basic properties in $\mathcal{B}(\mathcal{H})$:

Definition 1.3.1 *1. mapping T from $\mathcal{H}$ to $\mathcal{H}$ is said to be a linear operator if it satisfies the following:*
(1) Additive : $T(x+y) = Tx + Ty \quad \forall x, y \in \mathcal{H}$
(2) Homogeneous : $T(\lambda x) = \lambda Tx \quad \forall x \in \mathcal{H}$ *and* $\forall \lambda \in \mathbb{C}$

2. A linear operator T on $\mathcal{H}$ is said to be bounded if it satisfies:

$$\exists c > 0 \quad \forall x \in \mathcal{H} : \quad \|Tx\| \leqslant c\|x\|$$

Definition 1.3.2 *The set of all bounded linear operators on $\mathcal{H}$ is an unitary algebra over $\mathbb{C}$ denoted by $\mathcal{B}(\mathcal{H})$, where the addition, the external product and the product are defined as follows : Let $T, S \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$:*
(1) $\forall x \in \mathcal{H} \quad (T+S)x = Tx + Sx$
(2) $\forall x \in \mathcal{H} \quad (\lambda T)x = \lambda Tx$
(3) $\forall x \in \mathcal{H} \quad (TS)\, x = T(Sx)$

And the unitary element is I the identity ($Ix = x, \forall x \in \mathcal{H}$), moreover $\mathcal{B}(\mathcal{H})$ is a Banach algebra when we equip it with the following norm: Let $T \in \mathcal{B}(\mathcal{H})$, then

$$\|T\| = \inf\{c > 0 : \|Tx\| \leq c\|x\| \quad \textit{for all } x \in \mathcal{H}\}$$

Theorem 1.3.1 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\} = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in \mathcal{H}\backslash\{0\}\right\} = \sup\{|\langle Tx, y\rangle| : \|x\| = \|y\| = 1\}$$

Proposition 1.3.1 *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$. Then:*
(1) $\forall x \in \mathcal{H} \quad \|Tx\| \leq \|T\|\|x\|$
(2) $\|TS\| \leq \|T\|\|S\|$
(3) $\|T^n\| \leq \|T\|^n$

Definition 1.3.3 *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is said to be invertible if there exists $S \in \mathcal{B}(\mathcal{H})$ such that $TS = ST = I$, in which case S is the inverse of T and it is denoted by T^{-1}, and $\mathcal{I}(\mathcal{H})$ denotes to the set of all invertible operators in $\mathcal{B}(\mathcal{H})$.*

Lemma 1.3.1 *Let $T, S \in \mathcal{I}(\mathcal{H})$, and $n \in \mathbb{N}$. Then:*
(1) The inverse of T^{-1} is T.
(2) The inverse of ST is $T^{-1}S^{-1}$.
(3) The inverse of T^n is T^{-n} $\left(T^{-n} = (T^{-1})^n\right)$.

Lemma 1.3.2 (Banach's Isomorphism Theorem) *Let $T \in \mathcal{B}(\mathcal{H})$. If T is bijective, then T invertible. The result of Banach's Isomorphism Theorem is that in $\mathcal{B}(\mathcal{H})$, when T^{-1} exists, then it is bounded.*

Theorem 1.3.2 ((Neumann series)) *Let $T \in \mathcal{B}(\mathcal{H})$ such that $\|T\| < 1$. Then $I - T$ is invertible and the inverse given by*

$$(I-T)^{-1} = \sum_{n=0}^{\infty} T^n \quad \textit{and} \quad \left\|(I-T)^{-1}\right\| \leq \frac{1}{1 - \|T\|}$$

Theorem 1.3.3 *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent :*
(1) T is invertible.
(2) $R(T)$ is dense in $\mathcal{H}$ and there exists $\alpha > 0$ such that $\|Tx\| \geq \alpha\|x\|$ for all $x \in \mathcal{H}$.

☛ **Proof**
(1) $\Longrightarrow$ (2) Since T is invertible, then $R(T) = \mathcal{H}$ which means that $R(T)$ is dense in $\mathcal{H}$. And $\forall y \in \mathcal{H} \quad \|T^{-1}y\| \leq \|T^{-1}\| \|y\|$, let $x \in \mathcal{H}$ setting $y = Tx$, and since T is invertible $\|T^{-1}\| \neq 0$, we get :

$$\|x\| = \left\|T^{-1}Tx\right\| \leq \left\|T^{-1}\right\| \|Tx\| \Longrightarrow \|Tx\| \geq \left\|T^{-1}\right\|^{-1} \|x\| \quad \text{for all } x \in \mathcal{H}.$$

we have that: $\quad \forall x \in \mathcal{H}: \quad \|Tx\| \geq \|T^{-1}\|^{-1} \|x\|$
(2) $\Longrightarrow$ (1) Since $R(T)$ is dense in $\mathcal{H} \Longleftrightarrow \overline{R(T)} = \mathcal{H}$, and since there exists $\alpha > 0$ such that $\|Tx\| \geq \alpha\|x\|$ for all $x \in \mathcal{H}$, and Let $(y_n)_{n\in\mathbb{N}} \subset R(T)$ and $y \in \mathcal{H}$ such that $y_n \underset{n\to\infty}{\longrightarrow} y$, since $(y_n)_{n\in\mathbb{N}} \subset R(T)$, there exists $(x_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ such that $\forall n \in \mathbb{N} \quad y_n = Tx_n$, based on the assumption, we have: $\forall n, m \in \mathbb{N}$: $\|y_n - y_m\| = \|Tx_n - Tx_m\| \geq \alpha \|x_n - x_m\|$, since $(y_n)_{n\in\mathbb{N}}$ is convergent, then $\lim_{n,m\to\infty} \|y_n - y_m\| = 0$ $\Longrightarrow \lim_{n,m\to\infty} \|x_n - x_m\| = 0 \Longleftrightarrow (x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $(\mathcal{H}, \|\|)$ is a Banach space, then there exists $x \in \mathcal{H}$ such that $x_n \underset{n\to\infty}{\longrightarrow} x$ We know that $\forall n \in \mathbb{N} \quad y_n = Tx_n$, using the boundedness of T, we obtain:

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = T\left(\lim_{n\to\infty} x_n\right) = Tx \Longrightarrow y = Tx, \text{ then } y \in R(T).$$

Hence $\overline{R(T)} = R(T) \Longleftrightarrow R(T)$ is closed. $\Longrightarrow R(T)$ is closed, then $R(T) = \mathcal{H}$. On the other hand, let $x \in N(T) \Longleftrightarrow Tx = 0 \Longrightarrow 0 = \|Tx\| \geq \alpha\|x\| \Longrightarrow \|x\| = 0 \Longleftrightarrow x = 0$ Then $N(T) \subset \{0\} \Longrightarrow N(T) = \{0\}$ Hence T is bijective, we conclude that T is invertible.

1.3.1 The adjoint of a linear operator

Theorem 1.3.4 *Let $T \in \mathcal{B}(\mathcal{H})$. Then there exists a unique $T^* \in \mathcal{B}(\mathcal{H})$ such that*

$$\forall x, y \in \mathcal{H} \quad \langle Tx, y\rangle = \langle x, T^* y\rangle$$

Let $T \in \mathcal{B}(\mathcal{H})$. The operator T^* is called the adjoint of T.

☛ **Proof** Consider $f(x) = \langle Tx, y\rangle \quad \forall x \in \mathcal{H}$. First of all let's prove that $f \in \mathcal{H}'$ for every arbitrary $y \in \mathcal{H}$. f is linear : Let $x, z \in \mathcal{H}$ and $\lambda \in \mathbb{C}$: $\quad f(\lambda x+z) = \langle T(\lambda x+z), y\rangle = \langle \lambda Tx+Tz, y\rangle = \lambda\langle Tx, y\rangle + \langle Tz, y\rangle = \lambda f(x) + f(z)$ f is bounded for every arbitrary $y \in \mathcal{H}$: Let $x \in \mathcal{H}$, using the boundedness of T and Cauchy-Schwarz inequality, we obtain:

$$|f(x)| = |\langle Tx, y\rangle| \leq \|Tx\|\|y\| \leq \|T\|\|y\|\|x\|$$

Therefore

$$\exists c = \|T\|\|y\| \geq 0 \quad \forall x \in \mathcal{H} \quad |f(x)| \leq c\|x\|$$

Then $f \in \mathcal{H}'$ for every arbitrary $y \in \mathcal{H}$. Applying Riesz's representation theorem on f, we get For every arbitrary $y \in \mathcal{H}$, there exists a unique $z \in \mathcal{H}$ such that $f(x) = \langle Tx, y\rangle = \langle x, z\rangle$ We set for every $y \in \mathcal{H} \quad T^* y = z$ (the uniqueness of the element z for every y enables us to say T^* is well-defined) Hence

$$\forall x, y \in \mathcal{H} \quad \langle Tx, y\rangle = \langle x, T^* y\rangle$$

Now we need to prove that $T^* \in \mathcal{B}(\mathcal{H})$ Let $\lambda \in \mathbb{C}$ and $y_1, y_2 \in \mathcal{H}$, then $\langle Tx, y_1\rangle = \langle x, T^* y_1\rangle$ and $\langle Tx, y_2\rangle = \langle x, T^* y_2\rangle \quad$ for every $x \in \mathcal{H}$

$$\langle Tx, \lambda y_1 + y_2\rangle = \langle x, T^*(\lambda y_1 + y_2)\rangle \qquad \text{for every } x \in \mathcal{H}$$
$$\langle Tx, \lambda y_1 + y_2\rangle = \bar{\lambda}\langle Tx, y_1\rangle + \langle Tx, y_2\rangle = \bar{\lambda}\langle x, T^* y_1\rangle + \langle x, T^* y_2\rangle = \langle x, \lambda T^* y_1 + T^* y_2\rangle$$

Then $\forall x \in \mathcal{H} \quad \langle x, \lambda T^* y_1 + T^* y_2\rangle = \langle x, T^*(\lambda y_1 + y_2)\rangle$, which gives us

$$\forall x \in \mathcal{H} \quad \langle x, \lambda T^* y_1 + T^* y_2 - T^*(\lambda y_1 + y_2)\rangle = 0$$

Hence $\quad T^*(\lambda y_1 + y_2) = \lambda T^* y_1 + T^* y_2$, then T^* is linear. Let's prove that T^* is bounded, using the boundedness of T and Cauchy-Schwarz inequality we get

$$\forall y \in \mathcal{H} \quad \|T^* y\|^2 = \langle T^* y, T^* y\rangle = \langle TT^* y, y\rangle \leq \|TT^* y\|\,\|y\| \leq \|T\|\,\|T^* y\|\,\|y\|$$

Thus $\quad \forall y \in \mathcal{H} \quad \|T^* y\| \leq \|T\|\|y\|$, therefore T^* is bounded. Then $T^* \in \mathcal{B}(\mathcal{H})$. We just still need to prove the uniqueness of T^*. Assume that there exists $S \in \mathcal{B}(\mathcal{H})$ such that $\forall x, y \in \mathcal{H} \quad \langle Tx, y\rangle = \langle x, Sy\rangle$, then:

$$\forall x, y \in \mathcal{H}: \quad \langle x, T^* y\rangle = \langle x, Sy\rangle$$
$$\forall x, y \in \mathcal{H}: \quad \langle x, T^* y - Sy\rangle = 0$$
$$\forall y \in \mathcal{H}: \quad T^* y = Sy$$

Hence $T^* = S$, so T^* is unique.

Proposition 1.3.2 *Proposition 2.1. Let $T, S \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. Then:*
(1) $(T+S)^* = T^* + S^*$
(2) $(\lambda T)^* = \bar{\lambda} T^*$
(3) $(TS)^* = S^* T^*$
(4) $(T^*)^* = T$
(5) $\|T^*\| = \|T\|$
(6) $\|T^* T\| = \|T\|^2$
(7) The function $f : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ by $f(T) = T^*$ is bounded (continuous).
(8) $T = 0$ in and only if $T^* T = 0$. .
(9) Let $n \in \mathbb{N}$, then $(T^n)^* = (T^*)^n$.
(10) $N(T) = (R(T^*))^{\perp}$.
(11) $N(T^*) = (R(T))^{\perp}$.
(12) $N(T^*) = \{0\}$ if and only if $R(T)$ dense in $\mathcal{H}$.

Definition 1.3.4 *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is called:*

1. *Self-adjoint operator : $T^* = T$.*
2. *Normal operator : $TT^* = T^*T$.*
3. *Positive operator : $\forall x \in \mathcal{H} : \langle Tx, x\rangle \geq 0$, and we denote $T \geq 0$.(T is strictly positive operator if $\forall x \in \mathcal{H}\backslash\{0\} : \quad \langle Tx, x\rangle > 0$, and we denote $T > 0$)*
4. *Unitary operator : $TT^* = T^*T = I$.*
5. *Isometry operator : $T^*T = I$.*
6. *Projection operator : $T^2 = T$.*
7. *Orthogonal projection operator : $T^2 = T = T^*$.*

Theorem 1.3.5 *Let $T \in \mathcal{B}(\mathcal{H})$. Then: (1) T is self-adjoint if and only if $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$.*
(2) T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in \mathcal{H}$.
(3) T is unitary if and only if $\|Tx\| = \|T^*x\| = \|x\|$ for all $x \in \mathcal{H}$.

☛ **Proof** (1) Assume that T is self-adjoint, then $\forall x \in \mathcal{H} \langle Tx, x\rangle = \langle x, T^*x\rangle = \langle x, Tx\rangle$. Then $\langle Tx, x\rangle = \overline{\langle Tx, x\rangle}$, therefore $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$. Conversely, suppose that $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$, then $\langle Tx, x\rangle = \langle x, Tx\rangle = \langle T^*x, x\rangle \implies \langle (T - T^*)\, x, x\rangle = 0$ for all $x \in \mathcal{H}$. we get $T - T^* = 0$, hence T is self-adjoint.
(2) T is normal $\iff T^*T - TT^* = 0 \iff \forall x \in \mathcal{H} : \quad \langle (T^*T - TT^*)\, x, x\rangle = 0$

$$\begin{aligned}
&\iff \forall x \in \mathcal{H} : \quad \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle \\
&\iff \forall x \in \mathcal{H} : \quad \langle Tx, Tx\rangle = \langle T^*x, T^*x\rangle \\
&\iff \forall x \in \mathcal{H} : \quad \|Tx\|^2 = \|T^*x\|^2 \\
&\iff \forall x \in \mathcal{H} : \quad \|Tx\| = \|T^*x\|
\end{aligned}$$

(3) Assume that T is unitary $\iff TT^* = T^*T = I$, then T is normal $\iff \|Tx\| = \|T^*x\| \, \forall x \in \mathcal{H}$. So we just need to prove that $\forall x \in \mathcal{H} : \quad \|Tx\| = \|x\|$. $\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle = \langle x, x\rangle = \|x\|^2$, therefore $\|Tx\| = \|T^*x\| = \|x\|$ for all $x \in \mathcal{H}$. Conversely, suppose that $\|Tx\| = \|T^*x\| = \|x\|$ for all $x \in \mathcal{H}$, then $TT^* = T^*T$ (using (2)). So we just need to prove that $T^*T = I$, we have that $\|Tx\| = \|x\|$ for all $x \in \mathcal{H}$. Then: $\|Tx\|^2 = \langle Tx, Tx\rangle = \|x\|^2 = \langle x, x\rangle \implies \langle T^*Tx, x\rangle = \langle x, x\rangle \implies \langle (T^*T - I)\, x, x\rangle = 0$ for all $x \in \mathcal{H} \implies T^*T = I$. Thus T is unitary.

Theorem 1.3.6 *If $T \in \mathcal{B}(\mathcal{H})$ is self-adjoint, then $\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$.*

☛ **Proof**
Let $\|x\| = 1$, we know that $|\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq \|T\|\|x\|^2 = \|T\|$. Then

$$\sup_{\|x\|=1} |\langle Tx, x\rangle| \leq \|T\|$$

Setting $\beta_T = \sup_{\|x\|=1} |\langle Tx, x\rangle|$, then $\forall x \in \mathcal{H} : \quad |\langle Tx, x\rangle| \leq \beta_T\|x\|^2$. Since T is self-adjoint, we have $\Re\langle Tx, y\rangle = \frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle)$ for all $x, y \in \mathcal{H}$. Then:

$$\begin{aligned}
\forall x, y \in \mathcal{H} : \quad |\Re\langle Tx, y\rangle| &\leq \frac{1}{4}(|\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle|) \\
&\leq \frac{1}{4}\beta_T \left(\|x+y\|^2 + \|x-y\|^2\right) \\
&= \frac{1}{2}\beta_T \left(\|x\|^2 + \|y\|^2\right)
\end{aligned}$$

Set $y = \frac{1}{\|Tx\|}$ Tx for all $\|x\| = 1$ such that $\|Tx\| \neq 0$, we obtain:

$$\begin{aligned}
\tfrac{1}{\|Tx\|}|\Re\langle Tx, Tx\rangle| &\leq \beta_T \\
\|Tx\| &\leq \beta_T \\
\sup_{\|x\|=1} \|Tx\| &\leq \beta_T
\end{aligned}$$

Then

$$\|T\| \le \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

we get that $\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$.

1.3.2 Spectrum of a bounded operator

Definition 1.3.5 *The spectrum of the operator T is the set*

$$\sigma(T) := \mathbb{C} - \rho(T) = \{\lambda \in \mathbb{C}; (T - \lambda I) \text{ is not invertible } \}$$

The set $\sigma(T)$ is divided on three parts:

1. *Punctual spectrum*

$$\sigma_p(T) = \{\lambda \in \mathbb{C}; (T - \lambda I) \text{ is not injective } \}$$

2. *Continuous spectrum*

$$\sigma_c(T) = \left\{ \begin{array}{c} \lambda \in \mathbb{C}; (T - \lambda I) \text{ is injective} \\ \operatorname{Im}(T - \lambda I) \neq E \text{ and is dense in } X \end{array} \right\}.$$

3. *Residual spectrum*

$$\sigma_r(T) = \left\{ \begin{array}{c} \lambda \in \mathbb{C}; (T - \lambda I) \text{ is injective} \\ \operatorname{Im}(T - \lambda I) \neq E \text{ and is not dense in } X \end{array} \right\}.$$

Exempel 1.3.1

Let $\mu \in C$ and $T = \mu I$. We have $T - \lambda I = (\mu - \lambda)I$. Thus, $T - \lambda I$ is invertible if and only if $\lambda \neq \mu$, then $\sigma(T) = \{\mu\}$ et $\rho(T) = C - \{\mu\}$.

Theorem 1.3.7 *Let H be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$, then,*
1) If $|\lambda| > \|T\|, \lambda \notin \sigma(T)$.
2) $\sigma(T)$ is closed in C.

☛ **Proof**
1) If $|\lambda| > \|T\|$, then $\|\lambda^{-1}T\| < 1$ consequently, $I - \lambda^{-1}T$ is invertible, and $T - \lambda I = \frac{1}{\lambda}(\lambda^{-1}T - I)$ is also invertible, then $\lambda \notin \sigma(T)$.
2) Define $F : C \longrightarrow \mathcal{B}(\mathcal{H})$ by $F(\lambda) = T - \lambda I$. We have $\|F(\lambda) - F(\mu)\| = \|(\mu - \lambda)I\| = |\lambda - \mu|$, then F is continuous (it suffices to choose $\alpha = \varepsilon$ in the definition of continuity). Let C be the set of all non-invertible operators. It is a closed set because the set of all invertible operators is open. Consequently, $\sigma(T) = F^{-1}(\mathcal{C})$ is closed.

Remark 1.3.1 . *The spectrum of an operator T is a closed bounded set, then compact set included in C. It is in the circle of center at the origin and radius $\|T\|$.*

Remark 1.3.2 *Let $T \in \mathcal{B}(\mathcal{H})$ then*

$$\rho\left(T^*\right) = \{\bar{\lambda} \in \mathbb{C} : \lambda \in \rho(T)\},$$
$$\sigma\left(T^*\right) = \{\bar{\lambda} \in \mathbb{C} : \lambda \in \sigma(T)\}.$$

☛ **Proof**
If $\lambda \in \rho(T)$, then $T - \lambda I$ is invertible, consequently, $(T - \lambda I)^* = T^* - \bar{\lambda} I$ is invertible. Thus, $\bar{\lambda} \in \rho\left(T^*\right)$. Similarly, if $\bar{\lambda} \in \rho\left(T^*\right)$ then $\bar{\bar{\lambda}} = \lambda \in \rho(T)$. Therefore, $\lambda \in \rho(T) \Longleftrightarrow \bar{\lambda} \in \rho\left(T^*\right)$ which is equivalent to $\lambda \in \sigma(T) \Longleftrightarrow \bar{\lambda} \in \sigma\left(T^*\right)$.

Theorem 1.3.8 *Let H be a Hilbert space over C, and $T \in \mathcal{B}(\mathcal{H})$ then,*
1) For any polynomial p, we have $\sigma(p(T)) = p(\sigma(T)) = \{p(\lambda); \lambda \in \sigma(T)\}$.
2) If T is invertible $\sigma\left(T^{-1}\right) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$.

Proof

1) Let $q(z) = p(z) - p(\lambda)$, since $q(\lambda) = 0$, then $q(z) = (z-\lambda)r(z)$ and $q(T) = (T - \lambda I)r(T)$, where $r(z)$ is a polynomial.

If $\lambda \in \sigma(T)$ then $(T - \lambda I)$ is not invertible, consequently, $q(T) = p(T) - p(\lambda)I = (T - \lambda I)r(T)$ is not invertible, therefore, $p(\lambda) \in \sigma(p(T))$.

Reciprocally, let $\lambda \in \sigma(p(T))$ and define the polynomial $q(z) = p(z) - \lambda$. The polynomial $p(z)$ can be written $q(z) = c\,(z-\mu_1)\,(z-\mu_2)\cdots(z-\mu_n)$ for some $c \neq 0$ and $\mu_1, \mu_2, \cdots, \mu_n \in C$. Since $\lambda \in \sigma(p(T))$ then $q(T) = p(T) - \lambda I$ is not invertible, accordingly, there exists $1 \leq i \leq n$ such that $T - \mu_i I$ is not invertible. Therefore, $\mu_i \in \sigma(T)$. On the other hand, $q(\mu_i) = p(\mu_i) - \lambda = 0$, which gives $\lambda = p(\mu_i) \in P(\sigma(T))$.
2) Since T is invertible, then $0 \notin \sigma(T)$. Thus, every $\lambda \in \sigma(T)$ can be written $\lambda = \mu^{-1}$, and we have

$$T^{-1} - \mu I = -\mu T^{-1}(T - \lambda I)$$
$$T - \lambda I = -\lambda T\left(T^{-1} - \mu I\right)$$

where $-\mu T^{-1}$ and λT are invertibles. Consequently, $T^{-1} - \mu I$ is not invertible if and only if $T - \lambda I$ is not invertible. Thus,

$$\mu = \lambda^{-1} \in \sigma\left(T^{-1}\right) \Longleftrightarrow \lambda \in \sigma(T)$$

and $\sigma\left(T^{-1}\right) = \{\lambda^{-1} \in \mathbb{C} : \lambda \in \sigma(T)\}$.

1.3.3 Spectral radius and Numerical range and Numerical radius of a linear operator in $\mathcal{B}(\mathcal{H})$

Spectral radius

Definition 1.3.6 . *Let $T \in \mathcal{B}(\mathcal{H})$. The spectral radius of T is denoted by $r(T)$, and it is defined as follows :*

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

Needless to say that $\sup_{\lambda \in \sigma(T)} |\lambda|$ exists and finite. (remember that $\sigma(T)$ is a compact non-empty set)

Remark 1.3.3 *Let $T \in \mathcal{B}(\mathcal{H})$, then $r(T) \leq \|T\|$.*

Proof

If $|\lambda| > \|T\| \Longrightarrow \lambda \in \rho(T)$, then $\forall \lambda \in \sigma(T) : |\lambda| \leq \|T\|$. Thus $\sup_{\lambda \in \sigma(T)} |\lambda| \leq \|T\|$, therefore $r(T) \leq \|T\|$.

Proposition 1.3.3 *Let $T, S \in \mathcal{B}(\mathcal{H}), \alpha \in \mathbb{C}$ and $n \in \mathbb{N}$. Then:*
(1) $r(\alpha T) = |\alpha| r(T)$.
(2) $r\left(T^n\right) = r(T)^n$.
(3) $r(TS) = r(ST)$.
(4) $r\left(T^*\right) = r(T)$.

Theorem 1.3.9 (Gelfand's formula) *Let $T \in \mathcal{B}(\mathcal{H})$. Then :*

$$r(T) = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|T^n\|^{\frac{1}{n}}$$

Theorem 1.3.10 *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that T commutes with S i.e. $TS = ST$. Then :*

$$r(TS) \leq r(T)r(S)$$
$$r(T + S) \leq r(T) + r(S)$$

Proposition 1.3.4 *Let $T \in \mathcal{B}(\mathcal{H})$ be normal. Then $r(T) = \|T\|$.*

☛ **Proof**

we have $\forall n \in \mathbb{N} : \|T^n\| = \|T\|^n \Longrightarrow \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \|T\|$.
Since $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \Longrightarrow r(T) = \|T\|$.

☛ **Numerical range**

Definition 1.3.7 *Let $T \in \mathcal{B}(\mathcal{H})$. The numerical range of T is denoted by $W(T)$, and it is defined as follows :*

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$$

Proposition 1.3.5 *Let $T, S \in \mathcal{B}(\mathcal{H}), \alpha, \beta \in \mathbb{C}$ and $U \in \mathcal{B}(\mathcal{H})$ be unitary. Then:*
(1) $W(\alpha T + \beta I) = \alpha W(T) + \beta = \{\alpha z + \beta : z \in W(T)\}$.
(2) $W(T + S) \subset W(T) + W(S)$.
(3) $W(T^*) = \{\bar{z} : z \in W(T)\}$.
(4) $W(\Re(T)) = \Re W(T) = \{\Re z : z \in W(T)\}$ and $W(\Im(T)) = \Im W(T) = \{\Im z : z \in W(T)\}$.
(5) $W(U^*TU) = W(T)$.

Proof.

☛ **Proof**

(1) Let $x \in \mathcal{H}$ such that $\|x\| = 1$, then:

$$\langle(\alpha T + \beta I)x, x\rangle = \alpha\langle Tx, x\rangle + \beta\langle x, x\rangle = \alpha\langle Tx, x\rangle + \beta \Longrightarrow W(\alpha T + \beta I) = \alpha W(T) + \beta$$

(2) Let $x \in \mathcal{H}$ such that $\|x\| = 1$, then:

$$\langle(T + S)x, x\rangle = \langle Tx, x\rangle + \langle Sx, x\rangle \in W(T) + W(S) \Longrightarrow W(T + S) \subset W(T) + W(S)$$

(3) Let $x \in \mathcal{H}$ such that $\|x\| = 1$, then:

$$\langle T^*x, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle} \Longrightarrow W(T^*) = \{\bar{z} : z \in W(T)\}$$

(4) Let $x \in \mathcal{H}$ such that $\|x\| = 1$, we know that $\Re(T) = \frac{1}{2}(T + T^*)$ and $\Im(T) = \frac{1}{2i}(T - T^*)$, then

$$\langle\Re(T)x, x\rangle = \left\langle \frac{1}{2}(T + T^*)x, x\right\rangle = \frac{1}{2}\langle Tx, x\rangle + \frac{1}{2}\langle T^*x, x\rangle = \frac{1}{2}(\langle Tx, x\rangle + \overline{\langle Tx, x\rangle}) = \Re(\langle Tx, x\rangle)$$

Thus $W(\Re(T)) = \Re W(T)$, and processing the same the way, we obtain $W(\Im(T)) = \Im W(T)$.
(5) Let $x \in \mathcal{H}$ such that $\|x\| = 1$, and since U is unitary, we have $\|Ux\| = \|x\| = 1$, then:

$$\langle U^*TUx, x\rangle = \langle TUx, Ux\rangle \in W(T) \Longrightarrow W(U^*TU) \subset W(T)$$

Since U is surjective, then for all $\|x\| = 1, \exists y \in \mathcal{H}$ such that $x = Uy$, we have $\|y\| = \|Uy\| = \|x\| = 1$, then:

$$\langle Tx, x\rangle = \langle TUy, Uy\rangle = \langle U^*TUy, y\rangle \in W(U^*TU) \Longrightarrow W(T) \subset W(U^*TU)$$

Hence $W(U^*TU) = W(T)$.

Proposition 1.3.6 *Let $T \in \mathcal{B}(\mathcal{H})$ and $\alpha \in \mathbb{C}$, then:*
1) $W(T) = \{\alpha\}$ if and only if $T = \alpha I$.
2) If $\mathcal{H}$ is a finite dimensional space, then $\omega(T)$ is compact.

☛ **Proof**

1) If $T = \alpha I$, it is obvious that $W(T) = \{\alpha\}$. Assume that $W(T) = \{\alpha\} \Longrightarrow \forall x \in \mathcal{H}$ where $\|x\| = 1:\ \langle Tx, x\rangle = \alpha$. Then $\forall x \in \mathcal{H}\quad \langle Tx, x\rangle = \alpha\|x\|^2 = \alpha\langle x, x\rangle \Longrightarrow \forall x \in \mathcal{H}\quad \langle (T-\alpha I)x, x\rangle = 0$. Thus $T = \alpha I$ as desired.

2) Since $\mathcal{H}$ is a finite dimensional space, then the unit sphere $\mathrm{S}_{\mathcal{H}}$ is compact. ($\mathrm{S}_{\mathcal{H}} = \{x \in \mathcal{H} : \|x\| = 1\}$)
Define $f : \mathrm{S}_{\mathcal{H}} \longrightarrow \mathbb{C}$ by $f(x) = \langle Tx, x\rangle$, then f is continuous, since: $\forall x, y \in\ \mathrm{S}_{\mathcal{H}}$:

$$\begin{aligned} |f(x) - f(y)| = |\langle Tx, x\rangle - \langle Ty, y\rangle + \langle Ty, x\rangle - \langle Ty, x\rangle| &= |\langle Tx - Ty, x\rangle + \langle Ty, x - y\rangle| \\ &\leq |\langle Tx - Ty, x\rangle| + |\langle Ty, x - y\rangle| \\ &\leq \|T\|\|x - y\|\|x\| + \|T\|\|x - y\|\|y\| = 2\|T\|\|x - y\| \end{aligned}$$

Therefore f is continuous, and since $W(T) = f(\mathcal{S}_{\mathcal{H}})$, then $W(T)$ is compact.

Theorem 1.3.11 *Let $T \in \mathcal{B}(\mathcal{H})$ is self adjoint iff $W(T)$ is real.*

☛ **Proof**

If T is self adjoint, we have, for all $f \in H$, $\langle Tf, f\rangle = \langle f, Tf\rangle = \overline{\langle Tf, f\rangle}$, and hence $W(T)$ is real. Conversely, if $\langle Tf, f\rangle$ is real for all $f \in H$, we have $\langle Tf, f\rangle - \langle f, Tf\rangle = 0 = \langle (T - T^*) f, f\rangle$. Thus the operator $T - T^*$ has only $\{0\}$ in its numerical range. As will be shown in the next section, such an operator has to be the null operator. So $T - T^* = 0$ and $T = T^*$.

Theorem 1.3.12 (Toeplitz-Hausdorff Theorem) . *The numerical range $W(T)$ of any operator T is a convex set in $\mathbb{C}$.*

☛ **Numerical radius**

Definition 1.3.8 *Let $T \in \mathcal{B}(\mathcal{H})$. The numerical radius of T is denoted by $\omega(T)$, and it is defined as follows :*

$$\omega(T) = \sup_{\lambda \in W(T)} |\lambda| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

Proposition 1.3.7 *The numerical radius ω defines a norm on $\mathcal{B}(\mathcal{H})$.*

☛ **Proof**

(1) If $T = 0$, it is clear that $\omega(T) = 0$. Now assume that $\omega(T) = 0$, then:

$$\forall \lambda \in W(T): \quad \lambda = 0 \Longleftrightarrow \forall x \in \mathcal{H} \text{ where } \|x\| = 1 : \langle Tx, x\rangle = 0 \Longrightarrow \forall x \in \mathcal{H} \quad \langle Tx, x\rangle = 0.$$

Then $T = 0$. Hence $T = 0 \Longleftrightarrow \omega(T) = 0$.

(2)Let $T \in \mathcal{B}(\mathcal{H})$ and $\alpha \in \mathbb{C}$. Using the fact that $W(\alpha T) = \alpha W(T)$, we get:

$$\omega(\alpha T) = \sup_{\lambda \in W(\alpha T)} |\lambda| = \sup_{\lambda \in W(T)} |\alpha\lambda| = |\alpha| \sup_{\lambda \in W(T)} |\lambda| = |\alpha| W(T)$$

Thus $\omega(\alpha T) = |\alpha| W(T)$.

(3)Let $T, S \in \mathcal{B}(\mathcal{H})$. Using the fact that $W(T + S) \subset W(T) + W(S)$, we obtain:

$$\omega(T+S) = \sup_{\lambda \in W(T+S)} |\lambda| \leq \sup_{\lambda \in W(T)+W(S)} |\lambda| = \sup_{\lambda_1 \in W(T), \lambda_2 \in W(S)} |\lambda_1 + \lambda_2| \leq \sup_{\lambda_1 \in W(T)} |\lambda_1| + \sup_{\lambda_2 \in W(S)} |\lambda_2|$$

Then $\quad \omega(T + S) \leq \omega(T) + \omega(S)$. Therefore ω is a norm on $\mathcal{B}(\mathcal{H})$.

Proposition 1.3.8 *Let $T \in \mathcal{B}(\mathcal{H})$. Then:*

$$\omega(T) \leq \|T\| \leq 2\omega(T)$$

☛ **Proof**

- Let $x \in \mathcal{H}$ such that $\|x\| = 1$, then:

$$|\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq \|T\|\|x\|^2 = \|T\|$$

Thus $\sup_{\|x\|=1} |\langle Tx, x\rangle| \leq \|T\|$, therefore $\omega(T) \leq \|T\|$. - Let $x, y \in \mathcal{H}$, then:

$$\left|\left\langle T\frac{x}{\|x\|}, \frac{x}{\|x\|}\right\rangle\right| \leq \omega(T) \Longrightarrow |\langle Tx, x\rangle| \leq \omega(T)\|x\|^2$$

We have that $\langle Tx, y\rangle = \frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle + i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), x-iy\rangle)$. Therefore

$$\begin{aligned}
|\langle Tx, y\rangle| &\leq \frac{1}{4}(|\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle| + |\langle T(x+iy), x+iy\rangle| + |\langle T(x-iy), x-iy\rangle|) \\
&\leq \frac{\omega(T)}{4}\left(\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2\right) \\
&= \omega(T)\left(\|x\|^2 + \|y\|^2\right)
\end{aligned}$$

Thus $\quad |\langle Tx, y\rangle| \leq \omega(T)\left(\|x\|^2 + \|y\|^2\right)$ for all $x, y \in \mathcal{H}$, then:

$$\|T\| = \sup_{\|x\|=\|y\|=1} |\langle Tx, y\rangle| \leq \sup_{\|x\|=\|y\|=1} \omega(T)\left(\|x\|^2 + \|y\|^2\right) = 2\omega(T)$$

Hence $\|T\| \leq 2\omega(T)$.

Chapter 2

Elementary Spectral Theory

In this chapter we cover the basic resulte of spectral theory. The most important of these are the non-emptiness of spectrum,Beurlings spectral radius formula,And the Gelfand representation theory for commutative Banach algebra.
throughout this capter the ground field for all vector spaces and algebras is the the complaxe field $\mathbb{C}$,unless the cntrary is explicitly indicated in a particuler context.

2.1 Banach Algebras

We begin by setting up the basic vecabulary needed to discuss Banach algebras and by giving some exampals.

Definition 2.1.1 *1. An algebra is a vector space A together with a bilinear map:*

$$A^2 \to A \quad (a,b) \to ab$$

such that

$$\forall a,b,c \in A \quad a(bc) = (ab)c$$

2. A norm $\|.\|$ A is said to sulmultiplicative if

$$\|ab\| \leqslant \|a\|\|b\| \quad \forall a,b \in A.$$

In this case the pair $(A.\|.\|)$ is called a normed algebra.

3. If A admits a unit $1(a1 = 1a = a, \forall a \in A)$ and $\|1\| = 1$, we say that A is a unital normed algebra.

4. A complete normed algebra is called a Banach algebra.

5. A complete unital normed algebra is called a unital Banach algelra.

Remark 2.1.1 *is A is a normed algebra, then it is ereident from the ins quality*

$$\|ab - a'b'\| \leqslant \|a\| \, \|b - b'\| + \|a - a'\| \, \|b'\|$$

that the multiplication operation $(a,b) \to ab$ is jontly continuous.

Definition 2.1.2 *1. A subalgebra of A is a vector subspace B such that*

$$\forall b, b' \in B \Rightarrow bb' \in B$$

2. Endowed with the multiplication got by restriction B is itself an algebra.

3. A subalgebra of a normed algebra is obviously itself a normed algebra with the norm got by restriction.

4. *The closure of a subalyetra is a subalgebra.*
5. *A closed subalgebra of a Banach algeha is a Banach algebra.*

Exempel 2.1.1

- If Ω is a set, $\ell^\infty(\Omega)$ the set of all bounded complex valued functions on Ω, is a unital Banach algebra where the operations are defined pointwise:

$$(f+g)(x) = f(x) + g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$(\lambda f)(x) = \lambda f(x)$$

  and the norm is the sup-norm

$$\|f\|_\infty = \sup_{x\in\Omega} |f(x)|$$

- If Ω is a set, the set $C_b(\Omega)$ of all bounded continuous complex-valued functions on Ω is a closed subalgebra of $\ell^\infty(\Omega)$. Thus, $C_b(\Omega)$ is a unital Banach algebra.

  Remarek
  If Ω is compact set, $C(\Omega)$, the set of continuous functions from Ω to $\mathbb{C}$, is of course equal to $C_b(\Omega)$.

- If Ω is a locally compact Hausdorff space, we say that a continuous function f from Ω to $\mathbf{C}$ vanishes at infinity, if for each positive number ε the set $\{\omega \in \Omega || f(\omega)| \geq \varepsilon\}$ is compact. We denote the set of such functions by $C_0(\Omega)$. It is a closed subalgebra of $C_b(\Omega)$, and therefore, a Banach algebra. It is unital if and only if Ω is compact, and in this case $C_0(\Omega) = C(\Omega)$. The algebra $C_0(\Omega)$ is one of the most important examples of a Banach algebra, and we shall see it used constantly in C^*-algebra theory (the functional calculus).

- If (Ω, μ) is a measure space, the set $L^\infty(\Omega, \mu)$ of (classes of) essentially bounded complex-valued measurable functions on Ω is a unital Banach algebra with the usual (pointwise-defined) operations and the essential supremum norm $f \mapsto \|f\|_\infty$.

- If Ω is a measurable space, let $B_\infty(\Omega)$ denote the set of all bounded complex-valued measurable functions on Ω. Then $B_\infty(\Omega)$ is a closed subalgebra of $\ell^\infty(\Omega)$, so it is a unital Banach algebra. This example will be used in connection with the spectral theorem in Chapter 3.

  Remark 2.1.2 *All of the above examples are of course abelian-that is, $ab = ba$ for all elements a and b,but the following examples are not in general.*

- If E is a normed vector space, denote by $B(E)$ the set of all bounded linear maps from E to itself (the operators on E). It is routine to show that $B(E)$ is a normed algebra with the pointwise-defined operations for addition and scalar multiplication, multiplication given by $(T, S) \mapsto T \circ S$, and norm the operator norm:

$$\|T\| = \sup_{x\neq 0} \frac{\|T(x)\|}{\|x\|} = \sup_{\|x\|\leq 1} \|T(x)\|$$

  If E is a Banach space, then $B(E)$ is complete (In fact, there is equivalence.) and is therefore a Banach algebra.

- The algebra $M_n(\mathbb{C})$ of $n \times n$-matrices with entries in $\mathbb{C}$. It is therefore a unital Banach algebra

☛ Recall that an upper triangular matrix is one of the form

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \dots & \lambda_{1n} \\ 0 & \lambda_{22} & \dots & \dots & \lambda_{2n} \\ 0 & 0 & \lambda_{33} & \dots & \lambda_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \lambda_{nn} \end{pmatrix}$$

(all entries below the main diagonal are zero). These matrices form a subalgebra of $M_n(\mathbf{C})$.

Proposition 2.1.1 *If $(B_\lambda)_{\lambda\in I}$ is a family of subalgebras of an algebra A, then $\cap_{\lambda\in I} B_\lambda$ is a subalgebra, also.*

Definition 2.1.3 *1. A left (respectively, right) ideal in an algebra A is a vector subspace I of A such that*

$$a \in A \text{ and } b \in I \Rightarrow ab \in I \text{ (respectively, right } ba \in I\text{)}$$

2. An ideal in A is a vector subspace that is simultaneously a left and a right ideal in A. (0 and A are ideals in A called the trivial ideals.)

3. A maximal ideal in A is a proper ideal (that is, it is not A) that is not contained in any other proper ideal in A.

4. Maximal left (on right) ideals are defined similarly.

5. An ideal I is modular if there is an element u in A such that:

$$\forall a \in A \quad a - au \in I \text{ and } a - ua \in I.$$

Exempel 2.1.2

☛ If ω is an element of a locally compact Hausdorff space Ω, and $M_\omega = \{f \in C_0(\Omega) \mid f(\omega) = 0\}$, then M_ω is a modular ideal in the algebra $C_0(\Omega)$. This is so because there is an element $u \in C_0(\Omega)$ such that $u(\omega) = 1$, and hence, $f - uf \in M_\omega$ for all $f \in C_0(\Omega)$. Since M_ω is of codimension one in $C_0(\Omega)$ (as $M \oplus \mathbf{C}u = C_0(\Omega)$), it is a maximal ideal.

Proposition 2.1.2 *1. If I is an ideal of A, then A/I is an algebra with the multiplication given by*

$$(a + I)(b + I) = ab + I$$

2. If I is modular, then A/I is unital (if $a - au, a - ua \in I$ for all $a \in A$, then $u + I$ is the unit). Conversely, if A/I is unital then I is modular.

3. If A is unital, then obviously all its ideals are modular, and therefore, A posesses maximal ideals.

4. If $(I_\lambda)_{\lambda\in\Lambda}$ is a family of ideals of an algebra A, then $\cap_{\lambda\in\Lambda} I_\lambda$ is an ideal of A.

Theorem 2.1.1 *If I is a closed ideal in a normed algebra A, then A/I is a normed algebra when endowed with the quotient norm*

$$\|a + I\| = \inf_{b\in I} \|a + b\|$$

☛ **Proof**

Let $\varepsilon > 0$ and suppose that a, b belong to A. Then $\varepsilon + \|a + I\| > \|a + a'\|$ and $\varepsilon + \|b + I\| > \|b + b'\|$ for some $a', b' \in I$. Hence,

$$(\varepsilon + \|a + I\|)(\varepsilon + \|b + I\|) > \|a + a'\| \, \|b + b'\| \geq \|ab + c\|$$

where $c = a'b + ab' + a'b' \in I$.
Thus, $(\varepsilon + \|a + I\|)(\varepsilon + \|b + I\|) \geq \|ab + I\|$. Letting $\varepsilon \to 0$, we get $\|a + I\|\|b + I\| \geq \|ab + I\|$. that is the quotient norm is submultiplicative.

Definition 2.1.4 *1. A homomorphism from an algebra A to an algebra B is a linear map $\varphi : A \to B$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$.*

2. We say φ is unital if A and B are unital and $\varphi(1) = 1$.

Proposition 2.1.3 *1. Its kernel $\ker(\varphi)$ is an ideal in A and its image $\varphi(A)$ is a subalgebra of B*

2. If I is an ideal in A, the quotient map $\pi : A \to A/I$ is a homomorphism.

2.2 The Spectrum and Spectral Radius

Definition 2.2.1 *Let $\mathbb{C}[z]$ denote the algebra of all polynomials in an indeterminate z with complex coefficients. If a is an element of a unital algebra A and $p \in \mathbb{C}[z]$ is the polynomial*

$$p = \lambda_0 + \lambda_1 z^1 + \cdots + \lambda_n z^n$$

we set

$$p(a) = \lambda_0 1 + \lambda_1 a^1 + \cdots + \lambda_n a^n$$

The map

$$\mathbb{C}[z] \to A, \quad p \mapsto p(a)$$

is a unital homomorphism.

Definition 2.2.2 *We say that $a \in A$ is invertible if there is an element b in A such that $ab = ba = 1$. In this case b is unique and written a^{-1}. The set*

$$\mathrm{Inv}(A) = \{a \in A \mid a \textit{ is invertible } \}$$

is a group under multiplication.

Definition 2.2.3 (the spectrum) *We define the spectrum of an element a to be the set*

$$\sigma(a) = \sigma_A(a) = \{\lambda \in \mathbb{C} \mid \lambda 1 - a \notin \mathrm{Inv}(A)\}$$

We shall henceforth find it convenient to write $\lambda 1$ simply as λ.

Exempel 2.2.1

☛ Let $A = C(\Omega)$, where Ω is a compact Hausdorff space. Then, the spectrum of a function $f \in A$ is given by $\sigma(f) = f(\Omega)$.

(a) **Spectrum Definition:**

$$\sigma(f) := \{\lambda \in \mathbb{C} \mid \lambda - f \notin \mathrm{Inv}(A)\}.$$

(b) **Invertibility Criterion:** For $g \in C(\Omega)$,

$$g \in \mathrm{Inv}(A) \iff \inf_{x \in \Omega} |g(x)| > 0.$$

(The inverse $g^{-1}(x) = \frac{1}{g(x)}$ must be continuous and bounded.)

(c) **Case Analysis:**

- **Case 1 ($\lambda \notin f(\Omega)$):** By compactness of $f(\Omega)$, $\exists \delta > 0$ such that:

$$\inf_{x \in \Omega} |\lambda - f(x)| \geq \delta > 0 \implies \lambda - f \in \mathrm{Inv}(A).$$

- **Case 2 ($\lambda \in f(\Omega)$):** $\exists x_0 \in \Omega$ with $\lambda = f(x_0)$. Thus:
$$\inf_{x \in \Omega} |\lambda - f(x)| = 0 \implies \lambda - f \notin \mathrm{Inv}(A).$$

(d) **Conclusion:**
$$\lambda \in \sigma(f) \iff \lambda \in f(\Omega).$$
Hence, $\sigma(f) = f(\Omega)$.

☛ Let $A = l^\infty(\Omega)$, where Ω is a non - empty set. then $\quad \sigma(f) = \overline{f(\Omega)} \quad \forall f \in A$.

(a) **Spectrum Definition:** The spectrum of f is:
$$\sigma(f) = \{\lambda \in \mathbb{C} \mid \lambda - f \notin \mathrm{Inv}(A)\}.$$

(b) **Invertibility Criterion:** A function $g \in \ell^\infty(\Omega)$ is invertible if and only if:
$$\inf_{x \in \Omega} |g(x)| > 0.$$
Thus, $\lambda - f$ is invertible precisely when:
$$\inf_{x \in \Omega} |\lambda - f(x)| > 0.$$

(c) **Non-Invertibility Analysis:**
- If $\lambda \in \overline{f(\Omega)}$, there exists a sequence $\{x_n\} \subseteq \Omega$ such that $\lim_{n\to\infty} f(x_n) = \lambda$. Consequently:
$$\inf_{x \in \Omega} |\lambda - f(x)| = 0,$$
making $\lambda - f$ non-invertible.
- If $\lambda \notin \overline{f(\Omega)}$, the distance $\mathrm{dist}(\lambda, f(\Omega)) = \delta > 0$, so:
$$\inf_{x \in \Omega} |\lambda - f(x)| \geq \delta > 0,$$
ensuring invertibility of $\lambda - f$.

(d) **Conclusion:** Combining these results:
$$\sigma(f) = \overline{f(\Omega)}.$$

☛ Let A be the algebra of upper triangular $n \times n$-matrices. If $a \in A$, say
$$a = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1n} \\ 0 & \lambda_{22} & \dots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_{nn} \end{pmatrix}$$
it is elementary that
$$\sigma(a) = \{\lambda_{11}, \lambda_{22}, \dots, \lambda_{nn}\}$$
Similarly, if $A = M_n(\mathbb{C})$ and $a \in A$, then $\sigma(a)$ is the set of eigenvalues of a.

Thus, one thinks of the spectrum as simultaneously a generalisation of the range of a function and the set of eigenvalues of a finite square matrix.

Remark 2.2.1 *1. If a, b are elements of a unital algebra A, then $1 - ab$ is invertible if and only if $1 - ba$ is invertible. This follows from the observation that if $1 - ab$ has inverse c, then $1 - ba$ has inverse $1 + bca$.*

2. A consequence of this equivalence is that $\sigma(ab)\backslash\{0\} = \sigma(ba)\backslash\{0\}$ for all $a, b \in A$.

Theorem 2.2.1 *Let a be an element of a unital algebra A. If $\sigma(a)$ is non-empty and $p \in \mathbb{C}[z]$, then*

$$\sigma(p(a)) = p(\sigma(a))$$

☛ **Proof**

We may suppose that p is not constant. If $\mu \in \mathbb{C}$, there are elements $\lambda_0, \ldots, \lambda_n$ in $\mathbb{C}$, where $\lambda_0 \neq 0$, such that

$$p - \mu = \lambda_0 (z - \lambda_1) \ldots (z - \lambda_n)$$

and therefore,

$$p(a) - \mu = \lambda_0 (a - \lambda_1) \ldots (a - \lambda_n)$$

It is clear that $p(a) - \mu$ is invertible if and only if $a - \lambda_1, \ldots, a - \lambda_n$ are. It follows that $\mu \in \sigma(p(a))$ if and only if $\mu = p(\lambda)$ for some $\lambda \in \sigma(a)$, and therefore, $\sigma(p(a)) = p(\sigma(a))$.

Theorem 2.2.2 *Let A be a unital Banach algebra and a an element of A such that $\|a\| < 1$. Then $1 - a \in \operatorname{Inv}(A)$ and*

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n$$

☛ **Proof**

Since $\sum_{n=0}^{\infty} \|a^n\| \leq \sum_{n=0}^{\infty} \|a\|^n = (1 - \|a\|)^{-1} < +\infty$, the series $\sum_{n=0}^{\infty} a^n$ is convergent, to b say, in A, and since $(1-a)(1 + \cdots + a^n) = 1 - a^{n+1}$ converges to $(1-a)b = b(1-a)$ and to 1 as $n \to \infty$, the element b is the inverse of $1 - a$.

☛ **Remark**

The series in **Theorem 2.2.2** is called the Neumann series for $(1-a)^{-1}$.

Theorem 2.2.3 *1. If A is a unital Banach algebra, then $\operatorname{Inv}(A)$ is open in A,*

2. the map

$$\operatorname{Inv}(A) \to A, \quad a \mapsto a^{-1}$$

is differentiable.

☛ **Proof**

1. Suppose that $a \in \operatorname{Inv}(A)$ and $\|b - a\| < \|a^{-1}\|^{-1}$. Then $\|ba^{-1} - 1\| \leq \|b - a\| \|a^{-1}\| < 1$, so $ba^{-1} \in \operatorname{Inv}(A)$, and therefore, $b \in \operatorname{Inv}(A)$. Thus, $\operatorname{Inv}(A)$ is open in A.

2. If $b \in A$ and $\|b\| < 1$, then $1 + b \in \operatorname{Inv}(A)$ and

$$\begin{aligned} \left\|(1+b)^{-1} - 1 + b\right\| &= \left\|\sum_{n=0}^{\infty}(-1)^n b^n - 1 + b\right\| = \left\|\sum_{n=2}^{\infty}(-1)^n b^n\right\| \\ &\leq \sum_{n=2}^{\infty} \|b\|^n = \|b\|^2/(1 - \|b\|)^{-1} \end{aligned}$$

Let $a \in \operatorname{Inv}(A)$ and suppose that $\|c\| < \frac{1}{2}\|a^{-1}\|^{-1}$. Then $\|a^{-1}c\| < 1/2 < 1$, so (with $b = a^{-1}c$),

$$\left\|\left(1 + a^{-1}c\right)^{-1} - 1 + a^{-1}c\right\| \leq \left\|a^{-1}c\right\|^2 / \left(1 - \left\|a^{-1}c\right\|\right)^{-1} \leq 2\left\|a^{-1}c\right\|^2$$

since $1-\|a^{-1}c\|>1/2$. Now define u to be the linear operator on A given by $u(b)=-a^{-1}ba^{-1}$. Then,

$$\begin{aligned}\left\|(a+c)^{-1}-a^{-1}-u(c)\right\| &= \left\|\left(1+a^{-1}c\right)^{-1}a^{-1}-a^{-1}+a^{-1}ca^{-1}\right\| \\ &\leq \left\|\left(1+a^{-1}c\right)^{-1}-1+a^{-1}c\right\|\left\|a^{-1}\right\| \leq 2\left(\left\|a^{-1}\right\|^{3}\|c\|^{2}\right)\end{aligned}$$

Consequently,

$$\lim_{c\to 0}\frac{\|(a+c)^{-1}-a^{-1}-u(c)\|}{\|c\|}=0$$

and therefore, the map $\sigma: b\mapsto b^{-1}$ is differentiable at $b=a$ with derivative $\sigma'(a)=u$.

Remark 2.2.2 *The algebra $\mathbb{C}[z]$ is a normed algebra where the norm is defined by setting*

$$\|p\|=\sup_{|\lambda|\leq 1}|p(\lambda)|$$

Observe that $\mathrm{Inv}(\mathbb{C}[z])=\mathbb{C}\backslash\{0\}$, *so the polynomials $p_n=1+z/n$ are not invertible. But* $\lim_{n\to\infty}p_n=1$, *which shows that* $\mathrm{Inv}(\mathbb{C}[z])$ *is not open in* $\mathbb{C}[z]$. *Thus, the norm on* $\mathbb{C}[z]$ *is not complete.*

Theorem 2.2.4 *1. Let A be a unital Banach algebra and let $a\in A$. The spectrum $\sigma(a)$ of a is a closed subset of the disc in the plane of centre the origin and radius $\|a\|$.*

2. the map

$$\mathbb{C}\backslash\sigma(a)\to A,\lambda\mapsto(a-\lambda)^{-1}$$

is differentiable.

☛ **Proof**

1. If $|\lambda|>\|a\|$, then $\|\lambda^{-1}a\|<1$, so $1-\lambda^{-1}a$ is invertible, and therefore, so is $\lambda-a$. Hence, $\lambda\notin\sigma(a)$. Thus, $\lambda\in\sigma(a)\Rightarrow|\lambda|\leq\|a\|$. The set $\sigma(a)$ is closed, that is, $\mathbb{C}\backslash\sigma(a)$ is open, because $\mathrm{Inv}(A)$ is open in A.
2. Differentiability of the map $\lambda\mapsto(a-\lambda)^{-1}$ follows from Theorem 2.2.2.

Definition 2.2.4 (the Spectral Radius) *If a is an element of a unital Banach algebra A, its spectral radius is defined to be*

$$r(a)=\sup_{\lambda\in\sigma(a)}|\lambda|.$$

Remark 2.2.3 *By Remark 2.2.1, $r(ab)=r(ba)$ for all $a,b\in A$.*

Exempel 2.2.2

- ☛ If $A=C(\Omega)$, where Ω is a compact Hausdorff space, then $r(f)=\|f\|_\infty(f\in A)$.
- ☛ Let $A=M_2(\mathbf{C})$ and

$$a=\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

Then $\|a\|=1$, but $r(a)=0$, since $a^2=0$.

Theorem 2.2.5 (Beurling) *If a is an element of a unital Banach algebra A, then*

$$r(a)=\inf_{n\geq 1}\|a^n\|^{1/n}=\lim_{n\to\infty}\|a^n\|^{1/n}$$

☛ **Proof**

If $\lambda\in\sigma(a)$, then $\lambda^n\in\sigma(a^n)$, so $|\lambda^n|\leq\|a^n\|$, and therefore, $r(a)\leq\inf_{n\geq 1}\|a^n\|^{1/n}\leq\liminf_{n\to\infty}\|a^n\|^{1/n}$.

Let Δ be the open disc in $\mathbb{C}$ centered at 0 and of radius $1/r(a)$ (we use the usual convention that $1/0=+\infty$). If $\lambda\in\Delta$, then $1-\lambda a\in\operatorname{Inv}(A)$. If $\tau\in A^*$, then the map

$$f:\Delta\to\mathbb{C},\lambda\mapsto\tau\left((1-\lambda a)^{-1}\right)$$

is analytic, so there are unique complex numbers λ_n such that

$$f(\lambda)=\sum_{n=0}^{\infty}\lambda_n\lambda^n\quad(\lambda\in\Delta)$$

However, if $|\lambda|<1/\|a\|(\leq 1/r(a))$, then $\|\lambda a\|<1$, so

$$(1-\lambda a)^{-1}=\sum_{n=0}^{\infty}\lambda^n a^n$$

and therefore,

$$f(\lambda)=\sum_{n=0}^{\infty}\lambda^n\tau(a^n)$$

It follows that $\lambda_n=\tau(a^n)$ for all $n\geq 0$. Hence, the sequence $(\tau(a^n)\lambda^n)$ converges to 0 for each $\lambda\in\Delta$, and therefore a fortiori, it is bounded. Since this is true for each $\tau\in A^*$, it follows from the principle of uniform boundedness that $(\lambda^n a^n)$ is a bounded sequence. Hence, there is a positive number M (depending on λ, of course) such that $\|\lambda^n a^n\|\leq M$ for all $n\geq 0$, and therefore, $\|a^n\|^{1/n}\leq M^{1/n}/|\lambda|$ (if $\lambda\neq 0$). Consequently, $\limsup_{n\to\infty}\|a^n\|^{1/n}\leq 1/|\lambda|$. We have thus shown that if $r(a)<|\lambda^{-1}|$, then $\limsup_{n\to\infty}\|a^n\|^{1/n}\leq|\lambda^{-1}|$. It follows that $\limsup_{n\to\infty}\|a^n\|^{1/n}\leq r(a)$, and since $r(a)\leq\liminf_{n\to\infty}\|a^n\|^{1/n}$, therefore $r(a)=\lim_{n\to\infty}\|a^n\|^{1/n}$.

Exempel 2.2.3

☛ Let A be the set of C^1-functions on the interval $[0,1]$. This is an algebra when endowed with the pointwise-defined operations, and a submultiplicative norm on A is given by

$$\|f\|=\|f\|_\infty+\|f'\|_\infty\quad(f\in A)$$

It is elementary that A is complete under this norm, and therefore, A is a Banach algebra.
Let $x:[0,1]\to\mathbb{C}$,$t\mapsto x(t)=t$.
so $x\in A$. Clearly, $\|x^n\|=1+n$ for all n, so $r(x)=\lim_{n\to\infty}(1+n)^{1/n}=1<2=\|x\|$.

Remark 2.2.4 *Let A be a normed algebra and suppose that $x,y\in A$ are such that $xy=yx$. Then*

1. $r(xy)\leq r(x)r(y)$
2. $r(x+y)\leq r(x)+r(y)$.

☛ **Proof**

1. Since $(xy)^n=x^ny^n$ for all $n\in\mathbb{N}$, For every $x\in A, r(x)=\lim_{n\to\infty}\|x^n\|^{1/n}$. yields that

$$r(xy)=\lim_{n\to\infty}\|x^ny^n\|^{1/n}\leq\lim_{n\to\infty}\|x^n\|^{1/n}\cdot\lim_{n\to\infty}\|y^n\|^{1/n}=r(x)r(y)$$

2. The proof of the second inequality requires much more effort. Take any $\alpha > r(x)$ and $\beta > r(y)$ and set $a = (1/\alpha)x$ and $b = (1/\beta)y$. Then $r(a) < 1$ and $r(b) < 1$.

   Because x and y commute, we have

$$\begin{aligned} \|(x+y)^n\|^{1/n} &= \left\| \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j} \right\|^{1/n} \\ &\leq \left(\sum_{j=0}^{n} \binom{n}{j} \alpha^j \beta^{n-j} \left\| a^j \right\| \cdot \left\| b^{n-j} \right\| \right)^{1/n}. \end{aligned}$$

   For each $n \in \mathbb{N}$, choose $n', n'' \in \mathbb{N}_0$ such that $n' + n'' = n$ and

$$\left\| a^{n'} \right\| \cdot \left\| b^{n''} \right\| = \max_{0 \leq j \leq n} \left\| a^j \right\| \cdot \left\| b^{n-j} \right\|$$

   With this choice of n' and n'' we have

$$\begin{aligned} r(x+y) &= \lim_{n\to\infty} \|(x+y)^n\|^{1/n} \\ &\leq (\alpha+\beta) \liminf_{n\to\infty} \left\| a^{n'} \right\|^{1/n} \left\| b^{n''} \right\|^{1/n} \end{aligned}$$

   Now, the sequence $(n'/n)_n \subseteq [0,1]$ has a convergent subsequence $(n'_k/n_k)_k$ with limit γ, say. If $\gamma \neq 0$, then $n'_k \to \infty$ and hence

$$\lim_{k\to\infty} \left\| a^{n'_k} \right\|^{1/n_k} = r(a)^\gamma < 1$$

   whereas if $\gamma = 0$, then

$$\limsup_{k\to\infty} \left\| a^{n'_k} \right\|^{1/n_k} \leq \lim_{k\to\infty} \|a\|^{n'_k/n_k} \leq 1$$

   Thus, in either case,

$$\limsup_{k\to\infty} \left\| a^{n'_k} \right\|^{1/n_k} \leq 1$$

   Similarly, since $(n''_k/n_k)_k$ converges,

$$\limsup_{k\to\infty} \left\| b^{n''_k} \right\|^{1/n_k} \leq 1$$

   The above upper estimate for $r(x+y)$ now shows that $r(x+y) \leq \alpha + \beta$. Since this holds for all $\alpha > r(x)$ and $\beta > r(y)$, it follows that $r(x+y) \leq r(x) + r(y)$.

Definition 2.2.5 (the holes) *Recall that if K is a non-empty compact set in $\mathbb{C}$, its complement $\mathbb{C} \backslash K$ admits exactly one unbounded component, and that the bounded components of $\mathbb{C} \backslash K$ are called the holes of K.*

Theorem 2.2.6 *Let B be a closed subalgebra of a unital Banach algebra A, containing the unit of A.*

1. *The set $\mathrm{Inv}(B)$ is a clopen subset of $B \cap \mathrm{Inv}(A)$.*

2. *For each $b \in B$,*

$$\sigma_A(b) \subseteq \sigma_B(b) \quad \textit{and} \quad \partial\sigma_B(b) \subseteq \partial\sigma_A(b)$$

3. *If $b \in B$ and $\sigma_A(b)$ has no holes, then $\sigma_A(b) = \sigma_B(b)$.*

☛ **Proof**

1. Clearly $\mathrm{Inv}(B)$ is an open set in $B \cap \mathrm{Inv}(A)$. To see that it is also closed, let (b_n) be a sequence in $\mathrm{Inv}(B)$ converging to a point $b \in B \cap \mathrm{Inv}(A)$. Then (b_n^{-1}) converges to b^{-1} in A, so $b^{-1} \in B$, which implies that $b \in \mathrm{Inv}(B)$. Hence, $\mathrm{Inv}(B)$ is clopen in $B \cap \mathrm{Inv}(A)$.
2. (a) If $b \in B$, the inclusion $\sigma_A(b) \subseteq \sigma_B(b)$ is immediate from the inclusion $\mathrm{Inv}(B) \subseteq \mathrm{Inv}(A)$.

   (b) If $\lambda \in \partial\sigma_B(b)$, then there is a sequence (λ_n) in $\mathbb{C}\backslash\sigma_B(b)$ converging to λ. Hence, $b - \lambda_n \in \mathrm{Inv}(B)$, and $b - \lambda \notin \mathrm{Inv}(B)$, so $b - \lambda \notin \mathrm{Inv}(A)$, by Condition (1). Also, $b - \lambda_n \in \mathrm{Inv}(A)$, so $\lambda_n \in \mathbb{C}\backslash\sigma_A(b)$. Therefore, $\lambda \in \partial\sigma_A(b)$. This proves Condition (2).
3. If $b \in B$ and $\sigma_A(b)$ has no holes, then $\mathbb{C}\backslash\sigma_A(b)$ is connected. Since $\mathbb{C}\backslash\sigma_B(b)$ is a clopen subset of $\mathbb{C}\backslash\sigma_A(b)$ by Conditions (1) and (2), it follows that $\mathbb{C}\backslash\sigma_A(b) = \mathbb{C}\backslash\sigma_B(b)$, and therefore, $\sigma_A(b) = \sigma_B(b)$.

Theorem 2.2.7 *Let A be a unital Banach algebra.*

1. *If $a \in A$ and $f : \mathbb{R} \to A$ is differentiable, $f(0) = 1$, and $f'(t) = af(t)$ for all $t \in \mathbb{R}$, then $f(t) = e^{ta}$ for all $t \in \mathbb{R}$.*
2. *If $a \in A$, then e^a is invertible with inverse e^{-a}, and if a, b are commuting elements of A, then $e^{a+b} = e^a e^b$.*

☛ **Proof**

1. First we observe that if $f : \mathbb{R} \to A$ is defined by $f(t) = e^{ta}$, then $f(t) = \sum_{n=0}^{\infty} t^n a^n/n!$, so differentiating term by term we get $f'(t) = af(t)$. Now suppose f, g are any pair of differentiable maps from $\mathbb{R}$ to A such that $f'(t) = af(t)$ and $g'(t) = ag(t)$ and $f(0) = g(0) = 1$. Then the map $h : \mathbb{R} \to A, t \mapsto f(t)g(-t)$, is differentiable with zero derivative (apply the product rule for differentiation). Hence, $h(t) = 1$ for all $t \in \mathbb{R}$. Applying this to the map $t \mapsto e^{ta}$, we get $e^{ta}e^{-ta} = 1$; in particular, $e^a e^{-a} = 1$.

   It follows that if $f : \mathbb{R} \to A$ is differentiable, $f(0) = 1$, and $f'(t) = af(t)$ for all t, then $f(t) = e^{ta}$ (set $g(t) = e^{ta}$ and get $f(t)e^{-ta} = 1$, so $f(t) = e^{ta}$).
2. Now suppose that a and b are commuting elements of A and set $f(t) = e^{ta}e^{tb}$. Then $f(0) = 1$ and $f'(t) = e^{ta}be^{tb} + ae^{ta}e^{tb}$ (by the product rule) $= (a + b)f(t)$. Hence, $f(t) = e^{t(a+b)}$ for all $t \in \mathbb{R}$, so, in particular, $e^{a+b} = f(1) = e^a e^b$.

Theorem 2.2.8 *Let A be a unital abelian Banach algebra.*

1. *For all $a, b \in A$.then $\sigma(a + b) \subseteq \sigma(a) + \sigma(b)$*
2. *For all $a, b \in A$.then $\sigma(ab) \subseteq \sigma(a)\sigma(b)$*

Remark this is not true for all Banach algebras.

Exempel 2.2.4

☛ **Counterexample:** $A = M_2(\mathbb{C})$

1. Violation of Spectral Inclusion for Sums:

Take $c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, d = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then:

$$\sigma(c + d) = \{1, -1\}, \quad \sigma(c) + \sigma(d) = \{0\}.$$

Thus, $\sigma(c + d) \not\subseteq \sigma(c) + \sigma(d)$.

2. Violation of Spectral Inclusion for Products:

Take $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then:

$$xy = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma(xy) = \{0, 1\}, \quad \sigma(x)\sigma(y) = \{0\}.$$

Thus, $\sigma(xy) \not\subseteq \sigma(x)\sigma(y)$.

Theorem 2.2.9 *Let A be a unital Banach algebra.*

1. *If a is invertible in A, than $\sigma\left(a^{-1}\right) = \{\lambda^{-1} \mid \lambda \in \sigma(a)\}$.*
2. *For any element $a \in A$, than $r\left(a^n\right) = (r(a))^n$.*

Theorem 2.2.10 *If a is a hermitian element of a C^*-algebra A, then $\sigma(a) \subseteq \mathbb{R}$.*

☛ **Proof**

We may suppose that A is unital. Since e^{ia} is unitary, $\sigma\left(e^{ia}\right) \subseteq \mathbb{T}$. If $\lambda \in \sigma(a)$ and $b = \sum_{n=1}^{\infty} i^n (a - \lambda)^{n-1}/n!$ then $e^{ia} - e^{i\lambda} = \left(e^{i(a-\lambda)} - 1\right) e^{i\lambda} = (a - \lambda) b e^{i\lambda}$. Since b commutes with a, and since $a - \lambda$ is non-invertible, $e^{ia} - e^{i\lambda}$ is non-invertible. Hence, $e^{i\lambda} \in \mathbb{T}$, and therefore $\lambda \in \mathbb{R}$. Thus, $\sigma(a) \subseteq \mathbb{R}$.

2.3 The Gelfand representation

The idea of this section is to represent an abelian Banach algebra as an algebra of continuous functions on a locally compact Hausdorff space. This is an extremely useful way of looking at these algebras, but in the case of the more "complicated" algebras, the picture it presents may be of limited accuracy.

Theorem 2.3.1 *Let A be an abelian Banach algebra.*
(1) If A is unital, then

$$\sigma(a) = \{\tau(a) \mid \tau \in \Omega(A)\} \quad (a \in A)$$

(2) If A is non-unital, then

$$\sigma(a) = \{\tau(a) \mid \tau \in \Omega(A)\} \cup \{0\} \quad (a \in A)$$

☛ **Proof**

If A is unital and a is an element of A whose spectrum contains λ, then the ideal $I = (a - \lambda)A$ is proper, so I is contained in a maximal ideal 1.3. The Gelfand Representation 15 $\ker(\tau)$, where $\tau \in \Omega(A)$. Hence, $\tau(a) = \lambda$. This shows that the inclusion $\sigma(a) \subseteq \{\tau(a) \mid \tau \in \Omega(A)\}$ holds, and the reverse inclusion is clear.

Now suppose that A is non-unital, and let $\tau_\infty : \tilde{A} \to \mathbf{C}$ be the canonical homomorphism. Then $\Omega(\tilde{A}) = \{\tilde{\tau} \mid \tau \in \Omega(A)\} \cup \{\tau_\infty\}$, where $\tilde{\tau}$ is the unique character on $\tilde{A}$ extending the character τ on A. Hence, by Condition (1), $\sigma(a) = \sigma_{\tilde{A}}(a) = \{\tau(a) \mid \tau \in \Omega(\tilde{A})\} = \{\tau(a) \mid \tau \in \Omega(A)\} \cup \{0\}$ for each $a \in A$.

Theorem 2.3.2 (Gelfand Representation) *Suppose that A is an abelian Banach algebra and that $\Omega(A)$ is non-empty. Then the map*

$$A \to C_0(\Omega(A)), a \mapsto \hat{a}$$

is a norm-decreasing homomorphism, and

$$r(a) = \|\hat{a}\|_\infty \quad (a \in A)$$

If A is unital, $\sigma(a) = \hat{a}(\Omega(A))$, and if A is non-unital, $\sigma(a) = \hat{a}(\Omega(A)) \cup \{0\}$, for each $a \in A$.

☛ **Proof**
By Theorem 2.3.1 the spectrum $\sigma(a)$ is the range of $\hat{a}$, together with $\{0\}$ if A is non-unital. Hence, $r(a) = \|\hat{a}\|_\infty$, which implies that the map $a \mapsto \hat{a}$ is norm-decreasing. That this map is a homomorphism is easily checked.

Theorem 2.3.3 (Gelfand) *If a is an element of a unital Banach algebra A, then the spectrum $\sigma(a)$ of a is non-empty.*

☛ **Proof** Suppose that $\sigma(a) = \emptyset$ and we shall obtain a contradiction.
If $|\lambda| > 2\|a\|$, then $\|\lambda^{-1}a\| < \frac{1}{2}$, and therefore, $1 - \|\lambda^{-1}a\| > \frac{1}{2}$. Hence,

$$\begin{aligned}\|(1- \lambda^{-1}a)^{-1} - 1\| &= \| \sum_{n=1}^{\infty} (\lambda^{-1}a)^n \| \\ &\leq \frac{\|\lambda^{-1}a\|}{1 - \|\lambda^{-1}a\|} \leq 2 \left\|\lambda^{-1}a\right\| < 1\end{aligned}$$

Consequently, $\left\|(1 - \lambda^{-1}a)^{-1}\right\| < 2$, and therefore,

$$\|(a - \lambda)^{-1}\| = \left\|\lambda^{-1} \left(1 - \lambda^{-1}a\right)^{-1}\right\| < 2/|\lambda| < \|a\|^{-1}$$

($a \neq 0$ since $\sigma(a) = \emptyset$). Moreover, since the map $\lambda \mapsto (a - \lambda)^{-1}$ is continuous, it is bounded on the (compact) disc $2\|a\|\mathbb{D}$. Thus, we have shown that this map is bounded on all of $\mathbb{C}$; that is, there is a positive number M such that $\|(a - \lambda)^{-1}\| \leq M(\lambda \in \mathbb{C})$.

If $\tau \in A^*$, the function $\lambda \mapsto \tau\left((a - \lambda)^{-1}\right)$ is entire, and bounded by $M\|\tau\|$, so by Liouville's theorem in complex analysis, it is constant. In particular, $\tau\left(a^{-1}\right) = \tau\left((a - 1)^{-1}\right)$. Because this is true for all $\tau \in A^*$, we have $a^{-1} = (a - 1)^{-1}$, so $a = a - 1$, which is a contradiction.

Chapter 3

C*-algebras

In this chapter we commence our study of C^*-algebras and of operators on Hilbert spaces. Hilbert spaces are very well-behaved compared with general Banach spaces, and the same is even more true of C^*-algebras as compared with general Banach algebras. The main results of this chapter are a theorem of Gelfand, which asserts that (up to isomorphism) all abelian C*-algebras are of the form $C_0(\Omega)$, where Ω is a locally compact Hausdorff space, and the spectral theorem. This theorem enables us to "synthesize" a normal operator from linear combinations of projections where the coefficients lie in the spectrum. It is a very powerful result.

3.1 Definition and Examples

We begin by setting up the basic vecabulary needed to discuss C*-algebras and by giving some exampals.

Definition 3.1.1 *1. An involution on an algebra A is a conjugate-linear map*

$$\begin{aligned} * : A &\longrightarrow A \\ a &\longrightarrow a^* \end{aligned}$$

such that

$$a^{**} = a \quad , \quad (ab)^* = b^*a^* \quad \forall a, b \in A.$$

*the pain $(A, *)$ is called an involutive algebra or a *-algebra.*

2. *A Banach *-algebra is a *-algebra A together with a complete sulemultaiplicative norm such that*

$$\|a^*\| = \|a\| \quad \forall a \in A$$

3. *if A has a unit such that $\|1\| = 1$, we coll A a unictal Banach $*$-algebra.*

4. *A C^*-algebra is a Banach*-algelna such that*

$$\|a^*a\| = \|a\|^2 \quad (a \in A)$$

Exempel 3.1.1

- The scalar field C is a unital C*-algebra with involution given by complex conjugation $\lambda \mapsto \bar{\lambda}$.
- If Ω is a locally compact Hausdorff space, then $C_0(\Omega)$ is a C^*-algebra with involution $f \mapsto \bar{f}$. Similarly, all of the following algebras are C^*-algebras with involution given by $f \mapsto \bar{f}$:
  (a) $\ell^\infty(S)$ where S is a set;
  (b) $L^\infty(\Omega, \mu)$ where (Ω, μ) is a measure space;
  (c) $C_b(\Omega)$ where Ω is a topological space;

(d) $B_\infty(\Omega)$ where Ω is a measurable space.

- Let $\mathcal{H}$ be a complex Hilbert space with inner product denoted $\langle \cdot, \cdot \rangle$. The collection of bounded linear operators on $\mathcal{H}$, denoted by $\mathcal{B}(\mathcal{H})$, is a C^*-algebra. The linear structure is clear. The product is by composition of operators. The $*$ operation is the adjoint; for any operator a on $\mathcal{H}$, its adjoint is defined by the equation $< a^*\xi, \eta >=< \xi$, a $\rangle$, for all ξ and η in $\mathcal{H}$. Finally, the norm is given by

$$\|a\| = \sup\{\|a\xi\| \mid \xi \in \mathcal{H}, \|\xi\| \leq 1\}$$

for any a in $\mathcal{B}(\mathcal{H})$.

- If n is any positive integer, we let $M_n(\mathbb{C})$ denote the set of $n \times n$ complex matrices. It is a C^*-algebra using the usual algebraic operations for matrices. The * operation is to take the transpose of the matrix and then take complex conjugates of all its entries. For the norm, we must resort back to the same definition as our last example

$$\|a\| = \sup\left\{\|a\xi\|_2 \mid \xi \in \mathbb{C}^n, \|\xi\|_2 \leq 1\right\}$$

where $\|\cdot\|_2$ is the usual ℓ^2-norm on $\mathbb{C}^n$. Of course, this example is a special case of the last using $\mathcal{H} = \mathbb{C}^n$, and using a fixed basis to represent linear transformations as matrices.

- Let X be a compact Hausdorff space and consider

$$C(X) = \{f : X \to \mathbb{C} \mid f \text{ continuous }\}$$

The algebraic operations of addition, scalar multiplication and multiplication are all point-wise. The $*$ is point-wise complex conjugation. The norm is the usual supremum norm

$$\|f\| = \sup\{|f(x)| \mid x \in X\}$$

for any f in $C(X)$. This particular examples has the two additional features that $C(X)$ is both unital and commutative.

Extending this slightly, let X be a locally compact Hausdorff space and consider

$$C_0(X) = \{f : X \to \mathbb{C} \mid f \text{ continuous, vanishing at infinity }\}$$

Recall that a function f is said to vanish at infinity if, for every $\epsilon > 0$, there is a compact set K such that $|f(x)| < \epsilon$, for all x in $X \backslash K$. The algebraic operations and the norm are done in exactly the same way as the case above. This example is also commutative, but is unital if and only if X is compact (in which case it is the same as $C(X)$).

- Suppose that A and B are C^*-algebras, we form their direct sum

$$A \oplus B = \{(a, b) \mid a \in A, b \in B\}$$

The algebraic operations are all performed coordinate-wise and the norm is given by

$$\|(a, b)\| = \max\{\|a\|, \|b\|\}$$

for any a in A and b in B. There is an obvious extension of this notion to finite direct sums. Also, if $A_n, n \geq 1$ is a sequence of C^*-algebras, their direct sum is defined as

$$\oplus_{n=1}^{\infty} A_n = \left\{(a_1, a_2, \ldots) \mid a_n \in A_n, \text{ for all } n, \lim_n \|a_n\| = 0\right\}$$

Aside from noting the condition above on the norms, there is not much else to add.

Proposition 3.1.1 *If $(A_\lambda)_{\lambda\in\Lambda}$ is a family of C^*-algebras, then the direct sum $\oplus_\lambda A_\lambda$ is a C*-algebra with the pointwise-defined involution.*

Proposition 3.1.2

1. *If $\varphi : A \to B$ is a homomorphism of $*$-algebras A and B and φ preserves adjoints, that is, $\varphi(a^*) = (\varphi(a))^*(a \in A)$, then φ is a *-homomorphism. If in addition φ is a bijection, it is a *-isomorphism. If $\varphi : A \to B$ is a *-homomorphism, then $\ker(\varphi)$ is a self-adjoint ideal in A and $\varphi(A)$ is a *-subalgebra of B.*

2. *An automorphism of a $*$-algebra A is a $*$-isomorphism $\varphi : A \to A$. If A is unital and u is a unitary in A, then*

$$\mathrm{Ad}\,u : A \to A, \quad a \mapsto uau^*$$

   is an automorphism of A. Such automorphisms are called inner.

3. *We say elements a, b of A are unitarily equivalent if there exists a unitary u of A such that $b = uau^*$. Since the unitaries form a group, this is an equivalence relation on A. Note that $\sigma(a) = \sigma(b)$ if a and b are unitarily equivalent..*

Next, we introduce some terminology for elements in a C^*-algebra. For a given a in A, the element a^* is usually called the adjoint of a. The first term in the following definition is then rather obvious. The second is much less so, but is used for historical reasons from operator theory. The remaining terms all have a geometric flavour. If one considers the elements in $\mathcal{B}(\mathcal{H})$, operators on a Hilbert space, each of these purely algebraic terms can be given an equivalent formulation in geometric terms of the action of the operator on the Hilbert space.

3.2 Properties of C*-algebras

Definition 3.2.1 *Let A be a C^*-algebra.*

1. *An element a is self-adjoint if $a^* = a$.*
   (the set of self -adjoint elements of A is den oted by A_{sa}.)

2. *An element a is normal if $a^*a = aa^*$.*

3. *An element p is a projection if $p^2 = p = p^*$ (that is, p is a self-adjoint idempotent).*

4. *Assuming that A is unital, an element u is a unitary if $u^*u = 1 = uu^*$; that is, u is invertible and $u^{-1} = u^*$.*

5. ***Assuming that** A **is unital**, an element u is an isometry if $u^*u = 1$.*

6. *An element u is a partial isometry if u^*u is a projection.*

7. *An element a is positive if it may be written $a = b^*b$, for some b in A. In this case, we often write $a \geq 0$ for brevity.*

Proposition 3.2.1 *For each $a \in A$ there exist unique self-adjoint $b, c \in A$ such that $a = b + ic$ called the real and imaginary parts of a,respectively.*

The real and imaginary parts of a are denoted $\mathrm{Re}\,a$ and $\mathrm{Im}\,a$.

☛ **Proof**

Simply check that $b := (a + a^*)/2$ and $c := (a - a^*)/2i$ are self-adjoint. The sum $b + ic$ is obviously a. Uniqueness follows from the fact that if $b + ic = 0$ for b, c self-adjoint then $b = 0 = c$ since

$$b = b^* = (-ic)^* = ic = -b$$

Proposition 3.2.2 *Let A be a C^*-algebra.*

1. *the elements a^*a and aa^* are self-adjoint.*
2. *If A is unital then $1^* = 1 \quad (1^* = (11^*)^* = 1)$*
3. *If $a \in \mathrm{Inv}(A)$, then $(a^*)^{-1} = (a^{-1})^*$*
4. *for any $a \in A$ then $\sigma(a^*) = \{\lambda \in C; \lambda \in \sigma(a)\}$*

Theorem 3.2.1 *If a is a self-adjoint element of a C^*-algebra A, then $r(a) = \|a\|$.*

☛ **Proof**
Clearly, $\|a^2\| = \|a\|^2$, and therefore by induction $\left\|a^{2^n}\right\| = \|a\|^{2^n}$.
so $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{n\to\infty} \left\|a^{2^n}\right\|^{1/2^n} = \|a\|$.

Lemma 3.2.1 1. *There is at most one norm on a *-algebra making it a C^*-algebra.*

2. *Let A be a Banach algebra endowed with an involution such that $\|a\|^2 \leq \|a^*a\| \, (a \in A)$. Then A is a C^*-algebra.*

☛ **Proof**

1. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on a $*$-algebra A making it a C^*-algebra, then

$$\|a\|_j^2 = \|a^*a\|_j = r(a^*a) = \sup_{\lambda \in \sigma(a^*a)} |\lambda| \quad (j = 1, 2),$$

so $\|a\|_1 = \|a\|_2$

2. The inequalities $\|a\|^2 \leq \|a^*a\| \leq \|a^*\| \, \|a\|$ imply that $\|a\| \leq \|a^*\|$ for all a. Hence, $\|a\| = \|a^*\|$, and therefore $\|a\|^2 = \|a^*a\|$.

We associate to each C*-algebra A a certain unital C*-algebra $M(A)$ which contains A as an ideal. This algebra is of great importance in more advanced aspects of the theory, especially in certain approaches to **K-theory.**

Definition 3.2.2 *A double centraliser for a C*-algebra A is a pair (L, R) of bounded linear maps on A, such that for all $a, b \in A$*

$$L(ab) = L(a)b, \quad R(ab) = aR(b) \quad \text{and} \quad R(a)b = aL(b)$$

Exempel 3.2.1

☛ if $c \in A$ and L_c, R_c are the linear maps on A defined by $L_c(a) = ca$ and $R_c(a) = ac$, then (L_c, R_c) is a double centraliser on A. It is easily checked that for all $c \in A$

$$\|c\| = \sup_{\|b\| \leq 1} \|cb\| = \sup_{\|b\| \leq 1} \|bc\|$$

and therefore $\|L_c\| = \|R_c\| = \|c\|$.

Lemma 3.2.2 1. *If (L, R) is a double centraliser on a C^*-algebra A, then $\|L\| = \|R\|$.*

☛ **Proof**
Since $\|aL(b)\| = \|R(a)b\| \leq \|R\|\|a\|\|b\|$, we have

$$\|L(b)\| = \sup_{\|a\| \leq 1} \|aL(b)\| \leq \|R\|\|b\|$$

and therefore $\|L\| \leq \|R\|$. Also, $\|R(a)b\| = \|aL(b)\| \leq \|L\|\|a\|\|b\|$ implies

$$\|R(a)\| = \sup_{\|b\| \leq 1} \|R(a)b\| \leq \|L\|\|a\|$$

and therefore $\|R\| \leq \|L\|$. Thus, $\|L\| = \|R\|$.

3.3 Positive Elements of C*-Algebras

In this section we introduce a partial order relation on the hermitian elements of a C*-algebra. The principal results are the existence of a unique positive square root for each positive element and Theorem 2.2.4, which asserts that elements of the form a^*a are positive.

Definition 3.3.1 *Let A be a unital algebra and B a subalgebra*

An element a of a C^*-algebra A is positive if a is hermitian and $\sigma(a) \subseteq \mathbb{R}^+$. We write $a \geq 0$ to mean that a is positive, and denote by $A^+$the set of positive elements of A. By the preceding observation $B^+ = B \cap A^+$for any C^*-subalgebra B of A.

Remark 3.3.1 *1. Let $A = C_0(\Omega)$, where Ω is a locally compact Hausdorff space. Then A_{sa} is the set of real-valued functions in A and there is a natural partial order on A_{sa} given by $f \leq g$ if and only if $f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$. An element $f \in A$ is positive, that is, $f \geq 0$, if and only if f is of the form $f = \bar{g}g$ for some $g \in A$, and in this case f has a unique positive square root in A, namely the function $\omega \mapsto \sqrt{f(\omega)}$. Note that if $f = \bar{f}$ we can also express the positivity condition in terms of the norm: If $t \in \mathbb{R}$, then f is positive if $\|f - t\| \leq t$, and in the reverse direction if $\|f\| \leq t$ and $f \geq 0$, then $\|f - t\| \leq t$. We shall presently define a partial order on an arbitrary C*-algebra that generalises that of $C_0(\Omega)$, and we shall obtain similar, and many other, results.*

Lemma 3.3.1 *Let a be a self-adjoint element of a unital C^*-algebra A.*
1. Let $f(x) = \max\{x, 0\}$ and $g(x) = \max\{-x, 0\}$, for x in $\mathbb{R}$. Then $f(a), g(a)$ are both positive. We have

$$\begin{aligned} a &= f(a) - g(a), \\ af(a) &= f(a)^2, \\ ag(a) &= -g(a)^2, \\ f(a)g(a) &= 0. \end{aligned}$$

2. If $\operatorname{spec}(a) \subseteq [0, \infty)$, then a is positive.

☛ **Proof**

1.It is clear that f and g are continuous functions on the spectrum of a, each is real-valued, $f(x)-g(x) = x, xf(x) = f(x)^2, xg(x) = -g(x)^2$ and $f(x)g(x) = 0$, for all x in $\mathbb{R}$. Corollary 1.4.9 then implies that $f(a), g(a)$ are self-adjoint elements of A, satisfying the desired equations. Since $f(x)$ is positive, it has a square root and it follows that $f(a) = (\sqrt{f}(a))^2 = (\sqrt{f}(a))^*(\sqrt{f}(a))$ is positive. The same argument shows $g(a)$ is positive.

2.For the second statement, it suffices to notice in the proof above that if $\operatorname{spec}(a) \subseteq [0, \infty)$, then $g(a) = 0$.

The following statement has a very easy proof which is a nice application of the spectral theorem. Moreover, the result is a very useful tool in dealing with elements with positive spectrum.

Lemma 3.3.2 *Let a be a self-adjoint element of a unital C^*-algebra A. The following are equivalent.*
1. $\operatorname{spec}(a) \subseteq [0, \infty)$.
2. For all $t \geq \|a\|$, we have $\|t - a\| \leq t$.
3. For some $t \geq \|a\|$, we have $\|t - a\| \leq t$.

☛ **Proof**

Since the spectrum of a is a subset of the reals and the spectral radius of a is its norm, $\operatorname{spec}(a) \subseteq [-\|a\|, \|a\|]$. We consider the function $f_t(x) = t - x$, for values of $t \geq \|a\|$. The function is positive and monotone decreasing on $[-\|a\|, \|a\|]$. Hence, the norm of the restriction of f_t to the spectrum of a is just its value at the infimum of $\operatorname{spec}(a)$. In view of, $\|t - a\|$ is also the value of f_t at the infimum of $\operatorname{spec}(a)$.

Finally, notice that $f_t(x) \leq t$ if and only if $x \geq 0$. Putting this together, we see that the minimum of $\operatorname{spec}(a)$ is negative if and only if $\|t-a\| > t$. Taking the negations of both statements, $\operatorname{spec}(a) \subseteq [0,\infty)$ if and only if $\|t-a\| \leq t$. This holds for all $t \geq \|a\|$.

Here is one very useful consequence of this result.

Lemma 3.3.3 *If a and b are self-adjoint elements of the unital C^* algebra A and* $\operatorname{spec}(a), \operatorname{spec}(b) \subseteq [0,\infty)$, *then* $\operatorname{spec}(a+b) \subseteq [0,\infty)$.

☛ **Proof**

Let $t = \|a\| + \|b\|$ which is evidently at least $\|a+b\|$. Then we have

$$\begin{aligned} \|t-(a+b)\| &= \|(\|a\|-a)+(\|b\|-b)\| \\ &\leq \|(\|a\|-a)\| + \|(\|b\|-b)\| \\ &\leq \|a\| + \|b\| \\ &= t \end{aligned}$$

where we have used the last lemma in moving from the second line to the third. The conclusion follows from another application of the last lemma.

Before getting to our main result, we will need the following very basic fact regarding the spectrum.

Lemma 3.3.4 *Let a, b be two elements of a unital algebra A. Then we have*

$$\operatorname{spec}(ab)\backslash\{0\} = \operatorname{spec}(ba)\backslash\{0\}.$$

☛ **Proof**

It clearly suffices to prove that if λ is a non-zero complex number such that $\lambda - ab$ is invertible, then $\lambda - ba$ is invertible also. Consider $x = \lambda^{-1} + \lambda^{-1}b(\lambda - ab)^{-1}a$. We see that

$$\begin{aligned} x(\lambda - ba) &= \left(\lambda^{-1} + \lambda^{-1}b(\lambda-ab)^{-1}a\right)(\lambda - ba) \\ &= 1 - \lambda^{-1}ba + \lambda^{-1}b(\lambda-ab)^{-1}(\lambda a - aba) \\ &= 1 - \lambda^{-1}ba + \lambda^{-1}b(\lambda-ab)^{-1}(\lambda - ab)a \\ &= 1 - \lambda^{-1}ba + \lambda^{-1}ba \\ &= 1. \end{aligned}$$

A similar computation which we omit shows that $(\lambda - ba)x = 1$ and we are done.

We are now ready to prove our main result.

Theorem 3.3.1 *Let a be a self-adjoint element of a unital C^*-algebra A. Then a is positive if and only if* $\operatorname{spec}(a) \subseteq [0,\infty)$.

☛ **Proof**

The 'if' direction has already been done in Lemma 1.6.1. Let us now assume that $a = b^*b$, for some b in A. Using the notation of 1.6.1, consider $c = bg(a)$. Then we have $c^*c = g(a)b^*bg(a) = g(a)ag(a) = -g(a)^3$. Write $c = d + ie$, where d, e are self-adjoint elements of A. A simple computation shows that

$$cc^* = d^2 + e^2 - c^*c = d^2 + e^2 + g(a)^3.$$

As the functions x^2 and $g(x)^3$ are positive, it follows from Corollary 1.5.4 that each of d^2, c^2 and $g(a)^3$ has spectrum contained in $[0,\infty)$. By Lemma 1.6.3, so does cc^*. On the other hand, again using Corollary 1.5.4, we have $\operatorname{spec}(c^*c) = \operatorname{spec}(-g(a)^3) \subseteq (-\infty, 0]$.

We now appeal to Lemma 1.6.4 (using $a = c, b = c^*$) to conclude that $\operatorname{spec}(c^*c) = \operatorname{spec}(cc^*) = \{0\}$. But this means that $-g(a)^3 = c^*c = 0$ and it follows that $g(a) = 0$. This implies that the restriction of g to the spectrum of a is zero, which means $\operatorname{spec}(a) \subseteq [0,\infty)$ and we are done.

Chapter 4

The numerical range and the numerical radius of operators In C*-algebras

4.1 Numerical Range

Definition 4.1.1 *1. Let $\mathfrak{A}$ be a unital C^*-algebra with unit denoted by e. Let $\mathfrak{A}'$ denote the dual space of $\mathfrak{A}$, and define the set of normalized states of $\mathfrak{A}$ by*

$$\mathcal{S}(\mathfrak{A}) = \{\varphi \in \mathfrak{A}' : \varphi(e) = \|\varphi\| = 1\}.$$

2. A linear functional $\varphi \in \mathfrak{A}'$ is said to be positive, and write $\varphi \geq 0$, if $\varphi(x^*x) \geq 0$ for all $x \in \mathfrak{A}$. Note that the set of normalized states $\mathcal{S}(\mathfrak{A})$ is

$$\mathcal{S}(\mathfrak{A}) = \{\varphi \in \mathfrak{A}' : \varphi \geq 0 \quad \text{and} \quad \varphi(e) = 1\}$$

3. A positive linear functional φ on $\mathfrak{A}$ is said to be pure if for every positive functional ψ on $\mathfrak{A}$ satisfying $\psi(x^*x) \leq \varphi(x^*x)$ for all $x \in \mathfrak{A}$, there is a scalar $0 \leq \mu \leq 1$ such that $\psi = \mu\varphi$. The set of pure states on $\mathfrak{A}$ is denoted by $\mathcal{P}(\mathfrak{A})$.

Definition 4.1.2 (Numerical Range) *Let $\mathfrak{A}$ be a unital C^*-algebra with unit denoted by e. The numerical range of an element $x \in \mathfrak{A}$ is*

$$W(\mathfrak{A}, x) = \{\varphi(x) : \varphi \in \mathcal{S}(\mathfrak{A})\}$$

It is a nonempty compact and convex set of the complex plane $\mathbb{C}$,

Exempel 4.1.1

- let $\mathcal{A} = \mathbb{B}(H)$ be the C*-algebra of all bounded linear operators on a complex Hilbert space H and $T \in \mathcal{A}$. where

$$W(\mathbb{B}(H), T) := \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$$

  enditemize

- Let K be a compact Hausdorff space. Then

$$W(C(K), f) = \operatorname{co} f(K)$$

  for all $f \in C(K)$.

  for Example $f(z) = z$ on the unit circle $\mathbb{T}$:

$$W(f) = \overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$$

☛ Although you can define the algebraic numerical range for a linear operator on any Banach algebra, the most common use of this definition is to let $\mathcal{A}$ be the set of bounded operators on a Banach space, which will be denoted $\mathcal{L}(E)$. In this form the definition for the algebraic numerical range is

$$W(\mathcal{L}(E), T) = \{\Phi(T) : \Phi \in \mathcal{L}(E)^*, \Phi(1) = 1 = \|\Phi\|\}$$

Theorem 4.1.1 *compact-convex For each $x \in A$, $W(A, x)$ is a compact convex subset of $\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$).*

☛ **Proof**
We have:

$$S(A) = \{\varphi \in A' : \|\varphi\| \leq 1 \text{ and } \varphi(e) = 1\}$$

Thus $S(A)$ is a convex weak* compact subset of A'. The set $W(A, x)$ is the image of $S(A)$ under the weak* continuous linear mapping:

$$\varphi \mapsto \varphi(x),$$

and therefore is a compact convex subset of $\mathbb{F}$. Hence $W(A, x)$ is a compact subset of $\mathbb{F}$.

Theorem 4.1.2 *Let B be a subalgebra of A containing the unit element. Then for each $b \in B$:*

$$W(B, b) = W(A, b)$$

☛ **Proof**
By the Hahn-Banach theorem, the restriction mapping:

$$\varphi \mapsto \varphi|_B$$

maps $S(A)$ onto $S(B)$. Therefore:

$$W(A, b) = W(B, b).$$

Theorem 4.1.3 *Let $(A, \|\cdot\|)$ be a unital Banach algebra with unit element e, and let $a \in A$. Then:*

$$\max\{\operatorname{Re}\lambda : \lambda \in W(A, a)\} = \inf_{\alpha>0} \frac{1}{\alpha}\left(\|e + \alpha a\| - 1\right) = \lim_{\alpha\to 0^+} \frac{1}{\alpha}\left(\|e + \alpha a\| - 1\right),$$

where $W(A, a)$ denotes the numerical range of a.

☛ **Proof**
Let $\mu = \max\{\operatorname{Re}\lambda : \lambda \in V(A, a)\}$. The proof consists of two parts:
Part 1: $\mu \leq \inf_{\alpha>0} \frac{1}{\alpha}\left(\|e + \alpha a\| - 1\right)$.
For any $\alpha > 0$ and any $f \in S(A, e)$ (the set of linear functionals with $f(e) = \|e\| = 1$), we have:

$$f(a) = \frac{1}{\alpha}\left(f(e + \alpha a) - 1\right).$$

Since $\|f\| \leq 1$, it follows that:

$$\operatorname{Re} f(a) \leq \frac{1}{\alpha}\left(\|e + \alpha a\| - 1\right).$$

Taking the supremum over all such f and the infimum over $\alpha > 0$ gives the desired inequality.
Part 2: $\inf_{\alpha>0} \frac{1}{\alpha}\left(\|e + \alpha a\| - 1\right) \leq \mu$.
Assume $a \neq 0$ (the case $a = 0$ is trivial). For $0 < \alpha < \|a\|^{-1}$ and $x \in B(A)$ (the unit ball), $f \in S(A, x)$, we have:

$$\|(e - \alpha a)x\| \geq \operatorname{Re} f((e - \alpha a)x) \geq 1 - \alpha\mu.$$

Applying this to $x = e + \alpha a$ yields:

$$\|e + \alpha a\| \leq \frac{1}{1 - \alpha\mu}\|e - \alpha^2 a^2\| \leq \frac{1 + \alpha^2\|a^2\|}{1 - \alpha\mu}.$$

Thus:

$$\frac{\|e+\alpha a\|-1}{\alpha} \leq \frac{\mu+\alpha\|a^2\|}{1-\alpha\mu}.$$

Taking the limit as $\alpha \to 0^+$ completes the proof.
Combining both parts establishes the equality.

Theorem 4.1.4 *Let A be a complete unital Banach algebra with unit element e. For each $a \in A$,*

$$\sigma(A,a) \cap \mathbb{F} \subset W(A,a),$$

where:

- *$\sigma(A,a)$ denotes the spectrum of a,*
- *$\mathbb{F}$ denotes the underlying field ($\mathbb{C}$ or $\mathbb{R}$),*
- *$W(A,a)$ denotes the numerical range of a.*

☛ **Proof**

Let $\lambda \in \sigma(A,a) \cap \mathbb{F}$. Then $\lambda e - a$ is a singular element in A. We consider two cases:
Case 1: $\lambda e - a$ has no left inverse.

Let $J = A(\lambda e - a)$ be the left ideal generated by $\lambda e - a$. Since $\lambda e - a$ is singular, J is a proper left ideal. From the Banach algebra properties, we have:

$$\|e - x\| \geq 1 \quad \text{for all } x \in J.$$

By the Hahn-Banach theorem, there exists a linear functional $f \in A^*$ such that:

$$f(e) = \|f\| = 1 \quad \text{and} \quad f(J) = \{0\}.$$

Since $J = A(\lambda e - a)$, we have $f((\lambda e - a)y) = 0$ for all $y \in A$. Taking $y = e$ gives:

$$f(\lambda e - a) = 0 \implies \lambda = f(a).$$

Thus $\lambda \in V(A,a)$.
Case 2: $\lambda e - a$ has no right inverse.

The proof is analogous using right ideals instead of left ideals.
Combining both cases proves the inclusion.

4.2 Numerical Radius

Definition 4.2.1 (Numerical Radius) *Let $x \in \mathcal{A}$ The numerical radius of x is denoted by $\omega(x)$ and it is defined as follow*

$$w(x) = \sup\{|z| : z \in W(x)\}$$

. It is well-known that $v(\cdot)$ define a norm on $\mathfrak{A}$, which is equivalent to the C^*-norm $\|\cdot\|$. In fact, the following inequalities are well-known:

$$\frac{1}{2}\|x\| \leq w(x) \leq \|x\| \quad (x \in \mathfrak{A})$$

Proposition 4.2.1 *Let $\mathfrak{A}$ be a unital C^*-algebra with unit denoted by e*

1. *It if x is normal, then $w(x) = \|x\|$.*
2. *since $\mathcal{P}(\mathfrak{A})$ coincides with the set of all extremal points of $\mathcal{S}(\mathfrak{A})$ thus for every $x \in \mathfrak{A}$ we have*

$$w(x) = \sup_{\varphi \in \mathcal{S}(\mathfrak{A})} |\varphi(x)| = \sup_{\varphi \in \mathcal{P}(\mathfrak{A})} |\varphi(x)|.$$

Now, let us introduce a new type of parallelism in C^*-algebra based on numerical radius

Definition 4.2.2 *An element $x \in \mathfrak{A}$ is called the numerical radius parallel to another element $y \in \mathfrak{A}$, denoted by $x\|_v y$, if abd only if $w(x + \lambda x) = w(x) + w(y)$ for some $\lambda \in \mathbb{C}$.Such that $|\lambda| = 1$*

Remark 4.2.1 *It is easy to see that the numerical radius parallelism is reflexive $(x\|_v x)$, symmetric $(x\|_v y$ if and only if $y\|_v x)$ and $\mathbb{R}$-homogenous $(x\,\|_v y \Rightarrow \alpha x\|_v\, \beta y$ for all $\alpha, \beta \in \mathbb{R}$). Notice that two linearly dependent elements are numerical radius parallel. The converse is however not true, in general.*

We start our work with the following Theorem.

Theorem 4.2.1 *Let $\mathfrak{A}$ be a C^*-algebra and let φ be a state over $\mathfrak{A}$. For $x \in \mathfrak{A}$ the following statements hold.*

1. $\sup_{\theta \in \mathbb{R}} \left|\operatorname{Re}\left(e^{i\theta}\varphi(x)\right)\right| = |\varphi(x)|$.
2. $\sup_{\theta \in \mathbb{R}} \left|\operatorname{Im}\left(e^{i\theta}\varphi(x)\right)\right| = |\varphi(x)|$.

☛ **Proof**

We may assume that $\varphi(x) \neq 0$ otherwise (1) and (2) trivially hold.

1. Put $e^{i\theta_0} = \frac{\overline{\varphi(x)}}{|\varphi(x)|}$. Then we have

$$|\varphi(x)| = \left|\operatorname{Re}\left(e^{i\theta_0}\varphi(x)\right)\right| \leq \sup_{\theta \in \mathbb{R}} \left|\operatorname{Re}\left(e^{i\theta}\varphi(x)\right)\right| \leq \sup_{\theta \in \mathbb{R}} \left|e^{i\theta}\varphi(x)\right| = |\varphi(x)|$$

   and hence $|\varphi(x)| = \sup_{\theta \in \mathbb{R}} \left|\operatorname{Re}\left(e^{i\theta}\varphi(x)\right)\right|$.

2. By replacing x in (1) by ix, we obtain

$$\sup_{\theta \in \mathbb{R}} \left|\operatorname{Im}\left(e^{i\theta}\varphi(x)\right)\right| = \sup_{\theta \in \mathbb{R}} \left|\operatorname{Re}\left(e^{i\theta}\varphi(ix)\right)\right| = |\varphi(ix)| = |\varphi(x)|$$

Now, we are in a position to state two useful characterizations of the numerical radius for elements of a C^*-algebra

Theorem 4.2.2 *Let $\mathfrak{A}$ be a C^*-algebra. For $x \in \mathfrak{A}$ the following statements hold.*

1. $\sup_{\theta \in \mathbb{R}} \left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| = w(x)$.
2. $\sup_{\theta \in \mathbb{R}} \left\|\operatorname{Im}\left(e^{i\theta}x\right)\right\| = w(x)$.

☛ **Proof**

1. Since $\operatorname{Re}\left(e^{i\theta}x\right)$ is self adjoint for any $\theta \in \mathbb{R}$, we have

$$\left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| = w\left(\operatorname{Re}\left(e^{i\theta}x\right)\right)$$

   Therefore, we get

$$\begin{aligned}
\sup_{\theta\in\mathbb{R}}\left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| &= \sup_{\theta\in\mathbb{R}} w\left(\operatorname{Re}\left(e^{i\theta}x\right)\right)\\
&= \sup_{\theta\in\mathbb{R}}\sup_{\varphi\in\mathcal{S}(\mathfrak{A})}\left|\varphi\left(\operatorname{Re}\left(e^{i\theta}x\right)\right)\right|\\
&= \sup_{\theta\in\mathbb{R}}\sup_{\varphi\in\mathcal{S}(\mathfrak{A})}\left|\operatorname{Re}\left(e^{i\theta}\varphi(x)\right)\right|\\
&= \sup_{\varphi\in\mathcal{S}(\mathfrak{A})}\sup_{\theta\in\mathbb{R}}\left|\operatorname{Re}\left(e^{i\theta}\varphi(x)\right)\right|\\
&= \sup_{\varphi\in\mathcal{S}(\mathfrak{A})}|\varphi(x)| \qquad \text{(by Theorem. 4.3.1 (1))}\\
&= w(x).
\end{aligned}$$

   Thus $\sup_{\theta\in\mathbb{R}}\left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| = w(x)$.

2. By replacing x in **(1)** by ix, we reach that

$$\sup_{\theta\in\mathbb{R}}\left\|\operatorname{Im}\left(e^{i\theta}x\right)\right\| = \sup_{\theta\in\mathbb{R}}\left\|\operatorname{Re}\left(e^{i\theta}(ix)\right)\right\| = w(ix) = w(x)$$

Theorem 4.2.3 *The Crawford number of $T \in \mathbb{B}(H)$ is defined by*

$$c(T) = \inf\{|\langle Tx, x\rangle| : x \in H, \|x\| = 1\}$$

This concept is useful in studying linear operators. The Crawford number of $z \in \mathfrak{A}$ can be defined by

$$c(x) = \inf\{|\varphi(x)| : \varphi \in \mathcal{S}(\mathfrak{A})\}$$

In the following theorem, we give new improvement of the inequalities.

$$\frac{1}{2}\|x\| \leq w(x) \leq \|x\| \quad (x \in \mathfrak{A})$$

Theorem 4.2.4 *Let $\mathfrak{A}$ be a C^*-algebra. For $x \in \mathfrak{A}$ the following statements hold.*

1. $\frac{1}{2}\|x\| \leq \frac{1}{2}\sqrt{\left\||x|^2 + |x^*|^2\right\| + 2c\left(x^2\right)} \leq w(x)$.
2. $w(x) \leq \frac{1}{2}\sqrt{\left\||x|^2 + |x^*|^2\right\| + 2w\left(x^2\right)} \leq \frac{1}{2}\left(\|x\| + \|x^2\|^{\frac{1}{2}}\right) \leq \|x\|$.

☛ **Proof**

1. Let $x \in \mathfrak{A}$. By there is a state φ over $\mathfrak{A}$ such that

$$\varphi\left(|x|^2 + |x^*|^2\right) = \left\||x|^2 + |x^*|^2\right\|$$

   Let θ_0 be a real number such that $|\varphi\left(x^2\right)| = e^{2i\theta_0}\varphi\left(x^2\right)$. Then, by Theorem 4.1.2 (1), we have

$$
\begin{aligned}
w(x) \geq \left\|\operatorname{Re}\left(e^{i\theta_0}x\right)\right\| &= \frac{1}{2}\left\|e^{i\theta_0}x + e^{-i\theta_0}x^*\right\| \\
&= \frac{1}{2}\sqrt{\left\|\left(e^{i\theta_0}x + e^{-i\theta_0}x^*\right)\left(e^{i\theta_0}x + e^{-i\theta_0}x^*\right)^*\right\|} \\
&= \frac{1}{2}\sqrt{\left\||x|^2 + |x^*|^2 + 2\operatorname{Re}\left(e^{2i\theta_0}x^2\right)\right\|} \\
&\geq \frac{1}{2}\sqrt{\left|\varphi\left(|x|^2 + |x^*|^2 + 2\operatorname{Re}\left(e^{2i\theta_0}x^2\right)\right)\right|} \\
&= \frac{1}{2}\sqrt{\left|\varphi\left(|x|^2 + |x^*|^2\right) + 2\operatorname{Re}\left(e^{2i\theta_0}\varphi\left(x^2\right)\right)\right|} \\
&= \frac{1}{2}\sqrt{\left\||x|^2 + |x^*|^2\right\| + 2\left|\varphi\left(x^2\right)\right|} \\
&\geq \frac{1}{2}\sqrt{\left\||x|^2 + |x^*|^2\right\| + 2c\left(x^2\right)} \geq \frac{1}{2}\|x\|,
\end{aligned}
$$

which proves the inequalities in (1).

2. By Theorem **4.3.2 (1)**, as in the proof of (1) we get

$$
\begin{aligned}
w(x) &= \sup_{\theta\in\mathbb{R}}\left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\sqrt{\left\||x|^2 + |x^*|^2 + 2\operatorname{Re}\left(e^{2i\theta}x^2\right)\right\|} \\
&\leq \frac{1}{2}\sup_{\theta\in\mathbb{R}}\sqrt{\left\||x|^2 + |x^*|^2\right\| + 2\left\|\operatorname{Re}\left(e^{2i\theta}x^2\right)\right\|} \\
&\leq \frac{1}{2}\sqrt{\left\||x|^2 + |x^*|^2\right\| + 2\sup_{\theta\in\mathbb{R}}\left\|\operatorname{Re}\left(e^{2i\theta}x^2\right)\right\|} \\
&= \frac{1}{2}\sqrt{\left\||x|^2 + |x^*|^2\right\| + 2v\left(x^2\right)} \\
&\leq \frac{1}{2}\sqrt{\|x\|^2 + \|x^2\| + 2v\left(x^2\right)} \\
&\leq \frac{1}{2}\sqrt{\|x\|^2 + 3\left\|x^2\right\|} \quad (\text{ by } (1.1)) \\
&\leq \frac{1}{2}\sqrt{\|x\|^2 + 2\|x\|\left\|x^2\right\|^{\frac{1}{2}} + \|x^2\|} \quad \left(\text{since } \left\|x^2\right\| = \left\|x^2\right\|^{\frac{1}{2}}\left\|x^2\right\|^{\frac{1}{2}} \leq \|x\|\left\|x^2\right\|^{\frac{1}{2}}\right) \\
&= \frac{1}{2}\left(\|x\| + \left\|x^2\right\|^{\frac{1}{2}}\right) \leq \|x\|,
\end{aligned}
$$

which proves the inequalities in (2).
As a consequence of Theorem 4.3.2, we have the following result.

Proposition 4.2.2 *Let $\mathfrak{A}$ be a C^*-algebra. If $x \in \mathfrak{A}$ is such that $x^2 = 0$, then $w(x) = \frac{1}{2}\|x\|$.*

☛ **Proof**
Since $x^2 = 0$, by **Theorem 4.1.3 (2)**, we obtain $w(x) \leq \frac{1}{2}\left(\|x\| + \|x^2\|^{\frac{1}{2}}\right) = \frac{1}{2}\|x\|$. We also have that $\frac{1}{2}\|x\| \leq w(x)$ for every $x \in \mathfrak{A}$. Thus $w(x) = \frac{1}{2}\|x\|$.

The following result is another consequence of Theorem 4.3.2

Proposition 4.2.3 *Let $\mathfrak{A}$ be a C^*-algebra. If $x \in \mathfrak{A}$ is such that $w(x) = \|x\|$, then $\|x^2\| = \|x\|^2$.*

☛ **Proof**
It follows from Theorem **4.3.1 (2)** that $w(x) = \|x\|$ implies $\|x\| \leq \frac{1}{2}(\|x\| + \ \|x^2\|^{\frac{1}{2}}\Big) \leq \|x\|$.

Thus $\|x\| = \|x^2\|^{\frac{1}{2}}$, or equivalently $\|x^2\| = \|x\|^2$.

Theorem 4.2.5 *Let $\mathfrak{A}$ be a C^*-algebra and let $x \in \mathfrak{A}$. Then for $\alpha, \beta \in \mathbb{R}$, the following statements hold.*

1. $\sup_{\alpha^2+\beta^2=1} \|\alpha \operatorname{Re}(x) + \beta \operatorname{Im}(x)\| = w(x)$.
2. $\max\{\|Re(x)\|, \|\operatorname{Im}(x)\|\} \leq w(x)$.

☛ **Proof**

1. Let $\theta \in \mathbb{R}$. Put $\alpha = \cos\theta$ and $\beta = -\sin\theta$. We have

$$\begin{aligned}\operatorname{Re}\left(e^{i\theta}x\right) &= \frac{e^{i\theta}x + e^{-i\theta}x^*}{2} \\ &= \frac{(\cos\theta + i\sin\theta)x + (\cos\theta - i\sin\theta)x^*}{2} \\ &= (\cos\theta)\frac{x+x^*}{2} - (\sin\theta)\frac{x-x^*}{2i} \\ &= \alpha\operatorname{Re}(x) + \beta\operatorname{Im}(x).\end{aligned}$$

   Therefore

$$\sup_{\theta\in\mathbb{R}} \left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| = \sup_{\alpha^2+\beta^2=1} \|\alpha\operatorname{Re}(x) + \beta\operatorname{Im}(x)\|,$$

   and hence by **Theorem 4.3.1 (1)** we obtain $w(x) = \sup_{\alpha^2+\beta^2=1} \|\alpha\operatorname{Re}(x) + \beta\operatorname{Im}(x)\|$.

2. By setting $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$ in (i), we get $\|\operatorname{Re}(x)\| \leq v(x)$ and $\|\operatorname{Im}(x)\| \leq w(x)$. Thus $\max\{\|\operatorname{Re}(x)\|, \|\operatorname{Im}(x)\|\} \leq v(x)$.

In the next result, we obtain a necessary and sufficient condition for $w(x) = \frac{1}{2}\|x\|$ to hold. We will need the following lemma.

Lemma 4.2.1 *From [* ***Bibliography 28, Corollary 4.4****] Let $\mathfrak{A}$ be a C^*-algebra and let $x, y \in \mathfrak{A}$. Then the following statements are equivalent:*

1. $x\|y$.
2. *There exists a state φ over $\mathfrak{A}$ such that $|\varphi(x^*y)| = \|x\|\|y\|$.*

Theorem 4.2.6 *Let $\mathfrak{A}$ be a C^*-algebra and let $x \in \mathfrak{A}$. Then the following statements are equivalent:*

1. $w(x) = \frac{1}{2}\|x\|$.
2. $\|x\| = \left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| + \left\|\operatorname{Im}\left(e^{i\theta}x\right)\right\|$ *for all $\theta \in \mathbb{R}$.*

☛ **Proof**

1. **(1)** $\Rightarrow$ **(2)** Suppose that $w(x) = \frac{1}{2}\|x\|$. Then for any $\theta \in \mathbb{R}$, we have

$$\begin{aligned}\|x\| = \left\|e^{i\theta}x\right\| &= \left\|\operatorname{Re}\left(e^{i\theta}x\right) + i\operatorname{Im}\left(e^{i\theta}x\right)\right\| \\ &\leq \left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| + \left\|\operatorname{Im}\left(e^{i\theta}x\right)\right\| \\ &\leq 2\max\left\{\left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\|, \left\|\operatorname{Im}\left(e^{i\theta}x\right)\right\|\right\} \\ &\leq 2w\left(e^{i\theta}x\right) \quad (\text{ by Theorem 4.1.2 (2) }) \\ &= 2w(x) = \|x\|,\end{aligned}$$

   and hence $\|x\| = \left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| + \left\|\operatorname{Im}\left(e^{i\theta}x\right)\right\|$.

2. (**2**) $\Rightarrow$ **(1)** Suppose (2) holds. Thus for all $\theta \in \mathbb{R}$,

$$\left\|\operatorname{Re}\left(e^{i\theta}x\right) + i\operatorname{Im}\left(e^{i\theta}x\right)\right\| = \left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| + \left\|\operatorname{Im}\left(e^{i\theta}x\right)\right\|$$

so $\operatorname{Re}\left(e^{i\theta}x\right) \parallel \operatorname{Im}\left(e^{i\theta}x\right)$. By **Lemma 4.1.2**, there exists a state φ over $\mathfrak{A}$ such that

$$\left|\varphi\left(\left(\operatorname{Re}\left(e^{i\theta}x\right)\right)^* \operatorname{Im}\left(e^{i\theta}x\right)\right)\right| = \left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| \left\|\operatorname{Im}\left(e^{i\theta}x\right)\right\|,$$

and hence

$$\left|\varphi\left(\operatorname{Re}\left(e^{i\theta}x\right)\operatorname{Im}\left(e^{i\theta}x\right)\right)\right| = \left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| \left\|\operatorname{Im}\left(e^{i\theta}x\right)\right\|.$$

From this it follows that $w\left(\operatorname{Re}\left(e^{i\theta}x\right)\operatorname{Im}\left(e^{i\theta}x\right)\right) = \left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| \left\|\operatorname{Im}\left(e^{i\theta}x\right)\right\|$, so by **Theorem 4.1.2 (2)** we reach that

$$\left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| \left\|\operatorname{Im}\left(e^{i\theta}x\right)\right\| = \left\|\operatorname{Im}\left(\left(\operatorname{Re}\left(e^{i\theta}x\right)\operatorname{Im}\left(e^{i\theta}x\right)\right)\right\|$$

On the other hand,

$$\begin{aligned}
&\operatorname{Im}\left(\operatorname{Re}\left(e^{i\theta}x\right)\operatorname{Im}\left(e^{i\theta}x\right)\right) = \operatorname{Im}\left(\left(\frac{e^{i\theta}x + e^{-i\theta}x^*}{2}\right)\left(\frac{e^{i\theta}x - e^{-i\theta}x^*}{2i}\right)\right) \\
&= \operatorname{Im}\left(\frac{e^{2i\theta}x^2 - e^{-2i\theta}x^{*2} - xx^* + x^*x}{4i}\right) \\
&= \frac{1}{2i}\left\{\frac{e^{2i\theta}x^2 - e^{-2i\theta}x^{*2} - xx^* + x^*x}{4i}\right. \\
&\quad \left. - \frac{e^{-2i\theta}x^{*2} - e^{2i\theta}x^2 - xx^* + x^*x}{-4i}\right\} \\
&= \frac{xx^* - x^*x}{4} = \operatorname{Im}(\operatorname{Re}(x)\operatorname{Im}(x))
\end{aligned}$$

and by **Theorem 4.1.2 (2)**we get

$$\left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| \left\|\operatorname{Im}\left(e^{i\theta}x\right)\right\| = \left\|\operatorname{Im}(\operatorname{Re}(x)\operatorname{Im}(x))\right\|$$

Thus for all $\theta \in \mathbb{R}$, by (2) we obtain

$$\left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| = \frac{\|x\| + \sqrt{\|x\|^2 - 4\|\operatorname{Im}(\operatorname{Re}(x)\operatorname{Im}(x))\|}}{2}$$

and

$$\left\|\operatorname{Im}\left(e^{i\theta}x\right)\right\| = \frac{\|x\| - \sqrt{\|x\|^2 - 4\|\operatorname{Im}(\operatorname{Re}(x)\operatorname{Im}(x))\|}}{2}$$

Since

$$\operatorname{Re}\left(e^{i\theta}x\right) = \frac{(\cos\theta + i\sin\theta)x + (\cos\theta - i\sin\theta)x^*}{2} = \cos\theta\operatorname{Re}(x) - \sin\theta\operatorname{Im}(x)$$

and

$$\operatorname{Im}\left(e^{i\theta}x\right) = \frac{(\cos\theta + i\sin\theta)x - (\cos\theta - i\sin\theta)x^*}{2i} = \sin\theta\operatorname{Re}(x) + \cos\theta\operatorname{Im}(x)$$

So, from relations

$$\left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| = \frac{\|x\| + \sqrt{\|x\|^2 - 4\|\operatorname{Im}(\operatorname{Re}(x)\operatorname{Im}(x))\|}}{2}$$

and

$$\left\|\operatorname{Im}\left(e^{i\theta}x\right)\right\| = \frac{\|x\| - \sqrt{\|x\|^2 - 4\|\operatorname{Im}(\operatorname{Re}(x)\operatorname{Im}(x))\|}}{2}$$

we conclude that the functions $\left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\|, \left\|\operatorname{Im}\left(e^{i\theta}x\right)\right\|$ are continuous on $\theta \in \mathbb{R}$ and therefore they must be constant, i.e.,

$$\left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| = \left\|\operatorname{Im}\left(e^{i\theta}x\right)\right\| = \frac{1}{2}\|x\| \quad (\theta \in \mathbb{R})$$

Thus $\sup_{\theta\in\mathbb{R}} \left\|\operatorname{Re}\left(e^{i\theta}x\right)\right\| = \frac{1}{2}\|x\|$. Now, by **Theorem 4.1.2 (1)** we conclude that $w(x) = \frac{1}{2}\|x\|$.

Theorem 4.2.7 *Let $\mathfrak{A}$ be a C^*-algebra. For $x \in \mathfrak{A}$ the following statements hold.*

1. $\frac{1}{2}\|x\| \leq \frac{1}{2}\sqrt{\|x^*x + xx^*\|} \leq w(x)$.
2. $w(x) \leq \frac{1}{\sqrt{2}}\sqrt{\|x^*x + xx^*\|} \leq \|x\|$.

☛ **Proof**

1. Let $x \in \mathfrak{A}$. Clearly, $\frac{1}{2}\|x\| \leq \frac{1}{2}\sqrt{\|x^*x + xx^*\|}$. But, by simple computations,

$$x^*x + xx^* = 2\operatorname{Re}^2(x) + 2\operatorname{Im}^2(x)$$

Consequently, by **Theorem 4.1.3 (2)** we get

$$\begin{aligned}\frac{1}{2}\sqrt{\|x^*x + xx^*\|} &= \frac{1}{2}\sqrt{\left\|2\operatorname{Re}^2(x) + 2\operatorname{Im}^2(x)\right\|} \\ &\leq \frac{1}{2}\sqrt{2\|\operatorname{Re}(x)\|^2 + 2\|\operatorname{Im}(x)\|^2} \\ &\leq \frac{1}{2}\sqrt{2w^2(x) + 2w^2(x)} = w(x)\end{aligned}$$

Therefore $\frac{1}{2}\sqrt{\|x^*x + xx^*\|} \leq w(x)$.

2. Obviously, $\frac{1}{\sqrt{2}}\sqrt{\|x^*x + xx^*\|} \leq \|x\|$. Now, let $\pi : \mathfrak{A} \to \mathbb{B}(H)$ be a nondegenerate faithful representation of $\mathfrak{A}$ on some Hilbert space H **(see Bibliography [7, Theorem 2.6.1])**. Let $\alpha, \beta \in \mathbb{R}$ satisfy $\alpha^2 + \beta^2 = 1$. Then for any unit vector $\xi \in H$, we have

$$\begin{aligned}
\|\pi(\alpha\operatorname{Re}(x)+\beta\operatorname{Im}(x))\xi\| &= \left\|\begin{bmatrix}\pi(\operatorname{Re}(x)) & \pi(\operatorname{Im}(x))\\ 0 & 0\end{bmatrix}\begin{bmatrix}\alpha\xi\\ \beta\xi\end{bmatrix}\right\| \\
&\leq \left\|\begin{bmatrix}\pi(\operatorname{Re}(x)) & \pi(\operatorname{Im}(x))\\ 0 & 0\end{bmatrix}\right\| \\
&= \left\|\begin{bmatrix}\operatorname{Re}(\pi(x)) & \operatorname{Im}(\pi(x))\\ 0 & 0\end{bmatrix}\right\| \quad \text{(since } \pi \text{ is representation)} \\
&= \left\|\begin{bmatrix}\operatorname{Re}(\pi(x)) & \operatorname{Im}(\pi(x))\\ 0 & 0\end{bmatrix}\begin{bmatrix}\operatorname{Re}(\pi(x)) & 0\\ \operatorname{Im}(\pi(x)) & 0\end{bmatrix}\right\|^{\frac{1}{2}} \\
&= \left\|\operatorname{Re}^2(\pi(x))+\operatorname{Im}^2(\pi(x))\right\|^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}}\left\|\pi(x)^*\pi(x)+\pi(x)\pi(x)^*\right\|^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}}\left\|\pi\left(x^*x+xx^*\right)\right\|^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}}\left\|x^*x+xx^*\right\|^{\frac{1}{2}}. \quad \text{(since } \pi \text{ is isometric)}
\end{aligned}$$

Hence we have $\|\pi(\alpha\operatorname{Re}(x)+\beta\operatorname{Im}(x))\xi\| \leq \frac{1}{\sqrt{2}}\sqrt{\|x^*x+xx^*\|}$ and so by taking the supremum over all $\xi \in H$ we obtain $\|\pi(\alpha\operatorname{Re}(x)+\beta\operatorname{Im}(x))\| \leq \frac{1}{\sqrt{2}}\sqrt{\|x^*x+xx^*\|}$. From this it follows that $\|\alpha\operatorname{Re}(x)+\beta\operatorname{Im}(x)\| \leq \frac{1}{\sqrt{2}}\sqrt{\|x^*x+xx^*\|}$ and hence

$$\sup_{\alpha^2+\beta^2=1}\|\alpha\operatorname{Re}(x)+\beta\operatorname{Im}(x)\| \leq \frac{1}{\sqrt{2}}\sqrt{\|x^*x+xx^*\|}$$

Now, by **Theorem 4.1.3 (1)** we conclude that $w(x) \leq \frac{1}{\sqrt{2}}\sqrt{\|x^*x+xx^*\|}$.

In what follows, $r(x)$ stands for the spectral radius of an arbitrary element x in a C^*-algebra $\mathfrak{A}$. It is well known that for every $x \in \mathfrak{A}$, we have $r(x) \leq \|x\|$ and that equality holds in this inequality if x is normal. In the following lemma we obtain a spectral radius inequality for sums of elements in C^*-algebras.

Lemma 4.2.2 *Let $\mathfrak{A}$ be a C^*-algebra and let $z, w \in \mathfrak{A}$. Then*

$$r(z+w) \leq \frac{1}{2}\left(\|z\|+\|w\|+\sqrt{(\|z\|-\|w\|)^2+4\min\{\|zw\|,\|wz\|\}}\right).$$

☛ **Proof**

We first recall that **Bibliography** [**13, Corollary 1**] tells us that

$$r(T+S) \leq \frac{1}{2}\left(\|T\|+\|S\|+\sqrt{(\|T\|-\|S\|)^2+4\min\{\|TS\|,\|ST\|\}}\right)$$

for all bounded linear operators T, S that acting on a Hilbert space.
Now, let $\pi : \mathfrak{A} \to \mathbb{B}(H)$ be a non-degenerate faithful representation of $\mathfrak{A}$ on some Hilbert space H **(see Bibliography** [**7, Theorem 2.6.1**]**).** Since π is isometric, by letting $T = \pi(z)$ and $S = \pi(w)$ in

$$r(T+S) \leq \frac{1}{2}\left(\|T\|+\|S\|+\sqrt{(\|T\|-\|S\|)^2+4\min\{\|TS\|,\|ST\|\}}\right)$$

we obtain

$$
\begin{aligned}
r(z+w) =& r(\pi(z)+\pi(w)) \\
\leq& \frac{1}{2}(\|\pi(z)\|+\|\pi(w)\| \\
&+\sqrt{(\|\pi(z)\|-\|\pi(w)\|)^2+4\min\{\|\pi(z)\pi(w)\|,\|\pi(w)\pi(z)\|\}}\Big) \\
=& \frac{1}{2}\left(\|z\|+\|w\|+\sqrt{(\|z\|-\|w\|)^2+4\min\{\|zw\|,\|wz\|\}}\right),
\end{aligned}
$$

and the statement is proved.

Now, we present a refinement of the triangle inequality for the numerical radius in C^*-algebras.

Theorem 4.2.8 *Let $\mathfrak{A}$ be a C^*-algebra. For $x, y \in \mathfrak{A}$ the following statements hold.*

1.

$$
\begin{aligned}
w(x+y) \leq& \frac{1}{2}(w(x)+w(y)) \\
&+\frac{1}{2}\sqrt{(w(x)-w(y))^2+4\sup_{\theta\in\mathbb{R}}\|\mathrm{Re}\,(e^{i\theta}x)\,\mathrm{Re}\,(e^{i\theta}y)\|} \\
\leq& w(x)+w(y).
\end{aligned}
$$

2.

$$
\begin{aligned}
w(x+y) \leq& \frac{1}{2}(w(x)+w(y)) \\
&+\frac{1}{2}\sqrt{(w(x)-w(y))^2+4\sup_{\theta\in\mathbb{R}}\|\mathrm{Im}\,(e^{i\theta}x)\,\mathrm{Im}\,(e^{i\theta}y)\|} \\
\leq& w(x)+w(y).
\end{aligned}
$$

☛ **Proof**

1. Since $\mathrm{Re}\left(e^{i\theta}(x+y)\right)$ is self adjoint for any $\theta\in\mathbb{R}$, we have

$$
\left\|\mathrm{Re}\left(e^{i\theta}(x+y)\right)\right\| = r\left(\mathrm{Re}\left(e^{i\theta}(x+y)\right)\right)
$$

So, by letting $z=\mathrm{Re}\left(e^{i\theta}x\right)$ and $w=\mathrm{Re}\left(e^{i\theta}y\right)$ in **Lemma 4.1.2,** we obtain

$$
\begin{aligned}
&\left\|\mathrm{Re}\left(e^{i\theta}(x+y)\right)\right\| = r\left(\mathrm{Re}\left(e^{i\theta}(x+y)\right)\right) \\
&= r\left(\mathrm{Re}\left(e^{i\theta}x\right)+\mathrm{Re}\left(e^{i\theta}y\right)\right) \\
&\leq \frac{1}{2}\left(\left\|\mathrm{Re}\left(e^{i\theta}x\right)\right\|+\left\|\mathrm{Re}\left(e^{i\theta}y\right)\right\|\right. \\
&\left.+\sqrt{\left(\|\mathrm{Re}\,(e^{i\theta}x)\|-\|\mathrm{Re}\,(e^{i\theta}y)\|\right)^2+4\left\|\mathrm{Re}\,(e^{i\theta}x)\,\mathrm{Re}\,(e^{i\theta}y)\right\|}\right) \\
&= \left\|\begin{bmatrix} \left\|\mathrm{Re}\left(e^{i\theta}x\right)\right\| & \sqrt{\left\|\mathrm{Re}\,(e^{i\theta}x)\,\mathrm{Re}\,(e^{i\theta}y)\right\|}\Big)\, \| \\ \sqrt{\left\|\mathrm{Re}\,(e^{i\theta}x)\,\mathrm{Re}\,(e^{i\theta}y)\right\|} & \left\|\mathrm{Re}\left(e^{i\theta}y\right)\right\| \end{bmatrix}\right\| \\
&\leq \left\|\begin{bmatrix} \sup_{\theta\in\mathbb{R}}\left\|\mathrm{Re}\left(e^{i\theta}x\right)\right\| & \sup_{\theta\in\mathbb{R}}\sqrt{\left\|\mathrm{Re}\,(e^{i\theta}x)\,\mathrm{Re}\,(e^{i\theta}y)\right\|} \\ \sup_{\theta\in\mathbb{R}}\sqrt{\left\|\mathrm{Re}\,(e^{i\theta}x)\,\mathrm{Re}\,(e^{i\theta}y)\right\|} & \sup_{\theta\in\mathbb{R}}\left\|\mathrm{Re}\left(e^{i\theta}y\right)\right\| \end{bmatrix}\right\|
\end{aligned}
$$

(by the norm monotonicity of matrices with nonnegative entries)

$$
= \left\|\begin{bmatrix} v(x) & \sup_{\theta\in\mathbb{R}}\sqrt{\left\|\mathrm{Re}\,(e^{i\theta}x)\,\mathrm{Re}\,(e^{i\theta}y)\right\|} \\ \sup_{\theta\in\mathbb{R}}\sqrt{\left\|\mathrm{Re}\,(e^{i\theta}x)\,\mathrm{Re}\,(e^{i\theta}y)\right\|} & v(y) \end{bmatrix}\right\|
$$

(by **Theorem 4.1.2 (1)**)

$$= \frac{1}{2}(w(x) + w(y)) + \frac{1}{2}\sqrt{(w(x) - w(y))^2 + 4\sup_{\theta\in\mathbb{R}} \|\mathrm{Re}\,(e^{i\theta}x)\,\mathrm{Re}\,(e^{i\theta}y)\|}$$

Therefore, for every $\theta \in \mathbb{R}$ we have

$$\left\|\mathrm{Re}\left(e^{i\theta}(x+y)\right)\right\| \leq \tfrac{1}{2}(w(x)\ ) + w(y)\big) . \quad + \frac{1}{2}\sqrt{(w(x) - w(y))^2 + 4\sup_{\theta\in\mathbb{R}} \|\mathrm{Re}\,(e^{i\theta}x)\,\mathrm{Re}\,(e^{i\theta}y)\|}.$$

and hence

$$\sup_{\theta\in\mathbb{R}} \left\|\mathrm{Re}\left(e^{i\theta}(x+y)\right)\right\| \leq \frac{1}{2}(w(x) + w(y)) + \frac{1}{2}\sqrt{(w(x) - w(y))^2 + 4\sup_{\theta} \|\mathrm{Re}\,(e^{i\theta}x)\,\mathrm{Re}\,(e^{i\theta}y)\|_{\theta\in\mathbb{R}}}\,.$$

Now, by **Lemma 4.1.2 (1)** and the above inequality we get

$$w(x+y) \leq \frac{1}{2}\left(w(x) + w(y) + \sqrt{(w(x) - w(y))^2 + 4\sup_{\theta\in\mathbb{R}} \|\mathrm{Re}\,(e^{i\theta}x)\,\mathrm{Re}\,(e^{i\theta}y)\|}\right).$$

Furthermore, by **Lemma 4.1.2 (1)** we have

$$\sup_{\theta\in\mathbb{R}} \left\|\mathrm{Re}\left(e^{i\theta}x\right)\mathrm{Re}\left(e^{i\theta}y\right)\right\| \leq \sup_{\theta\in\mathbb{R}} \left\|\mathrm{Re}\left(e^{i\theta}x\right)\right\| \sup_{\theta\in\mathbb{R}} \left\|\mathrm{Re}\left(e^{i\theta}y\right)\right\| = w(x)w(y)$$

Thus the inequalities **(1)** follow from

$$w(x+y) \leq \frac{1}{2}\left(w(x) + w(y) + \sqrt{(w(x) - w(y))^2 + 4\sup_{\theta\in\mathbb{R}} \|\mathrm{Re}\,(e^{i\theta}x)\,\mathrm{Re}\,(e^{i\theta}y)\|}\right).$$

and the above inequality.

2. It is enough to replace x and y in **(1)** by ix and iy, respectively.

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