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شنيبة عطية فاطمة

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NOTATIONS

- $\mathbb{R}$: The set of real numbers.
- $\mathbb{R}^n$: The Cartesian product of n copies of $\mathbb{R}$.
- $\mathbb{N}$: The set of natural numbers.
- $\mathbb{R}^+$: The set of positive real numbers.
- $\mathbb{C}$: The set of complex numbers.
- $L(E,F)$: The set of linear functions from E to F.
- $C(E,F)$: The set of continuous functions from E to F.
- $C(\Omega)$: The set of continuous functions from Ω to $\mathbb{R}$.
- $K(E,F)$: The set of all compact mappings from E to F.
- $d(x,y)$:The distance between x and y.
- $\|\cdot\|_p$: L^p norm where $1 \leq p \leq \infty$.
- $\sum$: The total
- $|.|$: Absolute value
- i.e.: That is.
- p.p.: Almost everywhere.

INTRODUCTION

The quadratic integral equation is considered one of the most complex and difficult types of integral equations to solve. These equations appear in many scientific and engineering fields, such as vibration theory, dynamics, thermodynamics, diffusion theory, physical plasmas, electromagnetic fields, and many others. Despite the challenges they face in solving, the study of these equations and the development of new methods for solving them are of paramount importance for understanding many natural phenomena and engineering processes. The quadratic integral equation is characterized by the presence of nonlinear terms.

Such as the squares and products of the carried function and its derivatives. These nonlinear terms make it difficult to obtain accurate solutions by traditional analytical methods, requiring the use of complex approximate or numerical methods.

The aim of this thesis is to study some common and important quadratic integral equations, which appear in many physical and engineering applications. This thesis is divided into three chapters:

- The first chapter: In this chapter, we covered all the basic concepts that we will use in this thesis.
- The second chapter: We reviewed the generalities about integral equations.
- The third chapter: We studied some quadratic integral equations and solved them by different methods (analytically, numerically, etc.).

CHAPTER I

PRELIMINARIES

I.1 Spaces

I.1.1 The Lebesgue spaces $L^p(\Omega)$

Definition I.1.1. (Lebesgue Space): Let $p \in \mathbb{R}$, $1 \leq p \leq \infty$. We call the Lebesgue space $L^p(\Omega)$ the set[9]:

$$L^p(\Omega) = \{v : \Omega \to \mathbb{R} \text{ measurable on } \Omega \text{ and } |v|^p \text{ Lebesgue integrable on } \Omega\},$$

It is a Banach space if it is equipped with the norm:

$$\|v\|_{L^p(\Omega)} = \left(\int_\Omega |v(x)|^p dx\right)^{\frac{1}{p}},$$

If $p = \infty$ and $v : \Omega \to \mathbb{R}$ is measurable, then we define $\|v\|_{L^\infty(\Omega)}$ by:

$$\|v\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} |v(x)| = \inf\{c; |v(x)| \leq c\},$$

The space $L^\infty(\Omega)$ is also a Banach space.

Theorem I.1.1. For any $p \in [1, +\infty[$, the spaces $L^p(\Omega)$ satisfy the following assertions:

1. The spaces $L^p(\Omega)$ are Banach spaces.

2. For any function $u \in L^p(\Omega)$, any $v \in L^p(\Omega)$, the Hölder inequality is satisfied; i.e.

$$\int_\Omega |u(x)v(x)|dx \leq \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)},$$

   with $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$.

3. The spaces $L^2(\Omega)$ are separable spaces for $[1, +\infty[$.

4. The space $L^2(\Omega)$ equipped with the inner product

$$(u, v) = \int_\Omega u(x)v(x)dx, \quad \forall u, v \in L^2(\Omega),$$

   is a Hilbert space. Moreover, the Cauchy-Schwarz inequality corresponding to the Hölder inequality is satisfied; i.e.

$$\int_{\Omega} |u(x)v(x)|dx \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)},$$

I.1.2 Banach Spaces

Definition I.1.2.1 A sequence $(x_k)_k$ of elements of a normed space E is called a Cauchy sequence if:

$$(\forall \varepsilon > 0) \quad ,(\exists N \geq 1) \quad ,(\forall k,l \geq 1) \Rightarrow \|x_k - x_l\| \leq \varepsilon,$$

Every convergent sequence is a Cauchy sequence[10].

Definition I.1.2.2 A normed space is said to be complete if every Cauchy sequence is convergent. A complete normed space is called a Banach space.

Fatou's Lemma States that if (f_n) is a sequence of non-negative measurable functions [16], then :

$$\liminf_{n \to +\infty} \int f_n \, d\mu \leq \int \liminf_{n \to +\infty} f_n \, d\mu,$$

Theorem I.1.2.1 (Riesz-Fisher Theorem). For any measured space $(S, \mathcal{I}, m)$, and for $1 \leq p < \infty$, the space $L^p(m)$ is a Banach space.

E. Fisher and F. Riesz actually demonstrated, independently, in 1907 that Ł$^2([0,1])$ is isomorphic to l^2; this essentially relies on the fact that $L^2(0,1)$ is complete; hence, the name Riesz-Fisher is given to this theorem, demonstrated in fact, for $L^p([0,1])$ and $1 < p < \infty$, by F. Riesz in 1910 (to distinguish it from the many other theorems due to F. Riesz).

Proof. Let $(F_n)_n$ be a Cauchy sequence in $L^p(m)$. Choose a representative $f_n \in L^p(m)$ of F_n.

a) Since the sequence is Cauchy, we can construct a subsequence (f_{n_k}) (with $n_i < n_j < \ldots$) such that:

$$\|f_{n_k+1} - f_{n_k}\|_p \leq \frac{1}{2^k} \quad \forall k \geq 1,$$

Let:

$$\begin{cases} g_k = \sum_{j=1}^{k} |f_{n_{j+1}} - f_{n_j}| \\ g = \sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}|, \end{cases}$$

Then f is measurable, and: $f(t) = \lim f_n(t)$ for almost every $t \in S$.

These functions are measurable, and we have:

$$\|g_k\|_p \leq \sum_{j=1}^{k} \||f_{n_{j+1}} - f_{n_j}|\|_p = \sum_{j=1}^{k} \|f_{n_{j+1}} - f_{n_j}\|_p \leq \sum_{j=1}^{k} \frac{1}{2^j} \leq 1,$$

Applying Fatou's Lemma to the sequence $(g_k^p)_{k\geq 1}$ gives:

$$\int_S g^p \, dm \leq \liminf_{k\to\infty} \int_S g_k^p \, dm \leq \liminf_{n\to\infty} \int_S \|g_k\|_p^p \, dm \leq 1,$$

Thus, the function g^p is integrable. In particular, it is finite almost everywhere, as is g . This means that the series $\sum_{k\geq 1}(f_{n_{k+1}}(t) - f_{n_k}(t))$converges absolutely for almost every $t \in S$. We can then define:

$$f(t) = \begin{cases} f_{n_1}(t) + \sum_{j=1}^{\infty}(f_{n_{j+1}}(t) - f_{n_j}(t)) & \text{if } g(t) < +\infty \\ 0, & \end{cases}$$

Then f is measurable, and: $f(t) = \lim f_{nk}(t)$ for almost every $t \in S$.

b) It remains to show that the sequence $(f_n)_n$ converges to f in $L^p(m)$, that is, for the semi-norm $||\cdot||_p$.

Let $\varepsilon > 0$. Since the sequence is Cauchy, there exists an integer $N \geq 1$ such that:

$$n, k \geq N \Rightarrow \|f_n - f_k\|_p \geq \varepsilon^p,$$

For $k \geq N$, Fatou's Lemma gives:

$$\int_S |f - f_k|^p \, dm \leq \liminf_{j\to\infty} \int_S |f_n - f_k|^p \, dm \leq \varepsilon^p,$$

We first deduce that $(f - f_k) \in L^p(m)$, hence $f = (f - f_k) + f_k \in L^p(m)$; and then, since $\varepsilon > 0$ is arbitrary, $\lim_{k\to\infty} ||f - f_k||_p = 0$.

c) Finally, if we denote by $f \in L^p(m)$ the equivalence class of f almost everywhere in $L^p(m)$, we have $\lim_{k\to\infty} ||F - F_k||_p = \lim_{k\to\infty} ||f - f_k||_p = 0$.

Remark. Note that in passing, we have proved the very important result following (we will no longer make a distinction between a function and its equivalence class almost everywhere)

Theorem I.1.1.2. If $f_n \underset{n\to\infty}{\longrightarrow} f \in L^p(m)$, with $1 \leq p < \infty$, then we can extract a subsequence $(f_{n_k})_k$ that converges almost everywhere to f.

I.1.3 Banach Algebras

Definition I.1.3.1. An algebra A is said to be

1. real or complex according to the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ respectively.
2. commutative if $(A, +, \circ)$ is commutative

Definition I.1.3.2. (Algebra) . Let A be a non-empty set. Then A is called an algebra if

1. $(A, +, \cdot)$ is a vector space over a field $\mathbb{F}$
2. $(A, +, \circ)$ is a ring and
3. $(\alpha a) \circ b = \alpha(a \circ b) = a \circ (\alpha b)$ for every $\alpha \in \mathbb{F}$, for every $a, b \in A$

Usually we write ab instead of $a \circ b$ for notational convenience [13].

Definition I.1.3.3. An algebra A is said to be unital if $(A, +, \circ)$ has a unit, usually denoted by 1. Let A be unital and $a \in A$. If there exists an element $b \in A$ such that $ab = ba = 1$, then b is called an inverse of a.

Remark I.1.3.1. The unit element in a Banach algebra is unique. Also if an element has an inverse, then it is unique.

Let $G(A) := \{a \in A : a \text{ is invertible in } A\}$. Then $1 \in G(A)$ and $0 \notin G(A)$. The set $G(A)$ is a multiplicative group.

Definition I.1.3.4. Let A be an algebra and $B \subseteq A$. Then B is said to be a subalgebra if B itself is an algebra with respect to the operations of A.

Definition I.1.3.5. Let $I \subseteq A$. Then I is called an ideal (two sided) if

1. I is a subspace i.e., if $a, b \in I$ and $\alpha \in F$, then $\alpha a + b \in I$
2. I is an ideal in the ring i.e., $a \in I$ and $c \in A$ implies that $ac, ca \in I$.

An ideal I is said to be maximal if $I \neq \{0\}$, $I \neq A$ and if J is any ideal of A such that $I \subseteq J$, then either $J = I$ or $J = A$.

Remark I.1.3.2. Every ideal is a subalgebra but a subalgebra need not be an ideal.

Definition I.1.2.6 (normed algebra). If A is an algebra and $\|\cdot\|$ is a norm on A satisfying

$$\|ab\| \leq \|a\| \cdot \|b\|, \text{ for all } a,b \in A,$$

then $\|\cdot\|$ is called an algebra norm and $(A, \|\cdot\|)$ is called a normed algebra. A complete normed algebra is called a Banach algebra.

Remark I.1.3.3. In a normed algebra, the multiplication is both left and right continuous with respect to the algebra norm. That is if $(a_n) \subset A$ is such that $a_n \to a$, then $a_n b \to ab$ and $ba_n \to ba$ as $n \to \infty$ for all $b \in A$. In fact, the multiplication is jointly continuous. That is $a_n b_n \to ab$ as $n \to \infty$. In fact, the other way is also true.

Lemma I.1.3.1. Let A be an algebra such that $(A, \|\cdot\|)$ is a Banach space and the multiplication is separately continuous. Then the multiplication is jointly continuous.

Remark I.1.3.4. We always denote the identity of a unital Banach algebra by 1 and assume that $\|1\| = 1$.

I.1.4 Hausdor spaces

Proposition 1.1.4.1. In a Hausdorff space, the intersection of all closed neighborhoods of a point contains only that point. Hence, singletons are closed.

Proof. Let $x \in X$, where X is a Hausdorff space. Denote by C the intersection of all closed neighborhoods of x. Suppose that there exists $y \in C$ such that $y \neq x$. By the definition of a Hausdorff space, there exist a neighborhood $U(x)$ of x and a neighborhood $V(y)$ of y such that $U(x) \cap V(y) = \emptyset$. Therefore, $y \notin U(x)$ because otherwise any neighborhood of y (in particular $V(y)$) would have non-empty intersection with $U(x)$. Hence, $y \notin C$, which contradicts the assumption. Therefore, C contains only the point x. Since x was arbitrary, this holds for all points in X. Thus, singletons are closed.

I.2 Gronwall

Theorem I.2.1. Let x, Ψ and χ be real continuous functions defined in $[a,b]$, $\chi(t) \geq 0$ for $t \in [a,b]$ [7]. We suppose that on $[a,b]$ we have the inequality

$$x(t) \leq \Psi(t) + \int_a^t \chi(s)x(s)ds,$$

Then

$$x(t) \leq \Psi(t) + \int_a^t \chi(s)\Psi(s)\exp\left(\int_t^s \chi(u)du\right)ds,$$

in $[a,b]$.

Proof. Let us consider the function $y(t) := \int_a^t \chi(u)x(u)du$, $t \in [a,b]$. Then we have $y(a) = 0$ and

$$\frac{dy}{dt} = \chi(t)x(t) \leq \chi(t)\Psi(t) + \chi(t)\int_b^a \chi(s)x(s)ds = \chi(t)\Psi(t) + \chi(t)y(t), \quad t \in (a,b),$$

By multiplication with $\exp\left(-\int_t^a \chi(s)ds\right) > 0$, we obtain

$$\frac{d}{dt}\left(y(t)\exp\left(-\int_t^a \chi(s)ds\right)\right) \leq \Psi(t)\chi(t)\exp\left(-\int_t^a \chi(s)ds\right),$$

By integration on $[a,t]$, one gets

$$y(t)\exp\left(-\int_t^a \chi(s)ds\right) \leq \int_t^a \Psi(u)\chi(u)\exp\left(-\int_u^a \chi(s)ds\right)du,$$

from where results

$$y(t) \leq \int_t^a \Psi(u)\chi(u)\exp\left(\int_t^u \chi(s)ds\right)du, \quad t \in [a,b],$$

Since $x(t) \leq \Psi(t) + y(t)$, the theorem is thus proved.

Next, we shall present some important corollaries resulting from Theorem 1.

Corollary I.2.1. If Ψ is differentiable, then from (1.1) it follows that

$$x(t) \leq \Psi(a)\left(\int_t^a \chi(u)du\right) + \int_t^a \exp\left(\int_t^s \chi(u)du\right)\Psi'(s)ds,$$

for all $t \in [a,b]$.

Proof. It is easy to see that

$$-\int_t^a \Psi(s)\frac{d}{dt}\left(\exp\left(\int_t^s \chi(u)du\right)\right)ds = -\Psi(s)\exp\left(\int_t^s \chi(u)du\right),$$

$$= -\Psi(t)+\Psi(a)\exp\left(\int_t^a \chi(u)du\right)+\int_t^a \exp\left(\int_t^s \chi(u)du\right)\Psi'(s)ds,$$

for all $t \in [a,b]$. Hence,

$$\Psi(t)+\int_t^a \Psi(u)\chi(u)\exp\left(\int_t^u \chi(s)ds\right)du,$$

$$= \Psi(a)\exp\left(\int_t^a \chi(u)du\right)+\int_t^a \exp\left(\int_t^s \chi(u)du\right)\Psi'(s)ds, \quad t \in [a,b],$$

and the corollary is proved.

Corollary I.2.2. If Ψ is constant, then from

$$x(t) \leq \Psi + \int_t^a \chi(s)x(s)ds,$$

it follows that

$$x(t) \leq \Psi \exp\left(\int_t^a \chi(u)du\right),$$

Another well-known generalisation of Gronwall's inequality is the following result due to I.

Theorem I.2.2. Let $x : [a,b] \to \mathbb{R}^+$ be a continuous function that satisfies the inequality:

$$x(t) \leq M + \int_t^a \Psi(s)\omega(x(s))ds, \quad t \in [a,b],$$

where $M \geq 0$, $\Psi : [a,b] \to \mathbb{R}^+$ is continuous and $\omega : \mathbb{R}^+ \to \mathbb{R}^*_+$ is continuous and monotone-increasing. Then the estimation

$$x(t) \leq \Phi^{-1}\left(\Phi(M)+\int_t^a \Psi(s)ds\right), \quad t \in [a,b],$$

holds, where $\Phi : \mathbb{R} \to \mathbb{R}$ is given by

$$\Phi(u) := \int_u^{u_0} \frac{ds}{\omega(s)}, \quad u \in \mathbb{R},$$

Proof. Putting

$$y(t) := \int_t^a \omega(x(s))\Psi(s)ds, \quad t \in [a,b],$$

we have $y(a) = 0$, and by the relation (1.6), we obtain

$$\frac{dy}{dt} \leq \omega(M + y(t))\Psi(t), \quad t \in [a,b],$$

By integration on $[a,t]$, we have

$$\sqrt{2y_\varepsilon(t)} \leq \sqrt{2y_\varepsilon(a)} + \int_t^a \Psi(s)ds, \quad t \in [a,b],$$

that is,

$$\Phi(y(t) + M) \leq \int_t^a \Psi(s)ds + \Phi(M), \quad t \in [a,b],$$

from where results the estimation.

Finally, we shall present another classical result which is important in the qualitative theory of differential equations for monotone operators in Hilbert spaces.

Theorem I.2.3. Let $x : [a,b] \to \mathbb{R}$ be a continuous function which satisfies the following relation:

$$\frac{1}{2}x^2(t) \leq \frac{1}{2}x_0^2 + \int_t^a \Psi(s)x(s)ds, \quad t \in [a,b], \tag{I.1}$$

where $x_0 \in \mathbb{R}$ and Ψ are nonnegative continuous in $[a,b]$. Then the estimation

$$|x(t)| \leq |x_0| + \int_t^a \Psi(s)ds, \quad t \in [a,b],$$

holds.

Proof. Let y_ε be the function given by

$$y_\varepsilon(t) := \frac{1}{2}x_0^2 + \varepsilon^2 + \int_t^a \Psi(s)x(s)ds, \quad t \in [a,b],$$

where $\varepsilon > 0$. By the relation (I.1), we have

$$x^2(t) \leq y_\varepsilon(t), \quad t \in [a,b],$$

Since $y'_{\varepsilon}(t) = \Psi(t)|x(t)|$, $t \in [a,b]$, we obtain

$$y'_{\varepsilon}(t) \leq \sqrt{2y_{\varepsilon}(a)} + \int_{t}^{a} \Psi(s)ds, \quad t \in [a,b],$$

By integration on the interval $[a,t]$, we can deduce that

$$|x(t)| \leq |x_0| + \varepsilon + \int_{t}^{a} \Psi(s)ds, \quad t \in [a,b],$$

for every $\varepsilon > 0$, which implies the lemma.

I.3 Banach Fixed Point Theorem

Banach established a theorem that applies to contractive functions defined on complete metric spaces. He asserts that a contraction on a Banach space has a unique fixed point. This theorem (also called the principle of the contractive mapping) forms a fascinating tool that facilitates the study of stability for delay differential equations[11].

Definition I.3.1.: Let f be a function on a set S. A fixed point of f is any point x satisfying $f(x) = x$. If such an x exists, we say that f has a fixed point, which is equivalent to saying that the equation $f(x) - x = 0$ has a null solution.

Definition I.3.2.: Let (S,d) be a complete metric space and $P : S \to S$ be a mapping. We say that P is a contraction if there exists a constant a in the interval $(0,1)$ such that for all x, y in S, $d(P(x),P(y)) \leq ad(x,y)$.

Theorem I.3.1. (Principle of the contractive mapping): Let (S,d) be a Banach space, and let $P : S \to S$ be a contraction. Then, there exists a unique x in S such that $f(x) = x$. Moreover,

$$x = \lim_{n \to \infty} x_n$$

, where $x_{n+1} = f(x_n)$ and x_1 is arbitrarily chosen in S.

I.4 Dominant convergence theorem

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions defined on an open set $\Omega \subset \mathbb{R}^n$ with values in $\mathbb{R}$. Suppose there exists a measurable function f such that:

$$\lim_{n\to+\infty} f_n(x) = f(x) \text{ for almost every } x \in \Omega$$
$$\lim_{n\to+\infty} |f_n(x)| \leq g(x) \text{ for almost every } x \in \Omega$$

and an integrable function $g : \mathbb{R} \to [0,+\infty]$ such that $|f(x)| \leq g(x)$ for almost every $x \in \Omega$. Then f is integrable, and we have:

$$\begin{aligned}\lim_{n\to+\infty} \int_\Omega f_n(x)\,dx &= \int_\Omega \lim_{n\to+\infty} f_n(x)\,dx \\ &= \int_\Omega f(x)\,dx.\end{aligned}$$

I.5 Homotopy perturbation method

Consider the following general differential equation

$$L(u) + N(u) = 0, \tag{I.2}$$

where L is the linear operator and N is the nonlinear operator. The simplest approach to construction of a homotopy equation is as follows[15]:

$$L(u) + pN(u) = 0, \tag{I.3}$$

or

$$L(u) - L(u_0) + pL(u) + N(u) = 0, \tag{I.4}$$

where p is a homotopy parameter $p \in [0,1]$ and u_0 is the initial solution.

The homotopy analysis method introduces an auxiliary parameter in the form:

$$(1-p)L(u) - L(u_0) + phL(u) + N(u) = 0, \tag{I.5}$$

where h is a non-zero parameter used to control the convergence. When $p = 1$, equation (I.5) becomes

$$hL(u)+N(u)=0, \tag{I.6}$$

For any non-zero h, equation (I.6) is equivalent to the original one. It requires skill to optimally choose the value of h because h can be any real number except zero. Though it can be freely chosen for an infinite series solution, e.g. $h=0.0001$ or $h=-0.0001$ or $h=10,000$, the choice of h will greatly affect the asymptotic property of a finite series solution. There are many publications on how to identify the value of h, and the prevailing method is the h-curve method.

For a nonlinear oscillator, equation (I.3) does not work. As an example, we consider the Duffing equation:

$$u''+u+eu^3=0; \quad u(0)=A, \quad u'(0)=0, \tag{I.7}$$

If a homotopy equation is constructed in the form:

$$u''+u+peu^3=0, \tag{I.8}$$

its zero-th order approximate equation is:

$$u_0''+u_0=0, \tag{I.9}$$

and only a $2p$-periodic solution can be obtained.

For absence of the linear term in an equation, for example:

$$u''+u^3=0; \quad u(0)=A, \quad u'(0)=0, \tag{I.10}$$

We can only obtain a series of solution instead of a periodic solution if the following homotopy equation is constructed:

$$u''+pu^3=0, \tag{I.11}$$

To solve the problem, the parameter-expansion method can be used. We re-write equation (I.10) in the form:

$$u''+0\cdot u+pu^3=0, \tag{I.12}$$

The coefficient, zero, of u in the missing linear term can be expanded in a series of p:

$$0 = a_0 + a_1 p + a_2 p^2 + \cdots, \tag{I.13}$$

where a_i can be determined in view of no secular term in u_i. The zero-order equation is obtained by setting $p = 0$:

$$u_0'' + a_0 u_0' = 0, \tag{I.14}$$

When $p = 1$, equation (I.13)returns to be the original one. So, the solution process is to deform a linear oscillator with an unknown frequency when $p = 0$ to the original one when $p = 1$. This process converges very fast, and only one iteration is always enough for most problems, though iteration can continue without any difficulty to obtain higher order approximate solutions.

For a nonlinear oscillator with a negative linear term, for example:

$$u'' - u + eu^3 = 0, \tag{I.15}$$

the parameter expansion technology can be also effectively used. The coefficient of u can be expanded into the form:

$$1 = a_0 + a_1 p + a_2 p^2 + \cdots, \tag{I.16}$$

The above remarks reflect the development of the homotopy perturbation method. The most important step in the homotopy perturbation method is to construct a suitable homotopy equation with a possible one free parameter. For equation (I.2) the general construction of the homotopy perturbation method is:

$$L\tilde{u}(t) + pLu(t) + Nu(t) = 0, \tag{I.17}$$

where $L\tilde{u}(t) = 0$ with the given boundary/initial condition can describe the basic property of the equation. It can be a linear equation with a free parameter or a nonlinear equation easy to be solved. For nonlinear oscillator, we always assume that:

$$L\tilde{u}(t) = u'' + x^2 u, \tag{I.18}$$

where x is the frequency to be further determined.

I.6 Functions Lipschitz

A Lipschitz function is a concept from mathematical analysis that describes certain types of functions and their behavior. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz if there exists a constant L such that for all pairs of points x and y in the domain [5]:

$$\|f(x) - f(y)\| \leq L\|x - y\|,$$

The Lipschitz Condition:

The Lipschitz condition, named after the mathematician Rudolf Lipschitz, is a property that characterizes certain functions in mathematical analysis. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to satisfy the Lipschitz condition if there exists a constant $L \geq 0$ such that for all pairs of points x, y in the domain:

$$\|f(x) - f(y)\| \leq L\|x - y\|,$$

Interpretation:

In simple terms, the Lipschitz condition states that the rate of change of the function f is bounded by a constant L. Geometrically, it implies that the function does not change too rapidly as we move between points in its domain.

I.7 Reminder on Operators

Let E and F be two normed vector spaces, and $\omega \supset R$ [6].

I.7.1 Bounded Linear Operators

Definition I.7.1.1: An operator A defined from E to F is termed linear if it satisfies the following conditions:

i) $Au \in F$ for every u, v in E and α, β in $\mathbb{R}$.

ii) $A(\alpha u + \beta v) = \alpha Au + \beta Av$.

Definition I.7.1.2: A linear operator A from E to F is considered bounded if there exists a positive constant C such that:

$$\|Au\|_F \leq C\|u\|_E, \text{ for all } u \in E.$$

Definition I.7.1.3: For a linear operator $A : E \to F$, the following properties are equivalent:

i) The operator A is continuous on E.

ii) The operator A is continuous at the point $0x$.

iii) The operator A is bounded.

I.7.2 Compact Part and Compact Operators

Definition I.7.2.1 (Compactness): Let U be a set in a normed space X. U is called compact if from any covering of U by open sets in U, one can extract a finite sub-covering, i.e., for all V_j, $j \in J$ (open sets); $U \subset \bigcup_{j \in J} V_j$, there exist $V_{j(k)}$, $j(k) = 1, 2, \ldots, n$ such that $U \subset \bigcup_{n=1}^{k} V_{j(k)}$.

Definition I.7.2.2: We say that the space E is compact if from any sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements in E, one can extract a convergent subsequence towards a point in E. A subset S of E is termed compact if the subspace $(S, \|\cdot\|)$ is compact. A subset S of E is termed relatively compact if its closure (i.e., S) is a compact subset of E.

Compact Applications

Let $T \in L(E, F)$, we say that T is a compact application if and only if one of the following equivalent propositions is satisfied:

i) The image of every bounded set in E is relatively compact in F.

ii) $T(B_E(0, 1))$ is relatively compact in F.

iii) From every bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ in E, one can extract a subsequence such that $\{Tx_{n_k}\}_{k\in\mathbb{N}}$ converges in F as $k \to \infty$.

The set of all compact applications is denoted by $K(E, F)$.

I.7.3 Integral Linear Operators

Definition I.7.3.1: Let

$$B_y(x) = \int_{\Omega} G(x, s) y(s) ds,$$

be an integral linear operator. The characteristics of this operator depend on the properties of the kernel $G(x,s)$. If $G(x,s)$ is continuous, then B is completely continuous in the space $C(\Omega)$.

I.7.4 Nemitskii Operator

Definition I.7.4.1: The Nemitskii operator N associated with the function $f : (x,y) \to f(x,y)$ is defined as:

$$Ny(x) = f(x,y(x)) \text{ for all } x \in \Omega.$$

The properties of N depend on the properties of f. If f is continuous, then N is continuous and bounded in the space of continuous functions.

I.7.5 Hammerstein Operator

Definition I.7.5.1 (Carathéodory Function): Let $I \subset \mathbb{R}$, we say that $f : I \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function if:

i) the mapping $x \mapsto f(x,y)$ is measurable for all $y \in \mathbb{R}^n$.

ii) the mapping $y \mapsto f(x,y)$ is continuous on $\mathbb{R}$ for almost every $x \in I$.

If for every real number $r > 0$, there exists $h_r \in L^p(I)$ such that $|f(x,y)| < h_r(x)$, for almost every $x \in I$ and for all $y \in \mathbb{R}^n$, with $\|y\| < r$, then f is called L^p-Carathéodory with $1 \leq p \leq \infty$.

Definition I.7.5.2: Let f be a Carathéodory function, and let Ω be compact. Consider a Green's kernel G associated with the function f. Then, the operator H defined as:

$$(Hu)(x) = \int_{\Omega} G(x,y) f(y,u(y)) dy.$$

is called the Hammerstein operator.

Remark.: Let

$$Hy(x) = \int_{\Omega} G(x,s) f(s,y(s)) ds.$$

We can express H as: $H = BN$, where

$$By(x) = \int_{\Omega} G(x,s) y(s) ds \quad \text{and} \quad Ny(x) = f(x,y(x)).$$

If f is continuous and B is completely continuous, then H is completely continuous in the space of continuous functions.

For more detailed properties of the operators B, N, and H, .

Example I.7.5.1: If Ω is a domain in $\mathbb{R}^n$, a solution u of the problem:

$$\Delta u = f(x,u) \text{ in } \Omega u = 0 \text{ on } \partial\Omega$$

can be expressed in integral form:

$$u(x) = \int_\Omega G(x,y) f(y,u(y)) dy,$$

where G is the Green's kernel.

I.7.6 Compactness Criterion (Ascoli-Arzela Theorem)

Let (E,d_E) be a compact metric space and (F,d_F) be a complete metric space. Let $C(E,F)$ denote the vector space of all continuous functions [6] $f : E \to F$.

Definition I.7.6.1: The family $D \subset C(E,F)$ is said to be equicontinuous if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d_F(f(x), f(y)) < \varepsilon$ for every $x,y \in E$ satisfying $d_E(x,y) < \delta$ and every $f \in D$.

Theorem I.7.6.1: The family $D \subset C(E,F)$ is relatively compact if and only if

1. D is equicontinuous.

2. For every $t \in E$, the set:

   $$M(t) = \{x(t) : x(\cdot) \in D\},$$

   is relatively compact in F.

Special Cases

1. If F is a Banach space of finite dimension, the second condition of the Ascoli-Arzela theorem is equivalent to D being uniformly bounded, i.e., there exists $M > 0$ such that for every $x \in D$:

   $$\|x\|_\infty = \sup_{t \in E} \|x(t)\| \leq M,$$

   In particular, for $F = \mathbb{R}$ (In this case, for every $t \in E$, $M(t)$ is a closed bounded set in a finite-dimensional space F).

2. If F is a compact metric space, then the Ascoli-Arzela theorem can be expressed as follows: the family $D \subset C(E,F)$ is relatively compact if and only if D is equicontinuous.

I.8 Taylor Series

Let $f(x)$ be a function with derivatives of all orders in an interval $[x_0, x_1]$ that contains an interior point a. The Taylor series of $f(x)$ generated at $x = a$ is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

or equivalently:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \quad ,$$

The Taylor series generated by $f(x)$ at $a = 0$ is called the Maclaurin series and is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

which is equivalent to:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \quad ,$$

In what follows, we will discuss a few examples for the determination of the Taylor series at $x = 0$.

I.9 Some Proposition and theories

Proposition I.9.1 Let $(t,x,y) \mapsto g(t,x,y)$ be a continuous function defined on a compact subset D of the space $\mathbb{B} = [t_1, t_2] \times B_R \times \mathbb{R}^n$, for some $R > 0$; then there exists a sequence of Lipschitzian functions $\{g_k\}$, leading from $\mathbb{B}$ into $\mathbb{R}^n$ [3] , such that

$$\lim_{k \to \infty} g_k(w) = g(w), \quad \text{for all } w = (t,x,y) \in D,$$

and

$$\|g_k\| = \sup\{|g_k(w)|, w \in D\} \leq \|g\| = \sup\{|g(w)|, w \in D\},$$

Proposition I.9.2 Let E be a Banach space, $V \subset E$ be a (suitable nonempty) open set, and S be a nonlinear compact continuous operator from the closure of V into E. Then, if there is a sufficiently small ε such that there exists a compact and continuous operator S (from the closure of V into E) satisfying $\|S(x) - S(x)\| < \varepsilon$, $\forall x$, and such that the equation $x - S(x) = b$ has at most one solution if $\|b\| < \varepsilon$, then the set of fixed points of it is an acyclic set.[3]

The well-known Gronwall Lemma, from the standard theory of Ordinary Differential Equations, will also be used:

If, for $t_0 \leq t \leq t_1$, $\varphi(t) \geq 0$ and $\psi(t) \geq 0$ are continuous functions such that the inequality $\varphi(t) \leq K + M\int_{t_0}^{t} \psi(s)\varphi(s)ds$ holds on $t_0 \leq t \leq t_1$, with K and M positive constants, then $\varphi(t) \leq K\exp\left(M\int_{t_0}^{t} \psi(s)ds\right)$ on $t_0 \leq t \leq t_1$.

Lemma I.9.1 Assume that Ω is a nonempty, bounded, convex, closed subset of $C[0,a]$ and the operators F and G transform continuously the set Ω into $C[0,a]$ in such a way that $F(\Omega)$ and $G(\Omega)$ are bounded. Moreover, assume that the operator $T = F \cdot G$ transforms Ω into itself. If the operators F and G each satisfy the Darbo condition on the set Ω (with respect to the measure of noncompactness w_0) with constant Q_1 and Q_2, respectively, then the operator T satisfies the Darbo condition on Ω with the constant

$$\|F(\Omega)\|Q_2 + \|G(\Omega)\|Q_1,$$

In particular, if $\|F(\Omega)\|Q_2 + \|G(\Omega)\|Q_1 < 1$ then T is a contraction with respect to w_0 and so has at least one fixed point in Ω.

Theorem I.9.1 Let Q be a nonempty, bounded, closed, and convex subset of E, and let $H : Q \to Q$ be a continuous transformation which is a contraction with respect to the Hausdorff measure of noncompactness v, i.e., there exists a constant $a \in [0,1)$ such that $v(H(X)) \leq av(X)$ for any nonempty subset X of Q. Then H has at least one fixed point in the set Q.

CHAPTER II

GENERALITIES ABOUT INTEGRAL EQUATIONS

II.1 Definition

Integral equations are equations where the unknown function $u(x)$ appears under the integral sign [14] . A common form is

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)\,dt, \tag{II.1}$$

where $K(x,t)$ is the kernel, and a and b are the limits of integration. These equations arise in various fields like physics, chemistry, biology, and engineering, often stemming from initial value problems over finite intervals. To solve such integral equations, techniques are employed to determine $u(x)$, which is the main focus of study. By converting initial value problems into integral equations, one can apply these techniques effectively, supported by practical examples.

II.2 Classification of Linear Integral Equations

The most frequently used linear integral equations fall under two main classes namely Fredholm and Volterra integral equations. However, in this text we will distinguish four more related types of linear integral equations in addition to the two main classes. In what follows, we will give a list of the Fredholm and Volterra integral equations, and the four related types [14].

II.2.1 Fredholm Linear Integral Equations

An integral equation is an equation in which the unknown function $u(x)$ to be determined appears under the integral sign. A typical form of an integral equation in $u(x)$ is of the form

$$\varphi(x)u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)\,dt, a \leq x,t \leq b, \tag{II.2}$$

where the kernel of the integral equation $K(x,t)$ and the function $f(x)$ are given in advance, and λ is a parameter. Equation (II.2) is called linear because the unknown function $u(x)$ under the integral sign occurs linearly, i.e., the power of $u(x)$ is one. The value of φx will give the following kinds of Fredholm linear integral equations:

1. When $\varphi(x) = 0$, Eq (II.2) becomes

$$f(x) + \lambda \int_a^b K(x,t)u(t)dt = 0, \tag{II.3}$$

and the integral equation is called a Fredholm integral equation of the first kind.

2. When $\varphi(x) = 1$, Eq.(II.2) becomes

$$u(x) = f(x) + \lambda \int_a^b K(1,t)u(t)dt, \tag{II.4}$$

and the integral equation is called a Fredholm integral equation of the second kind.

In fact, Equation (II.4) can be obtained from (II.2) by dividing both sides of (II.2) by $\phi(x)$ provided that $\varphi(x) \neq 0$.

In summary, the Fredholm integral equation is of the first kind if the unknown function $u(x)$ appears only under the integral sign. However, the Fredholm integral equation is of the second kind if the unknown function $u(x)$ appears inside and outside the integral sign.

II.2.2 Volterra Linear Integral Equations

The standard form of Volterra linear integral equations, where the limits of integration are functions of x rather than constants, are of the form

$$\varphi(x)u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)\,dt, \tag{II.5}$$

where the unknown function $u(x)$ under the integral sign occurs linearly as stated before. It is worth noting that (II.5) can be viewed as a special case of Fredholm integral equation when the kernel $K(x,t)$ vanishes for $t > x$, x is in the range of integration $[a,b]$. As in Fredholm equations, Volterra integral equations fall under two kinds, depending on the value of $\varphi(x)$, namely:

1. When $\varphi(x) = 0$, Eq. (II.5) becomes

$$f(x) + \lambda \int_a^x K(x,t)u(t)dt = 0, \tag{II.6}$$

and in this case the integral equation is called Volterra integral equation of the first kind.

2. When $\varphi(x) = 1$, Eq. (II.5) becomes

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt, \tag{II.7}$$

and in this case the integral equation is called Volterra integral equation of the second kind.

Examining the equations (II.1)–(II.7) carefully, the following remarks can be concluded: In summary, the Volterra integral equation is of the first kind if the unknown function $u(x)$ appears only under the integral sign. However, the Volterra integral equation is of the second kind if the unknown function $u(x)$ appears inside and outside the integral sign.

Remarks. Examining the equations ((II.1)–(II.7) carefully, the following remarks can be concluded.

1. **The structure of Fredholm and Volterra equations:** The unknown function $u(x)$ appears linearly only under the integral sign in linear Fredholm and Volterra integral equations of the first kind. However, the unknown function $u(x)$ appears linearly inside the integral sign and outside the integral sign as well in the second kind of both linear Fredholm and Volterra integral equations.

2. **The limits of integration:** In Fredholm integral equations, the integral is taken over a finite interval with fixed limits of integration. However, in Volterra integral equations, at least one limit of the range of integration is a variable, and the upper limit is the most commonly used with a variable limit.

3. **The origins of integral equations:** It is important to note that integral equations arise in engineering, physics, chemistry, and biology problems . Further, integral equations arise as representation forms of differential equations. Furthermore, Fredholm and Volterra integral equations arise from different origins and applications, such as boundary value problems as in Fredholm equations, and from initial value problems as in Volterra equations. Based on the fact that integral equations arise from distinct origins, different techniques and approaches will be used to determine the solution of each type of integral equations.

4. **The linearity property:** As indicated before, the unknown function $u(x)$ in Fredholm and Volterra integral equations (II.5)–(II.7) occurs to the first power wherever it exists. However, nonlinear Fredholm and Volterra integral equations arise if $u(x)$ is replaced by a nonlinear function $F(u(x))$, such as $u^2(x)$, $u^3(x)$, $e^{u(x)}$ and so on. The following are examples of nonlinear integral equations:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u^2(t)dt, \tag{II.8}$$

$$u(x) = f(x) + \lambda \int_a^b K(x,t)e^{u(t)}dt, \tag{II.9}$$

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\sin(u(t))dt, \tag{II.10}$$

where the linear function $u(x)$ in (II.1) has been replaced by the nonlinear functions $u^2(t)$, $e^{u(t)}$ and $\sin(u(t))$ respectively.

5. **The homogeneity property:** If we set $f(x) = 0$ in Fredholm or Volterra integral equation of the second kind given by (II.3) and (II.8), the resulting equation is called a homogeneous integral equation, otherwise it is called nonhomogeneous or inhomogeneous integral equation.

6. **The singular behavior of the integral equation:** An integral equation is called singular if the integration is improper. This usually occurs if the interval of integration is infinite, or if the kernel becomes unbounded at one or more points of the interval of consideration $a \leq t \leq b$. .

Four additional types of integral equations, which are closely related to the primary classes of Fredholm and Volterra integral equations, emerge frequently in science and engineering applications. Here, we introduce these significant equations as distinct types, contributing to a broader understanding of integral equation theory and its practical applications.

II.2.3 Integro-Differential Equations

A differential-integral equation is called linear when it is a combination of integration and differentiation and has the general form as follows:

$$u^{(n)}(x) = f(x) + \lambda \int_{a(x)}^{b(x)} K(x,t)u(t)\,dt, \tag{II.11}$$

where n is the order of differentiation of the unknown function $u(x)$, and the functions $a(x)$ and $b(x)$ are the limits of integration, which can be variables or constants depending on the type of equation. We distinguish three types in this category, which are:

- The linear Volterra differential-integral equation

$$u^{(n)}(x) = f(x) + \lambda \int_a^x K(x,t)u(t)\,dt \tag{II.12}$$

- The linear Fredholm differential-integral equation

$$u^{(n)}(x) = f(x) + \lambda \int_a^b K(x,t)u(t)\,dt \tag{II.13}$$

II.2.4 Singular Integral Equations

The integral equation of the first kind

$$f(x) = \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)\,dt, \tag{II.14}$$

or the integral equation of the second kind

$$u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)\,dt, \tag{II.15}$$

is called singular if the lower limit, the upper limit, or both limits of integration are infinite. Additionally, equation (II.14) or (II.15) is also called a singular integral equation if the kernel $K(x,t)$ becomes infinite at one or more points in the domain of integration. Examples of the first type of singular integral equations are given by the following examples:

$$u(x) = 2x + 6\int_0^\infty sin(x-t)u(t)\,dt, \tag{II.16}$$

$$u(x) = x + \frac{1}{3}\int_{-\infty}^0 cos(x+t)u(t)\,dt, \tag{II.17}$$

$$u(x) = 1 + x^2 + \frac{1}{6}\int_{-\infty}^\infty (x+t)u(t)\,dt, \tag{II.18}$$

where the singular behavior in these examples has resulted from the range of integration becoming infinite.

Examples of the second kind of singular integral equations are given by

$$x^2 = \int_0^x \frac{1}{\sqrt{x-t}}u(t)\,dt, \tag{II.19}$$

$$x = \int_0^x \frac{1}{(x-t)^\alpha}u(t)\,dt \quad 0 < \alpha < 1, \tag{II.20}$$

$$u(x) = 1 - 2\sqrt{x} - \int_x^0 \frac{1}{\sqrt{x}-t}u(t)dt, \tag{II.21}$$

where the singular behavior in this kind of equations has resulted from the kernel $K(x,t)$ becoming infinite as $t \to x$. It is important to note that integral equations similar to examples (II.19) and (II.20) are called Abel's integral equation and generalized Abel's integral equation respectively. Moreover these types of singular integral equations are among the earliest integral equations established by the Norwegian mathematician Niels Abel in 1823. Singular equations similar to example (II.21) are called the weakly-singular second-kind Volterra type integral equations. This type of equations usually arises in science and engineering applications like heat conduction, super-fluidity and crystal growth.

II.2.5 Volterra-Fredholm Integral Equations

The Volterra-Fredholm integral equation, which is a combination of disjoint Volterra and Fredholm integrals, appears in one integral equation. The Volterra-Fredholm integral equations arise from the modelling of the spatiotemporal development of an epidemic, from boundary value problems and from many physical and chemical applications. The standard form of the Volterra-Fredholm integral equation reads

$$u(x) = f(x) + \int_0^x K_1(x,t)u(t)dt + \int_a^b K_2(x,t)u(t)dt, \tag{II.22}$$

where $K_1(x,t)$ and $K_2(x,t)$ are the kernels of the equation.
Examples of the Volterra-Fredholm integral equations are

$$u(x) = 2x - \int_0^x (x-t)u(t)dt + \int_0^{\frac{\pi}{2}} u(t)dt, \tag{II.23}$$

$$u(x) = \sin(x) - \cos(x) + \int_0^x u(t)dt + \int_0^{\frac{\pi}{2}} u(t)dt, \tag{II.24}$$

Notice that the unknown function $u(x)$ appears inside the Volterra and Fredholm integrals and outside both integrals.

II.2.6 Volterra-Fredholm Integro-Differential Equations

The Volterra-Fredholm integro-differential equation, which is a combination of disjoint Volterra and Fredholm integrals and a differential operator, may appear in one integral equation. The Volterra-Fredholm integro-differential equations arise from many physical and chemical applications similar to the Volterra-Fredholm integral equations. The standard form of the Volterra-Fredholm integro-differential equation reads

$$u^{(n)}(x) = f(x) + \int_0^x K_1(x,t)u(t)dt + \int_a^b K_2(x,t)u(t)dt, \tag{II.25}$$

where $K_1(x,t)$ and $K_2(x,t)$ are the kernels of the equation, and n is the order of the ordinary derivative of $u(x)$. Notice that because this kind of equations contain ordinary derivatives, then initial conditions should be prescribed depending on the order of the derivative involved.
Examples of the Volterra-Fredholm integro-differential equations are

$$u'(x) = 1 + \int_0^x (x-t)u(t)dt + \int_0^1 xtu(t)dt, u(0) = 1, \tag{II.26}$$

$$u''(x) = -x - \frac{1}{6}x^3 + \int_0^x u(t)dt + \int_{-\pi}^{\pi} xt(t)dt, u(0) = 0, u'(0) = 2, \tag{II.27}$$

Notice that the unknown function $u(x)$ appears inside the Volterra and Fredholm integrals and outside both integrals. In closing this section, we illustrate the classifications and the basic concepts that were discussed earlier by the following examples.

II.3 Classification of Nonlinear Integral Equations

A nonlinear integral equation is an equation in which the unknown is a function, often denoted by symbols like φ, u, or another letter, and it appears both inside and outside the integration symbol. Its general form is[12]:

$$f(x) = \int_b^a k(x,t)F(u(t))dt, \tag{II.28}$$

or

$$u(x) = f(x) + \lambda \int_b^a k(x,t)F(u(t))dt, \tag{II.29}$$

If $b = a$ (variable), it is a nonlinear Volterra integral equation, but if b is a constant, it is a nonlinear Fredholm integral equation.

Definitions and Labels:

1. u is the unknown function to be determined.

2. f is a known function defined on the interval $[a,b]$, similar to the second-side function in differential equations.

3. k is called the kernel and is a two-variable function defined on the domain $[a,b]\times[a,b]=D$.

4. λ is a constant and is called the parameter of the integral equation.

5. F is a nonlinear function in terms of $u(t)$.

There are several forms of nonlinear integral equations, the most common of which are:

II.3.1 Hamrichstein Nonlinear Integral Equation

We call the nonlinear Hamrichstein integral equation every equation written in the form:

$$\mu u(x) = f(x) + \lambda \int_D k(x,t)F(t,u(t))dt, \quad D \in \mathbb{R}^n, \quad n \in \mathbb{N}^*,$$

1. Case (1): $\mu = 0$

$$f(x) = \lambda \int_a^x k(x,t)F(t,u(t))dt, \tag{II.30}$$

   It is called the Hammerstein-Volterra nonlinear integral equation of the first class.

2. Case (2): , $\mu = 1$

$$u(x) = f(x) + \lambda \int_a^x k(x,t)F(t,u(t))dt, \tag{II.31}$$

   It is called the Hammerstein-Volterra nonlinear integral equation of the second class.

II.3.2 Fredholm-Volterra Nonlinear Integral Equation

We call the nonlinear Friedholm-Volterra integral equation every integral equation written in the form

$$\begin{aligned} &\mu u(\bar{x},\bar{t}) + \lambda \int_\Omega k(\bar{x}-\bar{z},\bar{y}-\bar{s})F(t,u(\bar{z},\bar{s}))d\bar{z}d\bar{s} + \\ &\lambda \int_0^t G(t,T)u(\bar{x},\bar{y},T)dT = f(\bar{x},\bar{y},T) = \\ &(x_1,x_2,x_3...x_n),\bar{t} = (t_1,t_2,t_3...t_n), \end{aligned} \tag{II.32}$$

With Ω dependent on the integration curve.

II.3.3 Jurshon-Volterra Nonlinear Integral Equation

We call the nonlinear Jurshon-Volterra integral equation every equation written in the form

$$\mu u(x) = f(x) + \lambda \int_0^x k(x,t,u(t))\,dt \quad 0 \le x \le T < +\infty, \tag{II.33}$$

We distinguish here three different cases:

1. Case (1): $\mu = 0$,
$$f(x) = \lambda \int_x^0 k(x,t,u(t))dt, \tag{II.34}$$
It is the nonlinear Jurshon-Volterra integral equation of the first class.

2. Case (2): $\mu = cet \neq 0$
$$u(x) = f(x) + \lambda \int_0^x k(x,t,u(t))dt, \tag{II.35}$$
It is the nonlinear Jurshon-Volterra integral equation of the second class.

3. Case (3): $\mu = \mu(x)$
$$\mu(x)u(x) = f(x) + \lambda \int_0^x k(x,t,u(t))dt, \tag{II.36}$$
It is the nonlinear Jurshon-Volterra integral equation of the third class.

II.3.4 Cauchy's Nonlinear Integral Equation

Cauchy's anomalous nonlinear integral equation is called every integral equation written in the form

$$a(x)u(x) + b(x) \int_\Gamma \frac{u(t)}{t-x} dt + \int_\Gamma F(x,t,u(t))dt = f(x), \tag{II.37}$$

Where $a(x)$, $b(x)$, and $f(x)$ are given functions, and $u(x)$ is the unknown function.

II.3.5 Fredholm Nonlinear Integral Equation

We call the nonlinear Friedholm integral equation every integral equation written in the form

$$u(x) = f(x) + \lambda \int_a^b k(x,t)F(u(t))dt, \tag{II.38}$$

It is of the second type and is not homogeneous, but if $f = 0$, then it is homogeneous.

*The integral equation $f(x) = \int_a^b k(x,t)F(u(t))dt$, is the nonlinear Fredholm equation of the first class.

II.3.6 Quadratic integral equation

The quadratic integral equation has the following general form [11]:

$$u(x) = f(x) + \lambda \int_a^b K(x,t,u(t))u(t)dt$$

where:

- $u(x)$ is the unknown function to be found.
- $f(x)$ is a known function.
- λ is a known constant.
- $K(x,t,u(t))$ is the integration coefficient function that depends on the unknown function $u(t)$.

The reason it is considered a nonlinear equation is because there is a product of $u(t)$ with the integration factor function $K(x,t,u(t))$, which makes the equation nonlinear with respect to the unknown function $u(t)$.
Here are examples of quadratic integral equations:

a) Chandrasekhar's quadratic integral equation:

$$u(x) = 1 + \frac{x^2}{2} + \lambda \int_0^x (x-t)u^2(t)\,dt$$

This equation appears in the theory of perturbations in astrophysics.

b) Beju's quadratic integral equation:

$$u(x) = e^x + \lambda \int_0^x e^{(x-t)}u^2(t)\,dt$$

This is used in the study of light propagation in nonlinear media.

c) Coyne's quadratic integral equation:

$$u(x) = f(x) + \lambda \int_0^1 K(x,t)u^2(t)\,dt$$

Where $K(x,t)$ is a specified linear function, and this appears in models of nonlinear diffusion.

II.4 Classification of integral equations with respect to a nucleus

Integral equations can be classified and divided with respect to the kernel into [12]:

1. Integral equation with a continuous kernel $k(x,y)$ in the interval $[a,b]$ and having the condition $|\lambda| \geq |k(x,y)|$, where λ is a constant.

2. An integral equation with an abnormal kernel and the condition

$$\left(\int_a^b \int_a^b |k(x,y)|^2 \, dx\, dy \right)^{\frac{1}{2}} = C,$$

where C is a finite value. Hence, the integral equation is called a Fredholm-type equation.

Integral equations are classified according to the abnormal kernel as follows:

1. **If the kernel takes the form**

$$k(x,y) = \begin{cases} \frac{A(x,y)}{|x-y|^{\alpha}}, & 0 \leq \alpha < 1 \quad \text{(Carlman kernel)}, (2) \\ A(x,y) \ln |x-y|, & \text{(logarithmic kernel)}, (3) \end{cases}$$

where $A(x,y)$ is a continuous function and its derivatives, in this case, the integral equation is said to be weakly anomalous with respect to the Carlman kernel in (2) or logarithmic kernel in (3).

2. If the kernel is in the form

$$k(x,y) = \frac{B(x,y)}{x-y},$$

it is called the Cauchy kernel, where (x,y) is a continuous function and its derivatives.

3. **If the kernel is in the form**

$$k(x,y) = \frac{C(x,y)}{(x-y)^m}, \quad m \geq 2,$$

the integral equation is called a strongly anomalous equation when $m = 2$, but if $m > 2$, the integral equation is called a weakly anomalous equation, where (x,y) is a continuous function and its derivatives.

4. **If the kernel is in the form**

$$k(x,y) = \frac{D(x,y)}{(x-y)^{\alpha}}, \quad 0 \le \alpha \le 1,$$

where (x,y) is a continuous function and its derivatives, the integral equation is called Abel's formula.

5. **Separable kernels:** The kernel (x,y) is called a separable kernel if it can be expressed as the sum of a number of finite terms such that each term is a product of a function in x only and a function in y only, as follows:

$$k(x,y) = \sum_{i=1}^{n} a_i(x) b_i(y).$$

6. **Identical nuclei:** A composite function with the value (x,y) is called symmetric if: $k(x,y) = k(y,x)$ where $(*)$ expresses the permutation of function variables, and if the kernel is real, then $k(x,y) = k(y,x)$, but if $k(x,y) = -k(y,x)$, then this kernel is called skew-symmetric.

II.5 Some methods for solving integral equations

II.5.1 Approximations Successive The Method

The successive approximations method, also called the Picard iteration method, provides a scheme that can be used for solving initial value problems or integral equations. This method solves any problem by finding successive approximations to the solution by starting with an initial guess, called the zeroth approximation. As will be seen, the zeroth approximation is any selective real-valued function that will be used in a recurrence relation to determine the other approximations [2].

Given the linear Volterra integral equation of the second kind

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)\,dt, \tag{II.39}$$

where $u(x)$ is the unknown function to be determined, $K(x,t)$ is the kernel, and λ is a parameter. The successive approximations method introduces the recurrence relation

$$u_n(x) = f(x) + \lambda \int_0^x K(x,t)u_{n-1}(t)\,dt, \quad n \geq 1, \tag{II.40}$$

where the zeroth approximation $u_0(x)$ can be any selective real-valued function. We always start with an initial guess for $u_0(x)$, mostly we select 0, 1, x for $u_0(x)$, and by using (II.40), several successive approximations u_k, $k \geq 1$ will be determined as

$$\begin{aligned}
u_1(x) &= f(x) + \lambda \int_0^x K(x,t)u_0(t)\,dt, \\
u_2(x) &= f(x) + \lambda \int_0^x K(x,t)u_1(t)\,dt, \\
u_3(x) &= f(x) + \lambda \int_0^x K(x,t)u_2(t)\,dt, \\
&\vdots \\
u_n(x) &= f(x) + \lambda \int_0^x K(x,t)u_{n-1}(t)\,dt.
\end{aligned} \tag{II.41}$$

The question of convergence of $u_n(x)$ is justified by noting the following theorem.

Theorem II.5.1.1 If $f(x)$ in (II.40) is continuous for the interval $0 \leq x \leq a$, and the kernel $K(x,t)$ is also continuous in the triangle $0 \leq x \leq a$, $0 \leq t \leq x$, the sequence of successive approximations $u_n(x)$, $n \geq 0$ converges to the solution $u(x)$ of the integral equation under discussion.

It is interesting to point out that the variational iteration method admits the use of the iteration formula:

$$u_{n+1}(x) = u_n(x) + \lambda \int_0^x \left(\frac{\partial u_n(\xi)}{\partial \xi} - \tilde{u}_n(\xi) \right) d\xi, \tag{II.42}$$

whereas the successive approximations method uses the iteration formula

$$u_n(x) = f(x) + \lambda \int_0^x K(x,t)u_{n-1}(t)\,dt, \quad n \geq 1, \tag{II.43}$$

The difference between the two formulas can be summarized as follows:

1. The first formula contains the Lagrange multiplier λ that should be determined first before applying the formula. The successive approximations formula does not require the use of λ.

2. The first variational iteration formula allows the use of the restriction $\tilde{u}_n(\xi)$ where $\delta\tilde{u}_n(\xi) =$

0. The second formula does require this restriction.

3. The first formula is applied to an equivalent ODE of the integral equation, whereas the second formula is applied directly to the iteration formula of the integral equation itself.

The successive approximations method, or the Picard iteration method, will be illustrated by the following examples.

Examples

Example 01 : Using the method of successive approximations, find the solution of the following Volterra nonlinear integral equation[14]:

$$u(x) = e^x + \frac{1}{3}x\left(1 - e^{3x}\right) + \int_0^x u^3(t)\,dt, \tag{II.44}$$

Assuming a zero approximation equal to $u_0(x) = 1$, we obtain the following successive approximations:

$$\begin{aligned}
u_1(x) &= e^x + \frac{1}{3}x\left(1 - e^{3x}\right) + \int_0^x u_0^3(t)\,dt \\
&= 1 + x + \frac{1}{2!}x^2 - \frac{4}{3}x^3 - \frac{35}{24}x^4 - \frac{67}{60}x^5 + \cdots \\
u_2(x) &= e^x + \frac{1}{3}x\left(1 - e^{3x}\right) + \int_0^x u_1^3(t)\,dt \\
&= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{67}{60}x^5 + \cdots \\
u_3(x) &= e^x + \frac{1}{3}x\left(1 - e^{3x}\right) + \int_0^x u_2^3(t)\,dt \\
&= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots
\end{aligned}$$

By inserting the limit, we find the solution of equation (II.44):

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = e^x$$

Example 02:

Study the following Fredholm integral equation [14]

$$\begin{aligned}
u(x) &= e^x + e^{-1}\int_0^1 u(t)\,dt \\
u_0(x) &= 0 \\
u_1(x) &= e^x + e^{-1}\int_0^1 u_0(t)\,dt \\
u_1(x) &= e^x \\
u_2(x) &= e^x + e^{-1}\int_0^1 e^t\,dt \\
u_2(x) &= e^x + e^{-1} \\
u_3(x) &= e^x + e^{-1} \\
u_n(x) &= e^x + (1-e^{-1})^{-(n-1)}, \quad n \geq 1 \\
u(x) &= \lim_{n\to\infty} u_n(x) \\
&= \lim_{n\to\infty} (e^x + (1-e^{-1})^{-(n-1)}) \\
&= e^x + 1
\end{aligned}$$

II.5.2 The Series Solution Method

The method of solving with a series depends mainly on Taylor's diffusion, and to solve this equation, we will assume that it is Have an analytical solution and its general formula is as follows

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{II.45}$$

Examples

Example 01 : Using the Series solution method, find the solution to the following Volterra nonlinear integral equation

$$u(x) = 1 + x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{12}x^4 + \int_0^x (x-t)u(t)^2\,dt, \tag{II.46}$$

By employing relation (II.45) and substituting it into relation (II.46), we find

$$a_0 + a_1x + a_2x^2 + \cdots = 1 + x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{12}x^4 + \int_0^x (x-t)u(a_0 + a_1t + a_2t^2 + \ldots)^2\,dt, \tag{II.47}$$

Then by calculation we find

$$a_0+a_1x+a_2x^2+\cdots=1+x-\frac{1}{2}(a_0^2-1)x^2-\frac{1}{3}(a_0a_1-1)x^3-\frac{1}{12}(a_0^2+2a_0a_1-1)x^4, \quad \text{(II.48)}$$

Then, by matching both sides of equation (II.47) and (II.48) , we obtain the values of the coefficients

$$a_0=1, \quad a_1=1, \quad a_2=0 \quad \text{for } n\geq 2, \quad \text{(II.49)}$$

By substituting, we find the exact solution to equation (II.46)

$$u(x)=a_0+a_1x=x+1, \quad \text{(II.50)}$$

Example 02 : Using the series solution method, find the solution of the nonlinear Fredholm integral differential equation

$$u'(x)=1-\frac{2}{15}x-\frac{226}{105}x^2+\int_{-1}^{1}(xt+x^2t^2)u^2(t)\,dt, \quad u(0)=1, \quad \text{(II.51)}$$

First, we substitute the analytical solution(II.45) into equation (II.51) and we find

$$\left(\sum_{n=0}^{\infty}a_nx^n\right)'=1-\frac{2}{15}x-\frac{226}{105}x^2+\int_{-1}^{1}(xt+x^2t^2)\left(\sum_{n=0}^{\infty}a_nt^n\right)^2 dt, \quad \text{(II.52)}$$

Then by calculating the integral on the right side using $a_1=0$, we find

$$\begin{aligned} &a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+\ldots \\ &=1-\left(\frac{2}{15}+\frac{4}{3}a_1+\frac{4}{9}a_3a_4+\frac{4}{5}a_1a_2+\frac{4}{7}a_1a_4+\frac{4}{7}a_2a_3+\frac{4}{5}a_3\right)x \\ &-\left(\frac{52}{35}+\frac{2}{9}a_3^2+\frac{4}{9}a_2a_4+\frac{4}{7}a_1a_3+\frac{2}{7}a_2^2+\frac{4}{7}a_4+\frac{4}{5}a_2+\frac{2}{5}a_1^2+\frac{2}{11}a_4^2\right)x^2, \end{aligned} \quad \text{(II.53)}$$

Then by matching both sides we find

$$a_0=1; \quad a_1=1; \quad a_2=1; \quad a_3=0, \quad \text{for } n\geq 3, \quad \text{(II.54)}$$

The exact solution to equation(II.51) is as follows

$$u(x) = 1 + x + x^2, \tag{II.55}$$

Example 03 : Solve the Volterra-Fredholm integral equation by using the series solution method [2].

$$u(x) = -5 - x + 12x^2 - x^3 - x^4 + \int_0^x (x-t)u(t)\,dt + \int_0^1 (x+t)u(t)\,dt, \tag{II.56}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{II.57}$$

into both sides of Eq. (II.80) leads to

$$\sum_{n=0}^{\infty} a_n x^n = -5 - x + 12x^2 - x^3 - x^4 + \int_0^x (x-t) \sum_{n=0}^{\infty} a_n t^n\,dt + \int_0^1 (x+t) \sum_{n=0}^{\infty} a_n t^n\,dt, \tag{II.58}$$

Evaluating the integrals at the right side, using few terms from both sides, and collecting the coefficients of like powers of x, we find

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = & -5 + \frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 + \frac{1}{5}a_3 + \frac{1}{6}a_4 \\ & + \left(-1 + a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 + \frac{1}{4}a_3 + \frac{1}{5}a_4\right)x \\ & + \left(12 + \frac{1}{2}a_0\right)x^2 + \left(-1 + \frac{1}{6}a_1\right)x^3 \\ & + \left(-1 + \frac{1}{12}a_2\right)x^4 + \cdots . \end{aligned}, \tag{II.59}$$

Equating the coefficients of like powers of x in both sides of (II.59), and solving the resulting system of equations, we obtain

$$a_0 = 0, \quad a_1 = 6, \quad a_2 = 12, \quad a_3 = a_4 = a_5 = \cdots = 0,, \tag{II.60}$$

The exact solution is therefore given by

$$u(x) = 6x + 12x^2, \tag{II.61}$$

II.5.3 The Adomian decomposition method (ADM)

The Adomian decomposition method (ADM) was introduced and developed by George Adomian . A considerable amount of research work has been invested recently in applying this method to a wide class of linear and nonlinear ordinary differential equations, partial differential equations, and integral equations as well[2].

The Adomian decomposition method consists of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{II.62}$$

or equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots, \tag{II.63}$$

where the components $u_n(x)$, $n \geq 0$ are to be determined in a recursive manner. The decomposition method concerns itself with finding the components individually. As will be seen through the text, the determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated.

To establish the recurrence relation, we substitute (II.62) into the Volterra integral equation to obtain

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x K(x,t) \sum_{n=0}^{\infty} u_n(t)dt, \tag{II.64}$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = f(x) + \lambda \int_0^x K(x,t)[u_0(t) + u_1(t) + \cdots]dt, \tag{II.65}$$

The zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sign. Consequently, the components $u_j(x)$, $j \geq 1$ of the unknown function $u(x)$ are completely determined by setting the recurrence relation:

$$\begin{aligned} u_0(x) &= f(x), \\ u_{n+1}(x) &= \lambda \int_0^x K(x,t)u_n(t)dt, \quad n \geq 0, \end{aligned} \tag{II.66}$$

that is equivalent to

$$
\begin{aligned}
u_0(x) &= f(x), \\
u_1(x) &= \lambda \int_0^x K(x,t)u_0(t)dt, \\
u_2(x) &= \lambda \int_0^x K(x,t)u_1(t)dt, \\
u_3(x) &= \lambda \int_0^x K(x,t)u_2(t)dt, \\
&\vdots
\end{aligned}
$$

and so on for other components.

In view of (II.66), the components $u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x)$, $\ldots$ are completely determined. As a result, the solution $u(x)$ of the Volterra integral equation in a series form is readily obtained by using the series assumption in (II.62).

It is clearly seen that the decomposition method converted the integral equation into an elegant determination of computable components. It was formally shown by many researchers that if an exact solution exists for the problem, then the obtained series converges very rapidly to that solution. The convergence concept of the decomposition series was thoroughly investigated by many researchers to confirm the rapid convergence of the resulting series. However, for concrete problems, where a closed-form solution is not obtainable, a truncated number of terms is usually used for numerical purposes. The more components we use, the higher accuracy we obtain.

Examples

Example 01 : Solve the following Volterra integral equation:

$$u(x) = 1 - \int_0^x u(t)\,dt, \tag{II.67}$$

Given $f(x) = 1$, $\lambda = -1$, and $K(x,t) = 1$, the solution $u(x)$ is assumed to have a series form given in Equation (II.62). Substituting this decomposition series into both sides of Equation (II.67),we get

$$\sum_{n=0}^{\infty} u_n(x) = 1 - \int_0^x \sum_{n=0}^{\infty} u_n(t)\,dt, \tag{II.68}$$

Equivalently,

$$u_0(x)+u_1(x)+u_2(x)+\cdots = 1-\int_0^x [u_0(t)+u_1(t)+u_2(t)+\cdots]\,dt, \tag{II.69}$$

We identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation:

$$\begin{aligned} u_0(x) &= 1, \\ u_{k+1}(x) &= -\int_0^x u_k(t)\,dt, \quad k\geq 0. \end{aligned} \tag{II.70}$$

So, we have

$$\begin{aligned} u_0(x) &= 1, \\ u_1(x) &= -\int_0^x u_0(t)\,dt = -\int_0^x 1\,dt = -x, \\ u_2(x) &= -\int_0^x u_1(t)\,dt = -\int_0^x (-t)\,dt = \frac{1}{2!}x^2, \\ u_3(x) &= -\int_0^x u_2(t)\,dt = -\int_0^x \frac{1}{2!}t^2\,dt = -\frac{1}{3!}x^3, \\ u_4(x) &= -\int_0^x u_3(t)\,dt = -\int_0^x \frac{1}{3!}t^3\,dt = \frac{1}{4!}x^4, \\ &\vdots \end{aligned}$$

Using Equation (II.62), the series solution is given by:

$$u(x) = 1-x+\frac{1}{2!}x^2-\frac{1}{3!}x^3+\frac{1}{4!}x^4-\cdots, \tag{II.71}$$

This series converges to the closed form solution:

$$u(x) = e^{-x}, \tag{II.72}$$

Example 02 : We are given the Fredholm integral equation :

$$u(x) = e^x - x + x\int_0^1 tu(t)\,dt, \tag{II.73}$$

The Adomian decomposition method assumes that the solution $u(x)$ has a series form given in Equation (II.62). Substituting this decomposition series into both sides of Equation (II.73), we get

$$\sum_{n=0}^{\infty} u_n(x) = e^x - x + x\int_0^1 t\sum_{n=0}^{\infty} u_n(t)\,dt, \tag{II.74}$$

Equivalently,

$$u_0(x)+u_1(x)+u_2(x)+\cdots = e^x - x + x\int_0^1 t[u_0(t)+u_1(t)+u_2(t)+\cdots]\,dt, \tag{II.75}$$

We identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation:

$$\begin{aligned} u_0(x) &= e^x - x, \\ u_{k+1}(x) &= x\int_0^1 t u_k(t)\,dt, \quad k \geq 0. \end{aligned} \tag{II.76}$$

Consequently, we obtain

$$\begin{aligned} u_0(x) &= e^x - x, \\ u_1(x) &= x\int_0^1 t u_0(t)\,dt = x\int_0^1 t(e^t - t)\,dt = \frac{2}{3}x, \\ u_2(x) &= x\int_0^1 t u_1(t)\,dt = x\int_0^1 t\left(\frac{2}{3}t\right)\,dt = \frac{2}{9}x, \\ u_3(x) &= x\int_0^1 t u_2(t)\,dt = x\int_0^1 t\left(\frac{2}{9}t\right)\,dt = \frac{2}{27}x, \\ u_4(x) &= x\int_0^1 t u_3(t)\,dt = x\int_0^1 t\left(\frac{2}{27}t\right)\,dt = \frac{2}{81}x, \\ &\vdots \end{aligned}$$

Using Equation , the series solution is given by:

$$u(x) = e^x - x + \frac{2}{3}x\left(\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n\right), \tag{II.77}$$

Notice that the infinite geometric series at the right side has $a_1 = 1$ and the ratio $r = \frac{1}{3}$. The sum of the infinite series is therefore given by

$$S = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}, \tag{II.78}$$

The series solution (II.77) converges to the closed-form solution:

$$u(x) = e^x, \tag{II.79}$$

This is obtained by using (II.78) in (II.77).

Eample 03: Use the Adomian decomposition method to solve the following Volterra-Fredholm integral equation

$$u(x) = e^x + 1 + x + \int_0^x (x-t)u(t)\,dt - \int_0^1 e^{x-t}u(t)\,dt, \tag{II.80}$$

Using the decomposition series (II.62), and using the recurrence relation (II.76), we obtain

$$\begin{aligned} u_0(x) &= e^x + 1 + x, \\ u_1(x) &= \int_0^x (x-t)u_0(t)\,dt + \int_0^1 e^{x-t}u_0(t)\,dt \\ &= -x - 1 + \frac{1}{2}x^2 + \cdots, \end{aligned} \tag{II.81}$$

and so on. We notice the appearance of the noise terms ± 1 and $\pm x$ between the components $u_0(x)$ and $u_1(x)$. By canceling these noise terms from $u_0(x)$, the non-canceled term of $u_0(x)$ gives the exact solution

$$u(x) = e^x, \tag{II.82}$$

that satisfies the given equation.

It is to be noted that the modified decomposition method can be applied here. Using the modified recurrence relation

$$\begin{aligned} u_0(x) &= e^x, \\ u_1(x) &= 1 + x + \int_0^x (x-t)u_0(t)\,dt - \int_0^1 e^{x-t}u_0(t)\,dt = 0. \end{aligned} \tag{II.83}$$

The exact solution $u(x) = e^x$ follows immediately.

CHAPTER III

STUDY THE EXISTENCE OF A SOLUTION TO THE QUADRATIC INTEGRAL EQUATION

III.1 Quadratic integral equation

Quadratic integral equations are a special type of integral equations and pose a challenge due to their nonlinear nature. Solving them requires the use of various analytical and numerical methods. Choosing the appropriate method depends on the characteristics of the equation and the imposed conditions. This may require combining several approaches to arrive at an exact or approximate solution.

Equation 01

$$u(t) = h(t) + (Tu)(t) + \int_0^t \phi(t,s,u(s))\,ds, \quad t \in I = [0,L], [3] \tag{III.1}$$

Theorem. Let us assume that:

1. $\phi : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $|\phi(t,s,u)| \leq \alpha + \beta|u|$, for every $(t,s,u) \in I \times I \times \mathbb{R}$, $\alpha, \beta \in \mathbb{R}^+ = (0,+\infty)$.

2. $h : I \to \mathbb{R}$ is a continuous function;

3. T is a continuous operator from the Banach space $B(I,\mathbb{R})$ into itself such that there exists $a > 0$ with $|(Tu)(t)| \leq a|u(t)|$ for every $t \in I$;

4. $aL\alpha < 1$.

Then the integral equation (III.1)has at least one solution in the space $B(I,\mathbb{R})$.

Proof: Let M be a suitable ball $B_\rho(0)$ in the space $B(I,\mathbb{R})$. Consider, for every $q \in M$, the map $U : M \subset B(I,\mathbb{R}) \to B(I,\mathbb{R})$ defined as follows:

$$u = \phi(q) \quad \text{if and only if} \quad u(t) = h(t) + (T\phi(q))(t) + \int_0^t \phi(t,s,u(s))\,ds.$$

Any possible fixed point of the (as usual multivalued) function ϕ will be a solution of the integral problem.

To prove the theorem, the following steps in the proof have to be established:

i) ϕ is a (relatively) compact operator.

To obtain such a result, we prove (by using Ascoli's theorem) that ϕ is an equicontinuous and equibounded operator.

We take a function $t \to q(t) \in M \subset B(I,\mathbb{R})$; so $||q|| \leq \rho$. Now, using the Gronwall Lemma, we obtain:

$$|u(t)| \leq |h(t)| + a||q|| \int_0^t (\alpha + \beta|u(t)|)\,ds \leq (||h|| + a\rho\alpha L)\exp(a\rho\beta L).$$

Thus we can say that there is some constant ρ_0 such that $\phi(q) \subset B_{\rho_0}(0)$ for every $q \in M$. So the set $\phi(M)$ is equibounded.

ii) ϕ is equicontinuous.

Since $||q|| \leq \rho$, then $||Tq|| \leq a\rho$. Let $x \in \phi(q)$ and let us assume that $t_1, t_2 \in [0, L]$ are such that $|t_2 - t_1| < \delta$, for a given positive constant δ. Thus:

$$\begin{aligned}
||u(t_2) - u(t_1)|| &\leq ||(Tq)(t_2) \int_0^{t_2} \phi(t_2, s, u(s))\,ds - (Tq)(t_1) \int_0^{t_1} \phi(t_1, s, u(s))\,ds|| \\
&\leq ||(Tq)(t_2)|| \int_0^{t_2} ||\phi(t_2, s, u(s))||\,ds \\
&\quad - ||(Tq)(t_1)|| \int_0^{t_1} \phi(t_1, s, u(s))\,ds + ||(Tq)(t_2)|| \int_{t_2}^0 \phi(t_1, s, u(s))\,ds + \ldots \\
&\quad + ||(Tq)(t_2)|| \int_{t_2}^0 \phi(t_1, s, u(s))\,ds - ||(Tq)(t_2)|| \int_{t_1}^0 \phi(t_1, s, u(s))\,ds + \ldots \\
&\quad + ||(Tq)(t_2)|| \int_{t_1}^0 \phi(t_1, s, u(s))\,ds - ||(Tq)(t_1)|| \int_0^{t_1} \phi(t_1, s, u(s))\,ds|| \\
&\leq ||(Tq)(t_2)|| \int_0^{t_2} ||\phi(t_2, s, u(s)) - \phi(t_1, s, u(s))||\,ds + \ldots \\
&\quad + ||(Tq)(t_2)|| \int_{t_2}^{t_1} ||\phi(t_1, s, u(s))||\,ds + ||(Tq)(t_2) - (Tq)(t_1)|| \int_0^{t_1} ||\phi(t_1, s, u(s))||\,ds \\
&\leq a\rho L^2 1 + a\rho(\alpha + \beta\rho)||t_2 - t_1|| + L(\alpha + \beta\rho)^2 2 \leq \varepsilon
\end{aligned}$$

whenever $t_1, t_2 \in I$ are such that $|t_2 - t_1| < \delta$.

So ϕ is an equicontinuous operator.

iii)$\phi$is an upper semicontinuous operator.

Indeed, let $q_n \to q_0$ and let $u_n \in \phi(q_n)$, $u_n \to u_0$. We need to show that $u_0 \in \phi(q_0)$.

From $u_n(t) = h(t) + (Tq_n)(t) \int_0^t \phi(t, s, u_n(s))\,ds$, from the continuity of the operator T and the function ϕ (with respect to x), there follows $\lim_{n\to+\infty}(Tq_n)(t) = (Tq_0)(t)$ and

$$\lim_{n\to+\infty} \int_0^t \phi(t, s, u_n(s))\,ds = \int_0^t \lim_{n\to+\infty} \phi(t, s, u_n(s))\,ds,$$

from condition 1 and the Dominated Lebesgue Convergence Theorem. So we get

$$\lim_{n\to+\infty} u_n(t) = h(t) + (Tq_0)(t) \int_0^t \phi(t, s, u_0(s))\,ds.$$

The latter means that $\lim_{n\to+\infty} u_n(t) = u_0(t)$, or, equivalently, that $u_0 \in \phi(q_0)$. **iv)** $\phi(q)$ **is acyclic for every** $q \in M$**.**

To achieve this, we want to apply Proposition I.11.2. Let $\phi_n(t,s,\cdot) \to \phi(t,s,\cdot)$ be a sequence of Lipschitzean functions (with respect to the third variable) such that (see Proposition I.11.1 $\|\phi_n\| \leq \|\phi\|$, $\forall t \in I$, $u \in M$. Now, let us define $\phi_n : M \to B(I,\mathbb{R})$ and, as before, let $y \in \phi_n(q)$ if $y(t) = h(t) + (Tq)(t)\int_0^t \phi_n(t,s,y(s))\,ds$, $\forall t \in I$.

The operators U_n are compact and continuous (using the same argument as before). So, to apply Proposition I.11.2, we need to verify that the equation $u = \phi_n(q)$ admits at most one solution. To this end, let u and y be two solutions such that $u = \phi_n(q)$ and $y = \phi_n(q)$.

Simultaneously, we have:

$$u(t) = h(t) + (Tq)(t)\int_0^t \phi_n(t,s,u(t))\,ds,$$

and

$$y(t) = h(t) + (Tq)(t)\int_0^t \phi_n(t,s,y(s))\,ds.$$

But

$$||u(t)-y(t)|| \leq ||(Tq)(t)\int_0^t \phi_n(t,s,u(t))\,ds - (Tq)(t)\int_0^t \phi_n(t,s,y(s))\,ds|| \leq ||(Tq)(t)||\int_0^t k_0||u(t)-y(s)||\,ds,$$

where k_0 is the Lipschitz constant of the sequence of functions u_n. Thus, $\forall t \in I$, there is

$$||u(t)-y(t)|| \leq a\rho\int_0^t k_0||u(t)-y(s)||\,ds.$$

This means that $||u(t)-y(t)|| = 0$ for every $t \in I$.

Moreover, to prove that $\|\phi_n - \phi\| \leq \varepsilon$, it will be enough to observe that if $y(t)$ and $z(t)$ are solutions of $\phi_n(q)$ and $\phi(q)$, respectively, then

$$||y(t)-z(t)|| \leq ||(Tq)(t)\int_0^t ||\phi_n(t,s,u(t)) - \phi(t,s,u(s))||\,ds \leq a\rho L\varepsilon.$$

v) There is a ball $B_R(0)$ such that $U(B_R(0)) \subset B_R(0)$.

If $\|q\| \leq R$, we get:

$$\|u\| \leq a\|q\|\int_0^t (\alpha + \beta||u(t)||)\,ds \leq a\|q\|L(\alpha+\beta\|u\|) \leq \alpha\frac{aRL}{1-a\frac{\beta L}{\alpha}}.$$

Thus, $\|u\|(1-a\frac{\beta L}{\alpha}) \leq \alpha aRL$, hence we obtain $\|u\| \leq \frac{\alpha aRL}{1-a\frac{\beta L}{\alpha}}$.

The last condition of the theorem allows us to take $R \leq \frac{1-a}{\alpha L}\frac{\alpha a}{\beta}$, and so $\|u\| \leq R$, and consequently, $\phi(B_R(0)) \subset B_R(0)$.

Equation 02

$$u(t) = f\left(t, \int_0^t v(t,s,u(s))\,ds, u(\alpha(t))\right) \cdot g\left(t, \int_0^a u(t)\phi(t,s,u(s))\,ds, u(\beta(t))\right), \qquad \text{(III.2)}$$

Theorem. Let us assume that:

i) $f,g : [0,a]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous, and there exist nonnegative constants c_1,c_2,d_1,d_2 such that

$$|f(t,0,u)| \leq c_1 + d_1|u|$$
$$|g(t,0,u)| \leq c_2 + d_2|u|$$

ii) The functions $f(t,y,u)$ and $g(t,y,u)$ satisfy a Lipschitz condition with respect to the variables y and x with constants $k,k_0 \geq 0$ respectively, i.e.[4],

$$|f(t,y_1,u) - f(t,y_2,u)| \leq k|y_1-y_2|,$$
$$|g(t,y_1,u) - g(t,y_2,u)| \leq k|y_1-y_2|,$$

for all $t\in[0,a]$ and $y_1,y_2,u\in\mathbb{R}$, and

$$|f(t,y,u_1) - f(t,y,u_2)| \leq k_0|u_1-u_2|,$$
$$|g(t,y,u_1) - g(t,y,u_2)| \leq k_0|u_1-u_2|,$$

for all $t\in[0,a]$ and $u_1,u_2,y\in\mathbb{R}$.

iii) $\phi, v : [0,a]\times[0,a]\times\mathbb{R}\to\mathbb{R}$ are continuous.

iv) $\alpha,\beta : [0,a]\to[0,a]$ are continuous and satisfy,

$$|\alpha(t_1)-\alpha(t_2)| \leq |t_1-t_2|, \quad |\beta(t_1)-\beta(t_2)| \leq |t_1-t_2|,$$

for all $t_1,t_2\in[0,a]$.

v) (Sublinear nonlinearity) There exist nonnegative constants $\alpha_1, \beta_1, \alpha_2$, and β_2 such that

$$|v(t,s,u)| \leq \alpha_1 + \beta_1 |u|, \quad |u(t,s,u)| \leq \alpha_2 + \beta_2 |u|,$$

for all $t,s \in [0,a]$ and $u \in \mathbb{R}$.

vi) The inequality

$$[k(\tilde{\alpha} + \tilde{\beta} r) \cdot a + (c + dr)][k(\tilde{\alpha} + \tilde{\beta} r) \cdot r \cdot a + (c + dr)] \leq \tilde{r}$$

has a positive solution $\tilde{r}_0$, where $\tilde{\alpha} = \max\{\alpha_1, \alpha_2\}$, $\beta\tilde{r} = \max\{\beta_1, \beta_2\}$, $c = \max\{c_1, c_2\}$, and $d = \max\{d_1, d_2\}$.

vii)

$$k' \left(k\tilde{\alpha} + \tilde{\beta} r_0\right) \cdot a \cdot (1 + r_0) + 2(c + dr_0) < 1$$

Under the tacit assumptions (i)-(vii) above, the functional-integral equation (III.2) has at least one solution $x \in C[0,a]$.

Remark: Assumption (v) essentially imposes a sublinear nonlinearity on the kernels u and v appearing in (III.2). To address a quadratic type of nonlinearity, as can occur in, for example, the integral equation of Chandrasekhar,

$$u(t) = 1 + u(t) \int_0^1 \frac{t}{t+s} \phi(s) u(t)\, ds, \tag{III.3}$$

We need to show that our technique can be applied to include this important class of integral equations.

Typically, the existence of solutions for (III.3) is established under the additional assumption that the characteristic function u in (III.3) is an even polynomial in s. For such characteristic functions, it is known that the resulting solutions can be expressed in terms of Chandrasekhar's H-functions.

In our approach, we establish the existence of solutions for equation (III.3) under the much weaker assumption that ϕ is continuous and $\phi(0) = 0$. The condition $\phi(0) = 0$ is physically meaningful in certain radiative transfer scenarios. An intriguing question arises in this context: for a general characteristic function, can the solutions we obtain be represented as an infinite linear combination of classical H-functions?

Proof. To prove this result using Lemma I.9.1 as our main tool, we need to define operators

F and G on the space $C[0,a]$ in the following way:

$$(Fu)(t) = f(t, \int_0^t v(t,s,u(s))\,ds, u(\alpha(t))),$$

$$(Gu)(t) = g(t, \int_0^a u(t)\phi(t,s,u(s))\,ds, u(\beta(t))).$$

Next, we prove that the operators F and G transform the space $C[0,a]$ into itself. To this end, we are going to prove that F,G are compositions of continuous functions defined on $[0,a]$; that is, the operator F can be expressed as the composition of the following functions:

$$\begin{aligned}
&[0,a] \xrightarrow{\mathrm{Id}\times\int v\times(u\circ\alpha)} [0,a]\times\mathbb{R}\times\mathbb{R} \xrightarrow{\mathrm{f}} \mathbb{R} \\
&(f,t) \mapsto \left(t, \int_0^t v(t,s,u(s))\,ds, u(\alpha(t))\right) \mapsto Fu(t).
\end{aligned}$$

Now, considering assumptions (i), (iii), and (iv), it follows that the functions defined above are continuous. Consequently, F maps the Banach algebra $C[0,a]$ into itself. Similarly, it can be shown that the operator G also maps $C[0,a]$ into itself.

The desired operator T on $C[0,a]$ is defined as follows:

$$Tu = (Fu)\cdot(Gu).$$

Clearly, T maps $C[0,a]$ into itself. Furthermore, by applying assumptions (ii), (iv), and (v), we obtain that for every $t \in [0,a]$,

$$\begin{aligned}
|Fu(t)| &\leq \text{`}f\left|t, \int_0^t v(t,s,u(s))\,ds, u(\alpha(t))\right| \qquad \text{(III.4)}\\
&\leq \left|f\left(t, \int_0^t v(t,s,u(s))\,ds, u(\alpha(t))\right) - f(t,0,u(\alpha(t)))\right| + |f(t,0,u(\alpha(t)))| \\
&\leq k\left|\int_0^t v(t,s,u(s))\,ds\right| + c_1 + d_1|u(\alpha(t))| \\
&\leq k(\alpha_1+\beta_1||u||)\cdot a + (c_1 + d_1||u||).
\end{aligned}$$

On the other hand, by (ii), (iv), and (v) again, we have

$$\begin{aligned}
|Gu(t)| &\leq \left| g\left(t, \int_0^a u(t)\phi(t,s,u(s))\,ds, u(\beta(t))\right)\right| \\
&\leq \left| g\left(t, \int_a^0 u(t)\phi(t,s,u(s))\,ds, u(\beta(t))\right) - g(t,0,u(\beta(t)))\right| + |g(t,0,u(\beta(t)))| \\
&\leq k\left|\int_0^a u(t)\phi(t,s,u(s))\,ds\right| + c_2 + d_2|u(\beta(t))| \\
&\leq k||u||(\alpha_2+\beta_2||u||)\cdot a + (c_2+d_2||u||),
\end{aligned} \tag{III.5}$$

Linking (III.4) and (III.5) we obtain

$$\begin{aligned}
&|Tu(t)| \\
&\quad = |Fu(t)|\cdot|Gu(t)| \\
&\quad \leq [k(\alpha_1+\beta_1||u||)\cdot a + (c_1+d_1||u||)]k||u||(\alpha_2+\beta_2||u||)\cdot a + (c_2+d_2||u||).
\end{aligned}$$

Hence,

$$||Tx|| \leq [k(\tilde{\alpha}_1+\tilde{\beta}_1||u||)\cdot a + (c+d||u||)][k||u||(\tilde{\alpha}_2+\tilde{\beta}_2||u||)\cdot a + (c_2+d_2||u||)]$$

Taking into account assumption (vi), we deduce that the operator T maps the ball $B_{r_0} \subset C[0,a]$ into itself.

Next, we show that the operator F is continuous on B_{r_0}. To do this, fix $\varepsilon > 0$ and take $x, y \in B_{r_0}$ such that $||u-y|| \leq \varepsilon$. Then, for $t \in [0,a]$, we get

$$\begin{aligned}
|Fu(t)-Fy(t)| &= \left| f\left(t, \int_0^t v(t,s,u(t)\,ds, u(\alpha(t))\right) - f\left(t, \int_0^t v(t,s,y(s)\,ds, y(\alpha(t))\right)\right| \\
&\leq \left| f\left(t, \int_0^t v(t,s,u(t)\,ds, u(\alpha(t))\right) - f\left(t, \int_0^t v(t,s,y(s))\,ds, u(\alpha(t))\right)\right| \\
&+ \quad \leq \left| f\left(t, \int_0^t v(t,s,y(s)\,ds, u(\alpha(t))\right) - f\left(t, \int_0^t v(t,s,y(s))\,ds, y(\alpha(t))\right)\right| \\
&\leq k\int_0^t |v(t,s,u(s)) - v(t,s,y(s))|\,ds + k^{'}|u(\alpha(t)) - y(\alpha(t))| \\
&\leq k\cdot w(v,\varepsilon)\cdot a + k^{'}||u-y|| \\
&\leq k\cdot w(v,\varepsilon)\cdot a + k^{'}\varepsilon,
\end{aligned}$$

Where

$w(v,\varepsilon) = \sup\{|v(t,s,u_1) - v(t,s,u_2)| : t,s \in [0,a], u_1,u_2 \in [-r_0,r_0], |u_1 - u_2| \leq \varepsilon\}$. Using the fact that the function v is uniformly continuous on the bounded subset
$[0,a]\times[0,a]\times[-r_0,r_0]$, we infer that $w(v,\varepsilon) \to 0$ as $\varepsilon \to 0$.
Thus, the above estimate shows that the operator F is continuous on B_{r_0}. Similarly, one can infer that the operator G is continuous on B_{r_0}, and consequently, deduce that T is a continuous operator on B_{r_0}.

Now, we prove that the operators F and G satisfy the Darbo condition with respect to the measure w_0 in the ball B_{r_0}. Take a nonempty subset X of B_{r_0} and $u \in X$. Then, for a fixed $\varepsilon > 0$ and $t_1,t_2 \in [0,a]$ such that $t_1 \leq t_2$ and $t_2 - t_1 \leq \varepsilon$,we obtain

$$\begin{aligned}
|Fu(t_2)-Fu(t_1)| &= \left| f\left(t_2, \int_0^{t_2} v(t_2,s,u(t))\,ds, u(\alpha(t_2))\right) - f\left(t_1, \int_0^{t_1} v(t_1,s,u(t))\,ds, u(\alpha(t_1))\right) \right| \\
&\leq \left| f\left(t_2, \int_0^{t_2} v(t_2,s,u(t))\,ds, u(\alpha(t_2))\right) - f\left(t_2, \int_0^{t_1} v(t_1,s,u(t))\,ds, u(\alpha(t_2))\right) \right| \\
&\quad + \left| f\left(t_2, \int_0^{t_1} v(t_1,s,u(t))\,ds, u(\alpha(t_2))\right) - f\left(t_1, \int_0^{t_1} v(t_1,s,u(t))\,ds, u(\alpha(t_1))\right) \right| \\
&\leq k \left| \int_0^{t_2} v(t_2,s,u(t))\,ds - \int_0^{t_1} v(t_1,s,u(t))\,ds \right| \\
&\quad + \left| f\left(t_2, \int_0^{t_1} v(t_1,s,u(t))\,ds, u(\alpha(t_2))\right) - f\left(t_1, \int_0^{t_1} v(t_1,s,u(t))\,ds, u(\alpha(t_2))\right) \right| \\
&\leq k \left[\int_0^{t_1} |v(t_2,s,u(t)) - v(t_1,s,u(t))|\,ds + \int_{t_1}^{t_2} |v(t_2,s,u(t))|\,ds \right] \\
&\quad + \left| f\left(t_2, \int_0^{t_1} v(t_1,s,u(t))\,ds, u(\alpha(t_2))\right) - f\left(t_1, \int_0^{t_1} v(t_1,s,u(t))\,ds, u(\alpha(t_2))\right) \right| \\
&\quad + k'|u(\alpha(t_2)) - u(\alpha(t_1))|,
\end{aligned} \tag{III.6}$$

At this point, we introduce the notation:
$w_v(\varepsilon,\cdot,\cdot) = \sup\{|v(t,s,u) - v(t',s,u)| : t,t',s \in [0,a], |t-t'| \leq \varepsilon \quad \text{and} \quad u \in [-r_0,r_0]\}$,
$L = \sup\{|v(t,s,u)| : t,s \in [0,a], u \in [-r_0,r_0]\}$,
$w_f(\varepsilon,\cdot,\cdot) = \sup\{|f(t,u,y) - f(t',u,y)| : t,t' \in [0,a], |t-'| \leq \varepsilon, u \in [-Lr_0a, Lr_0a], y \in [-r_0,r_0]\}$.

Then, using (III.1) we obtain the estimate

$$|Fu(t_2) - Fu(t_1)| \leq k[w_v(\varepsilon,\cdot,\cdot)\cdot a + L\varepsilon] + w_f(\varepsilon,\cdot,\cdot) + k'|u(\alpha(t_2)) - u(\alpha(t_1))|.$$

Now, assumption (iv) allows us to deduce

$$w(Fu,\varepsilon) \leq k\left[w_v(\varepsilon,\cdot,\cdot)\cdot a + L\varepsilon\right] + w_f(\varepsilon,\cdot,\cdot) + k' w(u,\varepsilon),$$

Thus, taking the supremum in X, then the limit as $\varepsilon \to 0$, and taking into account the uniform continuity of the functions f and v in bounded sets, we can deduce that

$$w_0(FX) \leq k_0 w_0(X), \tag{III.7}$$

Similarly, one can prove that

$$w_0(GX) \leq k_0 w_0(X), \tag{III.8}$$

Finally, linking (III.4), (III.5), (III.7), (III.8)and keeping in mind lemma I.9 we infer that the operator T satisfies the Darbo condition on $B_r 0$ with respect to the measure w_0 with constant

$$Q = k_0\left(hk(\tilde{\alpha} + \beta\tilde{r})\cdot a\cdot(1+r_0) + 2(c+dr_0)\right),$$

(see assumption (vii))
Furthermore, based on assumption (vii), we can infer that the operator T acts as a contraction on B_{r_0}. Thus, applying Darbo's theorem, we establish the existence of at least one fixed point for T within B_{r_0}. As a result, it follows that the functional-integral equation (III.2) possesses at least one solution in the domain B_{r_0}.
This completes the proof.

Examples

Example 01. First, we note that equation (III.2) concerns the well-known functional equation of the first order with a possible delay of the form

$$u(t) = f_1(t, u(\alpha(t))),$$

To obtain this example, it is sufficient to put $f(t,y,u) = f_1(t,u)$ and $g(t,y,u) = 1$.
Example 02. Next, setting $g(t,y,u) \equiv 1$ and $f(t,y,u) = a(t) + y$, equation (III.2)reduces to the well-known nonlinear Volterra integral equation

$$u(t) = a(t) + \int_0^t v(t,s,u(s))\,ds.$$

Example 03. On the other hand, if we choose $f(t,y,u) \equiv 1$, $g(t,y,u) = 1+y$, $u(t,s,y) = \frac{t}{t+s}u(s)y$, and $\beta(t) = t$ in Theorem above , equation (III.2) now takes the form

$$u(t) = 1 + u(t) + \int_a^t \frac{t}{t+s}\phi(s)u(t)\,ds, \tag{III.9}$$

This integral equation is the famous quadratic integral equation of Chandrasekhar, extensively discussed in numerous papers and monographs.

Equation 03

Firstly, we study the existence of solutions of the quadratic integral equation[8]

$$u(t) = f_1(t,u(t)) + \int_0^t f_2(s,u(t))\,ds + f_2(t,u(t)) \cdot \int_0^t f_1(s,u(t))\,ds, \tag{III.10}$$

uLet the integral operator H_i be defined as

$$(H_i u)(t) = \int_0^t f_i(s,u(t))\,ds, \quad i = 1,2,$$

Then, Equation (III.10) may be written in operator form as:

$$(Au)(t) = (F_1u)(t) * (H_2u)(t) + (F_2u)(t) * (H_1u)(t),$$

where $(F_iu)(t) = f_i(t,u(t))$, $i = 1,2$.

Theorem. Let us assume that:

(i) Functions $f_i : I \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the Carathéodory condition (i.e., measurable in t for all $u \in \mathbb{R}_+$ and continuous in x for all $t \in [0,1]$). There exist two functions $a_1, a_2 \in L_1$ and constants $b_1, b_2 > 0$ such that

$$f_i(t,u) \leq a_i(t) + b_i|u| \quad \forall (t,u) \in I \times \mathbb{R}_+,$$

where $f_i(t,u)$ for $i = 1,2$, and $f_i(t,u)$ are almost everywhere nondecreasing in both variables.

(ii) Let $d > \sqrt{16b_1b_2||a_1|| \cdot ||a_2||}$, where $d = 1 - 2b_1||a_2|| - 2b_2||a_1||$.

Now let r be a positive root of the equation

$$2b_1b_2r^2\,(1-2b_1||a_2||-2b_2|a_1|)\,r+2||a_1||\cdot||a_2||=0.$$

Define the set

$$B_r=\{u\in L_1:\|u\|\leq r\}.$$

To establish the existence of at least one L_1-positive solution to the quadratic integral Equation (III.10), we present the following theorem:

Assume conditions (i) and (ii) are satisfied. If $2rb_1b_2<1$, then there exists at least one solution $u\in L_1$ for the quadratic integral equation (III.10). Moreover, this solution is positive and almost everywhere non-decreasing on I.

Proof. Take an arbitrary $u\in L1$, then we get

$$\begin{aligned}||(Au)(t)||\leq{}&(a_1(t)+b_1||u(t)||)\int_0^t(a_2(s)+b_2||u(t)||)\,ds\\&+(a_2(t)+b_2||u(t)||)\int_0^t(a_1(s)+b_1||u(t)||)\,ds,\end{aligned}$$

which implies that

$$
\begin{aligned}
||(Au)(t)|| = \int_0^1 ||Au(t)||\,dt \le & \int_0^1 a_1(t)\int_0^t a_2(s)\,ds\,dt + b_2\int_0^1 a_1(t)\int_0^t ||u(t)||\,ds\,dt \\
& + b_1\int_0^1 ||u(t)||\int_0^t a_2(s)\,ds\,dt + b_1b_2\times\int_0^1 ||u(t)||\int_0^t ||u(t)||\,ds\,dt \\
& + \int_0^1 a_2(t)\int_0^t a_1(s)\,ds\,dt + b_1\int_0^1 a_2(t)\int_0^t ||u(t)||\,ds\,dt \\
& + b_2\times\int_0^1 ||u(t)||\int_0^t a_1(s)\,ds\,dt + b_1b_2\times\int_0^1 ||u(t)||\int_0^t ||u(t)||\,ds\,dt \\
\le & \int_0^1 a_2(s)\int_s^1 a_1(t)\,dt\,ds + b_2\int_0^1 ||u(t)||\int_s^1 a_1(t)\,ds\,dt \\
& + b_1\int_0^1 a_2(s)\int_s^1 ||u(t)||\,ds\,dt + 2b_1b_2\times\int_0^1 ||u(t)||\int_t^1 ||u(t)||\,dt\,ds \\
& + \int_0^1 a_1(s)\int_s^1 a_2(t)\,dt\,ds + b_1\int_0^1 ||u(t)||\int_s^1 a_2(t)\,dt\,ds \\
& + b_2\int_0^1 a_1(s)\times\int_s^1 ||u(t)||\,dt\,ds \\
\le & \,||a_1||\int_0^1 a_2(s)\,ds + b_2||a_1||\int_0^1 ||u(t)||\,ds + b_1||u(t)||\int_0^1 a_2(s\,ds) \\
& + 2b_1b_2||u||\int_0^1 ||u(t)||\,ds \\
& + ||a_2||\int_0^1 a_1(s)\,ds + b_1||a_2||\int_0^1 ||u(t)||\,ds + b_2||u(t)||\int_0^1 a_1(s)\,ds \\
\le & \,2||a_1||||a_2|| + 2b_1||u||||a_2|| + 2b_2||a_1||||u|| + 2b_1b_2||u||^2 \\
\le & \,r.
\end{aligned}
$$

Based on this estimate, we demonstrate that the operator A maps the ball B_r into itself, where

$$r = \frac{d - \sqrt{d^2 - 16b_1b_2\|a_1\|\cdot\|a_2\|}}{2b_1b_2}.$$

From assumption (ii), we know

$$0 < d^2 - 16b_1b_2\|a_1\|\cdot\|a_2\| < d^2,$$

which implies

$$0 < \sqrt{d^2 - 16b_1b_2\|a_1\|\cdot\|a_2\|} < d.$$

Therefore, d is positive, ensuring r is a positive constant.

Now, denote by Q_r the subset of $B_r \subset L_1$ consisting of all functions that are almost everywhere non-decreasing on I.

The set Q_r is nonempty, bounded, convex, and closed. Moreover, Q_r is compact in measure.

From assumption (i), we conclude that the operator A maps Q_r into itself. Since the operator $F_i u(t) = f_i(t, u(t))$ is continuous, the operator H_i is continuous, and therefore, the product $F_i \circ H_i$ is continuous. Thus, the operator A is continuous on Q_r.

Consider u as a nonempty subset of Q_r. Fix $\varepsilon > 0$ and choose a measurable subset $D \subseteq I$ such that $\operatorname{meas} D \leq \varepsilon$. Then, for any $u \in X$,

$$
\begin{aligned}
||Au||_{L_1(D)} &= \int_D |(Au)(t)|\,dt \\
&\leq \int_D a_1(t) \int_0^t a_2(s)\,ds\,dt + \int_D a_2(t) \int_0^t a_1(s)\,ds\,dt + b_2 \times \int_D a_1(t) \int_0^t |u(t)|\,ds\,dt \\
&\quad + b_1 \int_D a_2(t) \int_0^t |u(t)|\,ds\,dt + b_1 \int_D |u(t)| \int_0^t a_2(s)\,ds\,dt + b_2 \int_D |u(t)| \int_0^t a_1(s)\,ds\,dt \\
&\quad + 2b_1 b_2 \int_D |u(t)| \int_0^t |u(t)|\,ds\,dt \\
&\leq \int_D a_2(s) \int_D a_1(t)\,dt\,ds + \int_D a_1(s) \int_D a_2(t)\,dt\,ds + b_2 \int_D |u(t)| \int_D a_1(t)\,dt\,ds \\
&\quad + b_1 \int_D |u(t)| \int_D a_2(t)\,dt\,ds + b_1 \int_D a_2(s) \int_D |u(t)|\,dt\,ds + b_2 \int_D a_1(s) \int_D |u(t)|\,dt\,ds \\
&\quad + 2b_1 b_2 \int_D |u(t)| \int_D |u(t)|\,dt\,ds \\
&\leq ||a_1||_{L_1(D)} \int_D a_2\,ds + ||a_2||_{L_1(D)} \int_D a_1\,ds + b_1 \int_D |u(t)|\,ds \int_D a_2\,dt \\
&\quad + b_2 \int_D |u(t)|\,ds \int_D a_1\,dt + b_2 ||a_1||_{L_1(D)} \int_D |u|\,ds \\
&\quad + b_1 ||a_2||_{L_1(D)} \int_D |u|\,ds + 2b_1 b_2 \int_D |u(t)|\,ds \int_D |u(t)|\,dt \\
&\leq 2||a_1||_{L_1(D)} ||a_2||_{L_1(D)} + 2b_1 ||u||_{L_1(D)} ||a_2||_{L_1(D)} \\
&\quad + 2b_2 ||u||_{L_1(D)} ||a_1||_{L_1(D)} + 2b_1 b_2 ||u||_{L_1(D)} ||u||_{L_1(D)} \\
&\leq 2||a_1||_{L_1(D)} ||a_2||_{L_1(D)} + r b_1 ||a_2||_{L_1(D)} + r b_2 ||a_1||_{L_1(D)} \\
&\quad + 2r b_1 b_2 ||u||_{L_1(D)} \\
&\leq 2||a_1||_{L_1(D)} ||a_2||_{L_1(D)} + r(b_1 ||a_2||_{L_1(D)} + b_2 ||a_1||_{L_1(D)} + 2b_1 b_2 ||u||_{L_1(D)}).
\end{aligned}
$$

Since

$$
\lim_{\varepsilon \to 0} \sup \left\{ \int_D |a_i(t)|\,dt \,:\, D \subset I, \operatorname{meas} D < \varepsilon \right\} = 0, \quad i = 1, 2,
$$

we obtain

$$\beta(Au(t)) \leq 2rb_1b_2\beta(u(t)), \quad \text{for all } t \in I.$$

This implies

$$\beta(Au) \leq 2rb_1b_2\beta(Au), \tag{III.11}$$

where β is the De Blasi measure of weak noncompactness.

We can write (III.11) in the form

$$\chi u(Au) \leq 2rb_1b_2\chi u(u),$$

Given that χu represents the Hausdorff measure of noncompactness,

Since $2rb_1b_2 < 1$, according to Theorem I.9.1, A is confirmed to be a contraction with respect to the measure of noncompactness χu. Consequently, A possesses at least one fixed point in Q_r, which thereby serves as a solution to the quadratic integral equation (III.10).

Equation 04

$$u(t) = \int_0^t f_1(s,u_0(s))\,ds + \int_0^t f_2(s,u_0(s))\,ds, \quad t \in (0,1), \tag{III.12}$$

with

$$u(0) = u_0, \tag{III.13}$$

Theorem Assuming the conditions of Theorem III.1 are met, there exists at least one solution $u \in AC(0,1]$ to the quadratic integro-differential equations (III.12) and (III.13) that is positive and non-decreasing on I [8].

Proof. Differentiating both sides of (III.12), we obtain

$$u'(t) = f_1(t,u'(t))\int_0^t f_2(s,u'(t))\,ds + f_2(t,u'(t))\int_0^t f_1(s,u'(t))\,ds.$$

Let $u'(t) = u(t) \in L_1$, then (III.12) will be similar to (III.10), and

$$u(t) = u(0) + \int_0^t u(s)\,ds \in AC(0,1].$$

From Theorem III.1, there exists at least one positive and non-decreasing solution of (III.12) and (III.13).

Examples

Example 01. Consider the problem

$$u(t) = \int_0^t f(s,u'(s))^2\,ds, \quad \text{a.e. } t \in (0,1], u(0) = u_0$$

with the initial condition $u(0) = u_0$.

Then, this problem has at least one positive and nondecreasing solution $u \in AC(0,1]$, by taking $f_1(t,u(t)) = f_2(t,u(t))$ in Eq(III.12) .

III.2 Quadratic integral equation with phase-lag-term

Quadratic integral equations with phase-lag-term represent a significant mathematical challenge due to their nonlinear and complex nature. However, several analytical and numerical methods have been developed to deal with these equations and obtain approximate solutions [1].

III.2.1 Using the numerical method

$$\begin{aligned} u(t) = a(t) + \frac{1}{q}\int_0^t u(\tau)\,d\tau \left(\int_0^1 k(t,s)f(s,u(s))\,ds - 1\right) \quad - \frac{1}{q}\int_0^t u(\tau)\,d\tau \int_0^t \int_0^1 k_\tau(\tau,s)f(s,u(s))\,ds\,d\tau \\ + (u(t)-u_0)\left(\int_0^1 k(t,s)f(s,u(s))\,ds \quad - \int_0^t \int_0^1 k_\tau(\tau,s)f(s,u(s))\,ds\,d\tau.\right), \end{aligned} \tag{III.14}$$

Equation (III.14) can be written in the following integral operator form:

$$\begin{aligned} (Hu)(t) &= a(t) + \frac{1}{q}(Vu)(t) \\ (Vu)(t) &= (Tu)(t)((KF)(t)-1) + (Tu)(t)(K_\tau F)(t) + q(u(t)-u_0)((KF)(t) - (K\tau F)(t)), \end{aligned} \tag{III.15}$$

where

$$(Tu)(t) = \int_0^t u(\tau)\,d\tau, \quad (KF)(t) = \int_0^1 k(t,s)f(s,u(s))\,ds,$$

and

$$(K\tau F)(t) = -\int_0^t \int_0^1 \frac{\partial}{\partial \tau} k(\tau,s)f(s,u(s))\,ds\,d\tau.$$

In order to discuss the existence of at least one solution of Eq. (III.14).

Theorem III.2.1 . Under the following assumptions:

(i) $a : I \to \mathbb{R}$ is a continuous, non-decreasing, and nonnegative function on I.

(ii) $k : I \times I \to \mathbb{R}$ is continuous, and both $s \mapsto k(t,s)$ and $t \mapsto k(t,s)$ are non-decreasing on I for fixed $t \in I$ and $s \in I$. Additionally, $k_\tau(\tau,s) : [0,1] \times [0,1] \to \mathbb{R}$ is continuous with respect to both variables τ and s, and satisfies $|-k_\tau(\tau,s)| < c$ for all $\tau \in I$, where $c > 0$ is a constant.

(iii) The operator $T : C(I) \to C(I)$ is continuous and satisfies the Darbo condition with a constant α for the measure of noncompactness μ, such that $|(Tu)(t)| \leq \|u\|$.

(iv) The function $f : I \times I \to \mathbb{R}$ satisfies conditions (a)-(d), and there exists a non-decreasing function $m : \mathbb{R}_+ \to \mathbb{R}_+$ such that $|f(s,u(s))| \leq m(|u|)$.

(v) The unknown function $u(t)$ satisfies $|u(t) - u_0| \leq \|u\|$ in the space $C(I)$.

(vi) The inequality
$$\|u\| + rm(r)(k^* + c)\left(1 + \frac{1}{q}\right) \leq r,$$
holds, where $k^* = \max\{k(t,s) : t,s \in I\}$, $\|u\|$ denotes the norm of $u(t)$ in $C(I)$, and $m(r_0)(k^* + c)(q+1) < q$.

Under assumptions (i)-(vi), Equation (III.14) has at least one solution $u = u(t)$ in the space $C(I)$, which is non-decreasing and nonnegative on the interval I.

Proof. Let us consider the operator H defined on the space $C(I)$ by the formula (III.15). Taking into account assumptions (i) – (v) and the properties of the superposition operator, we infer that the function Hu is continuous on I for any function $u \in C(I)$, i.e., the operator H transforms the space $C(I)$ into itself. Further, applying our assumptions, we derive the following estimate:

$$\begin{aligned}
|(Hu)(t)| \leq |a(t)| + \frac{1}{q}|(Tu)(t)| \quad &\times \int_0^1 |k(t,s)||f(s,u(s))|\,ds + \frac{1}{q}|(Tu)(t)| \\
&\times \int_0^t \int_0^1 |-k_\tau(\tau,s)||f(s,u(s))|\,ds\,d\tau + |u(t) - u_0| \\
&\times \left(\int_0^1 |k(t,s)||f(s,u(s))|\,ds + \int_0^t \int_0^1 |-k_\tau(\tau,s)||f(s,u(s))|\,ds\,d\tau\right) \\
&\leq \|a\| + \frac{1}{q}\|u\|k^* m(|u|) + \frac{1}{q}c\|u\|m(|u|) + \|u\|m(|u|)(k^* + c).
\end{aligned}$$

Hence, we obtain

$$|(Hu)(t)| \leq \|a\| + \|u\|m(|u|)(k^* + c)\left(1 + \frac{1}{q}\right).$$

From the above estimate and assumption (vi), we infer that there exists $r_0 > 0$ such that the operator H transforms the ball B_{r_0} into itself.

In what follows, we will consider the operator H on the subset $B_{r_0}^+$ of the ball B_{r_0} defined by:

$$B_{r_0}^+ = \{u \in B_{r_0} : u(t) \geq 0, \text{ for } t \in I\}.$$

Notice that the set $B_{r_0}^+$ is nonempty, bounded, closed, and convex. Hence, and in view of assumptions (i) – (v), we deduce easily that H transforms the set $B_{r_0}^+$ into itself.

Now, we show that H is continuous on the set $B_{r_0}^+$. To do this, let us fix $\varepsilon > 0$ and choose $\delta > 0$ according to the continuity of H. Further, take arbitrarily $u, \psi \in B_{r_0}^+$ such that $\|u - \psi\| \leq \delta$. Then, for $t \in I$, we derive the following estimates:

$$\begin{aligned}
|(Hu)(t) - (H\psi)(t)| &\leq \frac{1}{q}\|u\|k^* l \int_0^1 \|u(s) - \psi(s)\|\, ds + \frac{1}{q}\|u - \psi\|k^* \int_0^1 m(|\psi|)\, ds \\
&+ \frac{1}{q}\|u\|cl \int_0^t \int_0^1 \|u(s) - \psi(s)\|\, ds d\tau + \frac{1}{q}\|u - \psi\|c \int_0^t \int_0^1 m(|\psi|)\, ds d\tau \\
&+ \|u\| \left| k^* l \int_0^1 \|u(s) - \psi(s)\|\, ds + cl \int_0^t \int_0^1 \|u(s) - \psi(s)\|\, ds d\tau \right| \\
&+ \|u - \psi\| \left| k^* \int_0^1 m(|\psi|)\, ds + c \int_0^t \int_0^1 m(|\psi|)\, ds d\tau \right| \\
&\leq \frac{1}{q}(k^* l\|u\| + k^* m(|\psi|) + cl\|u\| + cm(|\psi|) + q\|u\|(k^* l + cl) \\
&+ qk^* m(|\psi|) + qcm(|\psi|))\|u - \psi\| \\
&\leq \frac{1}{q}(1 + q)(k^* l r_0 + k^* m(r_0) + cl r_0 + cm(r_0))\delta.
\end{aligned}$$

The above estimate allows us to deduce that the operator H is continuous on the set $B_{r_0}^+$. Now, let us take a nonempty set Φ, such that $\Phi \in B_{r_0}^+$. Further, fix arbitrarily $\varepsilon > 0$ and choose $u \in \Phi$

and $t_1, t_2 \in I$ such that $|t_2 - t_1| \leq \varepsilon$. Then, keeping in mind our assumptions, we obtain

$$\begin{aligned}
|(Hu)(t_2)-(Hu)(t_1)| \leq\ & |a(t_2)-a(t_1)| + \frac{1}{q}|(Tu)(t_2)-(Tu)(t_1)| \int_0^1 |k(t,s)f(s,u(s))|\,ds \\
& + \frac{1}{q}|(Tu)(t_2)-(Tu)(t_1)| \int_{t_1}^{t_2}\int_0^1 |-k_\tau(\tau,s)||f(s,u(s))|\,dsd\tau \\
& + \frac{1}{q}|(Tu)(t_2)| \int_{t_2}^{t_1}\int_0^1 |-k_\tau(\tau,s)||f(s,\psi(s))|\,dsd\tau \\
& + |u(t_2)-u_0| \int_{t_2}^{t_1}\int_0^1 |-k_\tau(\tau,s)||f(s,u(s))|\,dsd\tau \\
& + |u(t_2)-u(t_1)| \left| \int_0^1 |k(t,s)||f(s,u(s))|\,ds + \int_{t_1}^{t_2}\int_0^1 |-k_\tau(\tau,s)f(s,u(s))|\,dsd\tau \right|
\end{aligned}$$

$$\begin{aligned}
|(Hu)(t_2)-(Hu)(t_1)| \leq\ & \omega(a,\varepsilon) + \frac{1}{q}\omega(Tu,\varepsilon)k^* m(r_0) \\
& + \frac{1}{q}(t_2-t_1)cm(r_0)r_0 + \frac{1}{q}\omega(Tu,\varepsilon)cm(r_0) \\
& + (t_2-t_1)cm(r_0)r_0 + \omega(u,\varepsilon)(k^* m(r_0) + cm(r_0)) \\
\leq\ & \omega(a,\varepsilon) + \frac{1}{q}(k^* m(r_0) + cm(r_0))\omega(Tu,\varepsilon) \\
& + cm(r_0)r_0\varepsilon\left(1+\frac{1}{q}\right) + (k^* m(r_0) + cm(r_0))\omega(u,\varepsilon).
\end{aligned}$$

Hence, keeping in mind our assumptions and the above-established facts, we arrive at the following inequality

$$\omega_0(H\Phi) \leq \frac{1}{q} m(r_0)(k^*+c)\omega_0(T\Phi) + m(r_0)(k^*+c)\omega_0(\phi). \tag{III.16}$$

In what follows, fix arbitrary $u \in \Phi$ and $t_1, t_2 \in I$ with $t_2 \geq t_1$. Then, taking into account our assumptions, we have

$$
\begin{aligned}
&|(Hu)(t_2)-(Hu)(t_1)|-[(Hu)(t_2)-(Hu)(t_1)] \\
\leq &\frac{1}{q}\{|(Tu)(t_2)-(Tu)(t_1)|-[(Tu)(t_2)-(Tu)(t_1)]\} \quad \left|\int_0^1 k(t,s)f(s,u(s))\,ds\right| \\
&+\frac{1}{q}\{|(Tu)(t_2)-(Tu)(t_1)|\}\int_{t_1}^{t_2}\int_0^1 |-k_\tau(\tau,s)f(s,u(s))|\,dsd\tau \\
&+\{|(u)(t_2)|-[(u_0)|]\}\int_{t_2}^{t_1}\int_0^1 |-k_\tau(\tau,s)f(s,\psi(s))|\,dsd\tau \\
&+\{|(u)(t_2)-u_0|-[(u)(t_2)-u_0]\}\left|\int_{t_2}^{t_1}\int_0^1 -k_\tau(\tau,s)f(s,u(s))|\,dsd\tau\right| \\
&+\{|(u)(t_2)-(u)(t_1)|-[(u)(t_2)-(u)(t_1)]\} \\
&\times\left|\int_0^1 k(t,s)f(s,u(s))\,ds \quad +\int_{t_1}^{t_2}\int_0^1 |-k_\tau(\tau,s)f(s,u(s))|\,dsd\tau\right| \\
&\leq \frac{1}{q}m(r_0)(k^*+c)d(Tu)+m(r_0)(k^*+c)d(u).
\end{aligned}
$$

Hence, we get

$$d(Hu)\leq \frac{1}{q}m(r_0)(k^*+c)d(Tu)+m(r_0)(k^*+c)d(u),$$

and consequently,

$$d(H\Phi)\leq \frac{1}{q}m(r_0)(k^*+c)d(T\Phi)+m(r_0)(k^*+c)d(\Phi). \tag{III.17}$$

Finally, from Eqs (III.16)and (III.17) give us that

$$\omega_0(H\Phi)+d(H\Phi)\leq \frac{1}{q}m(r_0)(k^*+c)(\omega_0(T\Phi)+d(T\Phi))+m(r_0)(k^*+c)(\omega_0(\Phi)+d(\Phi)).$$

Following the concepts of the measure of noncompactness μ in Sec. 2, we get

$$\mu(H\Phi)\leq \frac{1}{q}m(r_0)(k^*+c)\mu(T\Phi)+m(r_0)(k^*+c)\mu(\Phi).$$

From assumption (iii), we obtain

$$\mu(H\Phi)\leq m(r_0)(k^*+c)\left(1+\frac{1}{q}\right)\mu(\Phi).$$

Now, keeping in mind the above inequality and the fact that $m(r_0)(k^*+c)(q+1)<q$, in view of Theorem III.2.1, then Eq(III.15) has at least one solution $\phi \in C(I)$. This completes the proof.

Examples

Example 01 : Consider the following quadratic integral equation with a phase-lag term:

$$u(t+0.01)=\frac{1}{20}t^{10}+u(t+0.01)\int_0^1 \frac{t+\tau}{51+9e^{2\tau}}\frac{u(\tau)}{1+15e^{1+u(\tau)}}d\tau, \quad u(0)=0. \tag{III.18}$$

Using numerical treatment of the equation (III.18), we obtained

$$\begin{aligned} u(t)=&\frac{5}{11}t^{11}+\frac{1}{0.01}\int_0^t u(\tau)d\tau\left(\int_0^1 \frac{t+s}{51+9e^{2s}}\frac{u(s)}{1+15e^{1+u(s)}}ds-1\right)\\ &-\frac{1}{0.01}\int_0^t u(\tau)d\tau\int_0^t\int_0^1 \frac{1}{51+9e^{2s}}\frac{u(s)}{1+15e^{1+u(s)}}dsd\tau\\ &+u(t)\left(\int_0^1 \frac{t+s}{51+9e^{2s}}\frac{u(s)}{1+15e^{1+u(s)}}ds-\int_0^t\int_0^1 \frac{1}{51+9e^{2s}}\frac{u(s)}{1+15e^{1+u(s)}}dsd\tau.\right) \end{aligned}$$

where $t \in I=[0,1]$.

In this example, comparing with (III.15), we have $a(t)=\frac{5t^{11}}{11}$ which is nonnegative and continuous on I with norm $\|a(t)\|=\max_{t\in I}\left|\frac{5t^{11}}{11}\right|=\frac{5}{11}$. The operator T is defined as $(Tu)(t)=\int_0^t u(\tau)d\tau$, which is also nonnegative and continuous with norm $\|T\|=1$. The kernel $k(t,\tau)=\frac{t+\tau}{51+9e^{2\tau}}$, which is continuous with respect to t and τ.Also, we have $|k(t,\tau)|=|\frac{t+\tau}{51+9e^{2\tau}}|\leq\frac{1}{4e^2}$, $k^*=\frac{1}{4e^2}$, and $k_\tau(\tau,s)=\frac{1}{51+9e^{2s}}$, where $|k_\tau(\tau,s)|=|\frac{t+\tau}{51+9e^{2\tau}}|\leq\frac{1}{4e^2}$ $(c=\frac{1}{4e^2})$. The function $f(\tau,u(\tau))=\frac{u(\tau)}{1+15e^{1+u(\tau)}}$ satisfies assumption (iv) with
$|f(\tau,u(\tau))|=\frac{|u(\tau)|}{15}$. Thus, we get $m(r)=\frac{r}{15}$.

Further, let us consider the inequality

$$\frac{5}{11}+\frac{101}{30e^2}r^2\leq r \quad , \tag{III.19}$$

or equivalently,

$$\frac{30e^2}{101}r-r^2\geq\frac{150e^2}{1111} \quad . \tag{III.20}$$

Using standard methods, we can verify that the function $\upsilon(r)=\frac{30e^2}{101}r-r^2$ attains its maximum at the point $r_0=0.7$, and $\upsilon(r_0)=\frac{30e^2}{101}(0.7)-(0.7)^2\geq\frac{150e^2}{1111}$. So, the number r_0 is a positive solution of the inequality for which $m(|r_0|)(k^*+c)(1.01)=3\times10^{-3}<0.01$.

Finally, taking into account all the above-established facts and Theorem III.2.1, we conclude that the equation (III.18) has at least one solution $u = u(t)$ defined, continuous, and nondecreasing on the interval I. Moreover, $\|u\| \leq r_0 = 0.7$.

III.2.2 Homotopy perturbation method

Equation (III.15) can be written in the following integral operator form:

$$\begin{aligned} L(v) &= v, \\ N(v) &= -\frac{1}{q}\int_0^t v(\tau)d\tau\left(\int_0^1 k(t,s)v(s)ds - 1\right) \quad + \frac{1}{q}\int_0^t v(\tau)d\tau \int_0^t\int_0^1 k_\tau(\tau,s)v(s)dsd\tau \\ &\quad - (v(t) - v_0)\left(\int_0^1 k(t,s)v(s)ds - \int_0^t\int_0^1 k_\tau(\tau,s)v(s)dsd\tau\right). \end{aligned} \tag{III.21}$$

W e obtain the homotopy operator for the Quadratic integral equation with Phase–lag term in time:

$$\begin{aligned} H(v,p) =& v(t) - u_0(t) + p\left(u_0(t) - \frac{1}{q}\int_0^t v(\tau)d\tau\left(\int_0^1 k(t,s)v(s)ds - 1\right)\right. \\ & + \frac{1}{q}\int_0^t v(\tau)d\tau\int_0^t\int_0^1 k_\tau(\tau,s)v(s)ds - (v(t)-v_0)\left(\int_0^1 k(t,s)v(s)ds\right. \\ & \left.\left. - \int_0^t\int_0^1 k_\tau(\tau,s)v(s)ds\right) - a(t)\right). \end{aligned} \tag{III.22}$$

According to the method, the next step involves searching for the solution of the operator equation $H(v,p) = 0$ in the form of a power series:

$$v(t) = \sum_{j=0}^{\infty} p^j v_j(t). \tag{III.23}$$

$$\begin{aligned} \sum_{j=0}^{\infty} p^j v_j(t) =& u_0(t) + p(a(t) - u_0(t)) + \\ & \frac{1}{q}\int_0^t \sum_{j=1}^{\infty} p^j v_{j-1}(\tau)d\tau\left(\int_0^1 k(t,s)\sum_{j=0}^{\infty} p^j v_j(s)ds - 1\right) \\ & - \frac{1}{q}\int_0^t \sum_{j=1}^{\infty} p^j v_{j-1}(\tau)d\tau\int_0^t\int_0^1 k_\tau(\tau,s)\sum_{j=0}^{\infty} p^j v_j(s)dsd\tau \\ & + \left(\sum_{j=0}^{\infty} p^j v_j(t) - v_0\right)\left(\int_0^1 k(t,s)\sum_{j=1}^{\infty} p^j v_{j-1}(s)ds + \int_0^t\int_0^1 k_\tau(\tau,s)\sum_{j=1}^{\infty} p^j v_{j-1}(s)dsd\tau\right). \end{aligned} \tag{III.24}$$

By comparing the expressions with the same powers of the parameter p, we receive the relations:

$$
\begin{aligned}
&p_0 : v_0(t) = u_0(t)\\
&p^1 : v_1(t) = a(t) - u_0(t) + \frac{1}{q}\int_0^t v_0(\tau)d\tau\left(\int_0^1 k(t,s)v_0(s)ds - 1\right)\\
&\quad -\frac{1}{q}\int_0^t v_0(\tau)d\tau\int_0^t\int_0^1 k_\tau(\tau,s)v_0(s)dsd\tau + (v_0(t) - v_0)\\
&\quad \left(\int_0^1 k(t,s)v_0(s)ds - \int_0^t\int_0^1 k_\tau(\tau,s)v_0(s)dsd\tau\right)\\
&p^i : v_i(t) = \frac{1}{q}\sum_{k=0}^{i-1}\left(\int_0^t v_k(\tau)d\tau\int_0^1 k(t,s)v(i-k-1)(s)ds\right)\\
&\quad -\frac{1}{q}\int_0^t v(i-1)(\tau)d\tau - \frac{1}{q}\sum_{k=0}^{i-1}\left(\int_0^t v_k(\tau)d\tau\int_0^t\int_0^1 k_\tau(\tau,s)v(i-k-1)(s)dsd\tau\right)\\
&\quad +\sum_{k=0}^{i-1}\left(v_{(i-k-1)}(t)\left(\int_0^1 k(t,s)vk(s)ds - \int_0^t\int_0^1 k_\tau(\tau,s)vk(s)dsd\tau\right)\right)\\
&\quad - v_0\left(\int_0^1 k(t,s)v_{(i-1)}(s)ds - \int_0^t\int_0^1 k_\tau(\tau,s)v_{(i-1)}(s)dsd\tau\right), \quad i \geq 2.
\end{aligned}
$$

Remark.01 It is worth noting that if we choose $u_0(t) = 0$ or $u_0(t) = a(t)$, then the Homotopy perturbation method under consideration becomes equivalent to the well-known method of successive approximations. In the first case, the first term, which is identically zero, is omitted.

Remark.02 If we cannot determine the sum of series (III.21) (for $p = 1$), we can accept the partial sum as an approximate solution of the equation under consideration. By taking the first $n+1$ components, we obtain what is known as the n-th order approximate solution.

$$\hat{u}_n = \sum_{j=0}^{n} v_j. \tag{III.25}$$

Examples

Example 01: Consider the following Quadratic integral equation with Phase–lag term of the form:

$$u(t+0.01) = (0.01-t)^2 - \frac{1}{4}t^2(0.01-t)^2 + u(t+0.01)\int_0^1 (\tau)d\tau; \quad (u(0) = 0) \tag{III.26}$$

with the exact solution $u(t) = t^2$.

Using numerical treatment of the equation (III.26), we obtained

$$
\begin{aligned}
u(t) =& a(t)+\frac{1}{0.01}\int_0^t u(\tau)d\tau\left(\int_0^1 su(s)ds-1\right)\\
&-\frac{2}{0.01}\int_0^t u(\tau)d\tau\int_0^t\int_0^1 \tau su(s)dsd\tau+u(t)\left(\int_0^1 su(s)ds-2\int_0^t\int_0^1 \tau su(s)dsd\tau\right);\\
&(t\in I=[0,1]).
\end{aligned}
\tag{III.27}
$$

where $a(t)=0.001t+t^2+0.08525t^3-0.015t^4-0.01t^5$.To solve Equation (III.27) using the Homotopy perturbation technique, we construct the homotopy. By comparing coefficients of terms with identical powers of p, we obtain:

$$
\begin{aligned}
&p^0: v_0(t)=u_0(t)\\
&p_1: v_1(t)=a(t)-u_0(t)+\frac{1}{0.001}\int_0^t v_0(\tau)d\tau\left(\int_0^1 t^2sv_0(s)ds-1\right)\\
&\quad -\frac{2}{0.01}\int_0^t v_0(\tau)d\tau\int_0^t\int_0^1 sv_0(s)d\tau+(v_0(t)-v_0)\\
&\quad \times\left(\int_0^1 t^2sv_0(s)ds-2\int_0^t\int_0^1 sv_0(s)d\tau\right).\\
&p_i: v_i(t)=\frac{1}{0.01}\sum_{k=0}^{i-1}\left(\int_0^t v_k(\tau)d\tau\int_0^1 t^2sv_{(i-k-1)}(s)ds\right)\\
&\quad -\frac{1}{0.01}\int_0^t v_{(i-1)}(\tau)d\tau-\frac{2}{0.01}\sum_{k=0}^{i-1}\left(\int_0^t v_k(\tau)d\tau\int_0^t\int_0^1 \tau sv_{(i-k-1)}(sds)d\tau\right)\\
&\quad +\sum_{k=0}^{i-1}\left(v_{(i-k-1)}(t)\left(\int_0^1 t^2sv_k(s)ds-2\int_0^t\int_0^1 \tau sv_k(s)d\tau\right)\right)\\
&\quad -v_0\left(\int_0^t\int_0^1 t^2sv_{(i-1)}(s)ds-2\int_0^t\int_0^1 \tau sv_{(i-1)}(s)d\tau\right).
\end{aligned}
$$

Let's choose $u_0(t)=0$. Then, by calculating the successive functions v_i determined by the last relations, we obtain successively:

$$
\begin{aligned}
&v_0(t)=0,\\
&v_1(t)=0.001t+t^2+0.08525t^3-0.015t^4-0.01t^5,\\
&v_2(t)=\frac{1}{10}\left(0.0005t^2+\frac{t^3}{3}+0.0213125t^4-0.003t^5-0.00166667t^6\right),
\end{aligned}
\tag{III.28}
$$

We have not managed to find the general form of the function v_j, but we can focus on the approximate solution $u^{\hat{n}}$ determined by partial sums from (III.26). The accuracy of the n-th order approximate solutions can be evaluated due to the existence of the exact solution.

In Table III.1, we present the absolute error $|u - u^{\hat{3}}|$ and relative errors $\delta = \left|\frac{u-u^{\hat{3}}}{u}\right| \times 100\%$ with which the n-th order approximate solutions reconstruct the exact solution. A plot of the error distribution over the entire interval $[0,1]$ is displayed in Fig. III.1. These results demonstrate that the method converges rapidly, and calculating just a few components of the series provides a very good approximation of the exact solution.

Table III.1: Comparison of the numerical results with the exact solution $u(t)$

t	$u(t)$	$u^{\hat{3}}(t)$	$E_3 = \|u(t) - u^{\hat{3}}(t)\|$	$\delta(\%)$
0.1	0.01	0.01021611	2.16092×10^{-4}	2.160921
0.2	0.04	0.04010982	1.09816×10^{-4}	0.274544
0.3	0.09	0.09321643	3.21644×10^{-3}	3.573822
0.4	0.16	0.16700542	7.00469×10^{-3}	4.377931
0.5	0.25	0.24872761	1.27233×10^{-3}	0.508932
0.6	0.36	0.35795785	2.04220×10^{-3}	0.567277
0.7	0.49	0.51990334	2.99026×10^{-2}	6.102571
0.8	0.64	0.68068262	4.06821×10^{-2}	6.356578
0.9	0.81	0.86195612	5.19558×10^{-2}	6.414296
1.0	1.00	1.06256211	6.25621×10^{-2}	6.256212

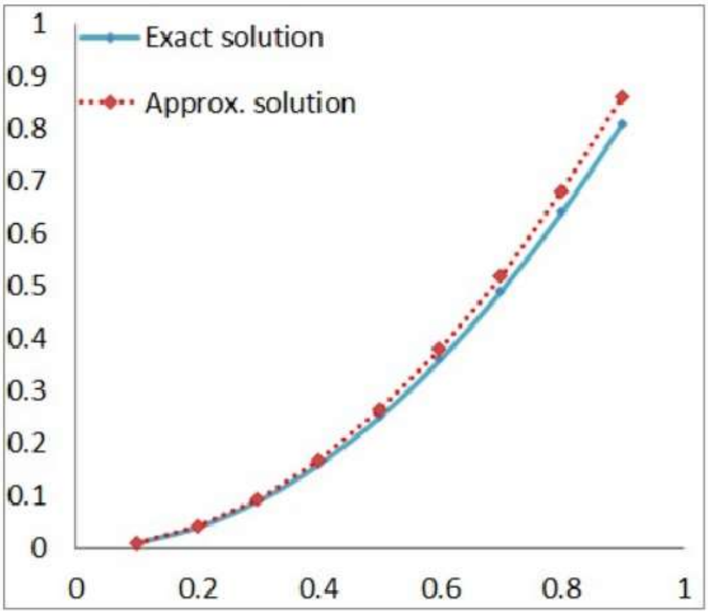

Figure III.1: Comparison of the approximate solution by HPM with the exact solution

GENERAL CONCLUSION

In this thesis , we studie a special type of nonlinear integral equation, namely the quadratic integral equation.

Some quadratic integral equations were highlighted. We reviewed the various analytical and numerical methods for solving these equations, including the homogeneity perturbation method, the numerical method.

These methods have been shown and how to apply them to specific examples to obtain approximate solutions to quadratic integral equations. Through this study, we hope that you have contributed to enhancing understanding of this special type of integral equations and developing new and more efficient methods for solving them, which will open new horizons in scientific research and engineering applications.

. We hope that we have achieved even a little of the desired goal, and like any other research, this research cannot be free of shortcomings. May God have mercy on someone who saw something disheveled and damaged it, or had a defect that spoiled it, or a defect, so he completed it.

We ask everyone who has benefited from a letter of this memorandum to pray for good for us And righteousness.

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ABSTRACT

The quadratic integral equation is a special type of integral equations, and poses a challenge due to its nonlinear nature.

Our main goal in this memo is to study some different analytical and numerical solution methods for certain quadratic integral equations. The choice of the appropriate method for the solution depends on the properties of the equation and the imposed conditions.

Keywords: Quadratic integral equation, quadratic integral equation with phase lag term, numerical method, homotopy perturbation method ,fixed point.

Résumé

L'équation intégrale quadratique est un type particulier d'équations intégrales et pose un défi en raison de sa nature non linéaire.

Notre objectif principal dans ce mémo est d'étudier différentes méthodes analytiques et numériques de résolution de certaines équations intégrales quadratiques. Le choix de la méthode appropriée pour la solution dépend des propriétés de l'équation et des conditions imposées.

Les mots clés : équation intégrale quadratique, équation intégrale quadratique avec terme de retard de phase, méthode numérique, méthode de perturbation d'homotopie,un point fixe.

الملخص:

المعادلة التكاملية التربيعية هي نوع خاص من المعادلات التكاملية، وتشكل تحديًا نظرًا لطبيعتها غير الخطية.

وهدفنا الرئيسي في هذه المذكرة هو دراسة بعض طرق الحل التحليلية والعددية المختلفة لبعض المعادلات التكاملية التربيعيةاذ يعتمد اختيار الطريقة المناسبة للحل على خصائص المعادلة والشروط المفروضة.

الكلمات الرئيسية: معادلة تكاملية تربيعية ، معادلة تكاملية تربيعية مع مصطلح تأخير الطور، الطريقة العددية، طريقة الاضطراب التجانس الهوموتوبي ، النقطة الثابتة.