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Theme

Fixed Point Theorems for Single-Valued and Multi-Valued Mappings in Metric Space.

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ABSTRACT

In this work, we study and present some fixed and coincidence point theorems for single-valued and multi-valued mappingm, satisfying the non-expansive type conditions in the context of asymptotically regular mappings and orbital completeness of the space. Also, we ivestigate some fixed and coincidence point theorems in complete, orbitally comlete and (T, f)-orbitally comlete partial metric space under the generalized contraction conditions. We concluded this work by proving the existence of solutions to some functional equations of dynamic programming and integral equations.

Keyswords: Metric space, fixed point theorem, contraction, non-expansive, coincidence point, dynamic programming and integral equations.

ملخص

في هذا العمل درسنا و قدمنا بعض نظريات النقطة الثابتة والمتطابقة للتطبيقات الوحيدة ومتعددة القيم و التي تحقق شروط عدم الاتساع في سياق التطبيقات المنتظمة التقاربية و الاتمام المداري للفضاء، بالإضافة الى تحققنا من بعض نظريات النقطة الثابتة والمتطابقة في الفضاء المتري الجزئي التام والتام المداري و $(T, f)-$ التام المداري تحت تعميم شروط التقليص. وختمنا هذا العمل بإثبات وجود حلول للمعادلات الدالية المبرمجة ديناميكيا والمعادلات التكاملية .

الكلمات المفتاحية: الفضاء المتري، النقطة الثابتة، تقليص، عدم الاتساع، النقطة المتطابقة، البرمجة الديناميكية و المعادلات التكاملية.

RESUMÉ

Dans ce travaille, nous étudions et présontons quelques theorémes de point fixe et de point coïncidence pour les applications univoques et multivoques, satisfaisant les conditions de type non expansive dans le contexte des applications asymptoticquement régulieres et de completude orbitale de l'espace. Aussi nous étudions quelques theorémes de point fixe et point de coïncidence dans un espace metrique partiel complet, orbitalement complet et $(T, f)-$ orbitalement complet sous des conditions contractives généralisées.
Nous avons conclu ce travaille en prouvons l' éxistence de solutions de certaines équations fonctionelle de programmation dynamique et d'équation intégrales.

Mots clés: Espace métrique, point fixe, contraction, non expansive, point coïncidence, programmation dynamique, équation intégrales.

Dedication

.... We would love to dedicate this modest work to :

★ ★ ★

Our beloved parents
have always been supporting
us and making massive sacrifices for us.

★ ★ ★

A huge dedication to our brothers
and sisters for their encouragements and love,
you are the best brothers and sisters in the whole
world.

★ ★ ★

Not to forget all our friends and relatives who
have been giving advices
and help us forme whenever needed,
especially our university friends with whom We
have had enjoyable and beautiful five years.

★ ★ ★

Acknowledgement

First of all we thank Allah who gave us the courage, the will, the strength and the patience during the difficult moments to accomplish this work.

⋆ ⋆ ⋆

We also express our profound gratitude and respectful framer:**Dr.DJEDIDI Mostepha** for his willingness to accept us to frame. We him let us also reserve our absolute respect for all the time he has spent directing us and to carry out work.

⋆ ⋆ ⋆

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⋆ ⋆ ⋆

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⋆ ⋆ ⋆

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CONTENTS

INTRODUCTION

the study of fixed point theorems has its origin in one the most fundamental results of functional analysis, the fixed point theorem of metric space. Over the years there have been many efforts to generalize this theorem for various classes of topological spaces and banach spaces.

Also, by fixed point theorem we shall mean a statement which asserts that under certain conditions a mapping T of a space X into a space Y admits one or more points x of X sunch that $Tx = x$. A good number of researchers have studied this theory due to its usefulness in the existence theory of differential equations, integral equations , and its applications in boundary value problems, approximation theory and non-linear analysis.

Fixed points and fixed point theorems have always been a major theoretical tool in fields as widely apart as differential equations, topology, economics, game theory, dynamics, optimal control, and functional analysis. Moreover, more or lessrecently, the usefulness of the conceptfor applications increased enormously by the development of accurate and efficient techniques for computing fixed points, making fixed point methods a major weapon in the arsenal of the applied mathematician.

Brouwer provides the first fixed-point theory (FPT). H. Poincare 1886 established the fixed point. A fixed point is sometimes called invariant because it does not change its resultwhen

applying certain transformations. There are three major categories of Fixed Point Theory: (1)- In 1912, Brouwer gives the FPT on Topology known as Topological Fixed Point Theory, (2)- In 1922, Banach gives the Fixed Point Theory on Metric space known as Metric Fixed Point Theory, and (3)- In 1955, Tarski gives Fixed Point Theory on Discretespace known as Discrete Fixed Point Theory. Its development has been rapidly accelerating in recent years it plays a significant role in modern mathematics.[38]
Fixed Points:(The Essence of Stability) Imagine a function that takes an input value and transforms it into another value. A fixed point of this function is a special input that, when fed into the function, remains unchanged. Think of it as an equilibrium point a value that the function back to itself. Visually, it's the point where the function's graph intersects the line $y = x$.
Expanding the Horizons: Multi-Valued Fixed Points Building on Banach's foundation, mathematicians like Wladimiro Nadler and GiacomoCaristi delved deeper. They explored the concept of multi-valued fixed points, where afunction might map an input to multiple outputs. Nadler's celebrated multi-valued fixed point theorem is a cornerstone in this area. Caristi, too, contributed significantly by demonistrating the existence of multi-valued fixed points, paving the way for further research. Beyond Metric Spaces: Exploring New Frontiers The initial focus was on metric spaces, where distances between points can be readily measured. However, mathematicians weren't content to stop there. Partial metric spaces andcone metric spaces, offering more general distance structures, were introduced. Researchers like Nadler and Caristi extended their theorems to these new spaces, demonstrating thepower and versatility of fixed point theory.
We divided this work into three chapters :

1. First, in the start of work, in (Chapter 1) we present some definitions revolve around metric space , basic mathematical concepts and some preliminary properties that we will use in this work.

2. Second, in chapter number two(Chapter 2), we offer some fixed and coincidence point theorems for single-valued and multi-valued mappings satisfying the non-expansive type conditions in the context of asymptotically regular mappings and orbital completeness of the space. and we present an investigate some fixed and coincidence point

theorems in complete, orbitally complete and $(T, f)-$orbitally complete partial metric spaces under the generalized contractive type conditions of mappings. Moreover, these results generalize andextend the several obtained results in the literature.

3. Finally, after study the theorems and propsitions in previous chapter we study the existence and uniqueness of the solution of the following functional equation, like that, we study the existence of the solution to a nonlinear integral equation by using a result obtained in the previous section.

NOTATIONS

The most frequently used notations, symbols and abbreviations are listed below:

$T : X \to Y$	:	T is a multivalued map from X to $P(Y)$.
$\mathbb{N}$	:	The set of natural numbers.
$\mathbb{R}$	:	The set of real numbers.
$\mathbb{R}_+$	:	The set of positive real numbers.
$H(A, B)$	:	The Hausdorff distance between A and B.
X, Y	:	Are metric spaces if no other indication is given.
$CB(X)$	:	The set of all closed and bounded subsets of X.
$CL(X)$	:	The set of closed subsets in X.
$K(X)$	:	The set of all compact subsets of X.
$C(K)$	:	The set of continuous funtions on K.
$B(W)$	:	The set of bounded funtions on W.
$\mathrm{Dom}(T)$	:	The domain of T.
$\mathrm{Graph}(T)$	:	The graph of T.
$\mathrm{Im}(T)$	:	The image of T.
$d(x, A)$	:	The distance between a point x and a set A.

$p(x, y)$	:	The distance between a point x and a set A in partiel metric space.

CHAPTER

1

PRELIMINARY NOTIONS

. In this chapter we will present some definitions revolve around metric space, basic mathematical concepts and some preliminary properties that we will use in this work.

1.1 Metric space

1.1.1 Distance

A metric space is just a set X equipped with a function d of two variables which measures the distance between points: $d(x, y)$ is the distance between two points x and y in X.

Definition 1.1.1. *[51] Let X be a none-empty set. A mapping $d : X \times X \longrightarrow \mathbb{R}$ is distance or metric and said to be a metric space on X if it satisfies the following conditions*

1. $d(x, y) \geq 0 \quad \forall x, y \in X$
2. $d(x, y) = 0 \Leftrightarrow x = y \quad \forall x, y \in X$

3. $d(x,y) = d(y,x), \quad \forall x, y \in X$ *(Symmetric)*

4. $d(x,y) \leq d(x,z) + d(z,y) \quad \forall x, y, z \in X$ *(Triangle Inequality)*

Exemple 1.1.1. *: Let X be any non-empty set .Define a mapping $d : X \times X \longrightarrow \mathbb{R}$ by*

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y. \end{cases} \tag{1.1}$$

Then d is a metric on X, and this metric is called discrete metric.

Exemple 1.1.2. *: $C([0,1])$ is the space of all continuous real-valued functions f over $[0,1]$. $C([0,1]) = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous on } [0,1]\}$ Define the mappings d_1, d_∞ : $C([0,1]) \times C([0,1]) \longrightarrow \mathbb{R}$ as follows: $d_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$ and $d_1(f,g) = \int_0^1 |f(x) - g(x)|dx$. Then d_1, d_∞ are metrics on $C([0,1])$.*

1.1.2 Open and closed sets

There are special types of sets that play a distinguished role in analysis; these are the open and closed sets. To expedite the discussion, it is helpful to have the notion of a neighbourhood in metric spaces

Definition 1.1.2. *(X,d) be a metric space. The set*

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}, \qquad \text{where } r > 0 \text{ and } x_0 \in X,$$

is called the open ball of radius r and centre x_0. The set

$$\bar{B}(x_0, r) = \{x \in X : d(x_0, x) \leq r\}, \qquad \text{where } r > 0 \text{ and } x_0 \in X,$$

is called the closed ball of radius r and centre x_0.

$$S(x_0, r) = \{x \in X : d(x_0, x) = r\}, \qquad \text{where } r > 0 \text{ and } x_0 \in X,$$

is called the sphere of radius r and centre x_0.

Definition 1.1.3. *. Let (X,d) be a metric space. A neighbourhood of the point $x_0 \in X$ is any open ball in (X,d) with centre x_0.*

Definition 1.1.4. *A subset G of a metric space (X, d) is said to be open if given any point $x \in G$, there exists $r > 0$ such that $B(x, r) \subseteq G$, i.e., each point of G is the centre of some open ball contained in G. Equivalently, every point of the set has a neighbourhood contained in the set.*

Theorem 1.1.1. *[51] In any metric space (X, d), each open ball is an open set.*

Result 1.1.1. *[3]*

Let (X, d) be a metric space. Let $E \subseteq X$. Then

1. *ϕ and X are both open and closed.*
2. *An arbitrary union of open sets in X and a finite intersection of open sets in X are open sets in X.*
3. *An arbitrary intersection of closed sets in X and a finite union of closed sets in X are closed sets in X.*
4. *G is open $\Leftrightarrow G = G^\circ$.*
5. *G is closed $\Leftrightarrow G = \bar{G}$.*

1.1.3 Continuity of Mappings

In this section, we discuss continuity of mappings with their properties.

Definition 1.1.5. *[43] Let T be a mapping from a metric space (X, d) to another metric space (Y, ρ). Then, T is said to be*

1. *continuous at $x_0 \in X$ if for given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x_0) > 0$ such that $\rho(Tx, Tx_0) < \varepsilon$ whenever $d(x, x_0) < \delta$ for all $x \in X$. In general, T is said to be continuous at $x_0 \in X$ if for any sequence $\{x_n\}$ in X such that $x_n \to x_0$ implies $Tx_n \to Tx_0$ in Y.*
2. *uniformly continuous on X if for given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho(Tx, Ty) < \varepsilon$ whenever $d(x, y) < \delta$ for all $x, y \in X$.*

Exemple 1.1.3. *1. Every polynomial function is continuous.*

2. *Let X be a discrete metric space and Y be any metric space. Then, any mapping $T : X \to Y$ is uniformly continuous.*

3. *Let C be a nonempty subset of a metric space (X, d). Then for each pair $x, y \in X$, we have $|d(x, C) - d(y, C)| \leqslant d(x, y)$. That is,*

$$|d(\cdot, C)(x) - d(\cdot, C)(y)| \leqslant d(x, y) for\ all x, y \in X.$$

In particular, we see that the function $x \to d(x, C)$ is uniformly continuous.

1.1.4 Convergent and Cauchy Sequence

A sequence (x_n) in a set X is a function $f : \mathbb{N} \to X$, where we write $x_n = f(n)$ for the nth term in the sequence.

Definition 1.1.6. *[51].Let (X, d) be a metric space. A sequence (x_n) in X converges to $x \in X$, written $x_n \to x$ as $n \to \infty$ or*

$$\lim_{n\to\infty} x_n = x,$$

if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n > N \textit{ implies that } d(x_n, x) < \epsilon.$$

That is, $x_n \to x$ if $d(x_n, x) \to 0$ as $n \to \infty$. Equivalently, $x_n \to x$ as $n \to \infty$ if for every neighborhood U of x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$.

Exemple 1.1.4. *Let $X = \mathbb{R}$ with $d(x, y) = |x - y|$ $x, y \in \mathbb{R}$. Let$\{X_n\}$ be a sequence of real numbers . It converges to $x \in \mathbb{R}$ in the sense of the metric space (X, d) if and only if*

$$\lim_{n\to\infty} d(X_n, x) = \lim_{n\to\infty} |X_n - x| = 0.$$

So convergence in $\mathbb{R}$ with usuel metric in the same as convergence of a sequence of real numbers in the familiar sence.

Definition 1.1.7. *[49] Let (X,d) be a metric space. A sequence (x_n) in X is a Cauchy sequence for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$m, n > N \text{ implies that } d(x_m, x_n) < \epsilon.$$

Completeness of a metric space is defined using the Cauchy condition.

Exemple 1.1.5. *: Let $x_n = \frac{1}{n}\sqrt{5}$ for each $n \in \mathbb{N}$. Note that each x_n is an irrational number (i.e., $x_n \in \mathbb{Q}^c$) and that $\{x_n\}$ converges to 0 . Thus, $\{x_n\}$ converges in $\mathbb{R}$ (i.e., to an element of $\mathbb{R}$). But 0 is a rational number (thus, $0 \notin \mathbb{Q}^c$), so although the sequence $\{x_n\}$ is entirely in $\mathbb{Q}^c$, it does not converge in $\mathbb{Q}^c$. Note, however, that $\{x_n\}$ is Cauchy.*

Exemple 1.1.6. *: Let $\bar{x}$ be an irrational number, and for each $n \in \mathbb{N}$ let $\{x_n\}$ be a rational number in the interval $]\bar{x} - \frac{1}{n}, \bar{x} + \frac{1}{n}[$. Then $\{x_n\}$ is a sequence of rational numbers that converges to the irrational number $\bar{x}-$ i.e., each $\{x_n\}$ is in $\mathbb{Q}$ and $\{x_n\} \to \bar{x} \notin \mathbb{Q}$.*
Thus, in a parallel to Example 1.1.5, $\{x_n\}$ here is a Cauchy sequence in $\mathbb{Q}$ that does not converge in $\mathbb{Q}$.

Examples 1.1.5 and 1.1.6 demonstrate that both the irrational numbers, $\mathbb{Q}^c$, and the rational numbers, $\mathbb{Q}$, are not entirely well-behaved metric spaces - they are not complete in that there are Cauchy sequences in each space that don't converge to an element of the space.

1.1.5 Complete Metric Spaces

Definition 1.1.8. *A metric space (X,d) is complete if every Cauchy sequence in X converges in X (i.e., to a limit that's in X).*

Exemple 1.1.7. *The real interval $]0,1[$ with the usual metric is not a complete space: the sequence $x_n = \frac{1}{n}$ is Cauchy but does not converge to an element of $]0,1[$.*

Exemple 1.1.8. *The metric space $\mathbb{Q}$ with the usual metric is not a complete space. but the metric space $\mathbb{R}$ with the usual metric is a complete space.*

Remark 1.1.1. *Every Cauchy sequence in a metric space is bounded.*

Remark 1.1.2. *If a Cauchy sequence has a subsequence that converges to x, then the sequence converges to x.*

1.2 Lipschitzian mapping

A mapping T from a metric space (X, d) into another metric space (Y, ρ) is said to satisfy the Lipschtz condition or is said to be Lipchtz continuous on X if there exists a constant $L > 0$ sunch that:

$$\rho(Tx, Ty) \leq Ld(x, y) \;\; \forall x, y \in X$$

If L is the least number for which the Lipschitzian condition is satisfied, then L is called Lipschitz constant . In this case, we say that$T$ is an $L - Lipschitz$ mapping or simply a Lipschitzian mapping with Lipschitz constant L. Otherwise it is called non-Lipschtz mapping.

Exemple 1.2.1. *Any $f : [a, b] \to \mathbb{R}$ with contunious is Lipschitz. As f' is continuous it is bounded on $[a, b]$, say $|f'(x)| \leq c$. the result then folows from the mean value theorem.*

$$\forall x, y \in [a, b] \;\; \exists \zeta \in (a, b), \quad f(x) - f(y) \leq f'(\zeta)(x - y)$$

1.3 Contractions

We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

Definition 1.3.1. *[2] Let (X, d) be a metric space and let $f : X \to X$ be a mapping.*

1. *A point $x \in X$ is called a fixed point of f if $x = f(x)$.*
2. *f is called contraction if there exists a fixed constant $h < 1$ such that*

$$d(f(x), f(y)) \leq hd(x, y), \qquad \text{for all } x, y \in X. \tag{1.2}$$

A contraction mapping is also known as Banach contraction. If we replace the inequality (1.2) with strict inequality and $h = 1$, then f is called contractive (or strict contractive). If (1.2) holds for $h = 1$, then f is called non-expansive; and if (1.2) holds for fixed $h < \infty$, then f is called Lipschitz continuous . Clearly, for the mapping f, the following obvious implications hold:

$$\text{contraction} \;\Rightarrow\; \text{contractive} \;\Rightarrow\; \text{non-expansive} \;\Rightarrow\; \text{Lipschitz continuous}$$

Exemple 1.3.1. *1. Consider the usual metric space* $(\mathbb{R}, d)$*. Define*

$$f(x) = \frac{x}{a} + b, \qquad \text{for all } x \in \mathbb{R}.$$

Then, f is contraction on $\mathbb{R}$ if $a > 1$ and the solution of the equation $x - f(x) = 0$ is $x = \frac{ab}{a-1}$.

2. *Consider the Euclidean metric space* $(\mathbb{R}^2, d)$*. Define*

$$f(x, y) = \left(\frac{x}{a} + b, \frac{y}{c} + b\right), \qquad \text{for all } (x, y) \in \mathbb{R}^2.$$

Then, f is contraction on $\mathbb{R}^2$ if $a, c > 1$. Now, solving the equation $f(x, y) = (x, y)$ for a fixed point, we get $x = \frac{ab}{a-1}$ and $y = \frac{cb}{c-1}$. Using induction, one can easily get the following concerning iterates of a contraction mapping. If f is a contraction mapping on a metric space (X, d) with contraction constant h, then f^n (where the superscript represents the nth iterate of f) is also a contraction on X with constant h^n. Consider then the situation in which $f : X \to X$ is not necessarily a contraction mapping, but f^n is a contraction for some n.

Exemple 1.3.2. .

1. *Let $f : [0, 2] \to [0, 2]$ be defined by*

$$f(x) = \begin{cases} 0, & x \in [0, 1], \\ 1, & x \in (1, 2]. \end{cases}$$

Then, $f^2(x) = 0$ for all $x \in [0, 2]$, and so, f^2 is a contraction on $[0, 2]$. Note that f is not continuous and thus not a contraction map.

2. *Define $f(x) = \cos x$ on $\mathbb{R}$. Then, f is not a contraction on $\mathbb{R}$. Indeed, suppose there exists $h \in (0, 1)$ such that*

$$\left|\frac{\cos x - \cos y}{x - y}\right| \leq h, \qquad \text{for all } x \neq y.$$

Letting $y \to x$, we get $|\sin x| \leq h$ for all x, which is false. Note that the iterated function $f^2(x) = \cos(\cos x)$ satisfies

$$\left|\frac{d}{dx}(\cos(\cos x))\right| = |\sin(\cos x)\sin(x)| < \sin 1 < 1,$$

and thus, by the mean-value theorem, f^2 is a contraction on $\mathbb{R}$.

3. *Define* $f(x) = \exp(-x)$ *on* $\mathbb{R}$. *Then,* f *is not a contraction on* $\mathbb{R}$. *But,* $f^2(x) = \exp(-\exp(-x))$ *is a contraction on* $\mathbb{R}$. *Indeed,*

$$\left|\frac{d}{dx}(\exp(-\exp(-x)))\right| = |\exp(-x - \exp(-x))| \leq e^{-1} < 1,$$

and thus, the conclusion follows by the mean-value theorem. For non-contractions, there are examples where f *has a unique fixed point but an iterate of* f *does not.*

Exemple 1.3.3. *Define*

$$f(x) = 1 - x, \quad \text{for all } x \in \mathbb{R}.$$

Then, f *is not a contraction, has a unique fixed point, but note that*

$$f^2(x) = x, \quad \text{for all } x \in \mathbb{R}$$

is rich with fixed points.

1.4 normed space

We introduce a notion of length for elements of a vector space. Note that a 'length' is something that a single element on its own should have, whereas 'distance' is something that only makes sense for a pair of element

Definition 1.4.1. *[43] Let* X *be a vector space over* $\mathbb{K}$. *A norm on* X *is a map* $\|\cdot\| : X \to [0, \infty)$ *that satisfies the following three properties.*

1. $\|x\| = 0$ *implies* $x = \mathbf{0}$. *(definiteness)*

2. $\|\lambda x\| = |\lambda| \cdot \|x\|$ *for all* $x \in X$ *and* $\lambda \in \mathbb{K}$. *(homogeneity)*

3. $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$. *(triangle inequality) A normed space is a pair* $(X, \|\cdot\|)$, *where* X *is a vector space and* $\|\cdot\|$ *is a norm on* X.

Exemple 1.4.1. *[43] The set of real numbers* $\mathbb{R}$ *with the absolute value norm* $\|x\| = |x|$ *is a one-dimensional real normed linear space. More generally,* $\mathbb{R}^n$, *where* $n = 1, 2, 3, \ldots$, *is an* n*-dimensional linear space. We define the Euclidean norm of a point* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *by*

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

and call $\mathbb{R}^n$ *equipped with the Euclidean norm n-dimensional Euclidean space. We can also define other norms on* $\mathbb{R}^n$*. For example, the sum or 1-norm is given by*

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n| .$$

Definition 1.4.2. *[43] Let* $(X, \|\cdot\|)$ *be a normed space. A sequence* $(x_n)_{n\in\mathbb{N}}$ *in* X *is said to converge to* $a \in X$ *if*

$$\forall \varepsilon > 0 \exists N_0 \in \mathbb{N} : \forall n \geq N_0 : \|x_n - a\| < \varepsilon.$$

In this case one writes $\lim_{n\to\infty} x_n = a$ *or* $x_n \to a$ *as* $n \to \infty$*. A sequence* $(x_n)_{n\in\mathbb{N}}$ *in* X *is called a Cauchy sequence if*

$$\forall \varepsilon > 0 \quad \exists N_0 \in \mathbb{N} : \forall n, m \geq N_0 : \|x_n - x_m\| < \varepsilon.$$

1.5 Banach Spaces

A normed linear(vector) space is a metric space with respect to the metric d derived from its norm, where $d(x, y) = \|x - y\|$.

Definition 1.5.1. *[43] A Banach space is a normed linear(vector) space that is a complete metric space with respect to the metric derived from its norm.*

Exemple 1.5.1. *[43] For* $1 \leq p < \infty$*, we define the p-norm on* $\mathbb{R}^n$ *(or* $\mathbb{C}^n$ *) by*

$$\|(x_1, x_2, \ldots, x_n)\|_p = \left(|x_1|^p + |x_2|^p + \ldots + |x_n|^p\right)^{1/p} .$$

For $p = \infty$*, we define the* ∞*, or maximum, norm by*

$$\|(x_1, x_2, \ldots, x_n)\|_\infty = \max\left\{|x_1|, |x_2|, \ldots, |x_n|\right\} .$$

Then $\mathbb{R}^n$ *equipped with the p-norm is a finite-dimensional Banach space for* $1 \leq p \leq \infty$*.*

Exemple 1.5.2. *[43] The space* $C([a, b])$ *of continuous, real-valued (or complex-valued) functions on* $[a, b]$ *with the sup-norm is a Banach space.*
More generally, the space $C(K)$ *of continuous functions on a compact metric space* K *equipped with the sup-norm is a Banach space.*

Exemple 1.5.3. *The space* ℓ^p *are Banach space for* $1 \leq p \leq \infty$

1.6 Partial Metric Spaces

Definition 1.6.1. *[39] Let X be a nonempty set and let $p : X \times X \to \mathbb{R}^+$be a function satisfy:*

p_1 $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$

p_2 $p(x, x) \leq p(x, y)$

p_3 $p(x, y) = p(y, x)$

p_4 $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$

for all $x, y, z \in X$. Then p is called partial metric on X and the pair (X, p) is called partial metric space. It is clear that if $p(x, y) = 0$, then from p_1 and p_2 we obtain $x = y$. But if $x = y, p(x, y)$ may not be zero. Various applications of this space has been extensively investigated by many authors [37], [53] .

Exemple 1.6.1. *Let $X = \mathbb{R}^+$and $p : X \times X \to \mathbb{R}^+$given by $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. Then $(\mathbb{R}^+, p)$ is a partial metric space.*

Exemple 1.6.2. *Let $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$. Then $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$ defines a partial metric p on X.*

Lemma 1.6.1. *A partial metric space (X, p) is first countable.*

Proof For each rational $r(> 0)$ let $B_r\left(x_0\right) = \{x \in X : p\left(x, x_0\right) < p\left(x_0, x_0\right) + r\}$. Then the family $\{B_r\left(x_0\right)\}, r > 0$ forms a neighbourhood base at x_0. So, (X, p) is first countable. We recall following well known definitions [18] and [32] From now on (X, p) will always denote a PMS.

Lemma 1.6.2. *[9] Let (X, p) be a partial metric space,$A, B \in CB(X)$ and $h > 1.\forall a \in A$, there exists $b = b(a) \in B$such that*

$$p(a, b) \leq hH_p(A, B). \tag{1.3}$$

Definition 1.6.2. *[34]*

1. *A sequence* $\{x_n\}$ *in a partial metric space* (X,p) *converges to* $x \in X$ *if and only if*

$$p(x,x)=\lim_{n\to+\infty} p\left(x,x_n\right).$$

2. *a sequence* $\{x_n\}$ *in a partial metric space* (X,p) *is said to be Cauchy if and only if*

$$\lim_{n,m\to+\infty} p\left(x_n,x_m\right) \text{ exists (and finite).}$$

3. *a partial metric space* (X,p) *is complete if any Cauchy sequence* $\{x_n\}$ *in* X *converges to a point* $x \in X$ *such that*

$$p(x,x)=\lim_{n,m\to+\infty} p\left(x_n,x_m\right)$$

Lemma 1.6.3. *[18] Let* (X,p) *be a partial metric space. Then the mapping* $p^s : X\times X \to [0,\infty)$ *given by*

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y).$$

for all $x,y \in X$, defines a metric on X. Thus, associated with any partial metric space (X,p) and (X,d) is a metric space, where $d=p^s$, induced by p.

Lemma 1.6.4. *[34]*

1. A sequence $\{x_n\}$ is Cauchy in a partial metric space (X,p) if and only if $\{x_n\}$ is Cauchy in the metric space (X,p^s).

2. A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete.

$$\lim_{n\to\infty} p^s\left(x,x_n\right)=0 \Leftrightarrow p(x,x)=\lim_{n\to\infty} p\left(x,x_n\right)=\lim_{n,m\to\infty} p\left(x_n,x_m\right)$$

Lemma 1.6.5. *Assume* $x_n \to x$ *as* $n\to\infty$ *in a partial metric space* (X,p) *such that* $p(x,x)=0$. *Then* $\lim_{n\to\infty} p\left(x_n,y\right)=p(x,y)$ *for every* $y\in X$.

1.7 δ Distance

Definition 1.7.1. *[41] Let $CB(X)$ denote the set of nonempty closed bounded subsets of X. More precisely*
$CB(X) = \{A : A$ is a nonempty closed and bounded subset of $X\}$.

Let (X, d) be a metric space, $x \in X$ and A, B be a tow subset of X. We defined the following distances:

1. The distance between x and A, denoted by $d(x, A)$ is defined as the smallest distance from x to elements of A, written as:
$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$
By convention, $d(x, \emptyset) = +\infty$. If on the contrary, A is not empty, then for all $\varepsilon > 0$, there exists an element $a \in A$ such that $d(x, a) \leq d(x, A) + \varepsilon$.

2. The distance from A to B denoted $D(A, B)$ is defined by:
$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

Exemple 1.7.1. *Let $X = R$, $A = [2, 4]$ and $B = [5, 6]$. Then*
$$D(A, B) = D([2, 4], [5, 6]) = \inf\{d(a, b) : a \in [2, 4], b \in [5, 6]\} = d(4, 5) = 1.$$

Definition 1.7.2. *Let (X, d) be a metric space and $A, B \in CB(X)$, we define the δ distance from A to B as follows:*
$$\delta(A, B) = \sup\{d(a, B) : a \in A\}.$$
By convention, $\delta(\emptyset, \emptyset) = +\infty$ and if $B \neq \emptyset$, we have $\delta(\emptyset, B) = 0$.

Exemple 1.7.2. *Let $X = \mathbb{R}$, $A = \left[0, \frac{5}{2}\right]$ and $B = [3, 4]$. Then*
$$\delta(A, B) = \delta\left(\left[0, \frac{5}{2}\right], [3, 4]\right) = \sup\left\{d(a, [3, 4]) : a \in \left(\left[0, \frac{5}{2}\right]\right\} = d(0, 3) = 3,$$
$$\delta(B, A) = \delta\left([3, 4], \left[0, \frac{5}{2}\right]\right) = \sup\left\{d\left(b, \left[0, \frac{5}{2}\right]\right) : b \in ([3, 4]\right\} = d\left(4, \frac{5}{2}\right) = \frac{3}{2}.$$

Remark 1.7.1. *[41] We observe that, $\delta(A, B) \neq \delta(B, A)$, imply that the δ distance is not symmetrical, so it is not a metric.*

1.8 Pompeiu-Hausdorff distance

The other notion of distance we will need is Hausdorff's. Pompeiu - Hausdorff distance between two sets A and B corresponds to the maximum between $\delta(A, B)$ and $\delta(B, A)$.

Definition 1.8.1. *[41] Let (X, d) be a metric space. Pompeiu-Hausdorff distance between two sets $A, B \in CB(X)$ is defined by:*

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\}.$$

Note that (1.7.1) can be rewritten as follows:

$$H(A, B) = \max\left\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A)\right\}.$$

The metric H defined on $CB(X)$ is called the Hausdorff distance, or Hausdorff metric, also called Pompeiu-Hausdorff distance, in $CB(X)$.

Exemple 1.8.1. *Let $X = R, A = \left[\frac{1}{2}, 3\right]$ and $B = [4, 5]$. Then*

$$\delta(A, B) = \delta\left(\left[\frac{1}{2}, 3\right], [4, 5]\right) = \sup\left\{d(a, [4, 5]) : a \in \left(\left[\frac{1}{2}, 3\right]\right\}\right. = d\left(\frac{1}{2}, 4\right) = \frac{7}{2}$$
$$\delta(B, A) = \delta\left([4, 5], \left[\frac{1}{2}, 3\right]\right) = \sup\left\{d\left(b, \left[\frac{1}{2}, 3\right]\right) : b \in ([4, 5]\right\} = d(5, 3) = 2$$

Hence

$$H(A, B) = H\left(\left[\frac{1}{2}, 3\right], [4, 5]\right) = \max\left\{\delta\left(\left[\frac{1}{2}, 3\right], [4, 5]\right), \delta\left([4, 5], \left[\frac{1}{2}, 3\right]\right)\right\} = 2.$$

Remark 1.8.1. *[41] If $A = \{a\}$ and $B = \{b\}$, then $H(A, B) = d(a, b)$. The metric H depends on the metric d. It is easy to see that the completeness of (X, d) implies the completeness of $(CB(X), H)$.*

Proposition 1.8.1.

1. $H(A, A) = 0$, for all $A \in \mathcal{P}(X)$.

2. $H(A, B) = H(B, A)$, for all $A, B \in \mathcal{P}(X)$.

3. $H(A, B) \leq H(A, C) + H(C, B)$, for all $A, B, C \in \mathcal{P}(X)$.

4. $H(A, B) = 0 \Longleftrightarrow \bar{A} = \bar{B}$.

Remark 1.8.2. *Let$A, B \in CB(X)$ and $a \in A$. Then for $\epsilon \succ 0$, there must exist a point $b \in B$ such that $d(a, b) \leq H(A, B) + \epsilon$.*

Definition 1.8.2. *Either (X, d) Given a metric space, let the sequence $\{A_n\} \in CB(X)$. the sequence $\{A_n\}$ is :*

1. *converges if exists $A \in CB(X)$ Such that*

$$H\left(A_n, A\right) \longrightarrow 0, \text{ when } n \longrightarrow \infty,$$

2. *from Cauchy If for every $m, n > 0$,*

$$H\left(A_m, A_n\right) \longrightarrow 0, \text{ when } m, n \longrightarrow \infty$$

Definition 1.8.3. *[41] A point of $x_0 \in X$ is said to be a fixed point of the multi-valued mappings $T : X \longrightarrow CB(X)$ if $x_0 \in Tx_0$.*

Exemple 1.8.2. *[41]*

1. *Let $X = [0;1]$ and let T defined by*

$$Tx = \begin{cases} \left[0; \frac{1}{2}\right], & x \neq \frac{1}{4}, \\ \left\{\frac{1}{4}\right\}, & x = \frac{1}{4}. \end{cases}$$

2. *Let $X = [-1;1]$ and let T defined by*

$$Tx = \begin{cases} \{-x\}, & x \notin \{-1, 0\}, \\ \{0, 1\}, & x = -1, \\ \{1\}, & x = 0. \end{cases}$$

Forward, we denote by $F(T)$ the set of all fixed points of a multi-valued mapping T that is,

$$F(T) = \{p \in X : p \in Tp\}$$

1.9 fixed point theorem in Metric space

1.9.1 Banach Contraction Principle

Theorem 1.9.1. *[2] Let (X, d)be a complete metric space, then each contraction map $f : X \to X$ has a unique fixed point.*

Proof. Let h be a contraction constant of the mapping f. We will explicitly construct a sequence converging to the fixed point. Let x_0 be an arbitrary but fixed element in X. Define a sequence of iterates $\{x_n\}$ in X by

$$x_n = f\left(x_{n-1}\right)\left(= f^n\left(x_0\right)\right), \quad \text{for all } n \geq 1.$$

Since f is a contraction, we have

$$d\left(x_n, x_{n+1}\right) = d\left(f\left(x_{n-1}\right), f\left(x_n\right)\right) \leq hd\left(x_{n-1}, x_n\right), \quad \text{for any } n \geq 1.$$

Thus, we obtain

$$d\left(x_n, x_{n+1}\right) \leq h^n d\left(x_0, x_1\right), \quad \text{for all } n \geq 1.$$

Hence, for any $m > n$, we have

$$d(x_n, x_m) \leq \left(h^n + h^{n+1} + \cdots + h^{m-1}\right) d(x_0, x_1) \leq \frac{h^n}{1-h} d(x_0, x_1)$$

We deduce that $\{x_n\}$ is Cauchy sequence in a complete space X. Let $x_n \to p \in X$. Now using the continuity of the map f, we get

$$p = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(p).$$

Finally, to show f has at most one fixed point in X, let p and q be fixed points of f. Then,

$$d(p,q) = d(f(p), f(q)) \leq hd(p,q).$$

Since $h < 1$, we must have $p = q$. The proof of the BCP yields the following useful information about the rate of convergence towards the fixed point.

Corollary 1.9.1. *[2] Let f be a contraction mapping on a complete metric space (X,d) with contraction constant h and fixed point p. For any $x_0 \in X$, with f-iterates $\{x_n\}$, we have the estimates*

$$d(x_n, p) \leq \frac{h^n}{1-h} d(x_0, f(x_0)) \tag{1.4}$$

$$d(x_n, p) \leq hd(x_n, p) \tag{1.5}$$

$$d(x_n, p) \leq \frac{h}{1-h} d(x_{n-1}, x_n). \tag{1.6}$$

In fact, the three inequalities in Corollary (1.9.1) serve different purposes. The inequality (1.4) tells us, in terms of the distance between x_0 and $f(x_0) = x_1$, how many times we need to iterate f starting from x_0 to be certain that we are within a specified distance from the fixed point. This is an upper bound on how long we need to compute. It is called an a priori estimate. Inequality (1.5) shows that once we find a term by iteration within some desired distance of the fixed point, all further iterates will be within that distance. However, (1.5) is not so useful as an error estimate since both sides of (1.5) involve the unknown fixed point. The inequality (1.6) tells us, after each computation, how much closer we are to the fixed point in terms of the previous two iterations. This kind of estimate, called an a posteriori estimate, is very important because if two successive iterations are nearly equal, (1.6) guarantees that we are very close to the fixed point.

Corollary 1.9.2. *[2] Let $f : X \to X$ be a contraction mapping on a complete metric space and $M \subseteq X$ be a closed subset such that $f(M) \subseteq M$. Then, the unique fixed point of f is in M.*

It may be the case that $f : X \to X$ is not a contraction on the whole space X, but rather a contraction on some neighborhood of a given point. In this case we have the following result:

Theorem 1.9.2. *[2] Let (X, d) be a complete metric space and let $B_r(y) = \{x \in X : d(x, y) < r\}$, where $y \in X$ and $r > 0$. Let $f : B_r(y) \to X$ be a contraction map with contraction constant $h < 1$. Further, assume that*

$$d(y, f(y)) < r(1 - h).$$

Then, f has a unique fixed point in $B_r(y)$.

Proof . Note that the uniform continuity of f allows us to extend f to a mapping defined on $\overline{B_r(y)}$ which is a contraction map having the same Lipschitz constant as the original map. We show that $f\left(\overline{B_r(y)}\right) \subseteq B_r(y)$. Let $x \in \overline{B_r(y)}$, then

$$d(y, f(x)) \leq d(y, f(y)) + d(f(y), f(x)) < r(1 - h) + hr = r,$$

and hence, $f : \overline{B_r(y)} \to B_r(y)$. Since $\overline{B_r(y)}$ is a complete metric space, using Theorem 2.1, f has a unique fixed point $p \in \overline{B_r(y)}$. Thus, $p \in B_r(y)$ because $p = f(p) \in B_r(y)$.

Remark 1.9.1. *If f is a contraction map on a complete metric space (X, d) with contraction constant h, then f^n is also a contraction on X with constant h^n and the unique fixed point of f is also the unique fixed point of any f^n.*

We observed in Example (1.3.2) that in some situations a function is not a contraction but its iterate is a contraction map. To get the conclusion of the BCP for the original function, the following early example of an extension of BCP is due Caccioppoli.

Theorem 1.9.3. *[2] Let (X, d) be a complete metric space and let $f : X \to X$ be a mapping such that for each $n \geq 1$, there exists a constant c_n such that*

$$d\left(f^n(x), f^n(y)\right) \leq c_n d(x, y), \quad \text{for all } x, y \in X,$$

where $\sum_{n=1}^{\infty} c_n < \infty$. Then, f has a unique fixed point.

Theorem 1.9.4. *[2] Let (X, d) be a complete metric space and let $f : X \to X$ be a mapping such that for some positive integer n, f^n is contraction on X. Then, f has a unique fixed point.*

Proof . By BCP, f^n has a unique fixed point, say $x \in X$ with $f^n(x) = x$. Since

$$f^{n+1}(x) = f\left(f^n(x)\right) = f(x),$$

it follows that $f(x)$ is a fixed point of f^n, and thus, by the uniqueness of x, we have $f(x) = x$, that is, f has a fixed point. Since, the fixed point of f is necessarily a fixed point of f^n, so is unique.

Remark 1.9.2. *1. Theorem (1.9.3) has importance in the scene that the mapping f is not even assumed to be continuous while the same result was appeared in the literature with continuity assumption on the mapping f.*

2. In some applications, it is the case that the mapping f is a Lipschitz which is not necessarily a contraction, whereas some power of f is a contraction mapping.

Exemple 1.9.1. *[2] Consider the metric space $X = C[a, b]$, of continuous real-valued functions defined on the compact interval $[a, b]$. This is a Banach space with respect to the sup norm. Define $f : X \to X$ by*

$$f(u)(t) = \int_a^t u(s)ds$$

Then,

$$|f(u) - f(v)| \leq (b - a)|u - v|.$$

Now, we compute

$$f^2(u)(t) = \int_a^t (t - s)u(s)ds$$

and inductively,

$$f^n(u)(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}u(s)ds.$$

Thus, we get

$$|f^n(u) - f^n(v)| \leq \frac{(b-a)^n}{n!}|u-v|.$$

Hence, f^n is a contraction map for all values of n for which $\frac{(b-a)^n}{n!} < 1$. It follows that f has the unique fixed point $u = 0$.

Remark 1.9.3. *It was important in the proof of BCP that the contraction constant h be strictly less than 1 . That gave us control over the rate of convergence of $f^n(x_0)$ to the fixed point since $h^n \to 0$ as $n \to \infty$. If we consider f is contractive mapping instead of a contraction, then we lose that control and indeed a fixed point need not exist.*

Exemple 1.9.2. *Let I be a closed interval in $\mathbb{R}$ and $f : I \to I$ be differentiable with $|f'(t)| < 1$ for all t. Then, the mean-value theorem implies $|f(x) - f(y)| < |x-y|$ for all $x \neq y$ in I. The following three functions satisfy this condition, where $I = [1, \infty)$ in the first case and $I = \mathbb{R}$ in the second and third cases:*

$$f(x) = x + \frac{1}{x}; f(x) = \sqrt{x^2+1}; f(x) = \log(1 + \exp(x)).$$

In each case, $f(x) > x$, so none of these functions has a fixed point in I.

Remark 1.9.4. *It is worth to mention that on the one hand, the BCP is very forceful and simple, and it became a classical tool in nonlinear analysis. But, on the other hand, Connell ,gave an example of a metric space X such that X is not complete and every contraction on X has a fixed point. Thus, Theorem (1.9.1) cannot characterize the metric completeness of X which means the notion of contractions is too strong from this point of view.*

A mapping f on a metric space (X, d) is called Kannan mapping if there exists $h \in \left[0, \frac{1}{2}\right)$ such that

$$d(f(x), f(y)) \leq h\{d(x, Tx) + d(y, Ty)\}, \quad \text{for all } x, y \in X$$

1.9.2 Nadler's fixed point theorem

We now introduce the class of multivalued contraction mappings and obtain a fixed point theorem for this class of mapping.

Let T be a mapping from a metric space (X, d) into $CB(X)$. Then T is said to be Lipschitzian if there exists a constant $k > 0$ such that

$$H(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X.$$

A multivalued Lipschitzian mapping T is said to be contraction (nonexpansive) if $k < 1 (k = 1)$. Let $F(T)$ denote the set of fixed points of T, i.e., $F(T) = \{x \in X : x \in Tx\}$.

Theorem 1.9.5. *[42] Let (X, d) be a complete metric space. If $T : X \to CB(X)$ is a m.v.c.m(multi-value contraction mapping), then T has a fixed point.*

Proof. Let $\alpha < 1$ be a Lipschitz constant for T, (we may assume $\alpha > 0$) and let $p_0 \in X$. Choose $p_1 \in Tp_0$. Since $Tp_0, Tp_1 \in CB(X)$ and $p_1 \in Tp_0$, there is a point $p_2 \in Tp_1$ such that:

$$d(p_1, p_2) \leq H(Tp_0, Tp_1) + \alpha$$

(see the remark which follows this proof).
Now, since

$$Tp_1, Tp_2 \in CB(X) \quad \textit{and} \quad p_2 \in Tp_1$$

there is a point $p_3 \in Tp_2$ such that

$$d(p_2, p_3) \leq H(Tp_1, Tp_2) + \alpha^2.$$

After that, continuing in this fashion we produce a sequence $\{p_i\}_{i=1}^{\infty}$ of points of X such that $p_{i+1} \in F(P_i)$ and

$$d(p_i p_{i+1}) \leq H(Tp_{i-1}, Tp_i) + \alpha^i \forall i \geq 1.$$

Also, we note that

$$\begin{aligned} d(p_i, p_{i+1}) \leq & H(Tp_{i-1}, Tp_i) + \alpha^i \leq \alpha d(p_{i-1}, p_i) + \alpha^i \\ & \leq \alpha[H(Tp_{i-2}, Tp_{i-1}) + \alpha^{i-1}] + \alpha^i \\ & \leq \alpha^3 d(p_{i-2}, p_{i-1}) + 2\alpha^i \leq ... \leq \alpha^i d(p_0, p_1) + i.\alpha^i \end{aligned}$$

for all $i \leq 1$. Hence

$$\begin{aligned} d(p_i, p_{i+j}) \leq & d(p_i, p_{i+1}) + d(p_{i+1}, p_{i+2}) + ... + d(p_{i+j-1}, p_{i+j}) \\ & \leq \alpha^i d(p_0, p_1) + i.\alpha^i + \alpha^{i+1} d(p_0, p_1) + (i+1).\alpha^{i+1} + ... \\ & + \alpha^{i+j-1} d(p_0, p_1) + (i+j-1).\alpha^{i+j-1} \\ & \leq \left(\sum_{n=i}^{i+j-1} \alpha^n \right) d(p_0, p_1) + \sum_{n=i}^{i+j-1} n\alpha^n \end{aligned}$$

for all $i, j \geq 1$. It follows that the sequence $\{p_i\}_{i=1}^{\infty}$ is a cauchy sequence .since (X, d) is complete, the sequence $\{p_i\}_{i=1}^{\infty}$ converges to some point $x_0 \in X$. Therefore, the sequence $\{Tp_i\}_{i=1}^{\infty}$ converge to $Tx_0$and, since $p_i \in Tp_{i-1}$for all i it follows that $x_0 \in Tx_0$.This completes the proof of theorem.

Remark 1.9.5. *Let $A, B \in CB(X)$ and let $a \in A$. If $\eta > 0$, then it is a simple consequence of the definition of $H(A, B)$ that there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \eta$(in the proof of the previous theorem the Lipschitz constantα and subsequently α^i play the role of such an η). However, there may not be a point $b \in B$ such that $d(a, b) \leq H(A, B)$(if B is compact, then such a point b does exist).*

Exemple 1.9.3. *let ℓ_2 denote the Hulbert space of all square summable sequences of $e_n$real numbers, let$a = (-1, -1/2, ..., -1/n, ...)$and, for each $n = 1, 2, ...$ lete_n be the vector in ℓ_2 with zeros in all its coordinates except the n^{th} coordinate which is equal to one. Let $A = \{a, e_1 e_2, ..., e_n, ...\}$and let $B = \{e_1 e_2, ..., e_n, ...\}$. Since $\|a - e_n\| = (\|a\|^2 + 1 + \frac{2}{n})^{\frac{1}{2}}$ for each $n = 1, 2, ...$, $H(A, B) = (\|a\|^2 + 1)^{\frac{1}{2}}$and there is no e_n in B such that $\|a - e_n\| \leq H(A, B)$.*

1.10 Fixed Point Theorems in Partial Metric Space

1.10.1 Banach Fixed Point Theorem

Theorem 1.10.1. *[26] Let (X, p) be a partial metric space and $f : X \to X$ satisfy the following*

$$p(f(x), f(y)) \leq \alpha p(x, f(x)) + \beta p(y, (f(y))) + \gamma p(x, y)$$

with $\alpha + \beta + \gamma < 1$ and $0 \leq \alpha, \beta, \gamma \; \forall x, y \in X$. Then f has a unique fixed point in X.
Proof. If x_0 is an arbitrary point in X and

$$x_n = f^n (x_0), n = 1, 2, \ldots\ldots\ldots, \left(x_0 = f^0 (x_0)\right).$$

Then,

$$\begin{aligned} p(x_2, x_1) &= p(f(x_1), f(x_2)) \\ &\leq \alpha p(x_1, x_2) + \beta p(x_0, x_1) + \gamma p(x_0, x_1). \\ &\leq \frac{\beta + \gamma}{1 - \alpha} p(x_0, x_1). \end{aligned}$$

Similarly,

$$p(x_3, x_2) \leq \left(\frac{\beta + \gamma}{1 - \alpha}\right)^2 p(x_0, x_1).$$

By induction,

$$p(x_n, x_{n+1}) \leq \left(\frac{\beta + \gamma}{1 - \alpha}\right)^n p(x_0, x_1).$$

Hence

$$p(x_n, x_{n+1}) = \delta^n p(x_0, f(x_0)), \tag{1.7}$$

where $\delta = \frac{\beta+\gamma}{1-\alpha} < 1$. Therefore,

$$\lim_{n \to \infty} \delta^n = 0.$$

Let h_k be a sequence of positive real numbers so that

$$\lim_{k \to \infty} h_k = 0.$$

Without loss of generality we assume $h_k \geq h_{k+1} \geq \ldots$.
Put $G_k = \{x \in X : p(x, f(x)) \leq h_k\}$. Now from 1.7 it follows that for large $k, G_k \neq \varphi$ for all $k \in \mathbb{N}$. Clearly $\{G_k\}$ forms a decreasing chain of nonempty sets in X. We show that f maps G_k into it self. If $x \in G_k$ then

$$p(f(x), f(f(x))) \leq \alpha p(x, f(x)) + \beta p(f(x), f(f(x))) + \gamma p(x, f(x)).$$

This implies that,

$$p(f(x), f(f(x)) \leq \frac{\alpha + \gamma}{1 - \alpha} p(x, f(x)) \leq \frac{\alpha + \gamma}{1 - \alpha} h_k < h_k.$$

Hence $p : G_k \to G_k$. Next we show that each G_k is closed. Let u be a limit point of G_k. Since a PMS is a first countable space, there is a sequence $\{x_j\}$ in G_k such that

$$\lim_j x_j = u.$$

Now we have

$$\begin{aligned}
p(u, f(u)) &\leq p(u, f(u)) + p\left(x_j, f(u)\right) - p\left(x_j, x_j\right) \\
&\leq p\left(u, x_j\right) + p\left(x_j, f\left(x_j\right)\right) + p\left(f\left(x_j\right), f(f(u)) - p\left(\left(f\left(x_j\right), f\left(x_j\right)\right)\right.\right. \\
&\leq p\left(u, x_j\right) + p\left(x_j, f\left(x_j\right)\right) + p\left(f\left(x_j\right), f(f(u))\right. \\
(1-\beta) p(u, f(u)) &\leq h_k + \alpha h_k + (1+\alpha) p\left(x_j, u\right) \\
&\leq (\alpha+1) h_k + (1+\gamma) p\left(x_j, u\right) \\
p(u, f(u)) &\leq \frac{\alpha+1}{1-\beta} h_k + \frac{1+\alpha}{1-\beta} p\left(x_j, u\right)
\end{aligned}$$

Passing on to the limit $j \to \infty$, we find $p(u, f(u)) \leq \frac{\alpha+1}{1-\beta} h_k$. Since

$$\lim_{j \to \infty} p\left(x_j, u\right) = 0 \text{ (using lemma 1.5 in [1]).}$$

Passing on to the limit $j \to \infty$, we find $p(u, f(u)) \leq \frac{\alpha+1}{1-\beta} h_k$. Since

$$\lim_{j \to \infty} p\left(x_j, u\right) = 0 \text{ (using lemma 1.5 in [1]).}$$

Now $0 \leq \alpha, \beta, \gamma < 1$ and $\alpha + \beta + \gamma < 1$ gives $\frac{\alpha+\gamma}{1-\beta} < 1$ and so

$$\sup_\gamma \left\{\frac{\alpha+\gamma}{1-\beta}\right\} \leq 1$$

and hence $\frac{\alpha+1}{1-\beta} \leq 1$. Thus $p(u, f(u)) \leq h_k$, and hence $u \in G_k$ and so G_k is closed. Finally we need to show G_k is bounded and diameter $(G_k) \to 0$ as $k \to \infty$. Let $u, v \in G_k$ then,

$$\begin{aligned}
p(u, v) &\leq p(u, f(u)) + p(f(u), v) - p(f(u), f(u)) \\
&\leq h_k + p(f(u), f(v)) + p(v, f(v)) - p(f(v), f(v)) \\
&\leq 2 h_k + \alpha p(u, f(u)) + \beta p(v, f(v)) + \gamma p(u, v).
\end{aligned}$$

Hence,

$$\begin{aligned}
p(u, v) &\leq (2+\alpha+\beta) h_k \\
&\leq \frac{\alpha+\beta+2}{1-\gamma} h_k.
\end{aligned}$$

Hence G_k is bounded and diameter $G_k \to 0$ as $k \to \infty$. Thus $\{G_k\}$ is a decreasing chain of nonempty closed sets in (X, p) with diameter $(G_k) \to 0$ as $k \to \infty$. By Cantor's type intersection theorem 1 ,

$$\bigcap_{k=1}^{\infty} G_k$$

is a singleton, say u for some $u \in X$. Hence $f(u) = u$. Uniqueness of u is clear from the argument and thus the proof is complete. We have a couple of interesting corollaries.

Corollary 1.10.1. . *If f is a contraction mapping from a complete PMS X into itself, then f has a unique fixed point in X*

(Taking $\alpha = \beta = 0$ and $0 < \gamma < 1$ in theorem 1.10.1 the above corollary will follow.)

1.10.2 Nadler's Fixed Point Theorem

Theorem 1.10.2. *[9] Let (X, p) be a complete partial metric space. If $T : X \longrightarrow CB^p(X)$ is a multifunction such that for all $x, y \in X$, we have*

$$H_p(Tx, Ty) \leq kp(x, y).$$

Where $k \in [0, 1[$. Then F admits a fixed point.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. By Lemma1.6.2 with $h = \frac{1}{\sqrt{k}}$, there exists $x_2 \in Tx_1$ such that

$$p(x_1, x_2) \leq \frac{1}{\sqrt{k}} H_p(Tx_0, Tx_1).$$

As, $H_p(Tx_0, Tx_1) \leq kp(x_0, x_1)$, then $p(x_1, x_2) \leq \sqrt{k}p(x_0, x_1)$.
For $x_2 \in Tx_1$, there exists $x_3 \in Tx_2$ such that

$$p(x_2, x_3) \leq \frac{1}{\sqrt{k}} H_p(Tx_1, Tx_2) \leq \sqrt{k}p(x_1, x_2)$$

Hence, we construct a sequence $\{x_n\}$ in X such that

$$x_{n+1} \in Tx_n \text{ and } p(x_{n+1}, x_n) \leq \sqrt{k}p(x_n, x_{n-1}) \text{ for all } n \geq 1. \tag{1.8}$$

Now by 1.8, we get

$$p(x_{n+1}, x_n) \leq (\sqrt{k})^n p(x_0, x_1) \text{ for all } n \in \mathbb{N}. \tag{1.9}$$

According to 1.9 and the property $(p4)$ of the partial metric.
for all $m \in \mathbb{N}^*$, we have

$$\begin{aligned} p(x_n, x_{n+m}) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{n+m-1}, x_{n+m}) \\ &\leq (\sqrt{k})^n p(x_0, x_1) + (\sqrt{k})^{n+1} p(x_0, x_1) + \cdots + (\sqrt{k})^{n+m-1} p(x_0, x_1) \\ &= \left((\sqrt{k})^n + (\sqrt{k})^{n+1} + \cdots + (\sqrt{k})^{n+m-1}\right) p(x_0, x_1) \\ &\leq (\sqrt{k})^n 1 - \sqrt{k} p(x_0, x_1) \xrightarrow[n\to+\infty]{} 0 \end{aligned}$$

since $0 < k < 1$. By definition of p^s, we obtain for all $m \in \mathbb{N}^*$,

$$p^s(x_n, x_{n+m}) \leq 2p(x_n, x_{n+m}) \xrightarrow[n\to+\infty]{\longrightarrow} 0.$$

This gives $\{x_n\}$ is a Cauchy sequence in (X, p^s). Since (X, p) is complete, then by Lemma 1.6.2 (X, p^s) is a complete metric space. Hence, the sequence $\{x_n\}$ converges to $x^* \in X$ with respect to the metric p^s, i.e.,

$$\lim_{n\to+\infty} p^s(x_n, x^*) = 0.$$

Moreover, 1.6.4, we have

$$p(x^*, x^*) = \lim_{n\to+\infty} p^s(x_n, x^*) = \lim_{n\to+\infty} p^s(x_n, x_n) \tag{1.10}$$

Since $H_p(Tx_n, Tx^*) \leq kp(x_n, x^*)$, then

$$\lim_{n\to+\infty} H_p(Tx_n, Tx^*) \tag{1.11}$$

Now $x_{n+1} \in Tx_n$ gives that

$$p(Tx_{n+1}, Tx^*) \leq \delta p(Tx_n, Tx^*) \leq H_p(Tx_n, Tx^*)$$

From 1.11, we get

$$\lim_{n\to+\infty} p(Tx_{n+1}, Tx^*) = 0 \tag{1.12}$$

On the other hand, we have

$$p(x^*, Tx^*) \leq p(x^*, Tx_{n+1}) + p(Tx_{n+1}, Tx^*)$$

We take $n \to +\infty$ and we use 1.10 and 1.12 , we get $p\left(x^{*}, Tx^{*}\right) = 0$.

$$p\left(x^{*}, x^{*}\right) = 0$$

so

$$p\left(x^{*}, x^{*}\right) = p\left(x^{*}, Tx^{*}\right).$$

By $a \in \bar{A} \Leftrightarrow p(a, A) = p(a, a)$ we find that $x^{*} \in \overline{Tx^{*}} = Tx^{*}$. Then T admits a fixed point.

CHAPTER

2

FIXED POINT THEOREM

In this chapter, we will present some fixed and coincidence point theorems for single-valued and multi-valued mappings satisfying the non-expansive type conditions in the context of asymptotically regular mappings and orbital completeness of the space.

2.1 Single valued contraction

2.1.1 Fixed Point Theorem for a Single Valued Mapping

Throughout this part, for all $x, y \in X$, we consider the following non-expansive type condition:

$$\begin{aligned} d(Tx,Ty) \leq & \alpha d(x,y) + \beta[d(x,Tx) + d(y,Ty)] \\ & + \gamma[d(x,Ty) + d(y,Tx)] \\ & + \delta[M(x,y) + hm(x,y)] \end{aligned} \tag{2.1}$$

where

$$\alpha \geq 0, \beta, \gamma, \delta > 0, \quad 0 < h < 1, \tag{2.2}$$

$$\alpha + 2\beta + 2\gamma + 2\delta = 1, \tag{2.3}$$

$$M(x, y) = \max\{d(x, Ty), d(y, Tx)\},$$

and

$$m(x, y) = \min\{d(x, Ty), d(y, Tx)\}.$$

Browder and Petryshyn [17] introduced the concept of asymptotically regular mapping at a point in the Hilbert spaces. A mapping T on a metric space (X, d) is said to asymptotically regular at $x \in X$,
if $\lim_{n\to\infty} d\left(T^n x, T^{n+1}x\right) = 0$.

Proposition 2.1.1. *Let T be a self-mapping on X satisfying the condition (2.1) with (2.2) and (2.3).*
Then T is asymptotically regular at each point in X.

Proof Let x_0 be any point in X, we define a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n = T^n x_0$, where $n \in \mathbb{N} \cup \{0\}$. From (2.1),

$$\begin{aligned} d\left(x_n, x_{n+1}\right) =& d\left(Tx_{n-1}, Tx_n\right) \\ \leq& \alpha d\left(x_{n-1}, x_n\right) + \beta\left[d\left(x_{n-1}, x_n\right) + d\left(x_n, x_{n+1}\right)\right] \\ &+ \gamma\left[d\left(x_{n-1}, x_{n+1}\right) + d\left(x_n, x_n\right)\right] \\ &+ \delta\left[M\left(x_{n-1}, x_n\right) + hm\left(x_{n-1}, x_n\right)\right] \end{aligned} \tag{2.4}$$

.

Here, $m\left(x_{n-1}, x_n\right) = 0$ and $M\left(x_{n-1}, x_n\right) = d\left(x_{n-1}, x_{n+1}\right) \leq d\left(x_{n-1}, x_n\right) + d\left(x_n, x_{n+1}\right)$. Suppose that $d\left(x_n, x_{n+1}\right) > d\left(x_{n-1}, x_n\right)$ for some n. Then by (2.4) and (2.3), we have

$$\begin{aligned} d\left(x_n, x_{n+1}\right) &< \alpha d\left(x_n, x_{n+1}\right) + 2\beta d\left(x_n, x_{n+1}\right)] + 2\gamma d\left(x_n, x_{n+1}\right) + 2\delta d\left(x_n, x_{n+1}\right) \\ &< (\alpha + 2\beta + 2\gamma + 2\delta) d\left(x_n, x_{n+1}\right) \\ &= d\left(x_n, x_{n+1}\right), \end{aligned}$$

which is a contradiction. Therefore $d\left(x_n, x_{n+1}\right) \leq d\left(x_{n-1}, x_n\right)$ for all n. Hence,

$$d\left(x_{n-1}, Tx_{n-1}\right) \leq d\left(x_0, Tx_0\right) \quad (n = 1, 2, 3, \ldots). \tag{2.5}$$

Now using (2.1), we have

$$\begin{aligned} d\left(x_1, Tx_2\right) =& d\left(Tx_0, Tx_2\right) \\ & \leq \alpha d\left(x_0, x_1\right) + \beta\left[d\left(x_0, Tx_0\right) + d\left(x_2, Tx_2\right)\right] + \gamma\left[\left(x_0, Tx_2\right) + d\left(x_2, Tx_0\right)\right] \\ & \quad + \delta\left[M\left(x_0, x_2\right) + hm\left(x_0, x_2\right)\right], \end{aligned}$$

from (2.5) and the triangle inequality, we get

$$d\left(x_1, Tx_2\right) \leq 2\alpha d\left(x_0, x_1\right) + 2\beta d\left(x_0, x_1\right) + 4\gamma d\left(x_0, x_1\right) + \delta(3+h) d\left(x_0, x_1\right).$$

Then

$$d\left(x_1, Tx_2\right) \leq [2(1-\beta) - \delta(1-h)] d\left(x_0, x_1\right). \tag{2.6}$$

By (2.1), we have

$$\begin{aligned} d\left(Tx_1, Tx_2\right) \leq& \alpha' d\left(x_1, x_2\right) + \beta'\left[d\left(x_1, Tx_1\right) + d\left(x_2, Tx_2\right)\right] \\ & + \gamma'\left[\left(x_1, Tx_2\right) + d\left(x_2, Tx_1\right)\right] + \delta'\left[M\left(x_1, x_2\right) + hm\left(x_1, x_2\right)\right]. \end{aligned}$$

Here, $M\left(x_1, x_2\right) = d\left(x_1, Tx_2\right), m\left(x_1, x_2\right) = 0$, and $\alpha', \beta', \gamma', \delta'$ are nothing but $\alpha, \beta, \gamma, \delta$ in such a way that $\alpha' + 2\beta' + 2\gamma' + 2\delta' = 1$. Then by using (2.5) and (2.6), we have

$$\begin{aligned} d\left(Tx_1, Tx_2\right) & \leq \alpha' d\left(x_0, x_1\right) + 2\beta' d\left(x_0, x_1\right) + \left(\gamma' + \delta'\right) d\left(x_1, Tx_2\right) \\ & \leq \left(\alpha' + 2\beta'\right) d\left(x_0, x_1\right) + \left(\gamma' + \delta'\right)(2(1-\beta) - \delta(1-h)) d\left(x_0, x_1\right) \\ & \leq \left(\alpha' + 2\beta' + 2\gamma' + 2\delta' - 2\gamma'\beta - \gamma'\delta(1-h) - 2\delta'\beta - \delta'\delta(1-h)\right) d\left(x_0, x_1\right) \\ \Rightarrow d\left(Tx_1, Tx_2\right) & \leq \left(1 - s^2(1-h)\right) d\left(x_0, x_1\right), \text{ where } s^2 = \delta\delta'. \end{aligned} \tag{2.7}$$

Again from (2.5) and (2.7) we have,

$$d\left(Tx_2, Tx_3\right) \leq d\left(Tx_1, Tx_2\right) \leq \left(1 - s^2(1-h)\right) d\left(x_0, x_1\right).$$

And from (2.1)and (2.5) we obtain,

$$\begin{aligned} d\left(Tx_2, Tx_4\right) \leq& \alpha d\left(x_2, x_4\right) + \beta\left[d\left(x_2, Tx_2\right) + d\left(x_4, Tx_4\right)\right] + \gamma\left[\left(x_2, Tx_4\right) + d\left(x_4, Tx_2\right)\right] \\ & + \delta\left[M\left(x_2, x_4\right) + hm\left(x_2, x_4\right)\right] \\ \leq& 2\alpha d\left(x_2, x_3\right) + 2\beta d\left(x_2, x_3\right) + 4\gamma d\left(x_2, x_3\right) + \delta(3+h) d\left(x_2, x_3\right) \\ \leq& (2(1-\beta) - \delta(1-h)) d\left(x_0, Tx_0\right). \end{aligned} \tag{2.8}$$

From (2.5), (2.8), we obtain

$$\begin{aligned}
d\left(Tx_3, Tx_4\right) \leq & \alpha' d\left(x_3, x_4\right)+\beta'\left[d\left(x_3, Tx_3\right)+d\left(x_4, Tx_4\right)\right]+\gamma'\left[\left(x_3, Tx_4\right)+d\left(x_4, Tx_3\right]\right. \\
& +\delta\left[M\left(x_3, x_4\right)+hm\left(x_3, x_4\right)\right] \\
\leq & \left(\alpha'+2\beta'\right) d\left(x_3, x_4\right)+\left(\gamma'+\delta'\right) d\left(x_3, Tx_4\right) \\
\leq & \left(\alpha'+2\beta'\right) d\left(x_3, x_4\right)+\left(\gamma'+\delta'\right)(2(1-\beta)-\delta(1-h)) d\left(x_2, x_3\right) \\
\leq & \left(1-s^2(1-h)\right)^2 d\left(x_0, x_1\right).
\end{aligned}$$

Moreover,

$$d\left(Tx_4, Tx_5\right) \leq d\left(Tx_3, Tx_4\right) \leq\left(1-s^2(1-h)\right)^2 d\left(x_0, x_1\right),$$

where $\alpha' \geq 0, \beta', \gamma', \delta'>0, \alpha'+2\beta'+2\gamma'+2\delta'=1$ and $s^2=\delta\delta'$. Analogously, $d\left(Tx_5, Tx_6\right) \leq \left(1-s^2(1-h)\right)^3 d\left(x_0, x_1\right)$, and continuing this process, we have

$$d\left(T^n x_0, T^{n+1} x_0\right) \leq\left(1-s^2(1-h)\right)^{\left[\frac{n}{2}\right]} d\left(x_0, Tx_0\right), \tag{2.9}$$

for all $n \in \mathbb{N}$, where $\left[\frac{n}{2}\right]$ denotes the greatest integer not exceeding $\frac{n}{2}$. Since $0<s \leq \frac{1}{2}$ and $h<1$, we have $\lim_{n \rightarrow \infty} d\left(T^n x_0, T^{n+1} x_0\right)=0$, i.e., T is asymptotically regular at x_0.

Proposition 2.1.2. *If T is a self-mapping on X satisfying the condition (2.1) with (2.2) and (2.3). If T has a fixed point (say p), then T is continuous at p*

Proof. Let $x_n \rightarrow p=Tp$. Then, by using (2.3) and triangular inequality in the condition (2.1), we get

$$\begin{aligned}
d\left(Tx_n, Tp\right) \leq & \alpha d\left(x_n, p\right)+\beta\left[d\left(x_n, Tx_n\right)+d(p, Tp)\right]+\gamma\left[\left(x_n, Tp\right)+d\left(p, Tx_n\right)\right] \\
& +\delta\left[M\left(x_n, p\right)+hm\left(x_n, p\right)\right] \\
\leq & \alpha d\left(x_n, p\right)+\beta\left[d\left(x_n, p\right)+d\left(p, Tx_n\right)\right]+\gamma\left[d\left(x_n, p\right)+d\left(p, Tx_n\right)\right] \\
& +\delta\left[\max \left\{d\left(x_n, p\right), d\left(p, Tx_n\right)\right\}+h \min \left\{d\left(x_n, p\right), d\left(p, Tx_n\right)\right\}\right] \\
< & (\alpha+2\beta+2\gamma+2\delta) d\left(x_n, p\right)=d\left(x_n, p\right).
\end{aligned}$$

Now, $Tx_n \rightarrow Tp$ whenever $n \rightarrow \infty$. Therefore, T is continuous at p.

Theorem 2.1.1. *Let (X, d) be a non-empty complete metric space, and let T be a self-mapping on X satisfying the condition (2.1) with (2.2) and (2.3). Then T has a unique fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X such that $x_{n+1} = T^n x_0$. Then, from (2.9), we obtain that $\{T^n x_0\}$ is a Cauchy sequence. Since X is complete, there is a $p \in X$ such that

$$\lim_{n\to\infty} T^n x = p.$$

Now, from (2.1), we have

$$\begin{aligned} d\left(T^n x, Tp\right) \leq & \alpha d\left(T^{n-1}x, p\right) + \beta\left[d\left(T^{n-1}x, T^n x\right) + d(p, Tp)\right] \\ & + \gamma\left[\left(T^{n-1}x, Tp\right) + d\left(p, T^n x\right)\right] + \delta\left[M\left(T^{n-1}x, p\right) + hm\left(T^{n-1}x, p\right)\right]. \end{aligned}$$

Taking the limit as $n \to \infty$,

$$d(p, Tp) \leq (\beta + \gamma + \delta) d(p, Tp).$$

Then $p = Tp$. Now, suppose that p and q are two fixed points of T. Then by (2.1), we get

$$\begin{aligned} d(p, q) =& d(Tp, Tq) \\ \leq & \alpha d(p, q) + \beta[d(p, Tp) + d(q, Tq)] + \gamma[d(p, Tq) + d(q, Tp)] \\ & + \delta[M(p, q) + hm(p, q)] \\ =& (\alpha + 2\gamma + \delta(1 + h)) d(p, q). \end{aligned}$$

Using (2.3), we get

$$d(p, q) \leq (1 - (2\beta + \delta(1 - h))) d(p, q)$$

which implies $p = q$. Hence, T has a unique fixed point. Now, we take the following examples for vindication of Theorem 2.1.1.

Exemple 2.1.1. *Let $X = [0, 10] \cup \{12\}$ be a usual metric space. Define $T : X \to X$ as:*

$$T(x) = \begin{cases} 10 & \text{if } 0 \leq x \leq 10, \\ 9 & \text{if } x = 12. \end{cases}$$

Then, it can be seen easily, T is satisfying the condition (2.1) for $\alpha = \frac{2}{10}, \beta = \frac{2}{10}, \gamma = \frac{1}{10}, \delta = \frac{1}{10}$, and 10 is only fixed point of T.

Exemple 2.1.2. *Let $X = \{1, 2, 4\}$ be a usual metric space. Define $T : X \to X$ as: $T1 = 2, T2 = 2, T4 = 1$. Then, it can be seen easily, T is satisfying the condition (2.1) for $\alpha = \frac{2}{10}, \beta = \frac{2}{10}, \gamma = \frac{1}{10}, \delta = \frac{1}{10}$, and 2 is only fixed point of T. Here, we also give an another example in support of Theorem 2.1.1.*

Exemple 2.1.3. *Let* $X = \{0, 1, 2\}$*. Define* $T : X \to X$ *as:* $T0 = 0, T1 = 0, T2 = 1$*. The metric* d *on* X *is defined by* $d(0, 0) = d(1, 1) = d(2, 2) = 0, d(0, 1) = d(1, 0) = 2, d(0, 2) = d(2, 0) = 1$ *and* $d(1, 2) = d(2, 1) = 1$*. Then* T *satisfying all the condition (1) for* $\alpha = \frac{2}{10}, \beta = \frac{2}{10}, \gamma = \frac{1}{10}, \delta = \frac{1}{10}$*, and 0 is only fixed point of* T*. However,* T *does not satisfy the condition of result of [27] for* $x = 0, y = 2$ *([27] .*

Remark 2.1.1. *2.1.1 is generalization of results of [16, 19, 22, 23, 27] and others.*

2.1.2 Coincidence Point Theorem for a Pair of Single Valued Mappings

First we recall some definitions which are very important to our work in this part.

Definition 2.1.1. *[31] Let* f *and* g *be two self-mapping of a metric space* X*. Then* f *and* g *are said to be compatible if* $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$*, whenever* $\{x_n\}$ *is a sequence such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in X$*. Let* T *and* f *be self-mappings on a metric space* (X, d) *such that* $T(X) \subset f(X)$*. Then, for any* $x_0 \in X$*, we choose a point* $x_1 \in X$ *such that* $fx_1 = Tx_0$*. Continuing this process, we can choose a sequence* $\{x_k\}$ *in* X *such that* $fx_{k+1} = Tx_k, k = 0, 1, 2, \ldots,$ *and the sequence* $\{fx_n\}$ *is called a* T*-sequence with initial point* x_0*.*

Definition 2.1.2. *[8] Let* T *and* f *be self-mappings on a metric space* (X, d) *such that* $T(X) \subset f(X)$*. Then the mapping* T *is said to be asymptotically* f*-regular at point* $x_0 \in X$*, if* $\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0$*, where* $\{fx_n\}$ *is a* T*-sequence with initial point* x_0*. In the same paper, Abbas et al. [8] obtained the following result on metric space under the notion of asymptotically* f*-regular mappings.*

Definition 2.1.3. *a coincidence point (or simply coincidence) of two mapping is a point in their common domain having the same image.*
formally, given two mappings $f, g : X \to Y$ *we say that a point* $x \in X$ *is a coincidence point of* f *and* g *if* $f(x) = g(x)$ *Coincidence theory (the study of coincidence points) is,in most settings , a generalisation of fixed point theory in the special case obtained from the above by letting* $X = Y$ *and taking* g *to be identity mapping.*

Theorem 2.1.2. *Let T and f be self-mappings on a metric space (X, d) such that $T(X) \subset f(X)$. Assume that, for all $x, y \in X$, the following condition holds.*

$$\begin{aligned} d(Tx, Ty) \leq & a_1 F_1[\min\{d(fx, Tx), d(fy, Ty)\}] + a_2 F_2[d(fx, Tx) d(fy, Ty)] \\ & + a_3 d(fx, fy) + a_4[d(fx, Tx) + d(fy, Ty)] + a_5[d(fx, Ty) + d(fy, Tx)], \end{aligned}$$

where for $i = 1, 2, 3, 4, 5, a_i \geq 0$ such that for every arbitrary fixed $k > 0$,
$0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$, we have $a_4 + a_5 \leq \lambda_1, a_3 + 2a_5 \leq \lambda_2$ and $a_1, a_2 \leq k$.
If $f(X)$ or $T(X)$ is a complete subspace of X and T is asymptotically f-regular at some point $x_0 \in X$, then T and f have a coincidence point. Now, in the direction of asymptotically f-regular mappings, we establish the following result.

Theorem 2.1.3. *Let T and f be self-mappings on a metric space (X, d) satisfying, for all $x, y \in X$, the following condition.*

$$\begin{aligned} d(Tx, Ty) \leq & \alpha d(fx, fy) + \beta[d(fx, Tx) + d(fy, Ty)] \\ & + \gamma[d(fx, Ty) + d(fy, Tx)] + \delta\left[M_f(x, y) + h m_f(x, y)\right], \end{aligned} \tag{2.10}$$

where $\alpha \geq 0, \beta, \gamma, \delta > 0$ and $0 < h < 1$ with $\alpha + 2\beta + 2\gamma + 2\delta = 1$,
$M_f(x, y) = \max\{d(fx, Ty), d(fy, Tx)\}$ and $m_f(x, y) = \min\{d(fx, Ty), d(fy, Tx)\}$.
If $T(X) \subset f(X)$ and T is asymptotically f-regular at some point x_0 in X, and one of the following holds.

1. *X is complete and f is surjective;*
2. *X is complete, f is continwous, and T and f are compatible;*
3. *$f(X)$ is complete subspace of X;*
4. *$T(X)$ is complete subspace of X.*

Then f and T have a coincidence point in X. Moreover, the coincidence value is unique, i.e., $fp = fq$, whenever $fp = Tp$ and $fq = Tq$.

Proof. Let $x_0 \in X$ be an arbitrary. Since $T(X) \subset f(X)$, we may construct a sequence $fx_{n+1} = Tx_n$ for $n \in \mathbb{N} \cup \{0\}$. Now, from (2.10), we get

$$\begin{aligned} d\left(fx_{n+1}, fx_{n+2}\right) \leq & \alpha d\left(fx_n, fx_{n+1}\right) + \beta\left[d\left(fx_n, Tx_n\right) + d\left(fx_{n+1}, Tx_{n+1}\right)\right] \\ & + \gamma\left[d\left(fx_n, Tx_{n+1}\right) + d\left(fx_{n+1}, Tx_n\right)\right] \\ & + \delta\left[M_f\left(x_n, x_{n+1}\right) + m_f\left(x_n, x_{n+1}\right)\right]. \end{aligned}$$

If for some $n, d\left(fx_{n+1}, fx_{n+2}\right) > d\left(fx_n, fx_{n+1}\right)$, then in the above inequality results as

$$d\left(fx_{n+1}, fx_{n+2}\right) < (\alpha + 2\beta + 2\gamma + 2\delta) d\left(fx_{n+1}, fx_{n+2}\right) = d\left(fx_{n+1}, fx_{n+2}\right),$$

which is a contradiction. Therefore,

$$d\left(fx_{n+1}, fx_{n+2}\right) \leq d\left(fx_n, fx_{n+1}\right) \text{ for all } n. \tag{2.11}$$

Without loss of generality, for $m > n$, we have

$$\begin{aligned} d\left(fx_n, fx_m\right) & \leq d\left(fx_n, fx_{n+1}\right) + d\left(fx_{n+1}, fx_{m+1}\right) + d\left(fx_{m+1}, fx_m\right) \\ & = d\left(fx_n, fx_{n+1}\right) + d\left(fx_{m+1}, fx_m\right) + d\left(Tx_n, Tx_m\right). \end{aligned}$$

Now from (2.10), (2.11) and triangular inequality, we obtain

$$\begin{aligned} d\left(fx_n, fx_m\right) \leq & d\left(fx_n, fx_{n+1}\right) + d\left(fx_{m+1}, fx_m\right) \\ & + \alpha d\left(fx_n, fx_m\right) + \beta\left[d\left(fx_n, Tx_n\right), d\left(fx_m, Tx_m\right)\right] \\ & + \gamma\left[d\left(fx_n, Tx_m\right) + d\left(fx_m, Tx_n\right)\right] + \delta\left[M_f\left(x_n, x_m\right) + m_f\left(x_n, x_m\right)\right] \\ \leq & d\left(fx_n, fx_{n+1}\right) + d\left(fx_{m+1}, fx_m\right) + \alpha d\left(fx_n, fx_m\right) \\ & + \gamma\left[d\left(fx_n, fx_m\right) + d\left(fx_m, fx_{m+1}\right) + d\left(fx_m, fx_n\right) + d\left(fx_n, fx_{n+1}\right)\right] \\ & + \beta\left[d\left(fx_n, fx_{n+1}\right) + d\left(fx_m, fx_{m+1}\right)\right] \\ & + \delta\left[\left(d\left(fx_n, fx_m\right) + d\left(fx_m, fx_{m+1}\right)\right) + \left(d\left(fx_m, fx_n\right) + d\left(fx_n, fx_{n+1}\right)\right)\right] \\ = & (\alpha + 2\gamma + 2\delta) d\left(fx_n, fx_m\right) + \beta\left[d\left(fx_n, fx_{n+1}\right) + d\left(fx_m, fx_{m+1}\right)\right] \\ & + (1 + \gamma + \delta)\left[d\left(fx_m, fx_{m+1}\right) + d\left(fx_n, fx_{n+1}\right)\right]. \end{aligned}$$

Then

$$d\left(fx_n, fx_m\right) = \left(\frac{1 + \gamma + \delta}{2\beta}\right)\left[d\left(fx_m, fx_{m+1}\right) + d\left(fx_n, fx_{n+1}\right)\right].$$

Since T is asymptotically f-regular, then the right hand side of the above inequality tends to zero, as $m, n \to \infty$. Thus, $\lim_{m,n\to\infty} d\left(fx_n, fx_m\right) = 0$. It follows that $\{fx_n\}$ is a Cauchy sequence in X. Now, we consider the following cases:

1. Suppose X is complete and f is surjective. Then there exists a point $p \in X$ with $\lim_{n\to\infty} fx_n = p$ and a point $z \in X$ such that $p = fz$, and from (2.10),
we get

$$\begin{aligned}
d(fz,Tz) \leq & d\left(fz, fx_{n+1}\right) + d\left(Tx_n, Tz\right) \\
& \leq d\left(fz, fx_{n+1}\right) + \alpha d\left(fx_n, fz\right) + \beta\left[d\left(fx_n, Tx_n\right) + d(fz,Tz)\right] \\
& \quad + \gamma\left[d\left(fx_n, Tz\right) + d\left(fz, Tx_n\right)\right] + \delta\left[M_f\left(x_{n+1}, z\right) + m_f\left(x_{n+1}, z\right)\right]. \\
& \leq d\left(fz, fx_{n+1}\right) + \alpha d\left(fx_n, fz\right) + \beta\left[d\left(fx_n, fx_{n+1}\right) + d(fz,Tz)\right] \\
& \quad + \gamma\left[d\left(fx_n, Tz\right) + d\left(fz, fx_{n+1}\right)\right] \\
& \quad + \delta \max\left\{d\left(fx_n, fx_{n+1}\right), d(fz,Tz)\right\} \\
& \quad + \delta \min\left\{d\left(fx_n, fx_{n+1}\right), d(fz,Tz)\right\},
\end{aligned}$$

making $n \to \infty$, we get $d(fz,Tz) \leq (\beta+\gamma+\delta)(fz,Tz)$ which implies $p = fz = Tz$. Hence, f and T have a coincidence point.

2. Suppose that X is complete, f is continuous, and f and T are compatible. Then, $\lim_{n\to\infty} fx_n = p$ implies $\lim_{n\to\infty} ffx_n = fp$, and from (2.10),
we get

$$\begin{aligned}
d(fp,Tp) \leq & d\left(fp, ffx_{n+1}\right) + d\left(fTx_n, Tp\right) \\
& \leq d\left(fp, ffx_{n+1}\right) + d\left(fTx_n, Tfx_n\right) + d\left(Tfx_n, Tp\right) \\
& \leq d\left(fp, ffx_{n+1}\right) + d\left(fTx_n, Tfx_n\right) + \alpha d\left(ffx_n, fp\right) \\
& \quad + \beta\left[d\left(ffx_n, Tfx_n\right) + d(fp,Tp)\right] + \gamma\left[d\left(ffx_n, Tp\right) + d\left(fp, Tfx_n\right)\right] \\
& \quad + \delta\left[M_f\left(fx_n, p\right) + m_f\left(fx_n, p\right)\right].
\end{aligned}$$

Note that, since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = p$, and f and T are compatible, $\lim_{n\to\infty} d\left(fTx_n, Tfx_n\right) = 0$. Taking the limit as $n \to \infty$, above inequality yields

$$d(fp,Tp) \leq (2\beta + 2\gamma + 2\delta) d(fp,Tp)$$

Then

$$d(fp,Tp) \leq (1-\alpha) d(fp,Tp).$$

It implies that $fp = Tp$.

3. If $f(X)$ is complete subspace of X, then $p \in f(X)$. Suppose $z \in f^{-1}p$, then $p = fz$ and the proof is completed by case (1).

4. If $T(X)$ is complete subspace of X, then $p \in T(X) \subset f(X)$, and the proof is completed by case (3). To establish uniqueness, suppose that q is another coincidence point of f and T, i.e., $fp = Tp$, $fq = Tq$. Then, from (2.10), we have

$$\begin{aligned} d(Tp, Tq) \leq \alpha d(fp, fq) + \beta[d(fp, Tp) + d(fq, Tq)] + \gamma[d(fp, Tq) + d(fq, Tp)] \\ + \delta\left[M_f(p, q) + m_f(p, q)\right] \\ \leq (\alpha + 2\gamma + 2\delta) d(Tp, Tq). \end{aligned}$$

Then

$$d(Tp, Tq) \leq (1 - 2\beta) d(Tp, Tq)$$

which implies that $Tp = Tq$, and hence $fp = fq$.

Remark 2.1.2. *Taking $f = I$ (identity mapping on X) in Theorem 2.1.3, we find an another version of Theorem 2.1.1 for $h = 1$.*

2.1.3 Coincidence point theorem in partial metric space

Recently, Kumar et al. have proved a fixed point result on a complete metric space (X, d) for a mapping $T : X \to X$ satisfying a non-expansive type condition, such that for all $x, y \in X$:

$$\begin{aligned} d(Tx, Ty) \leq & \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] \\ & + \gamma[d(x, Ty) + d(y, Tx)] + \delta[M(x, y) + hm(x, y)] \end{aligned}$$

where $\alpha \geq 0; \beta, \gamma, \delta > 0; 0 < h < 1; \alpha + 2\beta + 2\gamma + 2\delta = 1; M(x, y) = \max\{d(x, Ty), d(y, Tx)\}$ and $m(x, y) = \min\{d(x, Ty), d(y, Tx)\}$. In this section, we will present some fixed and coincidence point theorems for generalized type contractions in partial metric spaces. The approach is based on fixed point results obtained by Kumar et al. in the settings of the partial metric under similar type of contractive conditions. Our results are the extensions of Banach's contraction, Kannan's contraction, Chatterjea's contraction and Ćirić's contraction of metric spaces to partial metric spaces. Moreover, we show that our results are also true in orbitally complete and (T, f)-orbitally complete partial metric spaces, which generalize and extend the conclusions obtained in the literature. In addition, we have given some non-trivial examples to demonstrate our results.

Theorem 2.1.4. *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a mapping satisfying*

$$\begin{aligned} p(Tx, Ty) \leq & \alpha p(x, y) + \beta[p(x, Tx) + p(y, Ty)] \\ & + \gamma[p(x, Ty) + p(y, Tx)] \\ & + \delta\left[M_p(x, y) + hm_p(x, y)\right] \end{aligned} \tag{2.12}$$

for all $x, y \in X$, where $0 < h < 1$ and $\alpha \geq 0, \beta, \gamma, \delta > 0$ with $\alpha + 2\beta + 2\gamma + 2\delta < 1$, and

$$M_p(x, y) = \max\{p(x, Ty), p(y, Tx)\}$$

$$m_p(x, y) = \min\{p(x, Ty), p(y, Tx)\}.$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary. Now, we define a sequence $\{x_n \in X : n \in \mathbb{N} \cup \{0\}\}$ such that $x_{n+1} = Tx_n = T^n x_0$. Then, by (2.12), we have

$$\begin{aligned} p\left(x_n, x_{n+1}\right) = & p\left(Tx_{n-1}, Tx_n\right) \\ \leq & \alpha p\left(x_{n-1}, x_n\right) + \beta\left[p\left(x_{n-1}, x_n\right) + p\left(x_n, x_{n+1}\right)\right] \\ & + \gamma\left[p\left(x_{n-1}, x_{n+1}\right) + p\left(x_n, x_n\right)\right] \\ & + \delta\left[M_p\left(x_{n-1}, x_n\right) + hm_p\left(x_{n-1}, x_n\right)\right], \end{aligned}$$

which implies

$$\begin{aligned} p\left(x_n, x_{n+1}\right) \leq & \delta\left[\max\left\{p\left(x_{n-1}, x_{n+1}\right), p\left(x_n, x_n\right)\right\}\right. \\ & \left. + h \min\left\{p\left(x_{n-1}, x_{n+1}\right), p\left(x_n, x_n\right)\right\}\right] \\ & + (\alpha + \beta + \gamma) p\left(x_{n-1}, x_n\right) + (\beta + \gamma) p\left(x_n, x_{n+1}\right) \end{aligned} \tag{2.13}$$

If $p\left(x_n, x_{n+1}\right) > p\left(x_{n-1}, x_n\right)$ for some n, then (2.13) implies

$$\begin{aligned} p\left(x_n, x_{n+1}\right) < & \delta\left[\max\left\{p\left(x_{n-1}, x_{n+1}\right), p\left(x_n, x_n\right)\right\}\right. \\ & \left. + h \min\left\{p\left(x_{n-1}, x_{n+1}\right), p\left(x_n, x_n\right)\right\}\right] \\ & + (\alpha + 2\beta + 2\gamma) p\left(x_n, x_{n+1}\right). \end{aligned}$$

Now either $p(x_{n-1}, x_{n+1}) \geq p(x_n, x_n)$ or $p(x_{n-1}, x_{n+1}) \leq p(x_n, x_n)$ in above inequality. But, in both cases, we get $p(x_n, x_{n+1}) < p(x_n, x_{n+1})$, which is not true. Hence, $p(x_n, x_{n+1}) \leq p(x_{n-1}, x_n)$ for all n. Also, by (2.13), we have

$$\begin{aligned} p(x_n, x_{n+1}) \leq & \delta\left[\max\left\{p(x_{n-1}, x_{n+1}), p(x_n, x_n)\right\}\right. \\ & \left.+h\min\left\{p(x_{n-1}, x_{n+1}), p(x_n, x_n)\right\}\right] \\ & +(\alpha+2\beta+2\gamma)p(x_{n-1}, x_n). \end{aligned} \tag{2.14}$$

We consider the following two cases:

Case 1: If $\max\{p(x_{n-1}, x_{n+1}), p(x_n, x_n)\} = p(x_{n-1}, x_{n+1})$, then (2.14) implies

$$\begin{aligned} p(x_n, x_{n+1}) \leq & \delta\left[2p(x_{n-1}, x_n) - (1-h)p(x_n, x_n)\right] \\ & +(\alpha+2\beta+2\gamma)p(x_{n-1}, x_n) \\ \leq & (\alpha+2\beta+2\gamma+2\delta)p(x_{n-1}, x_n), \quad \text{since } h<1. \end{aligned}$$

Case 2: If $\max\{p(x_{n-1}, x_{n+1}), p(x_n, x_n)\} = p(x_n, x_n)$, then (2.14) implies

$$\begin{aligned} p(x_n, x_{n+1}) \leq & \delta\left[p(x_n, x_n) + hp(x_{n-1}, x_{n+1})\right] \\ & +(\alpha+2\beta+2\gamma)p(x_{n-1}, x_n) \\ \leq & \delta(1+h)p(x_n, x_n) + (\alpha+2\beta+2\gamma)p(x_{n-1}, x_n) \\ \leq & \delta(1+h)p(x_{n-1}, x_n) + (\alpha+2\beta+2\gamma)p(x_{n-1}, x_n) \quad (\text{ by } (p_2)) \\ < & (\alpha+2\beta+2\gamma+2\delta)p(x_{n-1}, x_n), \quad \text{since } h<1. \end{aligned}$$

Thus, in both cases, we conclude that $p(x_n, x_{n+1}) \leq kp(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, where $k = (\alpha+2\beta+2\gamma+2\delta) < 1$.

We now show that $\{x_n\}$ is a Cauchy sequence in X. Let $m, n > 0$ with $m > n$, then by(p_4) (the fourth condition of PMS), we have

$$\begin{aligned} p(x_n, x_m) \leq & p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \ldots + p(x_{n+m-1}, x_m) \\ & - p(x_{n+1}, x_{n+1}) - p(x_{n+2}, x_{n+2}) - \ldots - p(x_{n+m-1}, x_{n+m-1}) \\ \leq & k^n p(x_0, x_1) + k^{n+1}p(x_0, x_1) + \ldots + k^{n+m-1}p(x_0, x_1) \\ \leq & k^n\left(\frac{1-k^{m-1}}{1-k}\right)p(x_0, x_1). \end{aligned}$$

Making $n, m \to \infty$, we get $\lim_{n,m\to\infty} p\left(x_n, x_m\right) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in X, so by Lemma1.6.4, it is also a Cauchy sequence in the metric space (X, p^s). Thus the sequence $\{x_n\}$ converges to $x \in X$ because (X, p) is complete. Moreover,

$$\lim_{n,m\to\infty} p\left(x_n, x_m\right) = \lim_{n\to\infty} p\left(x_n, x\right) = p(x, x) = 0. \tag{2.15}$$

Further, we show that x is a fixed point of T. Also, for every n, we have

$$p(x, Tx) \leq p\left(x, Tx_n\right) + p\left(Tx_n, Tx\right) - p\left(Tx_n, Tx_n\right).$$

Taking $n \to \infty$ in this inequality, we get $p(x, Tx) \leq p(Tx, Tx)$. Hence, by(p_2) (the second condition of PMS), we get

$$p(x, Tx) = p(Tx, Tx). \tag{2.16}$$

Again from (2.12), we have

$$\begin{aligned}
p(x, Tx) \leq & p\left(x, Tx_n\right) + p\left(Tx_n, Tx\right) - p\left(Tx_n, Tx_n\right) \\
\leq & p\left(x, x_{n+1}\right) + \alpha p\left(x_n, x\right) + \beta\left[p\left(x_n, Tx_n\right) + p(x, Tx)\right] \\
& + \gamma\left[p\left(x_n, Tx\right) + p\left(x, Tx_n\right)\right] + \delta\left[M_p\left(x_n, x\right) + hm_p\left(x_n, x\right)\right] \\
\leq & p\left(x, x_{n+1}\right) + \alpha p\left(x_n, x\right) \\
& + \beta\left[p\left(x_n, x\right) + p\left(x, x_{n+1}\right) - p(x, x) + p(x, Tx)\right] \\
& + \gamma\left[p\left(x_n, x\right) + p(x, Tx) - p(x, x) + p\left(x, x_{n+1}\right)\right] \\
& + \delta\left[M_p\left(x_n, x\right) + hm_p\left(x_n, x\right)\right] \\
\leq & (1+\beta+\gamma) p\left(x, x_{n+1}\right) + (\alpha+\beta+\gamma) p\left(x_n, x\right) + (\beta+\gamma) p(x, Tx) \\
& + \delta\left[\max\left\{p\left(x_n, Tx\right), p\left(x, Tx_n\right)\right\} + h \min\left\{p\left(x_n, Tx\right), p\left(x, Tx_n\right)\right\}\right].
\end{aligned}$$

Again we consider the following cases.

Case 1: If $\max\left\{p\left(x_n, Tx\right), p\left(x, Tx_n\right)\right\} = p\left(x_n, Tx\right)$, then

$$p(x, Tx) \leq \left(\frac{1+\beta+\gamma+h\delta}{1-\beta-\gamma-\delta}\right) p\left(x, x_{n+1}\right) + \left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\delta}\right) p\left(x_n, x\right).$$

Case 2: If $\max\left\{p\left(x_n, Tx\right), p\left(x, Tx_n\right)\right\} = p\left(x, Tx_n\right)$, then

$$p(x,Tx) \le \left(\frac{1+\beta+\gamma+\delta}{1-\beta-\gamma-h\delta}\right) p\left(x, x_{n+1}\right) + \left(\frac{\alpha+\beta+\gamma+h\delta}{1-\beta-\gamma-h\delta}\right) p\left(x_n, x\right).$$

Thus, taking $n \to \infty$ in both cases, we get

$$p(x,Tx) = 0. \tag{2.17}$$

From (2.15), (2.16) and (2.17), we get

$$p(Tx,Tx) = p(x,Tx) = p(x,x) = 0. \tag{2.18}$$

Hence, by (p_1) (the first condition of PMS) and (2.18), we get $x = Tx$, i.e., x is a fixed point of T.

To show the uniqueness, we suppose that y is another fixed point of T. Then, by (2.12), we get

$$\begin{aligned} p(x,y) =& p(Tx,Ty) \\ \le& \alpha p(x,y) + \beta[p(x,Tx)+p(y,Ty)] + \gamma[p(x,Ty)+p(y,Tx)] \\ &+ \delta[\max\{p(x,Ty),p(y,Tx)\} + h\min\{p(x,Ty),p(y,Tx)\}] \\ \le& [\alpha + 2\gamma + (1+h)\delta]p(x,y), \end{aligned}$$

which implies $p(x,y) = 0$, that is, $x = y$. This completes the theorem.

We now consider the following example in the support of Theorem 2.1.4.

Exemple 2.1.4. *Let $X = [0,+\infty)$. Then (X,p), where $p(x,y) = \max\{x,y\}$, is a complete partial metric space. Define $T : X \to X$ by*

$$T(t) = \begin{cases} \frac{t}{4}, & \text{if } 0 \le t < 1 \\ \frac{t}{3+t}, & \text{if } t \ge 1 \end{cases}$$

Then T satisfies all the conditions of the Theorem 2.1.4 for $\alpha = \frac{1}{4}, \beta, \gamma, \delta \in \left(0, \frac{1}{8}\right)$ and $0 < h < 1$. Moreover, $0 \in X$ is the only fixed point of T.

Next we prove another result for a pair of mappings as follows.

Theorem 2.1.5. *Let (X,p) be a complete partial metric space and let $T, f : X \to X$ be mappings satisfying*

$$\begin{aligned} p(Tx,Ty) \leq &\alpha p(fx,fy) + \beta[p(fx,Tx) + p(fy,Ty)] \\ &+ \gamma[p(fx,Ty) + p(fy,Tx)] \\ &+ \delta\left[M_p^f(x,y) + hm_p^f(x,y)\right] \end{aligned} \tag{2.19}$$

for all $x, y \in X$, where $0 < h < 1$ and $\alpha \geq 0, \beta, \gamma, \delta > 0$,
with $\alpha + 2\beta + 2\gamma + 2\delta < 1$, and

$$\begin{aligned} M_p^f(x,y) &= \max\{p(fx,Ty), p(fy,Tx)\} \\ m_p^f(x,y) &= \min\{p(fx,Ty), p(fy,Tx)\}. \end{aligned}$$

If $T(X) \subset f(X)$ and $f(X)$ is a complete subspace of X, then T and f have a unique coincidence point.

Proof. Let $x_0 \in X$ be an arbitrary. Since $T(X) \subset f(X)$, we define a sequence $\{fx_n : n \in \mathbb{N} \cup \{0\}\}$ such that $fx_{n+1} = Tx_n$. Then, by (2.19), we have

$$\begin{aligned} p(fx_{n+1}, fx_{n+2}) \leq &\alpha p(fx_n, fx_{n+1}) + \beta\left[p(fx_n, Tx_n) + p(fx_{n+1}, Tx_{n+1})\right] \\ &+ \gamma\left[p(fx_n, Tx_{n+1}) + p(fx_{n+1}, Tx_n)\right] \\ &+ \delta\left[M_p^f(x_n, x_{n+1}) + hm_p^f(x_n, x_{n+1})\right] \end{aligned}$$

which implies

$$\begin{aligned} p(fx_{n+1}, fx_{n+2}) \leq &\delta\left[\max\left\{p(fx_n, fx_{n+2}), p(fx_{n+1}, fx_{n+1}\right)\right. \\ &\left. + h\min\left\{p(fx_n, fx_{n+2}), p(fx_{n+1}, fx_{n+1})\right\}\right] \\ &+ (\alpha + \beta + \gamma)p(fx_n, fx_{n+1}) \\ &+ (\beta + \gamma)p(fx_{n+1}, fx_{n+2}). \end{aligned} \tag{2.20}$$

If $p(fx_{n+1}, fx_{n+2}) > p(fx_n, fx_{n+1})$ for some n, then (2.20) implies

$$\begin{aligned} p(fx_{n+1}, fx_{n+2}) \leq &\delta\left[\max\left\{p(fx_n, fx_{n+2}), p(fx_{n+1}, fx_{n+1}\right)\right. \\ &\left. + h\min\left\{p(fx_n, fx_{n+2}), p(fx_{n+1}, fx_{n+1})\right\}\right] \\ &+ (\alpha + 2\beta + 2\gamma)p(fx_{n+1}, fx_{n+2}) \end{aligned}$$

Now, in above inequality, either $p(fx_n, fx_{n+2}) \geq p(fx_{n+1}, fx_{n+1})$
or $p(fx_n, fx_{n+2}) \leq p(fx_{n+1}, fx_{n+1})$. But, in both cases, we get $p(fx_{n+1}, fx_{n+2}) < p(fx_{n+1}, fx_{n+2})$, which is not true.

Hence, $p(fx_{n+1}, fx_{n+2}) \leq p(fx_n, fx_{n+1})$ for all n. Also, by (2.20), we have

$$\begin{aligned}
p(fx_{n+1}, fx_{n+2}) \leq& \delta [\max\{p(fx_n, fx_{n+2}), p(fx_{n+1}, fx_{n+1}) \\
& + h \min\{p(fx_n, fx_{n+2}), p(fx_{n+1}, fx_{n+1})\}] \\
& + (\alpha + 2\beta + 2\gamma) p(fx_n, fx_{n+1}).
\end{aligned} \tag{2.21}$$

We consider the following two cases:

Case 1: If $\max\{p(fx_n, fx_{n+2}), p(fx_{n+1}, fx_{n+1})\} = p(fx_n, fx_{n+2})$, then (2.21) implies

$$\begin{aligned}
p(fx_{n+1}, fx_{n+2}) \leq& \delta [2p(fx_n, fx_{n+1}) - (1-h) p(fx_{n+1}, fx_{n+1})] \\
& + (\alpha + 2\beta + 2\gamma) p(fx_n, fx_{n+1}) \\
\leq& (\alpha + 2\beta + 2\gamma + 2\delta) p(fx_n, fx_{n+1}), \quad \text{since } h < 1.
\end{aligned}$$

Case 2: If $\max\{p(fx_n, fx_{n+2}), p(fx_{n+1}, fx_{n+1})\} = p(fx_{n+1}, fx_{n+1})$, then (2.21) implies

$$\begin{aligned}
p(fx_{n+1}, fx_{n+2}) \leq& \delta [p(fx_{n+1}, fx_{n+1}) + hp(fx_n, fx_{n+2})] \\
& + (\alpha + 2\beta + 2\gamma) p(fx_n, fx_{n+1}) \\
\leq& \delta [(1+h) p(fx_n, fx_{n+1})] + (\alpha + 2\beta + 2\gamma) p(fx_n, fx_{n+1}) \\
\leq& (\alpha + 2\beta + 2\gamma + 2\delta) p(fx_n, fx_{n+1}), \quad \text{since } h < 1.
\end{aligned}$$

Thus, in both cases, we conclude that $p(fx_{n+1}, fx_{n+2}) \leq kp(fx_n, fx_{n+1})$, for all $n \in \mathbb{N} \cup \{0\}$, where $k = (\alpha + 2\beta + 2\gamma + 2\delta) < 1$.

We now show that $\{fx_n\}$ is a Cauchy sequence in X. Let $m, n > 0$ with $m > n$, then by (p_4), we have

$$\begin{aligned}
p(fx_n, fx_m) \leq& p(fx_n, fx_{n+1}) + p(fx_{n+1}, fx_{n+2}) + \cdots + p(fx_{n+m-1}, fx_m) \\
& - p(fx_{n+1}, fx_{n+1}) - \ldots - p(fx_{n+m-1}, fx_{n+m-1}) \\
\leq& k^n p(fx_0, fx_1) + k^{n+1} p(fx_0, fx_1) + \cdots + k^{n+m-1} p(fx_0, fx_1) \\
\leq& k^n \left(\frac{1 - k^{m-1}}{1-k} \right) p(fx_0, fx_1).
\end{aligned}$$

Making $n, m \to \infty$, we get $\lim_{n,m\to\infty} p(fx_n, fx_m) = 0$. Hence $\{fx_n\}$ is Cauchy sequence in X, so by Lemma1.6.4, it is also a Cauchy sequence in the metric space (X, p^s). Thus the sequence $\{fx_n\}$ will converge to $x \in X$ because (X, p) is complete. Moreover,

$$\lim_{n,m\to\infty} p\left(fx_n, fx_m\right) = \lim_{n\to\infty} p\left(fx_n, fx\right) = p(fx, fx) = 0 \tag{2.22}$$

Further, we show that x is a coincidence point of f and T. Also, for every n, we have

$$p(fx, Tx) \leq p\left(fx, Tx_n\right) + p\left(Tx_n, Tx\right) - p\left(Tx_n, Tx_n\right).$$

Taking $n \to \infty$ in this inequality, we get $p(fx, Tx) \leq p(Tx, Tx)$. Hence, by (p_2) (the second condition of PMS), we get

$$p(fx, Tx) = p(Tx, Tx). \tag{2.23}$$

Again from (2.19), we have

$$\begin{aligned}
p(fx, Tx) \leq & p\left(fx, Tx_n\right) + p\left(Tx_n, Tx\right) - p\left(Tx_n, Tx_n\right) \\
\leq & p\left(fx, fx_{n+1}\right) + \alpha p\left(fx_n, fx\right) + \beta\left[p\left(fx_n, Tx_n\right) + p(fx, Tx)\right] \\
& + \gamma\left[p\left(fx_n, Tx\right) + p\left(fx, Tx_n\right)\right] + \delta\left[M_p^f\left(x_n, x\right) + h m_p^f\left(x_n, x\right)\right]. \\
\leq & p\left(fx, fx_{n+1}\right) + \alpha p\left(fx_n, fx\right) \\
& + \beta\left[p\left(fx_n, fx\right) + p\left(fx, fx_{n+1}\right) - p(fx, fx) + p(fx, Tx)\right] \\
& + \gamma\left[p\left(fx_n, fx\right) + p(fx, Tx) - p(fx, fx) + p\left(fx, fx_{n+1}\right)\right] \\
& + \delta\left[M_p^f\left(x_n, x\right) + h m_p^f\left(x_n, x\right)\right] \\
\leq & (1+\beta+\gamma) p\left(fx, fx_{n+1}\right) + (\alpha+\beta+\gamma) p\left(fx_n, fx\right) \\
& + (\beta+\gamma) p(fx, Tx) + \delta\left[\max\left\{p\left(fx_n, Tx\right), p\left(fx, Tx_n\right)\right\}\right. \\
& \left. + h \min\left\{p\left(fx_n, Tx\right), p\left(fx, Tx_n\right)\right\}\right].
\end{aligned}$$

Again we consider the following cases:

Case 1: If $\max\left\{p\left(fx_n, Tx\right), p\left(fx, Tx_n\right)\right\} = p\left(fx_n, Tx\right)$, then

$$p(fx, Tx) \leq \left(\frac{1+\beta+\gamma+h\delta}{1-\beta-\gamma-\delta}\right) p\left(fx, fx_{n+1}\right) + \left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\delta}\right) p\left(fx_n, fx\right)$$

Case 2: If $\max\left\{p\left(fx_n, Tx\right), p\left(fx, Tx_n\right)\right\} = p\left(fx, Tx_n\right)$, then

$p(fx, Tx) \leq \left(\frac{1+\beta+\gamma+\delta}{1-\beta-\gamma-h\delta}\right) p\left(fx, fx_{n+1}\right) + \left(\frac{\alpha+\beta+\gamma+h\delta}{1-\beta-\gamma-h\delta}\right) p\left(fx_n, fx\right).$

Thus, taking $n \to \infty$ in both the cases, we get

$$p(fx, Tx) \leq 0 \Rightarrow p(fx, Tx) = 0. \tag{2.24}$$

Using equations (2.22), (2.23) and (2.24), we get

$$p(Tx, Tx) = p(fx, Tx) = p(fx, fx) = 0. \tag{2.25}$$

Hence, by (p_1) (the first condition of PMS) and (2.25), we get $fx = Tx$, i.e., x is a coincidence point of f and T.

To show the uniqueness, we suppose that y is another coincidence point of f and T, i.e., $fy = Ty, fx = Tx$. Then, from (2.19), we get

$$\begin{aligned}
p(Tx, Ty) \leq & \alpha p(fx, fy) + \beta[p(fx, Tx) + p(fy, Ty)] \\
& + \gamma[p(fx, Ty) + p(fy, Tx)] + [\max\{p(fx, Ty), p(fy, Tx)\} \\
& + h \min\{p(fx, Ty), d(fy, Tx)\}] \\
\Rightarrow \quad p(Tx, Ty) \leq & (\alpha + 2\gamma + (1 + h)\delta)d(Tx, Ty),
\end{aligned}$$

which implies $Tx = Ty$, and so $fx = fy$. This completes the proof.

We further provide the following example in the support of Theorem 2.1.5.

Exemple 2.1.5. *Let $X = \left[0, \frac{1}{2}\right]$. Then (X, p), where $p(x, y) = \max\{x, y\}$, is a complete partial metric space. Define mappings f and T on X by*

$$f(x) = x \quad \text{and} \quad T(x) = \frac{x}{2}.$$

Then f and T satisfy all the conditions of the Theorem 2.1.5 , and zero is the only coincidence point of f and T.

Remark 2.1.3. *Theorem 2.1.5 represents Theorem 2.1.4 for $f = I$ (identity mapping). Moreover, Theorem 5.3 of [40], Theorem 3 of [36], Theorem 1 of [30] and Theorem 2.1 of [19] are special cases of Theorem 2.1.5.*

Now we recall the notions of orbitally continuous mappings and orbitally complete partial metric spaces defined by Karapinar and Erhan [33].

Definition 2.1.4. *Let (X,p) be a partial metric space. A mapping $T : X \to X$ is called orbitally continuous if*

$$\lim_{i\to\infty} p\left(T^{n_i}x, z\right) = p(z,z) \Rightarrow \lim_{i\to\infty} p\left(TT^{n_i}x, Tz\right) = p(Tz,Tz), \text{ for each } x \in X.$$

Definition 2.1.5. *A partial metric space (X,p) is called orbitally complete if every Cauchy sequence $\{T^{n_i}x\}$ converges in (X,p), i.e., if*

$$\lim_{i,j\to\infty} p\left(T^{n_i}x, T^{n_j}x\right) = \lim_{i\to\infty} p\left(T^{n_i}x, z\right) = p(z,z), \quad \text{for each } x \in X$$

We further prove the following fixed point result under the assumption of orbitally continuous mapping on orbitally complete partial metric spaces.

Theorem 2.1.6. *Let (X,p) be an orbitally complete partial metric space and let $T : X \to X$ be an orbitally continuous mapping satisfying*

$$\begin{aligned} p(Tx,Ty) \leq & \alpha p(x,y) + \beta[p(x,Tx) + p(y,Ty)] \\ & + \gamma[p(x,Ty) + p(y,Tx)] \\ & + \delta\left[M_p(x,y) + hm_p(x,y)\right] \end{aligned} \tag{2.26}$$

for all $x, y \in X$, where $0 < h < 1$ and $\alpha \geq 0, \beta, \gamma, \delta > 0$ with $\alpha + 2\beta + 2\gamma + 2\delta < 1$, while $M_p(x,y)$ and $m_p(x,y)$ are defined as in Theorem 2.1.4. Then T has a unique fixed point.

Proof. Let x_0 be an arbitrary. We define a sequence $\{x_n \in X : n \in \mathbb{N} \cup \{0\}\}$ such that $x_{n+1} = Tx_n = T^n x_0$. If there exist $n \in \mathbb{N} \cup \{0\}$ such that $p\left(x_n, x_{n-1}\right) = 0$, then by (p_2), we have $p\left(x_{n-1}, x_{n-1}\right) = p\left(x_n, x_n\right)$. Thus, by (p_1)(the first condition of PMS), we get $x_{n-1} = x_n = Tx_{n-1}$. Suppose that $p\left(x_n, x_{n+1}\right) > 0$ for all $n \geq 0$. Now, using (2.26), we get

$$\begin{aligned} p\left(x_{n+1}, x_{n+2}\right) \leq & \alpha p\left(x_n, x_{n+1}\right) + \beta\left[p\left(x_n, x_{n+1}\right) + p\left(x_{n+1}, x_{n+2}\right)\right] \\ & + \gamma\left[p\left(x_n, x_{n+2}\right) + p\left(x_{n+1}, x_{n+1}\right)\right] \\ & + \delta\left[M_p\left(x_n, x_{n+1}\right) + hm_p\left(x_n, x_{n+1}\right)\right]. \end{aligned} \tag{2.27}$$

We consider the following two cases:

Case 1: If $\max\left\{p\left(x_n, x_{n+2}\right), p\left(x_{n+1}, x_{n+1}\right)\right\} = p\left(x_n, x_{n+2}\right)$, then (2.27) implies

$$\begin{aligned}
p\left(x_{n+1}, x_{n+2}\right) \leq & (\alpha+\beta+\gamma+\delta) p\left(x_n, x_{n+1}\right)+(\beta+\gamma+\delta) p\left(x_{n+1}, x_{n+2}\right) \\
& -(1-h) \delta p\left(x_{n+1}, x_{n+1}\right) \\
\leq & \left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\delta}\right) p\left(x_n, x_{n+1}\right) \quad (\text{ since } h<1) \\
= & k_1 p\left(x_n, x_{n+1}\right), \text{ where } k_1=\left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\delta}\right)<1 .
\end{aligned}$$

Case 2: If $\max \left\{p\left(x_n, x_{n+2}\right), p\left(x_{n+1}, x_{n+1}\right)\right\}=p\left(x_{n+1}, x_{n+1}\right)$, then (2.27) implies

$$\begin{aligned}
p\left(x_{n+1}, x_{n+2}\right) & \leq(\alpha+\beta+\gamma+h \delta) p\left(x_n, x_{n+1}\right)+(\beta+\gamma+\delta) p\left(x_{n+1}, x_{n+2}\right) \\
& \leq\left(\frac{\alpha+\beta+\gamma+h \delta}{1-\beta-\gamma-\delta}\right) p\left(x_n, x_{n+1}\right) \\
& =k_2 p\left(x_n, x_{n+1}\right), \text{ where } k_2=\left(\frac{\alpha+\beta+\gamma+h \delta}{1-\beta-\gamma-\delta}\right)<1 .
\end{aligned}$$

Thus, in both cases, we conclude that $p\left(x_{n+1}, x_{n+2}\right) \leq k p\left(x_n, x_{n+1}\right)$ for all $n \in \mathbb{N} \cup\{0\}$, where $k=\min \left\{k_1, k_2\right\}$.

Now, we show that $\left\{x_n\right\}$ is a Cauchy sequence in X. Without loss of generality, we assume that $n>m$. Then, using above inequality and (p_4)(the fourth condition of PMS), we have

$$\begin{aligned}
p\left(x_n, x_{n+m}\right. \leq & \leq p\left(x_n, x_{n+1}\right)+p\left(x_{n+1}, x_{n+2}\right)+p\left(x_{n+2}, x_{n+m}\right) \\
& -p\left(x_{n+1}, x_{n+1}\right)-p\left(x_{n+2}, x_{n+2}\right) \\
\leq & p\left(x_n, x_{n+1}\right)+p\left(x_{n+1}, x_{n+2}\right)+\ldots+p\left(x_{n+m-1}, x_{n+m}\right) \\
\leq & k^n p\left(x_0, x_1\right)+k^{n+1} p\left(x_0, x_1\right)+\ldots+k^{n+m-1} p\left(x_0, x_1\right) \\
\leq & k^n\left(\frac{1-k^{m-1}}{1-k}\right) p\left(x_0, x_1\right) .
\end{aligned}$$

Making $n, m \rightarrow \infty$, we get $\lim_{n, m \rightarrow \infty} p\left(x_n, x_m\right)=0$. Thus, by (p_2)(the second condition of PMS), we have $\lim_{n \rightarrow \infty} p\left(x_n, x_n\right)=0$ and $\lim_{m \rightarrow \infty} p\left(x_m, x_m\right)=0$. Hence $p^s\left(x_n, x_m\right)= 2 p\left(x_n, x_m\right)-p\left(x_n, x_n\right)-p\left(x_m, x_m\right) \rightarrow 0$ as $n \rightarrow \infty$. So, we conclude that $\left\{x_n\right\}=\left\{T^n x_0\right\}$ is a Cauchy sequence in $\left(X, p^s\right)$. Since (X, p) is orbitally complete, the sequence $\left\{T^n x_0\right\}$ converges in the metric space $\left(X, p^s\right)$ to $z \in X$, i.e., $\lim_{n \rightarrow \infty} p^s\left(T^n x_0, z\right)=0$. Again from Lemma 1.6.4 , we have

$$p(z, z)=\lim_{n \rightarrow \infty} p\left(T^n x_0, z\right)=\lim_{n, m \rightarrow \infty} p\left(T^n x_0, T^m x_0\right)=0. \tag{2.28}$$

Suppose that $p(z,Tz)>0$. Since T is orbitally continuous, by (2.28) we get

$$\lim_{n\to\infty} p\left(T^n x_0, z\right) = p(z,z) \Rightarrow \lim_{n\to\infty} p\left(TT^n x_0, Tz\right) = p(Tz,Tz) \tag{2.29}$$

Now, we have

$$\begin{aligned} p(z,Tz) &\leq p\left(z, T^{n+1}x_0\right) + p\left(T^{n+1}x_0, Tz\right) - p\left(T^{n+1}x_0, T^{n+1}x_0\right) \\ &\leq p\left(z, T^{n+1}x_0\right) + p\left(T^{n+1}x_0, Tz\right). \end{aligned}$$

Taking $n\to\infty$ and using Lemma 1.6.5 with (2.29), we get

$$p(z,Tz) \leq \lim_{n\to\infty} p\left(z, x_{n+2}\right) + \lim_{n\to\infty} p\left(T^{n+1}x_0, Tz\right) = p(Tz,Tz)$$

From (p_2)(the second condition of PMS) this is possible only if

$$p(z,Tz) = p(Tz,Tz) \tag{2.30}$$

Hence, by (2.26), we get

$$\begin{aligned} p(z,Tz) \leq & p\left(z, x_{n+1}\right) + p\left(x_{n+1}, Tz\right) - p\left(x_{n+1}, x_{n+1}\right) \\ \leq & p\left(z, x_{n+1}\right) + \alpha p\left(x_n, z\right) + \beta\left[p\left(x_n, x_{n+1}\right) + p(z,Tz)\right] \\ & + \gamma\left[p\left(x_n, Tz\right) + p\left(z, Tx_n\right)\right] + \delta\left[M_p(x,y) + h m_p(x,y)\right]. \end{aligned}$$

We also consider the following two cases:

Case 1: If $M_p(x,y) = p\left(x_n, Tz\right)$, then

$$p(z,Tz) \leq \left(\frac{1+\beta+\gamma+h\delta}{1-\beta-\gamma-\delta}\right) p\left(z, x_{n+1}\right) + \left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\delta}\right) p\left(x_n, z\right)$$

Case 2: If $M_p(x,y) = p\left(z, Tx_n\right)$, then

$$p(z,Tz) \leq \left(\frac{1+\beta+\gamma+\delta}{1-\beta-\gamma-h\delta}\right) p\left(z, x_{n+1}\right) + \left(\frac{\alpha+\beta+\gamma+h\delta}{1-\beta-\gamma-h\delta}\right) p\left(x_n, z\right).$$

Thus, taking $n\to\infty$ in both cases, we get

$$p(z,Tz) \leq 0 \Rightarrow p(z,Tz) = 0. \tag{2.31}$$

Using (2.28), (2.30) and (2.31), we have

$$p(Tz,Tz) = p(z,Tz) = p(z,z) = 0 \tag{2.32}$$

From (p_1)(the first condition of PMS) and (2.32), we get $z = Tz$, i.e., z is a fixed point of T.

Furthermore, let t be another fixed point of T, then from (2.26), we have

$$\begin{aligned} p(z,t) = p(Tz,Tt) \leq & \alpha p(z,t) + \beta[p(z,Tz) + p(t,Tt)] \\ & + \gamma[p(z,Tt) + p(t,Tz)] + \delta[\max\{p(z,Tt), p(t,Tz)\} \\ & + h\min\{p(z,Tt), p(t,Tz)\}] \\ \leq & (\alpha + 2\gamma + (1+h)\delta)p(z,t) \\ p(z,t) \leq & 0 \Rightarrow \quad p(z,t) = 0, \quad \text{that is } z = t, \end{aligned}$$

which completes the proof.

2.2 Multi-valued contraction

2.2.1 Coincidence Point Theorem for a Hybrid Pair of Mappings

Let (X,d) be a metric space. A mapping $T : X \to 2^X$ is multi-valued mapping, where 2^X is collection of all non-empty subsets of X. We denote by $CB(X)(C(X)$ respectively) the family of all nonempty closed bounded (compact respectively) subsets of X. Then a function $H : CB(X) \to \mathbb{R}$, defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\},$$

is Hausdorff metric for $A, B \in CB(X)\,(C(X)$ respectively), where $d(x,B) = \inf_{y\in B} d(x,y)$. Now, we recall some definitions which are the following.

Definition 2.2.1. *[21] An orbit of the multi-valued mapping T at a point $x_0 \in X$ is a sequence $\{x_n : x_n \in Tx_{n-1}\}$. A space X is T-orbitally complete if every Cauchy sequence of the form $\{y_n : y_n \in Ty_{n-1}\}$ converges in X.*

Definition 2.2.2. *[44] Let (X,d) be a metric space, $T : X \to C(X)$, and let S be a self-mapping of X. If, for a point $x_0 \in X$, there exists a sequence $\{x_n\} \subset X$ such that $Sx_{n+1} \in Tx_n, n \in \mathbb{N} \cup \{0\}$, then $O_S(x_0) = \{Sx_n : n = 1, 2, 3, \dots\}$ is an orbit of (T,S) at x_0. A space X is called (T,S)-orbitally complete if every Cauchy sequence of the form*

$\{Sx_n : Sx_n \in Tx_{n-1}\}$ *converges in* X. *Now, we prove the following result by using above mentioned concepts.*

Theorem 2.2.1. *Let* (X, d) *be a metric space, and* $T : X \to C(X)$. *Let* S *be a self-mapping of* X *such that for all* $x, y \in X$,

$$\begin{aligned} H(Tx, Ty) \leq &\alpha d(Sx, Sy) + \beta[(Sx, Tx) + d(Sy, Ty)] \\ &+ \gamma[(Sx, Ty) + d(Sy, Tx)] + \delta\left[M_s(x, y) + hm_s(x, y)\right], \end{aligned} \tag{2.33}$$

where $\alpha \geq 0, \beta, \gamma, \delta > 0$ and $0 < h < 1$ with $\alpha + 2\beta + 2\gamma + 2\delta = 1$ and $M_s(x, y) = \max\{(Sx, Ty), d(Sy, Tx)\}$, $m_s(x, y) = \min\{(Sx, Ty), d(Sy, Tx)\}$. If $T(X) \subset S(X)$, and one of the following holds:

a. X is (T, S)-orbitally complete and S is surjective,

b. $S(X)$ is (T, S)-orbitally complete,

c. $T(X)$ is (T, S)-orbitally complete.

Then S and T have a coincidence point in X, i.e., there exists $z \in X$ such that $Sz \in Tz$.

Proof. *Let* $x_0 \in X$ *and* $y_0 = Sx_0$. *Then, we construct sequences* $\{x_n\}$ *and* $\{y_n\}$ *as follows. Since* $T(X) \subset S(X)$, *we can choose* $y_1 = Sx_1 \in Tx_0$. *If* $Tx_0 = Tx_1$, *choose* $y_2 = Sx_2 \in Tx_1$ *such that* $y_1 = y_2$. *If* $Tx_0 \neq Tx_1$, *choose* $y_2 = Sx_2 \in Tx_1$ *such that* $d(y_1, y_2) \leq H(Tx_0, Tx_1)$, *such a choice is possible because* Tx *is compact for each* $x \in X$. *In general, choose* $y_{n+1} = \bar{S}x_{n+1} \in Tx_n$, *for each* $n \in \mathbb{N} \cup \{0\}$, *and* $y_{n+1} = y_{n+2}$ *if* $Tx_n = Tx_{n+1}$ *and* $d(y_{n+1}, y_{n+2}) \leq H(Tx_n, Tx_{n+1})$, *otherwise. Now, from (2.33) implies*

$$\begin{aligned} d(y_{n+1}, y_{n+2}) \leq &H(Tx_n, Tx_{n+1}) \\ \leq &\alpha d(Sx_n, Sx_{n+1}) + \beta\left[d(Sx_n, Tx_n) + d(Sx_{n+1}, Tx_{n+1})\right] \\ &+ \gamma\left[d(Sx_n, Tx_{n+1}) + d(Sx_{n+1}, Tx_n)\right] + \delta\left[M_s(x_n, x_{n+1}) + m_s(x_n, x_{n+1})\right] \\ \leq &\alpha d(y_n, y_{n+1}) + \beta\left[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})\right] \\ &+ \gamma\left[d(y_n, y_{n+2}) + d(y_{n+1}, y_{n+1})\right] \\ &+ \delta\left[\max\left\{d(y_n, y_{n+2}), d(y_{n+1}, y_{n+1})\right\} + \min\left\{d(y_n, y_{n+2}), d(y_{n+1}, y_{n+1})\right\}\right]. \end{aligned}$$

Suppose that $d(y_{n+1}, y_{n+2}) > d(y_n, y_{n+1})$ *for some* n*. Then, substituting in the above inequality, we have*

$$\begin{aligned} d(y_{n+1}, y_{n+2}) \leq& \alpha d(y_n, y_{n+1}) + \beta [d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})] \\ &+ \gamma [d(y_n, y_{n+2}) + d(y_{n+1}, y_{n+1})] + \delta [\{d(y_n, y_{n+2}) + d(y_{n+1}, y_{n+1})\}]. \end{aligned}$$

It implies that $d(y_{n+1}, y_{n+2}) < d(y_{n+1}, y_{n+2})$*, which is a contradiction. Therefore,*

$$d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1}) \text{ for all } n. \tag{2.34}$$

Now, if $d(Sx_{n-1}, Sx_{n+1}) \geq d(Sx_{n-1}, Tx_{n-1})$*, then the condition (2.33) implies (with* $h = 1$ *),*

$$\begin{aligned} d(y_n, y_{n+2}) \leq& H(Tx_{n-1}, Tx_{n+1}) \\ \leq& \alpha d(Sx_{n-1}, Sx_{n+1}) + \beta [d(Sx_{n-1}, Tx_{n-1}) + d(Sx_{n+1}, Tx_{n+1})] \\ &+ \gamma [d(Sx_{n-1}, Tx_{n+1}) + d(Sx_{n+1}, Tx_{n-1})] \\ &+ \delta [M_s(x_{n-1}, x_{n+1}) + m_s(x_{n-1}, x_{n+1})] \\ \leq& \alpha d(y_{n-1}, y_{n+1}) + \beta [d(y_{n-1}, y_n) + d(y_{n+1}, y_{n+2})] \\ &+ \gamma [d(y_{n-1}, y_{n+2}) + d(y_{n+1}, y_n)] + \delta [\max\{d(y_{n-1}, y_{n+2}), d(y_{n+1}, y_n)\} \\ &+ \min\{d(y_{n-1}, y_{n+2}), d(y_{n+1}, y_n)\}]. \end{aligned}$$

Using (2.34) and triangular, we obtain

$$d(y_n, y_{n+2}) \leq (2\alpha + 2\beta + 2\gamma + 2\delta) d(y_{n-1}, y_n) + (\gamma + \delta) d(y_n, y_{n+2})$$

$$\text{it implies } d(y_n, y_{n+2}) \leq \left(\frac{1+\alpha}{1-\gamma-\delta}\right) d(y_{n-1}, y_n).$$

Thus, $d(y_n, y_{n+2}) \leq k_1 d(y_{n-1}, y_n)$*, where* $k_1 = \frac{(1+\alpha)}{(1-\gamma-\delta)} < 2$*.*

Now, if $d(Sx_{n-1}, Sx_{n+1}) < d(Sx_{n-1}, Tx_{n-1})$*, then the condition (2.33) implies.*

$$\begin{aligned} d(y_n, y_{n+1}) \leq& H(Tx_{n-1}, Tx_n) \\ \leq& \alpha d(Sx_{n-1}, Sx_n) + \beta [d(Sx_{n-1}, Tx_{n-1}) + d(Sx_n, Tx_n)] \\ &+ \gamma [d(Sx_{n-1}, Tx_n) + d(Sx_n, Tx_{n-1})] \\ &+ \delta [M_s(x_{n-1}, x_n) + m_s(x_{n-1}, x_n)] \\ \leq& \alpha d(y_{n-1}, y_n) + \beta [d(y_{n-1}, y_n) + d(y_n, y_{n+1})] \\ &+ \gamma [d(y_{n-1}, y_{n+1}) + d(y_n, y_n)] + \delta [\max\{d(y_{n-1}, y_{n+1}), d(y_n, y_n)\} \\ &+ \min\{d(y_{n-1}, y_{n+1}), d(y_n, y_n)\}]. \end{aligned}$$

It implies that

$$d(y_n, y_{n+1}) < \left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta}\right) d(y_{n-1}, y_n).$$

So

$$\begin{aligned} d(y_n, y_{n+2}) &\leq d(y_n, y_{n+1}) + (y_{n+1}, y_{n+2}) \leq 2d(y_n, y_{n+1}) \\ &< \frac{2(\alpha+\beta+\gamma+\delta)}{(1-\beta)} d(y_{n-1}, y_n) = \frac{(1+\alpha)}{(1-\beta)} d(y_{n-1}, y_n). \end{aligned}$$

Thus $d(y_n, y_{n+2}) \leq k_2 d(y_{n-1}, y_n)$*, where* $k_2 = \frac{(1+a)}{(1-\beta)} < 2$.

Taking $k = \max\{k_1, k_2\}$*, we get* $0 < k < 2$*, and*

$$d(y_n, y_{n+2}) \leq kd(y_{n-1}, y_n), \text{ for all } n \in \mathbb{N}. \tag{2.35}$$

From inequalities (2.33),(2.34) and (2.35), we get

$$\begin{aligned} d(y_{n+1}, y_{n+2}) \leq &\operatorname{cd}(y_n, y_{n+1}) + \beta[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})] \\ &+ \gamma[d(y_n, y_{n+2}) + d(y_{n+1}, y_{n+1})] + \delta[\max\{d(y_n, y_{n+2}), d(y_{n+1}, y_{n+1})\} \\ &+ \min\{d(y_n, y_{n+2}), d(y_{n+1}, y_{n+1})\}] \\ \leq &(\alpha + 2\beta + k(\gamma+\delta)) d(y_{n-1}, y_n). \end{aligned}$$

Thus,

$$d(y_{n+1}, y_{n+2}) \leq \lambda d(y_{n-1}, y_n), \forall n \in \mathbb{N}, \tag{2.36}$$

where $\lambda = (\alpha + 2\beta + k(\gamma+\delta)) < 1$*. Using induction and (2.34) and (2.36), we obtain* $d(y_n, y_{n+1}) \leq \lambda^{\left\lfloor \frac{1}{2} \right\rfloor} d(y_0, y_1)$*, for all* $n \in \mathbb{N}$*, where* $\left[\frac{n}{2}\right]$ *denotes the greatest integer value. Hence,* $\{y_n\}$ *is a Cauchy sequence in* X*, and it is convergent to a point* $p \in X$ *in cases (a)-(c). Now, if* S *is surjective, then there exists a point* $z \in X$ *such that* $p = Sz$*. This is obviously true in cases (b) and (c). From (2.33),*

$$\begin{aligned} d(Sz, Tz) \leq &d(Sz, Sx_{n+1}) + H(Tx_n, Tz) \\ \leq &d(Sz, Sx_{n+1}) + \alpha d(Sx_m, Sz) + \beta[d(Sx_n, Tx_n) + d(Sz, Tz)] \\ &+ \gamma[d(Sx_n, Tz) + d(Sz, Tx_n)] + \delta[M_s(x_n, z) + m_n(x_n, z)] \end{aligned}$$

Then

$$\begin{aligned} d(p, Tz) \leq &d(p, y_{n+1}) + \operatorname{cod}(y_n, p) + \beta[d(y_n, y_{n+1}) + d(p, Tz)] \\ &+ \gamma[d(y_n, Tz) + d(p, y_{n+1})] + [\max\{d(y_n, Tz), d(p, y_{n+1})\} \\ &+ \min\{d(y_n, Tz), d(p, y_{n+1})\}]. \end{aligned}$$

Taking the limit as $n \to \infty$, *we have* $d(p, Tz) \leq (\beta + \gamma + \delta)d(p, Tz)$ *which implies that* $Sz \in Tz$. *Hence,* S *and* T *have a coincidence point.*

2.2.2 Partial hausdorff metric and a space orbitally complete

In 2012, Aydi et al[9]. introduced the concept of a partial Hausdorff metric with $CB^p(X)$, *the collection of all non-empty closed and bounded subsets of* X *with respect to partial metric* p. *Thereafter, in 2016, Abdessalem Benterki [15] used the concept of collection of closed subsets of* X *with respect to partial metric* p *say,* $CL^p(X)$, *instead of closed and bounded subsets of* X. *For any* $A, B, C \in CL^p(X)$, *resp.* $CB^p(X)$.

1. $p(a, C) = \inf\{p(a, x) : x \in C\}$.

2. $\delta_p(A, B) = \sup\{p(a, B) : a \in A\}$.

3. $\delta_p(B, A) = \sup\{p(b, A) : b \in B\}$.

4. $\rho_p(A, B) = \sup\{p(a, b) : a \in A, b \in B\}$.

Moreover, for all $A, B \in CL^p(X)$ *(resp.* $CB^p(X)$ *), we have*

$$H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\}.$$

Then H_p *is called the generalized partial Hausdorff metric, resp. partial Hausdorff metric, induced by* p.

We also cite the following definitions which are very useful to our next result.

Definition 2.2.3. *Let* T *be a self mapping of a partial metric space* X. *Then, an orbit of* T *at a point* $x_0 \in X$ *is a sequence* $\{x_n : x_n \in Tx_{n-1}\}$.

Definition 2.2.4. *Let* T *be a multi-valued mapping on a partial metric space* X. *Then* X *is said to be* T*-orbitally complete if every Cauchy sequence of the form* $\{x_n : x_n \in Tx_{n-1}\}$ *is convergent in* X *in the sense of the partial metric.*

Definition 2.2.5. *Let* f *and* T *be a self mapping and a multi-valued mapping on a partial metric space* (X, p), *respectively. If, for* $x_0 \in X$, *there exists a sequence* $\{x_n\}$ *such that* $fx_n \in$

$Tx_{n-1}, n \in \mathbb{N}$, then $O_f(x_0) = \{fx_n : n \in \mathbb{N}\}$ is called the orbit of (T, f) at x_0. Furthermore, $O_f(x_0)$ is called a regular orbit of (T, f) at x_0, if $p(fx_n, fx_{n+1}) \leq H_p(Tx_{n-1}, Tx_n)$, for every $n \in \mathbb{N}$.

Definition 2.2.6. *Let f and T be a self mapping and a multi-valued mapping on a partial metric space (X, p), respectively. Then X is said to be (T, f)-orbitally complete if every Cauchy sequence of the form $\{fx_n : fx_n \in Tx_{n-1}\}$ is convergent in the sense of the partial metric.*

Lemma 2.2.1. *[9] Let (X, p) be a partial metric space. Then,*

1. *(X, p) is T-orbitally complete if and only if (X, p^s) is T-orbitally complete;*
2. *(X, p) is (T, f)-orbitally complete if and only if (X, p^s) is (T, f) orbitally complete.*

Lemma 2.2.2. *[10] . Let $A, B \in CL^p(X)$ and $a \in A$. Then, for any $\epsilon > 0$ there exists a point $b \in B$ such that $p(a, b) \leq H_p(A, B) + \epsilon$.*

Furthermost, we prove the following result for a pair of mappings under the assumption of (T, f)-orbitally complete as follows.

Theorem 2.2.2. *Let (X, p) be a partial metric spaces. Suppose that $T : X \to CL^p(X)$ and $f : X \to X$ are such that $T(X) \subseteq f(X)$. If $f(X)$ is (T, f) orbitally complete and, for all $x, y \in X$, the following condition is satisfying:*

$$\begin{aligned} H_p(Tx, Ty) \leq & \alpha p(fx, fy) + \beta[p(fx, Tx) + p(fy, Ty)] \\ & + \gamma[p(fx, Ty) + p(fy, Tx)] \\ & + \delta\left[M_p^f(x, y) + h m_p^f(x, y)\right] \end{aligned} \tag{2.37}$$

where $0 < h < 1$ and $\alpha \geq 0, \beta, \gamma, \delta > 0$ with $\alpha + 2\beta + 2\gamma + 2\delta \leq k < 1$, while $M_p^f(x, y)$ and $m_p^f(x, y)$ are defined as in Theorem 2.1.5. Then T and f have a coincidence point, i.e., there exists a point $z \in X$ such that $fz \in Tz$.

Proof. *Let $\epsilon > 0$ such that $\mu = k + \epsilon < 1$, and x_0 be an arbitrary point of X.*
Since $T(X) \subset f(X)$, we choose a point $x_1 \in X$ such that $fx_1 \in Tx_0$.
Clearly $p(fx_1, Tx_1) \geq 0$. If $p(fx_1, Tx_1) = 0$, then nothing to prove because in this case

$fx_1 \in Tx_1$.

Assume $p(fx_1, Tx_1) > 0$, *then by Lemma 2.2.2, there exists* $fx_2 \in Tx_1$ *such that*

$$p(fx_1, fx_2) \leq H_p(Tx_0, Tx_1) + \epsilon p(fx_1, Tx_1)$$

. *Similarly, if* $p(fx_2, Tx_2) > 0$, *then there exists* $fx_3 \in Tx_2$ *such that* $p(fx_2, fx_3) \leq H_p(Tx_1, Tx_2) + \epsilon p(fx_2, Tx_2)$. *Continuing this process, we construct a sequence* $\{fx_n\} \subset f(X)$ *such that* $p(fx_{n+1}, fx_{n+2}) \leq H_p(Tx_n, Tx_{n+1}) + \epsilon p(fx_n, Tx_n)$ *and* $fx_{n+1} \in Tx_n$ *for all* $n \in \mathbb{N} \cup \{0\}$. *Using (2.37), we get*

$$\begin{aligned}
p(fx_{n+1}, fx_{n+2} \leq & H_p(Tx_n, Tx_{n+1}) + \epsilon p(fx_n, Tx_n) \\
\leq & \alpha p(fx_n, fx_{n+1}) + \beta[p(fx_n, Tx_n) + p(fx_{n+1}, Tx_{n+1})] \\
& + \gamma[p(fx_n, Tx_{n+1}) + p(fx_{n+1}, Tx_n)] \\
& + \delta[\max\{p(fx_n, Tx_{n+1}), p(fx_{n+1}, Tx_n)\} \\
& + h\min\{p(fx_n, Tx_{n+1}), p(fx_{n+1}, Tx_n)\}] \\
& + \epsilon p(fx_n, Tx_n) \\
\leq & \alpha p(fx_n, fx_{n+1}) + \beta[p(fx_n, fx_{n+1}) + p(fx_{n+1}, fx_{n+2})] \\
& + \gamma[p(fx_n, fx_{n+2}) + p(fx_{n+1}, Tx_n)] + \delta[p(fx_n, fx_{n+2}), \\
& + hp(fx_{n+1}, Tx_n)] + \epsilon p(fx_n, fx_{n+1}) \\
\leq & \alpha p(fx_n, fx_{n+1}) + \beta[p(fx_n, fx_{n+1}) + p(fx_{n+1}, fx_{n+2})] \\
& + \gamma p(fx_n, fx_{n+2}) + \delta p(fx_n, fx_{n+2}) + \epsilon p(fx_n, fx_{n+1}).
\end{aligned}$$

Now, if $p(fx_n, fx_{n+1}) < p(fx_{n+1}, fx_{n+2})$ *for some* $n \in \mathbb{N}$,
then $p(fx_{n+1}, fx_{n+2}) \leq \alpha p(fx_{n+1}, fx_{n+2}) < p(fx_{n+1}, fx_{n+2})$,
which is a contradiction.

Hence $p(fx_{n+1}, fx_{n+2}) \leq p(fx_n, fx_{n+1})$ *for all* $n \in \mathbb{N} \cup \{0\}$,
and so

$$p(fx_{n+1}, fx_{n+2}) \leq \mu p(fx_n, fx_{n+1})$$

Continuing in this way, we get

$$p(fx_{n+1}, fx_{n+2}) \leq \mu^n p(fx_0, fx_1).$$

Since $\mu < 1$, taking $n \to \infty$, we get $p(fx_{n+1}, fx_{n+2}) \to 0$ implies $\{fx_n\}$ is a Cauchy sequence in $f(X)$. Since $f(X)$ is (T, f)-orbitally complete, the sequence $\{fx_n\}$ is convergent to a point $a \in f(X)$, that is,

$$p(fx_{n+1}, fx_{n+2}) = p(a, a) = 0. \tag{2.38}$$

Also, $a \in f(X)$ implies that there exists a point $z \in X$ such that $fz = a$.

Now, we get

$$\begin{aligned} p(fz, Tz) \leq & p(fz, fx_{n+1}) + p(fx_{n+1}, Tz) \\ \leq & p(fz, fx_{n+1}) + H_p(Tx_n, Tz) \\ \leq & p(fz, fx_{n+1}) + \alpha p(fx_n, fz) + \beta [p(fx_n, Tx_n) + p(fz, Tz)] \\ & + \gamma [p(fx_n, Tz) + p(fz, Tx_n)] + \delta [M_p(x, y) + hm_p(x, y)]. \end{aligned}$$

We now consider the following cases.

***Case 1**: If $M_p(x, y) = p(fx_n, Tz)$, then*

$$\begin{aligned} p(fz, Tz) \leq & p(fz, fx_{n+1}) + \alpha p(fx_n, fz) + \beta [p(fx_n, fx_{n+1}) + p(fz, Tz)] \\ & + \gamma [p(fx_n, Tz) + p(fz, fx_n) + p(fx_n, f_{n+1})] + \delta [p(fx_n, Tz) \\ & + h \{p(fz, fx_n) + p(fx_n, fx_{n+1})\}]. \end{aligned}$$

Making $n \to \infty$, we get $p(fz, Tz) \leq (\beta + \gamma + \delta) p(fz, Tz)$ which implies $p(fz, Tz) = 0$, since $\beta + \gamma + \delta < 1$.

***Case 2**: If $M_p(x, y) = p(fz, Tx_n)$, then*

$$\begin{aligned} p(fz, Tz) \leq & p(fz, fx_{n+1}) + \alpha p(fx_n, fz) + \beta [p(fx_n, fx_{n+1}) + p(fz, Tz)] \\ & + \gamma [p(fx_n, Tz) + p(fz, fx_n) + p(fx_n, fx_{n+1})] + \delta [p(fz, fx_n) \\ & + p(fx_n, fx_{n+1}) + hp(fx_n, Tz)]. \end{aligned}$$

Making $n \to \infty$, we get $p(fz, Tz) \leq (\beta + \gamma + h\delta) p(fz, Tz)$ which implies $p(fz, Tz) = 0$, since $\beta + \gamma + h\delta < 1$.

Thus, in both cases, we conclude that $fz \in Tz$, i.e., z is the coincidence point of f and T.

Remark 2.2.1. *Theorem2.2.2 is a proper generalization of Theorem2.1.6 whenever* $f = I$ *identity mapping and* X *is* T*-orbitally complete.*

CHAPTER

3

APPLICATIONS

In this chapter we will applicate some properties and theorems that were presenting in previous chapter .

3.1 Application to Dynamic Programming

In this paragraph, we consider the functional equation given by Bellman and Lee [14] and find solution using our condition.

Definition 3.1.1. *A functional is a generalization of the notion of a function of a finite number of numerical variables, defined by Aristotle D. Michal.*

Definition 3.1.2. *Functional equations are equations in which the unknowns are functions. Some examples of functional equations are*

1. $f(x + y) = f(x)g(y) + g(x)f(y)$.

2. $f(x+y) + f(x-y) = 2f(x)f(y)$.

3. $f(x+y) + f(x-y) = 2f(x) + 2f(y)$.

4. $f(x + y) = f(x) + f(y) + f(x)f(y)$.

3.1.1 Dynamic programing

A dynamical system is a function with an attitude. A dynamical system is doing the same thing over and over again. A dynamical system is always knowing whatyou are going to do next. The difficulty is that virtually anything that evolvesover time can be thought of as a dynamical system. So let us begin by describingmathematical dynamical systems and then see how many physical situations arenicely modeled by mathematical dynamical systems. A dynamical system has two parts: a state vector which describes exactly thestate of some real or hypothetical system, and a function (i.e., a rule) which tellsus, given the current state, what the state of the system will be in the next instant of time.

Definition 3.1.3. *A dynamical system is a pair* (U, T) *were* U *is any set and* T *is a self-map on* U.

Definition 3.1.4. *Dynamic Programming (DP) is an optimization technique based on decomposition of acomplex optimization problem into a sequence of simpler problems in such a way that thetotal time needed to solve them is smaller than the time needed to solve the original problem*

3.1.2 Fixed point theorem for a solution to functional equation of dynamic programming

Let U *and* V *be Banach spaces and* $W \subset U, D \subset V$. *Let* $B(W)$ *denote the set of all bounded real valued functions on* W,

$$B(W) = \{f \quad is \quad function \quad suchthat \quad f : W \to \mathbb{R} \quad and|f(x)| < \infty\}$$

. *It is well known that* $B(W)$ *endowed with the metric,*

$$d_B(h, k) = \sup_{x \in W} |h(x) - k(x)|, \quad h, k \in B(W) \tag{3.1}$$

is a complete metric space. Bellman and Lee ([14])introduced the following basic form of the functional equation of dynamic programming:

$$p(x) = \sup_{y} H(x, y, p(\tau(x, y))), \tag{3.2}$$

where x and y respresent the state and decision vectors respectively, and $\tau : W \times D \to W$ represents the transformation of the process and $p(x)$ represents the optimal return function with initial state x. Now, we will study the existence and uniqueness of the solution of the following functional equation:

$$p(x) = \sup_{y}[g(x, y) + G(x, y, p(\tau(x, y)))], \quad x \in W \tag{3.3}$$

where $g : W \times D \to \mathbb{R}$ and $G : W \times D \times \mathbb{R} \to \mathbb{R}$ are bounded functions. Let $T : B(W) \to B(W)$ be a mapping defined by

$$T(h(x)) = \sup_{y}\{g(x, y) + G(x, y, h(\tau(x, y)))\}, \tag{3.4}$$

where $h \in B(W)$ and $x \in W$.

Theorem 3.1.1. *If there exist $\alpha \geq 0, \beta, \gamma, \delta > 0$ and $0 < h < 1$ such that $\alpha+2\beta+2\gamma+2\delta = 1$, and*

$$|G(x, y, h(t)) - G(x, y, k(t))| \leq M(h(t), k(t)), \tag{3.5}$$

for every $(x, y) \in W \times D, h, k \in B(W)$ and $t \in W$, where

$$\begin{aligned} M(h(t), k(t)) =& \alpha|h(t) - k(t)| + \beta\{|h(t) - T(h(t))| + |k(t) - T(k(t))|\} \\ &+ \gamma\{|h(t) - T(k(t))| + |k(t) - T(h(t))|\} \\ &+ \delta\{\max\{|h(t) - T(k(t))|, |k(t) - T(h(t))|\} \\ &+ h \min\{|h(t) - T(k(t))|, |k(t) - T(h(t))|\}\} \end{aligned}$$

then the functional equation (3.3) has a unique bounded solution in $B(W)$.

Proof. *Let $h, k \in B(W)$. Then for $x \in W$, we can choose $y_1, y_2 \in D$ so that*

$$T(h(x)) = \sup_{y_1}[g(x, y_1) + G(x, y_1, h(\tau(x, y_1)))]. \tag{3.6}$$

$$T(k(x)) = \sup_{y_2}[g(x, y_2) + G(x, y_2, h(\tau(x, y_2)))]. \tag{3.7}$$

we will applicate the properities of supermum, so: $\exists \quad \epsilon > 0]$ *such that*

$$T(h(x)) < g(x, y_1) + G(x, y_1, h(\tau(x, y_1))) + \epsilon, \tag{3.8}$$

$$T(k(x)) < g(x, y_2) + G(x, y_2, k(\tau(x, y_2))) + \epsilon. \tag{3.9}$$

Also from (3.4),

$$T(h(x)) \geq g(x, y_2) + G(x, y_2, h(\tau(x, y_2))), \tag{3.10}$$

$$T(k(x)) \geq g(x, y_1) + G(x, y_1, k(\tau(x, y_1))). \tag{3.11}$$

If the inequality (3.5) holds, from inequalities (3.8) and (3.11) we obatine

$$-T(k(x)) \leq -g(x, y_1) - G(x, y_1, k(\tau(x, y_1))). \tag{3.12}$$

we add(3.8) with (3.12), so we produce for us

$$\begin{aligned} T(h(x)) - T(k(x)) &< G(x, y_1, h(\tau(x, y_1))) - G(x, y_1, k(\tau(x, y_1))) + \epsilon \\ &\leq |G(x, y_1, h(\tau(x, y_1))) - G(x, y_1, k(\tau(x, y_1)))| + \epsilon \\ &\leq M(h(x), k(x)) + \epsilon. \end{aligned} \tag{3.13}$$

Similarly from (3.9) and (3.10), we obatin

$$T(k(x)) - T(h(x)) \leq M(h(x), k(x)) + \epsilon. \tag{3.14}$$

From (3.13) and (3.14), we establish

$$|T(h(x)) - T(k(x))| \leq M(h(x), k(x)) + \epsilon,$$

for each $x \in W$, *and for arbitrary* $\epsilon > 0$.
Thus, $d_B(T(h), T(k)) \leq M(h, k)$, *where* $\alpha, \delta > 0, \beta, \gamma \geq 0$ *such that* $\alpha + 2\beta + 2\gamma + 2\delta = 1$.
Moreover, the conditions of Theorem2.1.1 are satisfied for the mapping T.
Hence, the functional equation (3.3) has a unique bounded solution.

3.2 Application to integral equation

In this paragraph, we study the existence of the solution to a nonlinear integral equation by using a result obtained in the previous chapter.

3.2.1 Integral equation

([52]) J. Fourier is the initiator of the theory of integral equations. A term integral equation first suggested by Du Bois-Reymond in 1888. Du BoisReymond define an integral equation is understood an equation in which the unknown function occurs under one or more signs of definite integration. Late eighteenth and early ninetieth century Laplace, Fourier, Poission, Liouville and Able studies some special type of integral equation. The pioneering systematic investigations goes back to late nineteenth and early twentieth century work of volterra, Fredholm and Hilbert. In 1887, Volterra published a series of famous papers in which he singled out the notion of a functional and pioneered in the development of a theory of functional s in theory of linear integral equation of special type. Fredholm presented the fundamentals of the Fredholm integral equation theory in a paper published in 1903 in the Acta Mathematica. This paper became famous almost overnight and soon took its rightful place among the gems of modern mathematics. Hilbert followed Fredholm's famous paper with a series of papers in the Nachrichten of the GUttingen Academy.

Definition 3.2.1. *([52]) Integral equation is an equation in which the unknown, say a function of anumerical variable, occurs under an integral. That means a functional equationinvolving the unknown function under one or more integrals.*

Exemple 3.2.1.

$$u(x) = f(x) + 5\int_0^1 e^{x+t}u(t)dt$$

3.2.2 Solution of integral equation

A solution of integral equation is a function that satisfies the original integral equation. Verify that the given functions are solutions of the corresponding integral equations and check it kinds.

Exemple 3.2.2. *we consider integral equation.*

$$u(x) = x + \int_0^x \sin(x-t)u(t)dt$$

this solution is.

$$u(x) = x - \frac{x^3}{6}$$

3.2.3 Fixed point theorem for integral equation

First, we consider the following Hammerstein integral equation.

$$u(t) = \int_0^K G(t,s)f(s,u(s))ds, \quad \text{for all } t \in [0,K] \tag{3.15}$$

where $K > 0, f : [0,K] \times \mathbb{R}^+ \to \mathbb{R}^+$ *and* $G : [0,K] \times [0,K] \to \mathbb{R}^+$ *are continuous functions. Let* $X = C([0,K])$ *be the set of positive real continuous functions on* $[0,K]$.

Remark 3.2.1. *It is a clear that* $u : C[0,K] \to \mathbb{R}^+$ *because* $u(s) \in \mathbb{R}^+$ *conditions of* f *and set of start for* u *is* $[0,K]$ *such as* $t,s \in [0,K]$
Other side G *and* f *is continuous imply* u *is continuous .*

We endow X *with partial metric* D_p *defined by*

$$D_p(u,v) = \max_{t\in[0,K]}\{u(t),v(t)\}, \quad \text{for all } u,v \in X$$

Obviously, (X, D_p) *is a complete partial metric space.*
Let $(\alpha,\beta) \in X\times X, (\alpha_0,\beta_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ *we have* α *and* β *is function but* α_0, β_0 *it is real values be such that*

$$\alpha_0 \leq \alpha(t) \leq \beta(t) \leq \beta_0, \quad \text{for all } t \in [0,K]. \tag{3.16}$$

We can writing on the form

$$\alpha(t) = \int_0^K G(t,s)f(s,\alpha(s))ds$$

and

$$\beta(t) = \int_0^K G(t,s)f(s,\beta(s))ds$$

Assume that for all $t \in [0, K]$, we have

$$\alpha(t) \leq \int_0^K G(t,s)f(s,\beta(s))ds \tag{3.17}$$

and

$$\beta(t) \geq \int_0^K G(t,s)f(s,\alpha(s))ds \tag{3.18}$$

Let $f(s,\cdot)$ be a decreasing function for all $s \in [0, K]$, that is,

$$x, y \in \mathbb{R}^+ \text{with } y \leq x \Rightarrow f(s,x) \leq f(s,y). \tag{3.19}$$

Assume that

$$\max_{t\in[0,K]} \int_0^K G(t,s)ds \leq 1 \tag{3.20}$$

Moreover, for any mapping $T : X \to X$, we suppose that for all $s \in [0, K]$, and for all $x, y \in \mathbb{R}^+$with $(x \leq \beta_0$ and $y \geq \alpha_0)$ or $(x \geq \alpha_0$ and $y \leq \beta_0)$,

$$\begin{aligned} \max\{f(s,x), f(s,y)\} \leq & \alpha \max\{x,y\} \\ & + \beta(\max\{x,Tx\} + \max\{y,Ty)\}) \\ & + \gamma(\max\{x,Ty\} + \max\{y,Tx\}) \\ & + \delta(M(x,y) + hm(x,y)), \end{aligned} \tag{3.21}$$

where $0 < h < 1$ and $\alpha \geq 0, \beta, \gamma, \delta > 0$ such that $\alpha + 2\beta + 2\gamma + 2\delta < 1$, and

$$M(x,y) = \max\{\max\{x,Ty\}, \max\{y,Tx\}\},$$
$$m(x,y) = \min\{\max\{x,Ty\}, \max\{y,Tx\}\}.$$

Theorem 3.2.1. *Under assumptions (3.16)-(3.21), the integral equation (3.15) has a solution in $\{u \in C([0,K]) : \alpha \leq u(t) \leq \beta,$ for all $t \in [0,K]\}$.*

Proof. *We consider the closed subsets A_1 and A_2 of X, defined by*

$$A_1 = \{u \in X : u \leq \beta\}$$

and

$$A_2 = \{u \in X : u \geq \alpha\}$$

Also, we define a mapping $T : X \to X$ *by*

$$Tu(t) = \int_0^K G(t,s)f(s,u(s))ds, \quad \text{for all } t \in [0,K]$$

Let us prove that

$$T(A_1) \subset A_2 \quad \text{and} \quad T(A_2) \subset A_1. \tag{3.22}$$

Part one *(proof*$T(A_1) \subset A_2$*): Suppose that* $u \in A_1$*, that is,* $u(s) \leq \beta(s)$ *for all* $s \in [0,K]$*. Since* $G(t,s) \geq 0$ *for all* $t, s \in [0,K]$*, and* $f(s,.)$*is a function decreasing 3.19 we obtain*

$$G(t,s)f(s,u(s)) \geq G(t,s)f(s,\beta(s)), \quad \text{for all } t,s \in [0,K]$$

Now ,we enter the integration in the side of inequality.

$$\int_0^K G(t,s)f(s,u(s))ds \geq \int_0^K G(t,s)f(s,\beta(s))ds, \quad \text{for all } t \in [0,K].$$

On other side we know that.

$$\int_0^K G(t,s)f(s,\beta(s))ds \geq \alpha(t), \quad \text{for all } t \in [0,K].$$

with all this result we get.

$$\int_0^K G(t,s)f(s,u(s))ds \geq \int_0^K G(t,s)f(s,\beta(s))ds \geq \alpha(t), \quad \text{for all } t \in [0,K].$$

Hence, $Tu \in A_2$*.*

Part two $(proof T(A_2) \subset A_1)$

Similarly, let $u \in A_2$*, that is,* $u(s) \geq \alpha(s)$*, for all* $s \in [0, K]$*. Since* $G(t, s) \geq 0$ *for all* $t, s \in [0, K]$*, and* $f(s, .)$*is a function decreasing by 3.19 we obtain*

$$G(t,s)f(s,u(s)) \leq G(t,s)f(s,\alpha(s)), \qquad \textit{for all } t,s \in [0,K].$$

Now ,we enter the integration in the side of inequality.

$$\int_0^K G(t,s)f(s,u(s))ds \leq \int_0^K G(t,s)f(s,\alpha(s))ds, \qquad \textit{for all } t \in [0,K].$$

On other side we know that.

$$\int_0^K G(t,s)f(s,\alpha(s))ds \leq \beta(t), \qquad \textit{for all } t \in [0,K].$$

with all this result we get.

$$\int_0^K G(t,s)f(s,u(s))ds \leq \int_0^K G(t,s)f(s,\alpha(s))ds \leq \beta(t), \quad \forall\ t \in [0,K]$$

.

Hence $Tu \in A_1$*, and therefore condition (3.22) holds.*

Part three

Now, let $(u, v) \in A_1 \times A_2$*, that is,* $u(t) \leq \beta(t), v(t) \geq \alpha(t)$*, for all* $t \in [0, K]$*. Thus condition (3.16) implies that*

$$u(t) \leq \beta_0, \quad v(t) \geq \alpha_0, \qquad \textit{for all } t \in [0,K]$$

By conditions (3.20) and (3.21), for all $t \in [0, K]$*, we have*

$$\begin{aligned}\max\{Tx,Ty\} =& \max\left\{\int_0^T G(t,s)f(s,x(s))ds, \int_0^T G(t,s)f(s,y(s))ds\right\}\\ &\leq \int_0^T G(t,s)\max\{f(s,x(s)), f(s,y(s))\}ds\\ &\leq \int_0^T G(t,s)[\alpha \max\{x,y\}\\ &\quad + \beta(\max\{x,Tx\} + \max\{y,Ty)\})\\ &\quad + \gamma(\max\{x,Ty\} + \max\{y,Tx\})\\ &\quad + \delta(M(x,y) + hm(x,y))]ds\\ &= \int_0^T G(t,s)[\alpha(\max\{x,y)\})\Big] ds\\ &\quad + \int_0^T G(t,s)[\beta(\max\{x,Tx\} + \max\{y,Ty)\})\Big] ds\\ &\quad + \int_0^T G(t,s)[\gamma(\max\{x,Ty\} + \max\{y,Tx\})]ds\\ &\quad + \int_0^T G(t,s)[\delta(M(x,y) + hm(x,y))]ds.\end{aligned}$$

Thus, we have

$$\begin{aligned}D_P(Tx,Ty) \leq& [\alpha D_p(x,y)\\ &+ \beta\left(D_p(x,Tx) + D_p(y,Ty)\right) + \gamma\left(D_p(x,Ty) + D_p(y,Tx)\right)\\ &+ \delta(M(x,y) + hm(x,y))] \times \int_0^T G(t,s)ds\\ \leq& \alpha D_p(x,y) + \beta\left[D_p(x,Tx) + D_p(y,Ty)\right]\\ &+ \gamma\left[D_p(x,Ty) + D_p(y,Tx)\right] + \delta[M(x,y) + hm(x,y)].\end{aligned}$$

Part four

Now, if $M(x,y) = \max\{x,Ty\}$*, we have*

$$\begin{aligned}D_P(Tx,Ty) \leq& \alpha D_p(x,y) + \beta\left[D_p(x,Tx) + D_p(y,Ty)\right]\\ &+ \gamma\left[D_p(x,Ty) + D_p(y,Tx)\right]\\ &+ \delta\left[D_p(x,Ty) + hD_p(y,Tx)\right],\end{aligned}$$

and if $M(x,y) = \max\{y,Tx\}$*, we have*

$$\begin{aligned} D_P(Tx, Ty) \leq & \alpha D_p(x, y) + \beta \left[D_p(x, Tx) + D_p(y, Ty) \right] \\ & + \gamma \left[D_p(x, Ty) + D_p(y, Tx) \right] \\ & + \delta \left[D_p(y, Tx) + h D_p(x, Ty) \right]. \end{aligned}$$

By a similar method, we can show that the above inequality holds if $(u, v) \in A_2 \times A_1$. Thus, all the conditions of Theorem 2.1.4 hold, and therefore, T has a unique fixed point z in the set

$$A_1 \cap A_2 = \{u \in C([0, K]) : \alpha \leq u(t) \leq \beta, \quad \textit{for all } t \in [0, K]\}.$$

That is, $z \in A_1 \cap A_2$ is the solution to (3.15).

CONCLUSION

Our study of some theorems and propositions of fixed and coincidence point in single valued and multi valued mapping ,and its generalization in partial metric space, its some fixed and coincidence point results for generalized contractive conditions in complete partial metric spaces. Also, we prove the results are valid for analogous contractive conditions on orbitally complete and (T, f)-orbitally in complete partial metric spaces. These results generalize several theorems in partial metric spaces, and were used for study the existence of the solution for the basic form of the functional equation of dynamic programming also the solution of nonlinear integral equation.

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