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On the existence of solutions for some boundary Value problems of fractional differential equations

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Dedication:

F *or the one who spent her life and gave me all she can to make my life happier and easier, my great mother*

F *or my super hero and the big hearted man, my wonderful father*

F *or the one I have chosen to spend the rest of my life with, Ouadjih*

F *or those who were there whenever it is needed, my lovely brothers Youcef, Soulaimani and Marouan*

T *ous les membres de ma famille .*

F *or my sweethearts and soulmates, my gorgeous sisters Asia, Khaoula, Samiha, Isra and Houda*

F *or the special girl whome I was lucky to have as a friend, Chifa and all of my sweet friends .*

F *or all of my mates and the researchers everywhere.*

I *dedicate this humble work, wishing from Allah acceptance and blessings.*

Betta safa

Dedication:

F*or my caring, generous parents Meriem and Mohammed Sayah Dahdi.*

F*or his pure soul my brother Ahmed may God have mercy on him.*

F*or my dear sisters: Abir, Sabrine , Nour el houda ,Fatima zahra,and to the whole family.*

F*or my supportative friends: Fatiha, Botaina, and all my best freinds in university.*

F*or all my teachers of the faculty of exact sciences of the university of Hamma lakhdhar.*

Bouba(nada)

Contents

Notations

Ω	:= un ouvert borné de $\mathbb{R}^n$.
$\mathbb{R}$	: Space of real number.
$\mathbb{N}$	: Space of natural number.
$\mathbb{C}$	: Space of complex number.
$C([a,b],\mathbb{R})$	: contunous functions set in $[a,b] in \mathbb{R}$.
$conv\Omega$	: l'enveloppe the sets convexes Ω.
$\vert.\vert$	: absolute valeur.
$\\| \cdot \\|_\infty$	$:= \sup\{\\|x(t)\\| : t \in I\}$.
$AC([a,b])$	:absoulutely contunous in [a, b].
$AC^n([a,b])$	$:= \left\{ f : \Omega \to \mathbb{C} : f^{(k)} \in C([a,b]), k = 0 \ldots n-1, f^{(n-1)} \in AC([a,b]) \right\}$.
$\Gamma(\alpha)$	: Gamma function.
I^α	: fractional integral.
${}^{RL}D^\alpha$	: RiemanLioville fractionl derevative.
${}^{c}D^\alpha$	: Caputo fractionl derevative.
${}^{H}D^\alpha$	: Hadamard fractional derevative.
D_1	: Caputo-Hadamardfractional derevative.

Introduction:

Fractional calculus is one of important tool to study many problems and phenomenons from fields of science and engineering. as in physics, chemistry, hydrology, biophysics, thermodynamics, blood flow problems, statistical mechanics and control theory. Recently, it has known a significant development in fractional differential and integral equations, for example see [1, 3, 4, 11].

Differential equations with integral boundary conditions have different applications in applied science such as underground water flow, thermoelasticity, population dynamics and blood flow problems, some results in this way are given for instance see [6, 8, 10].

Fixed point theory play an important role in the study of the existence and uniqueness of various boundary value problems and several fixed point theorems were used in this way, as of Banach,o Schauder, Krasnoselskii, Schafer, Leray-Schauder alternative and others.

This memory contains an introduction, three chapters and a list of references.

In the first chapter, we present some basic definitions and notions of functional analysis, as metric spaces, Banach spaces, the campactness, convexity. We give also some definitions and properties concerning fractional calculus, like fractional integral and fractional derivative for different sense, in the sense of Riemann-Liouville, Caputo, Hadamard and Caputo-Hadamard. Some fixed point theorems are given, as Banach principle, Schauder fixed point theorem and Leray-Schauder alternative, which have been used in sequel of this work.

The second chapter is reserved to our main result, in fact, we prove the existence and the uniqueness of the solution for a boundary value problem of Caputo fractional differential equations with integral boundary conditions, by using Banach's fixed point theorem, an other existence theorem is given for the same problem, which based on Schauder fixed point theorem. Finally, two examples are given to demonstrate our main result.

In the third chapter, we focus on the study of a boundary value problem of fractional differential equations in the sense of Caputo-Hadamard with separated integral boundary conditions, the used technique in the proof based on Leray-Schauder alternative, we offer an example to illustrate our second main result.

Chapter 1

Preliminaries

In this chapter, we will recall some basic definitions and theorems, around metric spaces, complete spaces, normed spaces, Banach spaces, contractions, Compactness, Banach's principle theorem and Schauder's theorem, and Leray-Schauder.

1.1 Metric spaces

Définition 1.1.1.

We call distance on a set X any function d defined on the product $X^2 = X \times X$ and with values in the set $\mathbb{R}^+$ of the positive reals:

$$d : X \times X \longrightarrow \mathbb{R}^+$$

Checking the following properties:

1. $\forall x, y \in X,\ d(x,y) = 0$ *and* $d(x,y) = 0$ *if and only if* $x = y$,
2. $d(x,y) = d(y,x)$ *(symmetry),*
3. $\forall x, y, z \in X,\ d(x,z) \leq d(x,y) + d(y,z)$ *(triangle inequality).*

Exemple 1.1.

$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$, $x, y \in X = \mathbb{R}^*$ *be a distance on* $\mathbb{R}^*$ *for:*

1. $d(x,y) = 0 \Longleftrightarrow \left|\frac{1}{x} - \frac{1}{y}\right| = 0 \Longleftrightarrow \frac{1}{x} = \frac{1}{y} \Longleftrightarrow x = y.$
2. $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|(-1)\left(\frac{1}{y} - \frac{1}{x}\right)\right| = |-1|\left|\frac{1}{y} - \frac{1}{x}\right| = d(y,x).$
3. $d(x,z) = \left|\frac{1}{x} - \frac{1}{z}\right| = \left|\frac{1}{x} - \frac{1}{y} + \frac{1}{y} - \frac{1}{z}\right|.$

$$\leq \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right|$$

$$= d(x,y) + d(y,z).$$

Then d *is a distance on* $\mathbb{R}^*$.

Remark 1.

A metric space is an ordered pair (X,d), *where* X *is a set and* d *a metric.*

1.1.1 The sequences in metric spaces

Définition 1.1.2. (***Convergence of a sequence,limit***).

A sequence (x_n) *in a metric space* (X,d) *is said to be convergent if :*

$$\forall \epsilon > 0 \text{ ; } \exists N(\epsilon) \in \mathbb{N}^* : \forall n \geq N(\epsilon),\ d(x_n, x) < \epsilon.$$

is called the limit of (x_n) and we write

$$\lim_{n\to\infty} x_n = x.$$

We say that (x_n) converges to x or has the limit x.

Définition 1.1.3. (***Cauchy sequences***)

A sequence (x_n) *in a metric space* (X,d) *is said to be-Cauchy (or fundamental) if for* *every $\epsilon > 0$ there is an $N = N(\epsilon)$ such that:*

$$d(x_m, x_n) < \epsilon \quad \textit{for every} \quad m, n > N.$$

Remark 2.

all convergent sequence is Cauchy sequence.

Proof.

Let $x_n \longrightarrow x$, $\epsilon > 0$, and $N(\epsilon) \in \mathbb{N}^*$ be such that $n \geq N(\epsilon) \Longrightarrow d(x_m, x) < \dfrac{\epsilon}{2}$, and $m, n > N(\epsilon)$.

Then $d(x_m, x) < \dfrac{\epsilon}{2}$ and $d(x_n, x) < \dfrac{\epsilon}{2}$ and the triangle inequality yields

$$d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Let U be an interval of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of functions with $f_n : U \longrightarrow \mathbb{R}^p$. Let $|.|$ be any norm from $\mathbb{R}^p$.

Définition 1.1.4.

$\{f_n\}$ is uniformly bounded on U if there exists $M > 0$ such that:

$$|f_n(t)| \leq M \quad \textit{for all } n \textit{ and all } t \in U.$$

1.1.2 Complete metric spaces

Définition 1.1.5.

A metric space (X, d) is said to be complete if each Cauchy sequence $\{x_n\}$ in X has a limit (converges).

Exemple 1.2. (*The metric transform ϕ*)

Let (M, d)be a metric space, define the metric space (M, d_ϕ) by taking for $x, y \in M$;

$$d_\phi(x, y) = \phi(d(x, y)),$$

where $\phi : [0, \infty) \longrightarrow [0, \infty)$ is increasing, concave downward,

$$(\phi(at + (1-t)b \geq t\phi(a) + (1-t)\phi(b))).$$

and satisfies $\phi(0) = 0$.

It is complete metric space.

1.1.3 The continuity in metric spaces

Définition 1.1.6. (***Continuous mapping***)

Let $X = (X, d_1)$ and $Y = (Y, d_2)$ be metric spaces.

A mapping $T : X \longrightarrow Y$ is said to be continuous at a point $x_0 \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that:

$$d(Tx, Tx_0) < \epsilon \quad \text{for all } x \text{ satisfying} \quad d(x, x_0) < \delta.$$

T is said to be continuous on X if it is continuous at every point $x \in X$.

Définition 1.1.7.

$\{f_n\}$ is equicontinuous if for any $\epsilon > 0$ it exists $\delta > 0$, such that :

if $t_1, t_2 \in U$ *and* $|t_1 - t_2| \leq \delta$ *so* $|f(t_1) - f(t_2)| \leq \epsilon$.

Exemple 1.3.

Let $\mathcal{H} = \{f \in C([a,b], \mathbb{R}); \mid f(t) \mid \leq h\} \forall f \in \mathcal{H}$.

We apply mean value theorem,

if $t_0 \in [a,b]$ *fixe* , $\forall t \in]a,b[: \exists c \in]t, t_0[$,

$$\begin{aligned} |f(t) - f(t_0)| &\leq |f'(c)| \, |t - t_0| \, , \\ &\leq k \, |t - t_0| \, . \end{aligned}$$

taking $\delta = \frac{\epsilon}{k}$

if

$$|t - t_0| \leq \delta = \frac{\epsilon}{k} \Longrightarrow |f(t) - f(t_0)| \leq \epsilon$$

so f *equicontinuous an* t_0.

1.2 Compactness

1.2.1 Compact metric spaces

Définition 1.2.1.

Let X *be a metric space,we say that* X *is compact if every sequence of points of* E *has a convergent subsequence.*

1.2.2 Compact parts

Définition 1.2.2.

A subset M of a metric space (X, d) is said to be compact if any $(X_n)_{n\in N}$ of M admits a subsequence converging to a limit belonging to M.

Exemple 1.4.

Any closed and bounded part of $\mathbb{R}$ is compact.

Exemple 1.5.

$\{\frac{1}{n}, n \in \mathbb{N}^*\} \cup \{0\}$ *in the metric space $(\mathbb{R}, |.|)$, it is a bounded closed of a normed vector space of finite dimension.*

It is closed because convergent with values in this set is either stationary or converges to 0, since the points $\frac{1}{n}$ are isolated.

Définition 1.2.3. ***(Relatively compact parts)***

X is relatively compact if every sequence of X admits a subsequence converging to a limit belonging to X, That is to say, if the closure of X is compact.

Exemple 1.6.

The relatively compact parts of $\mathbb{R}^n$ are the bounded parts.

1.2.3 Compact mappings

Définition 1.2.4.

E and F two vector spaces normed and $u : E \longrightarrow F$ a linear mapping, u is said to be compact if,

1. *the image of each bounded set in E is relatively compact in F.*

2. *$T(BE(0; 1))$ is relatively compact in F.*

3. *for each sequence (x_n) bounded in E one can extract a subsequence (x_{n_k}) such that $u(x_{n_k})$ converges in F.*

Theorem 1.2.1. (***Arzela-Ascoli***) *[11]*

Let (X,d) be compect metrique space , (X',d') be complet metrique space, the part A in (X,X')is relativly compact if and only if

1. *A is uniformily bounded, there exists a constant $k > 0$ such that :*

$$\| f(x) \| \leq k \quad for \quad each \quad x \in X and f \in A.$$

2. *A is equicontinouce, for each $\varepsilon > 0$ ther exists $\delta > 0$ such that*

$$| x_1 - x_2 | < \delta \Rightarrow \| f(x_1) - f(x_2) \| < \varepsilon.$$

   $\forall x_1, x_2 \in X$ *et* $\forall f \in A$.

3. *for each $x \in X$, the set $A(x) = \{f(x); f \in A\}$ is relativly compact*

Exemple 1.7.

Let $f : ([0,1]) \times \mathbb{R} \longrightarrow \mathbb{R}$ a continuous application, consider the following integral equation:

$$G : u(t) \longrightarrow \int_0^t f(s, u(s))\, ds; \qquad t \in [0,1],$$

Then the operator of Hammerstein

$$G : C([0,1]) \longrightarrow C([0,1]).$$

$$u \longrightarrow Gu,$$

such that

$$Gu(t) = \int_0^t f(s, u(s))\, ds,$$

is compact.

Assume the set $A = \{f \in C([0,1]), \|f\| \leq M\}$, *Since* f *is continuous and bounded so:*

$$\begin{aligned} |Gu(t)| &= \left| \int_0^t f(s, u(s))\, ds \right|, \\ &\leq \int_0^t |f(s, u(s))|\, ds, \\ &\leq M \int_0^t ds, \\ &= Mt, \\ &\leq M \quad \forall t \in [0,1]. \end{aligned}$$

Therefore G *is bounded.*

We will show G *is equicontinuous, for all* $t_1, t_2 \in [0,1]$ *(suppose* $t_1 < t_2$*)we have:*

$$\begin{aligned} |Gu(t_1) - \ Gu(t_2)| &\leq \int_0^{t_1} |f(s, u(s))|\, ds - \int_0^{t_2} |f(s, u(s))|\, ds, \\ &= \int_{t_1}^{t_2} |f(s, u(s))|\, ds, \\ &\leq M \int_{t_1}^{t_2} ds \\ &\leq M |t_1 - t_2|. \end{aligned}$$

So, for all $\epsilon > 0$, *does it exist* $\delta \leq \frac{\epsilon}{M}$, *such that:*

for all $t_1, t_2 \in [0,1] : |t_2 - t_1| \leq \delta$. *so:*

$$|Gu(t_1) - \ Gu(t_2)| \leq M\delta \leq \epsilon$$

Hence the equicontinuity of G.

From the Ascoli-Arzela theorem, G *is compact in* A.

1.3 Banach spaces

1.3.1 Normed linear spaces

Définition 1.3.1. *(norm)*

A norm on a linear space E *is a mapping* $\|.\| : E \longrightarrow \mathbb{R}^+$ *which satisfies for each* $x, y \in E$; $\lambda \in \mathbb{R}$:

(1) $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda| \, \|x\|$.

(3) $\|x + y\| \leq ||x|| + \|y\|$.

Définition 1.3.2.

A linear space with a norm called a normed linear space.

Exemple 1.8.

(*The space* $C[0, 1]$) *This space consists of all countined real valued functions defined on* $[0, 1]$ *with the norme* $\|f\|$ *for* $f, g \in C[0, 1]$ *taken as above:*

$$\|f\| = \int_0^1 f(t) \, dt,$$

it is normed space.

Remark 3.

The $(E, \| \, . \, \|)$ *is said a normed space.*

Remark 4.

if $(E, \|.\|)$ *normed space so it is metric space, with the metric*

$$d(x, y) = \|x - y\| \, .$$

Définition 1.3.3. (***Convex set***)

A subset M of a vector space E is said to be convex if x, $y \in M$ implies:

$$K = \{z \in E \mid z = \alpha x + (1-\alpha) y, \quad 0 \leq \alpha \leq 1\} \subset M.$$

Définition 1.3.4.

A Banach space is a normed linear space $(E, \|.\|)$ which is complete relative to the metric d defined above.

Exemple 1.9.

$l^{\infty}(M)$ *This is the space of all bounded real-valued functions $f : M \longrightarrow \mathbb{R}$ where M is a complete metric space and*

$$\|f\| = \sup_{x \in M} |f(x)|.$$

The completeness of $l^{\infty}(M)$ follows from the fact that if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $l^{\infty}(M)$ then $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in M for each $x \in M$.

The function defined by $f(x) = \lim_{n\to\infty} f_n(x), x \in M$, exists since M is complete.

It is quite easy to show that $\lim_{n\to\infty} \|f_n - f\| = 0$.

1.4 Contractive conditions

1.4.1 Contraction

Définition 1.4.1.

Let f be a self-mapping of a Banach space $(E, \| \,.\, \|)$.

Then f is said to be contraction if there exists a real number $h < 1$ such that:

$$\| fx - fy \| \leq h \| x - y \| \quad \textit{for all} \quad x, y \in E. \tag{1.1}$$

A contraction mapping is also known as Banach contraction.

If we replace the inequality (1.1) with strict inequality and $h = 1$, then f is called contractive (or strict contractive).

If (1.1) holds for $h = 1$, then f is called nonexpansive, and if (1.1) holds for fixed $h < \infty$, then f is called Lipschitz continuous.

Clearly, for the mapping f , the following obvious implications hold:

contraction $\Longrightarrow$contractive $\Longrightarrow$ nonexpansive $\Longrightarrow$ Lipschitz continuous.

Exemple 1.10.

Consider the usual metric space $(\mathbb{R}, d)$. Define

$$f(x) = \frac{x}{a} + b, \qquad \textit{for all } x \in \mathbb{R},$$

$$\begin{aligned} d(fx, fy) &= \left| \frac{x}{a} + b - \frac{y}{a} - b \right|, \\ &= \left| \frac{1}{a}(x - y) \right|, \\ &\leq \left| \frac{1}{a} \right| |x - y|. \end{aligned}$$

Then, f is contraction on $\mathbb{R}$ if $a > 1$.

Définition 1.4.2.

Let f be a self-mapping of a Banach space $(E, \| \, . \, \|)$.

Then f is said to be ϕ-contractive if there exists a continuous mapping:

$$\phi : [0, \infty[\to [0, \infty[$$

with $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$ such that:

$$\| fx - fy \| \leq \phi \ (\max \ \{\| x - y \|, \| x - fx \|, \| y - fy \|, [\| x - fy \| + \| fx - y \|]/2 \}) . \forall x, y \in E$$

Définition 1.4.3.

Let f be a self-mapping of a Banach space $(E, \| \,.\, \|)$.

Then f is said to be ϕ-weakly contraction if there exists a continuous mapping:

$$\phi : [0, \infty[\to [0, \infty[$$

with $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$ such that:

$$\| fx - fy \| \leq \| x - y \| - \phi(\| x - y \|) \ \ \textit{for all} \ \ x, y \in E.$$

1.5 Calcul fractionaire:

We give an over view around the fractional derevative and the fractional integral with some properties.

1.5.1 Gamma Function:

Définition 1.5.1.

The gama function $\Gamma(z)$ *is defined by the integral :*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

z *is complex number :* $Re(z) > 0.$

Proposition 1.

we have some basic properties

1 *The function* $\Gamma(z)$ *is continuous for* $p > 0$.

2 *The function* $\Gamma(z)$ *obeys the property:*

$$\Gamma(z+1) = z\Gamma(z).$$

3 *The following relations are also valid:*

$$\begin{aligned} \Gamma(z+1) =& (z+n-1)\cdots(z+1)z\Gamma(z), \\ \Gamma(1) =& 1, \\ \Gamma(n+1) =& n!, \\ \Gamma(0) =& +\infty. \end{aligned}$$

1.5.2 Fractional Integral of Order α

Définition 1.5.2. *For every* $\alpha > 0$ *and a local integrable function* $f(t)$*, the right of order* α *is defined:*

$$I^\alpha f(t) = I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \lim \int_a^t (t-s)^{\alpha-1} f(s) ds, \qquad -\infty \leq a < t < \infty. \tag{1.2}$$

Alternatively, it can be defined also the left as:

$$I_b^\alpha = \frac{1}{\Gamma(\alpha)} \lim \int_t^b (t-s)^{\alpha-1} f(s)ds, \qquad -\infty < t < b \leqslant \infty. \tag{1.3}$$

Proposition 2.

we have the fallowing proprities:

i) $I^0 f(t) = f(t)$.

ii) $I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$.

iii) *Operateur intégral* I^α *is:*

$$I^\alpha(\lambda f(t) + g(t)) = \lambda I^\alpha f(t) + I^\alpha g(t), \lambda \in \mathbb{R}_+, \lambda \in \mathbb{C}.$$

iiii) $\frac{d}{dt}(I^\alpha f)(t) = I^{\alpha-1} f(t)$.

1.5.3 Riemann-Liouville fractional derivatives

For a continuous function $f : (0,\infty) \to \mathbb{R}$ $f \in L^1$, the Riemann-Liouville derivative of fractional order $\alpha > 0$, n = [a] + 1 ([a] denotes the integer part of the real number a) is defined as:

$$\begin{aligned} {}^{RL}D^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}^n\right) \int_0^t (t-s)^{n-\alpha-1} f(s)ds, \\ &= \left(\frac{d}{dt}\right)^n) I^{n-\alpha} f(t). \end{aligned}$$

In particular, if $\alpha = 0$, then:

$$\left(D^0 f\right)(t) = I^0 f(t) = f(t).$$

If $\alpha = n \in \mathbb{N}$, then:

$$\left(D^n f\right)(t) = I^n f(t) = f^n(t).$$

If $0 < \alpha < 1$, alors $n = 1$,

$$^{RL}D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int\limits_a^t (t-s)^{-\alpha} f(s) ds, t > a.$$

1.5.4 Caputo Fractional Derivatives

Définition 1.5.3.

The Caputo derivative of order α for a function $f \in AC^n(J, E)$ absolutely continuous is defined by:

$$\begin{aligned} ^{C}D^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \int\limits_0^t (t-s)^{n-\alpha-1} f^n(s) ds, \\ &= I^{n-\alpha} f^n(t), \qquad t > 0, n-1 < \alpha < n. \end{aligned}$$

If $0 < \alpha < 1$, then :

$$^{C}D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t (t-s)^{-\alpha} f'(s) ds.$$

Lemma 1.1. *[8]*

If $f \in AC^n[0,1]$, then the Caputo derivative $^{C}D^\alpha f(t)$ exists almost everywhere on [0,1], *where* $AC^n[0,1] = \left\{ f \in C^{n-1}[0,1] | f^{(n-1)} \text{is absolutely continuous} \right\}$ *and* n *is the smallest integer greater than or equal to* α.

Lemma 1.2. *[8]*

Let $\alpha \geq 0$ *and* $n = [\alpha] + 1$. *Then:*

$$I^\alpha({}^cD^\alpha f(t)) = f(t) + \sum_{k=0}^{n-1} \frac{f^k(0)}{k!} t^k.$$

Lemma 1.3. *[12]*

Let $\alpha > 0$. *Then the differential equation:*

$${}^cD^\alpha f(t) = 0,$$

has a solutions :

$$f(t) = c_0 + c_1 t + c_2 t^2 \cdots + c_{n-1} t^{n-1},$$

where

$$c_i \in \mathbb{R}, i = 1, 2, 3 \cdots n-1, n = [\alpha] + 1.$$

Lemma 1.4. *[12]*

Let $\alpha > 0$ *and* $n = [\alpha] + 1$.

Then:

$$D^\alpha x(t) = f(t),$$

equivalant to:

$$I^\alpha({}^cD^\alpha x(t)) = I^\alpha f(t) + c_0 + c_1 t + c_2 t^2 \cdots + c_{n-1} t^{n-1}, \tag{1.4}$$

where :

$$c_i \in \mathbb{R}, i = 1, 2, 3 \cdots n-1.$$

1.6 Relationship between fractional derivatives :

1) Let $\alpha \in \mathbb{R}_+, n \in \mathbb{N}^*$, and $n = [\alpha] + 1$, if ${}^cD^\alpha f(t))$ and ${}^{RL}D^\alpha f(t))$ exist,so:

$$\textbf{i)}\ {}^CD_x^\alpha f(x) =^{RL} D^\alpha f(x) - \sum_{i=0}^{n-1} \frac{f^i(a)(x-a)^{i-\alpha}}{\Gamma(i-\alpha+1)},$$

we deduce that if $f^i(a) = 0$ for all $i = 0, 1, \ldots n-1$, we will have

$${}^CD^\alpha f(x) =^{RL} D^\alpha f(x).$$

$$\textbf{ii)}\ {}^CD^\alpha f(x) =^{RL} D^\alpha \left(f(x) - \sum_{i=0}^{n-1} \frac{f^i(a)}{i!}(x-a)^i \right).$$

2) Let $0 < \alpha < 1$, the fractional derivative of Riemann-Liouville and that of Caputo are defined respectively by:

$${}^{RL}D^\alpha f(x) = \frac{d}{dx}\left({}^{RL}D^{-(1-\alpha)} f(x)\right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t)dt$$

$${}^CD^\alpha f(x) =^{RL} D^{-(1-\alpha)} \left(\frac{df(x)}{dx}\right) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} f'(t)dt$$

1.7 Caputo-Hadamard

Définition 1.7.1.

The Hadamard fractional integral of order $\alpha > 0$ for a function $f \in L^1([1, +\infty[, \mathbb{R})$ is *defined as:*

$$^{H}I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s)}{s}ds,$$

provided the integral exists.

Exemple 1.11.

Let $\beta > 0$. Then:

$$^{H}I_{1}^{\beta}\ln t = \frac{1}{\Gamma(2+\beta)}(\ln t)^{1+\beta};\ \textit{for a.e. } t \in [1,+\infty[.$$

Définition 1.7.2. .

The Hadamard fractional derivative of order $\alpha > 0$ applied to the function $h \in AC_{\delta}^{n}([1,+\infty[,\mathbb{R})$ is defined as:

$$\left(D_{1}^{\beta}f\right)(t) = \delta^{n}\left(^{H}I_{1}^{n-\alpha}f\right)(t)$$

where $n-1 < \alpha < n, n = [\alpha]+1$, and $[\alpha]$ is the integer part of α.

Lemma 1.5. *[8]*

If $\beta - 1 > \gamma > 0$, then:

(1) $^{H}T^{\gamma}\log(t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\gamma)}(\log t)^{\beta+\gamma-1}$,

(2) $^{H}D^{\gamma}\log(t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}(\log t)^{\beta-\gamma-1}$.

Lemma 1.6. *[8] For $\alpha > 0, n = [\alpha]+1$ and $x \in C(J)\cap L^{1}(J)$, the solution of Hadamard fractional differential equation $^{Hc}D^{\alpha}x(t) = 0$ is $x(t) = \sum_{i=1}^{n}c_{i}(\log t)^{\alpha-i}$, where $c_{i} \in \mathbb{R}(i = 1,2,\ldots,n)$,*

Définition 1.7.3.

For a given function $h \in AC_\delta^n([a,b],\mathbb{R})$*, such that* $0 < a < b$*, the Caputo-Hadamard fractional derivative of order* $\alpha > 0$ *is defined as follows:*

$$D_1^\alpha x(t) = {}^H D^\alpha \left[x(s) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left(\log \frac{s}{a} \right)^k \right](t),$$

where $\mathrm{Re}(\alpha) \geq 0$ *and* $n = [\mathrm{Re}(\alpha)] + 1$.

Lemma 1.7. *[7]*

Let $x \in AC_\delta^n([a,b],\mathbb{R})$ *or* $C_\delta^n([a,b],\mathbb{R})$ *and* $\alpha \in \mathbb{C}$*. Then:*

$$HI^\alpha \left(D_1^\alpha x \right)(t) = x(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left(\log \frac{t}{a} \right)^k.$$

Lemma 1.8. *[7]*

If $u \in C(J)$ *and* ${}^H D^\alpha u \in L^1(J)$*, then:*

$${}^H I^\alpha \left({}^H D^\alpha f \right)(t) = f(t) + c_1 (\log t)^{\alpha-1} + c_2 (\log t)^{\alpha-2} + \cdots + c_n (\log t)^{\alpha-n}$$

where $c_i \in \mathbb{R}(i = 1, 2, 3 \ldots, n), n = [\alpha] + 1$.

Theorem 1.7.1. ***(Banach's fixed point theorem)*** *[4]*

Let C *be a non-empty closed subset of a Banach space* X*.Then any contraction mapping* T *of* C *into itself has a unique fixed point.*

Theorem 1.7.2. ***(Schauder)*** *[4]*

Let X *be a Banach space.* C *be a closed, convex and nonempty subset of* X*. Let* $N : C \to C$ *be a continuous mapping such that* $N(C)$ *is a relatively compact subset of* X*. Then* N *has at least one fixed point in* C*.*

Theorem 1.7.3. *(**Nonlinear Alternative of Leray-Schauder type**). [4]*

Let X be a Banach space with $C \subset X$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$ and $N : \bar{U} \to C$ is a compact map. Then either,

(i) *N has a fixed point in $\bar{U}$, or*

(ii) *there is a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda N(u)$.*

Chapter 2

Existence of solutions for boundary value problem of Caputo fractional differential equations

In this chapiter, we will study the existence and uniqueness of the solution of a fractional boundary value problem for differential equation with integral boundary conditions, by using Schauder fixed point theorem.

An example is given to illustrate our main results.

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t)), & 0 \leq t \leq 1, \\ ax(0) - bx^{'}(0) = 0, & \\ x(1) = \int\limits_{0}^{1} g(s, x(s))ds. & \end{cases} \tag{2.1}$$

where $1 < \alpha \leq 2$ is a real number, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and D^{α} is Caputo derivative.

Where G is the Green function given by:

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}(t-s)^{\alpha-1}-\dfrac{at+b}{a+b}(1-s)^{\alpha-1}, & 0\leq s\leq t,\\ \dfrac{at+b}{a+b}(1-s)^{\alpha-1}, & t\leq s\leq 1\end{cases}. \tag{2.2}$$

Proof.

by lemma (**??**) we can reduce the problem (2.1) to the following forme :

$$\begin{aligned}x(t)&=I^{\alpha}f(t)+c_0+c_1t\\ &=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds+c_0+c_1t\end{aligned}$$

where the Constants $c_0,\ c_1,\in\mathbb{R}$.

From the boundary conditions of the problem (2.1), we find get:

$$c_0=\frac{b}{a+b}\left[\int_0^1 g(s,x(s)ds-\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}ds\right],$$

$$c_1=\frac{a}{a+b}\left[\int_0^1 g(s,x(s)ds-\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}ds\right].$$

Then the solution of the problem 2.1 is given by

$$\begin{aligned}x(t)=&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds+\frac{1}{\Gamma(\alpha)}\frac{(at+b)}{a+b}\int_0^1(1-s)^{\alpha-1}f(s)ds\\ &+\frac{(at+b)}{a+b}\int_0^1 g(s)ds,\\ =&\frac{1}{\Gamma(\alpha)}\Big[\int_0^t(t-s)^{\alpha-1}f(s)ds+\frac{(at+b)}{a+b}\int_0^1(1-s)^{\alpha-1}f(s)ds\Big]+\frac{(at+b)}{a+b}\int_0^1 g(s)ds.\end{aligned}$$

Hence we get (2.2). □

Define an operator:

$$T : C([0,1],\mathbb{R}) \to C([0,1]\mathbb{R}),$$
$$Tx(t) = \frac{(at+b)}{a+b} \int\limits_0^1 g(s)ds + \int\limits_0^1 G(t,s)f(s,x(s))ds,$$

the problem (2.1) has a solution if and only if T has a fixed point. Assume the following hypotheses hold:

H1) The function $f : I \times \mathbb{R} \times \mathbb{R} \to$ is continuous.

H2) There exists constant $K > 0$ where

$$|f(t,x(t)) - f(t,y(t))| < K|x - y.|$$

H3) There exists constant $M > 0$, where

$$|g(t,x(t)) - g(t,y(t))| < M|x - y|,$$

if

$$KG_0 + M < 1. \tag{2.3}$$

Proof.

We will apply the Banach theorem fixed point on the mapping $T \in X$ The proof will be given in several steps:

$TX \subset X$, and $T : X \to X$, it is clear, if

$$x \in C[0,1],$$

so

$$Tx \in C[0,1].$$

Now, we will prove T is continuous:

$$\begin{aligned}
\Big|T(x)(t) - T(y)(t)\Big| = \Big| & \int_0^1 G(t,s)f(s,x(s))ds + \frac{(at+b)}{a+b}\int_0^1 g(s,x(s))ds, \\
& - \int_0^1 G(t,s)f(s,y(s))ds + \frac{(at+b)}{a+b}\int_0^1 g(s,y(s))ds\Big|, \\
& \leqslant \int_0^1 \Big|G(t,s)\Big|\Big|f(s,x(s)) - f(s,y(s))\Big|ds, \\
& + \frac{at+b}{a+b}\int_0^1 \Big|g(s,x(s)) - g(s,y(s))\Big|ds,
\end{aligned}$$

by $[\mathbf{H}_1]$, $[\mathbf{H}_2]$ we have:

$$\begin{aligned}
& \leqslant \int_0^1 |G(t,s)K|x-y|ds + \frac{at+b}{a+b}\int_0^1 M|x-y|ds, \\
& \leqslant G_0K|x-y|ds + \frac{at+b}{a+b}\int_0^1 M|x-y|ds, \\
& \leqslant (G_0K + M)|x-y|.
\end{aligned}$$

Then

$$\|Tx(t) - Ty(t)\| \leqslant (G_0K + M)\|x - y\|_\infty.$$

So, the operator T is a contraction. Hence, by Banach contraction principle, T has a unique fixed point which is a unique solution of the problem (2.1). □

Now, we will utilize another fixed point theorem to prove the existence of the solutions.

Theorem 2.0.1. *the problem (2.1) has a solution if and only if T has a fixed point. We assume the following hypotheses hold:*

H1) *The function $f : I \times \mathbb{R} \times \mathbb{R} \to$ is continuous.*

H2) *There exists a constant K,*

where :

$$\|f(t, x(t)) - f(t, y(t))\|_\infty < K\|x - y\|_\infty,$$

$$\sup_{0 \leq t \leq 1} |f(t, 0)| = f_0.$$

H3) *There exists a constant M, where :*

$$\Big\|g(t, x(t)) - g(t, y(t))\Big\|_\infty < M\Big\|x - y\Big\|_\infty,$$

$$\sup_{0 \leq t \leq 1} |g(t, 0)| = g_0.$$

H4)

$$\frac{G_0 f_0 + g_0}{1 - G_0 f_0 - M} < R, \tag{2.4}$$

where

$$G_0 = \int_0^1 \mid G(t,s) \mid ds.$$

Proof. The proof will be given in several steps:

step 1: T is continuous.

Let (x_n) be a sequence such that $x_n \to x \in C([0,1], X)$. Then for each $t \in [0,1]$, we have:

$$\begin{aligned}
\mid T(x_n)(t) - T(x)(t) \mid =& \mid \int_0^1 G(t,s) f(s, x_n(s)) ds + \frac{at+b}{a+b} \int_0^1 g(s, x_n(s)) \\
& - \int_0^1 G(t,s) f(s, x(s)) ds + \frac{at+b}{a+b} \int_0^1 g(s, x(s)) \mid ds \\
& \leq \int_0^1 \|G(t,s)\| \|f(s, x_n(s)) - f(s, x(s))\| ds \\
& + \frac{at+b}{a+b} \int_0^1 \|g(s, x_n(s) - g(s, x(s))\| ds, \\
& \leq G_0 K \|x_n - x\| + \frac{at+b}{a+b} M \|x_n - x\|, \\
& \leq (G_0 K + M) \|x_n - x\|.
\end{aligned}$$

Then the Lebesgue dominated convergence theorem imply that

$$|T(x_n))(t) - T(x)(t)| \to 0, \quad as \;\; n \to \infty,$$

and hence:

$$\|T(x_n)) - T(x)\|_\infty \to 0, \quad as \quad n \to \infty,$$

Consequently, T is continuous.

Let b :

$$R \geq \frac{G_0 f_0 + g_0}{1 - G_0 f_0 - M}$$

$\sup\limits_{0 \leq t \leq 1} |f(t,0)| = f_0, \quad \sup\limits_{0 \leq t \leq 1} |g(t,0)| = g_0,$

we define:

$$B = \{x \in (C[0,1], \mathbb{R}), \|x\| \leq R\},$$

we will prove B is convex.

Proof. For $x, y \in B$, and $\lambda x - (1-\lambda)y \in B$, and $0 \leq \lambda \leq 1$, we have:

$$\|x\| \leq R, \qquad \|y\| \leq R,$$

so,

$$\begin{aligned} \|\lambda x + (1-\lambda)y\| &\leq |\lambda| \|x\| + |(1-\lambda)| \|y\|, \\ &\leq \lambda R + (1-\lambda)R, \\ &= R. \end{aligned}$$

□

It is clear B is bounded, closed of $C([0,1], \mathbb{R})$.

step 2: $T(B) \subset B$. Let $X \in T(B)$ we show that $T(x) \in T(B)$. For each $t \in [0,1]$, we have:

$$
\begin{aligned}
\|T(x)(t)\| = &\Big| \int_0^1 G(t,s)\left[f(s,x(s)) - f(s,0)\right] ds \\
&+ \int_0^1 G(t,s) f(s,0) ds \\
&+ \frac{at+b}{a+b} \int_0^1 [g(s,x(s)) - g(s,0)] ds + \int_0^1 g(s,0) ds \Big|, \\
&\leq \int_0^1 \|G(t,s)\| \|f(s,x(s)) - f(s,0)\| ds \\
&+ \frac{at+b}{a+b} \int_0^1 \|g(s,x(s)) - g(s,0)\| ds \\
&+ \int_0^1 \|G(t,s)\| \|f(s,0)\| ds + \int_0^1 \|g(s,0)\| ds, \\
&\leq \int_0^1 |G(t,s)| K \|x - 0\| ds \\
&+ \frac{at+b}{a+b} M \int_0^1 \|x - 0\| ds \\
&+ \int_0^1 |G(t,s)| \|f(s,0)\| ds + \int_0^1 \|g(s,0)\| ds, \\
&\leq G_0 K \|x\| + \frac{at+b}{a+b} M \|x\| + G_0 f_0 + g_0, \\
&\leq G_0 K R + \frac{at+b}{a+b} M R + G_0 f_0 + g_0, \\
&\leq (G_0 K + M) R + G_0 f_0 + g_0, \\
&\leq R.
\end{aligned}
$$

Then $T(B) \subseteq B$.

step 3: $T(B)$ is relatively compact.

Let t_1, t_2, $\in [0,1]$, with $t_1 < t_2$ and $x \in T(B)$. Then

$$\begin{aligned}
\|T(x)(t_2) - T(x)(t_1)\| &= \Big\| \int_0^1 [G(t_2,s) - G(t_1,s)] f(s,x(s)) ds \\
&+ \frac{a(t_2+b)}{a+b} \int_0^1 g(s,x(s)) ds \\
&- \frac{a(t_1+b)}{a+b} \int_0^1 g(s,x(s)) \Big\|, \\
&\leq \| \int_0^1 [G(t_2,s) - G(t_1,s)] f(s,x(s)) \| ds \\
&+ \frac{a(t_2-t_1)}{a+b} \int_0^1 g(s,x(s)) \| ds, \\
&\leq \int_0^1 \|[G(t_2,s) - G(t_1,s)]\| \|f(s,x(s))\| ds, \\
&+ \frac{a(t_2-t_1)}{a+b} \int_0^1 \|g(s,x(s))\| ds, \\
&\leq \int_0^1 \|[G(t_2,s) - G(t_1,s)]\| K \|x\| ds \\
&+ \frac{a(t_2-t_1)}{a+b} M \|x\|, \\
&\leq KR \int_0^1 [G(t_2,s) - G(t_1,s)] ds + \frac{a(t_2-t_1)}{a+b} MR.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of claims 1 to 3 T is equicontinous with the Arzela-Ascoli theorem, we conclude that:

$$T : C([0,1], x) \to C([0,1], x),$$

is continuous and compact. As a consequence of Schauder fixed point Theorem, we deduce that T has a fixed point which is a solution of the problem (2.1).

□

2.1 Examples:

Exemple 2.1.

$$^{c}D^{\frac{3}{2}}x(t) = \frac{1}{3e^{2t+1}}(t^2 \sin x + x \cos t),$$

$$x^{'}(0) + x(0) = 0,$$

$$x(1) = \int_0^1 \frac{1}{5}x(t)ds,$$

$$\alpha = \frac{3}{2}, a = b = 1, \quad g(t, x(t)) = \frac{1}{5}x(t).$$

1)

$$\begin{aligned}
|f(t,x(t)) - f(t,y(t)| &= \Big|\frac{1}{3e^{2t+1}}(t^2 \sin x + x \cos t) - \frac{1}{3e^{2t+1}}(t^2 \sin y + y \cos t)\Big|,\\
&= \Big|\frac{1}{3e^{2t+1}}\left[(t^2 \sin x - t^2 \sin y) + (x \cos t - y \cos t\right]\Big|,\\
&\leq \Big|\frac{1}{3e^{2t+1}}\Big||t^2|\|\sin x - \sin y\| + |\cos t|\|x - y\|,\\
&\leq \Big|\frac{1}{3e^{2t+1}}\Big||t^2|\|x - y\| + |\cos t|\|x - y\|,\\
&\leq \frac{2}{3e}\|x - y\|.
\end{aligned}$$

$$k = \frac{2}{3e}.$$

2)

$$\begin{aligned}
|g(t,x(t)) - g(t,y(t))| &= \Big| \int_0^1 \frac{1}{5}x(t)ds - \int_0^1 \frac{1}{5}y(t)ds \Big|, \\
&\leq \int_0^1 \frac{1}{5}ds||x-y||, \\
&\leq \frac{1}{5}||x-y||.
\end{aligned}$$

$$K = \frac{2}{3e}, \; M = \frac{2}{5}, \quad G_0 = \frac{2}{\Gamma(\frac{5}{2})},$$

$$KG_0 + M = \frac{4}{3e\Gamma(\frac{5}{2})} + \frac{2}{5} < 1.$$

Exemple 2.2.

$${}^cD^{3\backslash 2}x(t) = \frac{2+|x|}{9e^{t^2}(1+|x|)},$$

$$x^{'}(0) + x(0) = 0$$

$$x(1) = \int_0^1 \frac{1}{9e^{t^2}} \sin x ds,$$

1)

$$\begin{aligned}
|f(t,x(t)) - f(t,y(t))| &= |\frac{2+|x|}{9e^{t^2}(1+|x|)} - \frac{2+|y|}{9e^{t^2}(1+|y|)}|, \\
&\leq \|\frac{1}{9e^{t^2}}\|\|\frac{(2+|x|)(1+|y|)-(2+|y|)(1+|x|)}{(1+|x|)(1+|y|)}\|, \\
&\leq | \frac{1}{9e^{t^2}} | \, \|\frac{|y|-|x|}{(1+|x|)(1+|y|)}\|, \\
&\leq \frac{1}{9}\|x-y\|.
\end{aligned}$$

2)

$$\begin{aligned}
|g(t,x(t)) - g(t,y(t))| &= |\int_0^1 \frac{1}{9e^{t^2}} \sin x - \int_0^1 \frac{1}{9e^{t^2}} \sin y ds|,\\
&\leq \Big| \int_0^1 \frac{1}{9e^{t^2}} ds \Big| \|\sin x - \sin y\|,\\
&\leq \frac{1}{9}\|x-y\|.
\end{aligned}$$

$$K = \frac{1}{9}, \quad M = \frac{1}{9}, \quad G_0 = \frac{2}{\Gamma(\dfrac{5}{2})}, \quad f_0 = \frac{2}{9} \quad g_0 = 0.$$

$$\frac{G_0 f_0 + g_0}{1 - G_0 k - M} = \frac{\frac{2}{\Gamma(\frac{5}{2})} \times \frac{2}{9} + 0}{1 - \frac{2}{\Gamma(\frac{5}{2})} \times \frac{2}{9} - \frac{1}{9}} = \frac{\frac{4}{9\Gamma(\frac{5}{2})}}{1 - \frac{4}{9\Gamma(\frac{5}{2})} - \frac{1}{9}}.$$

$$\frac{4}{9\Gamma(\frac{5}{2})}(1 - \frac{2}{9\Gamma(\frac{5}{2})} - \frac{1}{9}) < 1.$$

Chapter 3

Existence of solutions for boundary value problem of fractional differential equations involving Caputo-Hadamard derivative

3.1 Position of the problem

Consider the following:

$$\begin{cases} D_1 x^{\alpha}(t) = f\big(t, x(t), D^{\beta} x(t)\big), \qquad 1 \leq t \leq e, \\ a_1 x(1) + b_1 \left(D^{\gamma} x(1)\right) = \int_1^e g\left(s, x(s)\right) ds, \\ a_2 x(e) + b_2 \left(D^{\gamma} x(e)\right) = \int_1^e h\left(s, x(s)\right) ds, \end{cases} \tag{3.1}$$

where D^{α},D^{β}, D^{γ} of the Caputo-Hdamard, $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $0 < \gamma < 1$ real number, and $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

Proposition 3.

The space $(X, \|.\|_\infty)$ *define* $X = \{x \in C[1,e], D^\beta x(t) \in C[1,e]\}$ *this space endowed with the norm*

$$\|x\|_X = \|x\|_\infty + \|D^\beta x(t)\|_\infty.$$

It is a Banach space.

Lemma 3.1. *The function* $x \in X$ *is a solution of the proplem:*

$$\begin{cases} D_1 x^\alpha(t) = f(t)\,, & 1 \le t \le e, \\ a_1 x(1) + b_1\left(D^\gamma x(1)\right) = \int_1^e g\,(s)\,ds, \\ a_2 x(e) + b_2\left(D^\gamma x(e)\right) = \int_1^e h\,(s)\,ds. \end{cases} \tag{3.2}$$

if and only if it is a solution of the integral équation:

$$x(t) = \int_1^e G\,(t,s)\,f(s)ds + \frac{\log t}{a_1}(1 - v_1)\int_1^e g\,(s)\,ds - \frac{v_2}{a_2}\int_1^e h\,(s))\,ds,$$

where G *is a Green function given by:*

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha)} \log\left(\frac{t}{s}\right)^{\alpha-1} + \left(v_1 \frac{\log t}{\Gamma(\alpha-\gamma)} \log\left(\frac{e}{s}\right)^{\alpha-1} + v_2 \frac{\log t}{\Gamma(\alpha-\gamma)} \log\left(\frac{e}{s}\right)^{\alpha-\gamma-1}\right), \\ \qquad\qquad 1 < s \le t < e, \\ \\ v_1 \frac{\log t}{\Gamma(\alpha-\gamma)} \log\left(\frac{e}{s}\right)^{\alpha-1} + v_2 \frac{\log t}{\Gamma(\alpha-\gamma)} \log\left(\frac{e}{s}\right)^{\alpha-\gamma-1}, \quad 1 < t \le s < e, \end{cases}$$

where

$$v_1 = \frac{a_2}{\left(a_2 - \frac{b_2}{\Gamma(2-\gamma)}\right)}, \qquad v_2 = \frac{b_2}{\left(a_2 - \frac{b_2}{\Gamma(2-\gamma)}\right)},$$

Proof. by lemma (3.1) we can reduce the problem to integral equation is equivalent :

$$x(t) = I^{\alpha} f(t) + c_0 + c_1 \log t,$$
$$= \frac{1}{\Gamma(\alpha)} \int_1^t \log \left(\frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds + c_0 + c_1 \log t,$$

where $c_0, c_1 \in \mathbb{R}$. On the other hand ,we use the following properties:

$$D^{\alpha} I^{\alpha} x(t) = x(t)$$

and

$$I^{\alpha} I^{\beta} x(t) = I^{\alpha+\beta} x(t)$$

, such that $\alpha, \beta > 0$.

For each $x \in X$ we have

$$x'(t) = \frac{d}{dt} I^{\alpha} f(t) + c_1$$
$$= D^1 I^{\alpha-1+1} f(t) + c_1$$
$$= D^1 I^1 I^{\alpha-1} f(t) + c_1$$
$$= I^{\alpha-1} f(t) + c_1.$$

As a conditions of the problem (2.2), we have

$$x(1) = c_0, \qquad D^{\gamma} x(1) = 0,$$

$$x(e) = \frac{1}{\Gamma(\alpha)} \int_1^e \log\left(1 - \log s\right)^{\alpha-1} \frac{f(s)}{s} ds + c_0 + c_1,$$
$$D^{\gamma} x(e) = \frac{1}{\Gamma(\alpha-\gamma-1)} \int_1^e \log\left(1 - \log s\right)^{\alpha-1} \frac{f(s)}{s} ds + \frac{1}{\Gamma(2-\beta)} c_1 = \int_1^e h\left(s\right) ds,$$

so

$$\begin{cases} a_1 x(1) + b_1 \left(D^{\gamma} x(1) \right) = \int_1^e g\left(s, x(s)\right) ds, \\ \\ a_2 x(e) + b_2 \left(D^{\gamma} x(e) \right) = \int_1^e h\left(s\right) ds, \end{cases}$$

$$\begin{cases} a_1 c_0 = \int_1^e g\left(s, x(s)\right) ds, \\ \\ a_2 \frac{1}{\Gamma(\alpha)} \int_1^e \log\left(1 - \log s\right)^{\alpha-1} \frac{f(s)}{s} ds + c_0 \\ \\ + c_1 + b_2 \frac{1}{\Gamma(\alpha-\gamma-1)} \int_1^e \left(1 - \log s\right)^{\alpha-1} \frac{f(s)}{s} ds \\ \\ + \frac{1}{\Gamma(2-\beta)} c_1 = \int_1^e h\left(s\right) ds, \end{cases}$$

$$\begin{cases} c_0 = \dfrac{1}{a_1} \int_1^e g(s, x(s)) ds, \\ c_1 = \dfrac{a_2}{\Gamma(\alpha)\left(a_2 - \frac{b_2}{\Gamma(2-\gamma)}\right)} \int_1^e \left(1 - \log s\right)^{\alpha-1} \frac{f(s)}{s} ds \\ + \dfrac{b_2}{\Gamma(\alpha-\gamma)(a_2 - \frac{b_2}{\Gamma(2-\gamma)})} \displaystyle\int_1^e \left(1 - \log s\right)^{\alpha-1} \frac{f(s)}{s} ds \\ - \frac{a_2}{a_1\left(a_2 - \frac{b_2}{\Gamma(2-\gamma)}\right)} \int_1^e g(s, x(s)) ds - \frac{1}{\left(a_2 - \frac{b_2}{\Gamma(2-\gamma)}\right)} \int_1^e h(s, x(s)) ds. \end{cases}$$

Then a solution the problem (3.1) is given by

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\left(\frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds + \frac{1}{a_1} \int_1^e g(s, x(s)) ds \right.$$
$$+ \frac{a_2 \log t}{\Gamma(\alpha-\gamma)\left(a_2 - \frac{b_2}{\Gamma(2-\gamma)}\right)} \int_1^e \log\left(\frac{e}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds$$
$$+ \frac{b_2 \log t}{\Gamma(\alpha-\gamma)\left(a_2 - \frac{b_2}{\Gamma(2-\gamma)}\right)} \int_1^e \log\left(\frac{e}{s}\right)^{\alpha-\gamma-1} \frac{f(s)}{s} ds$$
$$- \frac{a_2 \log t}{a_1\left(a_2 - \frac{b_2}{\Gamma(2-\gamma)}\right)} \int_1^e g(s, x(s)) ds - \frac{\log t}{(a_2 - \frac{b_2}{(\Gamma(2-\gamma)})} \int_1^e h(s, x(s)) ds,$$
$$= \int_1^t \left[\frac{1}{\Gamma(\gamma)} \log\left(\frac{t}{s}\right)^{\alpha-1} + v_1 \frac{\log t}{\Gamma(\alpha-\gamma)} \log\left(\frac{e}{s}\right)^{\alpha-1} + v_2 \frac{\log t}{\Gamma(\alpha-\gamma)} \log\left(\frac{e}{s}\right)^{\alpha-\gamma-1} \right] f(s) ds$$
$$+ \int_t^e \left[v_1 \frac{\log t}{\Gamma(\alpha-\gamma)} \log\left(\frac{e}{s}\right)^{\alpha-1} + v_2 \frac{\log t}{\Gamma(\alpha-\gamma)} \log\left(\frac{e}{s}\right)^{\alpha-1} \right] f(s) ds$$
$$+ \frac{\log t}{a_1} (1 - v_2) \int_1^e g\left(s_1 x(s)\right) ds - \frac{v_2}{a_2} \int_1^e h(s, x(s)) ds.$$

□

3.2 Existence of the Solutions

The problem (3.1) has a solution if and only if T has a fixed point we assume the following hypotheses:

Define an operator:

$$T : X \to X$$
$$Tx(t) = \int_1^e G(t,s) F(s, x(s)) ds + \frac{(\log t)}{a_1} (1 - v_1) \int_1^e g(s, x(s)) ds$$
$$- \frac{(v_1 \log t)}{a_2} \int_1^e h(s, x(s)) ds,$$

and

$$D_1^{\beta} T(x)(t) = \int_1^e (\log \frac{t}{s})^{\beta-\alpha-1} \frac{F(s)}{s} ds + \frac{c_1 (\log t)^{1-\beta}}{\Gamma(2-\beta)},$$

where $F \in X, F(t) = f(t, x(t), D_1^{\beta} x(t)),\ \int_1^e |G(t,s)| ds = G_0$.

(H_1) : f is continuous.

(H_2) : There exist continuous functions $\varphi_1, \varphi_2 : [1,e] \to \mathbb{R}_+$ such that

$$|f(t,x,u) - f(t,y,v)| \leqslant \varphi_1(t)|x-y| + \varphi_2(t) \mid u - v \mid,$$

where

$$\varphi_1^* = \sup_{0 \leqslant t \leq 1} |\varphi_1(t)|, \qquad \varphi_1^* = \sup_{1 \leqslant t \leq e} |\varphi_2(t)|.$$

(H_3) : There exists continuous function $\psi_1[1,e] \to \mathbb{R}_+$ such that

$$|g(t,x) - g(t,y)| \leqslant \psi_1^*(t)|x - y \mid,$$

where

$$\psi_1^* = \sup_{1 \leqslant t \leq e} |\psi_1(t)|.$$

(H_4) : There exists continuous function $\psi_2 \in [1,e] \to \mathbb{R}_+$ such that

$$|h(t,x) - h(t,y)| \leqslant \psi_2^*(t)|x - y \mid,$$

where

$$\psi_2^* = \sup_{1 \leqslant t \leq e} |\psi_2(t)|.$$

Theorem 3.2.1.

under the hypotheses $(H_1) - (H_4)$ the problem (3.1) has a solution provided

$$KG_0 + M < 1$$

Proof.

The problem (3.1) has a solution if and only if T has a fixed point.

We present the proof in four steps:

step 1 T is continuous.

Let $x_n(t)$ be a sequence in X such that $x_n \to x \in X$, if $t \in [1, e]$ we have

$$
\begin{aligned}
|T(x_n)(t) - T(x)(t)| &= \Big| \int_1^e G(t,s)\left(F_n(s) - F(s)\right) ds, \\
&+ \frac{(\log t)}{a_1}(1 - v_1) \int_1^e \left(g(s, x_n(s)) - g(s, x(s))\right) ds \\
&- \frac{(v_1 \log t)}{a_2} \int_1^e \left(h(s, x_n(s)) - h(s, x(s)\right) ds \Bigg|, \\
&\leq \int_1^e |G(t,s)||F_n(s) - F(s)| ds \\
&+ \frac{(1 - v_1)}{a_1} \int_1^e |g(s, x_n(s)) - g(s, x(s))| ds \\
&+ \frac{v_1}{a_2} \int_1^e |h(s, x_n(s)) - h(s, x(s))| ds \\
\|T(x_n)(t) - T(x)(t)\|_\infty &\leq \int_1^e |G(t,s) ds| \|F_n(s) - F(s)\|_\infty \\
&+ \frac{(1 - v_1)}{a_1} \int_1^e ds \psi_1^* \|x_n - x\|_\infty \\
&+ \frac{v_1}{a_2} \int_1^e ds \psi_2^* \|x_n - x\|_\infty \\
&\leq \int_1^e |G(t,s)| ds \|F_n(s) - F(s)\|_\infty \\
&+ \frac{(1 - v_1)}{a_1}(e - 1)\psi_1^* \|x_n - x\|_\infty \\
&+ \frac{v_1}{a_2}(e - 1)\psi_2^* \|x_n - x\|_\infty
\end{aligned}
$$

where $F_n(t), F(t) \in X$ such that $F_n(t) = f_n(t, x_n(t), D_1^\beta x_n(t)$ and $F(t) = f(t, x(t), D_1^\beta x(t)$, by (**H1**), we find

$$
\begin{aligned}
\|(F_n(s) - F(s)\|_\infty &= \|f_n(t, x_n(t), D_1^\beta x_n(t) - f(t, x(t), D_1^\beta x(t)\|_\infty \\
&\leq \varphi_1^* \|x_n - x\|_\infty + \varphi_2^* \|D_1^\beta x_n(t) - D_1^\beta x(t)\|_\infty
\end{aligned}
$$

and

$$\begin{aligned}
\|D_1^\beta x_n(t) - D_1^\beta x(t)\|_\infty &= \left\| \int\limits_1^e (\log\frac{t}{s})^{\beta-\alpha-1}\frac{F_n(s)}{s}ds + \frac{c_1(\log t)^{1-\beta}}{\Gamma(2-\beta)} \right. \\
&\left. - \int\limits_1^e (\log\frac{t}{s})^{\beta-\alpha-1}\frac{F(s)}{s}ds - \frac{c_1(\log t)^{1-\beta}}{\Gamma(2-\beta)} \right\|_\infty \\
&\leq \int\limits_1^e \left|(\log\frac{t}{s})^{\beta-\alpha-1}\frac{1}{s}\right|ds\|F_n(s) - F(s)\|_\infty \\
&\leq \int\limits_1^e \left|(\log\frac{t}{s})^{\beta-\alpha-1}\frac{1}{s}\right|ds\|f_n(t, x_n, D_1^\beta x_n(t) - f(t, x, D_1^\beta x(t)\|_\infty \\
&\leq A\varphi_1^*\|x_n - x\|_\infty + A\varphi_2^*\|D_1^\beta x_n(t) - D_1^\beta x(t)\|_\infty \\
&\leq \frac{A\varphi_1^*}{1 - A\varphi_2^*}\|x_n - x\|_\infty
\end{aligned}$$

where

$$A = \int\limits_1^e (\log\frac{t}{s})^{\beta-\alpha-1}\frac{1}{s}ds$$

so

$$\begin{aligned}
\|T(x_n)(t) - T(x)(t)\|_X &\leq \int_1^e |G(t,s)ds|\|(F_n(s) - F(s)\|_\infty \\
&+ \frac{(1-v_1)}{a_1}\int\limits_1^e ds\psi_1^*\|x_n - x\|_\infty \\
&+ \frac{v_1}{a_2}\int\limits_1^e ds\psi_2^*\|x_n - x\|_\infty \\
&\leq \varphi_1^*\|x_n - x\|_\infty + \varphi_2^*\|D_1^\beta x_n(t) - D_1^\beta x(t)\|_\infty \\
&\leq \varphi_1^*\|x_n - x\|_\infty + \frac{A\varphi_1^*}{1 - A\varphi_2^*}\|x_n - x\|_\infty \\
&\leq \left(\varphi_1^* + \frac{A\varphi_1^*}{1 - A\varphi_2^*}\right)\|x_n - x\|_\infty
\end{aligned}$$

Since $x_n \to x$ for each $t \in [1, e]$

$$\|T(x)_n(t) - T(x)(t)\|_X \longrightarrow 0, \quad \text{whene} \quad n \longrightarrow \infty.$$

Consequently, T is continuous,

let be

$$R > \frac{\Lambda(e-1) + G_0 f_0}{1 - G_0\varphi_1^* + (\Theta + \chi)(e-1) + \nu},$$

where

$$\Lambda = \frac{1-v_1}{a_1} g_0 + \frac{v_1}{a_2} h_0, \qquad \Theta = \frac{1-v_1}{a_1} + \frac{v_1}{a_1(1 - A\varphi_2^*)\Gamma(2-\beta)}\psi_1^*,$$

$$\chi = \frac{v_1}{a_1}(1 + \frac{1}{(1 - A\varphi_2^*)\Gamma(2-\beta)})\psi_2^*, \qquad \nu = \frac{\mu\varphi_1^* + \eta\varphi_1^*}{1 - A\varphi_2^*)\Gamma(2-\beta)} + G_0\varphi_2 * \frac{A\varphi_1^*}{1 - A\varphi_2^*},$$

and

$$\mu = \frac{v_1}{\Gamma(\alpha)} \int_1^e (\log\frac{t}{s})^{\alpha-1} \frac{f(s)}{s} ds, \qquad \eta = \frac{v_2}{\Gamma(\alpha-\beta)} \int_1^e (\log\frac{t}{s})^{\alpha-\beta_1} \frac{f(s)}{s} ds,$$

and $f_0 = \sup\limits_{t\in[1,e]} |f(t,0,0)|$,$g_0 = \sup\limits_{t\in[1,e]} |g(t,0)|$,$h_0 = \sup\limits_{t\in[1,e]} |h(t,0)|$,

we define

$$B = \{x \in X : \|x\|_\infty \le R\}.$$

It is clear that B is bounded, closed and convex subset of X.

step 2 T we will prove $T(B) \subseteq B$.

Let x be $\in B$ we will prove $T(x) \in B$,if $t \in [1, e]$, we have

$$
\begin{aligned}
|T(x)(t)| &= \left| \int_1^e G(t,s)F(t)ds + \frac{(\log t)}{a_1}(1-v_1)\int_1^e g(s,x(s))ds \right. \\
&\left. - \frac{(v_1 \log t)}{a_2}\int_1^e \ h(s,x(s))ds \right| \\
&\leq \int_1^e |G(t,s)||F(t)|ds + \frac{(1-v_1)}{a_1}\int_1^e |g(s,x(s))|ds, \\
&+ \frac{v_1}{a_2}\int_1^e |h(s,x(s))|ds, \\
\|T(x)(t)\|_\infty &\leq \int_1^e |G(t,s)|ds\|F(t)\|_\infty \\
&+ \frac{(1-v_1)}{a_1}\int_1^e \big(\|g(s,x(s)) + g(t,0) - g(t,0)\|_\infty\big)ds \\
&+ \frac{v_1}{a_2}\int_1^e \big(\|h(s,x(s) + h(t,0) - h(t,0)\|_\infty\big)ds, \\
&\leq G_0\|F(t)\|_\infty \\
&+ \frac{(1-v_1)}{a_1}\int_1^e ds\psi_1^*\|x-0\|_\infty + \frac{v_1}{a_2}\int_1^e ds\psi_2^*\|x-0\|_\infty \\
&+ \frac{(1-v_1)}{a_1}\int_1^e ds\|g(t,0\|_\infty + \frac{v_1}{a_2}\int_1^e ds\|h(t,0)\|_\infty \\
&\leq G_0\|F(t)\|_\infty \\
&+ \frac{(1-v_1)}{a_1}\psi_1^*(e-1)\|x\|_\infty + \frac{v_1}{a_2}(e-1)\psi_2^*\|x\|_\infty \\
&+ \frac{(\log t)}{a_1}(1-v_1)(e-1)g_0 + \frac{(v_1)}{a_2}(e-1)h_0,
\end{aligned}
$$

by (**H1**) for each $t \in [1,e]$,

we have

$$\begin{aligned}\|F(t)\|_\infty &= \|f(t,x(t),D_1^\beta x(t)-f(t,0,0)+f(t,0,0)\|_\infty,\\ &\le \|f(t,x,D_1^\beta x(t))-f(t,0,0)\|_\infty+\|f(t,0,0)\|_\infty,\\ &\le \varphi_1^*\|x-0\|_\infty+\varphi_2^*\left\|D_1^\beta x(t)-0\right\|_\infty+f_0,\end{aligned}$$

and

$$\begin{aligned}\|D_1^\beta x(t)-0\|_\infty &\le (\frac{A\varphi_1^*}{1-A\varphi_2^*})\|x-0\|_\infty+\|\frac{\log t-1}{(1-A\varphi_2^*)\Gamma(2-\beta)}c_1\|_\infty\\ &\le (\frac{A\varphi_1^*}{1-A\varphi_2^*})R+\frac{1}{(1-A\varphi_2^*)\Gamma(2-\beta)}c_1,\end{aligned}$$

so we find

$$\begin{aligned}\|T(x(t)\|_X &\le G_0(\varphi_1^*R+f_0)\\ &+\frac{(1-v_1)}{a_1}\psi_1^*(e-1)R+\frac{v_1}{a_2}(e-1)\psi_2^*R\\ &+\frac{(1-v_1)}{a_1}(e-1)g_0+\frac{v_1}{a_2}(e-1)h_0\\ &+G_0\varphi_2^*\left((\frac{A\varphi_1^*}{1-A\varphi_2^*})R+\frac{1}{(1-A\varphi_2^*)\Gamma(2-\beta)}c_1\right),\end{aligned}$$

Consequently, for each $t\in[1,e]T(B)\subseteq B$.

step 3 T(B) is bounded and equicontinuous.

Let $t_1,t_2\in[1,e], witht_1<t_2$ and $x\in B$, then:

$$\begin{aligned}
|T(x)(t_2) - T(x)(t_1)| &= \Bigg| \int_1^e G\left(t_2, s\right) F(s) ds \\
&+ \frac{(\log t_2)}{a_1}(1 - v_1) \int\limits_1^e g(s, x(s)) ds \\
&- \frac{(v_1 \log t_2)}{a_2} \int\limits_1^e h(s, x(s)) ds - \int_1^e G\left(t_1, s\right) F(s) ds \\
&- \frac{(\log t_1)}{a_1}(1 - v_1) \int\limits_1^e g(s, x(s)) ds + \frac{(v_1 \log t_2)}{a_2} \int\limits_1^e \ h(s, x(s)) ds \Bigg| \\
&\leq \int_1^e [G\left(t_2, s\right) - G\left(t_1, s\right)] ds |F(s)| \\
&+ \frac{|\log t_2 - \log t_1|}{a_1}(1 - v_1) \int\limits_1^e |g(s, x(s))| ds \\
&+ \frac{v_2 \big| \log t_1 - \log t_2 \big|}{a_2} \int\limits_1^e |h(s, x(s))| ds, \\
\|T(x)(t_2) - T(x)(t_1)\|_\infty &\leq \int\limits_1^e \left[G\left(t_2, s\right) - G\left(t_1, s\right)\right] ds \|F(s)\|_\infty \\
&+ \frac{|\log t_2 - \log t_1|}{a_1}(1 - v_1)(e - 1)(\psi_1^* R + g_0) \\
&+ \frac{v_2 |\log t_1 - \log t_2|}{a_2}(e - 1)(\psi_2^* R + h_0),
\end{aligned}$$

and

$$
\begin{aligned}
\|D_1^{\beta}x(t_2) - D_1^{\beta}x(t_1)\|_\infty &= \| \int_1^e ((\log\frac{t_2}{s})^{\beta-\alpha-1} - (\log\frac{t_1}{s})^{\beta-\alpha-1})\frac{F(s)}{s}ds \\
&+ \frac{c_1(\log t_2)^{1-\beta} - (\log t_1)^{1-\beta}}{\Gamma(2-\beta)}\Bigg\|_\infty, \\
&\leq \int_1^e \left|\frac{(\log\frac{t_2}{s})^{\beta-\alpha-1} - (\log\frac{t_1}{s})^{\beta-\alpha-1}}{s}\right| ds\|F(s)\|_\infty \\
&+ \left|\frac{(\log t_2)^{1-\beta} - (\log t_1)^{1-\beta}}{\Gamma(2-\beta)}\right| \Bigg(\frac{v_1}{\Gamma(\alpha)}\int_1^e \frac{\log{(1-\log s)}^{\alpha-1}}{s}|ds\|F(s)\|_\infty \\
&+ |\frac{v_2}{\Gamma(\alpha-\beta)}\int_1^e \frac{(1-\log s)^{\alpha-\beta-1}}{s}ds\|F(s)\|_\infty \\
&+ \frac{v_1}{a_1}\int_1^e \|g(s,x(s))\|ds + \frac{v_1}{a_2}\int_1^e \|h(s,x(s))\|ds\Bigg), \\
&\leq \int_1^e \left|\frac{(\log\frac{t_2}{s})^{\beta-\alpha-1} - (\log\frac{t_1}{s})^{\beta-\alpha-1})}{s}\right| ds\|F(s)\|_\infty \\
&+ \left|\frac{(\log t_2)^{1-\beta} + (\log t_1)^{1-B}}{\Gamma(2-\beta)}\right| (\mu+\eta)\|F(s)\| \\
&+ \frac{(v_1)}{a_1}(e-1)(\psi_1^* R + g_0) + \frac{v_1}{a_2}(e-1)(\psi_2^* R + h_0),
\end{aligned}
$$

$$\begin{aligned}\|T(x)(t_2)-T(x)(t_1)\|_X &= \int_1^e \left[G\left(t_2,s\right)-G\left(t_1,s\right)\right]ds\|F(s)\|_\infty \\ &+ \frac{|\log t_2-\log t_1|}{a_1}(1-v_1)(e-1)(\psi_1^*R+g_0) \\ &+ \frac{v_2|\log t_1-\log t_2|}{a_2}(e-1)(\psi_2^*R+h_0) \\ &+ \int_1^e \left|\frac{(\log\frac{t_2}{s})^{\beta-\alpha-1}-(\log\frac{t_1}{s})^{\beta-\alpha-1})}{s}\right|ds\|F(s)\|_\infty \\ &+ \left|\frac{(\log t_2)^{1-\beta}-(\log t_1)^{1-\beta}}{\Gamma(2-\beta)}\right|(\mu+\eta)\,\|F(s)\|_\infty \\ &+ \frac{(v_1)}{a_1}(e-1)(\psi_1^*R+g_0)+\frac{v_1}{a_2}(e-1)(\psi_2^*R+h_0),\end{aligned}$$

where

$$\begin{aligned}\|F(s)\|_X &\leq G_0(\varphi_1^*R+f_0) \\ &+ \frac{(1-v_1)}{a_1}\psi_1^*(e-1)R+\frac{v_1}{a_2}(e-1)\psi_2^*R \\ &+ \frac{(1-v_1)}{a_1}(e-1)g_0+\frac{v_1}{a_2}(e-1)h_0 \\ &+ G_0\varphi_2^*\left((\frac{A\varphi_1^*}{1-A\varphi_2^*})R+\frac{1}{(1-A\varphi_2^*)\Gamma(2-\beta)}c_1\right),\end{aligned}$$

since the function $t\to\log t$ is uniformly, when $|\log t_2-\log t_1|\to 0$ then $|t_2-t_1\to 0|$ also G is a uniformly,so $|G(s.t_2-G(s.t_2)|\to 0$ As $t_1\to t_2$, the right-hand side of the above inequality tends to zero.

Then TB is equicontinouaus.

step 4 a priori bounds. We now show there exists an open set I with $x\neq\lambda T(x)$,for

$\lambda \in [1, e]$ and $x \in \partial X$.

Let $x \in X$, and $x = \lambda T(X)$, for some $0 < \lambda < 1$.Thus for each $t \in [1, e]$,

we have

$$x(t) = \lambda\Bigg[\int_1^e G(t,s)F(s,x(s))ds + \frac{(\log t)}{a_1}(1-v_1)\int_1^e g(s,x(s))ds - \frac{(v_2 \log t)}{a_2}\int_1^e h(s,x(s))ds\Bigg],$$

$$= \lambda\int_1^e G(t,s)F(s,x(s))ds + \lambda\frac{(1-v_1)}{a_1}\int_1^e g(s,x(s))ds - \lambda\frac{(v_2)}{a_2}\int_1^e h(s,x(s))ds,$$

$$\|x(t)\|_\infty \leq \int_1^e |G(t,s)|ds\|F(s,x(s))\|_\infty + \frac{(1-v_1)}{a_1}\int_1^e \|g(s,x(s))\|_\infty ds + \frac{v_1 \log t)}{a_2}\int_1^e \|h(s,x(s))\|_\infty ds,$$

by (**H1**) we have

$$\|F(x(t))\|_\infty \leq (\varphi_1^*\|x\|_\infty + \varphi_2^*\|D_1^\beta x(t)\| + f_0),$$

then:

$$\begin{aligned}
\|x(t)\|_\infty &\leq G_0(\varphi_1^*\|x\|_\infty + \varphi_2^*\|D_1^\beta x(t)\|_\infty + f_0) \\
&+ \frac{(1-v_1)}{a_1}\psi_1^*(e-1)\|x\|_\infty \\
&+ \frac{(v_1)}{a_2}(e-1)\psi_2^*\|x\|_\infty, \\
&\leq (G_0\varphi_1^*\|x\|_\infty + G_0\varphi_2^*\|D_1^\beta x(t)\|_\infty + G_0 f_0), \\
&+ \frac{(1-v_1)}{a_1}\psi_1^*(e-1)\|x\|_\infty + \frac{(v_2)}{a_2}(e-1)\psi_2^*\|x\|_\infty \\
&+ \frac{(1-v_1)}{a_1}(e-1)g_0 + \frac{v_1}{a_2}(e-1)h_0,
\end{aligned}$$

consequntly, we get

$$\begin{aligned}
\|x(t)\|_\infty &\leq (G_0\varphi_1^*\|x\|_\infty + G_0\varphi_2^*\|D_1^\beta x(t)\|_\infty + G_0 f_0) \\
&+ \frac{(1-v_1)}{a_1}\psi_1^*(e-1)\|x\|_\infty + \frac{(v_2)}{a_2}(e-1)\psi_2^*\|x\|_\infty, \\
&\leq \left(G_0\varphi_1^* + \frac{(1-v_1)}{a_1}\psi_1^*(e-1) + \frac{v_2}{a_2}(e-1)\psi_2^*\right)\|x\|_\infty \\
&+ \left(G_0 f_0 + \frac{(1-v_1)}{a_1}(e-1)g_0\frac{v_2}{a_2}(e-1)h_0\right) + G_0\varphi_2^*\|D_1^\beta x(t)\|_\infty, \\
&\leq \frac{G_0 f_0 + \frac{(1-v_1)}{a_1}(e-1)g_0 - \frac{v_2}{a_2}(e-1)h_0}{(1(-G_0\varphi_1^* + \frac{(1-v_1)}{a_1}\psi_1^*(e-1) + \frac{v_2}{a_2}(e-1)\psi_2^*)} \\
&+ \frac{G_0\varphi_2^*\|D_1^\beta x(t)\|_\infty}{G_0\varphi_1^* + \frac{(1-v_1)}{a_1}\psi_1^*(e-1) + \frac{v_2}{a_2}(e-1)\psi_2^*},
\end{aligned}$$

so

$$\|x\|_X \leq \frac{G_0 f_0 + \frac{(1-v_1)}{a_1}(e-1)g_0 + \frac{v_2}{a_2}(e-1)h_0}{(1(-G_0\varphi_1^* + \frac{(1-v_1)}{a_1}\psi_1^*(e-1) + \frac{v_2}{a_2}(e-1)\psi_2^*)},$$

where

$$M = \frac{G_0 f_0 + \frac{(1-v_1)}{a_1}(e-1)g_0 + \frac{v_2}{a_2}(e-1)h_0}{(1 - G_0\varphi_1^* + \frac{(1-v_1)}{a_1}\psi_1^*(e-1) + \frac{v_2}{a_2}(e-1)\psi_2^*)},$$

let $x = x \in X : \|x\|_X < M + \varepsilon$. We have $\|x\|_x \leq M$ for each $x \in U \subset X$

$$U = \{x \in X, \|x\| \leq M + \varepsilon\},$$

suppose the exists $x \in \partial U$ such that

$$x = \lambda T x,$$

so we find

$$M + \varepsilon = \|x\|_x < M < M + \varepsilon,$$

which is a contradiction.

By our choice of X,there is no ∂U such that $x \in \lambda T(x)$,for $\lambda \in [0, 1]$.

Hence, by the theorem of "Leray-Schauder" T has a fixed point.

□

Exemple 3.1.

$$D_1^{3/2} = \frac{\sin t^2}{(e^{-t} + 2)^3(|x| + |D^{1/2}x(t)|)},$$

$$x(1) + D^{1/2}x(1) = \int_1^e \frac{t}{10}\cos x ds,$$
$$x(2) + D^{1/2}x(e) = \int_1^e \frac{t}{10}\sin x ds,$$

$$\alpha = \frac{3}{2}, \quad a_1 = a_2 = b_1 = b_2 = 1,$$

$$g(t,x(t)) = \frac{t}{10}\cos x, \quad h(t,x(t)) = \frac{t}{10}\sin x,$$

$$\begin{aligned}
\left|f(t,x,u)-f(t,y,v)\right| &= \left|\frac{\sin t^2 + (|x|+|D^{1/2}x(t)|)}{(e^{-t}+2)^3} - \frac{\sin t^2 + (|y|+|D^{1/2}y(t)|)}{(e^{-t}+2)^3}\right|,\\
&= \left\|\frac{\sin t^2}{(e^{-t}+2)^3}\left(\frac{|x-y|+|D^{1/2}x-D^{1/2}y|}{\Big(|x|+|D^{1/2}x|\Big)\Big(|y|+D^{1/2}y|\Big)}\right)\right\|,\\
&\leq \left|\frac{\sin t^2}{(e^{-t}+2)^3}\right|\left[\|x-y\|+\|D^{1/2}x-D^{1/2}y\|\right],\\
&\leq \frac{1}{(e^{-1}+2)^3}\|x-y\|+\|D^{1/2}x-D^{1/2}y\|,\\
&\leq \frac{1}{(e^{-1}+2)^3}\|x-y\|+\|u-v\|.
\end{aligned}$$

$$\varphi_1^* = \varphi_2^* = \frac{1}{(e^{-1}+2)^3},$$

$$\begin{aligned}
|g(t,x(t))-g(t,y(t))| &= \left|\int_1^e (\frac{t}{10}\cos x - \frac{t}{10}\cos y)ds\right|,\\
&\leq \int_1^e \left|\frac{t}{10}\right| ds\|x-y\|,\\
&\leq \frac{e^2-1}{20}\|x-y\|.
\end{aligned}$$

and

$$\begin{aligned}
|h(t,x(t))-h(t,y(t))| &= \left|\int_1^e (\frac{t}{10}\sin x - \frac{t}{10}\sin y)ds\right|,\\
&\leq \int_1^e \left|\frac{t}{10}\right| ds|\sin x-\sin y|,\\
&\leq \frac{e^2-1}{20}|x-y|.
\end{aligned}$$

$$\Psi_1^* = \Psi_2^* = \frac{e^2-1}{20}.$$

Conclusion and perspectives

In this memory, we have studied the existence and uniqueness of two fractional -differential problems, by using fixed point theorems(Banach principle, Schauder theorem and Leray-SChauder alternative). As perspectives, we will try to study the generalized fractional boundary value problem with a conformable fractional derivative of Caputo and its fractional integral. This generalized version represents a challenge from the analytical point of view, i.e. the existence and uniqueness of the solution. However, the numerical side remains the same.

Bibliography

[1] Ahmad. B and Nieto. J. J, *Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions*, Boundary Value Problems, vol. 2009, 11 pages, 2009.

[2] Banach. S, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., 3 (1922), 133-181.

[3] M. Benchohra, J.R. Graef and S. Hamani, *Existence results for boundary value problems with nonlinear fractional differential equations*, Appl. Anal. 87 (7) (2008), 851-863

[4] X. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[5] A. Hamrouni , S. Beloul, *Existence of solutions for boundary value problems of fractional integro-differential equations with integral boundary conditions on Banach spaces*,Nonlinear Stud, 26(3)(2019), 693-701.

[6] R. Hilfer. R, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore (2000).

[7] F. Jarad, T. Abdeljawad and D. Beleanu, *Caputo-type modification of the Hadamard fractional derivative*, Advances in Difference Equations. 2012, 142, (2012).

[8] A. A. Kilbas, H M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

[9] K. S. Miller. and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York (1993).

[10] Podulbny. I, *Fractional Diffrential Equations*, Academic Press, San Diego, 1999.

[11] A. Schwartz, *Analyse Topologie générale et analyse fonctionnelle*,Paris,1993.

[12] Zhang. S, *Positive solutions for boundary-value problems of nonlinear fractional differential equations*, Electron. J. Differential Equations, No. 36 (2006), 1-12.

الملخص

يعتبر مبدأ النقطة الثابتة ذا أهمية كبيرة في حل كثير المعادلات التفاضلية غير الخطية خاصة في دراسة وجود ووحدانية الحلول وفي هذه المذكرة تطرقنا الى دراسة وجود حلول بعض أنواع المعادلات التفاضلية ذات رتبة ناطقة بمفهوم كابيتو، و كابيتو هادمارد باستعمال تقنية نظريات النقطة الثابتة لبناخ، و شودر ، ولاري شودر.

الكلمات والجمل المفتاحية :المعادلات التفاضلية ذات رتبة ناطقة، وجود الحلول، الاشتقاقية ذات رتبة ناطقة حسب كابيتو ، الاشتقاقية ذات رتبة ناطقة حسب كابيتوهادمارد، التكامل ذو رتبة ناطقة، النقطة الثابتة.

Abstract

The fixed point principle is so important in the study of several non linear differential equations, particulary, problems of existence and uniqeness. In this memory, we present several existence results for certain classes of differential equations of fractional order in the sense of Caputo, Caputo-Hadamard, These results were obtained by using the fixed point theorems (Banach, schauder, leray schauder).

Key words and phrases : fractional differential equations, existence of solutions,caputo fractional derivative, , dérivée fractionnaire de type Caputo-Hadamard, the fractional order integral, fixed point.

Résume

Le principe de point fixe est très important dans la résolution de plusieurs équations différentielles non linéaires, en particulier, dans l'étude de l'existence et de l'unicité. Dans ce mémoire, nous présentons plusieurs résultats d'existence pour certaines classes d'équations différentielles d'ordre fractionnaire au sens de Caputo, Caputo-Hadamard Ces résultats ont été obtenus par l'utilisation du théorèmes de point fixe (Banach, schauder, leray schauder).

Phrases et mots clés : équations différentielles fractionnaires, existence de solutions, dérivée fractionnaire de type Caputo, dérivée fractionnaire de type Caputo-Hadamard, intégral fractionnaire, point fixe.