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The sub-supersolution method for partial differential equations

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بسم الله الرحمن الرحيم

الإهداء

لقلوبٍ علَّمتني الصبر:

أسرتي.. حبّكم بحرٌ لا ساحل له،

أنتم شمسُ نجاحاتي.

لأستاذي الموقَّر **زاوش المهدي**:

عطاؤك كان شمعةً أضاءت دربي،

فشكرًا لمنحني حروف المعرفة.

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هذا **العمل** ثمرة جهد وتعب اهديه لكم.

زغومة نجاة

الإهــــــــــــــــــــــــــداء

إلى من غرسوا في قلبي معنى الصبر،..
وغمروني بحب لا يذبل،

إلى **أسرتي** العظيمة،

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هذا العمل، ثمرة جهد وتعب، أهديه لكم... بمحبة وامتنان.

حوري محمد

Abstract

This memory presents a study of the sub-supersolution method as a robust tool for proving the existence of solutions to nonlinear differential equations, with a focus on its application to the Kirchhoff model derived from elastic string

vibrations. The methodology revolves around constructing

a pair of functions (supersolution and subsolution) that bound the desired solution, followed by a generating monotonic sequence converges to it.

Keywords: sub-supersolution method, nonlinear differential equations, monotonic sequence, existence.

ملخص

تقدم هذه المذكرة دراسة لطريقة الحلول السفلية والعلوية كأداة في إثبات وجود حلول للمعادلات التفاضلية الغير خطية, مع تطبيقها على نموذج كيرشوف الناشئ عن نمذجة إهتزازات الأوتار المرنة.

تعتمد المنهجية على بناء زوج من الدوال (حل سفلي وحل علوي) يحصران الحل المطلوب بينهما, ثم إنشاء متتالية رتيبة تتقارب نحوه.

كلمات مفتاحية: طريقة الحلول السفلية والعلوية. معادلات تفاضلية غير خطية, متتالية رتيبة, وجود.

Résumé

Cette mémoire présente une étude de la méthode des sous-super solutions comme un outil pour prouver l'existence de solutions à des équations différentielles non linéaires, en mettant l'accent sur l'étude de l'exitence d'une solution positive d'un problème elliptique de type Kirchhoff mo délisant les vibrations de cordes élastiques. La méthodologie tourne autour de la construction d'une paire de fonctions (super solution et sous solution) qui encadrent la solution souhaitée, vient d'une suite monotone générée convergente.

Mots-clés : méthode des sous-super solutions, équations différentielles non linéaires, suite mono tone, existence.

Contents

Notation

- $\mathbb{N}$: the set of natural numbers.
- $\mathbb{R}$: the set of real numbers.
- $\mathbb{R}^N$: represents a n-dimensional Euclidean space with real-number coordinates.
- $\Omega \subset \mathbb{R}^N$: open set in $\mathbb{R}^N$.
- $\overline{\Omega}$: closure of Ω.
- $\partial\Omega$: boundary of Ω.
- $C(\Omega)$: space of continuous functions on Ω.
- $C^k(\Omega)$: space of k times continuously differentiable functions on Ω.
- $\mathcal{D}(\Omega)$: the set of test functions.
- $mes(\Omega)$: is the measure of Ω.
- $\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right)$: gradient of the function u.
- $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$: Laplacian of u.
- $\|\nabla u\|^2_{L^2(\Omega)} = \int_\Omega |\nabla u|^2 \, dx \quad \text{with } |\nabla u|^2 = \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}\right)^2$.
- $|\gamma| = |(\gamma_1, \dots, \gamma_N)| = \sum_{i=1}^{N} \gamma_i$: is the order of γ with $\gamma_i \in \mathbb{N}$.
- $D^\gamma f = \frac{\partial^{|\gamma|} f}{\partial x_1^{\gamma_1} \dots \partial x_N^{\gamma_N}}$: is the partial derivative with respect to a multi-index.
- $\|u\| = \left(\int_\Omega |\nabla u|^2 \, dx\right)^{\frac{1}{2}}$: is the norm of u in $H_0^1(\Omega)$.

Introduction

The sub-supersolution method is a powerful analytical tool in the study of nonlinear partial differential equations (PDEs) originating from classical techniques in elliptic PDE theory. This method has evolved into a cornerstone for proving the existence and regularity of solutions to complex boundary value problems. Its core idea lies in constructing ordered pairs of functions subsolutions (lower bounds) and supersolutions (upper bounds) that satisfy differential inequalities. These functions act as "barriers," confining the solution within their range and enabling the construction of monotonic sequences that converge to the desired solution.

The roots of the sub-supersolution method trace back to the early 20th century, with foundational contributions from mathematicians like Perron, Picard, and Schauder. Initially applied to linear equations, the method gained prominence in the 1960s and 1970s through the work of Amann, Sattinger, and Lions, who extended it to nonlinear elliptic and parabolic equations. These developments were pivotal in addressing problems where direct variational approaches were insufficient, particularly for equations lacking coercivity or symmetry. The method's versatility was further demonstrated in the study of reaction-diffusion systems, free boundary problems, and equations modeling physical phenomena such as heat conduction and fluid dynamics. Over time, it has been refined to accommodate nonlocal operators, fractional derivatives, and systems with critical growth, solidifying its role as a fundamental technique in modern PDE analysis.

This memory explores the sub-supersolution method and its application to nonlinear elliptic equations of Kirchhoff type. The work is structured as follows. The first Chapter covers elements of functional analysis to support subsequent arguments such as Hölder spaces, Lebesgue spaces, Sobolev spaces, Poincaré inequality, the Lax-Milgram Theorem and Schauder estimates. The second Chapter presents the sub-supersolution method through the following boundary-value problem for the nonlinear Poisson equation,

$$\begin{cases} -\Delta u = g(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a regular function. The last Chapter applies the sub-supersolution method to the following Kirchhoff-type problem modeling the vibrations of elastic strings to prove the existence of a positive solution,

$$\begin{cases} -M(\|u\|^2)\Delta u = |u|^{p-1}u + \alpha g(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where p, α are positive real numbers and $M : [0, \infty[\to \mathbb{R}$, $g : \overline{\Omega} \to \mathbb{R}$ are given functions.

Chapitre 1

Preliminaries

In this chapter covers of functional analysis used throught this memory. The content of this chapter can be foind in [3, 5, 7, 8]

1.1 Hölder spaces

Definition 1.1.1.

Let Ω be an open set in $\mathbb{R}^N$. A function $f : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}$ is called Hölder continuous with exponent β $(0 < \beta \leq 1)$ if there exists a constant $C > 0$ such that:

$$|f(x) - f(y)| \leq C|x - y|^\beta \quad \forall x, y \in \Omega.$$

If $\beta = 1$, the function is Lipschitz continuous. We note that f is continuous and we set

$$\|f\|_{C^0(\overline{\Omega})} = \sup_{x\in\overline{\Omega}} |f(x)|, \quad |f|_{C^{0,\beta}(\overline{\Omega})} = \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^\beta}.$$

Generally, we define the Hölder space $C^{k,\beta}(\overline{\Omega})$ as the space of functions in C^k whose all k-th order partial derivatives in $C^{0,\beta}(\Omega)$.

Theorem 1.1.1.

The space $C^{0,\beta}(\overline{\Omega})$ is a Banach space when equipped with the norm

$$\|f\|_{C^{0,\beta}(\overline{\Omega})} = \|f\|_{C^0(\overline{\Omega})} + |f|_{C^{0,\beta}(\overline{\Omega})}.$$

The space $C^{k,\beta}(\overline{\Omega})$ is a Banach space with respect to the norm

$$\|f\|_{C^{k,\beta}(\overline{\Omega})} = \|f\|_{C^k(\overline{\Omega})} + \sum_{|\gamma|=k} |D^\gamma f|_{C^{0,\alpha}(\overline{\Omega})},$$

where $\gamma = (\gamma_1, ..., \gamma_N) \in \mathbb{N}^N$ *and*

$$\|f\|_{C^k(\overline{\Omega})} = \sum_{|\gamma| \leq k} \sup_{x \in (\overline{\Omega})} |D^\gamma f(x)|.$$

Example 1.1.1.

We define $g :]0,1[\to \mathbb{R}$ *by* $g(x) = \sqrt{x}$*. Then* $g \in C^{0,\frac{1}{2}}([0,1])$*. Indeed,*

$$\begin{aligned} \forall x, y \in]0,1[: \ |\sqrt{x} - \sqrt{y}| \leq \sqrt{x - y} &\Leftrightarrow x + y - 2\sqrt{xy} \leq x + y - 2xy \\ &\Leftrightarrow \sqrt{xy} \geq xy \\ &\Leftrightarrow xy \geq x^2 y^2. \end{aligned}$$

This last inegualty is true because $x, y \in]0,1[$*. Thus,* $g \in C^{0,\frac{1}{2}}([0,1])$*. But* $g \notin C^{1,\frac{1}{2}}([0,1])$ *because it is not differertiable at* 0*.*

Theorem 1.1.2.

Let Ω *be a bounded set in* $\mathbb{R}^N$*,* $k \in \mathbb{N}$ *and* $0 < \beta \leq 1$*. Then, the embedding*

$$C^{k,\beta}(\overline{\Omega}) \hookrightarrow C^k(\overline{\Omega})$$

is compact.

1.2 The Lebesgue spaces $\mathbf{L}^p(\Omega)$

Definition 1.2.1.

Let Ω *be an open set in* $\mathbb{R}^N$ *and* $p \in \mathbb{R}$ *with* $1 \leq p < \infty$*. We define the Lebesgue space* $L^p(\Omega)$ *by*

$$L^p(\Omega) = \left\{ v : \ \Omega \to \mathbb{R} \ \middle| \ \int_\Omega |v(x)|^p \ dx < \infty \right\}.$$

Theorem 1.2.1.

The function

$$\|.\|_{L^p(\Omega)} : \ L^p(\Omega) \to \mathbb{R}, \ v \mapsto \|v\|_{L^p(\Omega)} = \left(\int_\Omega |v(x)|^p \ dx \right)^{\frac{1}{p}}$$

is a norm on $L^p(\Omega)$*. The space* $(L^p(\Omega), \|.\|_{L^p(\Omega)})$ *is a Banach space.*

Theorem 1.2.2. ***(Hölder inequality)***

Let $p \in]1, \infty[$ *and* q *denote the conjugate exponent defined by* $\frac{1}{p} + \frac{1}{q} = 1$*. If* $u \in L^p(\Omega)$ *and*

$v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\int_\Omega |uv| \ dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

Moreover, the Cauchy-Schwarz inequality is a special case of the Hölder inequality satisfied

$$\int_\Omega |uv| \ dx \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}. \tag{1.1}$$

Theorem 1.2.3. ***(Dominated Convergence Theorem)***
Let Ω be an open set in $\mathbb{R}^N$. If $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions in $L^2(\Omega)$ such that
1. $f_n \to f$ pointwise almost everywhere in Ω.
2. There exists a function $g \in L^2(\Omega)$ such that $|f_n| \leq g$ for all $n \in \mathbb{N}$.
Then, $f \in L^2(\Omega)$ and

$$\lim_{n\to\infty} \int_\Omega |f_n - f|^2 \ dx = 0.$$

Theorem 1.2.4. ***(Monotone Convergence Theorem)***
Let $(u_n)_{n\in\mathbb{N}}$ be a monotone nondecreasing sequence (i.e., $u_{n+1} \geq u_n$ for all $n \in \mathbb{N}$) and there exists $M \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N}: \ u_n \leq M.$$

Then, $(u_n)_{n\in\mathbb{N}}$ converges to a function u satisfing $u \leq M$.

1.3 Sobolev spaces

Definition 1.3.1.
We call the Sobolev space of order 1 on Ω by

$$H^1(\Omega) = \Big\{u \in L^2(\Omega): \ \frac{\partial u}{\partial x_j} \in L^2(\Omega), \forall j = 1, ..., N\Big\}.$$

The derivation is to be understood in the sense of distributions. In other words, a function $u \in L^2(\Omega)$ such that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_j} dx = -\int_\Omega v_j \varphi \ dx, \ \forall \varphi \in \mathcal{D}(\Omega), \ \forall j = 1, ..., N,$$

which gives

$$v_j = \frac{\partial u}{\partial x_j}, \quad \forall j = 1, ..., N.$$

In a similar way, let $k \in \mathbb{N}$ and $1 \leq p < \infty$. The Sobolev space $W^{k,p}(\Omega)$ is defined by

$$W^{k,p}(\Omega) = \left\{ u \in L^p(\Omega) \,\middle|\, \forall \alpha \in \mathbb{N}^N,\ |\alpha| \leq k:\ D^\alpha u \in L^p(\Omega) \right\}.$$

Theorem 1.3.1.

The space $H^1(\Omega)$ is Hilbert space with the following inner product

$$\forall u, v \in H^1(\Omega):\ (u,v)_{H^1(\Omega)} = \int_\Omega uv\ dx + \int_\Omega \nabla u \cdot \nabla v\ dx$$

and the norm associed

$$\|u\|_{H^1(\Omega)} = \Big(\int_\Omega |\nabla u(x)|^2 + \int_\Omega |u(x)|^2 dx \Big)^{\frac{1}{2}}.$$

The space $W^{k,p}(\Omega)$ is a Banach space when equipped with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \Big(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u(x)|^p + \int_\Omega |u(x)|^p dx \Big)^{\frac{1}{p}}.$$

Definition 1.3.2.

Let Ω be an open bounded in $\mathbb{R}^N$. The space $H_0^1(\Omega)$ is defined by

$$H_0^1(\Omega) = \{ v \in H^1(\Omega):\ v = 0 \quad on\ \partial\Omega \},$$

which is equipped with the norm of $H^1(\Omega)$.

Theorem 1.3.2. *(**Poincaré inequality**)*

Let Ω be an open bounded in $\mathbb{R}^N$. There exists a constant C_Ω suth that

$$\forall u \in H_0^1(\Omega):\ \|u\|_{L^2(\Omega)} \leq C_\Omega \|u\|.$$

Remark 1.3.1.

By the Poincaré inequality, the space $H_0^1(\Omega)$ is a Hilbert space with the following inner product

$$\forall u, v \in H_0^1(\Omega):\ (u,v)_{H_0^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v\ dx$$

and the norm

$$\|v\| = \Big(\int_\Omega |\nabla u|^2\ dx \Big)^{1/2}.$$

This norm is equivalent to

$$\|u\|_{H_0^1(\Omega)} = \Big(\int_\Omega |\nabla u(x)|^2 + \int_\Omega |u(x)|^2 dx \Big)^{\frac{1}{2}}.$$

Theorem 1.3.3. ***(Sobolev embedding)***

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$ and let $k \in \mathbb{N}^*$, $1 \leq p < \infty$.

(i) If $kp < N$, then the embedding $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $1 \leq q < Np/(N-kp)$.

(ii) If $kp = N$, then the embedding $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leq q < \infty$ is compact.

(iii) If $kp > N$, then $W^{k,p}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$, where

$$\alpha = \begin{cases} k - N/p & \text{if } k - N/p < 1, \\ 1 & \text{if } k - N/p > 1. \end{cases}$$

If $k - N/p = 1$, the embedding holds for every $\alpha \in [0, 1[$.

We consider the linear Dirichlet boundary value problem

$$\begin{cases} -\Delta u(x) = h(x) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where h is a given function on Ω. If $h \in L^2(\Omega)$, then, a weak solution of (1.2), a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega hv \, \mathrm{d}x, \quad \forall v \in H_0^1(\Omega).$$

Theorem 1.3.4. ***(Schauder estimates)***

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. If Ω is of class $C^{2,\beta}$ and $h \in C^{0,\beta}(\overline{\Omega})$ then $u \in C^{2,\beta}(\overline{\Omega})$ is a classical solution of (1.2) and

$$\|u\|_{C^{2,\beta}(\overline{\Omega})} \leq C\|h\|_{C^{0,\beta}(\overline{\Omega})}.$$

Theorem 1.3.5. ***(Maximum principle)***

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and let h be a nonnegative function. Suppose that $u \in C^2(\Omega) \cup C(\bar{\Omega})$ satisfying (1.2). Then $u \geq 0$ in Ω. Moreover, either $u > 0$ in Ω or $u \equiv 0$ in Ω.

Theorem 1.3.6. ***(Lax-Milgram Theorem)***

Let $a : H \times H \to \mathbb{R}$ a bilinear function, for which there exist constants $\alpha, \beta > 0$ such that

$$\forall u, v \in H : \ |a(u, v)| \leq \alpha\|u\|\|v\|$$

and

$$\forall u \in H : \ \beta\|u\|^2 \leq a(u, u).$$

On the other hand, let $L : H \to \mathbb{R}$ be a bounded linear functional on H. Then, there exists a unique element $u \in H$ such that

$$\forall v \in H : \; a(u, v) = L(v).$$

Example 1.3.1.

Let $f \in C^{0,\beta}(\overline{\Omega})$ with $\beta \in]0, 1]$ and $0 \neq f \geq 0$. We show that the following linear problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

has a unique positive solution.

(1) Prove that the problem (1.3) has a unique solution $u \in H_0^1(\Omega)$. We use Lax-Milgram theory. Then, the variational formulation of (1.3) gives by

$$\int_\Omega \nabla u \cdot \nabla \varphi \; dx = \int_\Omega f\varphi \; dx \quad \forall \varphi \in H_0^1(\Omega).$$

Notice that $H_0^1(\Omega)$ is a Hilber space. We take

$$\forall u, \varphi \in H_0^1(\Omega) : \; a(u, \varphi) = \int_\Omega \nabla u \cdot \nabla \varphi \; dx$$

and

$$\forall \varphi \in H_0^1(\Omega) : \; L(\varphi) = \int_\Omega f\varphi \; dx.$$

• *a is a bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$. For all $u, \varphi, w \in H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$, we have*

$$\begin{aligned} a(\lambda u + \varphi, w) &= \int_\Omega \nabla(\lambda u + \varphi) \cdot \nabla w \; dx \\ &= \int_\Omega \nabla \varphi \cdot \nabla w + \lambda \nabla u \cdot \nabla w \; dx \\ &= \lambda \int_\Omega \nabla u \cdot \nabla w \; dx + \int_\Omega \nabla \varphi \cdot \nabla w \; dx \\ &= \lambda a(u, w) + a(\varphi, w). \end{aligned}$$

Then a is a bilinear form with respect to the first variable. Since a is symetric,

$$\begin{aligned} \forall u, \varphi \in H_0^1(\Omega) : \; a(u, \varphi) &= \int_\Omega \nabla u \cdot \nabla \varphi \; dx \\ &= \int_\Omega \nabla \varphi \cdot \nabla u \; dx \\ &= a(\varphi, u), \end{aligned}$$

so the function a is bilinear.

• *a is continuous. Let $u, \varphi \in H_0^1(\Omega)$. By using the Cauchy-Schwarz inequalities (1.1) and*

$$\forall y = (y_1, ..., y_N),\ z = (z_1, ..., z_N) \in \mathbb{R}^N : \qquad \Big|\sum_{i=1}^{N} y_i z_i\Big| \leq |y|\ |z|,$$

we obtain

$$\begin{aligned} |a(u,\varphi)| &= \left|\int_\Omega \nabla u \cdot \nabla \varphi \ dx\right| \\ &\leq \int_\Omega |\nabla u \cdot \nabla \varphi| \ dx \\ &\leq \int_\Omega |\nabla u|\ |\nabla \varphi| \ dx \\ &\leq \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2} \left(\int_\Omega |\nabla \varphi|^2 dx\right)^{1/2} \\ &\leq \|\nabla u\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)} \\ &\leq \|u\| \, \|\varphi\|, \end{aligned}$$

which gives the continuity of a.

• *a is coercive. For all $u \in H_0^1(\Omega)$, one can write*

$$\begin{aligned} a(u,u) &= \int_\Omega \nabla u \cdot \nabla u \ dx \\ &= \int_\Omega |\nabla u|^2 \ dx \\ &= \|u\|^2 . \end{aligned}$$

So the bilinear form a is coercive.

• *L is linear form on $H_0^1(\Omega)$. For all $\varphi, w \in H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$, we have*

$$\begin{aligned} L(\lambda\varphi + w) &= \int_\Omega (\lambda\varphi + w) f \ dx \\ &= \int_\Omega \lambda\varphi f + w f \ dx \\ &= \lambda \int_\Omega \varphi f dx + \int_\Omega w f \ dx \\ &= \lambda L(\varphi) + L(w), \end{aligned}$$

thus L is linear.

• *L is continuous. For all $\varphi \in H_0^1(\Omega)$, we have*

$$\begin{aligned} |L(v)| &= \left|\int_\Omega \varphi f \ dx\right| \\ &\leq \int_\Omega |\varphi|\ |f| \ dx. \end{aligned}$$

By the Cauchy-Schwarz inequality, we find

$$
\begin{aligned}
|L(v)| &\leq \left(\int_\Omega |f|^2 \, dx\right)^{1/2} \left(\int_\Omega |\varphi|^2 \, dx\right)^{1/2} \\
&\leq \|f\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} .
\end{aligned}
$$

Using Poincaré's inequality, one can write

$$
|L(\varphi)| \leq C_\Omega \;\; \|f\|_{L^2(\Omega)} \|\varphi\| ,
$$

hence, L is continuous.

From the Lax-Milgram Thorem, the problem (1.3) has a unique solution $u \in H_0^1(\Omega)$.

(2) We prove that the solution is positive using the weak maximum principle. Since $0 \neq f \geq 0$ is a regular function, the weak solution u of the problem (1.3) is nontrivial. Therefore, from Theorem 1.3.5, we get

$$
u > 0 \quad in \ \Omega.
$$

Finally, from (1) and (2) the problem (1.3) has a unique positive solution.

Theorem 1.3.7.

Let $\Omega \subseteq \mathbb{R}^N$ be an open. If $u \in H^1(\Omega)$, then the positive part $u^+ = \max(u, 0)$ and the negative part $u^- = \max(-u, 0)$ of u are in $H^1(\Omega)$ $(u^+, u^- \in H^1(\Omega))$. Furthermore, their weak derivatives of u^+ and u^- are given by

$$
\nabla u^+ = \begin{cases} \nabla u & a.e. \ on \ \{u > 0\}, \\ 0 & a.e. \ on \ \{u \leq 0\} \end{cases}
$$

and

$$
\nabla u^- = \begin{cases} 0 & a.e. \ on \ \{u \geq 0\}, \\ -\nabla u & a.e. \ on \ \{u < 0\}. \end{cases}
$$

Chapitre 2

METHOD OF SUBSOLUTIONS AND SUPERSOLUTIONS

This chapter illustrates a method for solving a nonlinear elliptic equation that involves to construct two global barrier functions a subsolution and a supersolution, which satisfy elliptic differential inequalities. While finding these barriers may be challenging, solving inequalities is often easier. Once these functions are identified, the maximum principle can be utilized to create a monotone sequence of functions that converge to the solution. This method is relatively straightforward and quite general, with the only limitations being the ability to solve the inequalities. The results of this Chapter can be found in [6].

2.1 Principle of the method

We will consider the following boundary-value problem for the nonlinear Poisson problem

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is smooth such that

$$\forall t \in \mathbb{R} : \ \left| g'(t) \right| \leq C, \tag{2.2}$$

for some positive constant C.

Let $v \in H_0^1(\Omega)$. We can obtain the variational formulation by multiplying (2.1) by v and integrating over Ω,

$$-\int_\Omega (\Delta u)\, v \; dx = \int_\Omega g(u) \; v \; dx.$$

Integrating by parts and using $v = 0$ on $\partial\Omega$, we obtain

$$-\int_\Omega (\Delta u)\, v \; dx = \int_\Omega \nabla u \cdot \nabla v \; dx = \int_\Omega g(u) \; v \; dx,$$

then

$$\forall v \in H_0^1(\Omega): \int_\Omega \nabla u \cdot \nabla v \; dx = \int_\Omega g(u) \; v \; dx.$$

Definition 2.1.1.

1. We say that $\bar{u} \in H^1(\Omega)$ is a weak supersolution of problem (2.1), if

$$\int_\Omega \nabla \bar{u} \cdot \nabla v \; dx \geq \int_\Omega g(\bar{u}) v \; dx \tag{2.3}$$

for all $v \in H_0^1(\Omega), v \geq 0$.

2. In the same way, $\underline{u} \in H^1(\Omega)$ is a weak subsolution of problem (2.1), if

$$\int_\Omega \nabla \underline{u} \cdot \nabla v \; dx \leq \int_\Omega g(\underline{u}) v \; dx \tag{2.4}$$

for all $v \in H_0^1(\Omega), v \geq 0$.

3. We say that $u \in H_0^1(\Omega)$ is a weak solution of problem (2.1), if

$$\int_\Omega \nabla u \cdot \nabla v \; dx = \int_\Omega g(u) v \; dx \tag{2.5}$$

for all $v \in H_0^1(\Omega)$.

Remark 2.1.1.

If $\bar{u}, \underline{u} \in C^2(\Omega)$, then from (2.3) and (2.4) it follows that

$$-\Delta \bar{u} \geq g(\bar{u}), \quad -\Delta \underline{u} \leq g(\underline{u}) \qquad \text{in } \Omega.$$

Theorem 2.1.1. *(existence theorem)*

Suppose there exist a weak supersolution $\bar{u}$ and a weak subsolution $\underline{u}$ of the problem (2.1) that satisfy the conditions

$$\underline{u} \leq 0, \quad \bar{u} \geq 0 \text{ on } \partial\Omega \text{ in the trace sense, } \underline{u} \leq \bar{u} \text{ a.e. in } \Omega. \tag{2.6}$$

So, there exists a weak solution u of the problem (2.1) such that

$$\underline{u} \leq u \leq \bar{u} \qquad \text{a.e. in } \Omega.$$

Proof. For $\alpha \in [0, \infty[$, set

$$\forall z \in \mathbb{R}: \; h(z) = g(z) + \alpha z.$$

The function h is differentiable and its derivative is given by

$$\forall z \in \mathbb{R}: \; h'(z) = g'(z) + \alpha.$$

If $\alpha \geq C$, it follows from (2.2) that

$$\begin{aligned} \forall z \in \mathbb{R}: \; h'(z) &= g'(z) + \alpha \\ &\geq g'(z) + |g'(z)| \\ &\geq 0. \end{aligned}$$

Hence, for all $\alpha \geq C$, the function h is nondecreasing.

Now, let us consider $u_0 = \underline{u}$. We prove that the following linear boundary-value problem

$$\begin{cases} -\Delta u_1 + \alpha u_1 = g(u_0) + \alpha u_0 & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega \end{cases} \tag{2.7}$$

has a unique weak solution $u_1 \in H_0^1(\Omega)$. We use the lax-Milgrame Theorem.

The variational formulation of (2.7) is

$$\int_\Omega \nabla u_1 \cdot \nabla v \; dx + \alpha \int_\Omega u_1 v \; dx = \int_\Omega (g(u_0) + \alpha u_0) v dx \quad \forall v \in H_0^1(\Omega).$$

Recalling that $H_0^1(\Omega)$ is a Hilber space with

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 \; dx + \int_\Omega |u|^2 \; dx \right)^{1/2}.$$

We take

$$\forall u, v \in H_0^1(\Omega): \; a(u, v) = \int_\Omega \nabla u \cdot \nabla v \; dx + \alpha \int_\Omega uv \; dx$$

and

$$\forall v \in H_0^1(\Omega): \; L(v) = \int_\Omega g(u_0)v + \alpha u_0 v \; dx.$$

- a is a bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$. For all $u, v, w \in H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned}
a(\lambda u + v, w) &= \int_\Omega \nabla(\lambda u + v) \cdot \nabla w \; dx + \alpha \int_\Omega (\lambda u + v) w \; dx \\
&= \int_\Omega \nabla v \cdot \nabla w + \lambda \nabla u \cdot \nabla w \; dx + \alpha \int_\Omega vw + \lambda uw \; dx \\
&= \int_\Omega \nabla v \cdot \nabla w \; dx + \lambda \int_\Omega \nabla u \cdot \nabla w \; dx + \alpha \int_\Omega vw \; dx + \alpha\lambda \int_\Omega uw \; dx \\
&= \lambda \left(\int_\Omega \nabla u \cdot \nabla w \; dx + \alpha \int_\Omega uw \; dx \right) + \left(\int_\Omega \nabla v \cdot \nabla w \; dx + \alpha \int_\Omega vw \; dx \right) \\
&= \lambda a(u, w) + a(v, w).
\end{aligned}$$

Then a is a bilinear form with respect to the first variable. Since a is symetric,

$$\begin{aligned}
\forall u, v \in H_0^1(\Omega) : \; a(u, v) &= \int_\Omega \nabla u \cdot \nabla v \; dx + \alpha \int_\Omega uv \; dx \\
&= \int_\Omega \nabla v \cdot \nabla u \; dx + \alpha \int_\Omega vu \; dx \\
&= a(v, u),
\end{aligned}$$

so the function a is bilinear.

- a is continuous. For all $u, v \in H_0^1(\Omega)$, we have

$$\begin{aligned}
|a(u, v)| &= \left| \int_\Omega \nabla u \cdot \nabla v \; dx + \alpha \int_\Omega uv \; dx \right| \\
&\leq \left| \int_\Omega \nabla u \cdot \nabla v \; dx \right| + \alpha \left| \int_\Omega uv \; dx \right| \\
&\leq \int_\Omega |\nabla u \cdot \nabla v| \; dx + \int_\Omega |u| \; |v| \; dx.
\end{aligned}$$

From the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|a(u, v)| &\leq \int_\Omega |\nabla u| \; |\nabla v| \; dx + \int_\Omega |u| \; |v| \; dx \\
&\leq \left(\int_\Omega |\nabla u|^2 \; dx \right)^{1/2} \left(\int_\Omega |\nabla v|^2 \; dx \right)^{1/2} \\
&+ \alpha \left(\int_\Omega |u|^2 \; dx \right)^{1/2} \left(\int_\Omega |v|^2 \; dx \right)^{1/2} \\
&\leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \alpha \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} .
\end{aligned} \tag{2.8}$$

We note

$$\forall v \in H_0^1(\Omega) : \; \|\nabla v\|_{L^2(\Omega)} \leq \|v\|_{H_0^1(\Omega)} , \; \|v\|_{L^2(\Omega)} \leq \|v\|_{H_0^1(\Omega)} .$$

So the inequality (2.8) becomes

$$\begin{aligned}|a(u,v)| &\leq \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} + \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \\ &\leq (\alpha+1) \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} .\end{aligned}$$

If we set $M = \alpha + 1$, we get

$$|a(u,v)| \leq M \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} ,$$

which gives the continuity of a.

• a is coercive. For all $u \in H_0^1(\Omega)$, one can write

$$\begin{aligned}a(u,u) &= \int_\Omega \nabla u \cdot \nabla u \ dx + \alpha \int_\Omega uu \ dx \\ &= \int_\Omega (\nabla u)^2 \ dx + \alpha \int_\Omega (u)^2 \ dx \\ &= \|\nabla u\|^2_{L^2(\Omega)} + \alpha \|u\|^2_{L^2(\Omega)} .\end{aligned}$$

Let $N = min(1, \alpha)$. Then,

$$\begin{aligned}a(u,u) &\geq N \|\nabla u\|^2_{L^2(\Omega)} + N \|u\|^2_{L^2(\Omega)} \\ &\geq N \left(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \right) \\ &\geq N \|u\|^2_{H_0^1(\Omega)} ,\end{aligned}$$

so a is coercive.

• L is linear form on $H_0^1(\Omega)$. For all $v, w \in H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned}L(\lambda v + w) &= \int_\Omega g(u_0)(\lambda v + w) + \alpha u_0(\lambda v + w) \ dx \\ &= \int_\Omega g(u_0)\lambda v + g(u_0)w + \alpha\lambda u_0 v + \alpha u_0 w \ dx \\ &= \int_\Omega \lambda(g(u_0)v + \alpha u_0 v) + (g(u_0)w + \alpha u_0 w) \ dx \\ &= \lambda \int_\Omega g(u_0)v + \alpha u_0 v \ dx + \int_\Omega g(u_0)w + \alpha u_0 w \ dx \\ &= \lambda L(v) + L(w),\end{aligned}$$

thus L is linear.

- L is continuous. For all $v \in H_0^1(\Omega)$, we have

$$
\begin{aligned}
|L(v)| &= \left| \int_\Omega g(u_0)v + \alpha u_0 v \; dx \right| \\
&\leq \int_\Omega |g(u_0)||v| + \alpha |u_0||v| \; dx \\
&\leq \int_\Omega |g(u_0)||v| \; dx + \int_\Omega \alpha |u_0||v| \; dx.
\end{aligned}
$$

By the Cauchy-Schwarz inequality, we find

$$
\begin{aligned}
|L(v)| \leq & \left(\int_\Omega |g(u_0)|^2 \; dx \right)^{1/2} \left(\int_\Omega |v|^2 \; dx \right)^{1/2} \\
& + \alpha \left(\int_\Omega |u_0|^2 \; dx \right)^{1/2} \left(\int_\Omega |v|^2 \; dx \right)^{1/2} \\
& \leq \|g(u_0)\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \alpha \|u_0\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
& \leq \|g(u_0)\|_{L^2(\Omega)} \|v\|_{H_0^1(\Omega)} + \alpha \|u_0\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \\
& \leq \left(\|g(u_0)\|_{L^2(\Omega)} + \alpha \|u_0\|_{H_0^1(\Omega)} \right) \|v\|_{H_0^1(\Omega)} \, .
\end{aligned}
$$

Setting $K = \|g(u_0)\|_{L^2(\Omega)} + \alpha \|u_0\|_{H_0^1(\Omega)}$, we get

$$
|L(v)| \leq K \|v\|_{H_0^1(\Omega)} \, ,
$$

hence, L is continuous.

Finally, by the Lax-Milgram theory the problem (2.7) has a unique weak solution $u_1 \in H_0^1(\Omega)$.

Let $n \in \mathbb{N}^*$ and $u_n \in H_0^1(\Omega)$. In view of the above, the following linear boundary-value problem

$$
\begin{cases}
-\Delta u_{n+1} + \alpha u_{n+1} = g(u_n) + \alpha u_n & \text{in } \Omega, \\
u_{n+1} = 0 & \text{on } \partial\Omega
\end{cases}
\tag{2.9}
$$

has a unique weak solution $u_{n+1} \in H_0^1(\Omega)$.

By induction on n, we prove that

$$
\underline{u} = u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \qquad \text{a.e. in } \Omega. \tag{2.10}
$$

For $n = 0$, we have

$$
\begin{aligned}
\int_\Omega \nabla u_1 \cdot \nabla v + \alpha u_1 v \; dx &= \int_\Omega \left(g\left(u_0\right) + \alpha u_0 \right) v \; dx \\
&= \int_\Omega \left(g\left(\underline{u}\right) + \alpha \underline{u} \right) v \; dx,
\end{aligned}
\tag{2.11}
$$

for all $v \in H_0^1(\Omega)$. By subtracting (2.11) from (2.4), we obtain

$$\int_\Omega \nabla \underline{u} \cdot \nabla v \; dx - \int_\Omega \nabla u_1 \cdot \nabla v + \alpha u_1 v \; dx \le \int_\Omega g(\underline{u}) v \; dx - \int_\Omega \left(g\left(\underline{u}\right) + \alpha \underline{u}\right) v \; dx,$$

which can be written as

$$\int_\Omega \nabla (u_0 - u_1) \cdot \nabla v + \alpha (u_0 - u_1) v \; dx \le 0. \tag{2.12}$$

Setting

$$v := (u_0 - u_1)^+ .$$

We notice that $v \in H_0^1(\Omega)$,

$$\nabla v = \begin{cases} \nabla \left(u_0 - u_1\right) & \text{a.e. on } \left\{u_0 \ge u_1\right\}, \\ 0 & \text{a.e. on } \left\{u_0 \le u_1\right\} \end{cases}$$

(see Theorem 1.3.7) and $v \ge 0$ a.e. in Ω. Taking $v := (u_0 - u_1)^+$ in (2.12), we get

$$\begin{aligned} &\int_\Omega \nabla \left(u_0 - u_1\right) \cdot \nabla \left(u_0 - u_1\right)^+ + \alpha \left(u_0 - u_1\right) \left(u_0 - u_1\right)^+ dx \\ &= \int_{\{u_0 \ge u_1\}} \nabla (u_0 - u_1) \cdot \nabla (u_0 - u_1) + \alpha (u_0 - u_1)(u_0 - u_1) \; dx \\ &+ \int_{\{u_0 \le u_1\}} \nabla (u_0 - u_1) \cdot 0 + \alpha (u_0 - u_1) 0 \; dx \\ &= \int_{\{u_0 \ge u_1\}} \nabla (u_0 - u_1) \cdot \nabla (u_0 - u_1) + \alpha (u_0 - u_1)(u_0 - u_1) \; dx \le 0, \end{aligned} \tag{2.13}$$

then

$$\int_{\{u_0 > u_1\}} \left|\nabla \left(u_0 - u_1\right)\right|^2 + \alpha \left(u_0 - u_1\right)^2 dx \le 0,$$

which implies $mes(\{u_0 > u_1\}) = 0$. Hence, $u_0 \le u_1$ a.e. in Ω.

Let us now suppose by induction that

$$u_{n-1} \le u_n \qquad \text{a.e. in } \Omega. \tag{2.14}$$

From (2.9), we find

$$\int_\Omega \nabla u_{n+1} \cdot \nabla v + \alpha u_{n+1} v \; dx = \int_\Omega \left(g\left(u_n\right) + \alpha u_n\right) v \; dx \tag{2.15}$$

and

$$\int_\Omega \nabla u_n \cdot \nabla v + \alpha u_n v \; dx = \int_\Omega \left(g\left(u_{n-1}\right) + \alpha u_{n-1}\right) v \; dx \tag{2.16}$$

for all $v \in H_0^1(\Omega)$. Using (2.14), (2.16) and the fact that g is nondecreasing, one can write

$$\begin{aligned}\int_\Omega \nabla u_n \cdot \nabla v + \alpha u_n v \; dx &= \int_\Omega \left(g\left(u_{n-1}\right) + \alpha u_{n-1}\right) v \; dx \\ &\leq \int_\Omega \left(g\left(u_n\right) + \alpha u_{n-1}\right) v \; dx.\end{aligned} \tag{2.17}$$

Subtracting (2.15) from and (2.17), we find

$$\int_\Omega \nabla(u_n - u_{n+1}) \cdot \nabla v + \alpha(u_n - u_{n+1}) v \; dx \leq 0. \tag{2.18}$$

Putting $v := (u_n - u_{n+1})^+$, we get

$$\begin{aligned}&\int_{\{u_n \geq u_{n+1}\}} \left|\nabla\left(u_n - u_{n+1}\right)\right|^2 + \alpha\left(u_n - u_{n+1}\right)^2 dx \\ &\quad = \int_\Omega \left[\left(g\left(u_{n-1}\right) + \alpha u_{n-1}\right) - \left(g\left(u_n\right) + \alpha u_n\right)\right]\left(u_n - u_{n+1}\right)^+ dx \leq 0.\end{aligned}$$

Thus $u_n \leq u_{n+1}$ a.e. in Ω.

Let us prove that

$$\forall n \in \mathbb{N}: \; u_n \leq \bar{u} \qquad \text{a.e. in } \Omega. \tag{2.19}$$

From the definiton of the supersolution $\bar{u}$ and subsolution $u_0 = \underline{u}$, we have $u_0 = \underline{u} \leq \bar{u}$. Then, (2.19) is valid for $n = 0$. Now for induction, suppose that

$$u_n \leq \bar{u} \qquad \text{a.e. in } \Omega. \tag{2.20}$$

Using (2.15) and (2.20), we get

$$\begin{aligned}\int_\Omega \nabla u_{n+1} \cdot \nabla v + \alpha u_{n+1} v \; dx &= \int_\Omega \left(g\left(u_n\right) + \alpha u_n\right) v \; dx \\ &\leq \int_\Omega \left(g\left(\bar{u}\right) + \alpha u_n\right) v \; dx.\end{aligned} \tag{2.21}$$

Subtracting (2.3) from (2.21), we find

$$\int_\Omega \nabla\left(u_{n+1} - \bar{u}\right) \cdot v + \alpha\left(u_{n+1} - \bar{u}\right) v \; dx \leq \int_\Omega [(g(u_n) + \alpha u_n) - (g(\bar{u}) + \alpha\bar{u}] v \; dx. \tag{2.22}$$

Taking $v := (u_{n+1} - \bar{u})^+$, we obtain

$$\begin{aligned}&\int_{\{u_{n+1} \geq \bar{u}\}} \left|\nabla\left(u_{n+1} - \bar{u}\right)\right|^2 + \alpha\left(u_{n+1} - \bar{u}\right)^2 dx \\ &\quad \leq \int_\Omega \left[\left(g\left(u_n\right) + \alpha u_n\right) - \left(g(\bar{u}) + \alpha\bar{u}\right)\right]\left(u_{n+1} - \bar{u}\right)^+ dx \leq 0.\end{aligned}$$

Using (2.20) and the fact that h is nondecreasing, one can write

$$\begin{aligned} &\int_{\{u_{n+1}\geq \bar{u}\}} |\nabla (u_{n+1}-\bar{u})|^2 + \alpha (u_{n+1}-\bar{u})^2 \; dx \\ &\leq \int_\Omega [(g(u_n)+\alpha u_n)-(g(\bar{u})+\alpha\bar{u})] (u_{n+1}-\bar{u})^+ \; dx \\ &\leq \int_\Omega [(g(\bar{u})+\alpha \bar{u})-(g(\bar{u})+\alpha\bar{u})] (u_{n+1}-\bar{u})^+ \; dx = 0, \end{aligned} \tag{2.23}$$

which gives

$$u_{n+1} \leq \bar{u} \quad \text{ a.e. in } \Omega.$$

Thus, (2.19) holds. In view of (2.10) and (2.19), we find

$$\underline{u} \leq \cdots \leq u_n \leq u_{n+1} \leq \cdots \leq \bar{u} \quad \text{ a.e. in } \Omega. \tag{2.24}$$

Since the sequence $(u_n)_{n\in\mathbb{N}}$ is nondecreasing and bounded above, there exists function $u : \Omega \longrightarrow \mathbb{R}$ (see theorem 1.2.4) suth that

$$u(x) := \lim_{n\to\infty} u_n(x) \text{ a.e. } x \in \Omega.$$

Also, from (2.24), one can write

$$\forall n \in \mathbb{N} : \; |u_n(x)| \leq max(|\bar{u}(x)|, |\underline{u}(x)|) \text{ a.e. } x \in \Omega, \tag{2.25}$$

with $max(|\bar{u}|, |\underline{u}|) \in L^2(\Omega)$. By the dominated convergence Theorem (see Theorem 1.2.3), $u \in L^2(\Omega)$ and

$$\lim_{n\to\infty} \|u_n - u\|_{L^2(\Omega)} = 0. \tag{2.26}$$

From (2.2), we have

$$-C \leq g'(t) \leq C. \tag{2.27}$$

Integrating (2.27) from 0 to $t > 0$, we get

$$-Ct \leq g(t) - g(0) \leq Ct,$$

which gives

$$-Ct + g(0) \leq g(t) \leq Ct + g(0).$$

This leads to

$$\forall t > 0 : \; |g(t)| \leq C|t| + |g(0)|. \tag{2.28}$$

If $t < 0$, we integrate (2.27) from $t > 0$ to 0 to get

$$Ct \leq g(0) - g(t) \leq -Ct,$$

which implies

$$\forall t < 0 : \ |g(t)| \leq C|t| + |g(0)|. \tag{2.29}$$

Combining (2.28) and (2.29), we get

$$\forall t \in \mathbb{R} : \ |g(t)| \leq C|t| + |g(0)|. \tag{2.30}$$

Taking $t = u_n(x)$ in (2.30), we obtain

$$\forall n \in \mathbb{N} : \ |g(u_n)| \leq C|u_n| + |g(0)|, \tag{2.31}$$

which gives

$$|g(u_n)|^2 \leq (C|u_n| + |g(0)|)^2.$$

Using the inequality

$$\forall a, b \in \mathbb{R} : \ a^2 + b^2 \geq 2ab,$$

we find

$$\begin{aligned} |g(u_n)|^2 &\leq (Cu_n)^2 + (|g(0)|)^2 + 2Cx\ |g(0)| \\ &\leq (Cu_n)^2 + (|g(0)|)^2 + (Cu_n)^2 + (|g(0)|)^2 \\ &= 2(Cu_n)^2 + 2\,(g(0))^2 . \end{aligned} \tag{2.32}$$

Integrating (2.32) over Ω, we get

$$\begin{aligned} \int_\Omega |g(u_n)|^2 \ dx &\leq 2C^2 \int_\Omega u_n^2 \ dx + 2g(0)^2 \int_\Omega \ dx \\ &= 2C^2 \left\| u_n \right\|^2_{L^2(\Omega)} + 2g(0)^2 mes(\Omega). \end{aligned}$$

This leads to

$$\begin{aligned} \|g(u_n)\|_{L^2(\Omega)} &\leq \left[2C^2 \left\| u_n \right\|^2_{L^2(\Omega)} + 2g(0)^2 mes(\Omega) \right]^{1/2} \\ &\leq \left[2C^2 \left\| u_n \right\|^2_{L^2(\Omega)} \right]^{1/2} + \left[2g(0)^2 mes(\Omega) \right]^{1/2} \\ &= \sqrt{2}C \left\| u_n \right\|_{L^2(\Omega)} + |g(0)| \sqrt{2mes(\Omega)}. \end{aligned} \tag{2.33}$$

Multiplying (2.9) by u_{n+1} and integrating over Ω, we obtain

$$\int_\Omega |\nabla u_{n+1}|^2 + \alpha (u_{n+1})^2 \ dx = \int_\Omega g\left(u_n\right) u_{n+1} + \alpha u_n u_{n+1} \ dx,$$

which gives

$$\begin{aligned}\int_\Omega |\nabla u_{n+1}|^2 + \alpha (u_{n+1})^2 \; dx &\leq \left| \int_\Omega g\left(u_n\right) u_{n+1} + \alpha u_n u_{n+1} \; dx \right| \\ &\leq \int_\Omega \left| g\left(u_n\right) u_{n+1} + \alpha u_n u_{n+1} \right| \; dx \\ &\leq \int_\Omega \left| g\left(u_n\right)\right| |u_{n+1}| + \alpha \left|u_n\right| |u_{n+1}| \; dx.\end{aligned}$$

By Cauchy-Schwarz inequality, we get

$$\int_\Omega |\nabla u_{n+1}|^2 + \alpha (u_{n+1})^2 \; dx \leq \left\| g\left(u_n\right)\right\|_{L^2(\Omega)} \left\|u_{n+1}\right\|_{L^2(\Omega)} + \alpha \left\|u_n\right\|_{L^2(\Omega)} \left\|u_{n+1}\right\|_{L^2(\Omega)}. \tag{2.34}$$

Using (2.25) and (2.33), we obtain from (2.34),

$$\begin{aligned}\beta \left\|u_{n+1}\right\|^2_{H^1_0(\Omega)} \leq &\left[\sqrt{2}C \left\|u_n\right\|_{L^2(\Omega)} + |g(0)|\sqrt{2mes(\Omega)}\right] \left\|u_{n+1}\right\|_{L^2(\Omega)} \\ &+ \alpha \left\|u_n\right\|_{L^2(\Omega)} \left\|u_{n+1}\right\|_{L^2(\Omega)} \\ &\leq \left[\sqrt{2}C \left\|w\right\|_{L^2(\Omega)} + |g(0)|\sqrt{2mes(\Omega)}\right] \left\|w\right\|_{L^2(\Omega)} \\ &+ \alpha \left\|w\right\|_{L^2(\Omega)} \left\|w\right\|_{L^2(\Omega)},\end{aligned} \tag{2.35}$$

where $\beta = min(1, \alpha)$ and $w = max(|\bar{u}|, |\underline{u}|)$. Thus, $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1_0(\Omega)$. Using the compact embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ and (2.26), there exist a subsequence $(u_{n_j})_{j\in\mathbb{N}} \subset (u_n)_{n\in\mathbb{N}}$ and $u \in H^1_0(\Omega)$ such that

$$(u_{n_j})_{j\in\mathbb{N}} \text{ converges weakly to } u \text{ in } H^1_0(\Omega). \tag{2.36}$$

By (2.2), one can write

$$\forall j \in \mathbb{N}: \; |g(u_{n_j}) - g(u)| \leq C|u_{n_j} - u|,$$

which implies

$$\forall j \in \mathbb{N}: \; \left\|g(u_{n_j}) - g(u)\right\|_{L^2(\Omega)} \leq C \left\|u_{n_j} - u\right\|_{L^2(\Omega)}. \tag{2.37}$$

Passing to the limit in (2.37) and taking into account (2.26), we get

$$\lim_{j\to\infty} \left\|g(u_{n_j}) - g(u)\right\|_{L^2(\Omega)} = 0. \tag{2.38}$$

Finally, letting $j \to \infty$ in

$$\int_\Omega \nabla u_{n_j+1} \cdot \nabla v + \alpha u_{n_j+1} v \; dx = \int_\Omega \left(g\left(u_{n_j}\right) + \alpha u_{n_j}\right) v \; dx$$

and using (2.26), (2.36) and (2.38), we obtain

$$\int_{\Omega} \nabla u \cdot \nabla v + \alpha uv \ dx = \int_{\Omega} \left(g\left(u\right) + \alpha u\right) v \ dx,$$

which can be written as

$$\forall v \in H_0^1(\Omega) : \int_{\Omega} \nabla u \cdot \nabla v \ dx = \int_{\Omega} g\left(u\right) v \ dx.$$

Thus, the limit u is a weak solution to the problem (2.1).

Chapitre 3

Existence of postive solution of a Kirchhoff problem by sub-supersolution method

In this chapter, we explore the existence of positive solutions for a nonlinear elliptic differential equation of Kirchhoff-type. We focus on a specific case where we prove the existence of positive solutions under certain conditions on the relevant parameters. The presented results rely on the sub-supersolution method, which provides deep insight into the behavior of the equation's solutions under different conditions. The research of this chapter are documented in references [2, 4].

3.1 Statement of the problem

The Kirchhoff equation originated in 1883 as a model for vibrations of elastic strings, where the stiffness of the string depends on the total deformation energy.
The original time-dependent equation is written as

$$\frac{\partial^2 u}{\partial t^2} - M\left(\int_\Omega |\nabla u(x,t)|^2 \, dx\right)\Delta u = g(x,u),$$

where

• $u(x,t)$ is displacement of the string at position x and time t.

• $M(s)$ is a function describing the string's stiffness, dependent on the total deformation energy $\int_\Omega |\nabla u|^2 \, dx$.

• Δu is the Laplacian operator (represents elastic restoring forces).

• $g(x,u)$ is a external forces or nonlinear interactions.

We need to study the steady-state (time-independent) case, where the system is in equilibrium (no temporal variation). So we have

$$\frac{\partial^2 u}{\partial t^2} = 0,$$

with Dirichlet boundary conditions and by adding nonlinear and external terms, the problem becomes

$$\begin{cases} -M\left(\|u\|^2\right)\Delta u = |u|^{p-1}u + \alpha g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{E}$$

where Ω is a bounded domain in $\mathbb{R}^N (N > 2)$ with smooth boundary $\partial\Omega$, α is a positive parameter, $p \in]1, \infty[$, M is a continuous function on $[0, \infty[$ such that

$$\exists m_0 > 0,\ \forall s \geq 0 :\ M(s) \geq m_0 \tag{M_0}$$

and

$$g(x) \in C^1(\overline{\Omega}) \text{ changes sign.} \tag{G}$$

3.2 Existence of postive solution by sub and supersolutions method

Theorem 3.2.1.
Assume that (G) holds and M is a noncreasing function satisfying M_0. Suppose further that the function

$$H(t) = tM(t^2) \tag{H}$$

is increasing on $\mathbb{R}$. Then there exists $\alpha_* > 0$ such that the problem (E) admits at least a positive solution for all $\alpha \in (0, \alpha_*)$.

Proof. First, we give an example of function M satisfying the assumptions of Theorem. We take

$$\begin{aligned} M : [0, \infty[&\to]0, \infty[\\ t &\mapsto \frac{at + b}{\mu t + \beta} + c, \end{aligned}$$

where a, b, μ, β and c are positive numbers such that

$$a\beta - b\mu < 0 \quad \text{and} \quad c\mu\beta + a\beta - b\mu \geq 0.$$

We show that M is noncreasing and satisfies the conditions (M_0) and (H). The function M differentiable and its derivative is given by

$$M'(t) = \frac{a\beta - b\mu}{(\mu t + \beta)^2}.$$

Since $a\beta - b\mu < 0$, it follows that $M' < 0$. So, M is noncreasing. Next,

$$\forall t \in [0, \infty[\, : \ M(t) = \frac{at+b}{\mu t + \beta} + c \geq c > 0.$$

Therefore, it suffices to choose $m_0 = c$ to have (M_0). Moreover,

$$\begin{aligned} H'(t) &= M\left(t^2\right) + 2t^2 M'\left(t^2\right) \\ &= \frac{at^2+b}{\mu t^2+\beta} + c + 2t^2 \frac{a\beta - b\mu}{\left(\mu t^2+\beta\right)^2} \\ &= \frac{\left(at^2+b\right)\left(\mu t^2+\beta\right) + c\left(\mu^2 t^4 + \beta^2\right) + 2t^2(c\mu\beta + a\beta - b\mu)}{\left(\mu t^2+\beta\right)^2} > 0 \end{aligned}$$

because $c\mu\beta + a\beta - b\mu \geq 0$. So, the function H is increasing on $\mathbb{R}$. Thus, (H) holds.

We will use the sub and supersolutions method. Let u_g be a nonnegative solution of problem (see Example 1.3.1),

$$\begin{cases} -\Delta u = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We determine a subsolution of (E) in the form

$$\underline{u} = \lambda u_g, \qquad \text{with } \lambda \in \,]0, \infty[\,.$$

We have

$$-M\left(\|\underline{u}\|^2\right) = -M\left(\|\lambda u_g\|^2\right) = -M\left(\lambda^2 \|u_g\|^2\right). \tag{3.1}$$

On the other hand,

$$\Delta \underline{u} = \Delta(\lambda u_g) = \lambda \Delta u_g = -\lambda g(x). \tag{3.2}$$

From (3.1) and (3.2), we get

$$\begin{aligned} -M\left(\|\underline{u}\|^2\right)\Delta\underline{u} &= -M\left(\lambda^2 \|u_g\|^2\right)\lambda\Delta u_g \\ &= M\left(\lambda^2 \|u_g\|^2\right)\lambda g(x). \end{aligned} \tag{3.3}$$

We say that $\underline{u}$ is subsolution of (E) if

$$-M\left(\|\underline{u}\|^2\right)\Delta\underline{u} \leq |\underline{u}|^{p-1}\underline{u} + \alpha g(x). \tag{3.4}$$

Insert $\underline{u} = \lambda u_g$ in (3.4) yields

$$M\left(\lambda^2 \|u_g\|^2\right)\lambda g(x) \leq \lambda\lambda^{p-1}|u_g|^{p-1}u_g + \alpha g(x),$$

which can be written as

$$g(x)(M\left(\lambda^2\left\|u_g\right\|^2\right)\lambda-\alpha)\leq\lambda^p|u_g|^{p-1}u_g.$$

We remember that *g(x)* changes sign, so to obtain the last inequality we must take λ such that

$$M\left(\lambda^2\left\|u_g\right\|^2\right)\lambda=\alpha$$

or

$$M\left(\lambda^2\left\|u_g\right\|^2\right)\frac{\lambda}{\alpha}=1. \tag{3.5}$$

Let us prove that the equation (3.5) admits at least a positive solution λ.

Setting $y=\lambda^2\left\|u_g\right\|^2$. Then, (3.5) becomes

$$\Phi(y):=\frac{M(y)\sqrt{y}}{\|u_g\|}=\alpha.$$

We have

$$\Phi(0)=\frac{M(0)\sqrt{0}}{\|u_g\|}=0.$$

On the other hand, by (M_0),

$$\forall y\in\left]0,\infty\right[:\ \Phi(y)\geq\frac{m_0\sqrt{y}}{\|u_g\|}.$$

Therefore,

$$\lim_{y\to+\infty}\Phi(y)=+\infty.$$

So, since Φ is continuous on $]0,+\infty[$, there exists $y_0=y_0(\alpha)>0$ such that $\Phi(y_0)=\alpha$. Hence, there exists $\lambda=\frac{\sqrt{y_0}}{\|u_g\|}>0$ satisfying (3.4). This gives the existence of the subsolution $\underline{u}$.

Now, let as find a supersolution $\bar{u}$ of (E) such that $\bar{u}(x)\geq\underline{u}(x)$ a.e. $x\in\Omega$. If $\bar{u}$ is supersolution of (E), then

$$-M\left(\|\bar{u}\|^2\right)\Delta\bar{u}\geq|\bar{u}|^{p-1}\bar{u}+\alpha g(x). \tag{3.6}$$

Let v be the unique positive solution of the linear Dirichlet problem

$$\begin{cases}-\Delta u=1 & \text{in }\Omega,\\ u=0 & \text{on }\partial\Omega.\end{cases}$$

We have to find $\bar{u}$ in the form

$$\bar{u}=Bv,\qquad\text{with }B\in\left]0,\infty\right[. \tag{3.7}$$

From (M_0), we obtain

$$-M\left(\|\bar{u}\|^2\right)\Delta\bar{u} = -M\left(\|\bar{u}\|^2\right)\Delta(Bv) = M\left(B^2\|v\|^2\right)B \geq m_0 B. \tag{3.8}$$

If we choose B suth that

$$m_0 B \geq B^p|v|_\infty^p + \alpha|g|_\infty, \tag{3.9}$$

then $\bar{u}$ is a supersolution of the problem (E). Set

$$\forall B \in \,]0,\infty[\,:\;\; h(B) = \frac{m_0 B - B^p|v|_\infty^p}{|g|_\infty}.$$

We have

$$h(0) = \frac{m_0 0 - 0^p|v|_\infty^p}{|g|_\infty} = 0$$

$$\lim_{B\to\infty} h(B) = -\infty.$$

On the other hand,

$$\forall B \in \,]0,\infty[\,:\;\; h'(B) = \frac{m_0 - pB^{p-1}|v|_\infty^p}{|g|_\infty},$$

then

$$h'(B) = 0 \Longleftrightarrow B_0 = \left(\frac{m_0}{p|v|_\infty^p}\right)^{1/(p-1)}.$$

We summarize the previous study on the function h in the following table

B	0		B_0		∞
$h'(B)$		$+$	0	-	
$h(B)$	0	$\nearrow$	$h(B_0)$	$\searrow$	$-\infty$

So,

$$\max_{B\in[0,\infty[} h(B) = h(B_0).$$

Therefore, if $0 < \alpha < \alpha_0 = h(B_0)$, we have

$$\frac{m_0 B_0 - B_0^p|v|_\infty^p}{|g|_\infty} > \alpha,$$

which equivalent to

$$m_0 B_0 > \alpha|g|_\infty + B_0^p|v|_\infty^p.$$

Thus, (3.9) holds and $\bar{u} = B_0 v$ is a supersolution of the problem (E). To complete the proof of the existence result we need the following lemma.

Lemma 3.2.1.

Under the hypotheses of Theorem, if v and w are nonnegative functions verifying

$$\begin{cases} -M\left(\|w\|^2\right)\Delta w \geq -M\left(\|v\|^2\right)\Delta v & \text{in } \Omega, \\ v = w = 0 & \text{on } \partial\Omega \end{cases} \tag{E*}$$

then

$$w \geq v \quad \text{in } \Omega.$$

proof.

By mutiltiplying both sides of (E*) by v and integrating over Ω, we get

$$\int_\Omega -M\left(\|w\|^2\right)\Delta w \cdot v\ dx \geq \int_\Omega -M\left(\|v\|^2\right)\Delta v \cdot v\ dx,$$

we use integration by parts, we find

$$\begin{aligned} M\left(\|w\|^2\right)\int_\Omega \Delta w \cdot v\ dx &\geq M\left(\|v\|^2\right)\int_\Omega \Delta v \cdot v\ dx \\ M\left(\|w\|^2\right)(w,v) &\geq M\left(\|v\|^2\right)\|v\|^2 \\ (w,v) &\geq \frac{M\left(\|v\|^2\right)\|v\|^2}{M\left(\|w\|^2\right)}, \end{aligned} \tag{3.10}$$

with

$$(w,v) = \int_\Omega \nabla w \cdot \nabla v\ dx.$$

In the same way, by mutiltiplying both sides of (E*) by w and integrating over Ω, we find

$$\frac{M\left(\|w\|^2\right)\|w\|^2}{M\left(\|v\|^2\right)} \geq (w,v). \tag{3.11}$$

From (3.10) and (3.11), we get

$$\frac{M\left(\|w\|^2\right)\|w\|^2}{M\left(\|v\|^2\right)} \geq (w,v) \geq \frac{M\left(\|v\|^2\right)\|v\|^2}{M\left(\|w\|^2\right)},$$

then

$$(w,v) \geq \frac{M\left(\|v\|^2\right)\|v\|^2}{M\left(\|w\|^2\right)},$$

by the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \|w\|\ \|v\| \geq (w,v) &\geq \frac{M\left(\|v\|^2\right)\|v\|^2}{M\left(\|w\|^2\right)} \\ \|w\|\ \|v\| &\geq \frac{M\left(\|v\|^2\right)\|v\|^2}{M\left(\|w\|^2\right)}, \end{aligned}$$

by mutiltiplying both sides by $M\left(\|w\|^2\right)$, we obtain

$$\begin{aligned} M\left(\|w\|^2\right)\|w\|\ \|v\| \geq M\left(\|v\|^2\right)\|v\|^2 \\ M\left(\|w\|^2\right)\|w\| \geq M\left(\|v\|^2\right)\|v\|. \end{aligned}$$

By (H), we get

$$H(\|w\|) \geq H(\|v\|). \tag{3.12}$$

From the increasing of function (H) and (3.12), we get

$$\|w\| \geq \|v\|,$$

from then and nonincreasing of function M, we obtain

$$M\left(\|w\|^2\right) \leq M\left(\|v\|^2\right). \tag{3.13}$$

From (E*) we have

$$\begin{aligned} -M\left(\|w\|^2\right)\Delta w \geq -M\left(\|v\|^2\right)\Delta v \\ -M\left(\|w\|^2\right)\Delta w + M\left(\|v\|^2\right)\Delta v \geq 0 \\ -\Delta\left(M\left(\|w\|^2\right)w\right) + \Delta\left(M\left(\|v\|^2\right)v\right) \geq 0 \\ -\Delta\left(M\left(\|w\|^2\right)w - M\left(\|v\|^2\right)v\right) \geq 0. \end{aligned}$$

Then, we can write (E*) as following

$$\begin{cases} -\Delta\left(M\left(\|w\|^2\right)w - M\left(\|v\|^2\right)v\right) \geq 0 & \text{in } \Omega, \\ v = w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.14}$$

By application of the maximum principle on (3.14), we get

$$\begin{aligned} M\left(\|w\|^2\right)w - M\left(\|v\|^2\right)v \geq 0 \\ M\left(\|w\|^2\right)w \geq M\left(\|v\|^2\right)v. \end{aligned} \tag{3.15}$$

From (3.13) and (3.15), we find

$$\begin{aligned} & M\left(\|w\|^2\right)w \geq M\left(\|v\|^2\right)v \\ & M\left(\|v\|^2\right)v \leq M\left(\|w\|^2\right)w \leq M\left(\|v\|^2\right)w \\ & M\left(\|v\|^2\right)v \leq M\left(\|v\|^2\right)w. \end{aligned} \tag{3.16}$$

Since M is a positive function and from (3.16), we get

$$w \geq v, \text{ in } \Omega.$$

This ends the proof of Lemma .

Because $g \in C^1(\overline{\Omega})$ (from (G)) and $\bar{u} = u_0 \in C^{0,\beta}(\overline{\Omega})$. Then, $|u_0|^{p-1}u_0 \in C^{0,\beta}(\overline{\Omega})$, which gives

$$f(., u_0) \in C^{0,\beta}(\overline{\Omega}), \tag{3.17}$$

where $f(x, u_0(x)) = |u_0(x)|^{p-1}u_0(x) + \beta g(x)$ for all $x \in \overline{\Omega}$. We prove that $H(\mathbb{R}) = \mathbb{R}$. From (M_0),

$$\forall t \geq 0: \ H(t) = tM(t^2) \geq m_0 t,$$

which implies

$$\lim_{t\to\infty} H(t) = \infty. \tag{3.18}$$

On the other hand, since M is a nonincreasing function, it flows that

$$\forall t < 0: \ M(t^2) \leq M(0),$$

which leads to

$$H(t) = tM(t^2) \geq tM(0), \ M(0) \geq m_0 > 0.$$

Thus,

$$\lim_{t\to-\infty} H(t) = -\infty. \tag{3.19}$$

Now, using (3.19), (3.18) and the fact that H is increasing function and continuous, we get $H(\mathbb{R}) = \mathbb{R}$. Therefore, since $f(., u_0) \in L^2(\Omega)$, the problem

$$\begin{cases} -M\left(\|u_1\|^2\right)\Delta u_1 = f(x, u_0) & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega \end{cases} \tag{E_1}$$

has a unique solution $u_1 \in H_0^1(\Omega)$ (see [1]). In addition, using (3.17), the solution u_1 is in $C^{2,\beta}(\overline{\Omega})$ (see Schauder estimates Thorem 1.3.4). In the same way, the problem

$$\begin{cases} -M\left(\|u_n\|^2\right)\Delta u_n = f(x, u_{n-1}) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega \end{cases} \tag{E_n}$$

has a unique solution $(u_n)_{n\in\mathbb{N}} \in H_0^1(\Omega) \cap C^{2,\beta}(\overline{\Omega})$, $n \geq 2$. Since u_0 is a supersolution, one can write

$$-M\left(\|u_0\|^2\right)\Delta u_0 \geq f\left(x,u_0\right) = -M\left(\|u_1\|^2\right)\Delta u_1.$$

From Lemma 3.2.1, we obtain $u_0 \geq u_1$. Also, using the fact that $u_0 \geq \underline{u} \geq 0$ and $t \longmapsto |t|^{p-1}t$ is monotone, we get

$$-M\left(\|u_1\|^2\right)\Delta u_1 = f\left(x,u_0\right) \geq f\left(x,u_1\right) = -M\left(\|u_2\|^2\right)\Delta u_2,$$

which gives $u_1 \geq \underline{u}$ (according to Lemma 3.2.1). For u_2, we have

$$-M\left(\|u_1\|^2\right)\Delta u_1 = f\left(x,u_0\right) \geq f\left(x,u_1\right) = -M\left(\|u_2\|^2\right)\Delta u_2.$$

Hence, $u_1 \geq u_2$. In the same way, since

$$-M\left(\|u_2\|^2\right)\Delta u_2 = f\left(x,u_1\right) \geq f(x,\underline{u}) \geq -M\left(\|\underline{u}\|^2\right)\Delta \underline{u},$$

we obtain $u_2 \geq \underline{u}$. Repeating this argument, one can construct a bounded monotone sequence $(u_n)_{n\in\mathbb{N}} \subset H_0^1(\Omega)$,

$$\bar{u} = u_0 \geq u_1 \geq u_2 \geq \cdots \geq u_n \geq \cdots \geq \underline{u} \geq 0, \quad \underline{u} \neq 0. \tag{3.20}$$

By using the arguments we show that $(u_n)_{n\in\mathbb{N}} \subset C^{0,\beta}(\bar{\Omega}), 0<\beta<1$, and since $g \in C^{0,\beta}(\bar{\Omega})$ we have $f\left(.,u_n\right) \in C^{0,\beta}(\bar{\Omega})$. From this and by using Schauder estimate we get $(u_n)_{n\in\mathbb{N}} \subset C^{2,\beta}$. Since Ω is bounded, it follows that $f(.,u_{n-1}) \in L^p(\Omega)$ for all $p \in]1,\infty[$. Therefore, from Thorem 1.3.3,

$$\begin{aligned}\|u_n\|_{W^{2,p}(\Omega)} &\leq C_p\|f(.,u_{n-1})\|_{L^p(\Omega)}\\ &\leq C_p\|f(.,u_0)\|_{L^p(\Omega)},\end{aligned}$$

where C_p is a constant independent of n. Let $p > \frac{N}{2}$ by the continuous embedding $W^{2,p}(\Omega) \hookrightarrow C^{0,\gamma}(\Omega)$, $\gamma \in]0,1]$, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $C^{0,\gamma}(\overline{\Omega})$, hence, by the Schauder estimates (see Thorem 1.3.4), $(u_n)_{n\in\mathbb{N}}$ is bounded in $C^{2,\gamma}(\overline{\Omega})$. Now since the embedding $C^{2,\gamma}(\overline{\Omega}) \hookrightarrow C^2(\overline{\Omega})$ is compact, there existe a subsequence $(u_{n_k})_{k\in\mathbb{N}} \subset (u_n)_{n\in\mathbb{N}}$ which converges in $C^2(\overline{\Omega})$ to a function u. From (3.20), it follows that $u \geq \underline{u} \geq 0$. Now, writting (E_n) for $n = n_k$ and letting $k \to \infty$ in (E_{n_k}), we see that u is a nonnegative nontrivial solution of the problem (E). The maximum principle gives $u > 0$.

Conclusion

In this memory, we studied the theoretical and applied aspects of the sub-supersolution method. It provided a systematic analysis of the sub-supersolution method as an innovative tool for addressing nonlinear differential equations, with a specific focus on its application to the Kirchhoff-type problem modeling the vibrations of elastic strings. The theoretical aspect is based on creating a monotonic sequence that progressively converges toward a solution. In the applied context, the methodology was successfully employed to investigate a Kirchhoff-type problem, demonstrating the existence of a positive solutions under specific conditions.

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