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# Approximation of the Bessel function of the first kind

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Finally, our thanks also to all the people who contributed directly or indirectly to the development of this work.

# Dedications

To my entire family, each in their own name.

All professors from the University of El-Oued.

My friends and everyone who encouraged me.

HAMEDANI Sabrine

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To my entire family, each in their own name.

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GADI Zouheira

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# Introduction

Bessel functions of the first kind $J_\eta$ [14] are essential to many applications in engineering and physics. These functions are some of the most significant special functions that must be evaluated using the proper mathematical formulas. They indicate solutions of several differential equations, including Bessel's equation [14]. Currently, the most popular method for approximating the values of $J_\eta$ is its expansion using a power series. However, the computation of the Gamma function complicates each of the many terms in this series that are required to obtain a good approximation for large values of the argument. Consequently, the corresponding rate of convergence will decline more slowly. This work presents a novel method for evaluating Bessel functions of the first kind using Haar wavelets [3, 6, 7, 8] in order to solve this problem and increase the convergence rate. Numerous issues have been resolved using this kind of wavelets, including those in integral equations [2, 4, 9, 15], image processing [1, 5, 10, 12, 13], and other areas of mathematics.

In this work, we examine the use of Haar wavelets to solve the Bessel's equation of order $\eta$ with initial conditions, which makes it simple to evaluate the first-kind Bessel functions. The primary benefit of our suggested approach is that its algorithm is simpler, quicker, and more precise.

This work is structured as follows :

The first chapter discusses the gamma function and its properties. The Bessel functions of the first kind is represented by power series and their properties are represented.

In the second chapter, we present the power series method for solving Bessel differential equations, examining their convergence and estimating the convergence error.

In the final chapter, we present the wavelet method to approximate the solution of first-kind Bessel equations, followed by examining its convergence and providing an estimated comparison between it and the power series method.

# Chapter 1
# Mathematical Preliminaries

Special functions are very important in mathematics, physics, and engineering. Among them, the Gamma function and Bessel functions are widely used because they help solve complex problems related to equations, integrals, and physical phenomena.

The Gamma function is a fundamental extension of the factorial function to real and complex numbers. Bessel functions, on the other hand, appear as solutions to certain differential equations called Bessel equations.

In this chapter, we will discover what these functions are, and their main properties.

## 1.1 Gamma Function

The Gamma function is denoted by $\Gamma$ and is defined for positive real numbers by the improper integral:

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt, \tag{1.1}$$

below are the key properties.

**Property 1.1.1.**

$$\Gamma(x+1) = x\Gamma(x) \tag{1.2}$$

**Proof.** From the defintion (1.1) of the function $\Gamma$, we write

$$\Gamma(x+1) = \int_0^{+\infty} t^x e^{-t} dt,$$

using integration by parts, let

$$u = t^x \Longrightarrow du = xt^{x-1}dt$$

and

$$dv = e^{-t}dt \Longrightarrow v = -e^{-t}.$$

Then

$$\Gamma(x+1) = -\left[t^x e^{-t}\right]_0^{+\infty} + \int_0^{+\infty} xt^{x-1}e^{-t}dt.$$

Boundary term :

$$\text{when } t \to +\infty, \text{ we find } t^x e^{-t} \to 0,$$

and

$$\text{at } t = 0, \text{ we have } t^x e^{-t} = 0.$$

So

$$\begin{aligned}\Gamma(x+1) &= \int_0^{+\infty} xt^{x-1}e^{-t}dt \\ &= x\int_0^{+\infty} t^{x-1}e^{-t}dt \\ &= x\Gamma(x).\end{aligned}$$

Therefore

$$\Gamma(x+1) = x\Gamma(x). \tag{1.3}$$

■

**Property 1.1.2**.

$$\forall n \in \mathbb{N}^* : \Gamma(n) = (n-1)!. \tag{1.4}$$

**Proof.** Using $(1.2)$, we have

$$\Gamma(n) = (n-1)\Gamma(n-1).$$

By applying this egality recursively, we have :

$$\Gamma(n) = (n-1)(n-2)...\Gamma(1).$$

Now, we compute $\Gamma(1)$,

$$\Gamma(1) = \int_0^{+\infty} e^{-t} dt = -\left[e^{-t}\right]_0^{+\infty} = 1,$$

thus

$$\Gamma(n) = (n-1)(n-2)...1 = (n-1)!.$$

■

**Property 1.1.3**.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{1.5}$$

**Proof.** We have

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-\frac{1}{2}} e^{-t} dt,$$

puting

$$t = x^2,$$

then

$$dt = 2xdx,$$

so

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^{+\infty} x^{-1} e^{-x^2}.2xdx \\ &= 2\int_0^{+\infty} e^{-x^2} dx.\end{aligned}$$

We use the Gaussian integral

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi},$$

thus

$$\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2},$$

hence

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^{+\infty} e^{-x^2} dx = \sqrt{\pi}.$$

■

**Example 1.1.4.** Using $(1.3)$, we compute $\Gamma\left(\frac{3}{2}\right)$ and $\Gamma\left(\frac{5}{2}\right)$.

1. We have

$$\begin{aligned}\Gamma\left(\frac{3}{2}\right) &= \Gamma\left(1+\frac{1}{2}\right)\\ &= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\\ &= \frac{1}{2}.\sqrt{\pi}\\ &= \frac{\sqrt{\pi}}{2}.\end{aligned}$$

2. We have

$$\begin{aligned}\Gamma\left(\frac{5}{2}\right) &= \Gamma\left(1+\frac{3}{2}\right)\\ &= \frac{3}{2}\Gamma\left(\frac{3}{2}\right)\\ &= \frac{3}{2}.\frac{\sqrt{\pi}}{2}\\ &= \frac{3\sqrt{\pi}}{4}.\end{aligned}$$

## 1.2 Bessel functions of the first kind

Recall the Bessel's equation of order $\eta$ [14] :

$$x^2y^{''} + xy^{'} + \left(x^2 - \eta^2\right) y = 0, \tag{1.6}$$

where $\eta$ is a nonnegative real constant called the order of the equation. This equation has two linearly independent solutions, one of the solutions is called a Bessel function of the first kind that is denoted by $J_\eta$. This solution is regular at $x = 0$ and defined by the following formula :

$$J_\eta\left(x\right) = \sum_{k\geq 0} \frac{\left(-1\right)^k}{k!\Gamma\left(k+\eta+1\right)}\left(\frac{x}{2}\right)^{\eta+2k}, \tag{1.7}$$

where $\Gamma$ is the Gamma function.

This series converges for all $x \in \mathbb{R}$ and it has the following properties.

**Property 1.2.1**.

$$x\dot{J}_\eta\left(x\right) + \eta J_\eta\left(x\right) = xJ_{\eta-1}\left(x\right), \tag{1.8}$$

**Proof.** We know that the Bessel's function is

$$J_\eta(x) = \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(k+\eta+1)} \left(\frac{x}{2}\right)^{\eta+2k}.$$

We differentiate both sides with respect to $x$

$$\dot{J}_\eta(x) = \frac{\partial J_\eta(x)}{\partial x} = \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(k+\eta+1)}.(\eta+2k).\left(\frac{x}{2}\right)^{\eta+2k-1}.\frac{1}{2},$$

then

$$\begin{aligned}
x\dot{J}_\eta(x) &= \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(k+\eta+1)}.(\eta+2k).\left(\frac{x}{2}\right)^{\eta+2k-1}.\left(\frac{x}{2}\right) \\
&= \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(k+\eta+1)}.(\eta+2k).\left(\frac{x}{2}\right)^{\eta+2k} \\
&= \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(k+\eta+1)}.(2\eta+2k-\eta).\left(\frac{x}{2}\right)^{\eta+2k},
\end{aligned}$$

thus

$$x\dot{J}_\eta(x) = \sum_{k\geq 0} \frac{(-1)^k.(2\eta+2k)}{k!\Gamma(k+\eta+1)}.\left(\frac{x}{2}\right)^{\eta+2k} - \eta\sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(k+\eta+1)}.\left(\frac{x}{2}\right)^{\eta+2k}. \tag{1.9}$$

Using the property

$$\Gamma(\eta+1) = \eta\Gamma(\eta),$$

and the definition of the function $J_\eta$, we get

$$\begin{aligned}
x\dot{J}_\eta(x) &= \sum_{k\geq 0} \frac{(-1)^k.2}{k!\Gamma(k+\eta)}.\left(\frac{x}{2}\right)^{\eta+2k} - \eta J_\eta(x) \\
&= x\sum_{k\geq 0} \frac{(-1)^k.\left(\frac{2}{x}\right)}{k!\Gamma(k+\eta)}.\left(\frac{x}{2}\right)^{\eta+2k} - \eta J_\eta(x) \\
&= x\sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(k+\eta)}.\left(\frac{x}{2}\right)^{\eta+2k-1} - \eta J_\eta(x) \\
&= x\sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(k+\eta-1+1)}.\left(\frac{x}{2}\right)^{\eta-1+2k} - \eta J_\eta(x) \\
&= xJ_{\eta-1}(x) - \eta J_\eta(x).
\end{aligned}$$

■

**Property 1.2.2**.

$$x\dot{J}_{\eta}(x) - \eta J_{\eta}(x) = -xJ_{\eta+1}(x).$$

**Proof.** We use (1.9) and (1.4), we find

$$x\dot{J}_{\eta}(x) = \sum_{k\geq 0} \frac{(-1)^k.(\eta+2k)}{\Gamma(k+1)\Gamma(k+\eta+1)}.\left(\frac{x}{2}\right)^{\eta+2k},$$

thus

$$\begin{aligned}
x\dot{J}_{\eta}(x) - \eta J_{\eta}(x) &= \sum_{k\geq 0} \frac{(-1)^k.(\eta+2k)}{\Gamma(k+1)\Gamma(k+\eta+1)}.\left(\frac{x}{2}\right)^{\eta+2k} \\
&\quad -\eta\sum_{k\geq 0} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\eta+1)}\left(\frac{x}{2}\right)^{\eta+2k} \\
&= \sum_{k\geq 0} \frac{(-1)^k.2k}{\Gamma(k+1)\Gamma(k+\eta+1)}.\left(\frac{x}{2}\right)^{\eta+2k} \\
&= \sum_{k\geq 1} \frac{(-1)^k.2k}{\Gamma(k+1)\Gamma(k+\eta+1)}.\left(\frac{x}{2}\right)^{\eta+2k},
\end{aligned}$$

from (1.3), we have

$$\begin{aligned}
x\dot{J}_{\eta}(x) - \eta J_{\eta}(x) &= \sum_{k\geq 1} \frac{(-1)^k.2}{\Gamma(k)\Gamma(k+\eta+1)}.\left(\frac{x}{2}\right)^{\eta+2k} \\
&= \sum_{k\geq 0} \frac{(-1)^{k+1}.2}{\Gamma(k+1)\Gamma(k+(\eta+1)+1)}.\left(\frac{x}{2}\right)^{\eta+2k+2} \\
&= (-1).2.\left(\frac{x}{2}\right)\sum_{k\geq 0} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+(\eta+1)+1)}.\left(\frac{x}{2}\right)^{(\eta+1)+2k} \\
&= -xJ_{\eta+1}(x).
\end{aligned}$$

■

**Property 1.2.3**.

$$J_{\eta-1}(x) + J_{\eta+1}(x) = \frac{2\eta}{x}J_{\eta}(x). \tag{1.10}$$

**Proof.** We have

$$x^{\eta}J_{\eta}(x) = \sum_{k\geq 0} \frac{(-1)^k 2^{-(\eta+2k)}}{k!\Gamma(k+\eta+1)}x^{(2\eta+2k)}$$

then

$$\begin{aligned}\frac{\partial\left(x^{\eta}J_{\eta}\left(x\right)\right)}{\partial x} &= \sum_{k\geq 0}\frac{\left(-1\right)^{k}\left(2\eta+2k\right).2^{-(\eta+2k)}}{k!\Gamma\left(k+\eta+1\right)}x^{(2\eta+2k-1)} \\ &= \sum_{k\geq 0}\frac{\left(-1\right)^{k}\left(\eta+k\right)}{k!\Gamma\left(k+\eta+1\right)}\left(\frac{x}{2}\right)^{(\eta+2k-1)}.x^{\eta},\end{aligned}$$

using the property of $\Gamma$, we find

$$\frac{\partial\left(x^{\eta}J_{\eta}\left(x\right)\right)}{\partial x}=x^{\eta}\sum_{k\geq 0}\frac{\left(-1\right)^{k}}{k!\Gamma\left(k+\eta\right)}\left(\frac{x}{2}\right)^{(\eta+2k-1)},$$

hence

$$\frac{\partial\left(x^{\eta}J_{\eta}\left(x\right)\right)}{\partial x}=x^{\eta}J_{\eta-1}\left(x\right).$$

Applying the product rule on the left side gives us :

$$\eta x^{\eta-1}J_{\eta}\left(x\right)+x^{\eta}\dot{J}_{\eta}\left(x\right)=x^{\eta}J_{\eta-1}\left(x\right),$$

thus

$$\dot{J}_{\eta}\left(x\right)=J_{\eta-1}\left(x\right)-\frac{\eta}{x}J_{\eta}\left(x\right) \tag{1.11}$$

On the other hand, we have

$$\begin{aligned}\frac{\partial\left(x^{-\eta}J_{\eta}\left(x\right)\right)}{\partial x} &= \sum_{k\geq 0}\frac{\left(-1\right)^{k}\left(2k\right).2^{-(\eta+2k)}}{k!\Gamma\left(k+\eta+1\right)}x^{(2k-1)} \\ &= \sum_{k\geq 1}\frac{\left(-1\right)^{k}}{(k-1)!\Gamma\left(k+\eta+1\right)}.\left(\frac{x}{2}\right)^{(\eta+2k-1)}.x^{-\eta} \\ &= x^{-\eta}\sum_{k\geq 0}\frac{\left(-1\right)^{k+1}}{(k)!\Gamma\left(k+(\eta+1)+1\right)}.\left(\frac{x}{2}\right)^{((\eta+1)+2k)} \\ &= -x^{-\eta}\sum_{k\geq 0}\frac{\left(-1\right)^{k}}{(k)!\Gamma\left(k+(\eta+1)+1\right)}.\left(\frac{x}{2}\right)^{((\eta+1)+2k)},\end{aligned}$$

then

$$\frac{\partial\left(x^{-\eta}J_{\eta}\left(x\right)\right)}{\partial x}=-x^{-\eta}J_{\eta+1}\left(x\right).$$

So

$$-\eta x^{-\eta-1}J_{\eta}\left(x\right)+x^{-\eta}\dot{J}_{\eta}\left(x\right)=-x^{-\eta}J_{\eta+1}\left(x\right),$$

therefore

$$\dot{J}_{\eta}(x) = \frac{\eta}{x} J_{\eta}(x) - J_{\eta+1}(x). \tag{1.12}$$

Combining (1.11) and (1.12), we see that

$$J_{\eta-1}(x) - \frac{\eta}{x} J_{\eta}(x) = \frac{\eta}{x} J_{\eta}(x) - J_{\eta+1}(x),$$

so

$$J_{\eta-1}(x) + J_{\eta+1}(x) = \frac{2\eta}{x} J_{\eta}(x).$$

■

**Property 1.2.4**.

$$J_{\eta-1}(t) - J_{\eta+1}(t) = 2\dot{J}_{\eta}(t). \tag{1.13}$$

**Proof.** By adding (1.11) and (1.12), we get

$$2\dot{J}_{\eta}(t) = J_{\eta-1}(x) - \frac{\eta}{x} J_{\eta}(x) + \frac{\eta}{x} J_{\eta}(x) - J_{\eta+1}(x)$$

then

$$2\dot{J}_{\eta}(t) = J_{\eta-1}(t) - J_{\eta+1}(t).$$

This is what needs to be proven.

■

# Chapter 2

# Power series solutions to the Bessel's equation

In this chapter, we develop the series solution for the Bessel differential equation and derive the standard form of the first kind Bessel functions. The focus is on the step-by-step construction of the series, determining the recurrence relations for the coeffcients, and the conditions under which the series converges.

## 2.1 Derivation of the power series representation

We shall use the method of Frobenius to solve the Bessel's equation. Thus, we seek solutions of the form

$$\mathrm{y}(x) = \sum_{k\geq 0} a_k x^{k+r}, \qquad x > 0, \tag{2.1}$$

with $a_0 \neq 0$.

Differentiation of (2.1) term by term yields

$$\mathrm{y}'(x) = \sum_{k\geq 0} (k+r) a_k x^{k+r-1},$$

Similarly, we obtain

$$\mathrm{y}''(x) = \sum_{k\geq 0} (k+r)(k+r-1) a_k x^{k+r-2},$$

Substituting these into (1.6), we obtain

$$\sum_{k\geq 0}(k+r)(k+r-1)a_k x^{k+r}+\sum_{k\geq 0}(k+r)a_k x^{k+r}$$

$$+\sum_{k\geq 0}a_k x^{k+r+2}-\eta^2\sum_{k\geq 0}a_k x^{k+r}=0.$$

This implies

$$x^r\left(\sum_{k\geq 0}\left[(k+r)^2-\eta^2\right]a_k x^k+\sum_{k\geq 0}a_k x^{k+2}\right)=0.$$

Now, cancel $x^r$, and try to determine $a_k$'s so that the coefficient of each power of $x$ will vanish.

For the constant term, we require $(r^2-\eta^2)\,a_0=0$.

Since $a_0\neq 0$, it follows that

$$r^2-\eta^2=0,$$

which is the indicial equation. The only possible values of $r$ are $\eta$ and $-\eta$.

We choose $r=\eta$, the equations for determining the coefficients are :

$$\left[(\eta+1)^2-\eta^2\right]a_1=0,$$

and

$$\left[(\eta+k)^2-\eta^2\right]a_k+a_{k-2}=0,\ \ k\geq 2.$$

Since $\eta\geq 0$, we have $a_1=0$. The second equation yields

$$a_k=-\frac{a_{k-2}}{(\eta+k)^2-\eta^2},$$

then

$$a_k=-\frac{a_{k-2}}{k\,(k+2\eta)}. \tag{2.2}$$

Since $a_1=0$, we immediately obtain

$$a_3=a_5=a_7=...=a_{2m+1}=...=0.$$

For the coefficients with even subscripts, we have

$$a_2=-\frac{a_0}{2\,(2+2\eta)}=-\frac{a_0}{2^2\,(1+\eta)},$$

$$a_4 = -\frac{a_2}{4(4+2\eta)} = (-1)^2 \cdot \frac{a_0}{2^4.2!\,(1+\eta)(2+\eta)},$$
$$a_6 = -\frac{a_4}{6(6+2\eta)} = (-1)^3 \cdot \frac{a_0}{2^6.3!\,(1+\eta)(2+\eta)(3+\eta)},$$

and, in general

$$a_{2k} = (-1)^k \cdot \frac{a_0}{2^{2k}.k!\,(1+\eta)(2+\eta)\ldots(k+\eta)}.$$

Therefore, the choice $r = \eta$ yields the solution

$$y(x) = a_0 x^\eta \left(1 + \sum_{k\geq 1} (-1)^k \cdot \frac{x^{2k}}{2^{2k}.k!\,(1+\eta)(2+\eta)\ldots(k+\eta)}\right). \tag{2.3}$$

Using (1.2), we can write

$$x = \frac{\Gamma(x+1)}{\Gamma(x)}.$$

Using this gamma function, we shall simplify the form of the solutions of the Bessel equation. We have

$$\begin{aligned}(1+\eta)(2+\eta)\ldots(k+\eta) &= \frac{\Gamma(2+\eta)}{\Gamma(1+\eta)} \cdot \frac{\Gamma(3+\eta)}{\Gamma(2+\eta)} \ldots \frac{\Gamma(k+\eta+1)}{\Gamma(k+\eta)} \\ &= \frac{\Gamma(k+\eta+1)}{\Gamma(1+\eta)}.\end{aligned}$$

When we choose

$$a_0 = \frac{2^{-\eta}}{\Gamma(1+\eta)}$$

the solution (2.3) for $x > 0$ can be written

$$J_\eta(x) = \left(\frac{x}{2}\right)^\eta \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(k+\eta+1)} \left(\frac{x}{2}\right)^{2k}. \tag{2.4}$$

The function $J_\eta$ defined above for $x > 0$ and $\eta \geq 0$ is called the Bessel function of the first kind of order $\eta$.

## 2.2 Convergence

We have

$$J_\eta(x) = \left(\frac{x}{2}\right)^\eta \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(k+\eta+1)} \left(\frac{x}{2}\right)^{2k}.$$

To determine where this generalized power series actually makes sense, we apply the Ratio Test to the absolute values of consecutive terms. Let the $k-th$ term of the power series component be :

$$\alpha_k = \frac{(-1)^k}{k!\Gamma(k+\eta+1)}\left(\frac{x}{2}\right)^{2k},$$

We evaluate the limit of the ratio of successive terms :

$$\lim_{k\to+\infty}\left|\frac{\alpha_{k+1}}{\alpha_k}\right| = \lim_{k\to+\infty}\left|\frac{\frac{(-1)^{k+1}}{(k+1)!\Gamma(k+\eta+2)}\left(\frac{x}{2}\right)^{2k+2}}{\frac{(-1)^k}{k!\Gamma(k+\eta+1)}\left(\frac{x}{2}\right)^{2k}}\right|,$$

Using (1.3) and simplifying the factorials, we get :

$$\lim_{k\to+\infty}\left|\frac{\alpha_{k+1}}{\alpha_k}\right| = \lim_{k\to+\infty}\frac{1}{(k+1)(k+\eta+1)}\left(\frac{x}{2}\right)^2.$$

For any finite, fixed real value of $x$ :

$$\begin{aligned}\lim_{k\to+\infty}\left|\frac{\alpha_{k+1}}{\alpha_k}\right| &= \lim_{k\to+\infty}\frac{x^2}{4(k+1)(k+\eta+1)}\\ &= 0.\end{aligned}$$

Because the limit is 0 for all $x$, the ratio is always less than 1. Therefore, by the Ratio Test, the power series for $J_\eta(x)$ converges absolutely for all real $x$. Its radius of convergence is infinite ($R=\infty$).

# Chapter 3

# Haar wavelet method for approximating Bessel Functions of the first kind

This chapter focuses on the application of the Haar wavelet method to approximate Bessel functions of the first kind. By transforming the problem into a system of algebraic equations using Haar wavelet expansions, the method provides an efficient framework for obtaining approximate solutions. The approach not only simplifies the computational process but also ensures good convergence properties.

The objective of this chapter is to develop the Haar wavelet formulation for Bessel functions, analyze its accuracy, and compare the results with known analytical or numerical solutions.

## 3.1 Haar wavelet basis

The Haar wavelet basis, defined as follows on the interval $[\alpha, \beta[$, forms a family of functions on subintervals of $[\alpha, \beta[$, generated from the scaling and wavelet functions through dilation and translation [8], these functions are derived from the scaling functions:

$$f_1(x) = \begin{cases} 1, & for\ x \in [\alpha, \beta[, \\ 0, & otherwise, \end{cases} \tag{3.1}$$

and the wavelet function :

$$f_2(x) = \begin{cases} 1, & for\ x \in \left[\alpha, \frac{\alpha+\beta}{2}\right[, \\ -1, & for\ x \in \left[\frac{\alpha+\beta}{2}, \beta\right[, \\ 0, & otherwise. \end{cases} \tag{3.2}$$

For the other wavelet functions $f_i$ :

$$f_i(x) = \begin{cases} 1, & for\ x \in [a, b[, \\ -1, & for\ x \in [b, c[, \\ 0, & otherwise, \end{cases} \tag{3.3}$$

where

$$a = \alpha + (\beta - \alpha)\frac{k}{m},$$
$$b = \alpha + (\beta - \alpha)\frac{k+0.5}{m},$$
$$c = \alpha + (\beta - \alpha)\frac{k+1}{m},$$
$$i = 2, 3, 4, ..., 2M.$$

And

$$m = 2^j, j = 0, 1, ..., J, \qquad M = 2^J,$$

$$k = 0, 1, ..., m-1, \qquad i = m + k + 1.$$

In [4] E. Babolian and A. Shahsavaran have proved that $g_M$ defined by :

$$g_M(x) = \sum_{i=1}^{2M} \lambda_i f_i(x), \ \lambda_i = \langle g, f_i \rangle, \ i = 1, ..., 2M, \tag{3.4}$$

converges to $g$ in $L^2([\alpha, \beta[)$.

So, any function $g \in L^2([\alpha, \beta[)$ can be decomposed as :

$$g(x) = \sum_{i \geq 1} \lambda_i f_i(x). \tag{3.5}$$

## 3.2 Our proposed approach

Consider the following second order linear differential equation :

$$x^2y^{''} + xy^{'} + \left(x^2 - \eta^2\right) y = 0, \ x \in [\alpha, \beta[ \, , \ \eta \in \mathbb{R}_+, \tag{3.6}$$

with initial conditions

$$y(\alpha) = d, \quad y^{'}(\alpha) = e.$$

To solve (3.6), using Haar wavelets, we must follow the following steps :

**Step 1**. We assume that

$$y^{''}(x) = \sum_{i=1}^{2M} \lambda_i f_i(x), \ x \in [\alpha, \beta[ \, , \text{ for a given level } M = 2^J. \tag{3.7}$$

**Step 2**. By integrating both sides of (3.7) from $\alpha$ to $x$, we find :

$$y^{'}(x) = y^{'}(\alpha) + \sum_{i=1}^{2M} \lambda_i R_{i,1}(x), \tag{3.8}$$

where

$$R_{i,1}(x) = \begin{cases} x - a, & for \ x \in [a, b[ \, , \\ c - x, & for \ x \in [b, c[ \, , \\ 0, & otherwise, \end{cases} \tag{3.9}$$

for $i = 2, ..., 2M$, and

$$R_{1,1}(x) = x - \alpha \text{ for all } x \in [\alpha, \beta[ \, .$$

**Step 3**. Also by integrating both sides of (3.8) from $\alpha$ to $x$, we get :

$$y(x) = y(\alpha) + y^{'}(\alpha)(x - \alpha) + \sum_{i=1}^{2M} \lambda_i R_{i,2}(x), \tag{3.10}$$

where

$$R_{i,2}(x) = \begin{cases} 0, & for \ x < a, \\ \frac{1}{2}(x-a)^2, & for \ a \leq x < b, \\ \frac{(\beta-\alpha)^2}{4m^2} - \frac{1}{2}(x-c)^2, & for \ b \leq x < c, \\ \frac{(\beta-\alpha)^2}{4m^2}, & for \ x \geq c, \end{cases} \tag{3.11}$$

for $i = 2, ..., 2M$, and

$$R_{1,2}(x) = \frac{1}{2}(x-\alpha)^2 \text{ for all } x \in [\alpha, \beta[ .$$

**Step 4**. By substituting (3.7) − (3.8) and (3.10) into (3.6), we obtain :

$$\sum_{i=1}^{2M} g_i(x)\lambda_i = h(x), \tag{3.12}$$

where

$$g_i(x) = x^2 f_i(x) + xR_{i,1}(x) + \left(x^2 - \eta^2\right) R_{i,2}(x) \text{ for } i = 1, ..., 2M$$

and

$$h(x) = -xy'(\alpha) - \left(x^2 - \eta^2\right)\left(y(\alpha) + y'(\alpha)(x-\alpha)\right).$$

**Step 5**. We introduce the collocation points

$$x_\ell = \alpha + (\beta - \alpha)\frac{\ell - 0.5}{2M}, \ell = 1, ..., 2M,$$

and by replacing them into (3.12), we obtain the following system of linear equations :

$$\sum_{i=1}^{2M} g_i(x_\ell)\lambda_i = h(x_\ell), \ 1 \le \ell \le 2M. \tag{3.13}$$

This system can be written in the matrix form as :

$$GA = H, \tag{3.14}$$

where

$$G(j,k) = g_k(x_j), 1 \le j \le 2M, 1 \le k \le 2M$$

and

$$A = (\lambda_1, \lambda_2, ..., \lambda_{2M})^T, \quad H = (h(x_1), h(x_2), ..., h(x_{2M}))^T.$$

**Step 6.** By solving (3.14), we obtain the wavelet coefficients $A = G^{-1}H$ to replace them into (3.10), then we get the numerical solution of (3.6).

## 3.3 Error estimation

**Lemma 3.3.1.** *The coefficients $\lambda_i$ that are given in* (3.7) *verify*

$$|\lambda_i| = \mathcal{O}\left(\frac{1}{2^{2j}}\right), i = 2^j + k + 1, \forall i \geq 2. \tag{3.15}$$

**Proof.** We have, $\forall i \geq 2$ :

$$\begin{aligned}
\lambda_i &= \int_\alpha^\beta y^{''}(x) f_i(x)\, dx \\
&= \int_a^b y^{''}(x)\, dx - \int_b^c y^{''}(x)\, dx \\
&= (b-a)\, y^{''}(\xi_1) - (c-b)\, y^{''}(\xi_2); \xi_1 \in [a,b], \xi_2 \in [b,c].
\end{aligned}$$

Since :

$$b - a = c - b = \frac{\beta - \alpha}{2m} = \frac{\beta - \alpha}{2^{j+1}},$$

then

$$\lambda_i = \frac{\beta - \alpha}{2^{j+1}} (\xi_1 - \xi_2)\, y^{'''}(\xi); \xi \in ]\xi_1, \xi_2[.$$

We have

$$a \leq \xi_1 \leq b \text{ and } b \leq \xi_2 \leq c,$$

then

$$0 \leq \xi_2 - \xi_1 \leq c - a = \frac{\beta - \alpha}{m} = \frac{\beta - \alpha}{2^j}.$$

Therefore

$$|\lambda_i| \leq \frac{(\beta - \alpha)^2}{2^{2j+1}} \left\| y^{'''} \right\|_\infty.$$

Consequently

$$|\lambda_i| = \mathcal{O}\left(\frac{1}{2^{2j}}\right).$$

■

**Lemma 3.3.2.** *The function $R_{i,2}$ for $i \geq 2$ verifies*

$$\exists \gamma \in \mathbb{R}_+^* : \|R_{i,2}\|_\infty \leq \frac{\gamma}{2^{2j}};\; i = 2^j + k + 1.$$

**Proof.** We have, $R_{i,2}$ is positive and increasing on $[a, b]$, then

$$\begin{aligned} \|R_{i,2}\|_\infty &\leq \frac{(b-a)^2}{2} \\ &= \frac{1}{2}\left(\frac{\beta-\alpha}{2m}\right)^2 \\ &\leq \frac{\gamma}{2^{2j}}. \end{aligned}$$

On the interval $[b, c]$, the function $R_{i,2}$ is also positive and increasing, therefore

$$\begin{aligned} \|R_{i,2}\|_\infty &\leq \frac{(\beta-\alpha)^2}{4m^2} \\ &\leq \frac{\gamma}{2^{2j}}. \end{aligned}$$

For $x \geq c$, the function $R_{i,2}$ is constant and verifies

$$\|R_{i,2}\|_\infty \leq \frac{\gamma}{2^{2j}}.$$

■

**Theorem 3.3.3.** *Let $y$ be the exact solution of problem* (3.6) *and $y_M$ its approximation formula, which is expressed in* (3.10), *then*

$$\|y - y_M\|_{L^2([\alpha,\beta[)} = \mathcal{O}\left(\frac{1}{M^2}\right); M = 2^J.$$

**Proof.** We have

$$y_M(x) = y(\alpha) + y^{'}(\alpha)(x-\alpha) + \sum_{i=1}^{2M} \lambda_i R_{i,2}(x),$$

and

$$y(x) = y(\alpha) + y^{'}(\alpha)(x-\alpha) + \sum_{i\geq 1} \lambda_i R_{i,2}(x),$$

we use the notation $\|.\|_{L^2([\alpha,\beta[)} = \|.\|_2$, we see that

$$\begin{aligned} \|y - y_M\|_2^2 &= \left\| \sum_{i\geq 2M+1} \lambda_i R_{i,2} \right\|_2^2 \\ &= \sum_{l\geq 2M+1} \sum_{i\geq 2M+1} \lambda_i \lambda_l \int_\alpha^\beta R_{i,2}(x) R_{l,2}(x)\, dt. \end{aligned}$$

If we put $i = 2^j + k + 1$ and $l = 2^r + s + 1$, we have

$$\begin{aligned}
\|y-y_M\|_2^2 &= \sum_{r\geq J+1}\sum_{s=0}^{2^r-1}\sum_{j\geq J+1}\sum_{k=0}^{2^j-1}\lambda_{2^j+k+1}\lambda_{2^r+s+1}\int_\alpha^\beta R_{2^j+k+1,2}\left(x\right)R_{2^r+s+1,2}\left(x\right)dt \\
&\leq \sum_{r\geq J+1}\sum_{j\geq J+1}\left|\lambda_{2^j+1}\lambda_{2^r+1}\right|\int_\alpha^\beta \left|R_{2^j+1,2}\left(x\right)R_{2^r+1,2}\left(x\right)\right|dt \\
&\quad+\sum_{r\geq J+1}\sum_{s=1}^{2^r-1}\sum_{j\geq J+1}\left|\lambda_{2^j+1}\lambda_{2^r+s+1}\right|\int_\alpha^\beta \left|R_{2^j+1,2}\left(x\right)R_{2^r+s+1,2}\left(x\right)\right|dt \\
&\quad+\sum_{r\geq J+1}\sum_{j\geq J+1}\sum_{k=1}^{2^j-1}\left|\lambda_{2^j+k+1}\lambda_{2^r+1}\right|\int_\alpha^\beta \left|R_{2^j+k+1,2}\left(x\right)R_{2^r+1,2}\left(x\right)\right|dt \\
&\quad+\sum_{r\geq J+1}\sum_{s=1}^{2^r-1}\sum_{j\geq J+1}\sum_{k=1}^{2^j-1}\left|\lambda_{2^j+k+1}\lambda_{2^r+s+1}\right|\int_\alpha^\beta \left|R_{2^j+k+1,2}\left(x\right)R_{2^r+s+1,2}\left(x\right)\right|dt.
\end{aligned}$$

Below, $c$ denotes a strictly positive constant and whose value may change from place to place.

By using Lemma (3.3.1) and Lemma (3.3.2), we have

$$\begin{aligned}
\|y-y_M\|_2^2 &\leq c\sum_{r\geq J+1}\sum_{j\geq J+1}\frac{1}{2^{2j}}.\frac{1}{2^{2r}} \\
&\quad+c\sum_{r\geq J+1}\sum_{j\geq J+1}\frac{1}{2^{2j}}.\frac{1}{2^{2r}}.\frac{(2^r-1)}{2^{2r}} \\
&\quad+c\sum_{r\geq J+1}\sum_{j\geq J+1}\frac{1}{2^{2j}}.\frac{1}{2^{2r}}.\frac{(2^j-1)}{2^{2j}} \\
&\quad+c\sum_{r\geq J+1}\sum_{j\geq J+1}\frac{1}{2^{2j}}.\frac{1}{2^{2r}}.\frac{(2^r-1)}{2^{2r}}.\frac{(2^j-1)}{2^{2j}} \\
&\leq c\left(\frac{1}{2^{2J}}\right)^2,
\end{aligned}$$

so

$$\|y-y_M\|_2\leq\frac{c}{2^{2J}}.$$

Therefore

$$\|y-y_M\|_{L^2([\alpha,\beta[)}\leq\mathcal{O}\left(\frac{1}{M^2}\right);\ M=2^J.$$

■

## 3.4 Numerical results

This section presents numerical examples to demonstrate the efficiency of the proposed Haar wavelet-based approach, implemented in Matlab. So, the numerical results of examples $(3.4.1, 3.4.2, 3.4.3)$ compare the values of $J_\eta$ for $\eta = 0, 1, 2$ on $[0, 1]$ by using their tabulated and their Haar wavelet approximations values at different levels $J$. For example 3.4.4, we compare the values of $J_{\frac{1}{2}}$ using its analytic and its Haar wavelet approach formula. All results are presented in tables.

**Example 3.4.1 :** Consider the Bessel function of order $\eta = 0$.

**Table 3.4.1**. Comparison of tabulated $J_0(x)$ using power series and wavelets method

| $x$ | $J_0(x)$ using power series | $J_0(x)$ using wavelets | | |
|---|---|---|---|---|
| | | $J = 5$ | $J = 6$ | $J = 7$ |
| 0 | 1 | 1 | 1 | 1 |
| 0.1 | 0.99750 | 0.99750 | 0.99750 | 0.99750 |
| 0.2 | 0.99002 | 0.99002 | 0.99002 | 0.99002 |
| 0.3 | 0.97762 | 0.97762 | 0.97762 | 0.97762 |
| 0.4 | 0.96039 | 0.96039 | 0.96039 | 0.96039 |
| 0.5 | 0.93846 | 0.93846 | 0.93846 | 0.93846 |
| 0.6 | 0.91200 | 0.91200 | 0.91200 | 0.91200 |
| 0.7 | 0.88120 | 0.88120 | 0.88120 | 0.88120 |
| 0.8 | 0.84628 | 0.84628 | 0.84628 | 0.84628 |
| 0.9 | 0.80752 | 0.80752 | 0.80752 | 0.80752 |
| 1 | 0.76519 | 0.76519 | 0.76519 | 0.76519 |

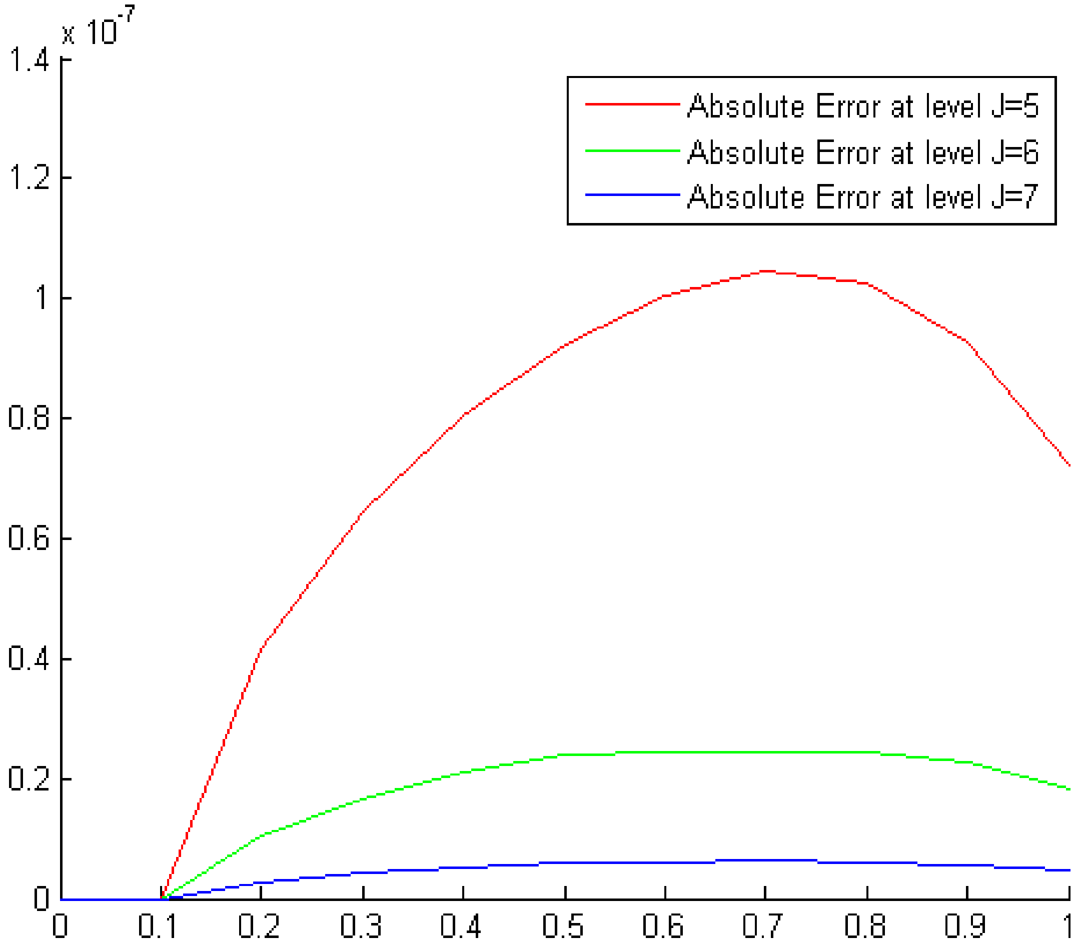


**Figure 3.4.1 :** Absolute Error for the Bessel function $J_0$ at levels $J = 5, 6$ and 7.

**Example 3.4.2 :** Consider the Bessel function of order $\eta = 1$.

**Table 3.4.2** : Comparison of tabulated $J_1(x)$ using power series and wavelets method

| $x$ | $J_1(x)$ using power series | $J_1(x)$ using wavelets | | |
|---|---|---|---|---|
| | | $J=5$ | $J=6$ | $J=7$ |
| 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.04993 | 0.04993 | 0.04993 | 0.04993 |
| 0.2 | 0.09950 | 0.09950 | 0.09950 | 0.09950 |
| 0.3 | 0.14831 | 0.14831 | 0.14831 | 0.14831 |
| 0.4 | 0.19602 | 0.19602 | 0.19602 | 0.19602 |
| 0.5 | 0.24226 | 0.24226 | 0.24226 | 0.24226 |
| 0.6 | 0.28670 | 0.28670 | 0.28670 | 0.28670 |
| 0.7 | 0.32899 | 0.32899 | 0.32899 | 0.32899 |
| 0.8 | 0.36884 | 0.36884 | 0.36884 | 0.36884 |
| 0.9 | 0.40594 | 0.40595 | 0.40595 | 0.40594 |
| 1 | 0.44005 | 0.44005 | 0.44005 | 0.44005 |

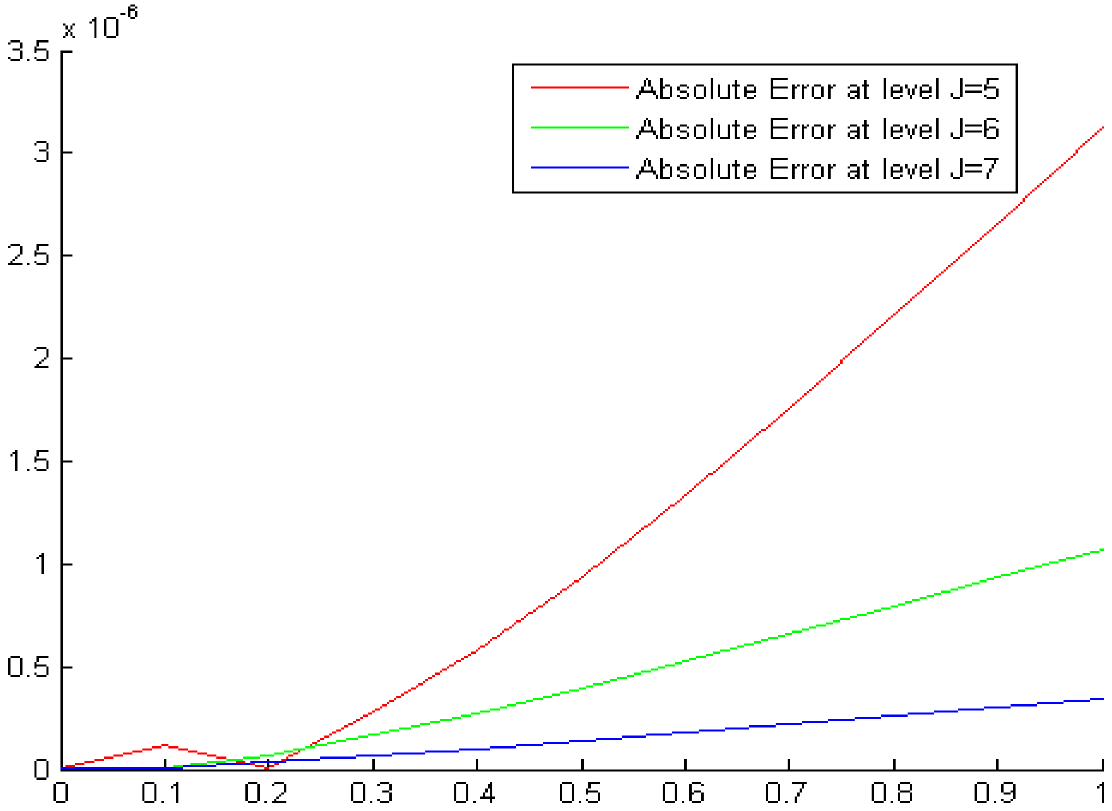


**Figure 3.4.2.** Absolute Error for the Bessel function $J_1$ at levels $J = 5, 6$ and $7$

**Example 3.4.3 :** Consider the Bessel function of order $\eta = 2$.

**Table 3.4.3**. Comparison of tabulated $J_2(x)$ using power series and wavelets method

| $x$ | $J_2(x)$ using power series | $J_2(x)$ using wavelets | | |
|---|---|---|---|---|
| | | $J = 5$ | $J = 6$ | $J = 7$ |
| 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.00124 | 0.00124 | 0.00124 | 0.00124 |
| 0.2 | 0.00498 | 0.00498 | 0.00498 | 0.00498 |
| 0.3 | 0.01116 | 0.01116 | 0.01116 | 0.01116 |
| 0.4 | 0.01973 | 0.01973 | 0.01973 | 0.01973 |
| 0.5 | 0.03060 | 0.03060 | 0.03060 | 0.03060 |
| 0.6 | 0.04366 | 0.04366 | 0.04366 | 0.04366 |
| 0.7 | 0.05878 | 0.05878 | 0.05878 | 0.05878 |
| 0.8 | 0.07581 | 0.07581 | 0.07581 | 0.07581 |
| 0.9 | 0.09458 | 0.09458 | 0.09458 | 0.09458 |
| 1 | 0.11490 | 0.11490 | 0.11490 | 0.11490 |

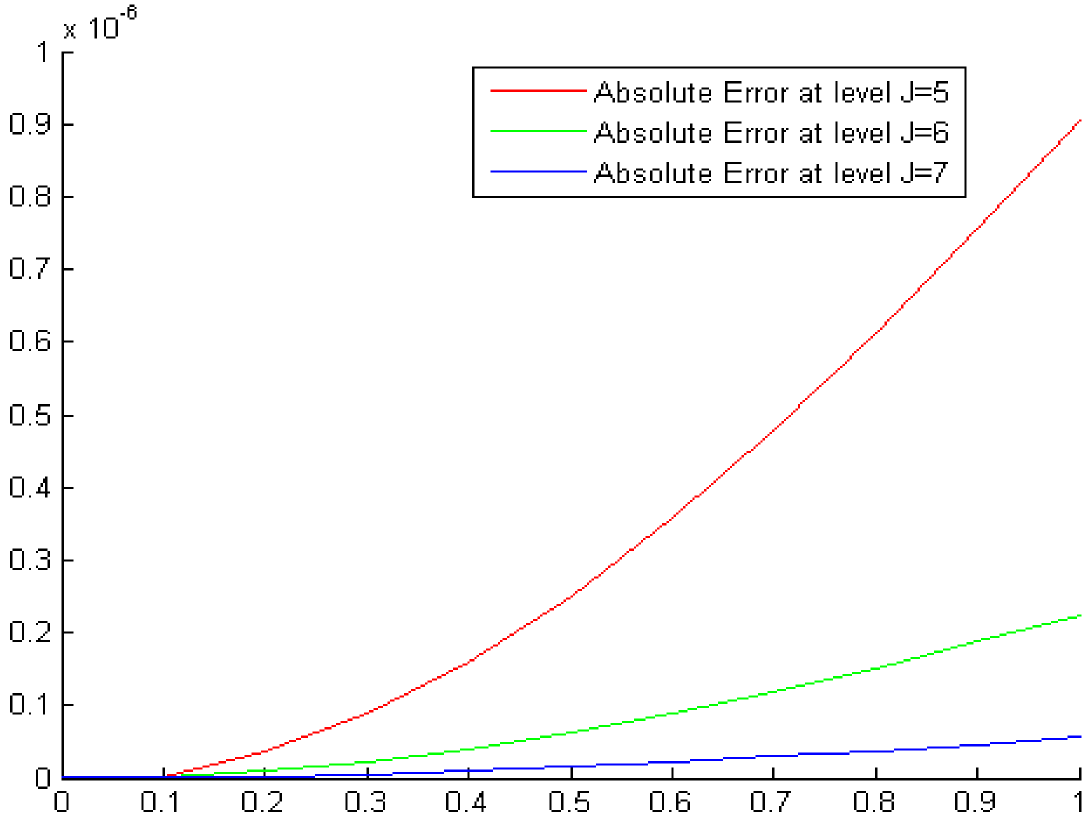


**Figure 3.4.3.** Absolute Error for the Bessel function $J_2$ at levels $J = 5, 6$ and 7

**Example 3.4.4 :** Consider the Bessel function of order $\eta = \frac{1}{2}$, where its analytic formula is $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi t}} \sin x$ for $x > 0$.

**Table 3.4.4.** Comparison of tabulated $J_{\frac{1}{2}}(x)$ using exact values and wavelets method

| $x$ | $J_{\frac{1}{2}}(x)$ using exact values | $J_{\frac{1}{2}}(x)$ using wavelets | | |
|---|---|---|---|---|
| | | $J=5$ | $J=6$ | $J=7$ |
| 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.25189 | 0.25189 | 0.25189 | 0.25189 |
| 0.2 | 0.35445 | 0.35451 | 0.35446 | 0.35445 |
| 0.3 | 0.43049 | 0.43056 | 0.43051 | 0.43049 |
| 0.4 | 0.49127 | 0.49135 | 0.49129 | 0.49128 |
| 0.5 | 0.54097 | 0.54105 | 0.54099 | 0.54097 |
| 0.6 | 0.58161 | 0.58169 | 0.58163 | 0.58162 |
| 0.7 | 0.61436 | 0.61444 | 0.61438 | 0.61436 |
| 0.8 | 0.63992 | 0.64000 | 0.63994 | 0.63993 |
| 0.9 | 0.65881 | 0.65889 | 0.65883 | 0.65881 |
| 1 | 0.67139 | 0.67147 | 0.67141 | 0.67140 |

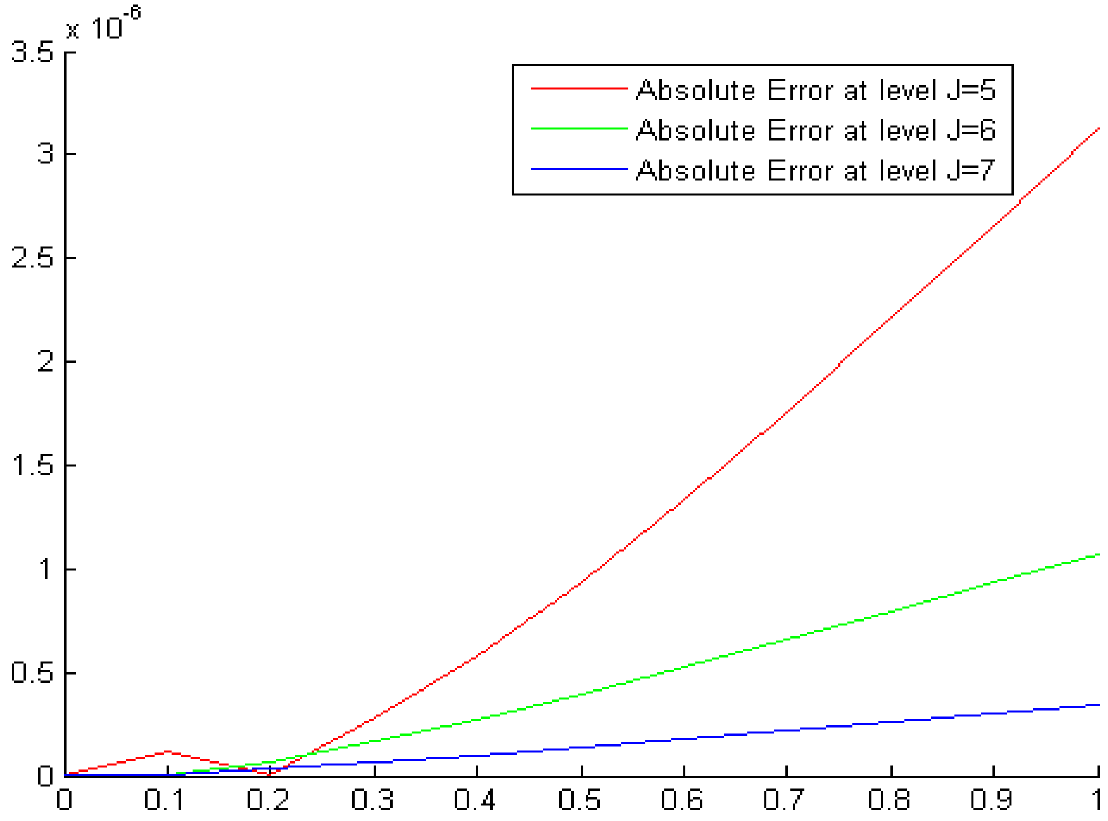


**Figure 3.4.4.** Absolute Error for the Bessel function $J_{\frac{1}{2}}$ at levels $J = 5, 6$ and 7.

Obviously by comparing columns of Tables $(3.4.1 - 3.4.4)$, one can easily see that our proposed approximations for Bessel functions $\eta = 0, 1, 2$ and $\eta = \frac{1}{2}$ are in excellent agreement with their tabulated and exact values. Where, the results of our approach are very significant and more accurate when the level $J$ increases.

# Conclusion

In this work, we investigated the approximation of the Bessel function of the first kind using two distinct approaches : the classical power series expansion and the Haar wavelet method. The study highlighted both the theoretical foundations and the practical performance of each technique.

The power series method, derived directly from the analytical definition of the Bessel function, provides highly accurate results in regions close to the origin. Its convergence is well understood and straightforward to implement, making it a reliable benchmark for comparison. However, its efficiency decreases for larger values of the argument, where more terms are required to maintain accuracy.

On the other hand, the Haar wavelet method offers a powerful alternative based on piecewise constant basis functions. This approach demonstrated strong adaptability and computational efficiency, particularly for approximating functions over larger intervals. Despite its simplicity, the Haar wavelet method achieved satisfactory accuracy with relatively low computational cost, making it attractive for numerical applications.

The comparison between the two methods shows that each has its own advantages depending on the context. While the power series is preferable for local and highly precise approximations, the Haar wavelet method is better suited for global approximations and problems requiring fast computation.

Overall, this study confirms that combining analytical and numerical techniques provides a deeper understanding of special functions such as the Bessel function.

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## Abstract:

In this work, we investigated the approximation of the Bessel function of the first kind using two distinct approaches: the classical power series expansion and the Haar wavelet method. The study highlighted both the theoretical foundations and the practical performance of each technique.

The power series method, derived directly from the analytical definition of the Bessel function, provides highly accurate results in regions close to the origin. Its convergence is well understood and straightforward to implement, making it a reliable benchmark for comparison. However, its efficiency decreases for larger values of the argument, where more terms are required to maintain accuracy.

On the other hand, the Haar wavelet method offers a powerful alternative based on piecewise constant basis functions. This approach demonstrated strong adaptability and computational efficiency, particularly for approximating functions over larger intervals. Despite its simplicity, the Haar wavelet method achieved satisfactory accuracy with relatively low computational cost, making it attractive for numerical applications.

The comparison between the two methods shows that each has its own advantages depending on the context. While the power series is preferable for local and highly precise approximations, the Haar wavelet method is better suited for global approximations and problems requiring fast computation. Overall, this study confirms that combining analytical and numerical techniques provides a deeper understanding of special functions such as the Bessel function.

**Key words**:
Power series method, Bessel’s Equation, Bessel Functions, Haar Wavelets, Cnvergence, Error estimation.

## ملخص

تُقدم طريقة متسلسلة القوى، المشتقة مباشرةً من التعريف التحليلي لدالة بيسل، نتائج دقيقة للغاية في المناطق القريبة من نقطة الأصل. تقاربها مفهوم جيدًا وسهل التطبيق، مما يجعلها معيارًا موثوقًا للمقارنة. مع ذلك، تنخفض كفاءتها مع ازدياد قيمة الوسيط، حيث يتطلب الأمر عددًا أكبر من الحدود للحفاظ على الدقة.

من جهة أخرى، تُقدم طريقة موجات هار بديلًا قويًا يعتمد على دوال أساسية ثابتة جزئيًا. أظهر هذا المنهج قدرة عالية على التكيف وكفاءة حسابية عالية، لا سيما لتقريب الدوال على فترات أوسع. على الرغم من بساطتها، حققت طريقة موجات هار دقة مرضية بتكلفة حسابية منخفضة نسبيًا، مما يجعلها جذابة للتطبيقات العددية.

تُظهر المقارنة بين الطريقتين أن لكل منهما مزاياها الخاصة حسب السياق. بينما تُفضّل متسلسلة القوى للتقريبات المحلية والدقيقة للغاية، بينما طريقة موجات هار تُعدّ أنسب للتقريبات الشاملة والمسائل التي تتطلب حسابًا سريعًا. بشكل عام، تؤكد هذه الدراسة أن الجمع بين التقنيات التحليلية والرقمية يُتيح فهمًا أعمق للدوال الخاصة، مثل دالة بيسل

**كلمات مفتاحية:**
طريقة متسلسلة القوى، معادلة بيسل، دوال بيسل، موجات هار، التقارب، تقدير الخطأ.