THE NUMERICAL SOLUTION OF THE HEAT DIFFUSION EQUATION VIA LATTICE BOLTZMANN METHOD

A.ATIA[12*] and K.MOHAMMEDI[2]

[1]*Université d'El-Oued, B.P 789, 39000 El-Oued, Algerie*

[2]*Laboratoire Energétique - Mécanique & Ingénieries (LEMI), Université M'Hamed Bougara, 35000 Boumerdes, Algerie*

*E-mail: maleksouf@yahoo.fr

Abstract. In the present work, the solution of the heat diffusion problem is presented using lattice Boltzmann method (LBM). The two dimensional task is considered and the different boundary conditions, specifically the Dirichlet and Neumann are taken into account. The D2Q4 lattice model is applied. To check the accuracy of the LBM algorithm, the same problems have been solved using the explicit variant of the finite difference method. In the final part of the paper, the results of computations are shown and the conclusions are formulated.

Keywords: lattice Boltzmann method, numerical solution, diffusion equation

INTRODUCTION

Over the last decade the lattice Boltzmann method (LBM) has been developed as a promising computational tool to analyze the large class of engineering problems, among others, the heat transfer problems. M.Bittagopal and C. Mishra [1] applied the LBM to solve the energy equations of conduction–radiation problems on non-uniform lattices. H. Shokouhmand et al [2] studied fully developed laminar flow and convective heat transfer between two parallel plates. R.Chaabane et al [3] investigated the solution of conduction problems with heat flux boundary condition.

In the present study, the simplest model D2Q4 BGK was chosen to solve the heat diffusion equation in 2D square domain with different boundary condition.

In the absence of convection and radiation, for a 2-D Cartesian geometry, energy equation is given by:

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{q}{\rho c} \tag{1}$$

Where $\alpha = \frac{k}{\rho c}$ is the thermal diffusivity. q, k, ρ and c are the rate of heat generation per unit volume, thermal conductivity of the medium, density and specific heat, respectively.

T, x, t denote the temperature, spatial co-ordinates and time, respectively. The equation (1) is supplemented by boundary conditions:

$$\begin{cases} Initial\ condition, t = 0\ ,\ T(x,y,0) = 0°C \\ Left\ side, x = 0,\ T(0,y,t) = 100°C \\ Right\ side, x = l,\ T(l,y,t) = 0°C \\ Bottom\ side, y = 0,\ \frac{\partial T}{\partial y}(x,0,t) = 0°C\ \ (adiabatic\ condition) \\ upper\ side, y = h,\ T(x,h,t) = 0°C \end{cases} \tag{2}$$

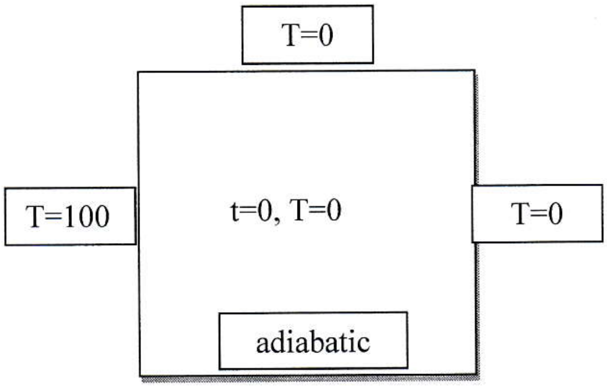

Fig. 1 A square (2D) domain with coordinate system

FORMULATION AND KINETIC EQUATION

The starting point of the LBM is the kinetic equation which for a 2-D geometry is given by [4]:

$$\frac{\partial f_i(r,t)}{\partial t} + \vec{e}_i \,.\, \nabla f_i(r,t) = \Omega_i \quad i = 1,2,3,\dots m \tag{3}$$

where f_i is the particle distribution function denoting the number of particles at the lattice node $\mathrm{r} = (\mathrm{r}(\mathrm{x},\mathrm{y},\mathrm{z}))$ at time t moving in direction i with velocitye $\vec{e}_i$ along the lattice link $\Delta r = e_i \Delta t$ connecting the nearest neighbors and m is the number of directions in a lattice through which the information propagates. The term collision operator Ω_i represents the rate of change of f_i due to collisions. The discrete Boltzmann equation with Bhatanagar–Gross–Krook (BGK) approximation is given by [4, 5]:

$$\frac{\partial f_i(r,t)}{\partial t} + \vec{e}_i \,.\, \nabla f_i(r,t) = -\frac{1}{\tau}[f_i(r,t) - f_i^{(eq)}(r,t)] \tag{4}$$

Where τ is the relaxation time and $f_i^{(eq)}$ is the equilibrium distribution function. For a given application, relaxation time τ is different for different lattices. The relaxation time τ for the D2Q4 lattice is computed from [6]:

$$\tau = \frac{2.\alpha.\Delta t}{\Delta x^2} + \frac{1}{2} \tag{5}$$

For the D2Q4 lattice, the four velocities $\vec{e}_i$ and their corresponding weights w_i are calculated from [6]:

$$\begin{cases} \vec{e}_1 = (1,0).c \\ \vec{e}_2 = (-1,0).c \\ \vec{e}_3 = (0,1).c \\ \vec{e}_4 = (0,-1).c \end{cases} \qquad and \quad w_1 = w_2 = w_3 = w_4 = 0.25 \tag{6}$$

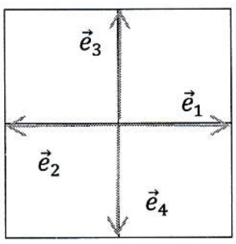

Fig. 2 D2Q4 model for thermal problem

It is to be noted that in the above equation, $c = \Delta x/\Delta t = \Delta y/\Delta t$ and the weights satisfy the relation $\sum_{i=1}^{m} w_i = 1$

After discretization in 2D, and considering heat generation, Eq. (3) can be written as:

$$f_i(x+\Delta x, y+\Delta y, t+\Delta t) = f_i(x,y,t)[1-\omega] + \omega f_i^{eq}(x,y,t) + w_i.\Delta t.\mathrm{S} \qquad (7)$$

Where $\mathrm{S} = \frac{q}{\rho c}$ is the force or source term, w_i is the weight in corresponding direction, and $\omega = \frac{\Delta t}{\tau}$ is non-dimensional relaxation time. Notice that the unit of S is the unit of temperature (°C) per unit of time.

This is the LB equation with BGK approximation that describes the evolution of the particle distribution function f_i. The solution of the above equation by LBM consists of two steps, collision and streaming [6]:

The collision step is:

$$f_i(x,y,t+\Delta t) = f_i(x,y,t)[1-\omega] + \omega f_i^{eq}(x,y,t) + w_i.\Delta t.\mathrm{S}$$

The streaming step is:

$$f_i(x+\Delta x, y+\Delta y, t+\Delta t) = f_i(x,y,t+\Delta t)$$

In case of heat transfer problems, the temperature is obtained after summing f_i overall direction [7]:

$$T(x,y,t) = \sum_{i=1}^{4} f_i\,(x,y,t) \qquad (8)$$

To process Eq. (7), an equilibrium distribution function is required. For heat conduction problems, this is given by:

$$f_i^{eq}(x,y,t) = w_i.T(x,y,t) \qquad (9)$$

From Eqs. (8) to (9), we also have

$$\sum_{i=1}^{4} f_i^{eq}(x,y,t) = \sum_{i=1}^{4} w_i.T(x,y,t) = T(x,y,t) = \sum_{i=1}^{4} f_i\,(x,y,t) \qquad (10)$$

Eq. (10), with definitions of temperature $T(x,y,t)$ and equilibrium function $f_i^{eq}(x,y,t)$ given in Eqs. (8) and (9), respectively, provide solution of a transient heat conduction problem in the LBM.

BOUNDARY CONDITIONS FOR LATTICE BOLTZMANN METHOD

In the following subsection, different boundary conditions will be discussed in detail.

1. The value of the function (temperatue) is given at the boundary:

For example, temperature at the left, upper and right boundaries are given, T=C1= 100°C at $x = 0$, and T=C2= 0°C at $y = h$ and $x = l$, (C1 and C2 are constants), so from Eq. (8):

$$\begin{cases} f_1(0,y,t) + f_2(0,y,t) + f_3(0,y,t) + f_4(0,y,t) = T = C1 = 100°\mathrm{C} \quad at\ x = 0 \\ f_1(x,h,t) + f_2(x,h,t) + f_3(x,h,t) + f_4(x,h,t) = T = C2 = 0 \quad at\ y = h \\ f_1(l,y,t) + f_2(l,y,t) + f_3(l,y,t) + f_4(l,y,t) = T = C2 = 0 \quad at\ x = l \end{cases} \qquad (11)$$

Then, the unknowns are $f_1(0,y,t)$ and $f_3(0,y,t)$ for left boundary, $f_1(x,h,t)$ and $f_4(x,h,t)$ for the upper boundary, and $f_2(l,y,t)$ and $f_3(l,y,t)$ for the right boundary, because the other distribution functions can be obtained from steaming step.

$f_1(0,y,t)$ and $f_3(0,y,t)$ can be calculated by using flux conservation equation [6]:

$$f_1^{eq}(0,y,t) - f_1(0,y,t) + f_2^{eq}(0,y,t) - f_2(0,y,t) = 0 \tag{12}$$

Since $w_i = 0.25$ for all steaming direction and $f_1^{eq}(0,y,t) = f_2^{eq}(0,y,t) = 0.$ 25T=0.25C1 (form Eq. (9)), then $f_1(0,y,t) = 0.25C1 + 0.25C1 - f_2(0,y,t)$, so:

$$f_1(0,y,t) = 0.5C1 - f_2(0,y,t) = 50 - f_2(0,y,t) \tag{13}$$

Similarly we can obtain:

$$f_4(x,h,t) = 50 - f_3(0,y,t) \tag{14}$$

Hence two unknowns are specified at the left boundary, similar method can be applied for other boundaries. The conditions for other boundaries will be given in the following equations without derivations, following the same procedure as before.

- **upper boundary**

$$\begin{cases} f_1(x,h,t) = 0.5C2 - f_2(x,h,t) \\ f_4(x,h,t) = 0.5C2 - f_3(x,h,t) \end{cases} \tag{15}$$

- **right boundary**

$$\begin{cases} f_2(l,y,t) = 0.5C2 - f_1(l,y,t) \\ f_3(l,y,t) = 0.5C2 - f_4(l,y,t) \end{cases} \tag{16}$$

2. **adiabatic boundary condition:** $\frac{\partial \mathrm{T}}{\partial \mathrm{y}} = 0$:

By using finite difference approach, $\frac{\partial \mathrm{T}}{\partial \mathrm{y}} = 0$, can be approximated as

$$\frac{T(x,y+1,t) - T(x,y,t)}{\Delta y} = 0 \tag{17}$$

It can simplified as $T(x,y+1,t) = T(x,y,t)$, hence for bottom boundary $y = 0$:

$$f_1(x,1,t) + f_2(x,1,t) + f_3(x,1,t) + f_4(x,1,t)$$
$$= f_1(x,0,t) + f_2(x,0,t) + f_3(x,0,t) + f_4(x,0,t) \tag{18}$$

It is rational to assume that $f_1(x,1,t) = f_1(x,0,t)$, $f_2(x,1,t) = f_2(x,0,t)$ and so on.

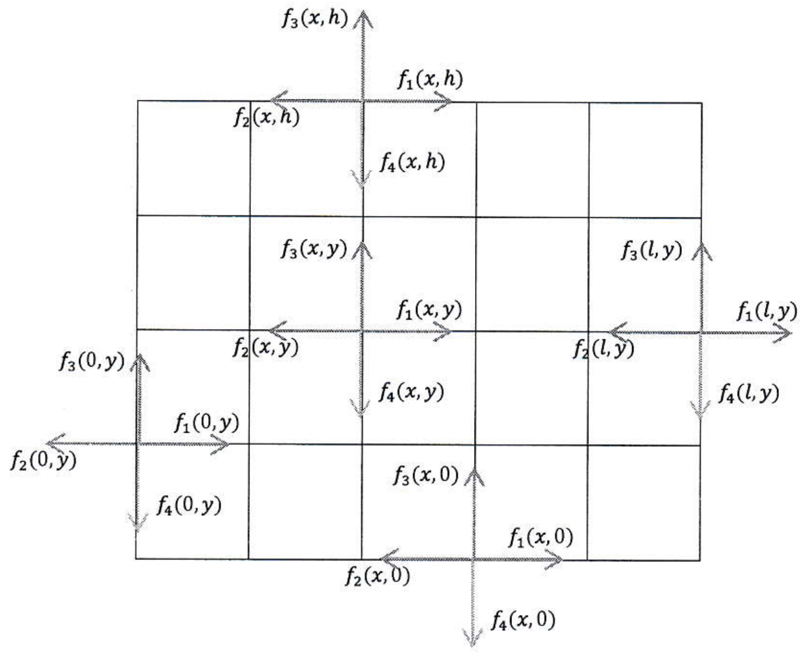

Fig. 3 Lattices for 2-D diffusion problem with distribution function at the boundary

RESULTS OF COMPUTATIONS

A two dimensional square slab shown in fig.1 subjected to a different boundary conditions. The length of the domain is 100 units. the temperature distribution in the slab obtained at time = 400 units (s), by using LB and FD methods. Thermal diffusivity is 0.25. Note that for stability conditions the time step for FDM is 0.2. There is no such a problem in using a time step of 1.0 with the LBM method. Hence, LBM is much faster and efficient than FDM.

Figure 2 illustrate the temperature distributions obtained by LBM algorithm (solid line) and by FDM algorithm (symbols) along the middle line (y=0.5 H), where H is the high of the square.

Figure 3 shows the isotherm plot at time =400 units (s)

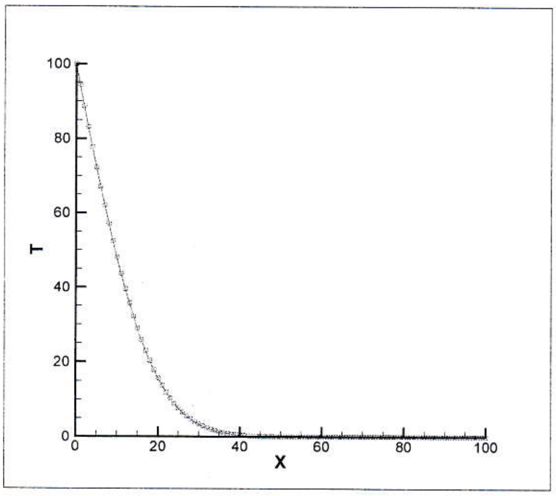

Figure 4 LBM algorithm (solid line) and by FDM algorithm (symbols)

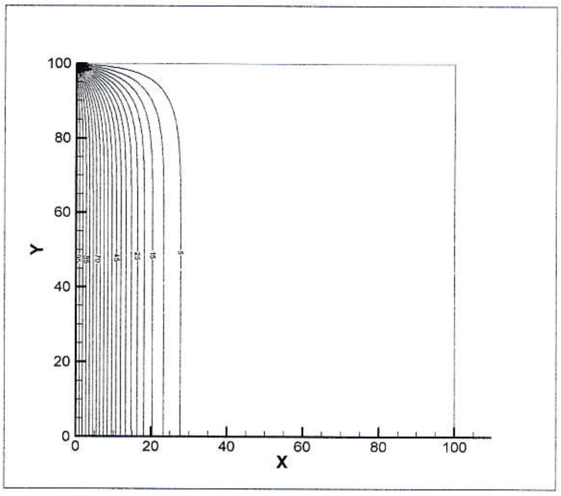

Figure 5 isotherm plot at time = 400 units by LBM algorithm

CONCLUSION

The lattice Boltzmann method for the 2D heat diffusion equation supplemented by different boundary conditions and initial condition has been presented. The exemplary tasks have been solved both by the lattice Boltzmann method and by the explicit scheme of the finite different method. The good agreement of the solutions obtained has been observed.

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