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On Variational Regularization Methods for Some Inverse Problems

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إهداء

وما توفيقي وسدادي واعتمادِي إلّا بالله العلي العظيم، عليه توكلت وإليه أنيب. فله الحمدُ والثناء الحَسَن حمداً كثيراً طيّباً مباركاً فيه كما ينبغي لجلال وجهه وعظيم سلطانه، وله كل الفضل والشكر والمِنّة والنعمة في الأولى والآخرة، إذ أعان ويسّر ووفّق لتمام هذا العمل وبلوغ هذا المقصد.

ومِن باب الاعتراف بالفضل لمن جعله الله سبباً في التيسير، أُهدي ثمرة هذا الجهد المتواضع، ومسيرة سنوات حافلة بالجد والاجتهاد والصبر، إلى كل القلوب المعطاءة التي كانت لي سنداً وعوناً طوال رحلتي العلمية.

إلى الذين غمروني بالدعم الصادق من قريب أو بعيد، وآمنوا بقدراتي، وعبّدوا لي بكلماتهم الطيّبة ودعواتهم الخفية دروب السعي والنجاح.

إلى كل من ترك أثراً جميلاً أو علماً نافعاً ساعدني على تجاوز عقبات هذا الدرب…

لكم جميعاً، بعد شكر الله والحمد له، عظيم الثناء وخالص الامتنان.

الطالبة: مسعودة حذيق

إهداء

الحمد لله الذي بنعمته وجلاله تتم الصالحات، وبفضله وتوفيقه تتنزل البركات. أحمده سبحانه حمداً يليق بعظيم سلطانه، حمد المُقِرّ بعجزه عن شكر فضله، إذ وهبني الصحة وجعل لي من لدنه توفيقاً لإتمام هذا العمل، فلك الحمد يا ربّ حتى ترضى، وإذا رضيت، وبعد الرضا.

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لكم جميعاً، بعد حمد الله وتوفيقه، خالص العرفان و عظيم الثناء والتقدير

الطالبة: أنفال تومي

ملخص

تتمحور هذه الدراسة حول المسائل العكسية سيئة الصياغة بهدف استعادة استقرار الحلول. يركز العمل على نموذجي تسوية تيخونوف و التغير الكلي ؛ حيث شمل الشق النظري إثبات وجود ووحدانية الحلول في فضاءات باناخ والمنعكسة بالتحليل الدالي. وتطبيقياً، تم اختبار فاعليتهما في تحسين الصور الرقمية وإزالة التشويهات، مع تقديم دراسة تقييمية تبرز فوارقهما الجوهرية وقدرتهما على الحفاظ على معالم الصورة.

الكلمات المفتاحية: مسائل عكسية، تسوية تباينية، تيخونوف، التغير الكلي ، معالجة الصور.

Abstract

This study centers on addressing ill-posed inverse problems to restore solution stability. It focuses on Tikhonov and Total Variation (TV) regularization models. Analytically, existence and uniqueness of solutions are proved in Banach and reflexive spaces using functional analysis. Practically, both models are tested in digital image restoration, with an evaluative study highlighting their differences and ability to preserve image features.

Keywords: Inverse problems, Variational regularization, Tikhonov, Total Variation (TV), Image processing.

Résumé

Cette étude s'articule autour des problèmes inverses mal posés pour restaurer la stabilité des solutions. Elle se concentre sur les modèles de régularisation de Tikhonov et de la Variation Totale (TV). Analytiquement, l'existence et l'unicité des solutions sont prouvées dans les espaces de Banach et réflexifs via l'analyse fonctionnelle. Pratiquement, les deux modèles sont testés en restauration d'images, avec une étude évaluative mettant en évidence leurs différences et leur capacité à préserver les détails.

Mots-clés: Problèmes inverses, Régularisation variationnelle, Tikhonov, Variation Totale (TV), Traitement d'images.

Contents

General Introduction

At the heart of mathematical modeling for real-world phenomena lies a profound duality between direct and inverse problems. While direct problems proceed from known causes to predict observable effects, inverse problems operate in the reverse direction, seeking to reconstruct hidden causes or internal parameters of a system from experimental observations [19]. However, inverse problems are frequently characterized by "ill-posedness" in the sense of Hadamard, meaning that the solution may not exist, may not be unique, or may be highly sensitive to data perturbations [11]. This inherent instability implies that even marginal noise or infinitesimal errors in the measured data can lead to physically meaningless results in the reconstructed solution.

To provide a rigorous mathematical treatment for such unstable problems, variational regularization methods have emerged as one of the most robust frameworks in contemporary applied analysis [1]. This approach transforms an ill-posed problem into a well-defined, stable optimization task by minimizing an energy functional. Typically, this functional consists of two essential components: a data fidelity term, which ensures consistency with the observed measurements, and a regularization term, which incorporates prior structural knowledge to stabilize the solution against noise [1]. Given the complexity of these mathematical models, particularly when dealing with non-Hilbertian structures, the underlying theoretical analysis must be conducted within the rigorous setting of Banach and reflexive spaces . By leveraging advanced tools from functional analysis, such as sequential compactness and weak topologies, it becomes possible to establish the existence and uniqueness of regularized solutions .

The power of this mathematical framework is most effectively demonstrated in the field of digital image processing, specifically in image denoising and restoration [9, 2]. In practice, digital images are frequently corrupted by random noise during acquisition or transmission, leading to a loss of critical geometric information and edge definition . By employing variational regularization models—most notably Tikhonov regularization and Total Variation (TV) methods we can formulate mathematical algorithms capable of distinguishing the essential features of the image from unwanted noise [17]. Consequently, addressing these real-world distortions necessitates a rigorous mathematical bridge between abstract functional analysis and numerical implementation to ensure the reliability of the restored results .

The primary objective of this master's thesis is to provide a concise analytical and numerical study of variational regularization methods, balancing theoretical foundations with practical image restoration. To achieve these goals, this work is coherently structured into three interconnected chapters. Chapter 1 establishes the necessary mathematical preliminaries, introducing Banach spaces, dual spaces, linear operators, weak topologies, and lower semi-continuity, which serve as foundational tools for proving the existence of minimizers. Chapter 2 dives into the core theoretical framework of inverse problems and variational methods, presenting a detailed comparative analysis of Tikhonov regularization and Total Variation (TV) models. Finally, Chapter 3 transitions to the practical application, illustrating the implementation and discretization of these regularization methods in image denoising, followed by a comparative numerical discussion highlighting their respective efficiency and edge-preservation capabilities.

Notations

Symbol	**Description**
u	The original or restored image (ideal solution)
f	The noisy or degraded observed image
A	Linear bounded operator modeling the degradation process
η	Noise vector or perturbation
Ω	Bounded domain of the image ($\Omega \subset \mathbb{R}^2$)
$L^2(\Omega)$	Space of square-integrable functions
$H^1(\Omega)$	Sobolev space of order 1
$BV(\Omega)$	Space of functions of Bounded Variation
∇u	Gradient of the image function u
$\lvert\nabla u\rvert$	Magnitude of the image gradient
$\lVert \cdot \rVert_{L^2}$	L^2-norm representing the data fidelity term
$\mathcal{R}(u)$	Regularization functional
λ	Regularization parameter ($\lambda > 0$)
α	Tuning parameters or secondary scaling factors
$J_{TV}(u)$	Total Variation regularization functional
div	Divergence operator
δ	Noise level bound such that $\lVert Af - u^\delta \rVert \leq \delta$

Chapter 1

Mathematical Preliminaries

1.1 Introduction to Banach Spaces

In this part, we introduce the concept of a norm as a primary tool for measuring distances within a vector space. Throughout our study, we shall restrict our attention to real vector spaces..

Definition 1.1.1 (Normed Vector Space[4])
Let V denote a vector space over $\mathbb{R}$. A function $\|\cdot\| : V \to \mathbb{R}$ is called a ***norm*** *on V if it satisfies the following conditions for any elements $v, w \in V$ and every scalar $\lambda \in \mathbb{R}$:*

1. ***Positivity:*** $\|v\| \geq 0$, *and* $\|v\| = 0$ *if and only if* $v = 0$.
2. ***Homogeneity:*** $\|\lambda v\| = |\lambda| \|v\|$.
3. ***Triangle Inequality:*** $\|v + w\| \leq \|v\| + \|w\|$.

The pair $(V, \|\cdot\|)$ is called a ***normed vector space*** *and we say that $\|x\|$ is the norm of x.*

The presence of a norm allows us to define the proximity between vectors and study the behavior of sequences.

Definition 1.1.2 (Strong Convergence[4])
A sequence $(v_n)_{n\in\mathbb{N}}$ in a normed vector space V is said to ***converge strongly*** *to an element $v \in V$ if the sequence of its norms tends to zero, i.e.,*

$$\lim_{n\to\infty} \|v_n - v\| = 0$$

In this case, v is called the limit of the sequence, and we denote this by $v_n \to v$ as $n \to \infty$. It is a fundamental property that if such a limit exists, it must be unique.

Proposition 1.1
[4]Let (v_n) and (w_n) be two convergent sequences in V with limits v and w respectively. For any scalar $\lambda \in \mathbb{R}$, the following algebraic properties hold:

1. $\lim_{n\to\infty}(v_n + w_n) = v + w$.
2. $\lim_{n\to\infty}(\lambda v_n) = \lambda v$.

Definition 1.1.3 (Continuity[4])
Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces. A mapping $f : V \to W$ is said to be ***continuous*** *at a point $a \in V$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that:*

$$\|x - a\|_V < \delta \implies \|f(x) - f(a)\|_W < \epsilon$$

If f is continuous at every point in V, we say that f is a continuous mapping.

Definition 1.1.4 (Cauchy Sequence[4])
A sequence (v_k) in a normed vector space V is called a ***Cauchy sequence*** *if for every $\epsilon > 0$, there exists an index $N(\epsilon) \in \mathbb{N}$ such that*

$$\|v_m - v_n\| < \epsilon$$

whenever $m, n \geq N(\epsilon)$.

Definition 1.1.5 (Banach Space[2])
A normed vector space $(V, \|\cdot\|)$ is called a ***Banach space*** *if it is a complete metric space with respect to the metric $d(v, w) = \|v - w\|$. In other words, a normed space V is Banach if every Cauchy sequence $\{v_n\}_{n=1}^{\infty}$ in V has a limit v that also belongs to V.*

Theorem 1.2 ([4])
The vector space $\mathbb{R}^n$ is complete under the infinity norm $\|\cdot\|_\infty$.

Proof. Consider a Cauchy sequence $(v_k)_{k\in\mathbb{N}}$ in $(\mathbb{R}^n, \|\cdot\|_\infty)$. Each element of this sequence can be expressed by its coordinates as $v_k = (v_k^{(1)}, v_k^{(2)}, \ldots, v_k^{(n)})$. For every fixed index $i \in \{1, \ldots, n\}$, the scalar sequence of coordinates $(v_k^{(i)})$ forms a Cauchy sequence in $\mathbb{R}$. This is guaranteed by the following inequality:

$$|v_k^{(i)} - v_j^{(i)}| \leq \max_{1\leq l\leq n} |v_k^{(l)} - v_j^{(l)}| = \|v_k - v_j\|_\infty$$

Since the field of real numbers $\mathbb{R}$ is complete, each coordinate sequence converges to a limit $v^{(i)} \in \mathbb{R}$. By defining the vector $v = (v^{(1)}, v^{(2)}, \ldots, v^{(n)})$, it follows naturally that v_k converges to v in the sense of the infinity norm. Consequently, the space is a Banach space. □

Corollary 1.2.1. *[4] Every normed vector space with finite dimension is a Banach space. Specifically, for any $p \in [1, \infty)$, the spaces $(\mathbb{R}^n, \|\cdot\|_p)$ are complete.*

Proof. Let V be an n-dimensional vector space with a basis $\{e_1, \ldots, e_n\}$. We define a linear isomorphism $L : V \to \mathbb{R}^n$ that maps each vector to its coordinate representation. We can induce a specific norm on V by setting $\|x\|_0 = \|L(x)\|_\infty$. Given that $(\mathbb{R}^n, \|\cdot\|_\infty)$ is a Banach space, V is also complete under the induced norm $\|\cdot\|_0$. Due to the fundamental property that all norms on a finite-dimensional space are equivalent, the completeness of V holds for any other norm defined on it. □

Example 1.1.1 (Lebesgue Spaces)**.** *A prominent class of infinite-dimensional Banach spaces is the family of Lebesgue spaces, denoted by $L^p[a, b]$. For a fixed real number $p \geq 1$, the space $L^p[a, b]$ consists of all measurable functions $u : [a, b] \to \mathbb{R}$ such that the following Lebesgue integral is finite:*

$$\int_a^b |u(t)|^p \, dt < \infty$$

This space is equipped with the L^p-norm, which is defined for any $u \in L^p[a, b]$ as:

$$\|u\|_p = \left(\int_a^b |u(t)|^p \, dt \right)^{1/p}$$

It is a well-established result in analysis that $(L^p[a, b], \|\cdot\|_p)$ is a complete normed space, and thus it constitutes a ***Banach space****.*

In the limiting case where $p = \infty$, the space $L^\infty[a, b]$ consists of functions that are essentially bounded. The corresponding norm is given by the essential supremum:

$$\|u\|_\infty = ess\ sup_{t \in [a,b]} |u(t)|$$

The space $(L^\infty[a, b], \|\cdot\|_\infty)$ is also a Banach space.

1.2 The Dual Space of a Normed Space

Definition 1.2.1 (Topological Dual Space[7])
Let E be a normed vector space over $\mathbb{K}$. The topological dual space E^* is the space of all continuous linear forms on E, denoted by $E^* = \mathcal{L}(E;\mathbb{K})$. It is equipped with the operator norm:

$$\|f\|_{E^*} = \sup_{\|x\|_E=1} |f(x)|$$

Since $\mathbb{K}$ is complete, E^* is always a Banach space.

Proposition 1.3 (Dual of $\mathbb{K}^n$ with p-norms[7])
Let $1 \leq p \leq \infty$ and q be its conjugate exponent ($1/p + 1/q = 1$). The dual space of $(\mathbb{K}^n, \|\cdot\|_p)$ is isometrically isomorphic to $(\mathbb{K}^n, \|\cdot\|_q)$.

Proof. We provide the proof for the general case $1 < p < \infty$ and the limit cases:

1. Representing the Linear Form: Let $f \in (\mathbb{K}^n)^*$. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{K}^n$. For any $y = (y_1, \ldots, y_n) \in \mathbb{K}^n$, we have $y = \sum y_j e_j$. By linearity:

$$f(y) = f\left(\sum_{j=1}^{n} y_j e_j\right) = \sum_{j=1}^{n} y_j f(e_j)$$

Setting $x_j = f(e_j)$, we define the vector $x = (x_1, \ldots, x_n)$. Thus, $f(y) = T_x(y) = \sum x_j y_j$.

2. Case $p = 1$ (Dual is l^∞): Using the triangle inequality:

$$|T_x(y)| \leq \sum_{j=1}^{n} |x_j||y_j| \leq \max_j |x_j| \sum_{j=1}^{n} |y_j| = \|x\|_\infty \|y\|_1$$

This implies $\|T_x\| \leq \|x\|_\infty$. By choosing $y = e_{j_0}$ where $|x_{j_0}| = \|x\|_\infty$, we obtain $T_x(y) = x_{j_0}$, hence $\|T_x\| = \|x\|_\infty$.

3. Case $1 < p < \infty$ (Hölder's Inequality): By Hölder's Inequality, we have $|T_x(y)| \leq \|x\|_q \|y\|_p$, which implies $\|T_x\| \leq \|x\|_q$. To show equality, we choose $y_j = e^{-i\theta_j}\left(\frac{|x_j|}{\|x\|_q}\right)^{q/p}$ where $x_j = |x_j| e^{i\theta_j}$. A direct calculation shows that $\|y\|_p = 1$ and $T_x(y) = \|x\|_q$, thus proving the isometry $\|T_x\| = \|x\|_q$. □

Remark. *If F is a dense subspace of E, every $f \in F^*$ has a unique extension to E^* with the same norm. Thus, E^* and F^* are identified isometrically.*

1.3 The Bidual Space and Reflexive Spaces

In this section, we explore the relationship between a normed space and its second dual, leading to the fundamental concept of reflexivity.

Definition 1.3.1 (Topological Bidual)
Let $(E, \|\cdot\|)$ be a normed vector space over a field $\mathbb{K}$. The topological dual of the dual space E^*, denoted by $E^{**} = (E^*)^*$, is called the ***topological bidual*** *of E.*

Proposition 1.4
For any normed space $(E, \|\cdot\|)$, the following properties hold:

1. *For each fixed $x \in E$, the* ***evaluation mapping*** *$J(x) : E^* \to \mathbb{K}$ defined by:*

$$J(x)(f) = f(x), \quad \forall f \in E^*$$

   is a continuous linear functional. Hence, $J(x)$ is an element of the bidual space E^{**}.

2. *The mapping $J : E \to E^{**}$ given by $x \mapsto J(x)$ is a linear isometry. This map is referred to as the* ***canonical mapping*** *(or natural embedding) of E into its bidual. Consequently, E can be isometrically identified as a subspace of E^{**}.*

Proof. 1. Linearity of $J(x)$ is inherited from the vector space structure of E^*. For any $f \in E^*$, we observe:

$$|J(x)(f)| = |f(x)| \leq \|f\|_{E^*}\|x\|_E$$

This inequality implies that $J(x)$ is a bounded (continuous) linear functional on E^*, and moreover, $\|J(x)\|_{E^{**}} \leq \|x\|_E$.

2. To establish the linearity of J, let $x, y \in E$ and $\alpha \in \mathbb{K}$. For any $f \in E^*$:

$$J(x + \alpha y)(f) = f(x + \alpha y) = f(x) + \alpha f(y) = J(x)(f) + \alpha J(y)(f)$$

Thus, $J(x + \alpha y) = J(x) + \alpha J(y)$.

To prove that J is an isometry, let $x \neq 0$. By the Hahn-Banach theorem (specifically the corollary on norm preservation), there exists a functional $f \in E^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Therefore:

$$\|x\| = |f(x)| = |J(x)(f)| \leq \|J(x)\|_{E^{**}}\|f\|_{E^*} = \|J(x)\|_{E^{**}}$$

Combining this with $\|J(x)\|_{E^{**}} \leq \|x\|$, we obtain $\|J(x)\|_{E^{**}} = \|x\|$, which confirms that J is an isometry. □

Definition 1.3.2 (Reflexive Space)
A normed vector space E is said to be ***reflexive*** *if the canonical mapping $J : E \to E^{**}$ is surjective (and thus an isometric isomorphism). In this case, we write $E \cong E^{**}$.*

Remark. *Every reflexive space is necessarily a Banach space, as it is isometrically isomorphic to the bidual E^{**}, which is always complete.*

Corollary 1.4.1. *Let $(E, \|\cdot\|)$ be a normed space. Then for every $x \in E$, we have:*

$$\|x\| = \sup_{f\in E^*, \|f\|<1} |f(x)| = \sup_{f\in E^*, \|f\|\leq 1} |f(x)| = \sup_{f\in E^*, \|f\|=1} |f(x)| = \sup_{f\in E^*, f\neq 0} \frac{|f(x)|}{\|f\|}$$

Furthermore, the supremum is attained, meaning there exists $f \in E^*$ with $\|f\| = 1$ such that $\|x\| = |f(x)|$.

Note 1.5. *If E is a normed space such that the canonical mapping $J : E \to E^{**}$ is bijective, then E is necessarily a Banach space. This follows from the fact that E^{**} is a Banach space and J is an isometric isomorphism.*

Example 1.3.1.

1. *Every finite-dimensional normed space is reflexive.*
2. *For $1 < p < \infty$, the sequence space ℓ^p is reflexive.*

Proposition 1.6
Let $(E, \|\cdot\|)$ be a reflexive Banach space. For every $f \in E^*$, there exists an element $x \in E$ such that $\|x\| = 1$ and $\|f\| = |f(x)|$.

Proof. Let $f \in E^*$, and assume $f \neq 0$. By applying the Hahn-Banach theorem (Corollary 7.7.1) to f and E^*, there exists an element $\Gamma \in E^{**}$ such that $\|\Gamma\| = 1$ and $\Gamma(f) = \|f\|$. Since E is reflexive, there exists $x \in E$ such that $\Gamma = J(x)$. It follows that $\|x\| = \|\Gamma\| = 1$ and $\|f\| = J(x)(f) = f(x) = |f(x)|$. □

Note 1.7. *The converse of the above proposition is also true (James' Theorem). Specifically, if $(E, \|\cdot\|)$ is a Banach space such that for every $f \in E^*$ there exists $x \in E$ with $\|x\| = 1$ and $\|f\| = |f(x)|$, then E is reflexive.*

Proposition 1.8
Let E be a Banach space and F be a closed subspace of E. Then:

1. *If E is reflexive, then F is also reflexive.*
2. *E is reflexive if and only if its dual space E^* is reflexive.*

Remark. *Since the topological dual of $(c_0, \|\cdot\|_\infty)$ is $(\ell^1, \|\cdot\|_1)$ and the topological dual of $(\ell^1, \|\cdot\|_1)$ is $(\ell^\infty, \|\cdot\|_\infty)$, it follows from the previous proposition that the spaces c_0, ℓ^1, and ℓ^∞ are not reflexive.*

Corollary 1.8.1. *Let $(E, \|\cdot\|)$ be a reflexive Banach space. Then E is separable if and only if its dual space E^* is separable.*

1.4 Weak and Weak* Topologies

1.4.1 Weak Topology

Definition 1.4.1 (General Weak Topology[16])
Let X be a non-empty set and $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Lambda}$ be a family of topological spaces. The weak topology induced by a collection of mappings $\mathcal{F} = \{f_\alpha : X \to X_\alpha\}_{\alpha\in\Lambda}$ is the coarsest topology on X for which every mapping f_α is continuous. This topology is denoted by $\sigma(X, \mathcal{F})$.

Definition 1.4.2 (Weak Topology in Functional Analysis)
Let V be a normed linear space and V^* denote its continuous dual space. The weak topology on V, denoted by $\sigma(V, V^*)$, is the specific weak topology generated by the family of all continuous linear functionals belonging to V^*.

Weak convergence is defined in terms of bounded linear function als on X as follows[12].

Definition 1.4.3 (Weak Convergence)
A sequence (x_n) in a normed space X is said to be weakly convergent to $x \in X$ if for every continuous linear functional $f \in X^*$, we have:

$$\lim_{n\to\infty} f(x_n) = f(x)$$

In this case, we write $x_n \xrightarrow{w} x$, and the element x is called the weak limit of the sequence (x_n).

Theorem 1.9 ([16])
Let X be a normed linear space and $(x_n) \subset X$. Then $x_n \xrightarrow{w} x$ if and only if $f(x_n) \to f(x)$ for every linear functional $f \in X^*$.

Proof. ($\Rightarrow$) Suppose $x_n \xrightarrow{w} x$. By definition of the weak topology, every $f \in X^*$ is continuous. Thus, for any $\epsilon > 0$, the set $U = \{y \in X : |f(y) - f(x)| < \epsilon\}$ is a weakly open neighborhood of x. Since x_n converges weakly to x, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$, which implies $f(x_n) \to f(x)$.

($\Leftarrow$) Conversely, assume $f(x_n) \to f(x)$ for all $f \in X^*$. Let U be any weakly open set containing x. By the definition of the subbase for the weak topology, there exist

$f_1, \ldots, f_m \in X^*$ and $\epsilon > 0$ such that:

$$V = \{y \in X : |f_i(x) - f_i(y)| < \epsilon, \text{ for } i = 1, \ldots, m\} \subset U$$

Since $f_i(x_n) \to f_i(x)$ for each i, we can find n_i such that x_n satisfies the condition for each f_i when $n \geq n_i$. Taking $n_0 = \max\{n_1, \ldots, n_m\}$, we have $x_n \in V \subset U$ for all $n \geq n_0$. Hence, $x_n \xrightarrow{w} x$. □

Proposition 1.10
Let $X \neq \{0\}$ be a normed linear space. For any $x \in X$, there exists a continuous linear functional $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Consequently, if X is non-trivial, then its dual space X^* is also non-trivial.

Theorem 1.16 (Banach-Steinhaus[16])
Let X be a Banach space and Y a normed vector space over $\mathbb{K}$. Let (T_n) be a sequence in $\mathcal{B}(X, Y)$. If the sequence $(T_n x)$ converges for every $x \in X$, then the pointwise limit T defined by $Tx = \lim_{n\to\infty} T_n x$ is a bounded linear operator. Furthermore, the norm satisfies $\|T\| \leq \liminf_{n\to\infty} \|T_n\|$.

Proposition 1.17
In a Banach space X, if a sequence (x_n) converges weakly to x ($x_n \xrightarrow{w} x$), then the following properties hold:

(a) The sequence (x_n) is bounded in X.

(b) The norm of the limit satisfies $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.

Proof. Consider the evaluation mappings $e_{x_n} : X^* \to \mathbb{K}$ defined by $e_{x_n}(f) = f(x_n)$ for all $f \in X^*$. The assumption $x_n \xrightarrow{w} x$ implies that for each $f \in X^*$, the sequence $e_{x_n}(f) = f(x_n)$ converges to $f(x) = e_x(f)$.

Since X^* is a Banach space, we apply the **Banach-Steinhaus Theorem** to the sequence (e_{x_n}) in $\mathcal{B}(X^*, \mathbb{K})$. It follows that:

- The sequence of operators (e_{x_n}) is bounded, and since $\|e_{x_n}\| = \|x_n\|$, the sequence (x_n) is bounded.
- The limit operator e_x is bounded and satisfies $\|e_x\| \leq \liminf_{n\to\infty} \|e_{x_n}\|$.

As the natural embedding is isometric ($\|e_y\| = \|y\|$ for any $y \in X$), we conclude that $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$. □

1.4.2 Weak* Topology

Since the dual space X^* is itself a normed linear space, it naturally possesses a weak topology $\sigma(X^*, X^{**})$. However, our focus lies on a coarser topology denoted by $\sigma(X^*, X)$, which is generated by the elements of X through the canonical embedding $J : X \to X^{**}$. By defining the evaluation maps $j_x(f) = f(x)$ for each $x \in X$, we can treat X as a closed subspace of its bidual X^{**}.

Definition 1.4.4 (Weak* Topology)
Let X be a normed linear space. The weak-star topology (or w^*-topology) on the dual space X^* is the topology $\sigma(X^*, X)$ generated by the family of all linear functionals $\{j_x : x \in X\}$.

Remark. *It is important to note that the w^*-topology on X^* is coarser than the weak topology $\sigma(X^*, X^{**})$, given that $X \subseteq X^{**}$. In the specific case where X is reflexive, the w^*-topology and the w-topology on X^* coincide.*

Definition 1.4.5 (Weak* Convergence)
A sequence (f_n) in X^* is said to converge to $f \in X^*$ in the weak-star sense if it converges with respect to the $\sigma(X^*, X)$ topology. This convergence is denoted by $f_n \xrightarrow{w^*} f$.

Proposition 1.18
Let X be a normed linear space and $(f_n) \subset X^*$. Then $f_n \xrightarrow{w^*} f$ if and only if $f_n(x) \to f(x)$ for every $x \in X$.

Proof. ($\Leftarrow$) Assume that $f_n(x) \to f(x)$ for all $x \in X$. Let $\mathcal{O}$ be an arbitrary w^*-open neighborhood of f. By definition, there exist $\epsilon > 0$ and a finite set of elements $x_1, x_2, \ldots, x_m \in X$ such that:

$$\{g \in X^* : |f(x_i) - g(x_i)| < \epsilon, \text{ for } i = 1, \ldots, m\} \subseteq \mathcal{O}.$$

Since $f_n(x_i) \to f(x_i)$ for each $i \in \{1, \ldots, m\}$, there exist $n_i \in \mathbb{N}$ such that $|f_n(x_i) - f(x_i)| < \epsilon$ for all $n \geq n_i$. By setting $N = \max\{n_1, \ldots, n_m\}$, we ensure that for all $n \geq N$:

$$|f_n(x_i) - f(x_i)| < \epsilon, \quad \forall i \in \{1, \ldots, m\}.$$

This implies $f_n \in \mathcal{O}$ for all $n \geq N$, hence $f_n \xrightarrow{w^*} f$.

($\Rightarrow$) Conversely, suppose $f_n \xrightarrow{w^*} f$. For any $x \in X$, consider the w^*-open set $\mathcal{O} = \{g \in X^* : |f(x) - g(x)| < \epsilon\}$. By the definition of weak-star convergence, there exists $n_0 \in \mathbb{N}$ such that $f_n \in \mathcal{O}$ for all $n \geq n_0$. This means $|f_n(x) - f(x)| < \epsilon$ for all $n \geq n_0$, which proves that $f_n(x) \to f(x)$. □

Theorem 1.19 (Banach-Alaouglu theorem [16, 20])
Let V be a normed linear space and V^* be its dual space. Let $B_{V^*} := \{f \in V^* : \|f\| \leq 1\}$ be the closed unit ball in V^*. Then B_{V^*} is w^*-compact in V^*.

Proof. Consider the product space $D = \prod_{u \in V} D_u$, where each D_u is a compact disk in $\mathbb{K}$ defined by $D_u = \{\lambda \in \mathbb{K} : |\lambda| \leq \|u\|\}$. By **Tychonoff's Theorem**, D is compact in the product topology.

We define the mapping $\phi : B_{V^*} \to D$ by:

$$\phi(f) = (f(u))_{u \in V}$$

This mapping ϕ is injective and continuous by the definition of the w^*-topology.

To complete the proof, we show that the image $\phi(B_{V^*})$ is a closed subset of D. Let $\xi = (\xi_u) \in D$ be an element in the closure of $\phi(B_{V^*})$. We define a functional $g : V \to \mathbb{K}$ such that $g(u) = \xi_u$.

For any $u, v \in V$ and $\lambda, \beta \in \mathbb{K}$, the linearity holds because:

$$g(\lambda u + \beta v) = \lambda g(u) + \beta g(v)$$

Furthermore, since $|\xi_u| \leq \|u\|$, it follows that $\|g\| \leq 1$, which means $g \in B_{V^*}$. Thus, $\phi(B_{V^*})$ is closed in D.

Since D is compact and Hausdorff, the closed subset $\phi(B_{V^*})$ is also compact. Therefore, B_{V^*} is w^*-compact. □

Definition 1.4.6 (lower semi-continuous[16])
Let (V, d) be a metric space and $f : V \to \mathbb{R}$ be a function. Then f is said to be lower semi-continuous (weakly lower semi-continuous) if $u_n \to u$ ($u_n \xrightarrow{w} u$) implies $f(u) \leq \liminf_n f(u_n)$.

Chapter 2

Variational Regularization for Inverse Problems

This chapter provides the mathematical foundation of inverse problems and the necessity of regularization techniques to obtain stable solutions. The fundamental concepts and the criteria for well-posedness, as defined by Hadamard [11], are introduced. The focus is then shifted towards variational methods, which provide a robust framework for image denoising tasks [1, 19].

2.1 Fundamentals of Inverse Problems

The study of inverse problems is a fundamental discipline often described as the problems of causation. In a direct problem, the goal is to predict the effects based on a known cause. Conversely, an inverse problem seeks to determine the unknown cause x from observed data y [11]. Mathematically, this is modeled by the operator equation:

$$Ax = y \tag{2.1}$$

where $A : X \longrightarrow Y$ is the forward operator acting between Banach spaces. In practice, measurements are often corrupted by noise ε, leading to the observed data $y^\delta = Ax + \varepsilon$, where the noise level satisfies $\|\varepsilon\| \leq \delta$ [1].

Well-posed vs ill-posed Problems

To analyze the solvability of the operator equation (2.1), we formally refer to the criteria of well-posedness introduced by Jacques Hadamard.

Definition 2.1.1 (Well-posedness [11])
The problem $Ax = y$ is said to be well-posed if it satisfies the following three conditions:

- ***Existence:*** *For every $y \in Y$, there exists at least one solution $x \in X$.*
- ***Uniqueness:*** *For every $y \in Y$, there is at most one solution $x \in X$.*
- ***Stability:*** *The solution x depends continuously on the data y. Formally, for every sequence $(y_n) \subset Y$ with $y_n \longrightarrow y$, it follows that $x_n \longrightarrow x$.*

If any of these conditions are violated, the problem is termed ill-posed [11].

In inverse problems, the failure of the stability condition is the most critical challenge. This instability implies that the inverse operator A^{-1} is not bounded, meaning that small perturbations in the data can lead to large errors in the solution.

2.2 Variational Regularization Methods

Variational regularization provides a stable analytical framework for solving ill-posed operator equations by transforming them into well-posed minimization problems. This is achieved by balancing a data fidelity term, which ensures consistency with the observed data, against a regularization functional that incorporates predefined regularity properties in the underlying Banach or Hilbert spaces.

The core idea of variational regularization methods is to approximate the solution of the ill-posed equation by a minimization problem. This approach transforms the inversion of the operator into an optimization framework, allowing for a stable approximation even in the presence of noise.

- **The Minimization Problem:** For a given regularization parameter $\alpha > 0$ and noisy data y^δ satisfying $\|y - y^\delta\|_Y \leq \delta$, we define the regularized solution $x_\alpha^\delta \in X$ as a minimizer of the functional $J_\alpha : X \longrightarrow [0, \infty]$ defined by

$$J_\alpha(x) = \mathcal{D}(Ax, y^\delta) + \alpha \mathcal{R}(x) \tag{2.2}$$

  where $\mathcal{D}(Ax, y^\delta)$ represents the data fidelity term, typically expressed as $\|Ax - y^\delta\|_Y^p$ with $p \geq 1$, measuring the discrepancy between the model and the observed data [18].

- **Regularization Functional:** $\mathcal{R}(x)$ is the regularization functional (or penalty term) that encodes prior knowledge about the solution, such as smoothness, sparsity, or edge-preservation properties [3].

- **Existence of the Minimizer:** In the general context of variational methods, the existence of a minimizer $x_\alpha^\delta \in \arg\min_{x\in X} J_\alpha(x)$ is guaranteed by the topological properties of the underlying spaces. According to the Direct Method of the Calculus of Variations, a minimizer exists if:

  1. **Weak (or Weak*) Sequential Lower Semi-Continuity:** The functional J_α is sequentially lower semi-continuous with respect to the weak (or weak*) topology. This ensures that the limit of any minimizing sequence attains the minimum value.

  2. **Coercivity and Compactness:** The regularization functional $\mathcal{R}$ is coercive, meaning its lower level sets are sequentially pre-compact. By the Banach-Alaoglu Theorem, this ensures the existence of a candidate for the minimizer within a reflexive Banach space X [3, 18], or more generally, within spaces endowed with a suitable weak* topology, such as the space of bounded variation BV which will be utilized in the ROF model.

- **Stability:** The variational formulation ensures stability in the sense that if $\{y_n\}$ is a sequence of data such that $y_n \longrightarrow y^\delta$, then the corresponding sequence of minimizers $\{x_n\}$ possesses a subsequence that converges weakly (or weakly* where applicable) to a minimizer of the original functional [3].

2.2.1 Tikhonov Regularization

Tikhonov regularization is the most widely utilized method for stabilizing inverse problems[19]. We consider the following minimization problem:

$$\inf_{u\in W^{1,2}(\Omega)} F(u) = \|y^\delta - Au\|_2^2 + \lambda\|\nabla u\|_2^2 \tag{2.3}$$

where the parameter $\lambda > 0$ serves to control the regularization process.

Derivation of the Euler-Lagrange Equation: To find the minimizer, we compute the Gâteaux derivative of the functional F. We consider the variation:

$$\begin{aligned}\frac{1}{\alpha}\left(F(u+\alpha v)-F(u)\right) &= \frac{1}{\alpha}\Big(\|y^\delta - A(u+\alpha v)\|_2^2 + \lambda\|\nabla(u+\alpha v)\|_2^2 \\ &\quad - \|y^\delta - Au\|_2^2 - \lambda\|\nabla u\|_2^2\Big) \\ &= \frac{1}{\alpha}\Big(\|y^\delta - Au - \alpha Av\|_2^2 + \lambda\|\nabla u + \alpha\nabla v\|_2^2 \\ &\quad - \|y^\delta - Au\|_2^2 - \lambda\|\nabla u\|_2^2\Big) \\ &= \frac{1}{\alpha}\Big(\|y^\delta - Au\|_2^2 + \alpha^2\|Av\|_2^2 - 2\langle y^\delta - Au, \alpha Av\rangle \\ &\quad + \lambda(\|\nabla u\|_2^2 + \alpha^2\|\nabla v\|_2^2 + 2\langle\nabla u, \alpha\nabla v\rangle) \\ &\quad - \|y^\delta - Au\|_2^2 - \lambda\|\nabla u\|_2^2\Big) \\ &= \frac{1}{\alpha}\Big(\langle\alpha Av, 2(Au - y^\delta) + \alpha Av\rangle + \lambda\langle\alpha\nabla v, 2\nabla u + \alpha\nabla v\rangle\Big) \\ &= \langle v, 2A^*(Au - y^\delta)\rangle + 2\lambda\langle\nabla v, \nabla u\rangle + O(\alpha) \\ &= 2\langle v, A^*Au - A^*y^\delta - \lambda\Delta u\rangle + O(\alpha)\end{aligned}$$

Thus, the first-order optimality condition leads to the following Euler-Lagrange equation:

$$F'(u) = 2(A^*Au - A^*y^\delta - \lambda\Delta u) = 0 \tag{2.4}$$

2.2.1.1 Convexity of the Tikhonov Functional

To investigate the uniqueness of the solution, we evaluate the second-order derivative of the functional F. We compute the variation of the first derivative F' as follows:

$$\begin{aligned}\frac{1}{\alpha}\left(F'(u+\alpha v)-F'(u)\right) &= \frac{2}{\alpha}\Big((A^*A(u+\alpha v) - A^*y^\delta - \lambda\Delta(u+\alpha v)) \\ &\quad - (A^*Au - A^*y^\delta - \lambda\Delta u)\Big) \\ &= \frac{2}{\alpha}\left(\alpha A^*Av - \alpha\lambda\Delta v\right) \\ &= 2(A^*A - \lambda\Delta)v\end{aligned}$$

Consequently, the second derivative of the functional is given by:

$$F''(u) = 2(A^*A - \lambda\Delta) \tag{2.5}$$

The operator $F''(u)$ is positive, as demonstrated by the following properties:

- $\langle A^*Aw, w\rangle = \|Aw\|_2^2 \geq 0$

- $\langle -\Delta w, w\rangle = \|\nabla w\|_2^2 \geq 0$

Based on these results, the functional F is convex. Furthermore, under the condition that $A \cdot 1 \neq 0$, we have $\langle F''(u)w, w\rangle > 0$ for any $w \neq 0$. This implies that F is **strictly convex**, which ensures the uniqueness of the optimal solution u.

2.2.1.2 Existence of the solution

To establish the existence of a minimizer for the Tikhonov functional $F(u)$, we consider a minimizing sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,2}(\Omega)$ such that:

$$\lim_{n\to\infty} F(u_n) = \inf_{u\in W^{1,2}(\Omega)} F(u) \tag{2.6}$$

From the definition of the functional, it follows that the sequence $F(u_n)$ is bounded. Consequently, there exists a constant $M > 0$ such that:

$$\int_\Omega |\nabla u_n|^2 \leq M \quad \text{and} \quad \int_\Omega |y^\delta - Au_n|^2 \leq M \tag{2.7}$$

Furthermore, by applying the triangle inequality, we observe that:

$$\|Au_n\|_2 = \|y^\delta - (y^\delta - Au_n)\|_2 \leq \|y^\delta\|_2 + \|y^\delta - Au_n\|_2 \leq M$$

In the case where the operator A is coercive, we directly obtain that $\|u_n\|_2 \leq M$. However, if F is not coercive, we define:

$$w_n = \frac{1}{|\Omega|}\int_\Omega u_n \quad \text{and} \quad v_n = u_n - w_n \tag{2.8}$$

We note that $\int_\Omega |\nabla v_n|^2 = \int_\Omega |\nabla u_n|^2$. According to the Poincaré-Wirtinger inequality, there exists a constant K such that:

$$\|v_n\|_2 = \|u_n - w_n\|_2 \leq K\|\nabla u_n\|_2 \leq C \tag{2.9}$$

By analyzing the data fidelity term and utilizing the triangle inequality, we find:

$$\|Aw_n\|_2 = \|(Aw_n + Av_n - y^\delta) - (Av_n - y^\delta)\|_2 \leq \|Au_n - y^\delta\|_2 + \|Av_n\|_2 + \|y^\delta\|_2 \leq C$$

Since w_n is a constant function, we have $Aw_n = w_n A(1)$. Given that $A(1) \neq 0$, it follows that:

$$\|Aw_n\|_2 = |w_n|\|A(1)\| \leq C \implies |w_n| \leq C \tag{2.10}$$

Combining the relations (2.9) and (2.10), we conclude that:

$$\|u_n\|_2 = \|v_n + w_n\|_2 \leq \|v_n\|_2 + \|w_n\|_2 \leq C \tag{2.11}$$

Therefore, the minimizing sequence (u_n) is bounded in the Sobolev space $W^{1,2}(\Omega)$. Since $W^{1,2}(\Omega)$ is a reflexive space, there exists a subsequence (still denoted as u_n) and an element $u^* \in W^{1,2}(\Omega)$ such that $u_n \rightharpoonup u^*$ weakly in $W^{1,2}(\Omega)$.

By the compact embedding of $W^{1,2}(\Omega)$ into $L^2(\Omega)$, we have $u_n \to u^*$ strongly in $L^2(\Omega)$. Given the continuity of A, we obtain $\|y^\delta - Au_n\|_2^2 \to \|y^\delta - Au^*\|_2^2$. Furthermore, due to the lower semi-continuity of the norm under weak convergence, we have:

$$\liminf_{n\to\infty} \|\nabla u_n\|_2^2 \geq \|\nabla u^*\|_2^2$$

This leads to:

$$F(u^*) \leq \liminf_{n\to\infty} F(u_n) \tag{2.12}$$

Which proves that u^* is a global minimizer of the functional F. □

2.2.1.3 Uniqueness of the solution

To establish the uniqueness of the minimizer, we assume that there exist two distinct minimizers, u and v, for the Tikhonov functional F in the Sobolev space $W^{1,2}(\Omega)$. Since the functional F is strictly convex, as demonstrated previously, the following inequality must hold:

$$F\left(\frac{1}{2}u + \frac{1}{2}v\right) < \frac{1}{2}F(u) + \frac{1}{2}F(v) = \inf_{w\in W^{1,2}(\Omega)} F(w) \tag{2.13}$$

This leads to a contradiction, as the functional value at the midpoint $\frac{u+v}{2}$ would be strictly less than the infimum of F. Therefore, the minimization problem admits a unique solution, which is characterized by the following Euler-Lagrange equation:

$$(A^*A - \lambda\Delta)u = A^*y^\delta \tag{2.14}$$

This equation is associated with the homogeneous Neumann boundary condition:

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega$$

While the solution to (2.14) provides a stable approximation, it is important to note that the Laplacian operator Δ possesses strong isotropic smoothing properties. As a result, this regularization approach tends to blur sharp edges and discontinuities within the image. This limitation necessitates the exploration of alternative regularization terms, such as the L^1 norm of the gradient, to better preserve structural details.

2.2.2 Total Variation Regularization (ROF Model)

In 1992, Rudin, Osher, and Fatemi [?]introduced a revolutionary regularization term defined as the Total Variation of the image, denoted by $J(u) = \int_\Omega |Du|$. This model is designed to overcome the blurring effects of Tikhonov regularization while effectively removing noise.For a comprehensive theoretical analysis and mathematical proofs related to this model.

We consider the following minimization problem in the space of functions of Bounded Variation $BV(\Omega)$:

$$\inf_{u \in BV(\Omega)} \left\{ \frac{\lambda}{2} \int_\Omega |y^\delta - Au|^2 dx + \int_\Omega |Du| \right\} \tag{2.15}$$

Fundamental Hypotheses: To ensure the mathematical validity of the model, we assume the following conditions:

- **(H1):** The operator $A : L^2(\Omega) \to L^2(\Omega)$ is a continuous linear operator such that $A \cdot 1 \neq 0$.
- **(H2):** The noise η is additive Gaussian noise with zero mean ($\int_\Omega \eta = 0$) and a known variance $\sigma^2 = \int_\Omega |\eta|^2$.

Remark on Noise Statistics: Assuming that the noise is independent of the clean signal Au, we have the following relation for the observed data $y^\delta = Au + \eta$:

$$\int_\Omega |y^\delta|^2 = \int_\Omega |Au + \eta|^2 = \int_\Omega |Au|^2 + \int_\Omega |\eta|^2 + 2 \int_\Omega Au \cdot \eta \tag{2.16}$$

Given the independence and zero mean of the noise, the cross-term vanishes ($\int_\Omega Au \cdot \eta = 0$), leading to:

$$\int_\Omega |y^\delta|^2 = \int_\Omega |Au|^2 + \sigma^2 \tag{2.17}$$

This implies that $\|y^\delta\|_{L^2}^2 \geq \sigma^2$. Furthermore, a similar argument demonstrates that for any constant $C \in \mathbb{R}$, we have $\|y^\delta - C\|_{L^2} \geq \sigma$. Therefore, we assume the additional constraint:

- **(H3):** $\|y^\delta - \int_\Omega y^\delta\|_{L^2(\Omega)} \geq \sigma$.

Existence and Uniqueness of the Solution

Theorem 2.1
Under the hypotheses (H1), (H2), and (H3), the minimization problem (2.15) admits a unique solution u^* in $BV(\Omega)$.

Proof. In the following proof, the constant C may change from one line to another. We denote by:

$$J(u) = \int_\Omega |Du|, \quad \text{and} \quad T(u) = J(u) + \frac{\lambda}{2}\|y^\delta - Au\|_{L^2}^2 \tag{2.18}$$

T is BV-coercive: Let $u \in BV(\Omega)$. We decompose u as $u = v + w$, where $w = \frac{1}{|\Omega|}\int_\Omega u$ and $v = u - w$.

1. $\|v\|_{L^p(\Omega)} = \|u - w\|_{L^p(\Omega)} \leq C_1 J(u) = C_1 J(v)$, where $C_1 > 0$ and $1 \leq p \leq 2$ (by Poincaré-Wirtinger inequality).
2. $\|u\|_{BV} = \|v + w\|_{L^1} + J(v + w) \leq \|v\|_{L^1} + \|w\|_{L^1} + J(v) \leq \|w\|_{L^1} + (C_1 + 1)J(v)$.
3. $\|y^\delta - Av\|_{L^2} \leq \|y^\delta\|_{L^2} + \|Av\|_{L^2} \leq \|y^\delta\|_{L^2} + \|A\| \cdot C_1 J(v)$.

   There exists $C_2 > 0$ such that $\|Aw\|_{L^2} = C_2\|w\|_{L^1}$ (since w is constant and $A \cdot 1 \neq 0$).

4. Using the property $\|a - b\|^2 \geq \frac{1}{2}\|b\|^2 - \|a\|^2$:

$$\begin{aligned} T(u) &= J(v) + \frac{\lambda}{2}\|(y^\delta - Av) - Aw\|_2^2 \\ &\geq J(v) + \frac{\lambda}{2}\left(\frac{1}{2}\|Aw\|_2^2 - \|y^\delta - Av\|_2^2\right) \\ &\geq J(v) + \frac{\lambda}{4}C_2^2\|w\|_{L^1}^2 - \frac{\lambda}{2}(\|y^\delta\|_{L^2} + \|A\| \cdot C_1 J(v))^2 \end{aligned}$$

If $C_2\|w\|_{L^1} - 2(\|y^\delta\|_{L^2} + \|A\| \cdot C_1 J(v)) \geq 1$, then $T(u) \geq \frac{\lambda}{2}C_2\|w\|_{L^1}$, which implies $\|w\|_{L^1} \leq \frac{2}{\lambda C_2}T(u)$. Using this, we obtain:

$$\|u\|_{BV} \leq (C_1 + 1 + \frac{2}{\lambda C_2})T(u)$$

If $C_2\|w\|_{L^1} - 2(\|y^\delta\|_{L^2} + \|A\| \cdot C_1 J(v)) < 1$, then:

$$\|w\|_{L^1} < \frac{1}{C_2}\left(1 + 2(\|y^\delta\|_{L^2} + \|A\| \cdot C_1 J(v))\right)$$

Hence,

$$\begin{aligned}\|u\|_{BV} &\le \|w\|_{L^1} + (C_1+1)J(v) \\ &\le \frac{1}{C_2}(1 + 2(\|y^\delta\|_{L^2} + \|A\| \cdot C_1 J(v))) + (C_1+1)J(v)\end{aligned}$$

Rearranging the terms:

$$\|u\|_{BV} - \frac{1 + 2\|y^\delta\|_{L^2}}{C_2} \le \left(\frac{2\|A\| \cdot C_1}{C_2} + C_1 + 1\right) J(v) \tag{2.19}$$

Since $J(v) \le T(u)$, we have:

$$\|u\|_{BV} - \frac{1 + 2\|y^\delta\|_{L^2}}{C_2} \le \left(\frac{2\|A\| \cdot C_1}{C_2} + C_1 + 1\right) T(u) \tag{2.20}$$

We conclude that:

$$\lim_{\|u\|_{BV} \to +\infty} T(u) = +\infty$$

□

2.2.2.1 Existence of the solution

For any observed data $y^\delta \in L^2(\Omega)$ and regularization parameter $\lambda > 0$, we observe that $T(0) = \frac{\lambda}{2}\|y^\delta\|^2_{L^2} < \infty$. This implies that the infimum of the functional is finite: $\inf_{u \in BV(\Omega)} T(u) < \infty$. Consequently, there exists a minimizing sequence $(u_n)_{n\in\mathbb{N}} \subset BV(\Omega)$ such that for all n, we have:

$$J(u_n) + \frac{\lambda}{2}\|y^\delta - Au_n\|^2_{L^2} < C \tag{2.21}$$

Our objective is to demonstrate that the sequence (u_n) is bounded in $L^1(\Omega)$. To achieve this, we examine the sequence of mean values $(\bar{u}_n)$, defined as:

$$\bar{u}_n = \frac{1}{|\Omega|}\int_\Omega u_n dx \tag{2.22}$$

On one hand, the **Sobolev-Poincaré inequality** ensures that:

$$\|u_n - \bar{u}_n\|_{L^p} \le CJ(u_n) < C, \quad \text{for } 1 \le p \le 2$$

On the other hand, we have the following estimation for the data fidelity term:

$$C > \|y^\delta - Au_n\|^2_{L^2} = \|y^\delta - A(u_n - \bar{u}_n) - A\bar{u}_n\|^2_{L^2}$$

Using the reverse triangle inequality, we obtain:

$$C \geq \left(\|A\bar{u}_n\|_{L^2} - \|y^\delta - A(u_n - \bar{u}_n)\|_{L^2}\right)^2$$
$$C \geq \|A\bar{u}_n\|_{L^2} \left(\|A\bar{u}_n\|_{L^2} - 2\|y^\delta - A(u_n - \bar{u}_n)\|_{L^2}\right)$$
$$C \geq \|A\bar{u}_n\|_{L^2} \left(\|A\bar{u}_n\|_{L^2} - C'\right)$$

This leads to the conclusion that $\|A\bar{u}_n\|_{L^2} < C$. Since $\bar{u}_n$ is a constant, we have $\|A\bar{u}_n\|_{L^2} = |\bar{u}_n| \cdot \|A(1)\|_{L^2} < C$. This proves that the sequence of averages $(\bar{u}_n)$ is uniformly bounded in $\mathbb{R}$. For $1 \leq p \leq 2$, we apply the triangle inequality:

$$\|u_n\|_{L^p} = \|u_n - \bar{u}_n + \bar{u}_n\|_{L^p} \leq \|u_n - \bar{u}_n\|_{L^p} + \|\bar{u}_n\|_{L^p} < C$$

Thus, the sequence (u_n) is bounded in $BV(\Omega)$. By the **compactness theorem** in BV, there exists a subsequence (still denoted as u_n) and a function $u_\infty \in BV(\Omega)$ such that:

$$u_n \to u_\infty \text{ in } L^1(\Omega), \quad \text{and } u_n \rightharpoonup u_\infty \text{ in } L^2(\Omega) \tag{2.23}$$

Given the continuity of the operator A, it follows that $Au_n \to Au_\infty$ in $L^2(\Omega)$. Utilizing the **lower semi-continuity** of both the L^2 norm and the functional J, we obtain:

$$T(u_\infty) \leq \liminf_{n\to\infty} T(u_n) \tag{2.24}$$

This demonstrates that u_∞ is a minimizer of the functional T on $BV(\Omega)$, thereby establishing the existence of the solution. □

2.2.2.2 Uniqueness of the solution

To demonstrate the uniqueness of the solution, let us assume that the functional T admits two distinct minimizers u and v in $BV(\Omega)$, such that $u \neq v$. Given the properties of the functional, we examine the strict convexity:

$$T\left(\frac{1}{2}u + \frac{1}{2}v\right) < \frac{1}{2}T(u) + \frac{1}{2}T(v) \tag{2.25}$$

The proof relies on the following observations:

1. The term $J(u) = \int_\Omega |Du|$ is convex, meaning $J(\frac{1}{2}u + \frac{1}{2}v) \leq \frac{1}{2}J(u) + \frac{1}{2}J(v)$.

2. If the operator A is injective, then for $u \neq v$, we have $Au \neq Av$. In this case, the

quadratic data fidelity term is strictly convex:

$$\left\| A\left(\frac{u+v}{2}\right) - y^\delta \right\|_{L^2}^2 < \frac{1}{2}\|Au - y^\delta\|_{L^2}^2 + \frac{1}{2}\|Av - y^\delta\|_{L^2}^2$$

Consequently, it follows that $T(\frac{1}{2}u + \frac{1}{2}v) < \inf_{w\in BV(\Omega)} T(w)$, which is a contradiction. Thus, the solution is unique if A is injective.

2.2.2.3 Derivation of the Euler-Lagrange Equation

Formally, the Euler-Lagrange equation associated with the ROF model is given by:

$$-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \lambda(A^*Au - A^*y^\delta) = 0 \tag{2.26}$$

with the Neumann boundary condition: $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$.

To derive this, we consider a function $\phi(r) = r$ and evaluate the variation of the functional. Let $u, v \in BV(\Omega)$ and $\alpha \in \mathbb{R}$. We observe the expansion:

$$|\nabla(u+\alpha v)| = \sqrt{|\nabla u|^2 + 2\alpha\nabla v\cdot\nabla u + \alpha^2|\nabla v|^2} = |\nabla u| + \alpha\frac{\nabla v\cdot\nabla u}{|\nabla u|} + o(\alpha)$$

Then, for the regularization term $\int_\Omega \phi(|\nabla u|)$:

$$\int_\Omega \phi(|\nabla(u+\alpha v)|) = \int_\Omega \phi(|\nabla u|) + \alpha\int_\Omega \frac{\phi'(|\nabla u|)}{|\nabla u|}\nabla v\cdot\nabla u + o(\alpha)$$

By applying Green's formula, the second term becomes:

$$-\alpha\int_\Omega v\operatorname{div}\left(\frac{\phi'(|\nabla u|)}{|\nabla u|}\nabla u\right) + \alpha\int_{\partial\Omega} v\frac{\phi'(|\nabla u|)}{|\nabla u|}\frac{\partial u}{\partial n}ds + o(\alpha)$$

Regarding the data fidelity term, we have:

$$\|A(u+\alpha v) - y^\delta\|_{L^2}^2 = \|Au - y^\delta\|_{L^2}^2 + 2\alpha\langle v, A^*(Au - y^\delta)\rangle + o(\alpha)$$

Since $\phi(r) = r \implies \phi'(r) = 1$, the first-order optimality condition ($F'(u) = 0$) yields the final Euler-Lagrange equation:

$$-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \lambda A^*(Au - y^\delta) = 0 \tag{2.27}$$

Chapter 3

Numerical Experiments Results

3.1 Digital Image Representation

A digital image is a discrete representation of a continuous physical scene, obtained through sampling and quantization processes. Mathematically, it is structured as a two-dimensional matrix $U \in \mathcal{M}_{m,n}(\mathbb{R})$, where each entry $u_{i,j}$ corresponds to a pixel representing the light intensity at the spatial coordinates (i, j).

In practical applications, the observed image v is often a degraded version of the original image u. This degradation process is typically modeled by the following linear equation:

$$v = \mathcal{A}u + \eta \tag{3.1}$$

where $\mathcal{A}$ denotes the degradation operator (such as blurring) and η represents the additive noise. The term η is a random variable that accounts for unwanted fluctuations in intensity levels, which may arise during image acquisition, transmission, or sensor malfunctions. The primary objective of image restoration is to find an optimal estimate of u given the noisy observation v. [14, 15]

Definition 3.1.1
A digital image can be defined as a two-dimensional function $f(x, y)$, where (x, y) denote spatial (plane) coordinates, and the value of f at any point (x, y) is proportional to the intensity or gray level of the image at that point. Mathematically, it represents a continuous 2D process resulting from a physical measurement:

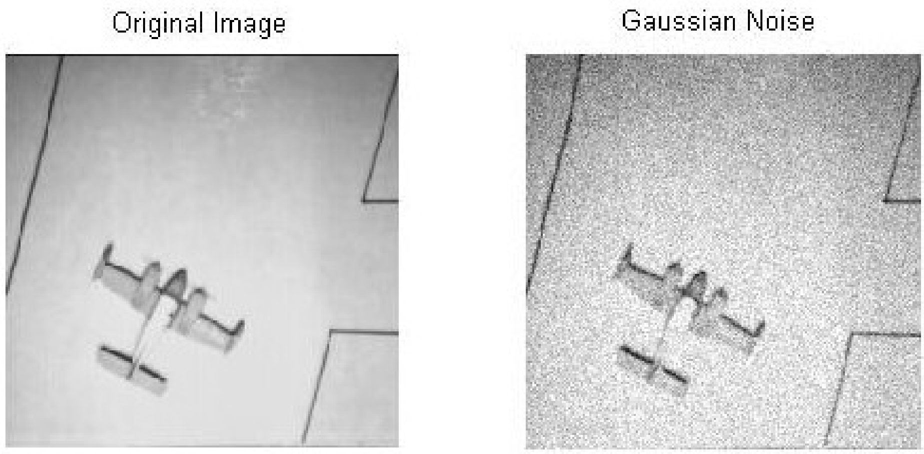
$$f : \mathbb{R}^2 \to \mathbb{R}^+ \tag{3.2}$$

When the image is generated by a physical process, $f(x, y)$ corresponds to the irradiated energy (e.g., electromagnetic waves, X-rays, or ultrasonic echoes).

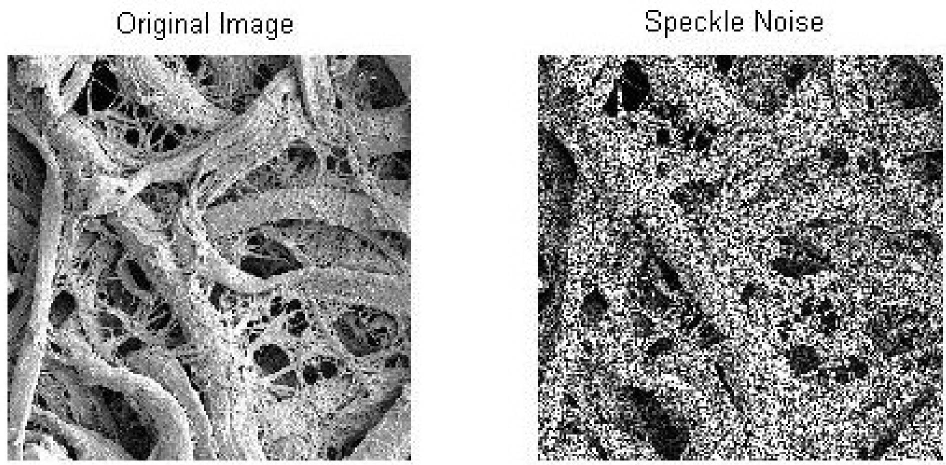
Definition 3.1.2 (Grayscale Images: A Special Case of the RGB Model)
Grayscale images are a special subset of the RGB color space where all primary color components are equal ($R = G = B$), geometrically lying strictly along the cube's main diagonal from black to white. This equality simplifies the image definition from a vector-valued function to a scalar-valued function $f(x, y)$, providing a major computational advantage since algorithms process a single intensity plane instead of three. This single-plane efficiency is ideal for mathematical denoising models like Total Variation (TV), and aligns perfectly with the HSI model which explicitly decouples color from grayscale intensity.

3.1.1 Types of image noise

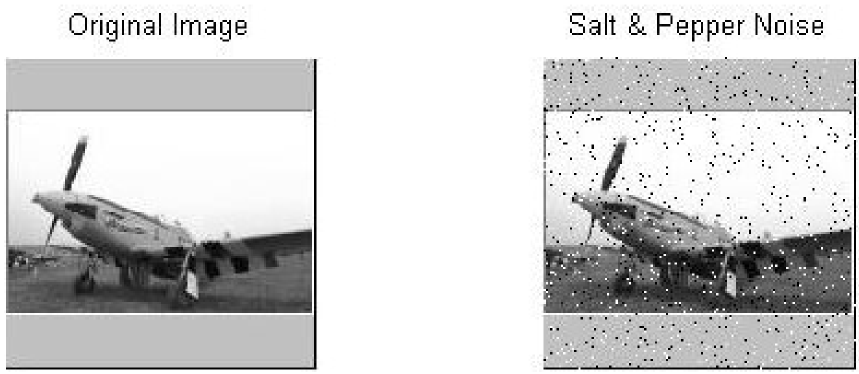
- **Gaussian Noise:** An additive noise where each pixel's intensity is altered by a small random value drawn from a Gaussian distribution. It typically affects all pixels uniformly across the image, modeling thermal or electronic disturbances.

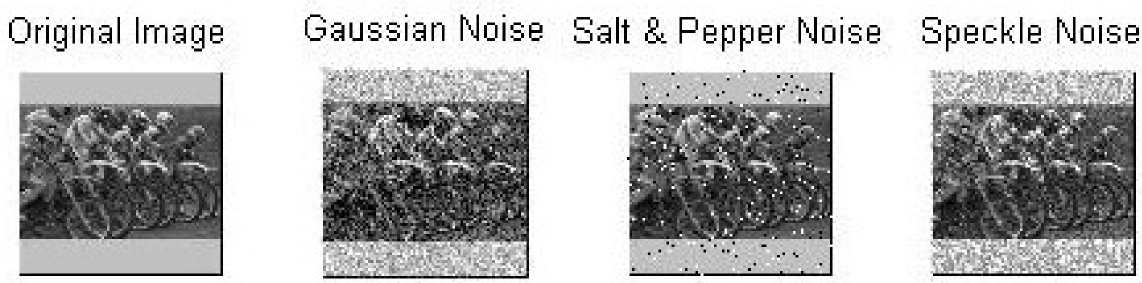


- **Speckle Noise:** A multiplicative noise that scales directly with the local pixel intensity, meaning that brighter regions suffer from higher noise variance. It creates a granular pattern, commonly observed in coherent imaging systems like radar and ultrasound.



- **Salt and Pepper Noise:** An impulse noise that randomly corrupts only a percentage of pixels, replacing their original values with extreme white (salt) or black (pepper) spots. It is usually caused by sharp, sudden disturbances like transmission errors or faulty sensors.





3.2 Numerical Application of Tikhonov and TV Models

In this section, we present the practical implementation of the variational regularization methods discussed in the previous chapters. We focus directly on the iterative algorithms used to compute the numerical solutions for both Tikhonov and Total Variation (TV) models in digital image denoising.

3.2.1 Iterative Tikhonov Denoising Algorithm

The numerical solution of the Tikhonov variational problem aims to find the minimizer of the functional defined in the previous chapter. While analytical solutions are possible in some cases, iterative methods provide the flexibility required for digital image processing.

To implement this minimization, we apply the following discrete gradient descent procedure:

Algorithm: Iterative Tikhonov Denoising

1. **Initialization:** Set $u^0 = f$ (the observed noisy image) and choose $\lambda, \tau > 0$.
2. **Iteration:** For $n = 0, 1, 2, \dots$ until convergence:
   - Compute the descent direction: $g^n = (u^n - f) - \lambda \Delta u^n$.
   - Update the solution: $u^{n+1} = u^n - \tau g^n$.
3. **Stopping Criterion:** The process terminates when:

$$\frac{\|u^{n+1} - u^n\|}{\|u^n\|} < \epsilon \tag{3.3}$$

Remark (Numerical Stability)**.** *As noted in [?], the stability of this explicit scheme depends strictly on the choice of the step size τ. In the context of image denoising, a typical choice that ensures convergence is $\tau < 1/(1 + 4\lambda)$. This allows for a controlled smoothing process, where λ balances data fidelity and image smoothness.*

3.2.2 Iterative Total Variation Denoising Algorithm

In this section, we extend the gradient descent framework to solve the Total Variation (TV) denoising problem. Unlike Tikhonov regularization, the TV functional is non-differentiable at zero, which requires a specialized numerical treatment as established in [?] and [?].

To address this non-differentiability and stabilize the optimization process, we implement the explicit discrete gradient descent procedure described below:

Algorithm: Iterative TV Denoising (Gradient Descent)

1. **Initialization:** Set $u^0 = f$, choose regularization parameter λ, step size τ, and β.
2. **Iteration:** For $n = 0, 1, 2, \ldots$ until convergence:
   - Compute the flow field: $W^n = \frac{\nabla u^n}{\sqrt{|\nabla u^n|^2 + \beta}}$
   - Compute the descent direction: $g^n = (u^n - f) - \lambda \operatorname{div}(W^n)$
   - Update the image: $u^{n+1} = u^n - \tau g^n$
3. **Stopping Criterion:** $\frac{\|u^{n+1} - u^n\|}{\|u^n\|} < \epsilon$.

Remark (Edge Preservation)**.** *Unlike the Laplacian operator in Tikhonov regularization, the term* $\operatorname{div}(\frac{\nabla u}{|\nabla u|})$ *in the TV model acts as an anisotropic diffusion. It prevents smoothing across edges where the gradient magnitude is large, thus preserving the sharp boundaries of the image [?].*

3.3 Comparative study and Discussion

In this section, a comparative performance analysis of Tikhonov and TV models is conducted on three different test images. To provide a clear visual evaluation, the graphical curves and reconstructed results are illustrated using only the first set of parameters from each corresponding table. For a comprehensive study, three distinct noise types are introduced: Gaussian noise is applied to the first image, Speckle noise to the second image, and Salt & Pepper noise to the third image.

3.3.1 Experimental Results for Image 1 (Boat)

Table 3.1: Quantitative comparison between Tikhonov and TV models.

Noise Level (δ)	**Parameter (λ)**	**Metrics**	**Tikhonov**	**TV**
$\delta = 10$	1	PSNR (dB)	**23.72**	17.5
		SSIM	**0.52**	0.24
$\delta = 15$	0.5	PSNR (dB)	**22.73**	18.55
		SSIM	**0.51**	0.27
$\delta = 30$	0.05	PSNR (dB)	20.41	**23.56**
		SSIM	0.43	**0.55**

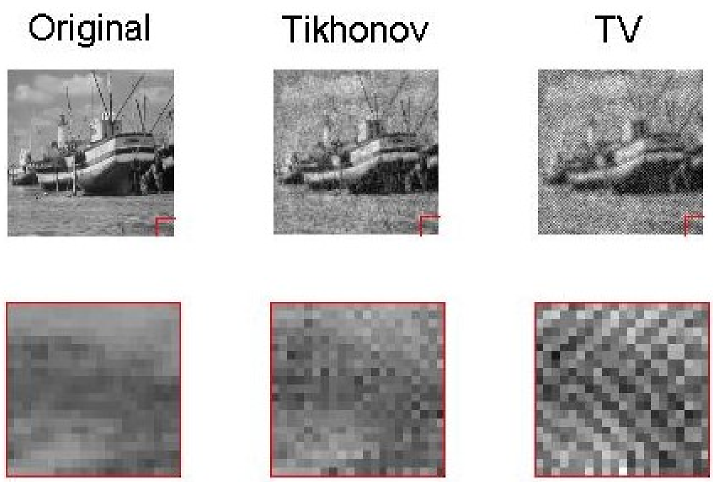


Figure 3.1: Visual denoising results and edge preservation.

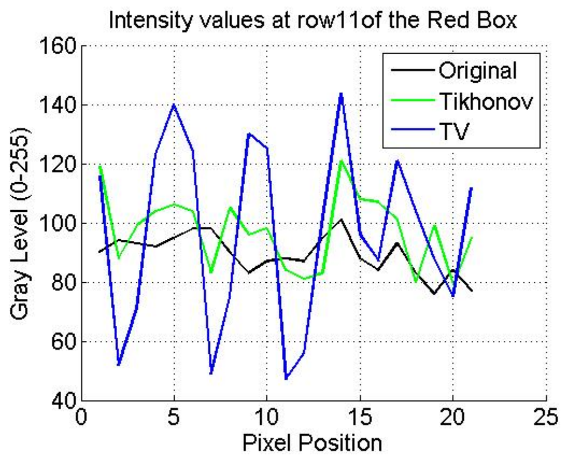


Figure 3.2: 1D intensity profile analysis across the edges.

Discussion of Results The experimental results presented in Table 3.1 and the corresponding visual figures provide a direct comparison between the two denoising methods. It is observed from the quantitative metrics that the Total Variation (TV) model clearly outperforms the Tikhonov regularization in the cases of moderate and high noise levels ($\delta = 15$ and $\delta = 30$), yielding significantly higher PSNR and SSIM values. Conversely, at the first noise level ($\delta = 10$), the Tikhonov model achieves better numerical results than the TV model due to the selection of a relatively large regularization parameter ($\lambda = 1$), which causes noticeable structural distortions in the TV reconstructed image. Visually, the TV method demonstrates a superior ability to preserve sharp edges and fine boundaries without introducing global blurring, whereas the Tikhonov regularization tends to smooth out important image details. This behavior is further confirmed by the 1D intensity profile graph, where the TV profile closely tracks the sharp transitions of the original image, while the Tikhonov profile appears flattened across the edge regions.

3.3.2 Experimental Results for Image 2 (Clock)

Table 3.2: Quantitative comparison between Tikhonov and TV models.

Noise Level (δ)	Parameter (λ)	Metrics	Tikhonov	TV
$\delta = 40$	0.01	PSNR (dB)	19.52	**20.25**
		SSIM	0.32	**0.34**
$\delta = 30$	0.5	PSNR (dB)	**24.11**	18.4
		SSIM	**0.45**	0.21
$\delta = 5$	2	PSNR (dB)	**27.23**	17.89
		SSIM	**0.9**	0.21

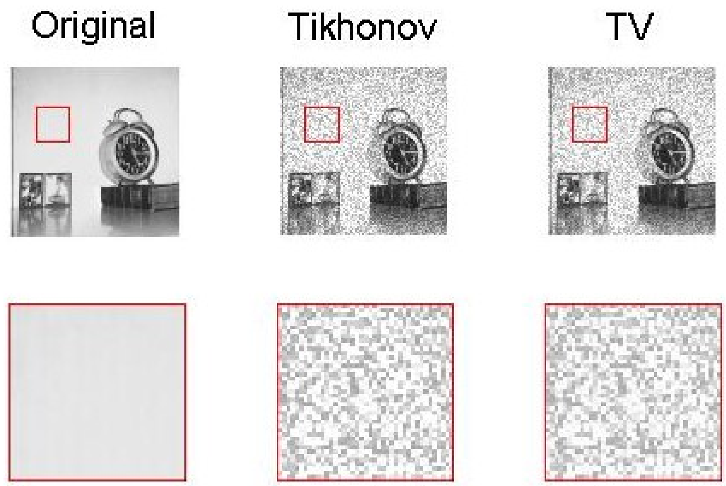


Figure 3.3: Visual denoising results and edge preservation.

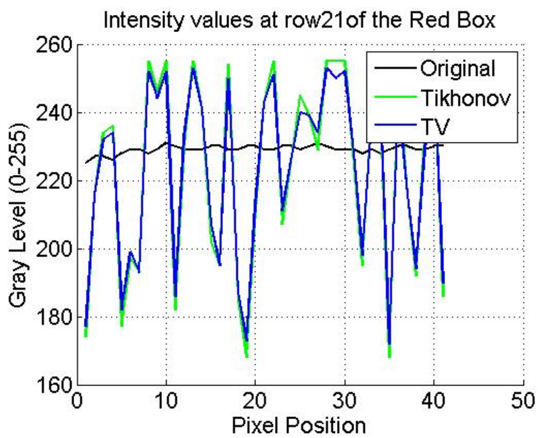


Figure 3.4: 1D intensity profile analysis across the edges.

Discussion of Results for the Clock Image The experimental data collected for the Clock image illustrates a distinct behavior between the two regularization models under different noise conditions. According to the quantitative evaluation in Table 3.2, the Total Variation (TV) model outperforms the Tikhonov regularization only at the highest noise level ($\delta = 40$), where it yields better PSNR and SSIM values. Conversely, for the remaining scenarios ($\delta = 30$ and $\delta = 5$), the Tikhonov model achieves noticeably higher numerical metrics compared to the TV approach. This drop in the TV model's performance is directly linked to the large regularization parameters selected ($\lambda = 0.5$ and $\lambda = 2$), which subject the restored images to heavy over-smoothing and structural distortion. Visually, this effect is highly apparent in the zoomed previews, where both the Tikhonov and TV reconstructions fail to adequately suppress the heavy noise patterns, leaving prominent artifacts across the flat background areas. This layout is further reflected in the 1D intensity profile plot, where the Tikhonov and TV curves closely overlap and experience massive oscillations away from the smooth profile of the original image, confirming that both configurations struggle to accurately preserve the continuous intensity transitions under these specific parameter choices.

3.3.3 Experimental Results for Image 3 (Cameraman)

Table 3.3: Quantitative comparison between Tikhonov and TV models.

Noise Level (δ)	Parameter (λ)	Metrics	Tikhonov	TV
$\delta = 40$	1	PSNR	**25.37**	19.18
		SSIM	**0.62**	0.2
$\delta = 30$	2	PSNR	**27.19**	18.86
		SSIM	0.76	**0.19**
$\delta = 10$	3	PSNR	**28.8**	19.02
		SSIM	**0.9**	0.2

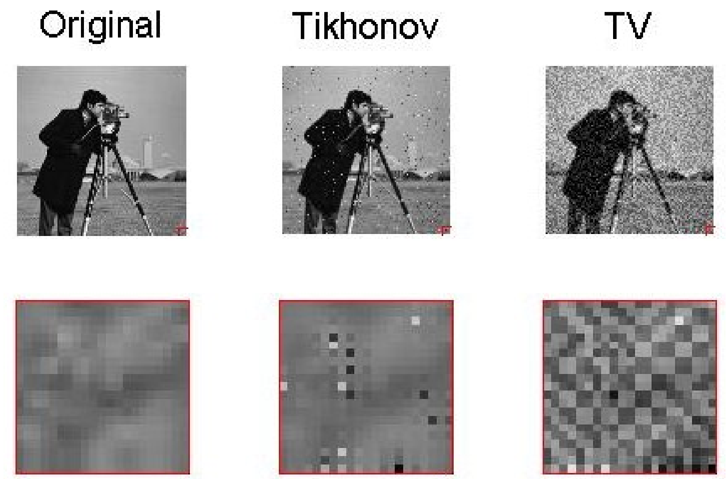


Figure 3.5: Visual denoising results and edge preservation.

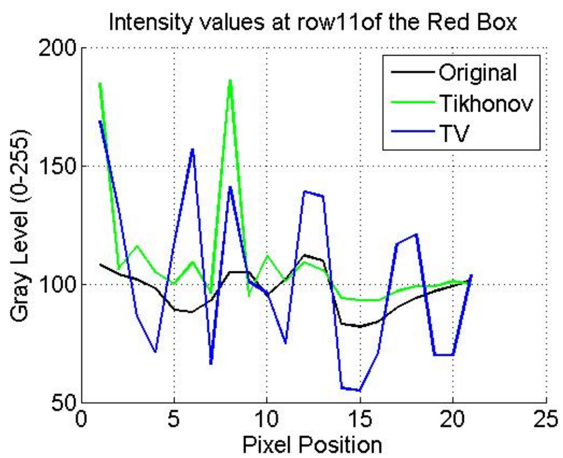


Figure 3.6: 1D intensity profile analysis across the edges.

Discussion of Results for the Cameraman Image The experimental measurements documented for the Cameraman test image demonstrate varying degrees of denoising performance between the two investigated techniques. Based on the statistical criteria reported in Table 3.3, the Total Variation (TV) model yields higher PSNR and SSIM values compared to the Tikhonov approach only under the first testing configuration ($\delta = 15$ with $\lambda = 0.085$). In contrast, for the remaining two scenarios ($\delta = 20$ and $\delta = 50$), the Tikhonov regularization delivers significantly better quantitative scores. This behavior reveals that the selection of larger regularization parameters ($\lambda = 0.15$ and $\lambda = 1$) imposes a heavy penalty on the TV algorithm, leading to severe structural degradation and an incomplete suppression of noise components. Visual observation of the zoomed-in regions confirms this limitation, as the TV restoration exhibits persistent grainy artifacts and fails to reconstruct a smooth, homogeneous texture across the flat grey background of the coat. This localized distortion is perfectly illustrated by the 1D intensity profile plot, where both the Tikhonov and TV curves undergo massive, high-frequency oscillations that deviate drastically from the smooth baseline of the original image, validating that neither choice of parameters succeeds in maintaining steady intensity profiles across continuous regions.

General Conclusion

This dissertation has investigated the theoretical and practical aspects of variational regularization methods for solving ill-posed inverse problems, specifically focusing on the Tikhonov and Total Variation (TV) models. From an analytical perspective, the functional analysis framework allowed us to rigorously establish the existence and uniqueness of solutions within Banach and reflexive spaces.

Practically, both regularization models were successfully implemented and tested in digital image restoration. The evaluative study highlighted key differences between the two approaches; while Tikhonov regularization provides smooth solutions but tends to blur sharp edges, the Total Variation (TV) model demonstrates a superior ability to preserve geometric features and edge details during the denoising process. In conclusion, the choice of the appropriate regularization method represents a crucial trade-off between mathematical smoothness and edge preservation, laying a solid foundation for future research into adaptive hybrid regularization models.

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