جامعة الشهيد حمّه لخضر - الوادي
UNIVERSITY OF ELOUED
Faculty of Exact Sciences - Department of Physics

General electronics

(2^{nd} year physics)

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Foreword

Level: 2nd year physics

Semester: 04

Module: General electronics

Goals:

Mastery and calculation of electrical networks and circuits of types RL, RC and RLC as well as quadrupoles and diodes accompanied by a set of application examples in the form of practical work.

Prior knowledges:

It is recommended to master the Physics 2 module (Electricity) taught in S2 and Mathematics, from the 1st year Matter Science.

Module contents:

<u>CHAPTER 01</u> – ELECTRICAL NETWORKS

Part I. Continuous regime.

Part II. Variable regime.

Part III. Sinusoidal regime.

Part IV. Study of simple R L and RC circuits in free and forced regime.

Part V. Study of complex RLC circuits in free and forced regime.

Part VI. Study of series and parallel resonant circuits.

<u>CHAPTER 02</u> – PASSIVE QUADRUPOLES

Part I. Representation of a passive network by a quadrupole.

Part II. Special passive quadrupoles (Γ, T, Π, etc.).

<u>CHAPTER 03</u> – DIODES

Part I. Semi-conductors.

Part II. Semi-conductor devices (Diode).

Part III. Diode circuits.

Part IV. Special-purpose diodes.

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Preface

This course brings together all the essential elements of electro kinetics, passive quadrupoles, and diodes. It is structured into three chapters which deal with the fundamental notions of electrical circuits in continuous, sinusoidal and transient regimes then review the passive quadrupoles and ending with the study of diodes.

Chapter: 01 Electrical networks

Part I. Continuous regime

This part aims to familiarize the student with the most fundamental tools, within the framework of the simplest operating regime: the continuous regime.

1. Introduction

The chapter begins with fundamental definitions, circuit elements including dependent sources, circuit laws and theorems, and analysis techniques. These theorems and methods are initially applied to DC-resistive circuits and then extended to RLC circuits.

2. Definitions and fundamental principles

Generally speaking, any electrical circuit can be represented in the form of an energy generator supplying a receiver responsible for transforming the electrical energy received into another usable form, the two devices being connected by conductors.

The operation of an electrical circuit is described by a transfer of charges between these two elements (figure 01). It is commonly accepted to represent this transfer by a flow of electrons which is modeled by an electric current passing through the conductors. This electric current (expressed in amperes) represents the quantity of charges q (in coulombs) passing through a given section of the conductor per unit of time, i.e.:

$$\boxed{i = \frac{\mathrm{d}q}{\mathrm{d}t}} \quad (01)$$

Since electrons have a negative charge, the convention is that the current *i* is represented in the opposite direction to the flow of electrons.

In a circuit made up of a single loop, the same current flows at all times throughout the circuit. Simple generators and receivers generally have two terminals. These are electric dipoles. The generating dipoles are called active, those which only consume energy are passive dipoles.

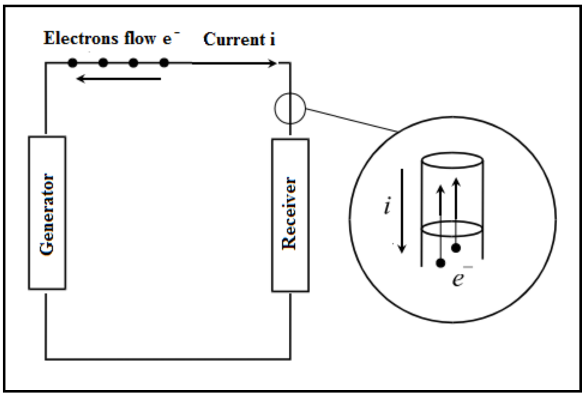


Fig. 01: Transfer of charges between generator and receiver

The most frequently encountered active dipoles (figure 02) are:

The Perfect voltage generator, which delivers a voltage e (in volts) and imposes it on the receiving dipole which therefore presents the same voltage e at its terminals. The current which then appears in the circuit depends on e and the receiver. This voltage e is the potential difference $V_A - V_B$. The arrow symbolizing this potential difference is directed towards the highest potential. As the electrons are attracted to the point corresponding to the highest potential (A), the current will be directed, leaving the generator, with an arrow pointing towards the highest potential.

The perfect current generator, which imposes a current *i* on the receiving dipole. The voltage which then appears across the receiver dipole depends on *i* and the receiver.

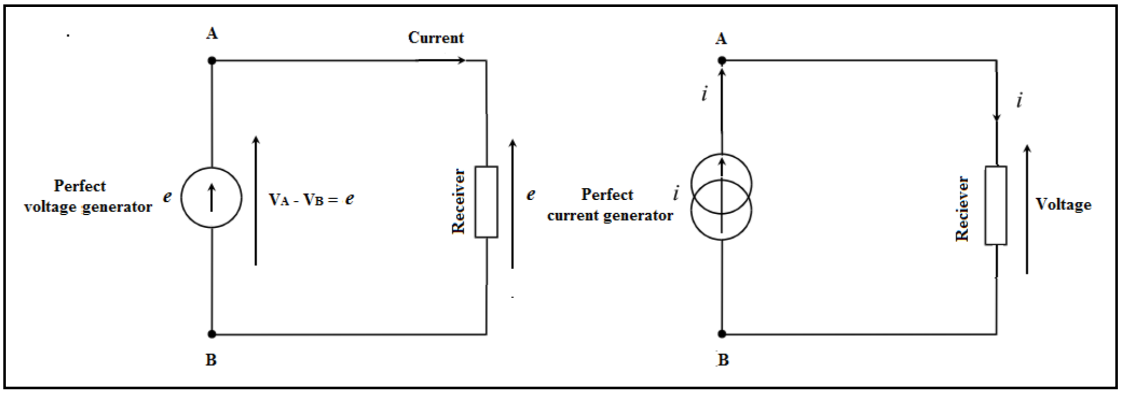


Fig. 02: Perfect active dipoles (Perfect voltage generator, perfect current generator)

For a circuit powered by a voltage generator, we generally consider that its terminal B constitutes the voltage reference for the entire circuit and is therefore at 0 V potential (also called ground).

Its terminal A is therefore at potential V_A = e. We therefore assimilate any difference in potential between any point x and this reference, to the potential of the point x.

Generators are said to be perfect in the sense that the voltage delivered by a perfect voltage generator does not depend on the rest of the circuit. Likewise, a perfect current generator delivers a current that does not depend on the rest of the circuit.

In reality, generators are not perfect and we consider that a model closer to reality consists of associating a resistance in series with a perfect voltage generator, or a resistance in parallel with a perfect current generator. These resistances are called internal resistances of the generators (figure 03).

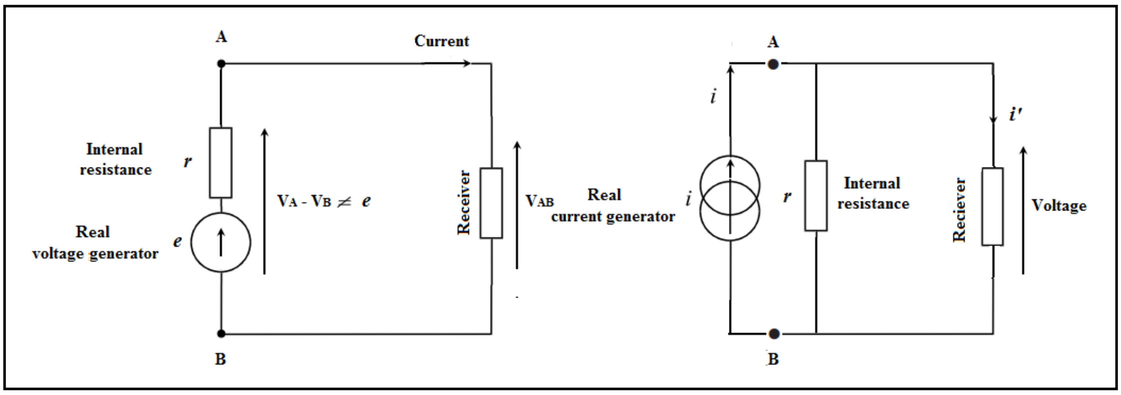


Fig. 03: Real active dipoles (Real voltage generator, real current generator)

3. Sign conventions

In a simple circuit composed of a voltage generator and a receiver dipole, taking into account the fact that the same voltage reigns at the terminals of the two elements, and that the same current circulates throughout the circuit, we note that on the side of the generator, current and voltage are represented by arrows directed in the same direction, while on the receiver side, they are directed in opposite directions (figure 04).

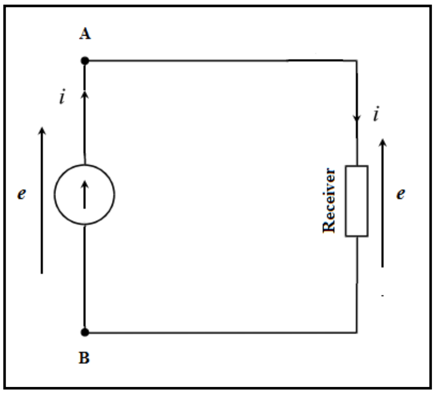


Fig. 04: Current and voltage representation in the generator, and the receiver

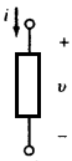
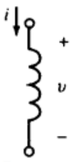
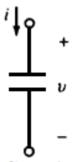
By convention, we will systematically direct the arrows of currents and voltages in the same direction for the generator (generator convention), and in opposite directions for any receiver (receiver convention).

Typically, a circuit includes a single generator. However, some may contain several. In this case, if a generator is considered to belong to the receiving part of the circuit, it is the receiver convention that we will use.

4. Voltage-current relations

The passive circuit elements resistance R, inductance L, and capacitance C are defined by the manner in which the voltage and current are related for the individual element. For example, if the voltage v and current i for a single element are related by a constant, then the element is a resistance, R is the constant of proportionality, and v = R i. Similarly, if the voltage is the time derivative of the current, then the element is an inductance, L is the constant of proportionality, and v = L di/dt. Finally, if the current in the element is the time derivative of the voltage, then the element is a capacitance, C is the constant of proportionality, and i = C dv/dt. Table (01) summarizes these relationships for the three passive circuit elements. Note the current directions and the corresponding polarity of the voltages.

Tab. 01 the relationships for the three passive circuit elements R, L, and C

Circuit element	Units	Voltage	Current	Power
 **Resistance**	ohms (Ω)	$v = Ri$ **(Ohms's law)**	$i = \frac{v}{R}$	$p = vi = i^2 R$
 **Inductance**	henries (H)	$v = L\frac{di}{dt}$	$i = \frac{1}{L}\int v\,dt + k_1$	$p = vi = Li\frac{di}{dt}$
 **Capacitance**	farads (F)	$v = \frac{1}{C}\int i\,dt + k_2$	$i = C\frac{dv}{dt}$	$p = vi = Cv\frac{dv}{dt}$

The operation law of a resistance is called Ohm's law.

4.1 Resistance

All electrical devices that consume energy must have a resistor (also called a resistance) in their circuit model. Inductors and capacitors may store energy but over time return that energy to the source or to another circuit element.

4.2 Inductance

The circuit element that stores energy in a magnetic field is an inductor (inductance). With time-variable current, the energy is generally stored during some parts of the cycle and then returned to the source during others. When the inductance is removed from the source, the magnetic field will collapse; in other words, no energy is stored without a connected source.

4.3 Capacitance

The circuit element that stores energy in an electric field is a capacitor (also called capacitance). When the voltage is variable over a cycle, energy will be stored during one part of the cycle and returned in the next. While an inductance cannot retain energy after removal of the source because the magnetic field collapses, the capacitor retains the charge and the electric field can remain after the source is removed. This charged condition can remain until a discharge path is provided, at which time the energy is released.

5. Dipoles associations

5.1 Series and parallels associations

Any two dipoles are said to be associated in series if one of the terminals of one is connected to one of the terminals of the other, the whole forming a new dipole. They are said to be associated in parallel if the pairs of terminals are connected two by two (figure 05).

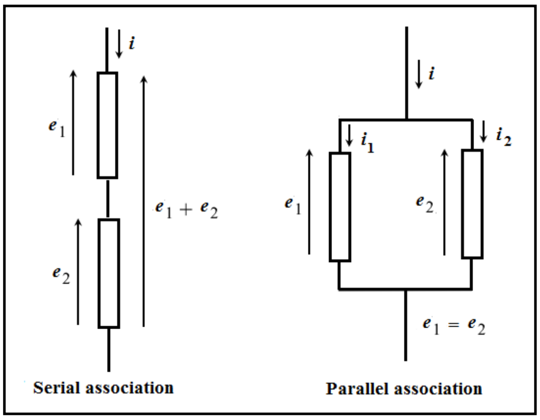


Fig. 05: Types of dipoles associations

In the case of the series association, the two dipoles carry the same current. The total voltage across the assembly is equal to the sum of the two potential differences across each of the two dipoles. In the case of the parallel association, the same potential difference reigns at the terminals of each of the two dipoles.

By combining resistances, we form a dipole which behaves like a resistance, the value of which is called equivalent resistance. It is the same when combining capacitors. Figure (06) shows some very simple usual associations.

It will be noted that the rules for associating resistors and those for associating capacitors are reversed.

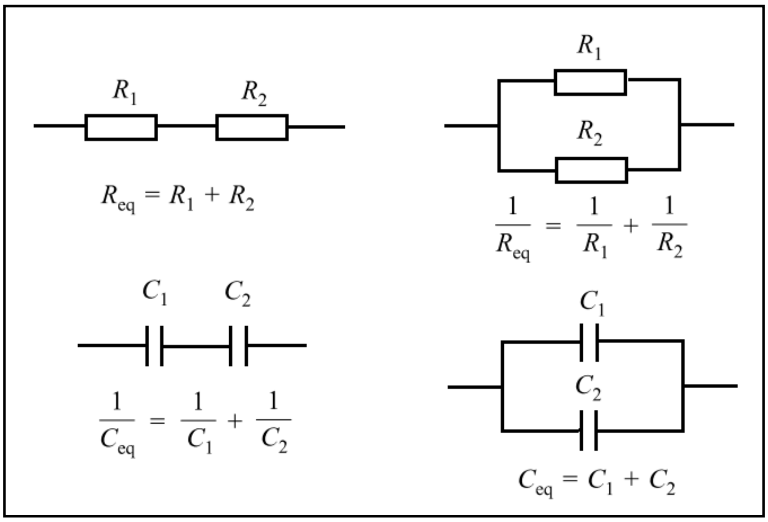
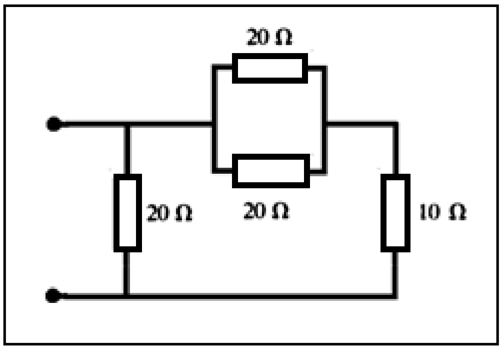


Fig. 06: Usual combinations of resistances and capacitors

Example: Find the equivalent resistance for the circuit shown in the following figure:



The two 20 Ω resistors in parallel have an equivalent resistance R_{eq} = [(20) (20) = (20 + 20)] = 10 Ω. This is in series with the 10 Ω resistor so that their sum is 20 Ω. This in turn is in parallel with the other 20 Ω resistor so that the overall equivalent resistance is 10 Ω.

5.2 Y and Delta networks

The network in Fig. (07) is called a T ("tee") or Y ("wye") network because of its shape. T and Y are different names for the same network, except that in the Y network the R_a and R_b arms form the upper part of a Y.

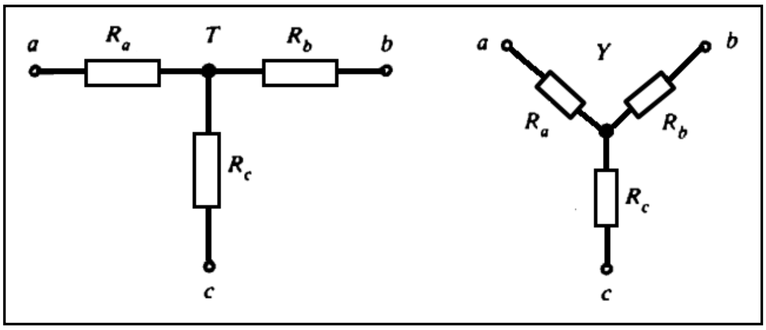


Fig. 07: Form of a T or Y network

The network in Fig. (08) is called π (pi) or Δ (delta) network because its shape resembles these Greek letters π and Δ are different names for the same network.

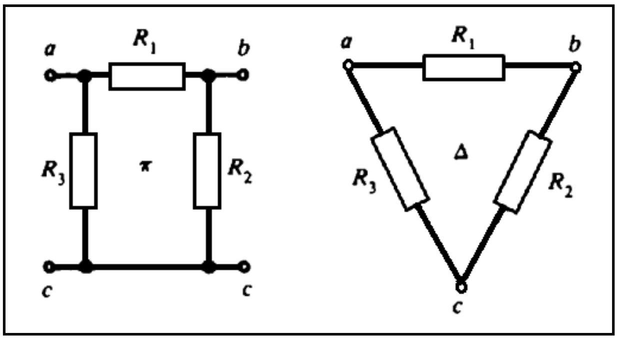


Fig. 08: Form of a π or Δ network

In analyzing networks, it is helpful to convert Y to Δ or Δ to Y to simplify the solution. The formulas for these conversions are derived from Kirchhoff's laws. Note that resistances in Y have subscript letters, R_a, R_b, and R_c, while the resistances in Δ are numbered R_1, R_2, and R_3.

Resistances are shown in a three-terminal network with three terminals a, b, and c. After the conversion formulas are used; one network is equivalent to the other because they have equivalent resistances across any one pair of terminals. The network in Fig. (09) allows the conversion between Y and Δ networks.

5.2.1 Δ to Y Conversion, or π to T

$$\boxed{R_a = \frac{R_1R_3}{R_1+R_2+R_3}; R_b = \frac{R_1R_2}{R_1+R_2+R_3}; R_c = \frac{R_2R_3}{R_1+R_2+R_3}} \quad (02)$$

The rule for Δ to Y conversion can be stated as follows: The resistance of any branch of the Y network is equal to the product of the two adjacent sides of the Δ network divided by the sum of the three Δ resistances.

5.2.2 Y to Δ Conversion, or T to π

$$\boxed{R_1 = \frac{R_aR_b+R_bR_c+R_cR_a}{R_c}; R_2 = \frac{R_aR_b+R_bR_c+R_cR_a}{R_a}; R_3 = \frac{R_aR_b+R_bR_c+R_cR_a}{R_b}} \quad (03)$$

The rule for Y to Δ conversion can be stated as follows: The resistance of any side of the Δ network is equal to the sum of the Y network resistances multiplied two at a time, divided by the resistance of the opposite branch of the Y network.

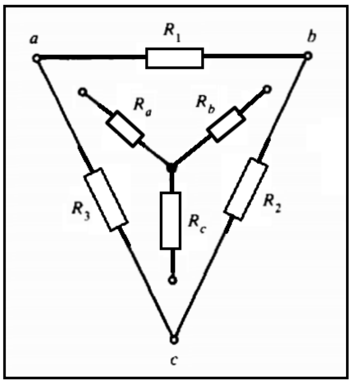
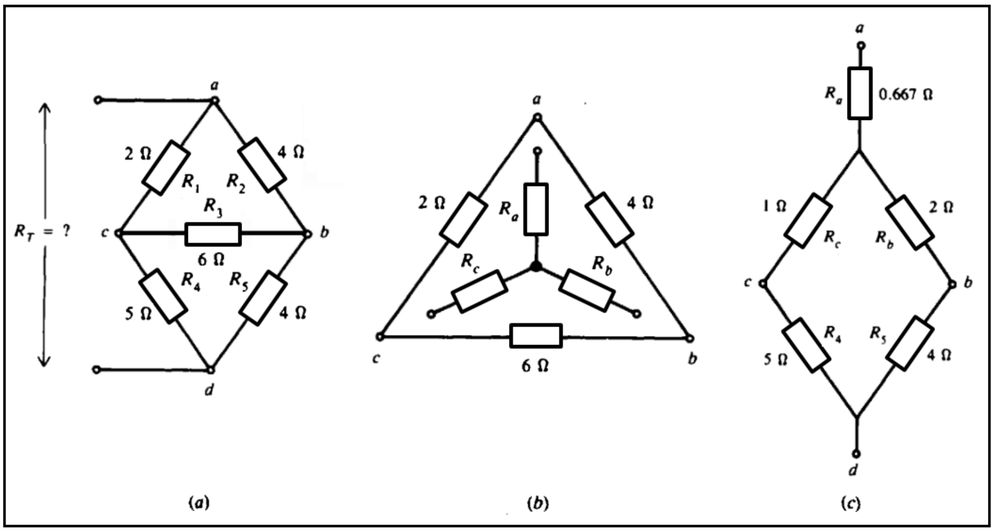
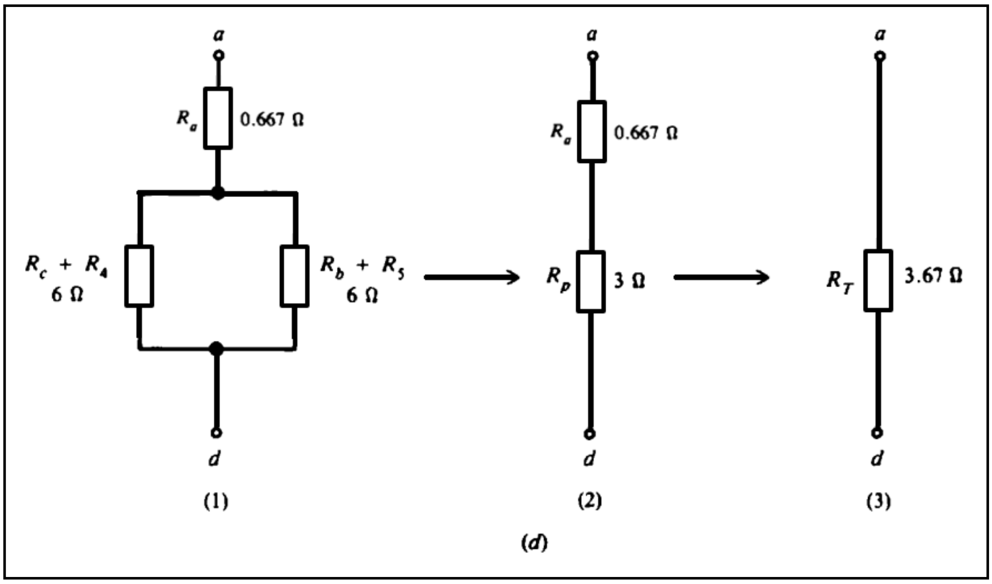


Fig. 09: Conversion between Y and Δ networks

<u>Example:</u> Use network conversion to find the equivalent or total resistance R_T between (a) and (d) in a bridge circuit consisting of two deltas in the next Fig. (a).

Reducing a bridge circuit by Δ to Y conversion





6. Electric regimes

Depending on the form of the voltage (or current) delivered by the generator which powers a circuit, we say that this circuit operates at a certain regime:

- If it delivers a constant voltage, the circuit operates in continuous mode. Continuous quantities will be noted with capital letters (E for voltage for example).
- If it delivers a variable voltage over time, we will be in the case of a variable regime and we will designate the quantities by lowercase letters: e(t), for example.
- If the voltage delivered is sinusoidal: $e(t) = E_0 \cos \omega t$, the regime will be said to be sinusoidal or harmonic.

Continuous and sinusoidal regimes are part of so-called permanent regimes. Often, variable regimes occur when a circuit goes from one permanent state to another. We then speak of transitional regimes. In a continuous regime circuit, the voltages and currents in the circuit are generally continuous.

In a sinusoidal circuit, voltages and currents are all sinusoidal, of the same frequency as the voltage source, but a priori presenting phase shifts.

7. Continuous regime

In continuous operation, an inductive element has no effect, Fig. (10). Its operating equation:

$$\boxed{u(t) = L\frac{\mathrm{d}i}{\mathrm{d}t}} \tag{04}$$

This equation clearly shows that, carried by any constant current, an inductance will always have a null potential difference at its terminals.

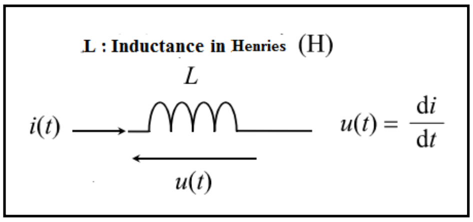


Fig. 10: Inductance with the operating equation

Likewise, for a capacitor, Fig. (11), the equation:

$$\boxed{u(t) = \frac{1}{C}\int i(t)\mathrm{d}t} \tag{05}$$

This equation shows that if u(t) = C^{te}, we have:

$$\boxed{i(t) = 0} \tag{06}$$

Therefore, in continuous mode, no current can pass through a capacitor. On the other hand, any capacitor which has a voltage U imposed has a stored charge Q such that:

$$\boxed{Q = CU} \tag{07}$$

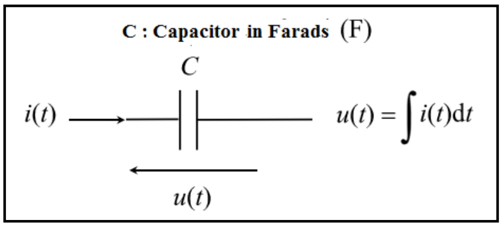


Fig. 11: Inductance with the operating equation

A perfect capacitor also has the property of keeping this stored charge, once the power supply U has been removed. This, of course, provided that it is isolated, that is to say that its two terminals are not connected to no other circuit.

For a resistance, Fig. (12), the operating equation is:

$$\boxed{u(t) = Ri(t)} \tag{08}$$

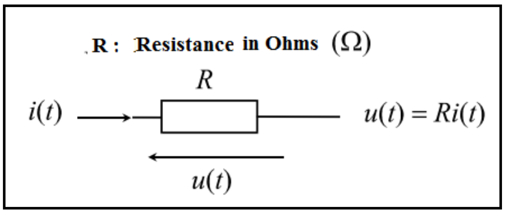


Fig. 12: Resistance with the operating equation

The law of operation of a resistor is called Ohm's law, Fig. (12).

8. Kirchhoff's laws

An electric circuit or network consists of a number of interconnected single circuit elements of the type described in Tab. (01). The circuit will generally contain at least one voltage or current source. The arrangement of elements results in a new set of constraints between the currents and voltages. These new constraints and their corresponding equations, added to the current-voltage relationships of the individual elements, provide the solution of the network.

8.1. Definitions

- **Electric network:** any simple or complex association of interconnected dipoles, powered by a generator.

- **Branch:** dipolar part of a network carried by the same current.

- **Node of a network:** Any point in the network common to more than two branches.

- **Mesh of a network**: Any path constituting a loop and made up of several branches.

In the diagram in figure (13), the association of resistances forming the AC dipole constitutes an electrical network powered by the voltage generator V. B, C, D and E are the nodes of this network. The diagram shows at least three meshes.

8.2 Law of nodes (Kirchhoff's first law or Kirchhoff's current law “KCL”)

The connection of two or more circuit elements creates a junction called a node. Kirchhoff’s current law (KCL) states that the algebraic sum of the currents at a node is zero (counting positively the currents directed to the node and negatively counting those leaving it). It may be stated alternatively that the sum of the currents entering a node is equal to the sum of the currents leaving that node. This law expresses the fact that there cannot be an accumulation of charges at any point on a network conductor. In the example, we can write among other equations:

$$\boxed{\begin{aligned} I_1 &= I_2 + I \\ I &= I_3 + I_4 \end{aligned}} \qquad (09)$$

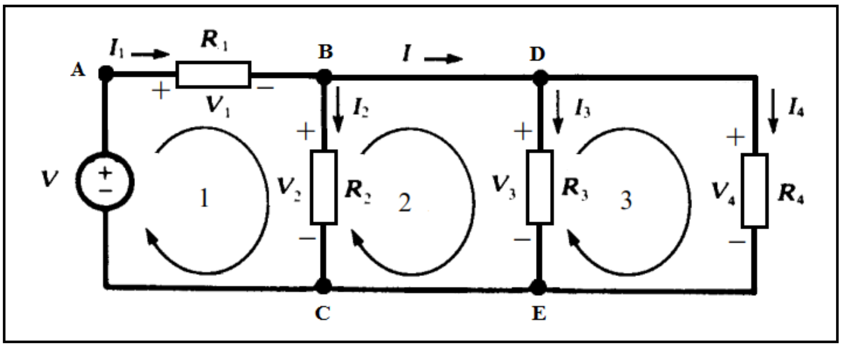


Fig. 13: Electrical network

8.3 Mesh law (Kirchhoff's second law)

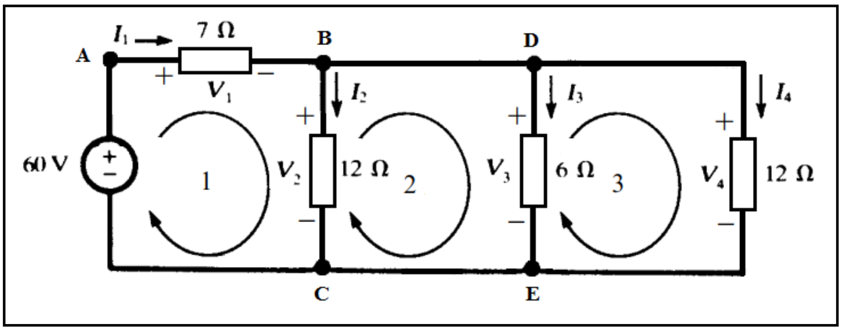
The algebraic sum of the potential differences along a mesh, obtained by traversing the mesh in a given direction, is zero. Potential differences oriented in the same direction as the direction of the mesh path are counted positively. Potential differences oriented in the opposite direction to the direction of the mesh path are counted negatively.

So, in the example for the meshes 1, 2 and 3:

$$\boxed{\begin{aligned} &\text{Mesh (1): } V - V_1 - V_2 = 0 \\ &\text{Mesh (2): } V_2 - V_3 = 0 \\ &\text{Mesh (3): } V_3 - V_4 = 0 \end{aligned}} \qquad (10)$$

These Kirchhoff laws are presented here in continuous mode (capital letters for voltages and currents). In reality, they remain valid whatever the regime. Like these Kirchhoff laws, most of the results presented in this course reminder from the first chapter are also valid regardless of the regime. However, the following exercises only concern continuous circuits.

Example: Use Kirchhoff laws (KVL and KCL) in the network shown in following figure to find the current supplied by the 60 V source.



KVL and KCL give:

$$\boxed{\begin{aligned} I_2(12) &= I_3(6) \\ I_2(12) &= I_4(12) \end{aligned}}$$

$$\boxed{\begin{aligned} 60 &= I_1(7) + I_2(12) \\ I_1 &= I_2 + I_3 + I_4 \end{aligned}}$$

$$\boxed{I_1 = I_2 + 2I_2 + I_2 = 4I_2}$$

$$\boxed{60 = I_1(7) + \tfrac{1}{4} I_1(12) = 10 I_1}$$

$I_1 = 6$ A	$I_2 = I_4 = 1.5$ A	$I_3 = 3$ A

9. Voltage Division

A set of series-connected resistors as shown in Fig. (14) is referred to as a voltage divider. The concept extends beyond the set of resistors illustrated here and applies equally to impedances in series.

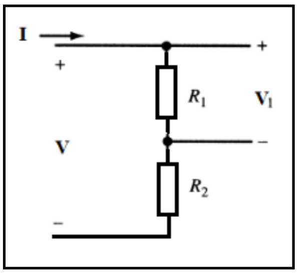
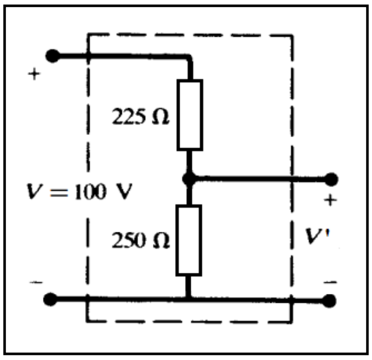


Fig. 14: Voltage Division

$$\boxed{\begin{aligned} V_1 &= I R_1 \\ V &= I(R_1 + R_2) \end{aligned}} \tag{10}$$

$$\boxed{V_1 = V\left(\frac{R_1}{R_1 + R_2}\right)} \tag{11}$$

Example: Find the voltage at the poles of the resistance 250 Ω in the network shown in the following figure.



$$\boxed{V' = V\frac{250}{225 + 250} = 100\frac{250}{225 + 250} = 52.6\ \text{V}}$$

10. Current Division

A parallel arrangement of resistors as shown in Fig. (15) results in a current divider. The ratio of the branch current I_1 to the total current I, illustrates the operation of the divider.

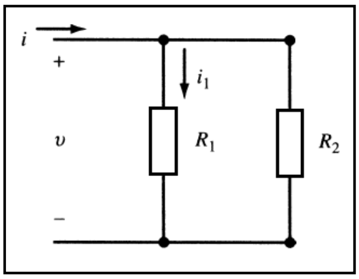
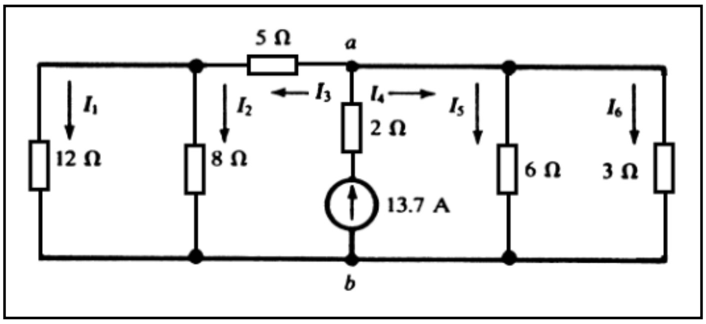


Fig. 15: Current Division

$$\boxed{\begin{aligned} I_1 &= \frac{V}{R_1} \\ I &= \frac{V}{R_1} + \frac{V}{R_2} \end{aligned}} \tag{12}$$

$$\boxed{I_1 = I\frac{R_2}{R_1 + R_2}} \tag{13}$$

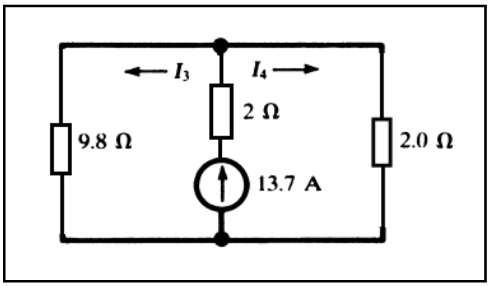
Example: Find all branch currents in the network shown in following figure.



The equivalent resistances to the left and right of nodes *a* and *b* are:

$$R_{\text{eq(left)}} = 5 + \frac{(12)(8)}{20} = 9.8\ \Omega \qquad R_{\text{eq(right)}} = \frac{(6)(3)}{9} = 2.0\ \Omega$$

Referring to the reduced network



$$I_3 = \frac{2.0}{11.8}(13.7) = 2.32\ \text{A} \qquad I_4 = \frac{9.8}{11.8}(13.7) = 11.38\ \text{A}$$

Then referring to the original network,

$$I_1 = \frac{8}{20}(2.32) = 0.93\ \text{A} \qquad I_2 = 2.32 - 0.93 = 1.39\ \text{A}$$

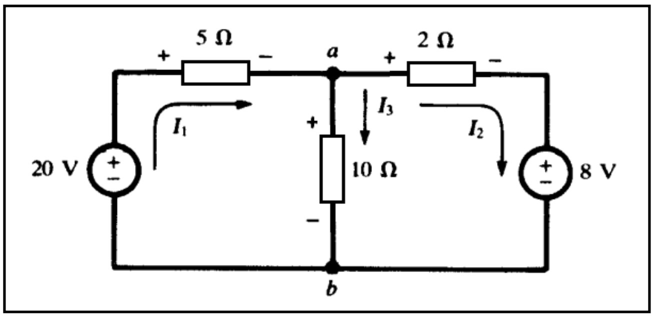
$$I_5 = \frac{3}{9}(11.38) = 3.79\ \text{A} \qquad I_6 = 11.38 - 3.79 = 7.59\ \text{A}$$

11. Analysis Methods

11.1 The branch current method

In the branch current method, a current is assigned to each branch in an active network. Then Kirchhoff's current law is applied at the principal nodes and the voltages between the nodes employed to relate the currents. This produces a set of simultaneous equations which can be solved to obtain the currents.

Example: Obtain the current in each branch of the network shown in the figure below using the branch current method.



Currents I_1; I_2, and I_3 are assigned to the branches as shown. Applying Kirchhoff's current law (KCL) at node a:

$$I_1 = I_2 + I_3$$

The voltage V_{ab} can be written in terms of the elements in each of the branches:

$$V_{ab} = 20 - I_1(5), \quad V_{ab} = I_3(10), \quad V_{ab} = I_2(2) + 8$$

Then the following equations can be written:

$$20 - I_1(5) = I_3(10)$$
$$20 - I_1(5) = I_2(2) + 8$$

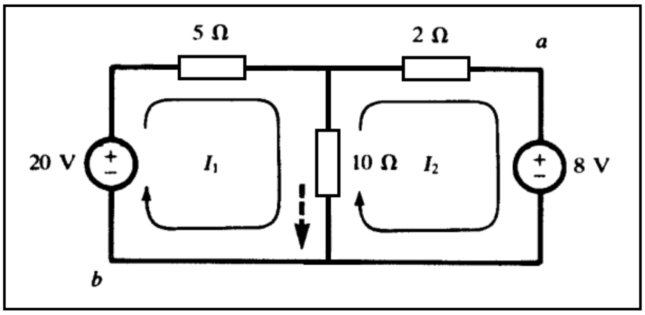
Solving the three equations (1), (2), and (3) simultaneously gives $I_1 = 2$ A, $I_2 = 1$ A, and $I_3 = 1$ A.

Other directions may be chosen for the branch currents and the answers will simply include the appropriate sign. In a more complex network, the branch current method is difficult to apply because it does not suggest either a starting point or a logical progression through the network to produce the necessary equations. It also results in more independent equations than either the mesh current or node voltage method requires.

11.2 The mesh current method

In the mesh current method, a current is assigned to each window of the network such that the currents complete a closed loop. They are sometimes referred to as loop currents. Each element and branch therefore will have an independent current. When a branch has two of the mesh currents, the actual current is given by their algebraic sum. The assigned mesh currents may have either clockwise or counterclockwise directions, although at the outset it is wise to assign to all of the mesh currents a clockwise direction. Once the currents are assigned, Kirchhoff's voltage law is written for each loop to obtain the necessary simultaneous equations.

<u>Example:</u> Obtain the current in each branch of the network shown in the figure below (same as the figure above) using the mesh current method.



The currents I_1 and I_2 are chosen as shown on the circuit diagram. Applying Kirchhoff's voltage law (KVL) around the left loop, starting at point a:

$$\boxed{-20 + 5I_1 + 10(I_1 - I_2) = 0}$$

and around the right loop, starting at point b:

$$\boxed{8 + 10(I_2 - I_1) + 2I_2 = 0}$$

Rearranging terms:

$$\boxed{\begin{aligned} 15I_1 - 10I_2 &= 20 \\ -10I_1 + 12I_2 &= -8 \end{aligned}}$$

Solving the last equations simultaneously results in $I_1 = 2$ A and $I_2 = 1$ A. The current in the center branch, shown dotted, is $I_1 - I_2 = 1$ A.

The currents do not have to be restricted to the windows in order to result in a valid set of simultaneous equations, although that is the usual case with the mesh current method. For example, where each of the currents passes through the source. In that problem they are called loop currents. The applicable rule is that each element in the network must have a current or a combination of currents and no two elements in different branches can be assigned the same current or the same combination of currents.

11.3 Matrices and determinants

The simultaneous equations of a three-mesh network can be written in matrix form:

$$\boxed{\begin{aligned} R_{11}\ I_1 + R_{12}\ I_2 + R_{13}\ I_3 &= V_1 \\ R_{21}\ I_1 + R_{22}\ I_2 + R_{23}\ I_3 &= V_2 \\ R_{31}\ I_1 + R_{32}\ I_2 + R_{33}\ I_3 &= V_3 \end{aligned}} \tag{14}$$

For three-mesh network, the elements of the matrices can be indicated in general form as follows:

$$\boxed{\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}} \tag{15}$$

The matrix equation arising from the mesh current method may be solved by various techniques.

One of these, the method of determinants (Cramer’s rule), will be presented here. It should be stated, however, that other techniques are far more efficient for large networks.

Solve matrix equation of figure above by the method of determinants.

The unknown current I_1 is obtained as the ratio of two determinants. The denominator determinant has the elements of resistance matrix. This may be referred to as the determinant of the coefficients and given the symbol Δ_R. The numerator determinant has the same elements as Δ_R except in the first column, where the elements of the voltage matrix replace those of the determinant of the coefficients. Thus,

$$\boxed{I_1 = \begin{vmatrix} V_1 & R_{12} & R_{13} \\ V_2 & R_{22} & R_{23} \\ V_3 & R_{32} & R_{33} \end{vmatrix} \Bigg/ \begin{vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{vmatrix} \equiv \frac{1}{\Delta_R} \begin{vmatrix} V_1 & R_{12} & R_{13} \\ V_2 & R_{22} & R_{23} \\ V_3 & R_{32} & R_{33} \end{vmatrix}} \tag{16}$$

Similarly,

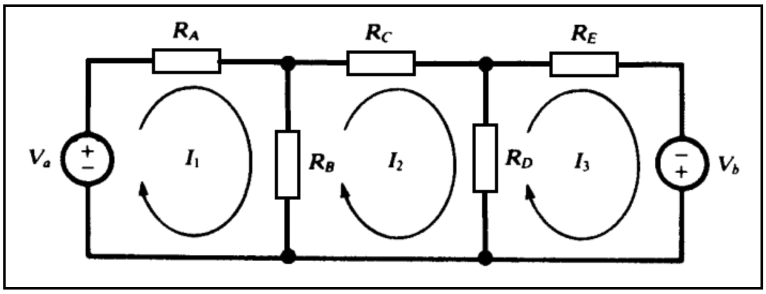
$$\boxed{I_2 = \frac{1}{\Delta_R} \begin{vmatrix} R_{11} & V_1 & R_{13} \\ R_{21} & V_2 & R_{23} \\ R_{31} & V_3 & R_{33} \end{vmatrix} \qquad I_3 = \frac{1}{\Delta_R} \begin{vmatrix} R_{11} & R_{12} & V_1 \\ R_{21} & R_{22} & V_2 \\ R_{31} & R_{32} & V_3 \end{vmatrix}} \tag{17}$$

An expansion of the numerator determinants by cofactors of the voltage terms results in a set of equations which can be helpful in understanding the network, particularly in terms of its driving-point and transfer resistances:

$$\boxed{\begin{aligned} I_1 &= V_1\left(\frac{\Delta_{11}}{\Delta_R}\right) + V_2\left(\frac{\Delta_{21}}{\Delta_R}\right) + V_3\left(\frac{\Delta_{31}}{\Delta_R}\right) \\ I_2 &= V_1\left(\frac{\Delta_{12}}{\Delta_R}\right) + V_2\left(\frac{\Delta_{22}}{\Delta_R}\right) + V_3\left(\frac{\Delta_{32}}{\Delta_R}\right) \\ I_3 &= V_1\left(\frac{\Delta_{13}}{\Delta_R}\right) + V_2\left(\frac{\Delta_{23}}{\Delta_R}\right) + V_3\left(\frac{\Delta_{33}}{\Delta_R}\right) \end{aligned}} \tag{18}$$

Here, Δ_{ij} stands for the cofactor of R_{ij} (the element in row i, column j) in Δ_R. Care must be taken with the signs of the cofactors.

Example: When KVL is applied to the three-mesh network of the following figure:

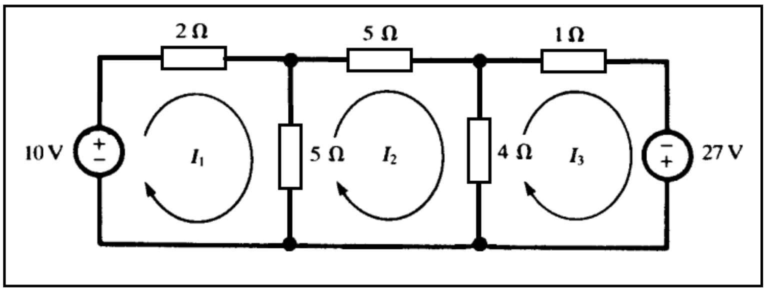


The following three equations are obtained:

$$\boxed{\begin{aligned} (R_A + R_B)I_1 \quad - R_BI_2 \quad &= V_a \\ -R_BI_1 + (R_B + R_C + R_D)I_2 \quad - R_DI_3 &= 0 \\ -R_DI_2 + (R_D + R_E)I_3 &= V_b \end{aligned}}$$

Placing the equations in matrix form,

$$\boxed{\begin{bmatrix} R_A + R_B & -R_B & 0 \\ -R_B & R_B + R_C + R_D & -R_D \\ 0 & -R_D & R_D + R_E \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_a \\ 0 \\ V_b \end{bmatrix}}$$



$$\begin{aligned} 7I_1 - 5I_2 + 0I_3 &= 10 \\ -5I_1 + 14I_2 - 4I_3 &= 0 \\ 0I_1 - 4I_2 + 5I_3 &= 27 \end{aligned}$$

$$\begin{bmatrix} 7 & -5 & 0 \\ -5 & 14 & -4 \\ 0 & -4 & 5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 27 \end{bmatrix}$$

$$I_1 = \begin{vmatrix} 10 & -5 & 0 \\ 0 & 14 & -4 \\ 27 & -4 & 5 \end{vmatrix} \Bigg/ \begin{vmatrix} 7 & -5 & 0 \\ -5 & 14 & -4 \\ 0 & -4 & 5 \end{vmatrix} = 3.55\,\text{A}$$

$$I_2 = \begin{vmatrix} 7 & 10 & 0 \\ -5 & 0 & -4 \\ 0 & 27 & 5 \end{vmatrix} \Bigg/ \begin{vmatrix} 7 & -5 & 0 \\ -5 & 14 & -4 \\ 0 & -4 & 5 \end{vmatrix} = -1.98\,\text{A}$$

$$I_3 = \begin{vmatrix} 10 & -5 & 10 \\ 0 & 14 & 0 \\ 27 & -4 & 27 \end{vmatrix} \Bigg/ \begin{vmatrix} 7 & -5 & 0 \\ -5 & 14 & -4 \\ 0 & -4 & 5 \end{vmatrix} = -2.98\,\text{A}$$

11.4 The node voltage method

The network shown in Fig. (16) contains five nodes, where 4 and 5 are simple nodes and 1, 2, and 3 are principal nodes. In the node voltage method, one of the principal nodes is selected as the reference and equations based on KCL are written at the other principal nodes. At each of these other principal nodes, a voltage is assigned, where it is understood that this is a voltage with respect to the reference node. These voltages are the unknowns and, when determined by a suitable method, result in the network solution.

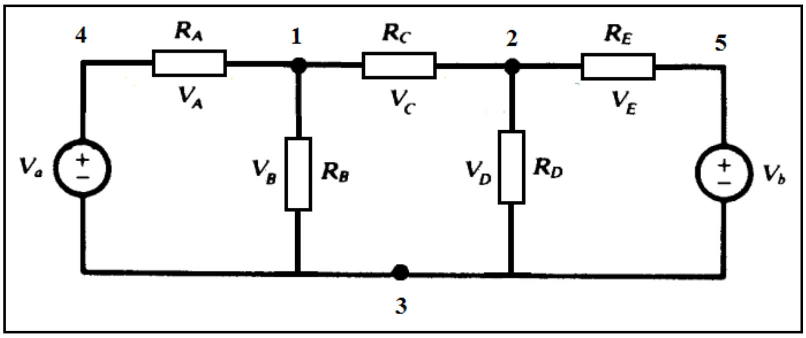


Fig. 16: The node voltage method

The network in Fig. (16) and node 3 selected as the reference for voltages V_B and V_D.

KCL requires that the total current out of node 1 be zero:

$$\boxed{\frac{V_B - V_a}{R_A} + \frac{V_B}{R_B} + \frac{V_B - V_D}{R_C} = 0} \qquad (19)$$

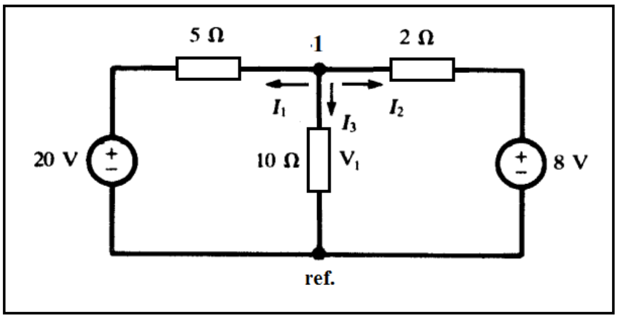
Similarly, the total current out of node 2 must be zero:

$$\boxed{\frac{V_D - V_B}{R_C} + \frac{V_D}{R_D} + \frac{V_D - V_b}{R_E} = 0} \qquad (20)$$

(Applying KCL in this form does not imply that the actual branch currents all are directed out of either node. Indeed, the current in branch 1–2 is necessarily directed out of one node and into the other.) Putting the two equations for V_B and V_D in matrix form:

$$\boxed{\begin{bmatrix} \frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C} & -\frac{1}{R_C} \\ -\frac{1}{R_C} & \frac{1}{R_C} + \frac{1}{R_D} + \frac{1}{R_E} \end{bmatrix} \begin{bmatrix} V_B \\ V_D \end{bmatrix} = \begin{bmatrix} V_a/R_A \\ V_b/R_E \end{bmatrix}} \qquad (21)$$

<u>Example:</u> Solve the circuit of the following figure using the node voltage method. In this circuit, with two principal nodes, only one equation is required. Assuming the currents are all directed out of the upper node and the bottom node is the reference,



$$\boxed{\frac{V_1 - 20}{5} + \frac{V_1}{10} + \frac{V_1 - 8}{2} = 0}$$

from which V_1 = 10 V. Then, I_1 = (10 – 20) / 5 = -2 A (the negative sign indicates that current I_1 flows into node 1); I_2 = (10 – 8) / 2 = 1 A; I_3 = 10 / 10 = 1 A.

11.5 Input and output resistances

In single-source networks, the input or driving-point resistance is often of interest. Such a network is suggested in Fig. (17), where the driving voltage has been designated as V_1 and the corresponding current as I_1. Since the only source is V_1, the equation for I_1 is:

$$\boxed{I_1 = \frac{V_1}{R_{\text{input}}}} \tag{22}$$

The input resistance is the ratio of V_1 to I_1:

$$\boxed{R_{\text{input}} = \frac{V_1}{I_1}} \tag{23}$$

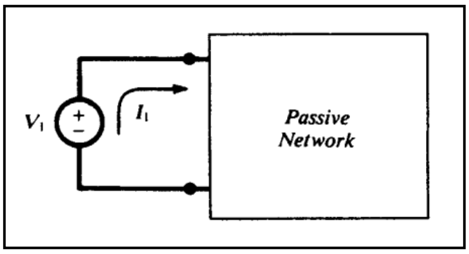
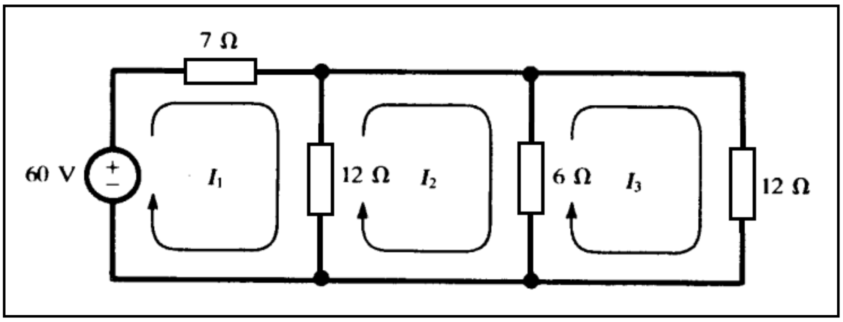


Fig. 17: The diagram to calculate the input resistance

Example: In the circuit of the following figure, obtain R_{input}:



Applying KVL to each mesh results in:

$$\begin{aligned} 60 &= 7I_1 + 12(I_1 - I_2) \\ 0 &= 12(I_2 - I_1) + 6(I_2 - I_3) \\ 0 &= 6(I_3 - I_2) + 12I_3 \end{aligned}$$

Rearranging terms and putting the equations in matrix form,

$$\begin{aligned} 19I_1 - 12I_2 \qquad &= 60 \\ -12I_1 + 18I_2 - \ 6I_3 &= 0 \\ -6I_2 + 18I_3 &= 0 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} 19 & -12 & 0 \\ -12 & 18 & -6 \\ 0 & -6 & 18 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 60 \\ 0 \\ 0 \end{bmatrix}$$

Using Cramer's rule to find I_1,

$$I_1 = \begin{vmatrix} 60 & -12 & 0 \\ 0 & 18 & -6 \\ 0 & -6 & 18 \end{vmatrix} \div \begin{vmatrix} 19 & -12 & 0 \\ -12 & 18 & -6 \\ 0 & -6 & 18 \end{vmatrix} = 17\,280 \div 2880 = 6\text{ A}$$

$$R_{\text{input}} = \frac{60\text{ V}}{6\text{ A}} = 10\ \Omega$$

Or

$$R_{\text{input}} = \frac{\Delta_R}{\Delta_{11}} = \frac{2880}{\begin{vmatrix} 18 & -6 \\ -6 & 18 \end{vmatrix}} = \frac{2880}{288} = 10\ \Omega$$

A voltage source applied to a passive network results in voltages between all nodes of the network. An external resistor connected between two nodes will draw current from the network and in general will reduce the voltage between those nodes. This is due to the voltage across the output resistance. The output resistance is found by dividing the open-circuited voltage to the short circuited current at the desired node (Fig. 18).

The output resistance is the ratio of V_{oc} to I_{sc}:

$$R_{\text{output}} = \frac{V_{oc}}{I_{sc}} \qquad (24)$$

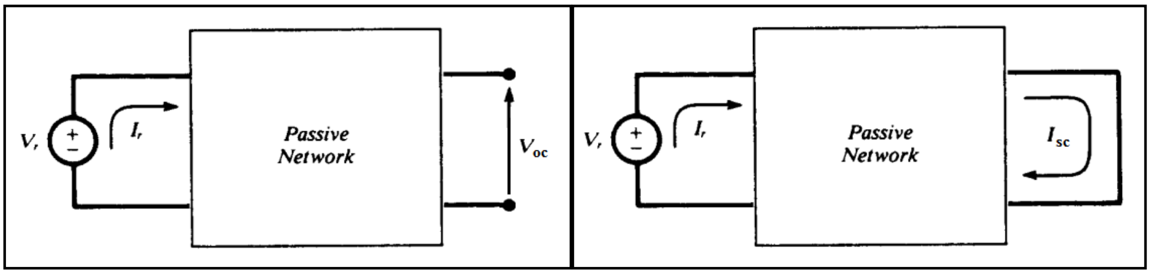
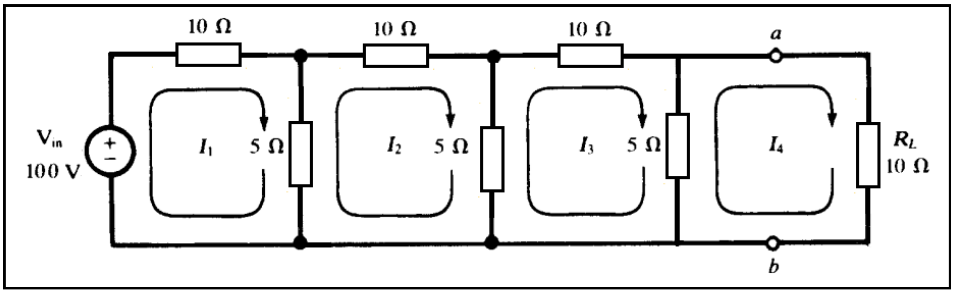
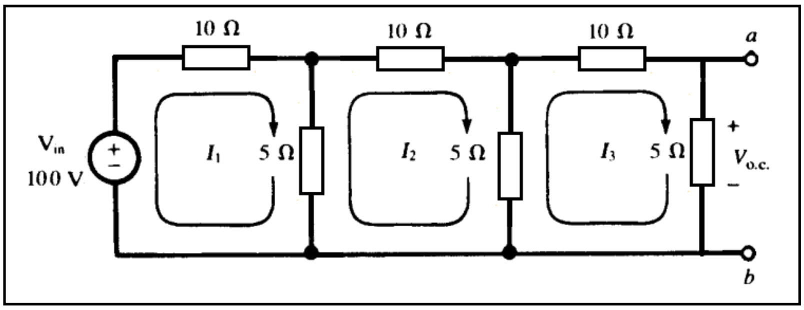


Fig. 18: The diagram to calculate the output resistance

<u>Example:</u> In the circuit of the following figure, obtain R_{output} (between a and b) as expressed by the ratio of V_{oc} to I_{sc}.



The open-circuit voltage V_{oc}: is the voltage across the 5 Ω resistor indicated in the following figure:



By inspection, the network equation is:

$$\begin{bmatrix} 15 & -5 & 0 \\ -5 & 20 & -5 \\ 0 & -5 & 20 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$$

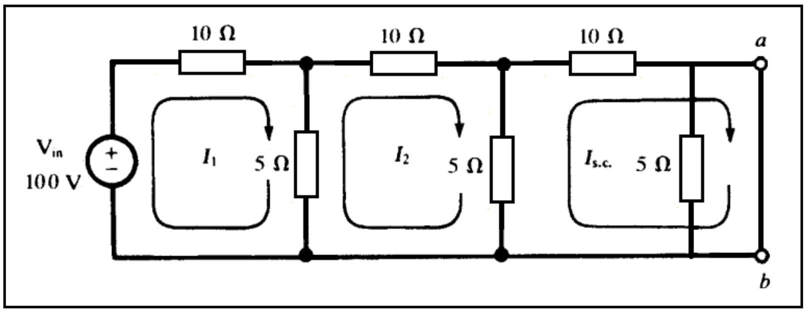
$$\Delta R = \begin{vmatrix} 15 & -5 & 0 \\ -5 & 20 & -5 \\ 0 & -5 & 20 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} 15 & -5 & 100 \\ -5 & 20 & 0 \\ 0 & -5 & 0 \end{vmatrix}$$

$$I_3 = \frac{\Delta_3}{\Delta_R} = 0.48\,(\text{A})$$

$$V_{\text{o.c.}} = I_3(5) = 0.48\,(5) = 2.4\text{ V}$$

The short-circuit current I_{sc} is obtained from the three-mesh circuit shown in the following figure:



By inspection, the network equation is:

$$\begin{bmatrix} 15 & -5 & 0 \\ -5 & 20 & -5 \\ 0 & -5 & 20 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_{\text{s.c.}} \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$$

$$\Delta R = \begin{vmatrix} 15 & -5 & 0 \\ -5 & 20 & -5 \\ 0 & -5 & 20 \end{vmatrix}$$

$$I_{\text{s.c.}} = \frac{100 \begin{vmatrix} -5 & 20 \\ 0 & -5 \end{vmatrix}}{\Delta_R} = 0.66\,(\text{A})$$

$$R_{\text{output}} = \frac{V_{\text{o.c.}}}{I_{\text{s.c.}}} = \frac{2.4\text{ V}}{0.66\text{ A}} = 3.64\ \Omega$$

11.6 Transfer resistance

A driving voltage in one part of a network results in currents in all the network branches. For example, a voltage source applied to a passive network results in an output current in that part of the network where a load resistance has been connected. In such a case the network has an overall transfer resistance. Consider the passive network suggested in Fig. (19), where the voltage source has been designated as V_r and the output current as I_s. The mesh current equation for I_s contains only one term, the one resulting from V_r in the numerator. The network transfer resistance is the ratio of V_r to I_s:

$$\boxed{R_{\text{Transfer}} = \frac{V_r}{I_s}} \qquad (26)$$

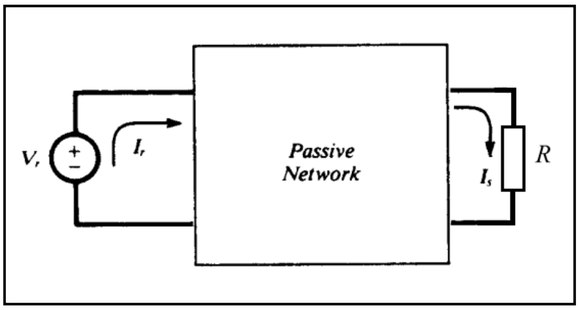
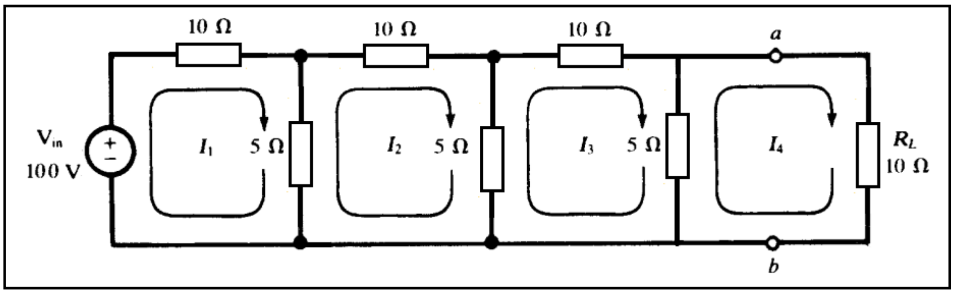


Fig. 19: The diagram to calculate the transfer resistance

<u>Example:</u> In the circuit of the following figure, obtain the transfer resistance ($R_{transfer}$) as expressed by the ratio of V_{in} to I_4.



By inspection, the network equation is:

$$\begin{bmatrix} 15 & -5 & 0 & 0 \\ -5 & 20 & -5 & 0 \\ 0 & -5 & 20 & -5 \\ 0 & 0 & -5 & 15 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Delta R = \begin{vmatrix} 15 & -5 & 0 & 0 \\ -5 & 20 & -5 & 0 \\ 0 & -5 & 20 & -5 \\ 0 & 0 & -5 & 15 \end{vmatrix} = 70000$$

$$\Delta_4 = \begin{vmatrix} 15 & -5 & 0 & 100 \\ -5 & 20 & -5 & 0 \\ 0 & -5 & 20 & 0 \\ 0 & 0 & -5 & 0 \end{vmatrix} = 12500$$

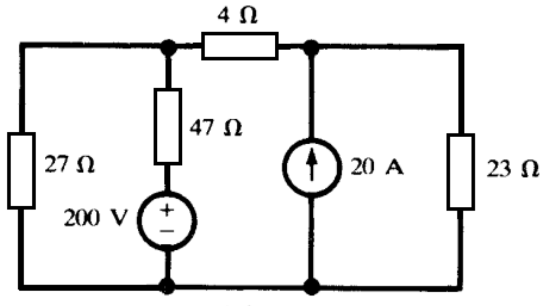
$$I_4 = \frac{\Delta_4}{\Delta_R} = \frac{12500}{70000} = 0.18 \text{ (A)}$$

$$R_{\text{transfer}} = \frac{V_{\text{in}}}{I_4} = \frac{100}{0.18} = 555.55 \ (\Omega)$$

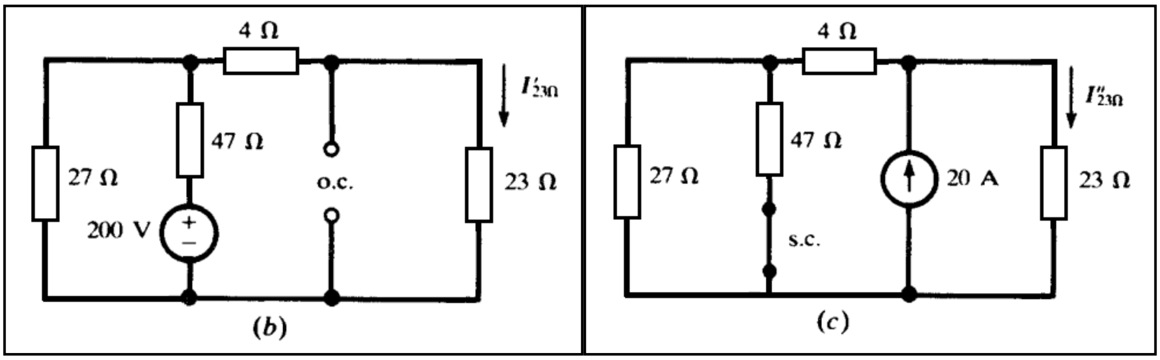
11.7 Superposition

A linear network which contains two or more independent sources can be analyzed to obtain the various voltages and branch currents by allowing the sources to act one at a time, then superposing the results. This principle applies because of the linear relationship between current and voltage. With dependent sources, superposition can be used only when the control functions are external to the network containing the sources, so that the controls are unchanged as the sources act one at a time. Voltage sources to be suppressed while a single source acts are replaced by short circuits; current sources are replaced by open circuits. Superposition cannot be directly applied to the computation of power, because power in an element is proportional to the square of the current or the square of the voltage, which is nonlinear.

Example: Compute the current in the 23Ω- resistor of Fig. (a) by applying the superposition principle. With the 200-V source acting alone, the 20-A current source is replaced by an open circuit, Fig. (b)



(a)



$$R_{eq} = 47 + \frac{(27)(4+23)}{54} = 60.5\,\Omega$$
$$I_T = \frac{200}{60.5} = 3.31\text{ A}$$
$$I'_{23\Omega} = \left(\frac{27}{54}\right)(3.31) = 1.65\text{ A}$$

When the 20-A source acts alone, the 200-V source is replaced by a short circuit, Fig. (c). The equivalent resistance to the left of the source is:

$$R_{eq} = 4 + \frac{(27)(47)}{74} = 21.15\,\Omega$$

Then

$$I''_{23\Omega} = \left(\frac{21.15}{21.15+23}\right)(20) = 9.58\text{ A}$$

The total current in the 23- resistor is:

$$I_{23\Omega} = I'_{23\Omega} + I''_{23\Omega} = 11.23\text{ A}$$

11.8 Thevenin's and Norton's theorems

A linear, active, resistive network which contains one or more voltage or current sources can be replaced by a single voltage source and a series resistance (Thevenin's theorem), or by a single current source and a parallel resistance (Norton's theorem). The voltage is called the Thevenin equivalent voltage, V_0, and the current the Norton equivalent current, I_0. The two resistances are the same, R_0. When terminals ab in Fig. (20 a) are open-circuited, a voltage will appear between them.

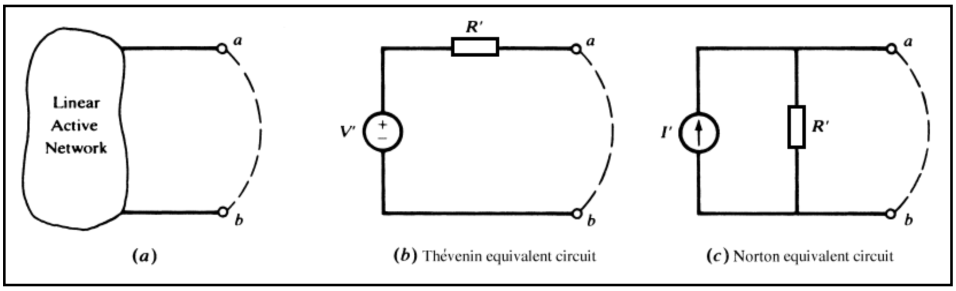
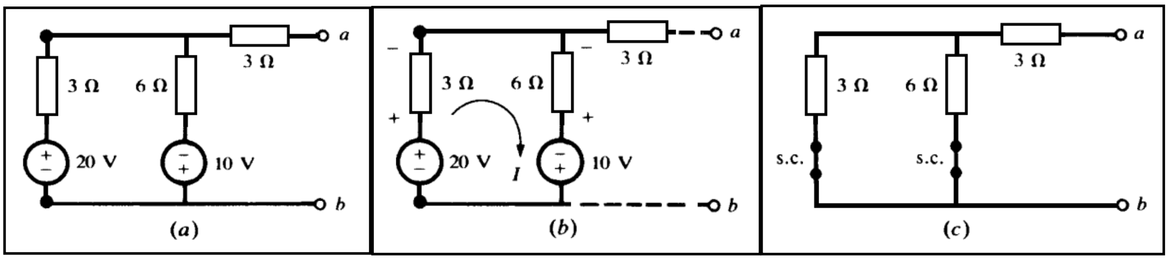


Fig. 20: The diagram to transform a linear active network by Thevenin's and Norton's theorems

From Fig. (20 b) it is evident that this must be the voltage V' of the Thevenin equivalent circuit. If a short circuit is applied to the terminals, as suggested by the dashed line in Fig. (20 a), a current will result. From Fig. (20 c) it is evident that this current must be I' of the Norton equivalent circuit. Now, if the circuits in (b) and (c) are equivalents of the same active network, they are equivalent to each other. It follows that I' = V' / R'. If both V' and I' have been determined from the active network, then R' = V'/ I'.

Example: Obtain the Thevenin and Norton equivalent circuits for the active network in Fig. (a). With terminals ab open, the two sources drive a clockwise current through the 3Ω and 6Ω resistors Fig. (b).



$$I = \frac{20+10}{3+6} = \frac{30}{9}\ \text{A}$$

Since no current passes through the upper right 3Ω resistor, the Thevenin voltage can be taken from either active branch:

$$V_{ab} = V' = 20 - \left(\frac{30}{9}\right)(3) = 10\,\text{V}$$

$$V_{ab} = V' = \left(\frac{30}{9}\right)6 - 10 = 10\,\text{V}$$

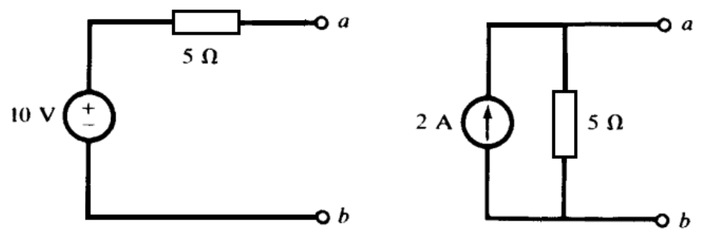
The resistance R' can be obtained by shorting out the voltage sources Fig. (c) and finding the equivalent resistance of this network at terminals ab:

$$\boxed{R' = 3 + \frac{(3)(6)}{9} = 5\,\Omega}$$

When a short circuit is applied to the terminals, current I_{sc} results from the two sources. Assuming that it runs through the short from a to b, we have, by superposition,

$$\boxed{I_{\text{s.c.}} = I' = \left(\frac{6}{6+3}\right)\left[\frac{20}{3+\frac{(3)(6)}{9}}\right] - \left(\frac{3}{3+3}\right)\left[\frac{10}{6+\frac{(3)(3)}{6}}\right] = 2\,\text{A}}$$

The following figure shows the two equivalent circuits. In the present case, V', R', and I' were obtained independently. Since they are related by Ohm's law, any two may be used to obtain the third.



(a) Thévenin Equivalent (b) Norton Equivalent

The usefulness of Thevenin and Norton equivalent circuits is clear when an active network is to be examined under a number of load conditions, each represented by a resistor.

11.9 Maximum power transfer theorem

At times it is desired to obtain the maximum power transfer from an active network to an external load resistor R_L. Assuming that the network is linear, it can be reduced to an equivalent circuit as in Fig. (21). Then:

$$\boxed{I = \frac{V'}{R' + R_L}} \tag{27}$$

and so the power absorbed by the load is:

$$\boxed{P_L = \frac{V'^2 R_L}{(R' + R_L)^2} = \frac{V'^2}{4R'}\left[1 - \left(\frac{R' - R_L}{R' + R_L}\right)^2\right]} \tag{28}$$

It is seen that P_L attains its maximum value, V' =4R', when R_L = R', in which case the power in R' is also V'^2 = 4R'. Consequently, when the power transferred is a maximum, the efficiency is 50 percent. It is noted that the condition for maximum power transfer to the load is not the same as the condition for maximum power delivered by the source. The latter happens when R_L = 0, in which case power delivered to the load is zero (i.e., at a minimum).

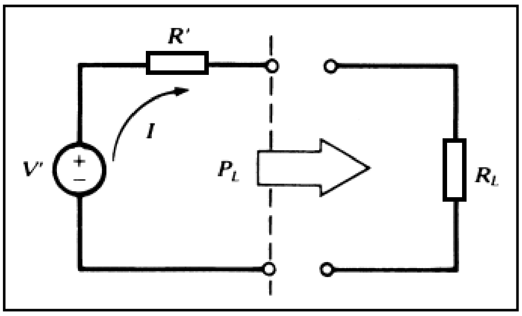
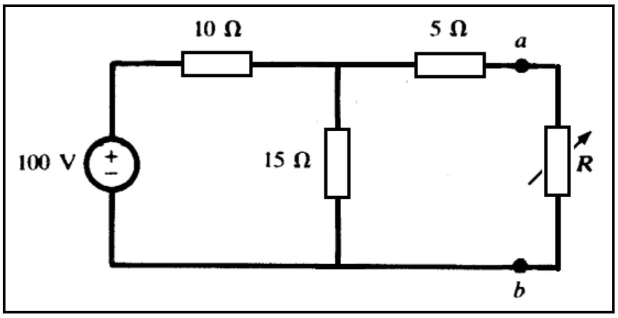


Fig. 21: The diagram of maximum power transfer

Example: Find the value of the adjustable resistance R which results in maximum power transfer across the terminals ab of the circuit shown in the following figure:



First a Thevenin equivalent is obtained, with V' = 60 V and R' = 11 Ω. Maximum power transfer occurs for R = R' = 11 Ω, with:

$$\boxed{P_{\max} = \frac{V'^2}{4R'} = 81.82 \text{ W}}$$

Part II. Variable electric regime

1. Introduction

The voltages and currents in electric circuits are described by four classes of time functions:

- Transitional functions;
- Periodic functions;

- Non-periodic functions;
- Random functions.

2. Transitional functions

Whenever a circuit is switched from one condition to another, either by a change in the applied source or a change in the circuit elements, there is a transitional period during which the branch currents and element voltages change from their former values to new ones. This period is called the transient.

After the transient has passed, the circuit is said to be in the steady state. Now, the linear differential equation that describes the circuit will have two parts to its solution, the complementary function (or the homogeneous solution) and the particular solution. The complementary function corresponds to the transient, and the particular solution to the steady state.

3. Periodic functions

A signal v(t) is periodic with period T for all (t) if:

$$\boxed{v(t) = v(t+T)} \qquad (01)$$

Four types of periodic functions which are specified for one period T and their corresponding graphs are as follows:

3.1 Sine wave:

$$\boxed{v_1(t) = V_0 \sin 2\pi t/T} \qquad (02)$$

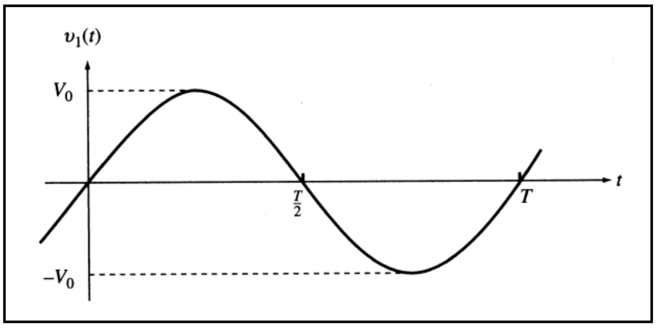


Fig. 01: Sine wave

3.2 Periodic pulse:

$$v_2(t) = \begin{cases} V_1 & \text{for } 0 < t < T_1 \\ -V_2 & \text{for } T_1 < t < T \end{cases} \tag{03}$$

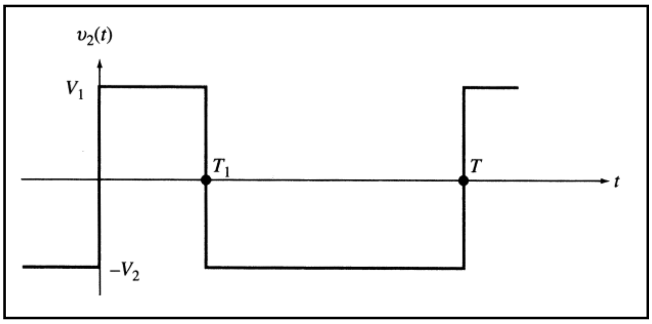


Fig. 02: Periodic pulse

3.3 Periodic tone burst:

$$v_3(t) = \begin{cases} V_0 \sin 2\pi t/\Lambda & \text{for } 0 < t < T_1 \\ 0 & \text{for } T_1 < t < T \end{cases} \tag{04}$$

Where T = k Λ and k is an integer.

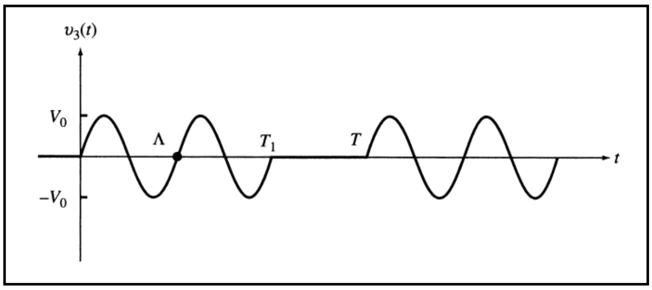


Fig. 03: Periodic tone burst

3.4 Repetition of a recording every T seconds:

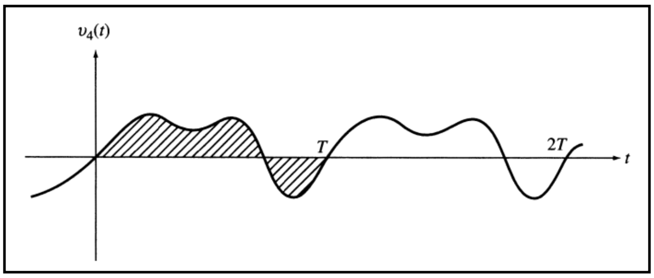


Fig. 04: Repetition of a recording every T seconds

3.5 Review of the sinusoidal function

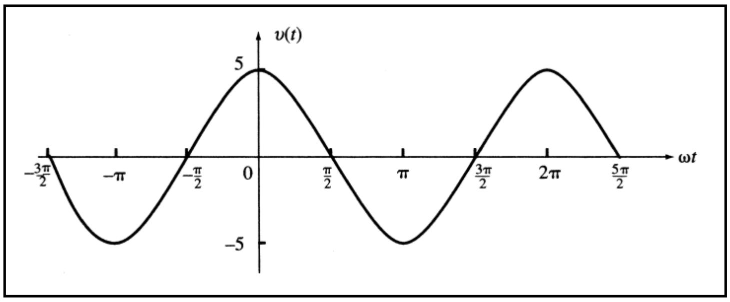
A sinusoidal voltage v(t) is given by:

$$\boxed{v(t) = V_0 \cos(\omega t + \theta)} \qquad (05)$$

where V_0 is the amplitude, ω is the angular velocity, or angular frequency, and θ is the phase angle. The angular velocity ω may be expressed in terms of the period T or the frequency f, where f ≡ 1/T. The frequency is given in hertz, Hz, or cycles/s. Since cos ω t = cos (ωt + 2π), ω and T are related by ω T = 2π. And since it takes T seconds for v(t) to return to its original value, it goes through 1/T cycles in 1 second. In summary, for sinusoidal functions we have:

$$\boxed{\omega = 2\pi/T = 2\pi f \qquad f = 1/T = \omega/2\pi \qquad T = 1/f = 2\pi/\omega} \qquad (06)$$

<u>Example:</u> Plot v(t) = 5cos ω t versus ω t.



3.5.1 Time shift and phase shift

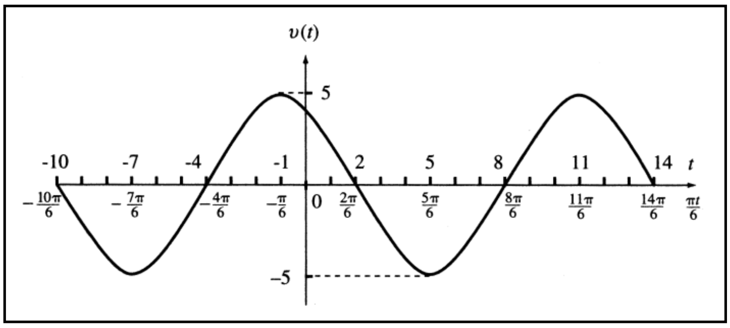
If the function v(t) = cos ω t is delayed by τ seconds, we get v (t − τ) = cos ω (t − τ) = cos (ω t − θ), where θ = ω τ. The delay shifts the graph of v(t) to the right by an amount of t seconds, which corresponds to a phase lag of θ = ω τ = 2πf τ. A time shift of τ seconds to the left on the graph produces v (t+ τ), resulting in a leading phase angle called an advance. Conversely, a phase shift of θ corresponds to a time shift of τ. Therefore, for a given phase shift, the higher the frequency, the smaller the required time shift.

<u>Example :</u> Plot v(t) = 5cos (πt/6 + 30°) versus t.

Rewrite the expression as :

$$\boxed{v(t) = 5\cos(\pi t/6 + \pi/6) = 5\cos[\pi(t+1)/6]}$$

This is a cosine function with a period of 12 s, which is advanced in time by 1 s. In other words, the graph is shifted to the left by 1 s or 30° as shown in the figure below:



3.5.2 The average and effective (RMS) values

A periodic function f(t), with a period T, has an average value F_{avg} given by:

$$\boxed{F_{\text{avg}} = \langle f(t) \rangle = \frac{1}{T}\int_{0}^{T} f(t)dt = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)dt} \tag{07}$$

The root-mean-square (rms) or effective value of f (t) during the same period is defined by:

$$\boxed{F_{\text{eff}} = F_{\text{rms}} = \left[\frac{1}{T}\int_{t_0}^{t_0+T} f^2(t)dt\right]^{1/2}} \tag{08}$$

It is seen that:

$$\boxed{F_{\text{eff}}^2 = \langle f^2(t) \rangle} \tag{09}$$

Average and effective values of periodic functions are normally computed over one period.

<u>Example:</u> Find the average and effective values of the cosine wave:

$$\boxed{v(t) = V_m \cos(\omega t + \theta)}$$

$$V_{avg} = \frac{1}{T}\int_0^T V_m \cos(\omega t + \theta)dt = \frac{V_m}{\omega T}[\sin(\omega t + \theta)]_0^T = 0$$

$$V_{eff}^2 = \frac{1}{T}\int_0^T V_m^2 \cos^2(\omega t + \theta)dt = \frac{1}{2T}\int_0^T V_m^2[1 + \cos 2(\omega t + \theta)]dt = V_m^2/2$$

$$V_{eff} = V_m/\sqrt{2} = 0.707\, V_m$$

The last equations show that the results are independent of the frequency and phase angle θ. In other words, the average of a cosine waves and its rms value are always 0 and 0.707 V_m, respectively.

$$v(t) = \begin{cases} V_m \sin\omega t & \text{when } \sin\omega t > 0 \\ 0 & \text{when } \sin\omega t < 0 \end{cases}$$

$$V_{avg} = \frac{1}{T}\int_0^{T/2} V_m \sin\omega t\, dt = \frac{V_m}{\omega T}[-\cos\omega t]_0^{T/2} = V_m/\pi$$

$$V_{eff}^2 = \frac{1}{T}\int_0^{T/2} V_m^2 \sin^2\omega t\, dt = \frac{1}{2T}\int_0^{T/2} V_m^2(1 - \cos 2\omega t)dt = V_m^2/4\ ,\quad V_{eff} = V_m/2$$

4. Nonperiodic Functions

A nonperiodic function cannot be specified for all times by simply knowing a finite segment. Examples of nonperiodic functions are:

$$v_1(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

$$v_2(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1/T & \text{for } 0 < t < T \\ 0 & \text{for } t > T \end{cases}$$

$$v_3(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-t/\tau} & \text{for } t > 0 \end{cases}$$

$$v_4(t) = \begin{cases} 0 & \text{for } t < 0 \\ \sin\omega t & \text{for } t > 0 \end{cases}$$

$$v_5(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-t/\tau}\cos\omega t & \text{for } t > 0 \end{cases}$$

$$v_6(t) = e^{-t/\tau} \quad \text{for all } t$$

$$v_7(t) = e^{-a|t|} \quad \text{for all } t$$

$$v_8(t) = e^{-a|t|}\cos\omega t \quad \text{for all } t$$

4.1 The Unit step function

The dimensionless unit step function, is defined by:

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases} \tag{10}$$

The function is graphed in Fig. (05). Note that the function is undefined at t = 0.

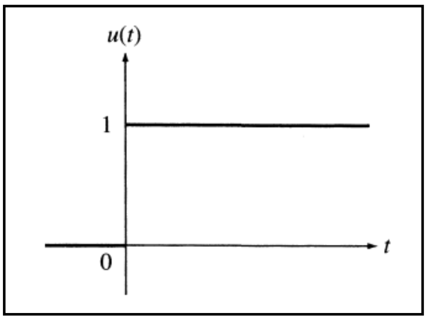


Fig. 05: The Unit step function

4.2 The unit impulse function

Consider the function $s_T(t)$ of Fig. (6 a), which is zero for t < 0 and increases uniformly from 0 to 1 in T seconds. Its derivative $d_T(t)$ is a pulse of duration T and height 1/T, as seen in Fig. (6 b).

$$d_T(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1/T & \text{for } 0 < t < T \\ 0 & \text{for } t > T \end{cases} \tag{11}$$

If the transition time T is reduced, the pulse in Fig. (6 b) becomes narrower and taller, but the area under the pulse remains equal to 1. If we let T approach zero, in the limit, the function s_T (t) becomes a unit step u(t) and its derivative d_T (t) becomes a unit pulse δ (t) with zero width and infinite height. The unit impulse δ (t) is shown in Fig. (6 c). The unit impulse or unit delta function is defined by: An impulse which is the limit of a narrow pulse with an area A is expressed by A δ (t). The magnitude δ is sometimes called the strength of the impulse. A unit impulse which occurs at $t = t_0$ is expressed by δ (t - t_0).

$$\delta(t) = 0 \qquad \text{for } t \neq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{12}$$

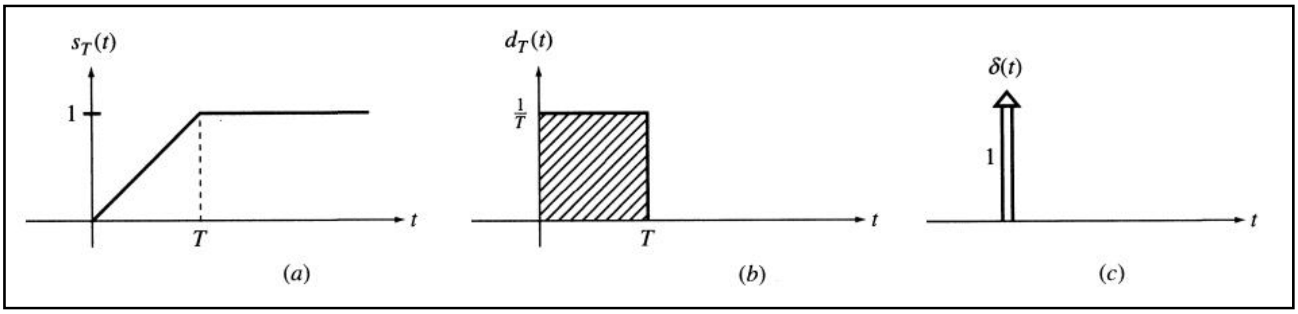


Fig. 06: The unit impulse function and unit delta function

<u>Example:</u> The voltage across the terminals of a 100 nF capacitor grows linearly, from 0 to 10 V, taking the shape of the function $s_T(t)$ in Fig. (6 a). Find:

- The charge across the capacitor at t = T.
- The current $i_C(t)$ in the capacitor for T = 1 s.

At t = T, v_C = 10 V. The charge across the capacitor is:

$$\boxed{Q = Cv_C = 10^{-7} \times 10 = 10^{-6}\ \text{C}}$$

$$\boxed{i_c(t) = C\frac{dv_C}{dt}}$$

$$\boxed{i_C(t) = \begin{cases} 0 & \text{for } t < 0 \\ I_0 = 10^{-6}/T\,(\text{A}) & \text{for } 0 < t < T \\ 0 & \text{for } t > T \end{cases}}$$

For T = 1s, $I_0 = 10^{-6}$A; for T = 1ms, $I_0 = 10^{-3}$A; and for T = 1μs, I_0 = 1A.

In all the preceding cases, the charge accumulated across the capacitor at the end of the transition period is:

$$\boxed{Q = \int_0^T i_C(t)\,dt = I_0 T = 10^{-6}\ \text{C}}$$

The amount of charge at t = T is independent of T. It generates a voltage v_C = 10 V across the capacitor.

4.3 The exponential function

The function f (t) = e^{st}, with s a complex constant, is called an exponential. It decays with time if the real part of s is negative and grows if the real part of s is positive. We will discuss exponentials e^{at} in which the constant a is a real number.

The inverse of the constant (a) has the dimension of time and is called the time constant $\tau = 1/a$. A decaying exponential $e^{-t/\tau}$ is plotted versus t as shown in Fig. (07). The function decays from one at $t = 0$ to zero at $t = \infty$.

After τ seconds, the function $e^{-t/\tau}$ is reduced to $e^{-1} = 0.368$. For $\tau = 1$, the function e^{-t} is called a normalized exponential which is the same as $e^{-t/\tau}$ when plotted versus t/τ.

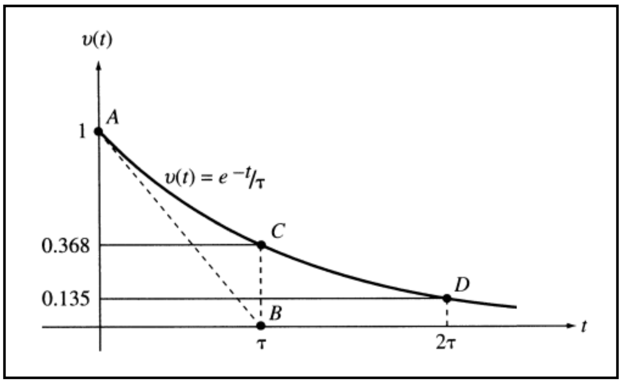


Fig. 07: The exponential function

4.4 The damped sinusoids

A damped sinusoid, with its amplitude decaying exponentially, has the form:

$$\boxed{v(t) = Ae^{-at}\cos(\omega t + \theta)} \qquad (13)$$

<u>Example:</u> The current $i = I_0 e^{-at}\cos\omega t$ passes through a series RL circuit. (a) Find v_{RL}, the voltage across this combination. (b) Compute v_{RL} for $I_0 = 3$ A, $a = 2$, $\omega = 40$ rad/s, $R = 5\ \Omega$ and $L = 0.1$ H. Sketch i as a function of time.

$$v_R = Ri = RI_0 e^{-at}\cos\omega t$$

$$v_L = L\frac{di}{dt} = -LI_0 e^{-at}(a\cos\omega t + \omega\sin\omega t)$$

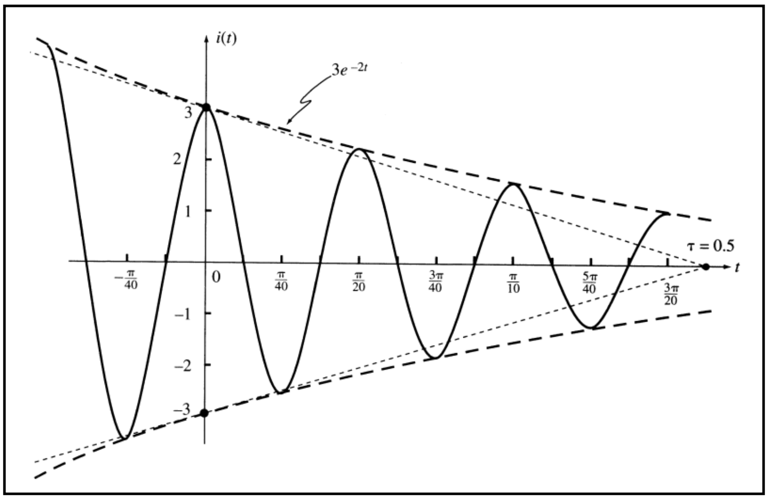
$$v_{RL} = v_R + v_L = I_0 e^{-at}[(R - La)\cos\omega t - L\omega\sin\omega t] = V_0 e^{-at}\cos(\omega t + \theta)$$

$$V_0 = I_0\sqrt{(R - La)^2 + L^2\omega^2} \qquad \text{and} \qquad \theta = \tan^{-1}[L\omega/(R - La)]$$

$$V_0 = I_0\sqrt{(R - La)^2 + L^2\omega^2}\ , \quad \theta = \tan^{-1}[L\omega/(R - La)]$$

$$V_0 = 18.75\ \text{V} \quad , \qquad \theta = 39.8°$$

$$i = 3e^{-2t}\cos 40t\ , \quad v_{RL} = 18.75e^{-2t}\cos(40t + 39.8°)$$



5. The random signals

So far we have dealt with signals which are completely specified. For example, the values of a sinusoidal waveform, such as the line voltage, can be determined for all times if its amplitude, frequency, and phase are known. Such signals are called deterministic.

There exists another class of signals which can be specified only partly through certain statistical measures such as their means, rms values, and frequency ranges. These are called random signals. Random signals can carry information and should not be mistaken with noise, which normally corrupts the information contents of the signal.

Part III. Sinusoidal electric regime

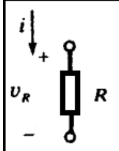
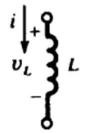
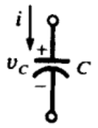
1. Alternating current circuits analysis

1.1 Voltage-current relations for the elements R, L, and C

Three passive dipoles are commonly used in electrical circuits. They have the particularity of having an operation which is expressed in the form of a simple, linear differential equation, with constant coefficients.

The operating equation of a dipole relates the voltage across its terminals and the current flowing through it. Assuming that, in the most general case, these two quantities are variable over time. the operating laws of the three usual passive dipoles are presented in Tab. (01).

Tab. 01: The voltage-current relationships for the single elements R, L, and C

	$i = I\cos\omega t$	$v = V\cos\omega t$
	$v_r = RI\cos\omega t$	$i_R = \dfrac{V}{R}\cos\omega t$
	$v_L = \omega LI\cos(\omega t + 90°)$	$i_L = \dfrac{V}{\omega L}\cos(\omega t - 90°)$
	$v_C = \dfrac{I}{\omega C}\cos(\omega t - 90°)$	$i_C = \omega CV\cos(\omega t + 90°)$

In Tab. (01) the responses of the three basic circuit elements are shown for applied current $i = I\cos\omega t$ and voltage $u = V\cos\omega t$. If sketches are made of these responses, they will show that for a resistance R, v and i are in phase. For an inductance L, i lags v by 90° or $\pi/2$ rad. And for a capacitance C, i leads v by 90° or $\pi/2$ rad.

The functions of v and i will be sines or cosines with the argument ωt. ω is the angular frequency and has the unit rad/s. Also, $\omega = 2\pi f$, where f is the frequency with unit cycle/s, or more commonly, hertz (Hz).

- Consider a resistance R with:

$$\boxed{i = I.\sin\omega t \ \text{(A)}} \tag{13}$$

The voltage is:

$$\boxed{v = Ri = R\,I.\sin\omega t \ \text{(V)}} \tag{14}$$

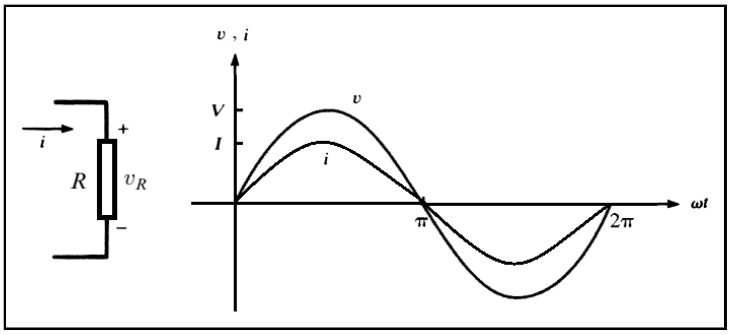


Fig. 08: The response of resistance R circuit driven by sinusoidal source

- Consider an inductance L with:

$$\boxed{i = I\cos\omega t} \tag{15}$$

The voltage is:

$$\boxed{v_L = L\frac{di}{dt} = \omega LI[-\sin\omega t] = \omega LI\cos(\omega t + 90°)} \quad (16)$$

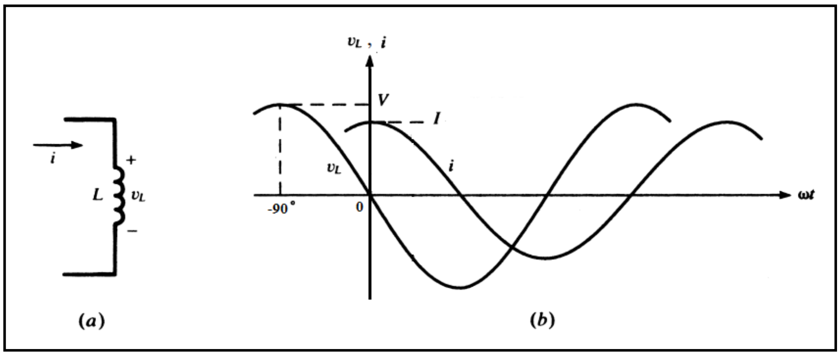


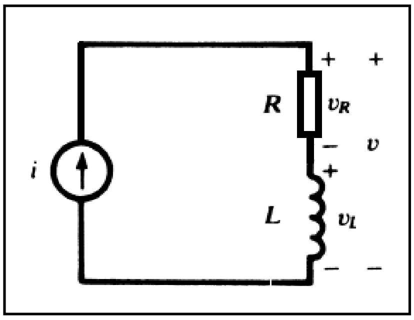
Fig. 09: The response of inductance L circuit driven by sinusoidal source

A comparison of v_L and i shows that the current lags the voltage by 90° or π/2 rad. The functions are sketched in Fig. (08 b). Note that the current function i is to the right of v, and since the horizontal scale is ω t, events displaced to the right occur later in time. This illustrates that i lags v. The horizontal scale is in radians, but note that it is also marked in degrees (− 90°, etc.). This is a case of mixed units just as with ω t. It is not mathematically correct but is the accepted practice in circuit analysis. The vertical scale indicates two different quantities, that is, v and i, so there should be two scales rather than one.

<u>Example:</u> The RL series circuit shown in figure below has a current i = I sin ω t. Obtain the voltage v across the two circuit elements and sketch v and i.

$$\boxed{v_R = RI\sin\omega t \qquad v_L = L\frac{di}{dt} = \omega LI\ \sin(\omega t + 90°)}$$

$$\boxed{v = v_R + v_L = RI\sin\omega t + \omega LI\ \sin(\omega t + 90°)}$$



Since the current is a sine function and:

$$\boxed{v = V \sin(\omega t + \theta) = V \sin \omega t \cos \theta + V \cos \omega t \sin \theta}$$

We have from the above:

$$\boxed{v = RI \sin \omega t + \omega LI \sin \omega t \cos 90° + \omega LI \cos \omega t \sin 90°}$$

Equating coefficients of like terms in the two above equations:

$$\boxed{V \sin \theta = \omega LI \qquad \text{and} \qquad V \cos \theta = RI}$$

Then

$$\boxed{v = I\sqrt{R^2 + (\omega L)^2}\, \sin[\omega t + \arctan(\omega L/R)]}$$

And

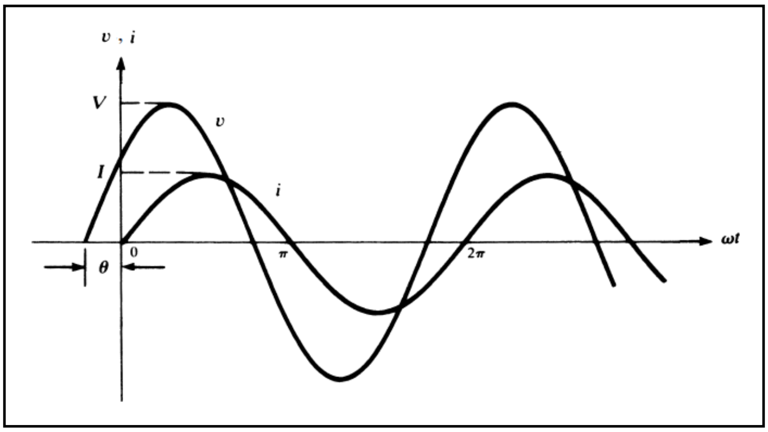
$$\boxed{V = I\sqrt{R^2 + (\omega L)^2} \qquad \text{and} \qquad \theta = \tan^{-1}\frac{\omega L}{R}}$$

The functions i and v are sketched in Fig. below. The phase angle θ, the angle by which i lags v, lies within the range $0° \leq \theta \leq 90°$, with the limiting values attained for $\omega L \ll R$ and $\omega L \gg R$, respectively. If the circuit had an applied voltage $v = V \sin \omega t$, the resulting current would be:

$$\boxed{i = \frac{V}{\sqrt{R^2 + (\omega L)^2}} \sin(\omega t - \theta)}$$

where, as before,

$$\boxed{\theta = \tan^{-1}(\omega L/R)}$$



- Consider a capacitance C with:

$$\boxed{i = I \sin \omega t} \tag{17}$$

The voltage is:

$$\boxed{v_C = (1/\omega C)\, I \sin(\omega t - 90°)} \tag{18}$$

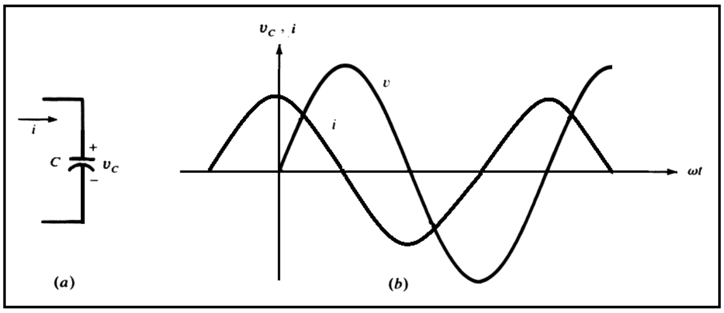
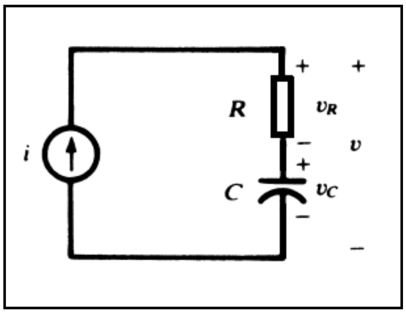


Fig. 10: The response of capacitance C circuit driven by sinusoidal source

<u>Example:</u> If the current driving a series RC circuit is given by i = I sin ω t, obtain the total voltage across the two elements.



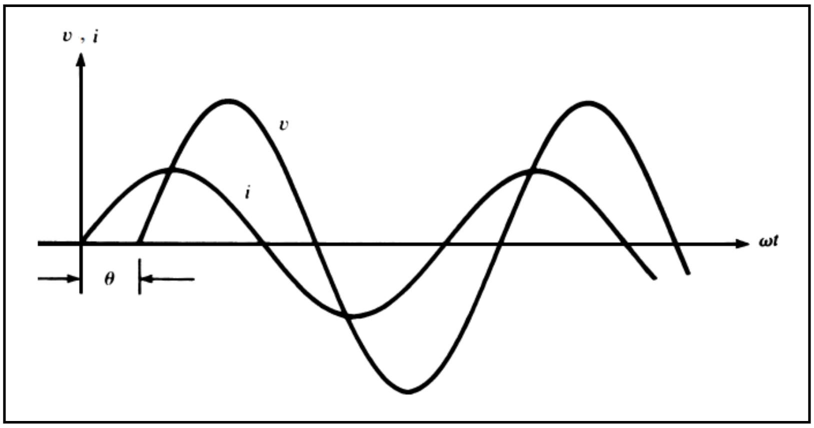
$$\boxed{v_R = RI \sin \omega t \qquad v_C = (1/\omega C)I \sin(\omega t - 90°)}$$

$$\boxed{v = v_R + v_C = V \sin(\omega t - \theta)}$$

Where

$$\boxed{V = I\sqrt{R^2 + (1/\omega C)^2} \qquad \text{and} \qquad \theta = \tan^{-1}(1/\omega CR)}$$

The negative phase angle shifts v to the right of the current i. Consequently, i leads v for a series RC circuit. The phase angle is constrained to the range $0° \le \theta \le 90°$. For $(1/\omega C) \ll R$, the angle $\theta = 0°$, and for $(1/\omega C) \gg R$, the angle $\theta = 90°$.



1.2 Power and energy consumed by a resistance R, and stored in an inductance L and a capacitance C

All electrical devices that consume energy must have a resistor (resistance) in their circuit model. Inductors and capacitors may store energy but over time return that energy to the source or to another circuit element.

1.2.1 Power and energy consumed by a resistance R

Power in the resistor:

$$\boxed{p = vi = i^2 R = v^2/R} \tag{19}$$

Energy is then determined as the integral of the instantaneous power:

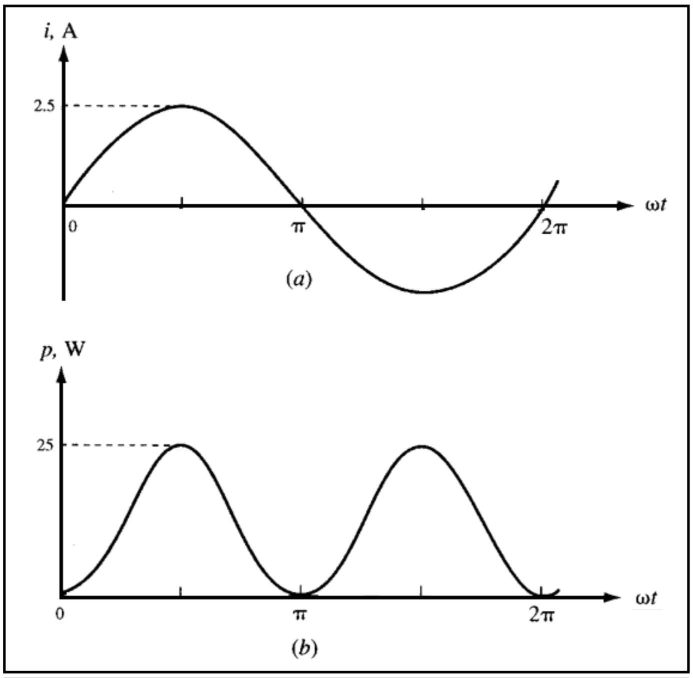
$$\boxed{w = \int_{t_1}^{t_2} p\,dt = R\int_{t_1}^{t_2} i^2\,dt = \frac{1}{R}\int_{t_1}^{t_2} v^2\,dt} \tag{20}$$

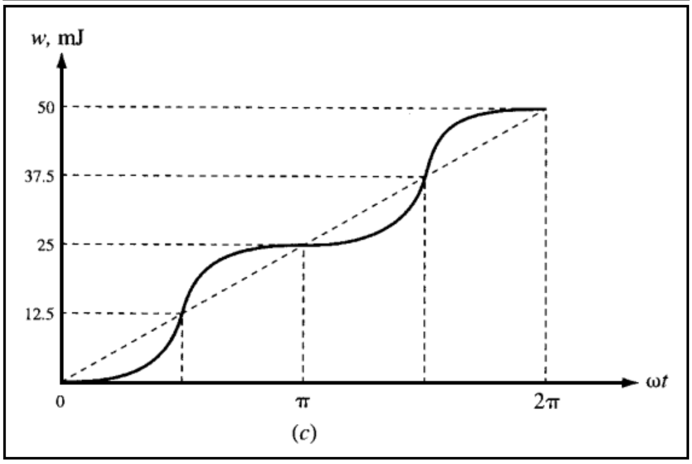
<u>**Example:**</u> A 4.0 Ω resistor has a current i = 2.5 sin ωt (A). Find the voltage, power, and energy over one cycle. Ω = 500 rad/s.

$$\boxed{v = Ri = 10.0 \sin \omega t \text{ (V)}}$$

$$\boxed{p = vi = i^2 R = 25.0 \sin^2 \omega t \text{ (W)}}$$

$$\boxed{w = \int_0^t p\, dt = 25.0\left[\frac{t}{2} - \frac{\sin 2\omega t}{4\omega}\right] \text{ (J)}}$$





1.2.2 Power and energy stored in an inductance L

The power and energy relationships for an inductance are as follows:

$$\boxed{p = vi = L\,\frac{di}{dt}\, i = \frac{d}{dt}\left[\frac{1}{2}\, Li^2\right]} \tag{21}$$

$$\boxed{w_L = \int_{t_1}^{t_2} p\,dt = \int_{t_1}^{t_2} Li\,dt = \frac{1}{2}\,L[i_2^2 - i_1^2]}\qquad(22)$$

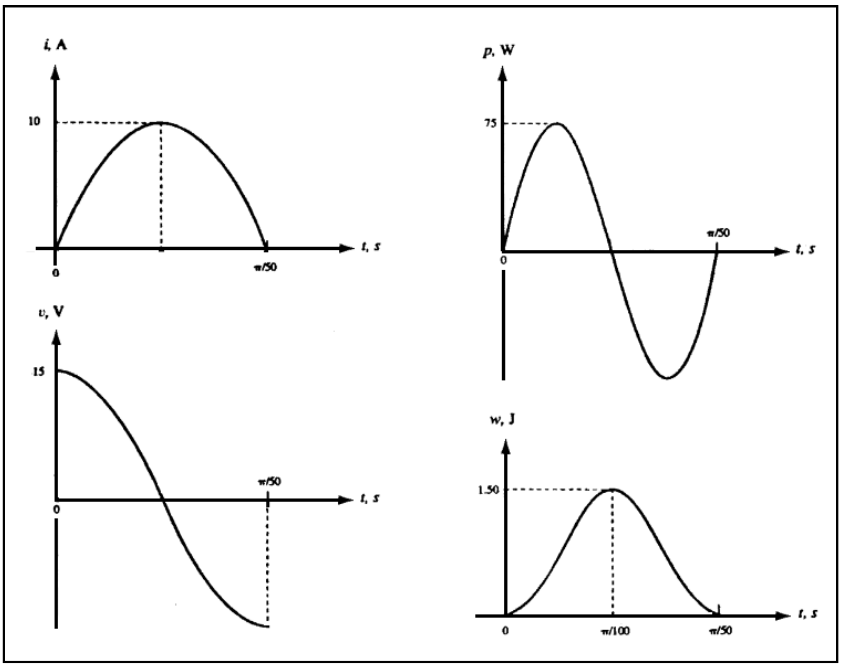
Energy stored in an inductance is:

$$\boxed{w_L = \tfrac{1}{2}Li^2}\qquad(23)$$

Example: In the interval 0 > t > (π / 50) s a 30-mH inductance has a current i = 10:0 sin 50t (A). Obtain the voltage, power, and energy for the inductance.

$$\boxed{v = L\,\frac{di}{dt} = 15.0\cos 50t\ (\mathrm{V}) \qquad p = vi = 75.0\sin 100t\ (\mathrm{W}) \qquad w_L = \int_0^t p\,dt = 0.75(1-\cos 100t)\ (\mathrm{J})}$$

As shown in the figure above, the energy is zero at t = 0 and t = (π = 50) s. Thus, while energy transfer did occur over the interval, this energy was first stored and later returned to the source.



1.2.3 Power and energy stored in a capacitance C

The power and energy relationships for a capacitance are as follows:

$$\boxed{p = vi = Cv\,\frac{dv}{dt} = \frac{d}{dt}\left[\frac{1}{2}\,Cv^2\right]}\qquad(24)$$

$$\boxed{w_C = \int_{t_1}^{t_2} p\,dt = \int_{t_1}^{t_2} Cv\,dv = \frac{1}{2}\,C[v_2^2 - v_1^2]} \tag{25}$$

The energy stored in the capacitance is:

$$\boxed{w_C = \tfrac{1}{2}Cv^2} \tag{26}$$

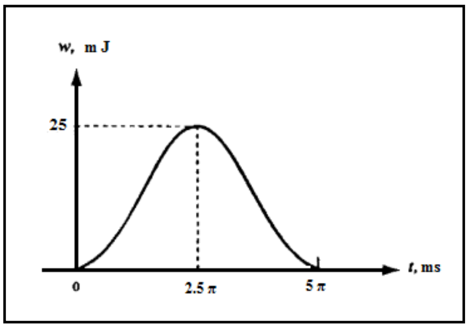
<u>Example:</u> In the interval 0 > t > 5π m s, a 20-mF capacitance has a voltage v = 50:0 sin 200t (V). Obtain the charge, power, and energy. Plot w_C assuming w = 0 at t = 0.

$$\boxed{q = Cv = 1000\sin 200t\ (\mu\text{C})}$$

$$\boxed{i = C\,\frac{dv}{dt} = 0.20\cos 200t\ (\text{A})}$$

$$\boxed{p = vi = 5.0\sin 400t\ (\text{W})}$$

$$\boxed{w_C = \int_{t_1}^{t_2} p\,dt = 12.5[1 - \cos 400t]\ (\text{mJ})}$$



1.3 Phasors

A brief look at the voltage and current sinusoids in the preceding examples shows that the amplitudes and phase differences are the two principal concerns. A directed line segment, or phasor, such as that shown rotating in a counterclockwise direction at a constant angular velocity ω (rad/s) in Fig. (10), has a projection on the horizontal which is a cosine function. The length of the phasor or its magnitude is the amplitude or maximum value of the cosine function. The angle between two positions of the phasor is the phase difference between the corresponding points on the cosine function.

Throughout this course, phasors will be defined from the cosine function. If a voltage or current is expressed as a sine, it will be changed to a cosine by subtracting 90° from the phase.

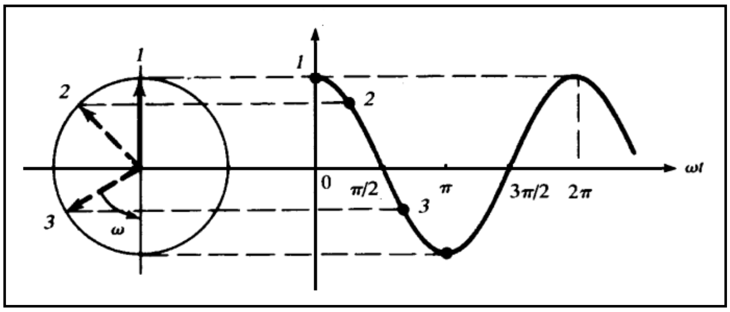
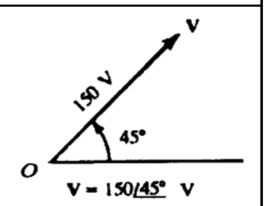
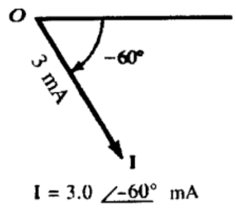


Fig. 10: Phasor diagram

Consider the examples shown in Table (02). Observe that the phasors, which are directed line segments and vectorial in nature, are indicated by boldface capitals, for example, V, I. The phase angle of the cosine function is the angle on the phasor. The phasor diagrams here and all that follow may be considered as a snapshot of the counterclockwise-rotating directed line segment taken at t = 0. The frequency f (Hz) and ω (rad/s) generally do not appear but they should be kept in mind, since they are implicit in any sinusoidal steady-state problem.

Tab. 02: Example of the phasor diagrams

Function	Phasor Representation
$v = 150\cos(500t + 45°) \quad (V)$	
$i = 3.0\sin(2000t + 30°) \quad (\text{mA})$ $= 3.0\cos(2000t - 60°) \quad (\text{mA})$	

<u>Example:</u> A series combination of R = 10 Ω and L = 20 mH has a current i = 5.0 cos (500t + 10°) (A). Obtain the voltages v and V, the phasor current I and sketch the phasor diagram.

Using the methods of example above (paragraph 9.1) of RL series circuit to calculate v_R, v_L and v:

$$\boxed{v_R = 50.0\cos(500t + 10°) \qquad v_L = L\frac{di}{dt} = 50.0\cos(500t + 100°)}$$

$$\boxed{v = v_R + v_L = 70.7\cos(500t + 55°) \quad (\text{V})}$$

The corresponding phasors are:

$$\boxed{\mathbf{I} = 5.0\ \underline{/10^\circ}\ \text{A} \qquad \text{and} \qquad \mathbf{V} = 70.7\ \underline{/55^\circ}\ \text{V}}$$

The phase angle of 45° can be seen in the time-domain graphs of i and v shown in the following figure on the left, and the phasor diagram with I and V shown in the same figure on the right.

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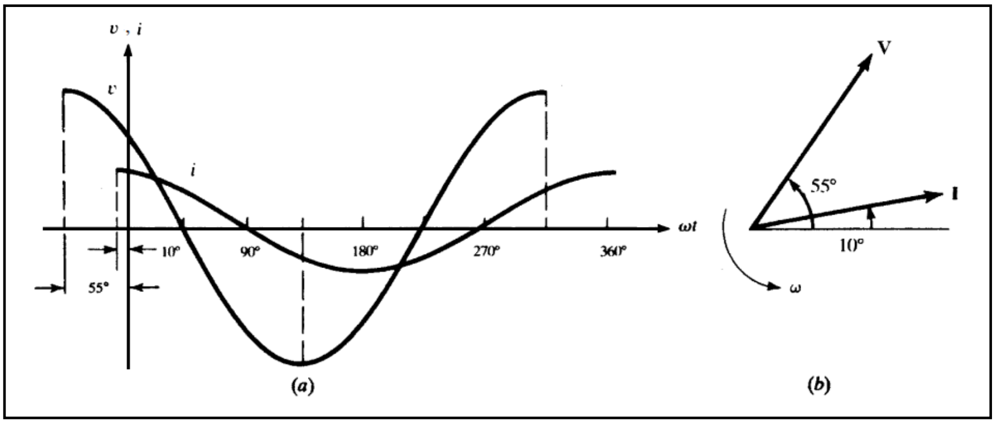


Phasors can be treated as complex numbers. When the horizontal axis is identified as the real axis of a complex plane, the phasors become complex numbers and the usual rules apply. In view of Euler's identity, there are three equivalent notations for a phasor.

Polar form $$\boxed{\mathbf{V} = V\ \underline{/\theta}} \tag{27}$$

Rectangular form $$\boxed{\mathbf{V} = V(\cos\theta + j\sin\theta)} \tag{28}$$

Exponential form $$\boxed{\mathbf{V} = Ve^{j\theta}} \tag{29}$$

The cosine expression may also be written as:

$$\boxed{v = V\cos(\omega t + \theta) = \mathrm{Re}\,[Ve^{j(\omega t+\theta)}] = \mathrm{Re}\,[\mathbf{V}e^{j\omega t}]} \tag{30}$$

The exponential form suggests how to treat the product and quotient of phasors.

Since,

$$\boxed{(V_1e^{j\theta_1})(V_2e^{j\theta_2}) = V_1V_2e^{j(\theta_1+\theta_2)},\ (V_1\ \underline{/\theta_1})(V_2\ \underline{/\theta_2}) = V_1V_2\ \underline{/\theta_1+\theta_2}} \tag{31}$$

And,

$$(V_1e^{j\theta_1})/(V_2e^{j\theta_2}) = (V_1/V_2)e^{j(\theta_1-\theta_2)}, \quad \frac{V_1\ \underline{/\theta_1}}{V_2\ \underline{/\theta}} = V_1/V_2\ \underline{/\theta_1-\theta_2} \tag{32}$$

The rectangular form is used in summing or subtracting phasors.

Example: Given V_1 and V_2, find the ratio V_1 / V_2 and the sum $V_1 + V_2$.

$$\mathbf{V}_1 = 25.0\ \underline{/143.13^\circ}\ ,\quad \mathbf{V}_2 = 11.2\ \underline{/26.57^\circ}$$

$$\mathbf{V}_1/\mathbf{V}_2 = \frac{25.0\ \underline{/143.13^\circ}}{11.2\ \underline{/26.57^\circ}} = 2.23\ \underline{/116.56^\circ} = -1.00 + j1.99$$
$$\mathbf{V}_1 + \mathbf{V}_2 = (-20.0 + j15.0) + (10.0 + j5.0) = -10.0 + j20.0 = 23.36\ \underline{/116.57^\circ}$$

1.4 Impedance and admittance

A sinusoidal voltage or current applied to a passive RLC circuit produces a sinusoidal response. With time functions, such as v(t) and i(t), the circuit is said to be in the time domain, Fig. (11 a); and when the circuit is analyzed using phasors, it is said to be in the frequency domain, Fig. (11 b). The voltage and current may be written, respectively,

$$\begin{aligned} v(t) &= V\cos(\omega t + \theta) = \mathrm{Re}[\mathbf{V}e^{j\omega t}]\ , \quad \mathbf{V} = V\ \underline{/\theta} \\ i(t) &= I\cos(\omega t + \phi) = \mathrm{Re}[\mathbf{I}e^{j\omega t}]\ , \quad \mathbf{I} = I\ \underline{/\phi} \end{aligned} \tag{33}$$

The ratio of phasor voltage V to phasor current I, is defined as impedance Z, that is, Z = V / I. The reciprocal of impedance is called admittance Y, so that Y = 1 / Z (S), where 1 S = 1 Ω^{-1} = 1 mho. Y and Z are complex numbers.

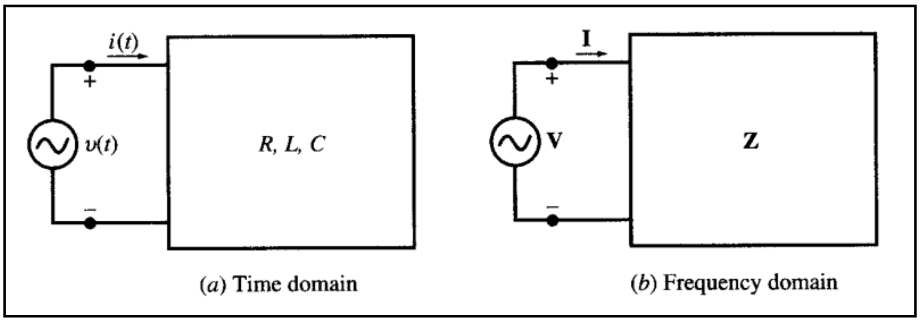


Fig. 11: Sinusoidal response of a passive RLC circuit in the time domain (a) and frequency domain (b)

When impedance is written in Cartesian form the real part is the resistance R and the imaginary part is the reactance X. The sign on the imaginary part may be positive or negative: When positive, X is called the inductive reactance, and when negative, X is called the capacitive reactance. When the admittance is written in Cartesian form, the real part is admittance G and the imaginary part is susceptance B. A positive sign on the susceptance indicates a capacitive susceptance, and a negative sign indicates an inductive susceptance. Thus,

$$\boxed{\begin{array}{l}\mathbf{Z} = R + jX_L \;\;,\;\; \mathbf{Z} = R - jX_C \\ \mathbf{Y} = G - jB_L \;\;,\;\; \mathbf{Y} = G + jB_C\end{array}} \tag{34}$$

The relationships between these terms follow from Z = 1 / Y. Then,

$$\boxed{\begin{array}{l}R = \dfrac{G}{G^2 + B^2} \;\;,\;\; X = \dfrac{-B}{G^2 + B^2} \\ G = \dfrac{R}{R^2 + X^2} \;\;,\;\; B = \dfrac{-X}{R^2 + X^2}\end{array}} \tag{35}$$

These expressions are not of much use in a problem where calculations can be carried out with the numerical values as in the following example.

Example: The phasor voltage across the terminals of a network such as that shown in Fig. (11 b) is $100.0\ \angle 45^\circ$ V and the resulting current is $5.0\ \angle 15^\circ$ A. Find the equivalent impedance and admittance.

$$\boxed{\begin{array}{l}\mathbf{Z} = \dfrac{\mathbf{V}}{\mathbf{I}}\dfrac{100.0\ \angle 45^\circ}{5.0\ \angle 15^\circ} = 20.0\ \angle 30^\circ = 17.32 + j10.0\,\Omega \\ \mathbf{Y} = \dfrac{\mathbf{I}}{\mathbf{V}} = \dfrac{1}{\mathbf{Z}} = 0.05\ \angle -30 = (4.33 - j2.50) \times 10^{-2}\ \mathrm{S}\end{array}}$$

$$\boxed{R = 17.32\,\Omega,\ X_L = 10.0\,\Omega,\ G = 4.33 \times 10^{-2}\ \mathrm{S},\ B_L = 2.50 \times 10^{-2}\ \mathrm{S}}$$

1.4.1 Combinations of impedances

The relation V = IZ (in the frequency domain) is formally identical to Ohm's law, v = i R, for a resistive network (in the time domain). Therefore, impedances combine exactly like resistances:

Impedances in series:

$$\boxed{\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots} \tag{36}$$

Impedances in parallel

$$\boxed{\frac{1}{\mathbf{Z}_{eq}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots} \tag{37}$$

In particular, for two parallel impedances,

$$\boxed{\mathbf{Z}_{eq} = \mathbf{Z}_1\mathbf{Z}_2/(\mathbf{Z}_1 + \mathbf{Z}_2)} \tag{38}$$

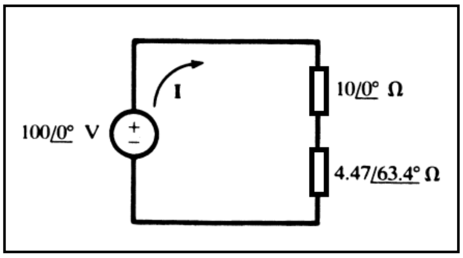
<u>Example:</u> For the circuit shown in figure below, obtain Z_{eq} and compute I.

For series impedances,

$$\boxed{\mathbf{Z}_{eq} = 10\ \underline{/0^\circ} + 4.47\ \underline{/63.4^\circ} = 12.0 + j4.0 = 12.65\ \underline{/18.43^\circ}\quad \Omega}$$

Then

$$\boxed{\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_{eq}} = \frac{100\ \underline{/0^\circ}}{12.65\ \underline{/18.43^\circ}} = 7.91\ \underline{/-18.43^\circ}\quad \text{A}}$$



1.4.2 Impedance Diagram

In an impedance diagram, an impedance Z is represented by a point in the right half of the complex plane. Figure (12) shows two impedances; Z_1, in the first quadrant, exhibits inductive reactance, while Z_2, in the fourth quadrant, exhibits capacitive reactance. Their series equivalent, $Z_1 + Z_2$, is obtained by vector addition, as shown. Note that the ''vectors'' are shown without arrowheads, in order to distinguish these complex numbers from phasors.

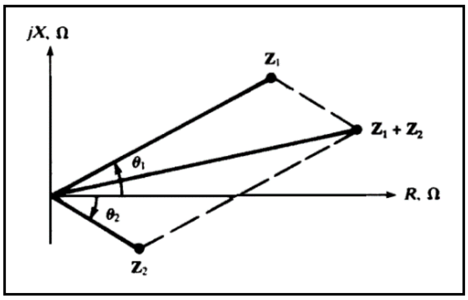
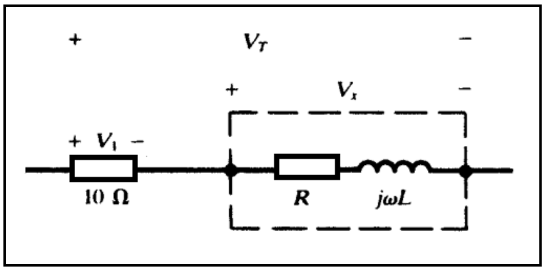


Fig. 12: Impedance diagram

Example: The constants R and L of a coil can be obtained by connecting the coil in series with a known resistance and measuring the coil voltage V_x, the resistor voltage V_1, and the total voltage V_T (See figure below). The frequency must also be known, but the phase angles of the voltages are not known.

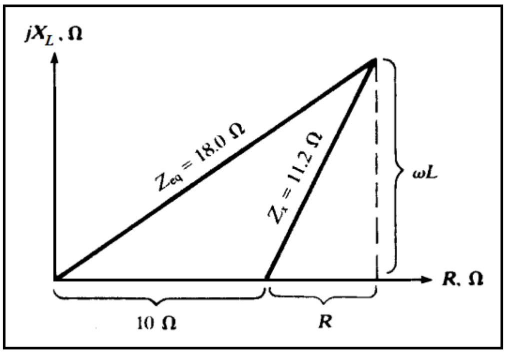


Given that f = 60 Hz, V_1 = 20 V, V_x = 22.4 V, and V_T = 36:0 V, find R and L.

The (effective) current is I = V_1 / 10 = 2.0 A. Then:

$$\boxed{Z_x = \frac{22.4}{2.0} = 11.2\,\Omega \qquad Z_{\text{eq}} = \frac{36.0}{2.0} = 18.0\,\Omega}$$

From the impedance diagram,



$$\boxed{\begin{aligned}(18.0)^2 &= (10+R)^2 + (\omega L)^2 \\ (11.2)^2 &= R^2 + (\omega L)^2\end{aligned}}$$

where ω = 2π60 = 377 rad/s. Solving simultaneously,

$$\boxed{R = 4.92\,\Omega \qquad L = 26.7\,\text{mH}}$$

1.4.3 Combinations of admittances

Replacing Z by 1/Y in the formulas above gives:

Admittances in series:

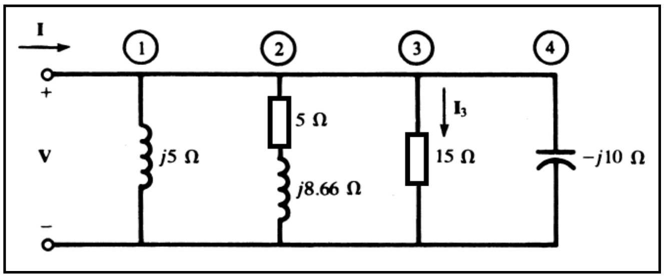
$$\boxed{\frac{1}{\mathbf{Y}_{\text{eq}}} = \frac{1}{\mathbf{Y}_1} + \frac{1}{\mathbf{Y}_2} + \cdots} \qquad (39)$$

Admittances in parallel:

$$\boxed{\mathbf{Y}_{\text{eq}} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots} \tag{40}$$

Thus, series circuits are easiest treated in terms of impedance; parallel circuits, in terms of admittance.

Example: Compute the equivalent impedance Z_{eq} and admittance Y_{eq} for the four-branch circuit shown in figure below.



Using admittances,

$$\mathbf{Y}_1 = \frac{1}{j5} = -j0.20\,\mathrm{S} \qquad \mathbf{Y}_3 = \frac{1}{15} = 0.067\,\mathrm{S}$$

$$\mathbf{Y}_2 = \frac{1}{5 + j8.66} = 0.05 - j0.087\,\mathrm{S} \qquad \mathbf{Y}_4 = \frac{1}{-j10} = j0.10\,\mathrm{S}$$

$$\mathbf{Y}_{\text{eq}} = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \mathbf{Y}_4 = 0.117 - j0.187 = 0.221\ \angle{-58.0^\circ}\ \mathrm{S}$$

$$\mathbf{Z}_{\text{eq}} = \frac{1}{\mathbf{Y}_{\text{eq}}} = 4.53\ \angle{58.0^\circ}\ \ \Omega$$

1.4.4 Admittance diagram

Figure (13), an admittance diagram, is analogous to Fig. (12) for impedance. Shown are an admittance Y_1 having capacitive susceptance and an admittance Y_2 having inductive susceptance, together with their vector sum, $Y_1 + Y_2$, which is the admittance of a parallel combination of Y_1 and Y_2.

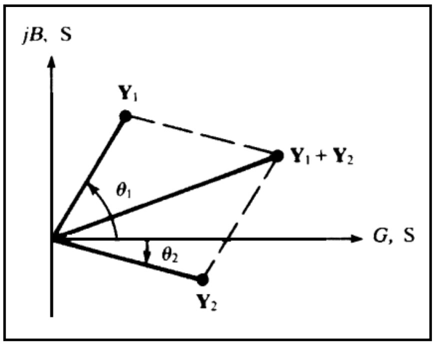
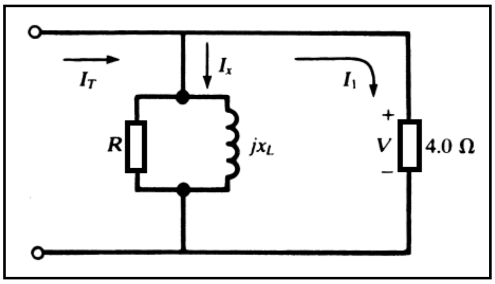


Fig. 13: Admittance diagram

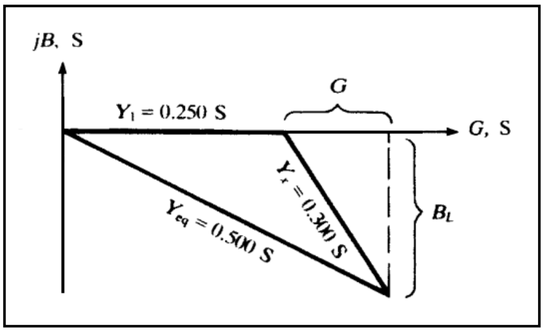
Example: In the parallel circuit shown in figure below, the effective values of the currents are: I_x = 18.0 A, I_1 = 15.0 A, I_T = 30.0 A. Determine R and X_L.



The (effective) voltage is V = I_1(4.0) = 60.0 V. Then:

$$Y_x = \frac{I_x}{V} = 0.300\,\text{S} \qquad Y_{eq} = \frac{I_T}{V} = 0.500\,\text{S} \qquad Y_1 = \frac{1}{4.0} = 0.250\,\text{S}$$

From the admittance diagram,



$$(0.500)^2 = (0.250 + G)^2 + B_L^2$$
$$(0.300)^2 = G^2 + B_L^2$$

which yield

$$G = 0.195\,\text{S},\ B_L = 0.228\,\text{S}$$

Then

$$R = \frac{1}{G} = 5.13\,\Omega \ \ , \ jX_L = \frac{1}{-jB_L} = j4.39\,\Omega$$

1.5 Voltage and current division in the frequency domain

In view of the analogy between impedance in the frequency domain and resistance in the time domain, Sections 8 and 9 of the first part of this course imply the following results.

(1) Impedances in series divide the total voltage in the ratio of the impedances, Fig. (14):

$$\boxed{\frac{\mathbf{V}_r}{\mathbf{V}_s} = \frac{\mathbf{Z}_r}{\mathbf{Z}_s} \quad , \quad \mathbf{V}_r = \frac{\mathbf{Z}_r}{\mathbf{Z}_{eq}} \mathbf{V}_T} \tag{41}$$

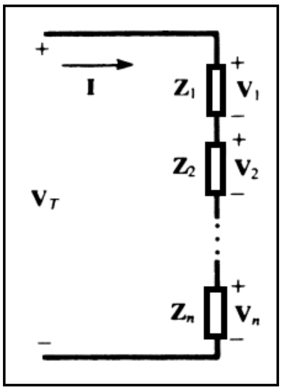


Fig. 14: Voltage division in the frequency domain

(2) Impedances in parallel (admittances in series) divide the total current in the inverse ratio of the impedances (direct ratio of the admittances), Fig. (15):

$$\boxed{\frac{\mathbf{I}_r}{\mathbf{I}_s} = \frac{\mathbf{Z}_s}{\mathbf{Z}_r} = \frac{\mathbf{Y}_r}{\mathbf{Y}_s} \quad , \quad \mathbf{I}_r = \frac{\mathbf{Z}_{eq}}{\mathbf{Z}_r} \mathbf{I}_T = \frac{\mathbf{Y}_r}{\mathbf{Y}_{eq}} \mathbf{I}_T} \tag{42}$$

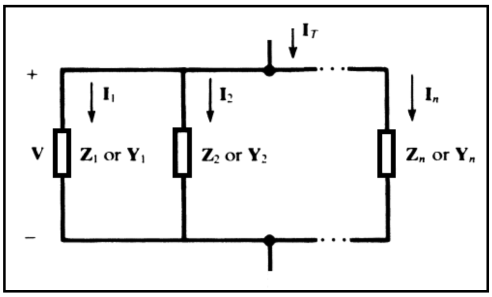


Fig. 15: Current division in the frequency domain

1.6 The mesh current method

Consider the frequency-domain network of Fig. (16). Applying KVL, as in Section 10.3 of the first part of this course, or simply by inspection, we find the matrix equation:

$$\boxed{\begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} & \mathbf{Z}_{13} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} & \mathbf{Z}_{23} \\ \mathbf{Z}_{31} & \mathbf{Z}_{32} & \mathbf{Z}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \end{bmatrix}} \tag{43}$$

for the unknown mesh currents I_1; I_2; I_3. Here, $Z_{11} \equiv Z_A + Z_B$, the self-impedance of mesh (1), is the sum of all impedances through which I_1 passes. Similarly, $Z_{22} \equiv Z_B + Z_C + Z_D$ and $Z_{33} \equiv Z_D + Z_E$ are the self-impedances of meshes (2) and (3).

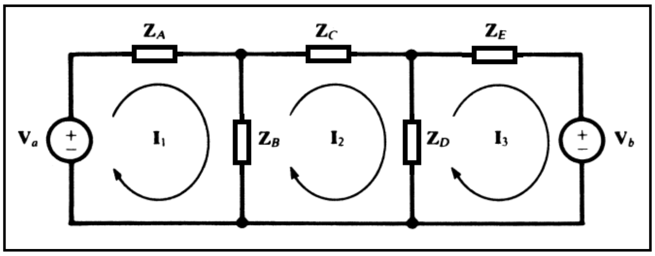


Fig. 16: The mesh current method

The 1,2-element of the Z-matrix is defined as:

$$\boxed{\mathbf{Z}_{12} \equiv \sum \pm \text{ (impedance common to } \mathbf{I}_1 \text{ and } \mathbf{I}_2)} \tag{44}$$

where a summand takes the plus sign if the two currents pass through the impedance in the same direction, and takes the minus sign in the opposite case. It follows that, invariably, $Z_{12} = Z_{21}$. In Fig. (16), I_1 and I_2 thread Z_B in opposite directions, whence

$$\boxed{\mathbf{Z}_{12} = \mathbf{Z}_{21} = -\mathbf{Z}_B} \tag{45}$$

Similarly,

$$\boxed{\begin{aligned} \mathbf{Z}_{13} = \mathbf{Z}_{31} &\equiv \sum \pm \text{ (impedance common to } \mathbf{I}_1 \text{ and } \mathbf{I}_3) = 0 \\ \mathbf{Z}_{23} = \mathbf{Z}_{23} &\equiv \sum \pm \text{ (impedance common to } \mathbf{I}_2 \text{ and } \mathbf{I}_3 = -\mathbf{Z}_D \end{aligned}} \tag{46}$$

The Z-matrix is symmetric.

In the V-column on the right-hand side of the equation, the entries V_k (k = 1; 2; 3) are defined exactly as in Section 10.3 of the first part of this course:

$$\boxed{\mathbf{V}_k \equiv \sum \pm \text{ (driving voltage in mesh } k)} \tag{47}$$

where a summand takes the plus sign if the voltage drives in the direction of I_k, and takes the minus sign in the opposite case. For the network of Fig. (16),

$$\boxed{\mathbf{V}_1 = +\mathbf{V}_a \qquad \mathbf{V}_2 = 0 \qquad \mathbf{V}_3 = -\mathbf{V}_b} \tag{48}$$

The preceding rules for writing the Z-matrix and the V-column have been formulated in such a way as to apply either to meshes or to loops. These rules are, of course, identical to those used in Section 10.3 of the first part of this course to write the R-matrix and V-column.

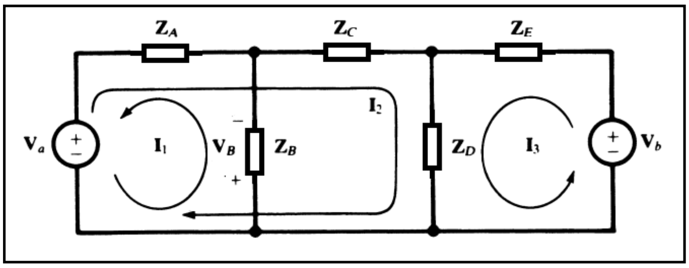
Example: Suppose that the phasor voltage across Z_B, with polarity as indicated in the figure below is sought. Choosing meshes as in Fig. (16) would entail solving for both I_1 and I_2, then obtaining the voltage as $V_B = (I_2 - I_1)\, Z_B$. In the figure below three loops (two of which are meshes) are chosen so as to make I_1 the only current in Z_B. Furthermore, the direction of I_1 is chosen such that $V_B = I_1\, Z_B$. Setting up the matrix equation:

$$\boxed{\begin{bmatrix} \mathbf{Z}_A + \mathbf{Z}_B & -\mathbf{Z}_A & 0 \\ -\mathbf{Z}_A & \mathbf{Z}_A + \mathbf{Z}_C + \mathbf{Z}_D & \mathbf{Z}_D \\ 0 & \mathbf{Z}_D & \mathbf{Z}_D + \mathbf{Z}_E \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} -\mathbf{V}_a \\ \mathbf{V}_a \\ \mathbf{V}_b \end{bmatrix}}$$

from which

$$\boxed{\mathbf{V}_B = \mathbf{Z}_B \mathbf{I}_1 = \frac{\mathbf{Z}_B}{\Delta_\mathbf{z}} \begin{vmatrix} -\mathbf{V}_a & -\mathbf{Z}_A & 0 \\ \mathbf{V}_a & \mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C & \mathbf{Z}_D \\ \mathbf{V}_b & \mathbf{Z}_D & \mathbf{Z}_D + \mathbf{Z}_E \end{vmatrix}}$$

where $\Delta_\mathbf{z}$ is the determinant of the Z-matrix.



1.7 Input and transfer impedances

The notions of input resistance (Section 10.5) and transfer resistance (Section 10.6) of the first part of this course have their exact counterparts in the frequency domain. Thus, for the single-source network of Fig. (17), the input impedance is:

$$\boxed{\mathbf{Z}_{\text{input},r} \equiv \frac{\mathbf{V}_r}{\mathbf{I}_r} = \frac{\Delta_z}{\Delta_{rr}}} \tag{49}$$

Where (rr) is the cofactor of Z_{rr} in Δ_z ; and the transfer impedance between mesh (or loop) r and mesh (loop) s is:

$$\boxed{\mathbf{Z}_{\text{transfer},rs} \equiv \frac{\mathbf{V}_r}{\mathbf{I}_s} = \frac{\Delta_z}{\Delta_{rs}}} \tag{50}$$

Where Δ_{rs} is the cofactor of $\mathbf{Z}_{rs}$ in Δ_z.

As before, the superposition principle for an arbitrary n-mesh or n-loop network may be expressed as:

$$\boxed{\mathbf{I}_k = \frac{\mathbf{V}_1}{\mathbf{Z}_{\text{transfer},1k}} + \cdots + \frac{\mathbf{V}_{k-1}}{\mathbf{Z}_{\text{transfer},(k-1)k}} + \frac{\mathbf{V}_k}{\mathbf{Z}_{\text{input},k}} + \frac{\mathbf{V}_{k+1}}{\mathbf{Z}_{\text{transfer},(k+1)k}} + \cdots + \frac{\mathbf{V}_n}{\mathbf{Z}_{\text{transfer},nk}}} \tag{51}$$

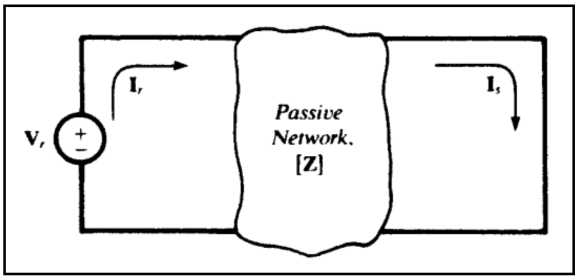
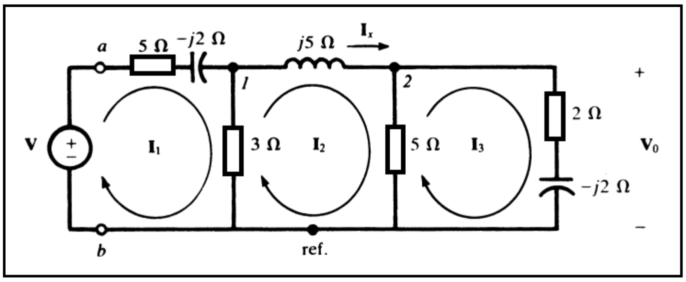


Fig. 17: The input and transfer impedances for the single-source network

Example: Find the input impedance at terminals ab for the network of the figure below:



With mesh current I_1 selected as shown on the diagram.

$$\boxed{\mathbf{Z}_{\text{input},1} = \frac{\Delta_z}{\Delta_{11}} = \frac{\begin{vmatrix} 8-j2 & -3 & 0 \\ -3 & 8+j5 & -5 \\ 0 & -5 & 7-j2 \end{vmatrix}}{\begin{vmatrix} 8+j5 & -5 \\ -5 & 7-j2 \end{vmatrix}} = \frac{315.5 \ \underline{/16.19^\circ}}{45.2 \ \underline{/24.86^\circ}} = 6.98 \ \underline{/-8.67^\circ} \quad \Omega}$$

For the same network, obtain the current in the inductor, I_x, by first obtaining the transfer impedance.

$$\mathbf{Z}_{\text{transfer},12} = \frac{\Delta_{\mathbf{z}}}{\Delta_{12}} = \frac{315.5\ \angle 16.19^\circ}{-\begin{vmatrix} -3 & -5 \\ 0 & 7-j2 \end{vmatrix}} = 14.45\ \angle 32.14^\circ\quad \Omega$$

$$\mathbf{I}_x = \mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_{\text{transfer},12}} = \frac{10\ \angle 30^\circ}{14.45\ \angle 32.14^\circ} = 0.692\ \angle -2.14^\circ\quad \text{A}$$

1.8 The node voltage method

The procedure is exactly as in Section 10.4 of the first part of this course, with admittances replacing reciprocal resistances. A frequency-domain network with n principal nodes, one of them designated as the reference node, requires n - 1 node voltage equations. Thus, for n = 4, the matrix equation would be:

$$\begin{bmatrix} \mathbf{Y}_{11} & \mathbf{Y}_{12} & \mathbf{Y}_{13} \\ \mathbf{Y}_{21} & \mathbf{Y}_{22} & \mathbf{Y}_{23} \\ \mathbf{Y}_{31} & \mathbf{Y}_{32} & \mathbf{Y}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} \tag{52}$$

in which the unknowns, V_1, V_2, and V_3, are the voltages of principal nodes 1, 2, and 3 with respect to principal node 4, the reference node.

Y_{11} is the self-admittance of node 1, given by the sum of all admittances connected to node 1. Similarly, Y_{22} and Y_{33} are the self-admittances of nodes 2 and 3.

Y_{12}, the coupling admittance between nodes 1 and 2, is given by minus the sum of all admittances connecting nodes 1 and 2. It follows that $Y_{12} = Y_{21}$. Similarly, for the other coupling admittances: $Y_{13} = Y_{31}$, $Y_{23} = Y_{32}$. The Y-matrix is therefore symmetric.

On the right-hand side of the equation, the I-column is formed just as in Section 10.4; i.e.,

$$\mathbf{I}_k = \sum \text{(current driving into node } k\text{)} \qquad (k = 1, 2, 3) \tag{53}$$

in which a current driving out of node k is counted as negative.

1.9 The input and transfer admittances

The matrix equation of the node voltage method,

$$[\mathbf{Y}][\mathbf{V}] = [\mathbf{I}] \tag{54}$$

is identical in form to the matrix equation of the mesh current method,

$$[\mathbf{Z}][\mathbf{I}] = [\mathbf{V}] \tag{55}$$

Therefore, in theory at least, input and transfer admittances can be defined by analogy with input and transfer impedances:

$$\mathbf{Y}_{\text{input},r} \equiv \frac{\mathbf{I}_r}{\mathbf{V}_r} = \frac{\Delta_{\mathbf{Y}}}{\Delta_{rr}}$$

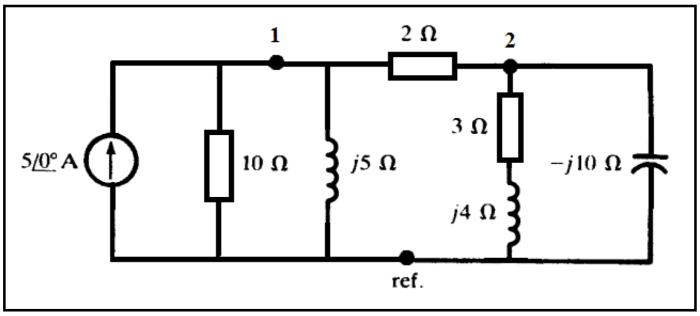
$$\mathbf{Y}_{\text{transfer},rs} \equiv \frac{\mathbf{I}_r}{\mathbf{V}_s} = \frac{\Delta_{\mathbf{Y}}}{\Delta_{rs}} \qquad (56)$$

where now (rr) and (rs) are the cofactors of Y_{rr} and Y_{rs} in $\Delta_{\mathbf{Y}}$. In practice, these definitions are often of limited use. However, they are valuable in providing an expression of the superposition principle (for voltages);

$$\mathbf{V}_k = \frac{\mathbf{I}_1}{\mathbf{Y}_{\text{transfer},1k}} + \cdots + \frac{\mathbf{I}_{k-1}}{\mathbf{Y}_{\text{transfer},(k-1)k}} + \frac{\mathbf{I}_k}{\mathbf{Y}_{\text{input},k}} + \frac{\mathbf{I}_{k+1}}{\mathbf{Y}_{\text{transfer},(k+1)k}} + \cdots + \frac{\mathbf{I}_{n-1}}{\mathbf{Y}_{\text{transfer},(n-\text{If})k}}$$

for k = 1; 2; . . .; n - 1. In words: the voltage at any principal node (relative to the reference node) is obtained by adding the voltages produced at that node by the various driving currents, these currents acting one at a time.

Example: For the network shown in figure below, obtain the input admittance and use it to compute node voltage V_1.



$$\mathbf{Y}_{\text{input},1} = \frac{\Delta_{\mathbf{Y}}}{\Delta_{11}} = \frac{\begin{vmatrix} \frac{1}{10}+\frac{1}{j5}+\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}+\frac{1}{3+j4}+\frac{1}{-j10} \end{vmatrix}}{\frac{1}{2}+\frac{1}{3+j4}+\frac{1}{-j10}} = 0.311 \;\underline{/-49.97^\circ} \quad \text{S}$$

$$\mathbf{V}_1 = \frac{\mathbf{I}_1}{\mathbf{Y}_{\text{input},1}} = \frac{5.0 \;\underline{/0^\circ}}{0.311 \;\underline{/-49.97^\circ}} = 16.1 \;\underline{/49.97^\circ} \quad \text{V}$$

For the same network, compute the transfer admittance $Y_{\text{transfer},12}$ and use it to obtain node voltage V_2.

$$\mathbf{Y}_{\text{transfer},12} = \frac{\Delta_{\mathbf{Y}}}{\Delta_{12}} = \frac{0.194 \;\underline{/-55.49^\circ}}{-(-0.50)} = 0.388 \;\underline{/-55.49^\circ} \quad \text{S}$$

$$\mathbf{V}_2 = \frac{\mathbf{I}_1}{\mathbf{Y}_{\text{transfer},12}} = 12.9 \ \angle 55.49^\circ \quad \text{V}$$

1.10 The Thevenin's and Norton's theorems

These theorems are exactly as given in Section 10.8 of the first part of this course, with the open-circuit voltage V', short-circuit current I', and representative resistance R' replaced by the open-circuit phasor voltage V', short-circuit phasor current I', and representative impedance Z'. See Fig. (18).

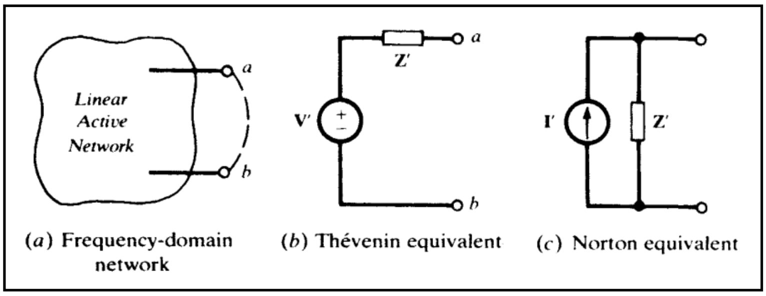


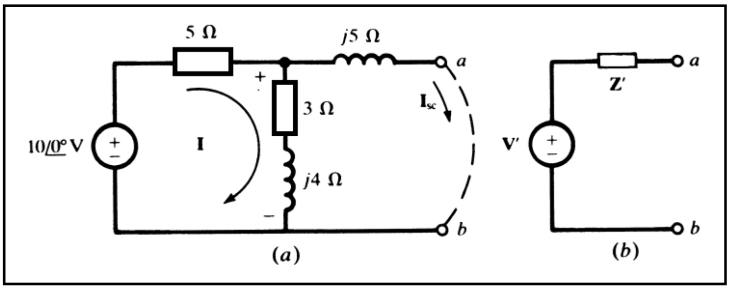
Fig. 18: The Thevenin and Norton equivalent network in frequency domain

Example: Replace the active network in figure below (a) at terminals ab with a Thevenin equivalent.

$$\mathbf{Z}' = j5 + \frac{5(3+j4)}{5+3+j4} = 2.50 + j6.25 \ \Omega$$

The open-circuit voltage V′ at terminals ab is the voltage across the 3 + j4 Ω impedance:

$$\mathbf{V}' = \left(\frac{10\angle 0^\circ}{8+j4}\right)(3+j4) = 5.59 \ \angle 26.56^\circ \ \text{V}$$

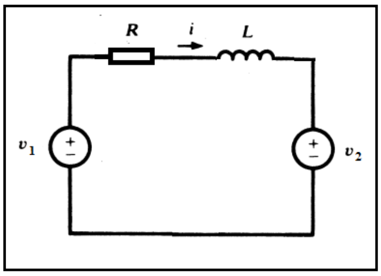


For the network above, obtain a Norton equivalent circuit (figure below). At terminals ab, I_{sc} is the Norton current I′. By current division,

1.11 Superposition of AC sources

How do we apply superposition to circuits with more than one sinusoidal source? If all sources have the same frequency, superposition is applied in the phasor domain. Otherwise, the circuit is solved for each source, and time-domain responses are added.

Example: Suppose a practical coil is connected in series between two voltage sources $v_1 = 5\cos\omega_1 t$ and $v_2 = 10\cos(\omega_2 t + 60°)$ such that the sources share the same reference node. See figure below. The voltage difference across the terminals of the coil is therefore $v_1 - v_2$. The coil is modeled by a 5-mH inductor in series with a 10-Ω resistor. Find the current i(t) in the coil for (a) $\omega_1 = \omega_2 = 2000$ rad / s and (b) $\omega_1 = 2000$ rad / s, $\omega_2 = 2\omega_1$.



(a) The impedance of the coil is:

$$\boxed{R + jL\omega = 10 + j10 = 10\sqrt{2}\ \angle 45°\ \Omega}$$

The phasor voltage between its terminals is:

$$\boxed{\mathbf{V} = \mathbf{V}_1 - \mathbf{V}_2 = 5 - 10\ \angle 60° = -j5\sqrt{3}\ \text{V}}$$

The current is:

$$\mathbf{I} = \frac{\mathbf{V}}{Z} = \frac{-j5\sqrt{3}}{10\sqrt{2}\ \angle 45°} \approx \frac{-j8.66}{14.14\ \angle 45°} = 0.61\ \angle -135°\ \text{A}$$
$$i = 0.61\cos(2000t - 135°)$$

(b) Because the coil has different impedances at $\omega_1 = 2000$ and $\omega_2 = 4000$ rad / s, the current may be represented in the time domain only. By applying superposition, we get $i = i_1 - i_2$, where i_1 and i_2 are currents due to v_1 and v_2, respectively.

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{Z_1} = \frac{5}{10 + j10} = 0.35\ \angle -45°\ \text{A}, \qquad i_1(t) = 0.35\cos(2000t - 45°)$$
$$\mathbf{I}_2 = \frac{\mathbf{V}_2}{Z_2} = \frac{10\ \angle 60°}{10 + j20} = 0.45\ \angle -3.4°\ \text{A}, \qquad i_2(t) = 0.45\cos(4000t - 3.4°)$$
$$i = i_1 - i_2 = 0.35\cos(2000t - 45°) - 0.45\cos(4000t - 3.4°)$$

Part IV. Study of simple R L and RC circuits in free and forced regime

1. Introduction

When one storage element is present (simple RL and RC circuit), the network equations will result in first-order differential equation that has the form:

$$\boxed{\frac{dx}{dt}(t) + a\,x(t) = f(t)} \qquad (01)$$

The function f (t) is called the forcing function and when it is zero the circuit is in a free regime and when f (t) ≠ 0 the forced response $x_p(t)$ depends on the forcing function f (t) and the circuit is in a forced regime.

In this part of the course we will find the response of simple RL and RC circuits in free and forced regime, given various initial conditions and sources. We will also present and solve important issues relating to natural, forced, step, and impulse responses, along with the DC steady state and the switching behavior of inductors and capacitors.

2. Capacitor discharge in a resistor

Assume a capacitor has a voltage difference V_0 between its plates. When a conducting path R is provided, the stored charge travels through the capacitor from one plate to the other, establishing a current i. Thus, the capacitor voltage v is gradually reduced to zero, at which time the current also becomes zero. In the RC circuit of Fig. (01),

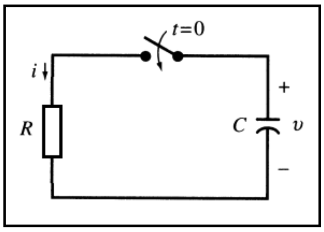


Fig. 01: The capacitor discharge in a resistor

R i = v and i = -C dv / dt. Eliminating i in both equations, gives:

$$\boxed{\frac{dv}{dt} + \frac{1}{RC}\,v = 0} \qquad (02)$$

The only function whose linear combination with its derivative can be zero is an exponential function of the form Ae^{st}. Replacing v by A e^{st} and dv / dt by sAe^{st} in (2), we get:

$$sAe^{st} + \frac{1}{RC} Ae^{st} = A\left(s + \frac{1}{RC}\right) e^{st} = 0 \tag{03}$$

from which:

$$s + \frac{1}{RC} = 0 \quad , \quad s = -\frac{1}{RC} \tag{04}$$

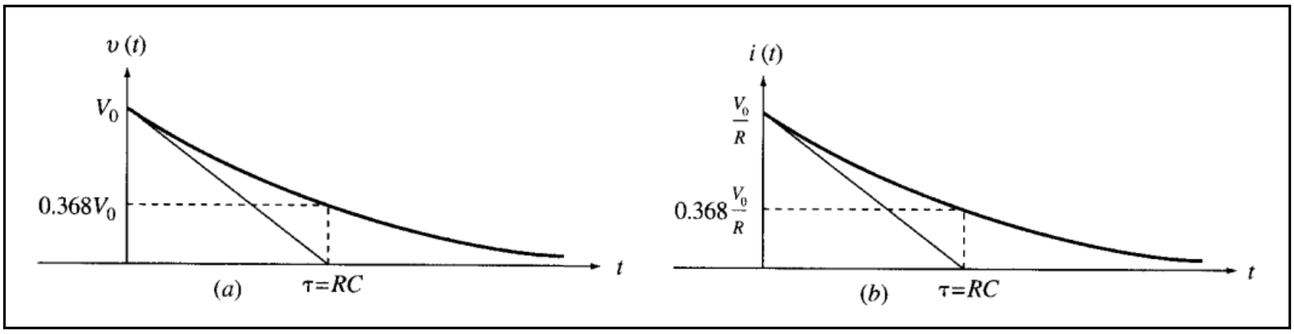
Given v (0) = A = V_0, v(t) and i(t) are found to be:

$$\begin{aligned} v(t) &= V_0 e^{-t/RC}, \quad t > 0 \\ i(t) &= -C\frac{dv}{dt} = \frac{V_0}{R} e^{-t/RC}, \quad t > 0 \end{aligned} \tag{05}$$

The voltage and current of the capacitor are exponentials with initial values of V_0 and V_0 / R, respectively. As time increases, voltage and current decrease to zero with a time constant of τ = RC. See Figs. (a) and (b).

<u>Example:</u> The voltage across a 1-μF capacitor is 10 V for t < 0. At t = 0, a 1-MΩ resistor is connected across the capacitor terminals. Find the time constant, the voltage v(t), and its value at t = 5 s.

$$\tau = RC = 10^6(10^{-6})\,\text{s} = 1\,\text{s} \;,\; v(t) = 10e^{-t}\,(\text{V}), t > 0 \;\;,\;\; v(5) = 10e^{-5} = 0.067\,\text{V}$$



3. Establishing a DC voltage across a capacitor

Connect an initially uncharged capacitor to a battery with voltage V_0 through a resistor at t = 0. The circuit is shown in Fig. (02).

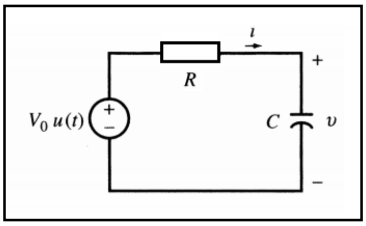


Fig. 02: The circuit for establishing a DC voltage across a capacitor

For t > 0, KVL around the loop gives R i + v = V_0 which, after substituting i = C (dv / dt), becomes:

$$\boxed{\frac{dv}{dt} + \frac{1}{RC}\,v = \frac{1}{RC}\,V_0 \qquad t > 0} \tag{06}$$

$$\boxed{v(0^+) = v(0^-) = 0} \tag{07}$$

The solution should satisfy both (06) and (07). The particular solution $v_p(t) = V_0$ satisfies (06) but not (07). The homogeneous solution $v_h(t) = Ae^{-t/RC}$ can be added and its magnitude A can be adjusted so that the total solution (08) satisfies both (06) and (07).

$$\boxed{v(t) = v_p(t) + v_h(t) = V_0 + Ae^{-t/RC}} \tag{08}$$

From the initial condition, v $(0^+) = V_0 + A = 0$ or $A = -V_0$. Thus the total solution is:

$$\boxed{v(t) = V_0(1 - e^{-t/RC})u(t)} \tag{09}$$

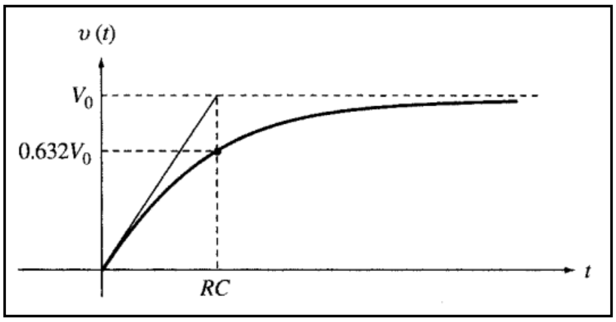


Fig. 03: The DC voltage across a capacitor

$$\boxed{i(t) = \frac{V_0}{R}\,e^{-t/RC}u(t)} \tag{10}$$

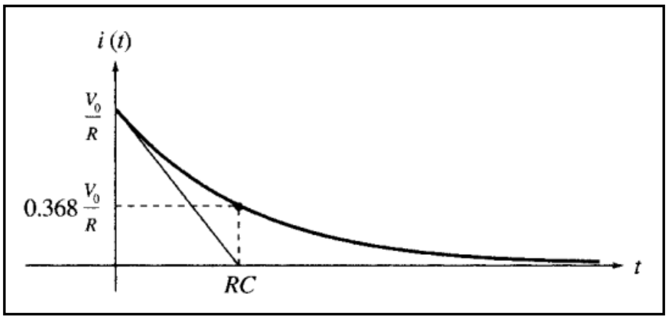


Fig. 04: The current in the capacitor

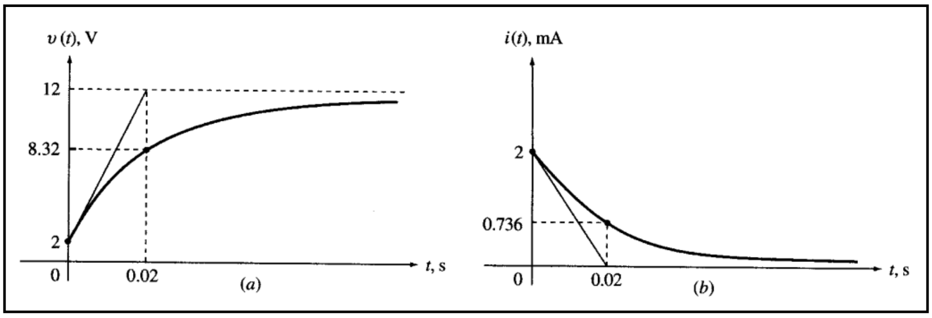
Example: A 4-µF capacitor with an initial voltage of v (0^-) = 2 V is connected to a 12-V battery through a resistor R = 5 kΩ at t = 0. Find the voltage across and current through the capacitor for t > 0. The time constant of the circuit is τ = RC = 0,02 s. Following the analysis of example above, we get:

$$\boxed{v(t) = 12 + Ae^{-50t}}$$

From the initial conditions, v (0^-) = v (0^+) = 12 + A = 2 or A = -10. Thus, for: t > 0,

$$v(t) = 12 - 10e^{-50t} \text{ (V)}$$
$$i(t) = (12 - v)/5000 = 2 \times 10^{-3} e^{-50t} A = 2e^{-50t} \text{ (mA)}$$

The current may also be computed from i = C (dv / dt). And so the voltage increases exponentially from an initial value of 2 V to a final value of 12 V, with a time constant of 20 ms, as shown in Fig. (a), while the current decreases from 2 mA to zero as shown in Fig. (b).



4. The source-free RL circuit

In the RL circuit of Fig. (02), assume that at t = 0 the current is I_0. For t > 0, i should satisfy $Ri + L(di/dt) = 0$, the solution of which is $i = Ae^{st}$. By substitution we find A and s:

$$\boxed{A(R + Ls)e^{st} = 0, \qquad R + Ls = 0, \qquad s = -R/L}$$

The initial condition I (0) = A = I_0. Then:

$$\boxed{i(t) = I_0 e^{-Rt/L} \qquad \text{for } t > 0} \tag{11}$$

The time constant of the circuit is L / R.

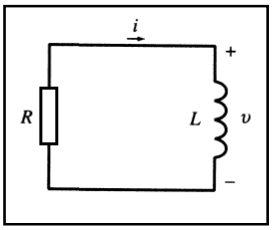
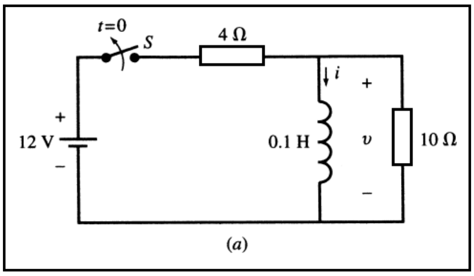
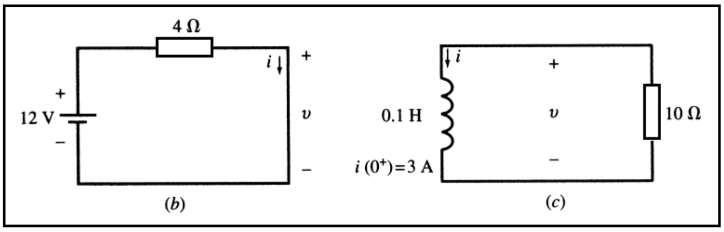


Fig. 05: The source-free RL circuit

Example: The 12-V battery in Fig. (a) is disconnected at t = 0. Find the inductor current and voltage v for all times.



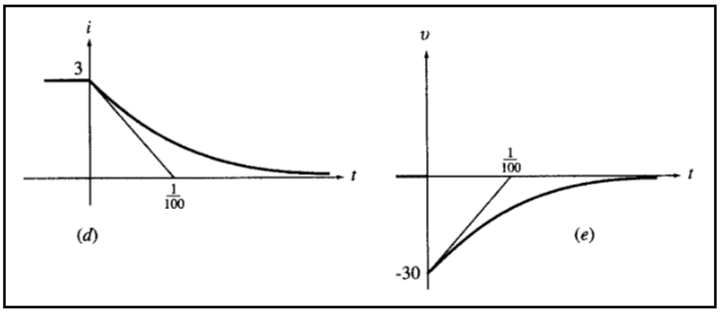
Assume the switch S has been closed for a long time. The inductor current is then constant and its voltage is zero. The circuit at t = 0^- is shown in Fig. (b) with i (0^-) = 12 / 4 = 3 A. After the battery is disconnected, at t > 0, the circuit will be as shown in Fig. (c). For t > 0, the current decreases exponentially from 3 A to zero. The time constant of the circuit is L / R = (1 / 100) s.



Using the results of example of paragraph (03), for, t > 0, the inductor current and voltage are, respectively,

$$\boxed{\begin{aligned} i(t) &= 3e^{-100t} \\ v(t) &= L(di/dt) = -30e^{-100t} \quad \text{(V)} \end{aligned}}$$

i(t) and v(t) are plotted in Figs. (d) and (e), respectively.



5. Establishing a DC current in an inductor

If a DC source is suddenly applied to a series RL circuit initially at rest, as in Fig. (06), the current grows exponentially from zero to a constant value with a time constant of L / R.

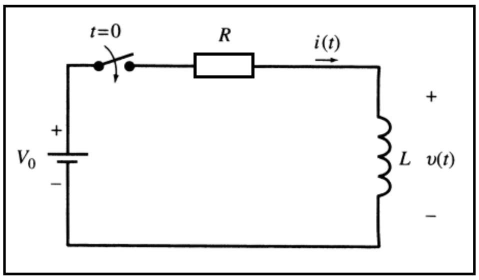


Fig. 06: The circuit for establishing a DC current in an inductor

The preceding result is the solution of the first-order differential equation (12) which is obtained by applying KVL around the loop. The solution follows.

$$\boxed{Ri + L\frac{di}{dt} = V_0 \qquad \text{for } t > 0, \qquad i(0^+) = 0} \tag{12}$$

Since $i = i_h(t) + i_p(t)$ where $i_h(t) = Ae^{-Rt/L}$ and $i_p(t) = V_0/R$ we have $i = Ae^{-Rt/L} + V_0/R$

The coefficient A is found from $i(0^+) = A + V_0/R = 0$ or $A = -V_0/R$. The current in the inductor and the voltage across it are given by (13) and (14) and plotted in Figs. (04 b) and (04 c), respectively.

$$\boxed{i(t) = V_0/R(1 - e^{-Rt/L}) \qquad \text{for } t > 0} \tag{13}$$

$$\boxed{v(t) = L\frac{di}{dt} = V_0 e^{-Rt/L} \qquad \text{for } t > 0} \tag{14}$$

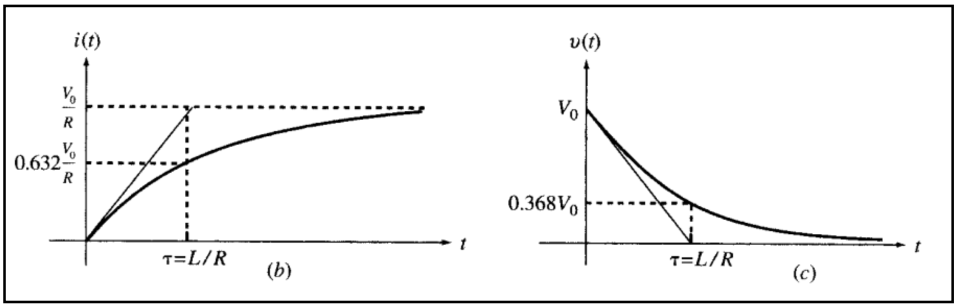


Fig. 07: The current in, and the voltage across, the inductor

6.The exponential decay function

The exponential decay function may be written in the form $e^{-t/\tau}$, where τ is the time constant (in, s). For the RC circuit, τ = RC; while for the RL circuit, τ = L / R. The general decay function:

$$\boxed{f(t) = Ae^{-t/\tau} \qquad (t > 0)} \tag{15}$$

is plotted in Fig. (08), with time measured in multiples of τ. It is seen that:

$$\boxed{f(\tau) = Ae^{-1} = 0.368A}$$

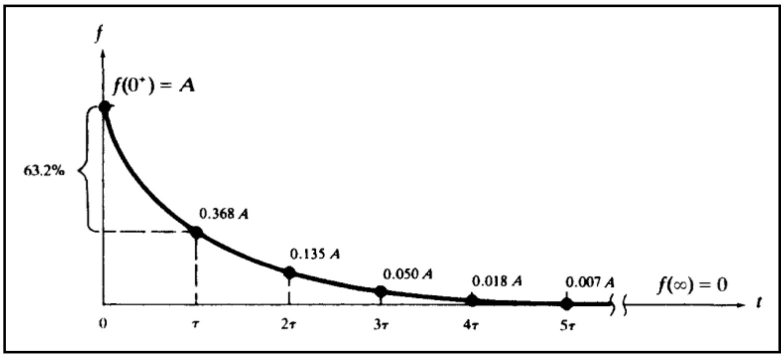


Fig. 08: The exponential decay function

that is, at t + τ the function is 36.8 percent of the initial value. It may also be said that the function has undergone 63.2 percent of the change from f (0^+) to f (∞). At t = 5τ, the function has the value 0.0067A, which is less than 1 percent of the initial value. From a practical standpoint, the transient is often regarded as over after t = 5τ.

The tangent to the exponential curve at t = 0^+ can be used to estimate the time constant. In fact, since

$$\boxed{\text{slope} = f'(0^+) = -\frac{A}{\tau}} \tag{16}$$

the tangent line must cut the horizontal axis at t = τ (see Fig. (09)). More generally, the tangent at

$t = t_0$ has horizontal intercept $t_0 + \tau$. Thus, if the two values $f(t_0)$ and $f'(t_0)$ are known, the entire curve can be constructed.

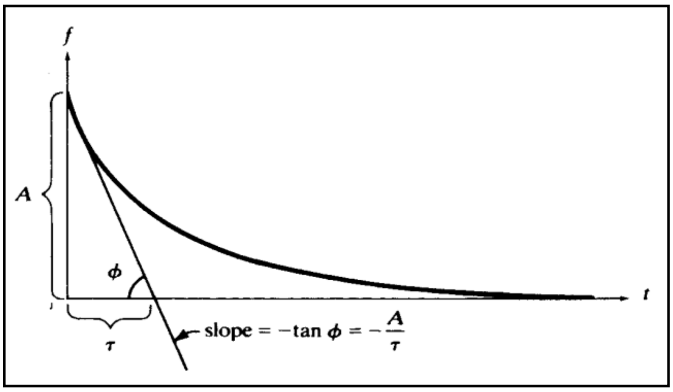


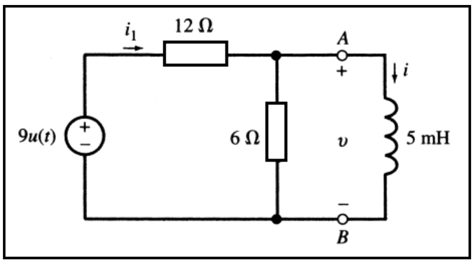
Fig. 09: The tangent to the exponential curve to estimate the time constant

7. Complex RL and RC circuits

A more complex circuit containing resistors, sources, and a single energy storage element may be converted to a Thevenin or Norton equivalent as seen from the two terminals of the inductor or capacitor. This reduces the complex circuit to a simple RC or RL circuit which may be solved according to the methods described in the previous sections.

If a source in the circuit is suddenly switched to a dc value, the resulting currents and voltages are exponentials, sharing the same time constant with possibly different initial and final values. The time constant of the circuit is either RC or L / R, where R is the resistance in the Thevenin equivalent of the circuit as seen by the capacitor or inductor.

Example: Find i, v, and i_1 in figure below.



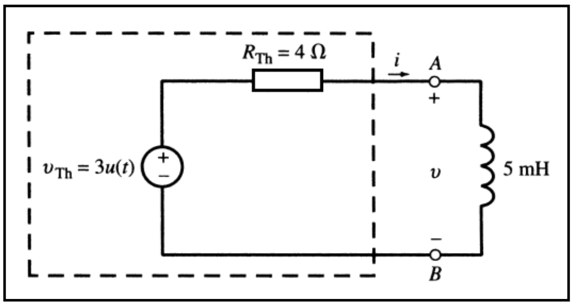
The Thevenin equivalent of the circuit to the left of the inductor is shown in figure below with

R_{Th} = 4 Ω and v_{Th} = 3u(t) (V). The time constant of the circuit is $\tau = L/R_{Th} = 5(10^{-3})/4\,\text{s} = 1.25\,\text{ms}$. The initial value of the inductor current is zero. Its final value is:

$$\boxed{i(\infty) = \frac{v_{Th}}{R_{Th}} = \frac{3\,\text{V}}{4\,\Omega} = 0.75\,\text{A}}$$

Therefore,

$$\boxed{i = 0.75(1 - e^{-800t})u(t) \quad \text{(A)} \qquad v = L\frac{di}{dt} = 3e^{-800t}u(t) \quad \text{(V)} \qquad i_1 = \frac{9 - v}{12} = \frac{1}{4}(3 - e^{-800t})u(t) \quad \text{(A)}}$$



v can also be derived directly from its initial value v (0^+) = (9 x 6) / (12 + 6) = 3 V, its final value v (∞) = 0 and the circuit's time constant.

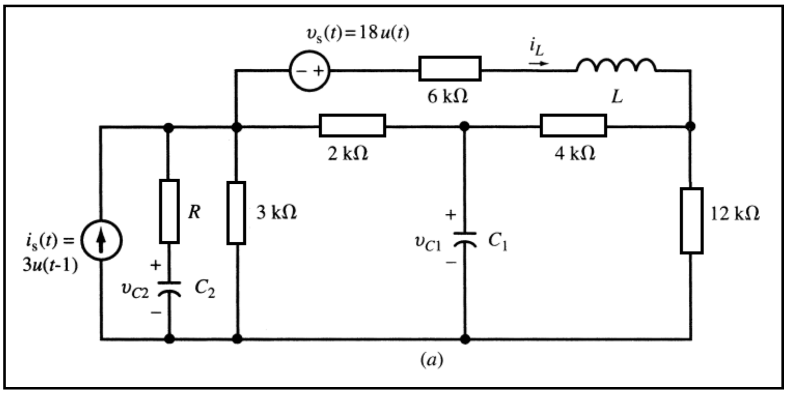
8. DC steady state in inductors and capacitors

As noted, the natural exponential component of the response of RL and RC circuits to step inputs diminishes as time passes. At t = ∞, the circuit reaches steady state and the response is made of the forced DC component only see Fig. (09).

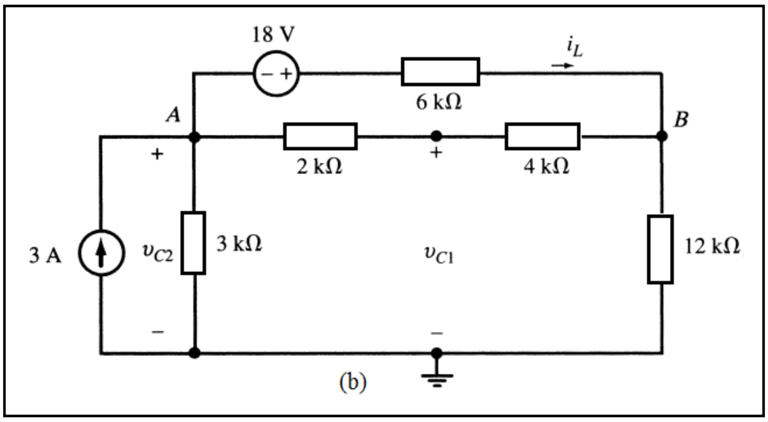
At the dc steady state of RLC circuits, assuming no sustained oscillations exist in the circuit, all currents and voltages in the circuit are constants. When the voltage across a capacitor is constant, the current through it is zero. All capacitors, therefore, appear as open circuits in the dc steady state. Similarly, when the current through an inductor is constant, the voltage across it is zero. All inductors therefore appear as short circuits in the dc steady state. The circuit will be reduced to a DC-resistive case from which voltages across capacitors and currents through inductors can be easily found, as all the currents and voltages are constants and the analysis involves no differential equations.

The DC steady-state behavior presented in the preceding paragraph is valid for circuits containing any number of inductors, capacitors, and DC sources.

Example: Find the steady-state values of i_L, v_{C1}, and v_{C2} in the circuit of Fig. (a).



When the steady state is reached, the circuit will be as shown in Fig. (b).



The inductor current and capacitor voltages are obtained by applying KCL at nodes A and B in Fig. (b). Thus,

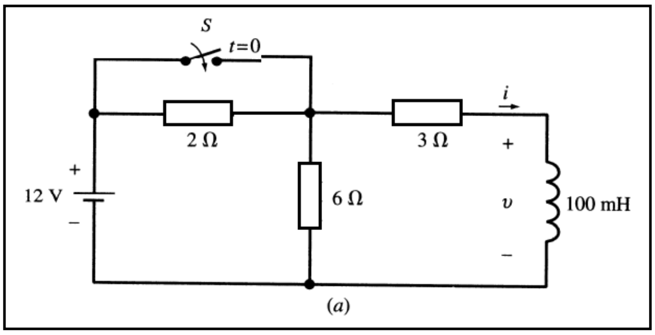
Node A:	$\dfrac{v_A}{3}+\dfrac{v_A-v_B}{6}+\dfrac{v_A+18-v_B}{6}=3$	or	$2v_A-v_B=0$
Node B:	$\dfrac{v_B}{12}+\dfrac{v_B-v_A}{6}+\dfrac{v_B-18-v_A}{6}=0$	or	$-4v_A+5v_B=36$

Solving for v_A and v_B we find $v_A = 6$ V and $v_B = 12$ V. By inspection of Fig. (b), we have $i_L = 2$ mA, $v_{C1} = 8$ V, and $v_{C2} = 6$ V.

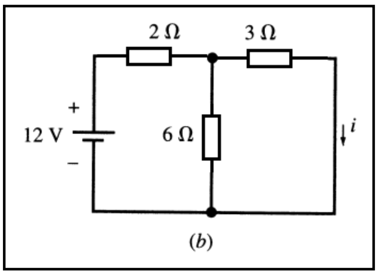
9. Transitions at switching time

A sudden switching of a source or a jump in its magnitude can translate into sudden jumps in voltages or currents in a circuit. A jump in the capacitor voltage requires an impulse current. Similarly, a jump in the inductor current requires an impulse voltage. If no such impulses can be present, the capacitor voltages and the inductor currents remain continuous. Therefore, the post-switching conditions of L and C can be derived from their pre-switching conditions.

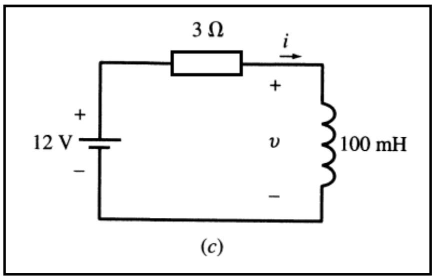
Example: In Fig. (a) the switch S is closed at t = 0. Find i and v for all times.



At t = 0^-, the circuit is at steady state and the inductor functions as a short with v (0^-) = 0 (see Fig. 7 (b)).



The inductor current is then easily found to be i (0^-) = 2 A. After S is closed at t = 0, the circuit will be as shown in Fig. (c).

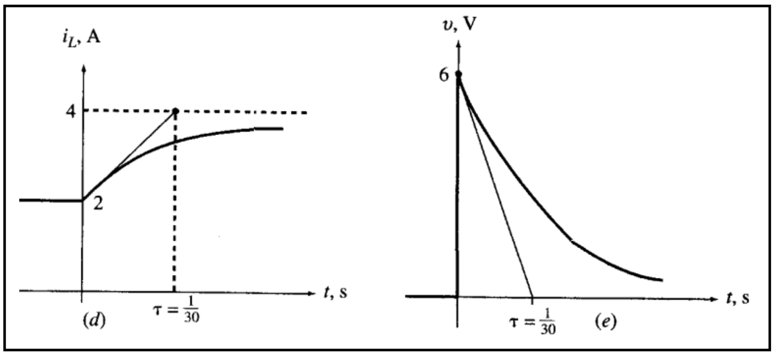


For t > 0, the current is exponential with a time constant of $\tau = L/R = 1/30\,\text{s}$, an initial value of $i(0^+) = i(0^-) = 2\,\text{A}$, and a final value of 12 / 3 = 4 A. The inductor's voltage and current are:

For $t < 0$, $\quad i = 2\,\text{A}$ and $v = 0$

For $t > 0$, $\quad i = 4 - 2e^{-30t}$ (A) and $v = L\dfrac{di}{dt} = 6e^{-30t}$ (V)

and plotted in Figs. (d) and (e).



10. Response of simple RL and RC circuits to a Pulse

In this section we will derive the response of a first-order circuit to a rectangular pulse. The derivation applies to RC or RL circuits where the input can be a current or a voltage. As an example, we use the series RC circuit in Fig. (10) with the voltage source delivering a pulse of duration T and height V_0. For $t < 0$, u and i are zero. For the duration of the pulse, we use (09) and (10) in Section (03):

$$\boxed{\upsilon = V_0(1 - e^{-t/RC}) \qquad (0 < t < T)} \tag{17}$$

$$\boxed{i = \frac{V_0}{R} e^{-t/RC} \qquad (0 < t < T)} \tag{18}$$

When the pulse ceases, the circuit is source-free with the capacitor at an initial voltage V_T.

$$\boxed{V_T = V_0(1 - e^{-T/RC})} \tag{19}$$

Using (05) in Section (02), and taking into account the time shift T, we have:

$$\boxed{\upsilon = V_T e^{-(t-T)/RC} \qquad (t > T)} \tag{20}$$

$$\boxed{i = -(V_T/R) e^{-(t-T)/RC} \qquad (t > T)} \tag{21}$$

The capacitor voltage and current are plotted in Figs. (11) and (12).

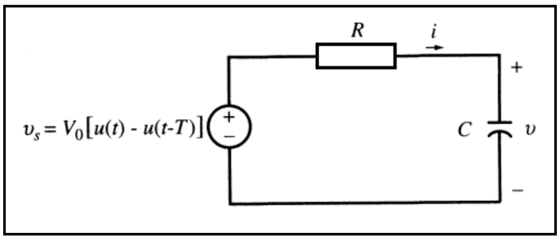


Fig. 10: The series RC circuit

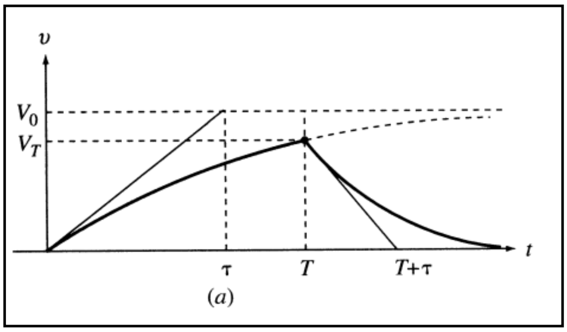


Fig. 11: The capacitor voltage

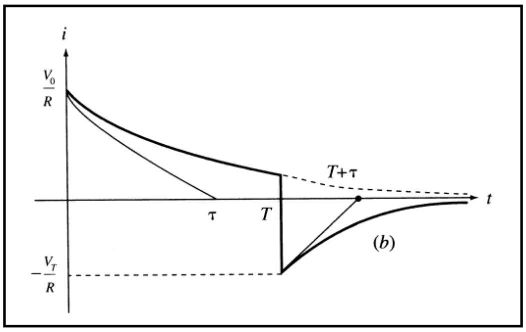


Fig. 12: The capacitor current

Example: In the circuit of last figure (section 10), let R = 1 kΩ and C = 1 μF and let the voltage source be a pulse of height V_0 and duration T. Find i and v for (a) V_0 = 1 V and T = 1 ms, (b) V_0 = 10 V and T = 0.1 ms, and (c) V_0 = 100 V and T = 0.01 ms.

We use (16), (17), (18), (19) and (20) with the time constant of t = RC = 1 ms. For convenience, time will be expressed in ms, voltages in V, and currents in mA. We also use the approximation $e^{-t} \approx 1 - t$ when t << 1.

(a) $V_0 = 1$ V, $T = 1$ ms.

For $0 < t < 1$ ms, $\upsilon = (1 - e^{-t})$, $i = e^{-t}$, and $V_T = (1 - e^{-1}) = 0.632$ V

For $t > 1$ ms, $\upsilon = 0.632e^{-(t-1)} = 1.72e^{-t}$, and $i = -1.72e^{-t}$

(b) $V_0 = 10$ V, $T = 0.1$ ms.

For $0 < t < 0.1$ ms, $\upsilon = 10(1 - e^{-t})$, $i = 10e^{-t}$, and $V_T = 10(1 - e^{-0.1}) = 0.95$ V

For $t > 0.1$ ms, $\upsilon = 0.95e^{-(t-0.1)} = 1.05e^{-t}$, and $i = -1.05e^{-t}$

(c) $V_0 = 100$ V, $T = 0.01$ ms.

For $0 < t < 0.01$ ms, $\upsilon = 100(1 - e^{-t}) \approx 100t$, $i = 100e^{-t} \approx 100(1 - t)$, and $V_T = 100(1 - e^{-0.01}) = 0.995$ V

For $t > 0.01$ ms, $\upsilon = 0.995e^{-(t-0.01)} = 1.01e^{-t}$ and $i = -1.01e^{-t}$

As the input voltage pulse approaches an impulse, the capacitor voltage and current approach $\upsilon = e^{-t}u(t)$ (V) and $i = \delta(t) - e^{-t}u(t)$, respectively.

11. Impulse response of RC and RL circuits

A narrow pulse can be modeled as an impulse with the area under the pulse indicating its strength. Impulse response is a useful tool in the analysis and synthesis of circuits. It may be derived in several ways: take the limit of the response to a narrow pulse, to be called the limit approach, as illustrated in Examples (section (10)) and (section (11)); take the derivative of the step response; or solve the differential equation directly. The impulse response is often designated by h(t).

Example: Find the limits of i and v of the circuit in figure (section 10) for a voltage pulse of unit area as the pulse duration is decreased to zero.

We use the pulse responses in (17), (18), (19), (20) and (21) with V_0 = 1/T and find their limits as T approaches zero. From (19) we have:

$$\boxed{\lim_{T\to 0} V_T = \lim_{T\to 0}(1 - e^{-T/RC})/T = 1/RC}$$

From (20) and (21) we have:

$$\begin{array}{llll}
\text{For } t < 0, & h_\upsilon = 0 & \text{and} & h_i = 0 \\
\text{For } 0^- < t < 0^+, & 0 \le h_\upsilon \le \dfrac{1}{RC} & \text{and} & h_i = \dfrac{1}{R}\delta(t) \\
\text{For } t > 0, & h_\upsilon(t) = \dfrac{1}{RC}e^{-t/RC} & \text{and} & h_i(t) = -\dfrac{1}{R^2C}e^{-t/RC}
\end{array}$$

Therefore,

$$\boxed{h_\upsilon(t) = \frac{1}{RC}e^{-t/RC}u(t) \qquad \text{and} \qquad h_i(t) = \frac{1}{R}\delta(t) - \frac{1}{R^2C}e^{-t/RC}u(\boldsymbol{t})}$$

Example: Find the impulse responses of the RC circuit in figure (section 10) by taking the derivatives of its unit step responses.

A unit impulse may be considered the derivative of a unit step. Based on the properties of linear differential equations with constant coefficients, we can take the time derivative of the step response to find the impulse response. The unit step responses of an RC circuit were found in (07), (08) and (09) to be:

$$\boxed{\upsilon(t) = (1 - e^{-t/RC})u(t) \qquad \text{and} \qquad i(t) = (1/R)e^{-t/RC}u(t)}$$

We find the unit impulse responses by taking the derivatives of the step responses. Thus:

$$\boxed{h_\upsilon(t) = \frac{1}{RC}e^{-t/RC}u(t) \qquad \text{and} \qquad h_i(t) = \frac{1}{R}\delta(t) - \frac{1}{R^2C}e^{-t/RC}u(\boldsymbol{t})}$$

12. Summary of step and impulse responses in RC and RL circuits

Responses of RL and RC circuits to step and impulse inputs are summarized in Tab. (01).

Table (01 a): Step and impulse responses in RC Circuits

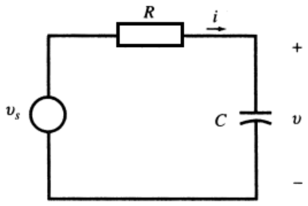
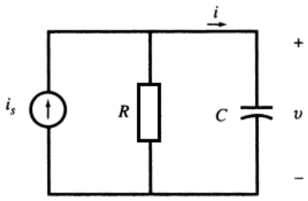
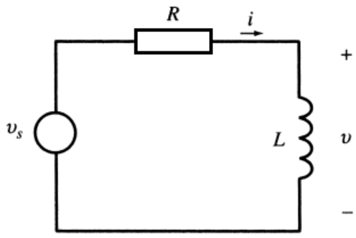
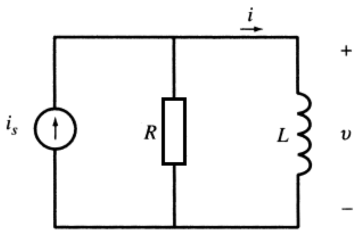
RC Circuit	Unit Step Response	Unit Impulse Response
	$\upsilon_s = u(t)$ $\begin{cases} \upsilon = (1 - e^{-t/Rc})u(t) \\ i = (1/R)e^{-t/Rc}u(t) \end{cases}$	$\upsilon_s = \delta(t)$ $\begin{cases} h_\upsilon = (1/RC)e^{-t/RC}u(t) \\ h_i = -(1/R^2C)e^{-t/RC}u(t) + (1/R)\delta(t) \end{cases}$
	$i_s = u(t)$ $\begin{cases} \upsilon = R(1 - e^{-t/RC})u(t) \\ i = e^{-t/RC}u(t) \end{cases}$	$i_s = \delta(t)$ $\begin{cases} h_\upsilon = (1/C)e^{-t/RC}u(t) \\ h_i = -(1/RC)e^{-t/RC}u(t) + \delta(t) \end{cases}$

Table (01 b): Step and Impulse Responses in RL Circuits

RL Circuit	Unit Step Response	Unit Impulse Response
	$\upsilon_s = u(t)$ $\begin{cases} \upsilon = e^{-Rt/L}u(t) \\ i = (1/R)(1 - e^{-Rt/L})u(t) \end{cases}$	$\upsilon_s = \delta(t)$ $\begin{cases} h_\upsilon = (R/L)e^{-Rt/L}u(t) + \delta(t) \\ h_i = -(1/L)e^{-Rt/L}u(t) \end{cases}$
	$i_s = u(t)$ $\begin{cases} \upsilon = Re^{-Rt/L}u(t) \\ i = (1 - e^{-Rt/L})u(t) \end{cases}$	$i_s = \delta(t)$ $\begin{cases} h_\upsilon = -(R^2/L)e^{-Rt/L}u(t) + R\delta(t) \\ h_i = (R/L)e^{-Rt/L}u(t) \end{cases}$

13. Summary response of RC and RL circuits to sudden exponential excitations

Consider the first-order differential equation which is derived from an RL combination in series with a sudden exponential voltage source $\upsilon_s = V_0 e^{st} u(t)$ as in the circuit of Fig. (13). The circuit is at rest for t < 0.

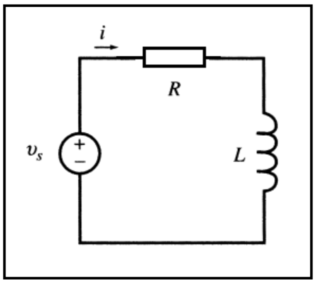


Fig. 13: The RL circuit for response to sudden exponential excitations

By applying KVL, we get:

$$\boxed{Ri + L\frac{di}{dt} = V_0 e^{st} u(t)} \tag{22}$$

For t > 0, the solution is:

$$\boxed{i(t) = i_h(t) + i_p(t) \qquad \text{and} \qquad i(0^+) = 0} \tag{23}$$

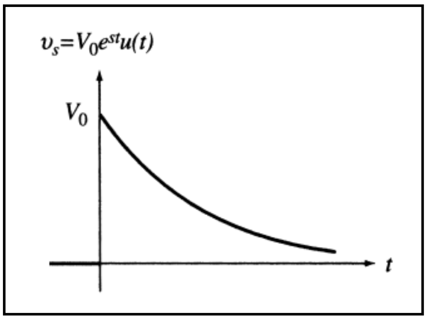


Fig. 14: The sudden exponential voltage source

The natural response $i_h(t)$ is the solution of $Ri + L(di/dt) = 0$, the case with a zero forcing function. Following an argument similar to that of Section (04) we obtain:

$$\boxed{i_h(t) = Ae^{-Rt/L}} \tag{24}$$

The forced response $i_p(t)$ is a function which satisfies (21) for t > 0. The only such function is:

$$\boxed{i_p(t) = I_0 e^{st}} \tag{25}$$

After substituting i_p in (22), I_0 is found to be $I_0 = V_0/(R + Ls)$. By choosing $A = -V_0/(R + Ls)$, the boundary condition $i(0^+) = 0$ is also satisfied. Therefore,

$$\boxed{i(t) = \frac{V_0}{R + Ls}(e^{st} - e^{-Rt/L})u(t)} \tag{26}$$

Special Case. If the forcing function has the same exponent as that of the natural response (s = −R/L), the forced response needs to be $i_p(t) = I_0 te^{-Rt/L}$. This can be verified by substitution in (22), which also yields $I_0 = V_0 / L$. The natural response is the same as (24). The total response is then

$$i(t) = i_p(t) + i_h(t) = (I_0 t + A)e^{-Rt/L} \tag{27}$$

From $i(0^-) = i(0^+) = 0$ we find A = 0, and so $i(t) = I_0 t e^{-Lt/R} u(t)$, where $I_0 = V_0 / L$.

14. Response of RC and RL circuits to sudden sinusoidal excitations

When a series RL circuit is connected to a sudden AC voltage $v_s = V_0 \cos \omega t$ (Fig. (15)) the equation of interest is:

$$Ri + L\frac{di}{dt} = V_0(\cos \omega t)u(t) \tag{28}$$

The solution is:

$$i(t) = i_h + i_p \quad \text{where} \quad i_h(t) = Ae^{-Rt/L} \quad \text{and} \quad i_p(t) = I_0 \cos(\omega t - \theta) \tag{29}$$

By inserting i_p in (28), we find I_0:

$$I_0 = \frac{V_0}{\sqrt{R^2 + L^2\omega^2}} \qquad \text{and} \qquad \theta = \tan^{-1}\frac{L\omega}{R} \tag{30}$$

Then,

$$i(t) = Ae^{-Rt/L} + I_0 \cos(\omega t - \theta) \qquad t > 0 \tag{31}$$

From $i(0^+) = 0$, we get $A = -I_0 \cos\theta$. Therefore,

$$i(t) = I_0[\cos(\omega t - \theta) - \cos\theta(e^{-Rt/L})] \tag{32}$$

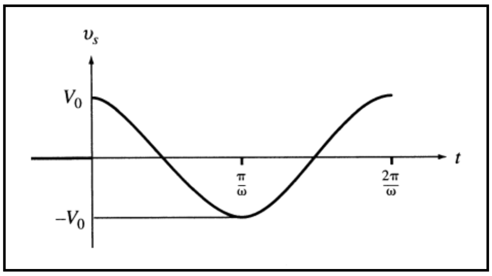


Fig. 15: The AC voltage source signal

15. Summary of forced response in simple R, L and C circuits

Table (02) summarizes some useful pairs of the forcing function and what should be guessed for $v_p(t)$. The responses are obtained by substitution in the differential equation. By a weighted linear combination of the entries in Tab. (02) and the appropriate time delay, the forced response to new functions may be deduced.

Table (02): Some useful pairs of the forcing function and the forced response

$f(t)$	$v_p(t)$		
1	$\frac{1}{a}$		
t	$\frac{t}{a} - \frac{1}{a^2}$		
$e^{st}, (s \neq -a)$	$\frac{e^{st}}{s+a}$		
e^{-at}	te^{-at}		
$\cos \omega t$	$A \cos (\omega t - \theta)$	where $A = \frac{1}{\sqrt{a^2 + \omega^2}}$	and $\tan \theta = \frac{\omega}{a}$
$e^{-bt} \cos \omega t$	$Ae^{-bt} \cos (\omega t - \theta)$	where $A = \frac{1}{\sqrt{(a-b)^2 + \omega^2}}$	and $\tan \theta = \frac{\omega}{a-b}$

Part V. Study of complex RLC circuits in free and forced regime

1. Introduction

When two or more storage elements are present, (complex RLC circuit), the network equations will result in second -order differential equations. In this in part of the course, several examples of second-order circuits in free and forced regime will be presented.

2. Series RLC circuit

The second-order differential equation, which will be examined shortly, has a solution that can take three different forms, each form depending on the circuit elements.

The series RLC circuit shown in Fig. (01) contains no voltage source. Kirchhoff's voltage law for the closed loop after the switch is closed is:

$$\boxed{\upsilon_R + \upsilon_L + \upsilon_C = 0 \quad \text{or} \quad Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = 0} \tag{01}$$

Differentiating and dividing by L yields

$$\boxed{\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0} \tag{02}$$

A solution of this second-order differential equation is of the form $i = A_1 e^{s_1 t} + A_2 e^{s_2 t}$. Substituting this solution in the differential equation results in:

$$\boxed{A_1 e^{s_1 t}\left(s_1^2 + \frac{R}{L}s_1 + \frac{1}{LC}\right) + A_2 e^{s_2 t}\left(s_2^2 + \frac{R}{L}s_2 + \frac{1}{LC}\right) = 0} \tag{03}$$

Therefore, if s_1 and s_2 must be the roots of $s^2 + (R/L)s + (1/LC) = 0,$

$$\boxed{s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \equiv -\alpha + \beta \qquad s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \equiv -\alpha - \beta} \tag{04}$$

Where

$$\boxed{\alpha \equiv R/2L,\ \beta \equiv \sqrt{\alpha^2 - \omega_0^2},\ \text{and}\ \omega_0 \equiv 1/\sqrt{LC}.} \tag{05}$$

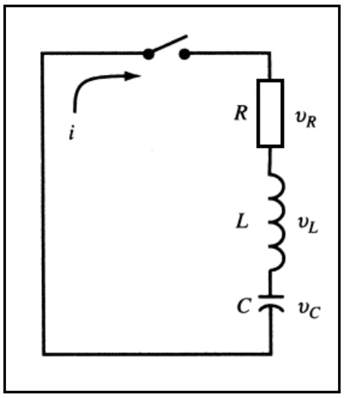


Fig. 01: The series RLC circuit

Overdamped case ($\alpha > \omega_0$)

In this case, both α and β are real positive numbers.

$$\boxed{i = A_1 e^{(-\alpha+\beta)t} + A_2 e^{(-\alpha-\beta)t} = e^{-\alpha t}(A_1 e^{\beta t} + A_2 e^{-\beta t})} \qquad (06)$$

<u>Example:</u> A series RLC circuit, with R = 200 Ω, L = 0.10 H, and C = 13.33 μF, has an initial charge on the capacitor of $Q_0 = 2.67 \times 10^{-3}$ C. A switch is closed at t = 0, allowing the capacitor to discharge. Obtain the current transient. (See Fig. below)

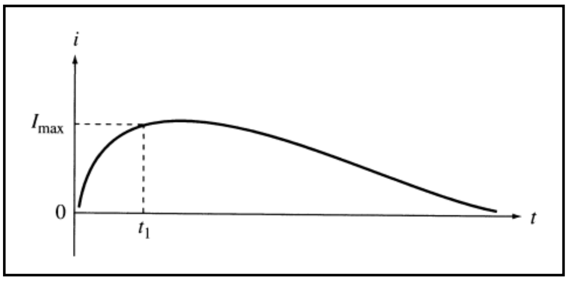
For this circuit,

$$\boxed{\alpha = \frac{R}{2L} = 10^3\,\mathrm{s}^{-1}, \qquad \omega_0^2 = \frac{1}{LC} = 7.5 \times 10^5\,\mathrm{s}^{-2}, \qquad \text{and} \qquad \beta = \sqrt{\alpha^2 - \omega_0^2} = 500\ \mathrm{s}^{-1}}$$

$$\boxed{i = e^{-1000t}(A_1 e^{500t} + A_2 e^{-500t})}$$

$$\boxed{\begin{array}{l} \text{The inductance: } i(0^+) = i(0^-) \\ \text{The capacitor: } Q(t=0^+) = Q(t=0^-) \ , \ v_C(0^-) = Q_0/C = 200\ \mathrm{V} \\ 0 = A_1 + A_2 \ , \ \pm 2000 = -500A_1 - 1500A_2 \\ A_1 = \pm 2, A_2 = \pm 2 \ , \ A_1 < 0 \end{array}}$$

$$\boxed{i = 2e^{-500t} - 2e^{-1500t} \ \text{(A)}}$$



Critically Damped Case (α = ω₀)

With $\alpha = \omega_0$, the differential equation takes on a different form and the two exponential terms suggested in the preceding will no longer provide a solution. The equation becomes:

$$\boxed{\frac{d^2i}{dt^2} + 2\alpha\frac{di}{dt} + \alpha^2 i = 0} \qquad (07)$$

and the solution takes the form

$$\boxed{i = e^{-\alpha t}(A_1 + A_2 t)} \qquad (08)$$

<u>Example:</u> Repeat the last example for C = 10 µF, which results in $\alpha = \omega_0$.

As in the example, the initial conditions are used to determine the constants. Since:

$$\boxed{i(0^-) = i(0^+),\ 0 = [A_1 + A_2(0)] \ \text{ and } A_1 = 0}$$

Then

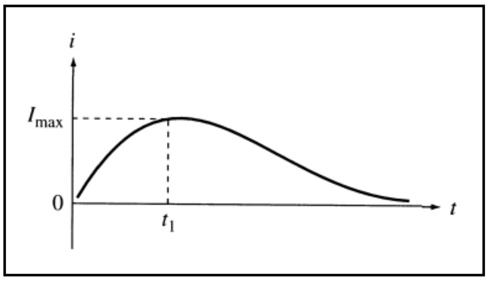
$$\boxed{\frac{di}{dt} = \frac{d}{dt}(A_2 t e^{-\alpha t}) = A_2(-\alpha t e^{-\alpha t} + e^{-\alpha t})}$$

from which

$$\boxed{A_2 = (di/dt)|_{0^+} = \pm 2000}$$

Hence,

$$\boxed{i = \pm 2000 t e^{-10^3 t}}$$



Underdamped or Oscillatory Case ($\alpha < \omega_0$)

When $\alpha < \omega_0$, s_1 and s_2 in the solution to the differential equation suggested in the preceding are complex conjugates $s_1 = \alpha + j\beta$ and $s_2 = \alpha - j\beta$, where b is now given by $\sqrt{\omega_0^2 - \alpha^2}$. The solution can be written in the exponential form

$$i = e^{-\alpha t}(A_1 e^{j\beta t} + A_2 e^{-j\beta t}) \tag{09}$$

or, in a readily derived sinusoidal form,

$$i = e^{-\alpha t}(A_3 \cos \beta t + A_4 \sin \beta t) \tag{10}$$

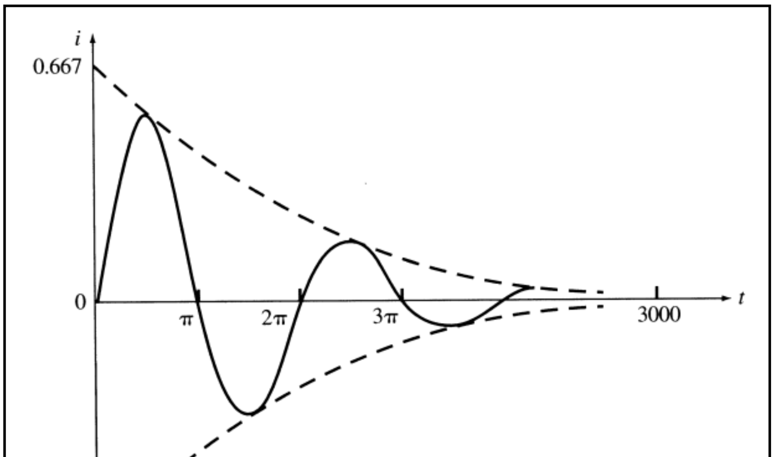
<u>Example:</u> Repeat the last example for C = 1 µF,

$$\alpha = \frac{R}{2L} = 1000\ \mathrm{s}^{-1}, \quad \omega_0^2 = \frac{1}{LC} = 10^7\ \mathrm{s}^{-2}, \quad \beta = \sqrt{10^7 - 10^6} = 3000\ \mathrm{rad/s}$$

$$i = e^{-1000t}(A_3 \cos 3000t + A_4 \sin 3000t)$$

$$i(0^+) = 0 \ , \quad \upsilon_c(0^+) = 200\ \mathrm{V} \ , \quad A_3 = 0 \ , \quad A_4 = \pm 0.667$$

$$i = \pm 0.667 e^{-1000t}(\sin 3000t)\ (\mathrm{A})$$



3. Parallel RLC circuit

The response of the parallel RLC circuit shown in Fig. (01) will be similar to that of the series RLC circuit, since a second-order differential equation can be expected. The node voltage method gives:

$$\boxed{\frac{\upsilon}{R} + \frac{1}{L}\int_0^t \upsilon\, dt + C\frac{d\upsilon}{dt} = 0} \qquad (11)$$

Differentiating and dividing by C yields

$$\boxed{\frac{d^2\upsilon}{dt^2} + \frac{1}{RC}\frac{d\upsilon}{dt} + \frac{\upsilon}{LC} = 0} \qquad (12)$$

A solution is of the form

$$\boxed{\upsilon = A_1 e^{s_1 t} + A_2 e^{s_2 t}} \qquad (13)$$

where

$$\boxed{\begin{aligned} s_1 &= -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\ s_2 &= -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \end{aligned}} \qquad (14)$$

with

$$\boxed{\alpha = 1/2RC \;,\; \omega_0 = 1/\sqrt{LC}} \qquad (15)$$

Note that α, the damping factor of the transient, differs from α in the series RLC circuit.

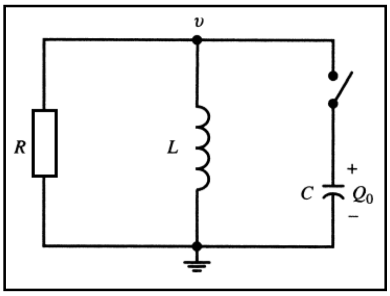


Fig. 02: The parallel RLC circuit

Overdamped Case ($\alpha^2 > \omega_0^2$)

In this case, the solution (13) applies.

<u>Example:</u> A parallel RLC circuit, with R = 1000 Ω, C = 0.167 μF, and L = 1.0 H, has an initial voltage V_0 = 50.0 V on the capacitor. Obtain the voltage v(t) when the switch is closed at t = 0.

We have:

$$\boxed{\alpha = \frac{1}{2RC} = 2994 \qquad \alpha^2 = 8.96 \times 10^6 \qquad \omega_0^2 = \frac{1}{LC} = 5.99 \times 10^6}$$

Since $\alpha^2 > \omega_0^2$, the circuit is overdamped and from (13) we have

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -1271 \quad \text{and} \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -4717$$

$$\text{At } t = 0, \qquad V_0 = A_1 + A_2 \quad \text{and} \quad \left.\frac{dv}{dt}\right|_{t=0} = s_1 A_1 + s_2 A_2$$

From the nodal equation (11), at t = 0 and with no initial current in the inductance L,

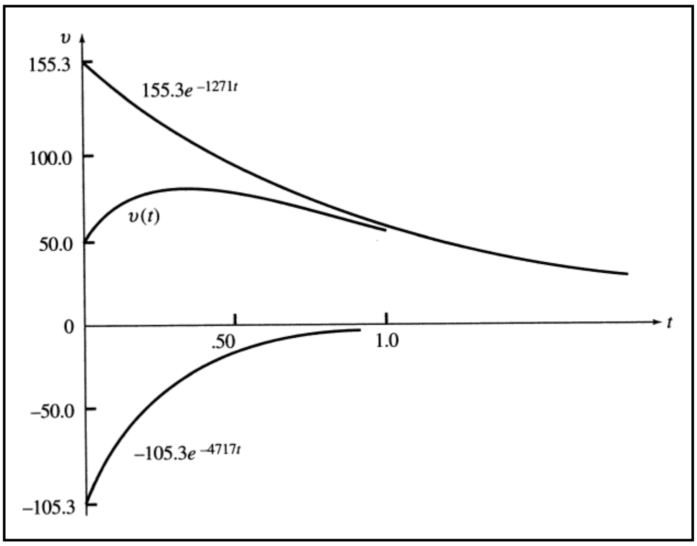
$$\boxed{\frac{V_0}{R} + C\frac{dv}{dt} = 0 \qquad \text{or} \qquad \left.\frac{dv}{dt}\right|_{t=0} = -\frac{V_0}{RC}}$$

Solving for A_1,

$$\boxed{A_1 = \frac{V_0(s_2 + 1/RC)}{s_2 - s_1} = 155.3 \qquad \text{and} \qquad A_1 = V_0 - A_1 = 50.0 - 155.3 = -105.3}$$

Substituting into (13)

$$\boxed{v = 155.3e^{-1271t} - 105.3e^{-4717t} \text{ (V)}}$$



Underdamped (Oscillatory) Case ($\alpha^2 < \omega_0^2$)

The oscillatory case for the parallel RLC circuit results in an equation of the same form as that of the underdamped series RLC circuit. Thus,

$$\boxed{v = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)} \tag{16}$$

$$\boxed{\alpha = 1/2RC \text{ and } \omega_d = \sqrt{\omega_0^2 - \alpha^2}} \tag{17}$$

ω_d is a radian frequency just as was the case with sinusoidal circuit analysis. Here it is the frequency of the damped oscillation. It is referred to as the damped radian frequency

<u>Example:</u> A parallel RLC circuit, with R = 200 Ω, L = 0.28 H, and C = 3.57 μF, has an initial voltage V_0 = 50.0 V on the capacitor. Obtain the voltage function when the switch is closed at t = 0.

$$\boxed{\alpha = \frac{1}{2RC} = \frac{1}{2(200)(3.57 \times 10^{-6})} = 700 \qquad \alpha^2 = 4.9 \times 10^5 \qquad \omega_0^2 = \frac{1}{LC} = \frac{1}{(0.28)(3.57 \times 10^{-6})} = 10^6}$$

Since $\alpha^2 < \omega_0^2$, the circuit parameters result in an oscillatory response.

$$\boxed{\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{10^6 - (4.9 \times 10^5)} = 714}$$

At t = 0, V_0 = 50.0; hence in (16) $A_1 = V_0 = 50.0$. From the nodal equation

$$\boxed{\frac{V_0}{R} + \frac{1}{L}\int_0^t v\,dt + C\frac{dv}{dt} = 0 \ , \ \left.\frac{dv}{dt}\right|_{t=0} = -\frac{V_0}{RC}}$$

Differentiating the expression for v and setting t = 0 yields

$$\boxed{\left.\frac{dv}{dt}\right|_{t=0} = \omega_d A_2 - \alpha A_1 \qquad \text{or} \qquad \omega_d A_2 - \alpha A_1 = -\frac{V_0}{RC}}$$

Since A_1 = 50.0,

$$\boxed{A_2 = \frac{-(V_0/RC) + V_0\alpha}{\omega_d} = -49.0 \quad \text{and} \quad v = e^{-700t}(50.0 \cos 714t - 49.0 \sin 714t) \ \text{(V)}}$$

The critically damped case will not be examined for the parallel RLC circuit, since it has little or no real value in circuit design. In fact, it is merely a curiosity, since it is a set of circuit constants whose response, while damped, is on the verge of oscillation.

4. Complex frequency

We have examined circuits where the driving function was a constant (e.g., V = 50.0 V), a sinusoidal function (e.g., v = 100.0 sin (500t + 30°) (V), or an exponential function, e.g., $v = 10e^{-5t}$ (V). In this section, we introduce a complex frequency, s, which unifies the three functions and will simplify the analysis, whether the transient or steady-state response is required.
We begin by expressing the exponential function in the equivalent cosine and sine form:

$$\boxed{e^{j(\omega t+\phi)} = \cos(\omega t+\phi) + j\sin(\omega t+\phi)} \tag{18}$$

We will focus exclusively on the cosine term $\cos(\omega t + \varphi) = \mathrm{Re}\, e^{j(\omega t + \varphi)}$ and for convenience drop the prefix Re. Introducing a constant A and the factor $e^{\sigma t}$,

$$\boxed{Ae^{\sigma t}e^{j(\omega t+\phi)} \Rightarrow Ae^{\sigma t}\cos(\omega t+\phi) \qquad Ae^{j\phi}e^{(\sigma+j\omega)t} = Ae^{j\phi}e^{\mathbf{s}t} \qquad \text{where } \mathbf{s} = \sigma + j\omega} \tag{19}$$

The complex frequency s = s + jω has units s^{-1}, and ω, as we know, has units rad/s. Consequently, the units on s must also be s^{-1}. This is the neper frequency with unit Np/s. If both s and ω are nonzero, the function is a damped cosine. Only negative values of s are considered. If s and ω are zero, the result is a constant. And finally, with ω = 0 and s nonzero, the result is an exponential decay function. In Tab. (01), several functions are given with corresponding values of s for the expression Ae^{st}. When Fig. (03) is examined for various values of s, the three cases are evident. If s = 0, there is no damping and the result is a cosine function with maximum values of $\pm V_m$ (not shown). If ω = 0, the function is an exponential decay with an initial value V_m. And finally, with both ω and s nonzero, the damped cosine is the result.

Tab. (01): Functions with corresponding values of s and A

$f(t)$	**s**	A
$10e^{-5t}$	$-5 + j\theta$	10
$5\cos(500t + 30°)$	$0 + j500$	5
$2e^{-3t}\cos(100t - 45°)$	$-3 + j100$	2
100.0	$0 + j0$	100.0

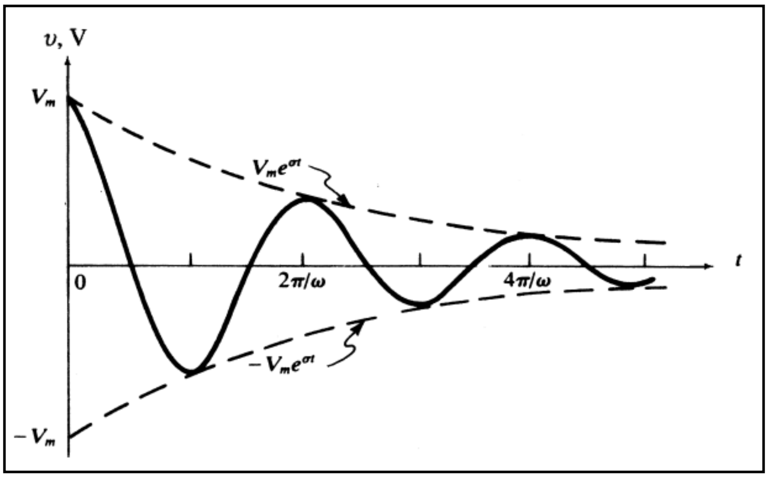


Fig. 03: The graph of the voltage v(t) for various values of s

5. Impedance (R, L, C) in s-domain

A driving voltage of the form $v = V_m e^{st}$ applied to a passive network will result in branch currents and voltages across the elements, each having the same time dependence e^{st}; e.g., $I_a e^{j\psi} e^{st}$, and $V_b e^{j\varphi} e^{st}$. Consequently, only the magnitudes of currents and voltages and the phase angles need be determined (this will also be the case in sinusoidal circuit analysis). We are thus led to consider the network in the s-domain (see Fig. (04))

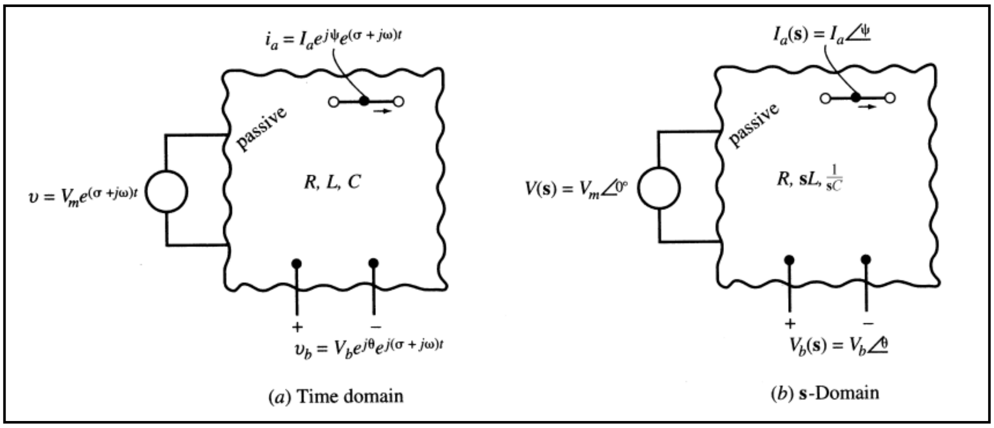


Fig. 04: The network in time-domain and s-domain

A series RL circuit with an applied voltage $v = V_m e^{j\phi} e^{st}$ will result in a current $i = I_m e^{j\psi} e^{st} = \mathbf{I}_m e^{st}$ which, substituted in the nodal equation

$$\boxed{Ri + L\frac{di}{dt} = V_m e^{j\phi} e^{st}} \tag{20}$$

will result in

$$\boxed{RI_m e^{st} = \mathbf{s}LI_m e^{st} = V_m e^{j\phi} e^{st}} \tag{21}$$

$$\boxed{I_m = \frac{V_m e^{j\phi}}{R + \mathbf{s}L}} \tag{22}$$

Note that in the s-domain the impedance of the series RL circuit is R + sL. The inductance therefore has an s-domain impedance sL.

Example: A series RL circuit, with R = 10 Ω and L = 2 H, has an applied voltage $v = 10\ e^{-2t}\cos(10t + 30°)$. Obtain the current i by an s-domain analysis.

$$\boxed{v = 10\,\underline{/30°}\,e^{st} = Ri + L\frac{di}{dt} = 10i + 2\frac{di}{dt}}$$

$$\boxed{i = Ie^{st},\quad 10\,\underline{/30°}\,e^{st} = 10Ie^{st} + 2\mathbf{s}Ie^{st} \quad \text{or} \quad I = \frac{10\,\underline{/30°}}{10 + 2\mathbf{s}}}$$

$$\boxed{\mathbf{s} = -2 + j10,\quad I = \frac{10\,\underline{/30°}}{10 + 2(-2 + j10)} = \frac{10\,\underline{/30°}}{6 + j20} = 0.48\,\underline{/-43.3°}}$$

$$\boxed{i = Ie^{st} = 0.48e^{-2t}\cos(10t - 43.3°)\ (\text{A})}$$

Example: A series RC circuit, with R = 10 Ω and C = 0.2 F, has the same applied voltage as in the last example. Obtain the current by an s-domain analysis.

As in the last example,

$$\boxed{v = 10\,\underline{/30°}\,e^{st} = Ri + \frac{1}{C}\int i\,dt = 10i + 5\int i\,dt}$$

$$\boxed{10\,\underline{/30°}\,e^{st} = 10Ie^{st} + \frac{5}{\mathbf{s}}Ie^{st}}$$

$$\boxed{I = \frac{10\,\underline{/30°}}{10 + \mathbf{5/s}} = 1.01\,\underline{/32.8°}}$$

$$\boxed{i = 1.01e^{-2t}\cos(10t + 32.8°)\ (\text{A})}$$

Note that the s-domain impedance for the capacitance is 1/(sC). Thus the s-domain impedance of a series RLC circuit will be Z(s) = R + sL + 1/(sC).

6. Network function and pole-zero plots

A driving voltage of the form v = Ve^{st} applied to a passive network will result in currents and voltages throughout the network, each having the same time function e^{st}; for example, $Ie^{j\psi}$ e^{st}. Therefore, only the magnitude I and phase angle y need be determined. We are thus led to consider an s-domain where voltages and currents are expressed in polar form, for instance, $V\angle\theta$, $I\angle\psi$, and so on. Figure (05) suggests the correspondence between the time-domain network, where s = σ + jω, and the s-domain where only magnitudes and phase angles are shown. In the s-domain, inductances are expressed by sL and capacitances by 1 / (sC). The impedance in the s-domain is Z(s) = V(s) / I(s). A network function H(s) is defined as the ratio of the complex amplitude of an exponential output Y(s) to the complex amplitude of an exponential input X(s) If, for example, X(s) is a driving voltage and Y(s) is the output voltage across a pair of terminals, then the ratio Y(s) / X(s) is non-dimensional. The network function H(s) can be:

$$\boxed{\mathbf{H}(\mathbf{s}) = \frac{\mathbf{Y}(\mathbf{s})}{\mathbf{X}(\mathbf{s})}} \tag{23}$$

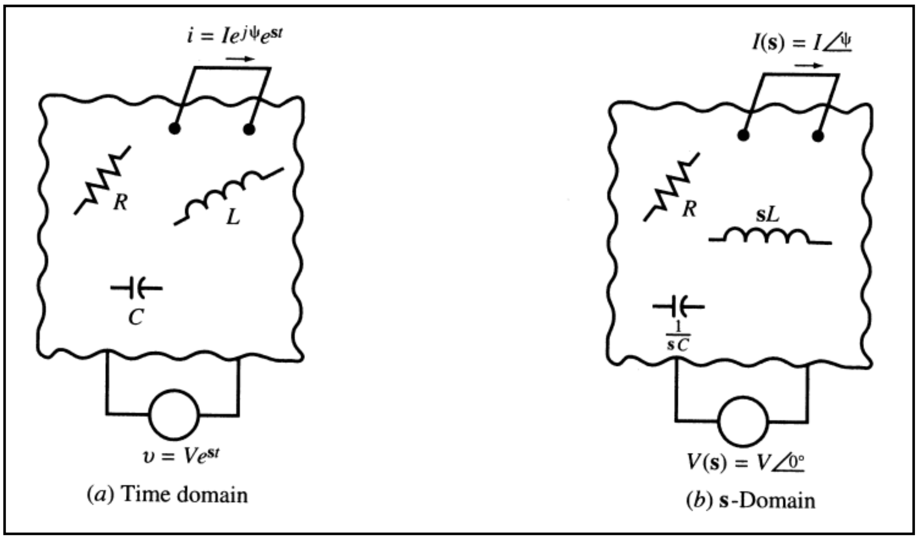


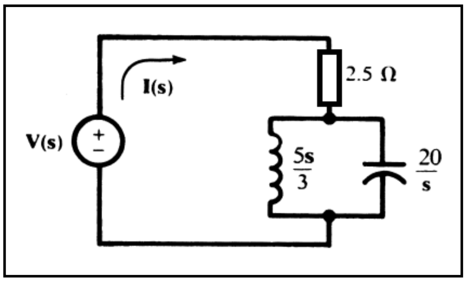
Fig. 05: The correspondence between the time-domain and the s-domain networks

In linear circuits made up of lumped elements, the network function H(s) is a rational function of s and can be written in the following general form:

$$\boxed{\mathbf{H}(\mathbf{s}) = k\frac{(\mathbf{s}-\mathbf{z}_1)(\mathbf{s}-\mathbf{z}_2)\cdots(\mathbf{s}-\mathbf{z}_\mu)}{(\mathbf{s}-\mathbf{p}_1)(\mathbf{s}-\mathbf{p}_2)\cdots(\mathbf{s}-\mathbf{p}_\nu)}} \tag{24}$$

where k is some real number. The complex constants z_m (m = 1, 2, . . ., μ), the zeros of H(s), and the p_n (n = 1, 2, . . ., u) the poles of H(s), assume particular importance when H(s) is interpreted as the ratio of the response (in one part of the s-domain network) to the excitation (in another part of the network). Thus, when s = z_m, the response will be zero, no matter how great the excitation; whereas, when s = p_n, the response will be infinite, no matter how small the excitation.

Example: A passive network in the s-domain is shown in the figure below. Obtain the network function for the current I(s) due to an input voltage V(s).



$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{I}(\mathbf{s})}{\mathbf{V}(\mathbf{s})} = \frac{1}{\mathbf{Z}(\mathbf{s})}$$

$$\mathbf{Z}(\mathbf{s}) = 2.5 + \frac{\left(\frac{5\mathbf{s}}{3}\right)\left(\frac{20}{\mathbf{s}}\right)}{\frac{5\mathbf{s}}{3} + \frac{20}{\mathbf{s}}} = (2.5)\frac{\mathbf{s}^2 + 8\mathbf{s} + 12}{\mathbf{s}^2 + 12}$$

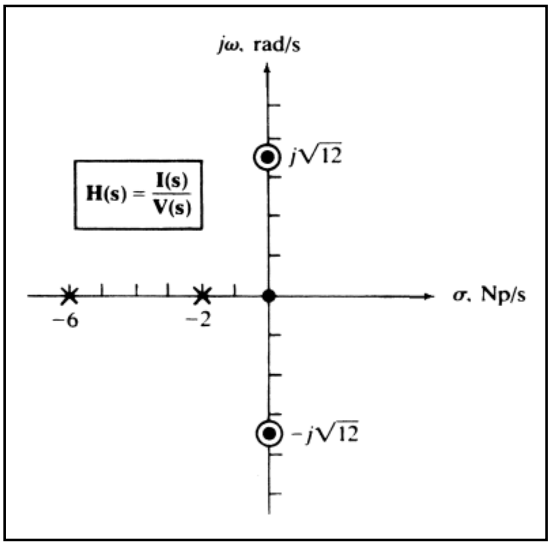
$$\mathbf{H}(\mathbf{s}) = (0.4)\frac{\mathbf{s}^2 + 12}{(\mathbf{s} + 2)(\mathbf{s} + 6)}$$

The numerator of H(s) in this example is zero when s = ± j 12. Consequently, a voltage function at this frequency results in a current of zero. In the next part where series and parallel resonance are discussed, it will be found that the parallel LC circuit is resonant at:

$$\boxed{\omega = 1/\sqrt{LC}}$$

$$\boxed{L = \tfrac{5}{3}\,\mathrm{H} \text{ and } C = \tfrac{1}{20}\,\mathrm{F},\ \omega = \sqrt{12}\ \mathrm{rad/s}}$$

The zeros and poles of a network function H(s) can be plotted in a complex **s**-plane. The figure below shows the poles and zeros of this example, with zeros marked ⊙ and poles marked x. The zeros occur in complex conjugate pairs, $\mathbf{s} = \pm j\sqrt{12}$, and the poles are s = -2 and s = -6.



jω, rad/s
j√12
H(s) = I(s)/V(s)
σ, Np/s
−6
−2
−j√12

Part VI. Study of series and parallel resonant circuits

1. Frequency response

The response of linear circuits to a sinusoidal input is also a sinusoid, with the same frequency but possibly a different amplitude and phase angle. This response is a function of the frequency. We have already seen that a sinusoid can be represented by a phasor which shows its magnitude and phase. The frequency response is defined as the ratio of the output phasor to the input phasor. It is a real function of jω and is given by:

$$\boxed{\mathbf{H}(j\omega) = \mathrm{Re}[\mathbf{H}] + j\,\mathrm{Im}[\mathbf{H}] = |\mathbf{H}|\, e^{j\theta}} \tag{01}$$

where Re[H] and I_m [H] are the real and imaginary parts of H(jω) and |H| and θ are its magnitude and phase angle. Re[H], I_m[H], |H|, and θ are, in general, functions of ω. They are related by:

$$\boxed{\begin{aligned} |\mathbf{H}|^2 &= |\mathbf{H}(j\omega)|^2 = \mathrm{Re}^2[\mathbf{H}] + \mathrm{Im}^2[\mathbf{H}] \\ \theta &= \underline{/\mathbf{H}(j\omega)} = \tan^{-1}\frac{\mathrm{Im}[\mathbf{H}]}{\mathrm{Re}[\mathbf{H}]} \end{aligned}} \tag{02}$$

The frequency response, therefore, depends on the choice of input and output variables. For example, if a current source is connected across the network of Fig. (01 a), the terminal current is the input and the terminal voltage may be taken as the output. In this case, the input impedance $Z = V_1/I_1$ constitutes the frequency response. Conversely, if a voltage source is applied to the input and the terminal current is measured, the input admittance $Y = I_1/V_1 = 1/Z$ represents the frequency response. For the two-port network of Fig. (01 b), the following frequency responses are defined:

- Input impedance $Z_{in}(j\omega) = V_1/I_1$
- Input admittance $Y_{in}(j\omega) = 1/Z_{in}(j\omega) = I_1/V_1$
- Voltage transfer ratio $H_v(j\omega) = V_2/V_1$
- Current transfer ratio $H_i(j\omega) = I_2/I_1$
- Transfer impedances V_2/I_1 and V_1/I_2

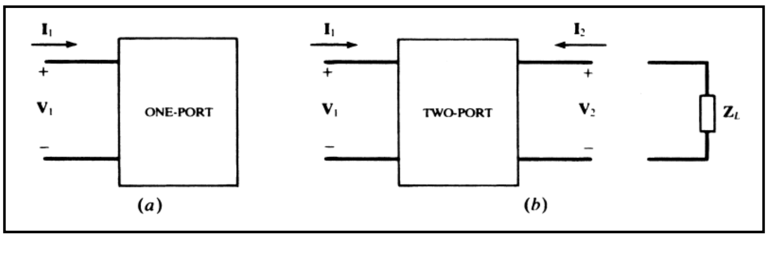


Fig. 01: The one-port and two-port networks

<u>Example:</u> Find the frequency response V_2/V_1 for the two-port circuit shown in figure below.

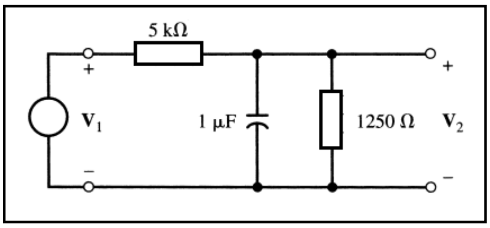
Let Y_{RC} be the admittance of the parallel RC combination. Then, $Y_{RC} = 10^{-6}\, j\omega + 1/1250$.

V_2/V_1 is obtained by dividing V_1 between Z_{RC} and the 5-kΩ resistor.

$$\mathbf{H}(j\omega) = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{\mathbf{Z}_{RC}}{\mathbf{Z}_{RC} + 5000} = \frac{1}{1 + 5000\,\mathbf{Y}_{RC}} = \frac{1}{5(1 + 10^{-3}\, j\omega)}$$

$$|\mathbf{H}| = \frac{1}{5\sqrt{1 + 10^{-6}\omega^2}} \qquad \theta = -\tan^{-1}(10^{-3}\omega)$$

Alternative solution: First we find the Thevenin equivalent of the resistive part of the circuit, $V_{Th} = V_1/5$ and $R_{Th} = 1$ kΩ, and then divide V_{Th} between R_{Th} and the 1-µF capacitor to obtain H(jω).



2. High-pass and low-pass networks

A resistive voltage divider under a no-load condition is shown in Fig. (02), with the standard two-port voltages and currents. The voltage transfer function and input impedance are:

$$\mathbf{H}_{v\infty}(\omega) = \frac{R_2}{R_1 + R_2} \qquad \mathbf{H}_{z\infty}(\omega) = R_1 + R_2 \tag{03}$$

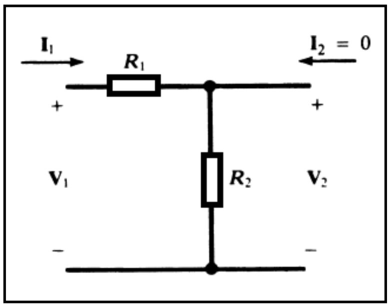
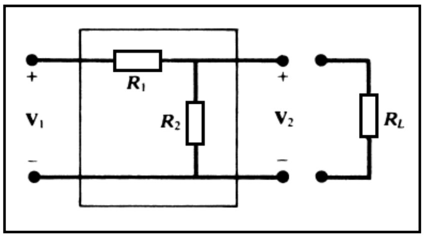


Fig. 02: The resistive voltage divider under a no-load condition

The ∞ in the subscripts indicates no-load conditions. Both $H_{v\infty}$ and $H_{z\infty}$ are real constants, independent of frequency, since no reactive elements are present.

Example: In the two-port network shown in figure below, R_1 = 7 kΩ and R_2 = 3 kΩ. Obtain the voltage ratio V_2/V_1 (a) at no-load, (b) for R_L = 20 kΩ.



(a) At no-load, voltage division gives:

$$\frac{V_2}{V_1} = \frac{R_2}{R_1 + R_2} = \frac{3}{7+3} = 0.30$$

(b) With R_L = 20 kΩ,

$$R_p = \frac{R_2 R_L}{R_2 + R_L} = \frac{60}{23}\,\text{k}\Omega$$

$$\frac{V_2}{V_1} = \frac{R_p}{R_1 + R_p} = \frac{60}{221} = 0.27$$

The voltage ratio is independent of frequency. The load resistance, 20 kΩ, reduces the ratio from 0.30 to 0.27.

If the network contains either an inductance or a capacitance, then $H_{v\infty}$ and $H_{z\infty}$ will be complex and will vary with frequency. If $|H_u\,\infty|$ decreases as frequency increases, the performance is called high-frequency roll-off and the circuit is a low-pass network. On the contrary, a high-pass network will have low-frequency roll-off, with $|H_{v\infty}|$ decreasing as the frequency decreases.
Four two-element circuits are shown in Fig. (03). Two high-pass and two low-pass.

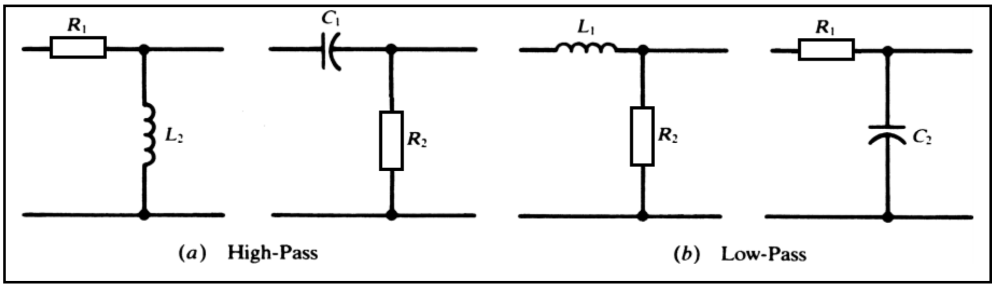


Fig. 03: The four two-element circuits, two high-pass and two low-pass

The RL high-pass circuit

- The input impedance frequency response of the RL high-pass circuit

The RL high-pass circuit shown in Fig. (04) is open-circuited or under a no-load condition. The input impedance frequency response is determined by plotting the magnitude and phase angle of

$$\boxed{\mathbf{H}_{z\infty}(\omega) = R_1 + j\omega L_2 \equiv |\mathbf{H}_z| \underline{/\theta_{\mathbf{H}}}} \qquad (04)$$

or, normalizing and setting $\omega_x \equiv R_1 / L_2$.

$$\boxed{\frac{\mathbf{H}_{z\infty}(\omega)}{R_1} = 1 + j(\omega/\omega_x) = \sqrt{1 + (\omega/\omega_x)^2}\ \underline{/\tan^{-1}(\omega/\omega_x)}} \qquad (05)$$

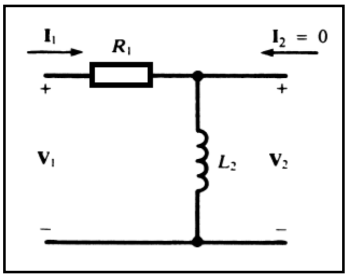


Fig. 04: The RL high-pass circuit

- The frequency response of the output-to-input voltage ratio of the RL high-pass circuit

Voltage division under a no-load condition gives:

$$\mathbf{H}_{v\infty}(\omega) = \frac{j\omega L_2}{R_1 + j\omega L_2} = \frac{1}{1 - j(\omega_x/\omega)}$$

$$|\mathbf{H}_v| = \frac{1}{\sqrt{1 + (\omega_x/\omega)^2}} \qquad \text{and} \qquad \theta_{\mathbf{H}} = \tan^{-1}(\omega_x/\omega) \qquad (06)$$

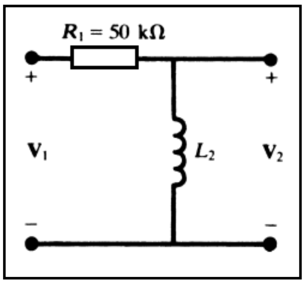
- The transfer impedance of the RL high-pass circuit under a no-load condition

$$\boxed{\mathbf{H}_{\infty}(\omega) = \frac{\mathbf{V}_2}{\mathbf{I}_1} = j\omega L_2 \qquad \text{or} \qquad \frac{\mathbf{H}_{\infty}(\omega)}{R_1} = j\frac{\omega}{\omega_x}} \qquad (07)$$

<u>**Example:**</u> (a) Find L_2 in the high-pass circuit shown in figure below, if $|H_v(\omega)| = 0.50$ at a frequency of 50 MHz. (b) At what frequency is $|H_v| = 0.90$?

(a) From Section (02), with $\omega_x \equiv R_1/L_2$

$$\boxed{|\mathrm{H}_v(\omega)| = \frac{1}{\sqrt{1 + (\omega_x/\omega)^2}}}$$



$$0.50 = \frac{1}{\sqrt{1 + (f_x/50)^2}} \qquad \text{or} \qquad f_x = 50\sqrt{3}\ \text{MHz}$$

$$L_2 = \frac{R_1}{2\pi f_x} = \frac{50 \times 10^3}{2\pi(50\sqrt{3} \times 10^6)} = 91.9\ \mu\text{H}$$

(b)

$$0.90 = \frac{1}{1 + (50\sqrt{3}/f)^2} \qquad \text{or} \qquad f = 179\ \text{MHz}$$

The RL low-pass circuit

Interchanging the positions of R and L results in a low-pass network with high-frequency roll-off (Fig. (05)). For the open-circuit condition,

$$\mathbf{H}_{v\infty}(\omega) = \frac{R_2}{R_2 + j\omega L_1} = \frac{1}{1 + j(\omega/\omega_x)} \tag{08}$$

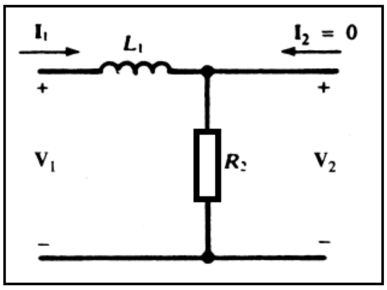


Fig. 05: The RL low-pass circuit

with $\omega_x \equiv R_2 / L_1$, that is,

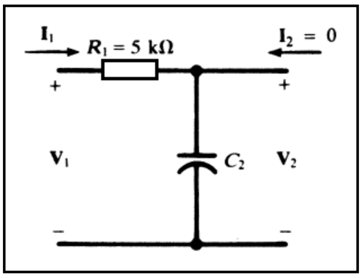
$$|\mathbf{H}_v| = \frac{1}{\sqrt{1 + (\omega/\omega_x)^2}} \qquad \text{and} \qquad \theta_{\mathbf{H}} = \tan^{-1}(-\omega/\omega_x) \tag{09}$$

Example: Obtain the voltage transfer function $H_{v\infty}$ for the open circuit shown in figure below. At what frequency, in hertz, does $|\mathbf{H}_v| = 1/\sqrt{2}$ if (a) C_2 = 10 nF, (b) C_2 = 1 nF?

$$\mathbf{H}_{v\infty}(\omega) = \frac{1/j\omega C_2}{R_1 + (1/j\omega C_2)} = \frac{1}{1 + j(\omega/\omega_x)} \quad , \quad |\mathbf{H}_v| = \frac{1}{\sqrt{1 + (\omega/\omega_x)^2}} \quad , \quad \omega_x \equiv \frac{1}{R_1 C_2} = \frac{2\times10^{-4}}{C_2} \text{ (rad/s)}$$

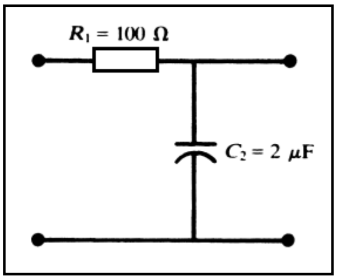
(a) $|\mathbf{H}_v| = 1/\sqrt{2}$ when $\omega = \omega_x = \dfrac{2\times10^{-4}}{10\times10^{-9}} = 2\times10^4$ rad/s or $f = (2\times10^4)/2\pi = 3.18$ kHz

(b) $f = \dfrac{10}{1}(3.18) = 31.8$ kHz



Comparing (a) with (b), it is seen that the greater the value of C_2, the lower the frequency at which |Hv| drops to 0.707 of its peak value, which is 1. Consequently, any stray shunting capacitance, in parallel with C_2, serves to reduce the response of the circuit.

Example: Find the frequency at which $|H_v| = 0.50$ for the low-pass RC network shown in figure below.



$$H_v(\omega) = \frac{1}{1 + j(\omega/\omega_x)} \qquad \text{where} \qquad \omega_x \equiv \frac{1}{R_1 C_2}$$

Then, $(0.50)^2 = \dfrac{1}{1 + (\omega/\omega_x)^2}$, $\dfrac{\omega}{\omega_x} = \sqrt{3}$

and $\omega = \sqrt{3}\left(\dfrac{1}{R_1 C_2}\right) = 8660$ rad/s or $f = 1378$ Hz

3. Half-power frequencies

The frequency ω_x calculated in the last example is the frequency at which:

$$\boxed{|\mathbf{H}_v| = 0.707|\mathbf{H}_v|_{max}} \tag{10}$$

and is called the half-power frequency.

Quite generally, any non-constant network function H(ω) will attain its greatest absolute value at some unique frequency ω_x. We shall call a frequency at which:

$$\boxed{|\mathbf{H}(\omega)| = 0.707|\mathbf{H}(\omega_x)|} \tag{11}$$

a half-power frequency, whether or not this frequency actually corresponds to 50 percent power.

4. Generalized two-port, two-element networks

The basic RL or RC network of the type examined in Section (02) can be generalized with Z_1 and Z_2, as shown in Fig. (06); the load impedance ZL is connected at the output port.

By voltage division,

$$\boxed{\mathbf{V}_2 = \frac{\mathbf{Z}'}{\mathbf{Z}_1 + \mathbf{Z}'}\mathbf{V}_1 \qquad \text{or} \qquad \mathbf{H}_v = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{\mathbf{Z}'}{\mathbf{Z}_1 + \mathbf{Z}'}} \tag{12}$$

where $\mathbf{Z}' = \mathbf{Z}_2\mathbf{Z}_L/(\mathbf{Z}_2 + \mathbf{Z}_L)$ is the equivalent impedance of Z_2 and Z_L in parallel. The other transfer functions are calculated similarly and are displayed in Tab. (01).

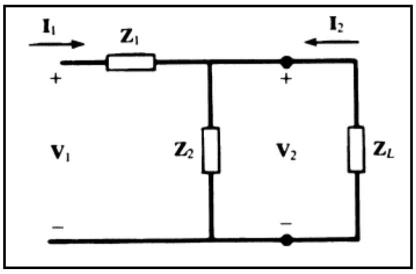


Fig. 06: The generalized two-port, two-element networks

Tab. (01): The transfer functions of the generalized two-port, two-element networks

Output Condition \ Network Function	$\mathbf{H}_z = \frac{\mathbf{V}_1}{\mathbf{I}_1}$ (Ω)	$H_\upsilon = \frac{V_2}{V_1}$	$\mathbf{H}_i = \frac{\mathbf{I}_2}{\mathbf{I}_1}$	$\mathbf{H}_\upsilon \mathbf{H}_z = \frac{\mathbf{V}_2}{\mathbf{I}_1}$ (Ω)	$\frac{\mathbf{H}_i}{\mathbf{H}_z} = \frac{\mathbf{I}_2}{\mathbf{V}_1}$ (S)
Short-circuit, $\mathbf{Z}_L = 0$	$\mathbf{Z}_1$	0	−1	0	$-\frac{1}{\mathbf{Z}_1}$
Open-circuit, $\mathbf{Z}_L = \infty$	$\mathbf{Z}_1 + \mathbf{Z}_2$	$\frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$	0	$\mathbf{Z}_2$	0
Load, $\mathbf{Z}_L$	$\mathbf{Z}_1 + \mathbf{Z}'$	$\frac{\mathbf{Z}'}{\mathbf{Z}_1 + \mathbf{Z}'}$	$\frac{-\mathbf{Z}_2}{\mathbf{Z}_2 + \mathbf{Z}_L}$	$\mathbf{Z}'$	$\frac{-\mathbf{Z}'}{\mathbf{Z}_L(\mathbf{Z}_1 + \mathbf{Z}')}$

5. Ideal and practical filters

In general, networks are frequency selective. Filters are a class of networks designed to possess specific frequency selectivity characteristics. They pass certain frequencies unaffected (the pass-band) and stop others (the stop-band). Ideally, in the pass-band, $H(j\omega) = 1$ and in the stop-band $H(j\omega) = 0$. We, therefore, recognize the following classes of filters: low-pass [Fig. (06 a)], high-pass [Fig. (06 b)], bandpass [Fig. (06 c)], and band-stop [Fig. (06 d)]. Ideal filters are not physically realizable, but we can design and build practical filters as close to the ideal one as desired. The closer to the ideal characteristic, the more complex the circuit of a practical filter will be.

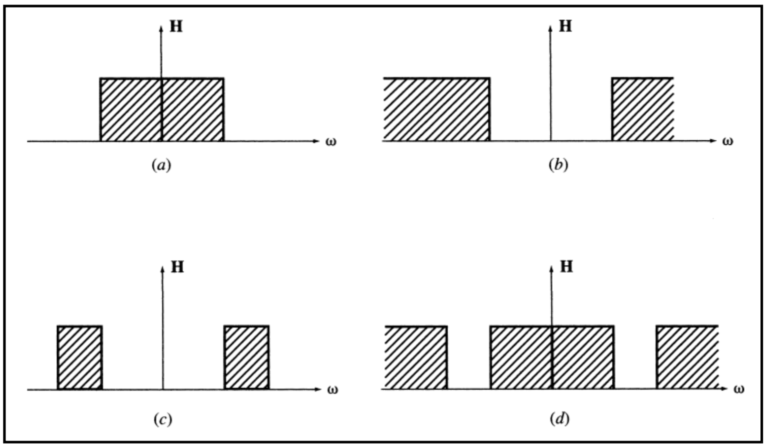


Fig. 06: The classes of filters

6. Passive and active filters

Filters which contain only resistors, inductors, and capacitors are called passive. Those containing additional dependent sources are called active. Passive filters do not require external energy sources and they can last longer. Active filters are generally made of RC circuits and amplifiers. The circuit in Fig. (07 a) shows a second-order low-pass passive filter. The circuit in Fig. (07 b) shows an active filter with a frequency response V_2/V_1 equivalent to that of the circuit in Fig. (07 a).

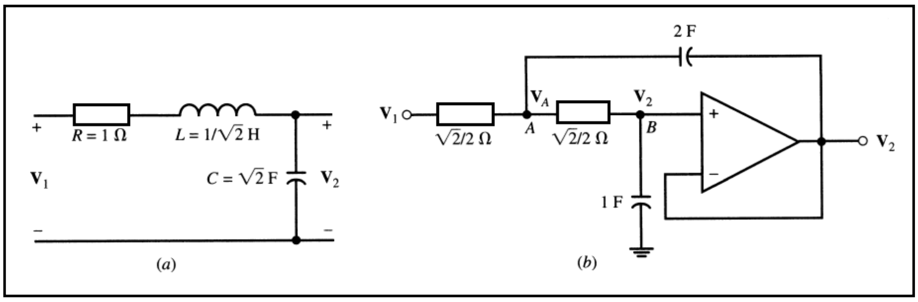


Fig. 07: A low-pass passive filter and an active filter

7. Bandpass filters and resonance

The following network function is called a bandpass function.

$$\boxed{\mathbf{H}(s) = \frac{k\mathbf{s}}{\mathbf{s}^2 + a\mathbf{s} + b} \quad , \quad a > 0 \quad , \quad b > 0 \quad , \quad k > 0} \tag{13}$$

The name is especially appropriate when the poles are complex, close to the jω axis, and away from the origin in the s-domain. The frequency response of the bandpass function is:

$$\boxed{\mathbf{H}(j\omega) = \frac{kj\omega}{b - \omega^2 + aj\omega} \qquad |\mathbf{H}|^2 = \frac{k^2\omega^2}{(b-\omega^2)^2 + a^2\omega^2} = \frac{k^2}{a^2 + (b-\omega^2)^2/\omega^2}} \tag{14}$$

The maximum of |H| occurs when $b - \omega^2 = 0$ or, $\omega = \sqrt{b}$ which is called the center frequency ω_0. At the center frequency, we have $|H|_{max} = |H(\omega_0)| = k/a$. The half-power frequencies are at ω_l and ω_h, where:

$$\boxed{|\mathbf{H}(\omega_l)|^2 = |\mathbf{H}(\omega_h)|^2 = \frac{1}{2}\,|\mathbf{H}(\omega_0)|^2} \tag{15}$$

$$\boxed{\omega_h - \omega_l = a \qquad \text{and} \qquad \omega_h\omega_l = b = \omega_0^2} \tag{16}$$

The bandwidth β is defined by:

$$\boxed{\beta = \omega_h - \omega_l = a} \tag{17}$$

The quality factor Q is defined by:

$$\boxed{Q = \omega_0/\beta = \sqrt{b}/a} \tag{18}$$

The quality factor measures the sharpness of the frequency response around the center frequency. This behavior is also called resonance.

Example: Consider the network function H(s) = 10s / (s^2 + 300s + 10^6). Find the center frequency, the lower and upper half-power frequencies, the bandwidth, and the quality factor.

Since $\omega_0^2 = 10^6$, the center frequency is ω_0 = 1000 rad/s.

The lower and upper half-power frequencies are, respectively,

$$\omega_l = \sqrt{a^2/4 + b} - a/2 = \sqrt{300^2/4 + 10^6} - 300/2 = 861.2 \text{ rad/s}$$
$$\omega_h = \sqrt{a^2/4 + b} + a/2 = \sqrt{300^2/4 + 10^6} + 300/2 = 1161.2 \text{ rad/s}$$

$$\text{The bandwidth is : } \beta = \omega_h - \omega_l = 1161.2 - 861.2 = 300 \text{ rad/s}$$
$$\text{The quality factor is : } Q = 1000/300 = 3.3$$

8. RLC series circuit; series resonance

The RLC circuit shown in Fig. (08) has, under open-circuit conditions, an input or driving-point impedance:

$$\mathbf{Z}_{\text{in}}(\omega) = R + j\left(\omega L - \frac{1}{\omega C}\right) \qquad (19)$$

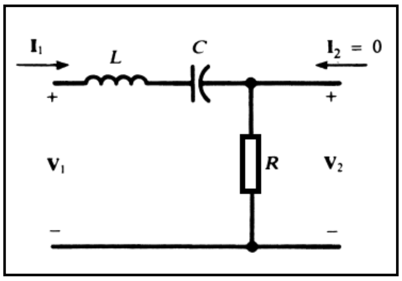


Fig. 08: RLC series circuit, (series resonance)

The circuit is said to be in series resonance (or low-impedance resonance) when $Z_{in}(\omega)$ is real (and so $|Z_{in}(\omega)|$ is a minimum); that is, when:

$$\omega L - \frac{1}{\omega C} = 0 \qquad \text{or} \qquad \omega = \omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad (20)$$

Figure (09) shows the frequency response. The capacitive reactance, inversely proportional to ω, is higher at low frequencies, while the inductive reactance, directly proportional to ω, is greater at the higher frequencies. Consequently, the net reactance at frequencies below ω_0 is capacitive, and the angle on Z_{in} is negative. At frequencies above ω_0, the circuit appears inductive, and the angle on Z_{in} is positive.

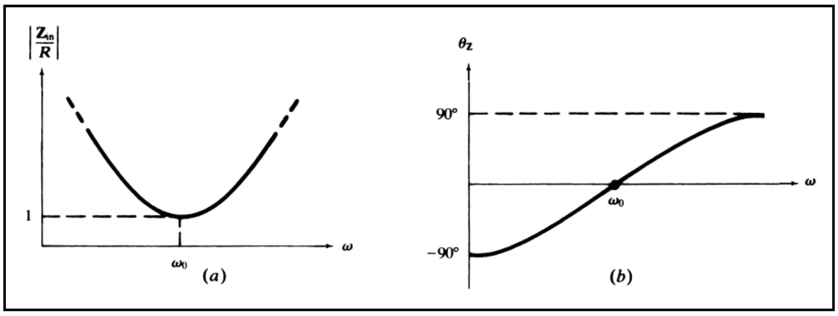


Fig. 09: The frequency response

By voltage division, the voltage transfer function for Fig. (08) is:

$$\boxed{\mathbf{H}_{v\infty}(\omega) = \frac{R}{\mathbf{Z}_{\text{in}}(\omega)} = R\mathbf{Y}_{\text{in}}(\omega)} \qquad (21)$$

The frequency response (magnitude only) is plotted in Fig. (10); the curve is just the reciprocal of that in Fig. (09 a). Note that roll-off occurs both below and above the series resonant frequency ω_0. The points where the response is 0.707, the half-power points, are at frequencies ω_l and wh. The bandwidth is the width between these two frequencies: $\beta = \omega_h - \omega_l$.

A quality factor, $Q_0 = \omega_0 L / R$, may be defined for the series RLC circuit at resonance. The half-power frequencies can be expressed in terms of the circuit elements, or in terms of ω_0 and Q_0, as follows:

$$\boxed{\begin{aligned} \omega_h &= \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = \omega_0\left(\sqrt{1 + \frac{1}{4Q_0^2}} + \frac{1}{2Q_0}\right) \\ \omega_l &= -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = \omega_0\left(\sqrt{1 + \frac{1}{4Q_0^2}} - \frac{1}{2Q_0}\right) \end{aligned}} \qquad (22)$$

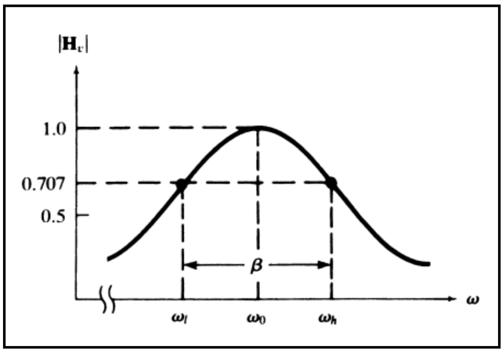
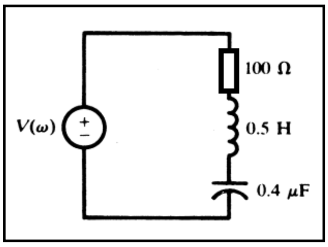


Fig. 10: The frequency response (magnitude only)
for RLC series circuit

$$\boxed{\beta = \frac{R}{L} = \frac{\omega_0}{Q_0}} \tag{23}$$

which suggests that the higher the "quality," the narrower the bandwidth.

Example: For the series RLC circuit shown in figure below, find the resonant frequency $\omega_0 = 2\pi f_0$. Also obtain the half-power frequencies and the bandwidth β.



$$\boxed{\mathbf{Z}_{\text{in}}(\omega) = R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

At resonance,

$$\boxed{\mathbf{Z}_{\text{in}}(\omega) = R \text{ and } \omega_0 = 1/\sqrt{LC} \quad , \quad \omega_0 = \frac{1}{\sqrt{0.5(0.4 \times 10^{-6})}} = 2236.1 \text{ rad/s} \quad , \quad f_0 = \frac{\omega_0}{2\pi} = 355.9 \text{ Hz}}$$

The power formula:

$$\boxed{P = I_{\text{eff}}^2 R = \frac{V_{\text{eff}}^2 R}{|\mathbf{Z}_{\text{in}}|^2}}$$

shows that $P_{\max} = V_{\text{eff}}^2/R$, which is achieved at $\omega = \omega_0$, and that $P = \frac{1}{2}P_{\max}$ when $|Z_{\text{in}}|^2 = 2R^2$; that is, when

$$\boxed{\omega L - \frac{1}{\omega C} = \pm R \qquad \text{or} \qquad \omega^2 \mp \frac{R}{L}\omega - \frac{1}{LC} = 0}$$

Corresponding to the upper sign, there is a single real positive root:

$$\boxed{\omega_h = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = 2338.3 \text{ rad/s} \qquad \text{or} \qquad f_h = 372.1 \text{ Hz}}$$

and corresponding to the lower sign, the single real positive root

$$\omega_l = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = 2138.3 \text{ rad/s} \qquad \text{or} \qquad f_l = 340.3 \text{ Hz}$$

9. Quality factor

A quality factor or figure of merit can be assigned to a component or to a complete circuit. It is defined as:

$$Q \equiv 2\pi\left(\frac{\text{maximum energy stored}}{\text{energy dissipated per cycle}}\right) \tag{24}$$

A practical inductor, in which both resistance and inductance are present, is modeled in Fig. (11). The maximum stored energy is $\frac{1}{2}LI_{\max}^2$, while the energy dissipated per cycle is:

$$(I_{\text{eff}}^2 R)\left(\frac{2\pi}{\omega}\right) = \frac{I_{\max}^2 R\pi}{\omega}, \quad Q_{\text{ind}} = \frac{\omega L}{R} \tag{25}$$

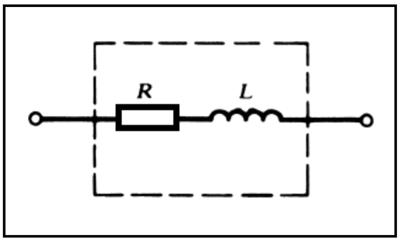


Fig. 11: The practical inductor

A practical capacitor can be modeled by a parallel combination of R and C, as shown in Fig. (12). The maximum stored energy is $\frac{1}{2}CV_{\max}^2$ and the energy dissipated per cycle is $V_{\max}^2\pi/R\omega$. Thus, $Q_{\text{cap}} = \omega CR$. The Q of the series RLC circuit is derived in Problem 12.6(a). It is usually applied at resonance, in which case it has the equivalent forms:

$$Q_0 = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{26}$$

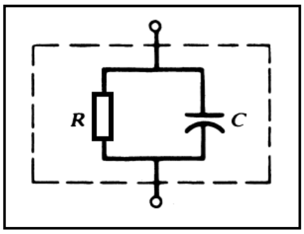


Fig. 12: The practical capacitor

10. RLC Parallel circuit; parallel resonance

A parallel RLC network is shown in Fig. (13). Observe that $V_2 = V_1$. Under the open-circuit condition, the input admittance is:

$$\boxed{\mathbf{Y}_{\text{in}}(\omega) = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C = \frac{1}{\mathbf{Z}_{\text{in}}(\omega)}} \tag{27}$$

The network will be in parallel resonance (or high-impedance resonance) when $Y_{in}(\omega)$, and thus $Z_{in}(\omega)$, is real (and so $|Y_{in}(\omega)|$ is a minimum and $|Z_{in}(\omega)|$ is a maximum); that is, when:

$$\boxed{-\frac{1}{\omega L} + \omega C = 0 \qquad \text{or} \qquad \omega = \omega_a \equiv \frac{1}{\sqrt{LC}}} \tag{28}$$

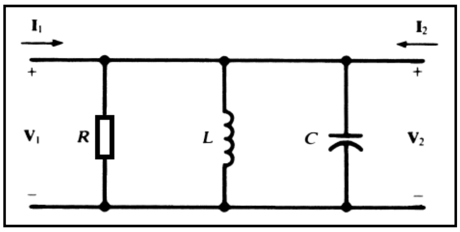


Fig. 13: The RLC Parallel circuit (parallel resonance)

The symbol ω_a is now used to denote the quantity $1/\sqrt{LC}$ in order to distinguish the resonance from a low-impedance resonance. Complex series-parallel networks may have several high-impedance resonant frequencies ω_a and several low-impedance resonant frequencies ω_0.

The normalized input impedance

$$\boxed{\frac{\mathbf{Z}_{\text{in}}(\omega)}{R} = \frac{1}{1 + jR\left(\omega C - \dfrac{1}{\omega L}\right)}} \tag{29}$$

is plotted (magnitude only) in Fig. (14). Half-power frequencies ω_l and ω_h are indicated on the plot. Analogous to series resonance, the bandwidth is given by:

$$\boxed{\beta = \frac{\omega_a}{Q_a}} \tag{30}$$

where Q_a, the quality factor of the parallel circuit at $\omega = \omega_a$, has the equivalent expressions

$$\boxed{Q_a = \frac{R}{\omega_a L} = \omega_a RC = R\sqrt{\frac{C}{L}}} \tag{31}$$

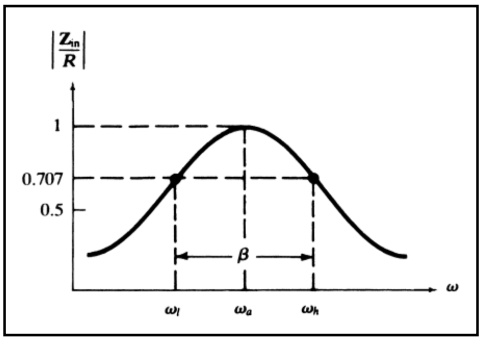
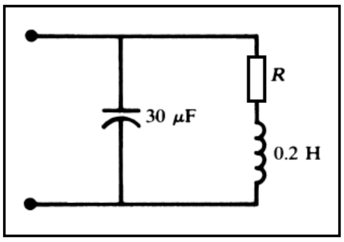


Fig. 14: The frequency response (magnitude only)
for RLC parallel circuit

Example: Compare the resonant frequency of the circuit shown in figure below for R = 0 to that for R = 50 Ω.

For R = 0, the circuit is that of an LC parallel tank, with:

$$\omega_a = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(0.2)(30\times 10^{-6})}} = 408.2 \text{ rad/s} \qquad \text{or} \qquad f_a = 65 \text{ Hz}$$



$$\text{For } R = 50\ \Omega, \quad \mathbf{Y}_{in} = j\omega C + \frac{1}{R + j\omega L} = \frac{R}{R^2 + (\omega L)^2} + j\left[\omega C - \frac{\omega L}{R^2 + (\omega L)^2}\right]$$

For resonance, $I_m\ Y_{in}$ is zero, so that

$$\omega_a = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{R^2 C}{L}}$$

Clearly, as R → 0, this expression reduces to that given for the pure LC tank. Substituting the numerical values produces a value of 0.791 for the radical; hence,

$$\omega_a = 408.2(0.791) = 322.9 \text{ rad/s} \qquad \text{or} \qquad f_a = 51.4 \text{ Hz}$$

11. Series-Parallel conversions

It is often convenient in the analysis of circuits to convert the series R_L combination to the parallel form (see Fig. (15)). Given R_s, L_s, and the operating frequency ω, the elements R_p, L_p of the equivalent parallel circuit are determined by equating the admittances:

$$\boxed{\mathbf{Y}_s = \frac{R_s - j\omega L_s}{R_s^2 + (\omega L_s)^2} \qquad \text{and} \qquad \mathbf{Y}_p = \frac{1}{R_p} + \frac{1}{j\omega L_p}} \tag{32}$$

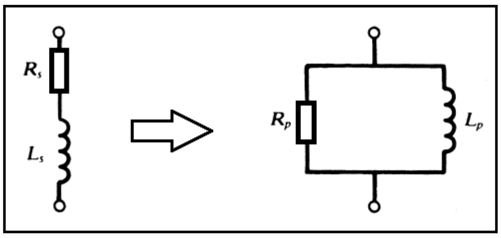


Fig. 15: The series-parallel conversions (RL)

$$\boxed{\begin{aligned} R_p &= R_s\left[1 + \left(\frac{\omega L_s}{R_s}\right)^2\right] = R_s(1 + Q_s^2) \\ L_p &= L_s\left[1 + \left(\frac{R_s}{\omega L_s}\right)^2\right] = L_s\left(1 + \frac{1}{Q_s^2}\right) \end{aligned}} \tag{33}$$

There are times when the RC circuit in either form should be converted to the other form (see Fig. (16)). Equating either the impedances or the admittances, one finds

$$\boxed{\begin{aligned} R_s &= \frac{R_p}{1 + (\omega C_p R_p)^2} = \frac{R_p}{1 + Q_p^2} \\ C_s &= C_p\left[1 + \frac{1}{(\omega C_p R_p)^2}\right] = C_p\left(1 + \frac{1}{Q_p^2}\right) \end{aligned}} \tag{34}$$

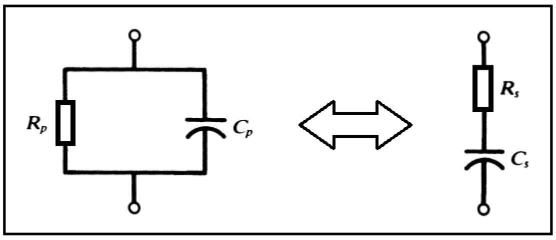


Fig. 16: The series-parallel conversions (RC)

$$\boxed{\begin{aligned} R_p &= R_s\left[1+\frac{1}{(\omega C_s R_s)^2}\right] = R_s(1+Q_s^2) \\ C_p &= \frac{C_s}{1+(\omega C_s R_s)^2} = \frac{C_s}{1+(1/Q_s)^2} \end{aligned}} \qquad (35)$$

Chapter: 02 Two-port networks

This chapter, using the theory of electric circuits, shows the definition of two- port networks and the practical way to use the equivalent circuit parameters to describe the passive linear circuits, like transfer coefficients matrix T, impedances matrix Z, admittances matrix Y and S-parameters matrix S.

1. Terminals and ports

In a two-terminal network, the terminal voltage is related to the terminal current by the impedance Z = V/I. In a four-terminal network, if each terminal pair (or port) is connected separately to another circuit as in Fig. (01), the four variables i_1, i_2, v_1, and v_2 are related by two equations called the terminal characteristics. These two equations, plus the terminal characteristics of the connected circuits, provide the necessary and sufficient number of equations to solve for the four variables.

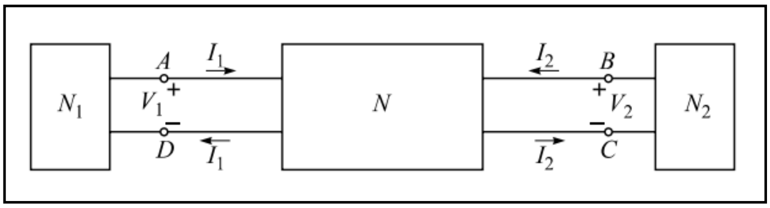


Fig. 01: The two-terminal network

2. Z-Parameters

The terminal characteristics of a two-port network, having linear elements and dependent sources, may be written in the s-domain as:

$$\boxed{\begin{aligned}\mathbf{V}_1 &= \mathbf{Z}_{11}\mathbf{I}_1 + \mathbf{Z}_{12}\mathbf{I}_2\\ \mathbf{V}_2 &= \mathbf{Z}_{21}\mathbf{I}_1 + \mathbf{Z}_{22}\mathbf{I}_2\end{aligned}} \qquad (01)$$

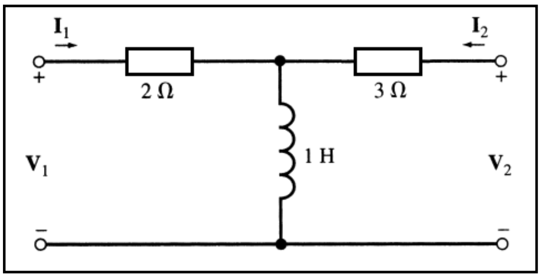
The coefficients Z_{ij} have the dimension of impedance and are called the Z-parameters of the network. The Z-parameters are also called open-circuit impedance parameters since they may be measured at one terminal while the other terminal is open. They are:

$$\boxed{\mathbf{Z}_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} \quad \mathbf{Z}_{12} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} \quad \mathbf{Z}_{21} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} \quad \mathbf{Z}_{22} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0}} \qquad (02)$$

<u>Example:</u> Find the Z-parameters of the two-port circuit in figure below.

Apply KVL around the two loops in this figure with loop currents I_1 and I_2 to obtain:

$$\boxed{\begin{aligned}\mathbf{V}_1 &= 2\mathbf{I}_1 + \mathbf{s}(\mathbf{I}_1 + \mathbf{I}_2) = (2 + \mathbf{s})\mathbf{I}_1 + \mathbf{s}\mathbf{I}_2\\ \mathbf{V}_2 &= 3\mathbf{I}_2 + \mathbf{s}(\mathbf{I}_1 + \mathbf{I}_2) = \mathbf{s}\mathbf{I}_1 + (3 + \mathbf{s})\mathbf{I}_2\end{aligned}}$$



The Z-parameters of the circuit are found to be, (note that in this example $Z_{12} = Z_{21}$)

$$\boxed{\begin{aligned} \mathbf{Z}_{11} &= \mathbf{s} + 2 \\ \mathbf{Z}_{12} &= \mathbf{Z}_{21} = \mathbf{s} \\ \mathbf{Z}_{22} &= \mathbf{s} + 3 \end{aligned}}$$

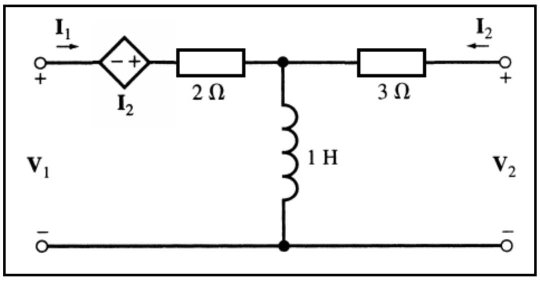
2.1 Reciprocal and nonreciprocal networks

A two-port network is called reciprocal if the open-circuit transfer impedances are equal: $Z_{12} = Z_{21}$. Consequently, in a reciprocal two-port network with current I feeding one port, the open-circuit voltage measured at the other port is the same, irrespective of the ports. The voltage is equal to $V = Z_{12}I = Z_{21}I$. Networks containing resistors, inductors, and capacitors are generally reciprocal. Networks that additionally have dependent sources are generally nonreciprocal (see the example below).

Example: The two-port circuit shown in figure below contains a current-dependent voltage source. Find its Z-parameters.

As in the last example, we apply Kirchhoff's Voltage Law (KVL) around the two loops:

$$\boxed{\begin{aligned} \mathbf{V}_1 &= 2\mathbf{I}_1 - \mathbf{I}_2 + \mathbf{s}(\mathbf{I}_1 + \mathbf{I}_2) = (2 + \mathbf{s})\mathbf{I}_1 + (\mathbf{s} - 1)\mathbf{I}_2 \\ \mathbf{V}_2 &= 3\mathbf{I}_2 + \mathbf{s}(\mathbf{I}_1 + \mathbf{I}_2) = \mathbf{s}\mathbf{I}_1 + (3 + \mathbf{s})\mathbf{I}_2 \end{aligned}}$$



The Z-parameters are

$$\boxed{\begin{aligned} \mathbf{Z}_{11} &= \mathbf{s} + 2 \\ \mathbf{Z}_{12} &= \mathbf{s} - 1 \\ \mathbf{Z}_{21} &= \mathbf{s} \\ \mathbf{Z}_{22} &= \mathbf{s} + 3 \end{aligned}}$$

With the dependent source in the circuit, $Z_{12} \neq Z_{21}$ and so the two-port circuit is nonreciprocal.

2.2 T-Equivalent of reciprocal networks

A reciprocal network may be modeled by its T-equivalent as shown in the circuit of Fig. (02). Z_a, Z_b, and Z_c are obtained from the Z-parameters as follows.

$$\boxed{\begin{aligned} \mathbf{Z}_a &= \mathbf{Z}_{11} - \mathbf{Z}_{12} \\ \mathbf{Z}_b &= \mathbf{Z}_{22} - \mathbf{Z}_{21} \\ \mathbf{Z}_c &= \mathbf{Z}_{12} = \mathbf{Z}_{21} \end{aligned}} \qquad (03)$$

The T-equivalent network is not necessarily realizable.

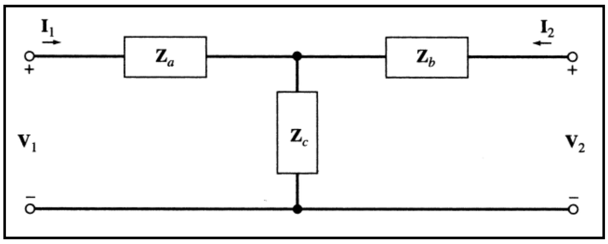


Fig. 02: The T-Equivalent of reciprocal networks

<u>Example:</u> Find the Z-parameters of Fig. (02). Again we apply KVL to obtain:

$$\boxed{\begin{aligned} \mathbf{V}_1 &= \mathbf{Z}_a\mathbf{I}_1 + \mathbf{Z}_c(\mathbf{I}_1 + \mathbf{I}_2) = (\mathbf{Z}_a + \mathbf{Z}_c)\mathbf{I}_1 + \mathbf{Z}_c\mathbf{I}_2 \\ \mathbf{V}_2 &= \mathbf{Z}_b\mathbf{I}_2 + \mathbf{Z}_c(\mathbf{I}_1 + \mathbf{I}_2) = \mathbf{Z}_c\mathbf{I}_1 + (\mathbf{Z}_b + \mathbf{Z}_c)\mathbf{I}_2 \end{aligned}}$$

By comparing (1) and the last equations the Z-parameters are found to be:

$$\boxed{\begin{aligned} \mathbf{Z}_{11} &= \mathbf{Z}_a + \mathbf{Z}_c \\ \mathbf{Z}_{12} &= \mathbf{Z}_{21} = \mathbf{Z}_c \\ \mathbf{Z}_{22} &= \mathbf{Z}_b + \mathbf{Z}_c \end{aligned}}$$

<u>Example:</u> The Z-parameters of a two-port network N are given by

$$\mathbf{Z}_{11} = 2\mathbf{s} + 1/\mathbf{s} \qquad \mathbf{Z}_{12} = \mathbf{Z}_{21} = 2\mathbf{s} \qquad \mathbf{Z}_{22} = 2\mathbf{s} + 4$$

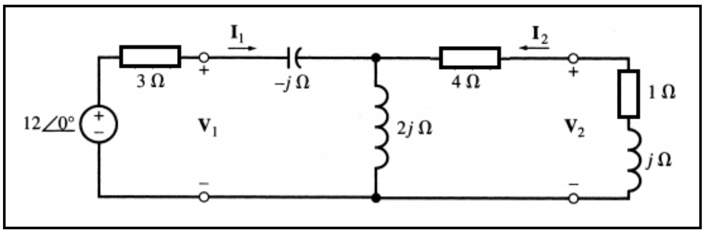
(a) Find the T-equivalent of N.

(b) The network N is connected to a source and a load as shown in the circuit of example of section (05) below. Replace N by its T-equivalent and then solve for i_1, i_2, v_1, and v_2.

(a) The three branches of the T-equivalent network (Fig. (02)) are

$$\begin{aligned}\mathbf{Z}_a &= \mathbf{Z}_{11} - \mathbf{Z}_{12} = 2\mathbf{s} + \frac{1}{\mathbf{s}} - 2\mathbf{s} = \frac{1}{\mathbf{s}}\\ \mathbf{Z}_b &= \mathbf{Z}_{22} - \mathbf{Z}_{12} = 2\mathbf{s} + 4 - 2\mathbf{s} = 4\\ \mathbf{Z}_c &= \mathbf{Z}_{12} = \mathbf{Z}_{21} = 2\mathbf{s}\end{aligned}$$

(b) The T-equivalent of N, along with its input and output connections, is shown in the phasor domain in figure below.



By applying the familiar analysis techniques, including element reduction and current division, to Last figure we find i_1, i_2, v_1, and v_2.

In the phasor domain	In the time domain:
$\mathbf{I}_1 = 3.29\ \underline{/-10.2^\circ}$	$i_1 = 3.29\cos(t - 10.2^\circ)$
$\mathbf{I}_2 = 1.13\ \underline{/-131.2^\circ}$	$i_2 = 1.13\cos(t - 131.2^\circ)$
$\mathbf{V}_1 = 2.88\ \underline{/37.5^\circ}$	$v_1 = 2.88\cos(t + 37.5^\circ)$
$\mathbf{V}_2 = 1.6\ \underline{/93.8^\circ}$	$v_2 = 1.6\cos(t + 93.8^\circ)$

3. Y-Parameters

The terminal characteristics may also be written as in (04), where I_1 and I_2 are expressed in terms of V_1 and V_2.

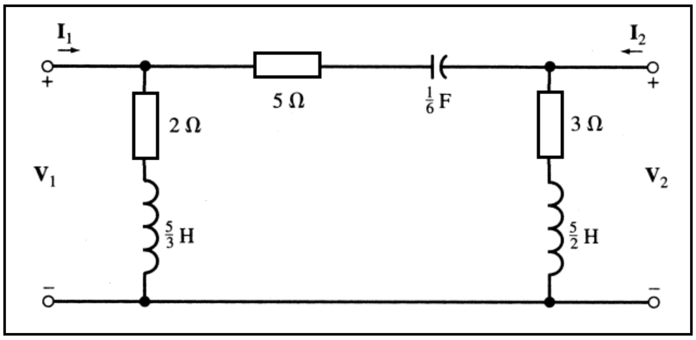
$$\begin{aligned}\mathbf{I}_1 &= \mathbf{Y}_{11}\mathbf{V}_1 + \mathbf{Y}_{12}\mathbf{V}_2\\ \mathbf{I}_2 &= \mathbf{Y}_{21}\mathbf{V}_1 + \mathbf{Y}_{22}\mathbf{V}_2\end{aligned} \qquad (04)$$

The coefficients Y_{ij} have the dimension of admittance and are called the Y-parameters or short-circuit admittance parameters because they may be measured at one port while the other port is short-circuited.

The Y-parameters are:

$$\mathbf{Y}_{11} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_1}\right|_{\mathbf{V}_2=0} \qquad \mathbf{Y}_{12} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{\mathbf{V}_1=0} \qquad \mathbf{Y}_{21} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_1}\right|_{\mathbf{V}_2=0} \qquad \mathbf{Y}_{22} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_2}\right|_{\mathbf{V}_1=0} \tag{05}$$

Example: Find the Y-parameters of the figure below.

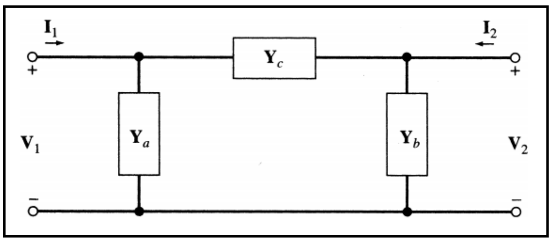


We apply Kirchhoff's Current Law (KCL) to the input and output nodes (for convenience, we designate the admittances of the three branches of the circuit by Y_a, Y_b, and Y_c as shown in figure below. Thus,

$$\mathbf{Y}_a = \frac{1}{2 + 5\mathbf{s}/3} = \frac{3}{5\mathbf{s} + 6}$$

$$\mathbf{Y}_b = \frac{1}{3 + 5\mathbf{s}/2} = \frac{2}{5\mathbf{s} + 6}$$

$$\mathbf{Y}_c = \frac{1}{5 + 6/\mathbf{s}} = \frac{\mathbf{s}}{5\mathbf{s} + 6}$$



The node equations are:

$$\boxed{\begin{aligned} \mathbf{I}_1 &= \mathbf{V}_1\mathbf{Y}_a + (\mathbf{V}_1 - \mathbf{V}_2)\,\mathbf{Y}_c = (\mathbf{Y}_a + \mathbf{Y}_c)\mathbf{V}_1 - \mathbf{Y}_c\mathbf{V}_2 \\ \mathbf{I}_2 &= \mathbf{V}_2\mathbf{Y}_b + (\mathbf{V}_2 - \mathbf{V}_1)\,\mathbf{Y}_c = -\mathbf{Y}_c\mathbf{V}_1 + (\mathbf{Y}_b + \mathbf{Y}_c)\mathbf{V}_2 \end{aligned}}$$

By comparing (04) with the last node equations, we get:

$$\boxed{\begin{aligned} \mathbf{Y}_{11} &= \mathbf{Y}_a + \mathbf{Y}_c \\ \mathbf{Y}_{12} &= \mathbf{Y}_{21} = -\mathbf{Y}_c \\ \mathbf{Y}_{22} &= \mathbf{Y}_b + \mathbf{Y}_c \end{aligned}}$$

Substituting Y_a, Y_b, and Y_c from the Kirchhoff's current law equations into the last equations, we find:

$$\boxed{\begin{aligned} \mathbf{Y}_{11} &= \frac{\mathbf{s}+3}{5\mathbf{s}+6} \\ \mathbf{Y}_{12} &= \mathbf{Y}_{21} = \frac{-\mathbf{s}}{5\mathbf{s}+6} \\ \mathbf{Y}_{22} &= \frac{\mathbf{s}+2}{5\mathbf{s}+6} \end{aligned}}$$

Since $Y_{12} = Y_{21}$, the two-port circuit is reciprocal.

4. Pi-equivalent of reciprocal networks

A reciprocal network may be modeled by its Pi-equivalent as shown in the last figure. The three elements of the Pi-equivalent network can be found by reverse solution. We first find the Y-parameters of the last figure. From (05) we have

$$\boxed{\begin{aligned} \mathbf{Y}_{11} &= \mathbf{Y}_a + \mathbf{Y}_c && \text{[Fig. 3 }(a)] \\ \mathbf{Y}_{12} &= -\mathbf{Y}_c && \text{[Fig. 3 }(b)] \\ \mathbf{Y}_{21} &= -\mathbf{Y}_c && \text{[Fig. 3 }(a)] \\ \mathbf{Y}_{22} &= \mathbf{Y}_b + \mathbf{Y}_c && \text{[Fig. 3 }(b)] \end{aligned}} \tag{06}$$

from which

$$\boxed{\mathbf{Y}_a = \mathbf{Y}_{11} + \mathbf{Y}_{12} \qquad \mathbf{Y}_b = \mathbf{Y}_{22} + \mathbf{Y}_{12} \qquad \mathbf{Y}_c = -\mathbf{Y}_{12} = -\mathbf{Y}_{21}} \tag{07}$$

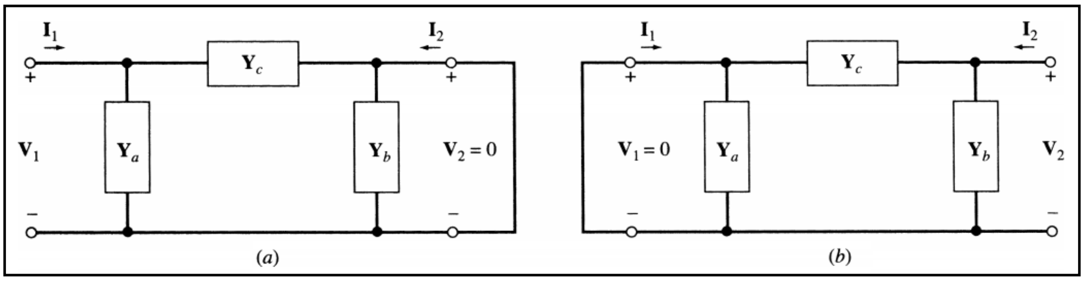


Fig. 03: The Pi-equivalent of reciprocal networks

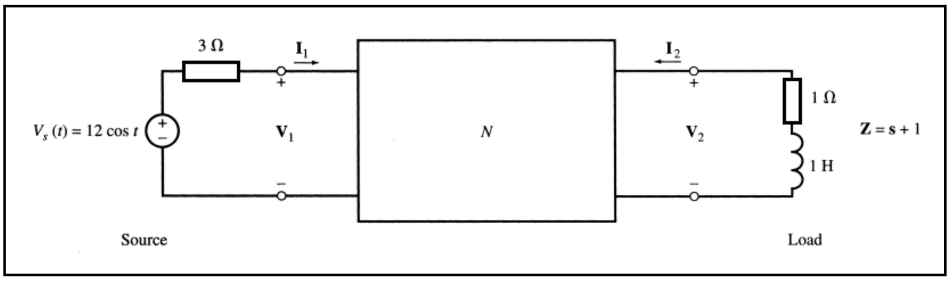
5. Application of terminal characteristics

The four terminal variables I_1, I_2, V_1, and V_2 in a two-port network are related by the two equations (01) or (04). By connecting the two-port circuit to the outside as shown in Fig. (01), two additional equations are obtained. The four equations then can determine I_1, I_2, V_1, and V_2 without any knowledge of the inside structure of the circuit.

<u>Example:</u> The Z-parameters of a two-port network are given by:

$$\boxed{\mathbf{Z}_{11} = 2\mathbf{s} + 1/\mathbf{s} \qquad \mathbf{Z}_{12} = \mathbf{Z}_{21} = 2\mathbf{s} \qquad \mathbf{Z}_{22} = 2\mathbf{s} + 4}$$

The network is connected to a source and a load as shown in the figure below. Find I_1, I_2, V_1, and V_2



The terminal characteristics are given by

$$\boxed{\begin{aligned} \mathbf{V}_1 &= (2\mathbf{s} + 1/\mathbf{s})\mathbf{I}_1 + 2\mathbf{s}\mathbf{I}_2 \\ \mathbf{V}_2 &= 2\mathbf{s}\mathbf{I}_1 + (2\mathbf{s} + 4)\mathbf{I}_2 \end{aligned}}$$

The phasor representation of voltage $v_s(t)$ is $V_s = 12$ V with s = j. From KVL around the input and output loops we obtain the two additional equations

$$\boxed{\begin{aligned} \mathbf{V}_s &= 3\mathbf{I}_1 + \mathbf{V}_1 \\ 0 &= (1 + \mathbf{s})\mathbf{I}_2 + \mathbf{V}_2 \end{aligned}}$$

Substituting s = j and Vs = 12 in the two last equations systems we get

$$\boxed{\begin{aligned} \mathbf{V}_1 &= j\mathbf{I}_1 + 2j\mathbf{I}_2 \\ \mathbf{V}_2 &= 2j\mathbf{I}_1 + (4 + 2j)\mathbf{I}_2 \\ 12 &= 3\mathbf{I}_1 + \mathbf{V}_1 \\ 0 &= (1 + j)\mathbf{I}_2 + \mathbf{V}_2 \end{aligned}}$$

from which

$$\mathbf{I}_1 = 3.29 \underline{/-10.2^\circ} \qquad \mathbf{I}_2 = 1.13 \underline{/-131.2^\circ}$$
$$\mathbf{V}_1 = 2.88 \underline{/37.5^\circ} \qquad \mathbf{V}_2 = 1.6 \underline{/93.8^\circ}$$

6. Conversion between Z- and Y-parameters

The Y-parameters may be obtained from the Z-parameters by solving (01) for I_1 and I_2. Applying Cramer's rule to (01), we get:

$$\mathbf{I}_1 = \frac{\mathbf{Z}_{22}}{\mathbf{D}_{ZZ}}\mathbf{V}_1 - \frac{\mathbf{Z}_{12}}{\mathbf{D}_{ZZ}}\mathbf{V}_2$$
$$\mathbf{I}_2 = \frac{-\mathbf{Z}_{21}}{\mathbf{D}_{ZZ}}\mathbf{V}_1 + \frac{\mathbf{Z}_{11}}{\mathbf{D}_{ZZ}}\mathbf{V}_2 \qquad (08)$$

Where, $\mathbf{D}_{ZZ} = \mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_{12}\mathbf{Z}_{21}$ is the determinant of the coefficient matrix in (01).

By comparing (08) with (04) we have

$$\mathbf{Y}_{11} = \frac{\mathbf{Z}_{22}}{\mathbf{D}_{ZZ}} \qquad \mathbf{Y}_{12} = \frac{-\mathbf{Z}_{12}}{\mathbf{D}_{ZZ}} \qquad \mathbf{Y}_{21} = \frac{-\mathbf{Z}_{21}}{\mathbf{D}_{ZZ}} \qquad \mathbf{Y}_{22} = \frac{\mathbf{Z}_{11}}{\mathbf{D}_{ZZ}} \qquad (09)$$

Given the Z-parameters, for the Y-parameters to exist, the determinant D_{zz} must be nonzero. Conversely, given the Y-parameters, the Z-parameters are

$$\mathbf{Z}_{11} = \frac{\mathbf{Y}_{22}}{\mathbf{D}_{YY}} \qquad \mathbf{Z}_{12} = \frac{-\mathbf{Y}_{12}}{\mathbf{D}_{YY}} \qquad \mathbf{Z}_{21} = \frac{-\mathbf{Y}_{21}}{\mathbf{D}_{YY}} \qquad \mathbf{Z}_{22} = \frac{\mathbf{Y}_{11}}{\mathbf{D}_{YY}} \qquad (10)$$

where, $\mathbf{D}_{YY} = \mathbf{Y}_{11}\mathbf{Y}_{22} - \mathbf{Y}_{12}\mathbf{Y}_{21}$ is the determinant of the coefficient matrix in (9). For the Z-parameters of a two-port circuit to be derived from its Y-parameters, D_{yy} should be nonzero.

Example: Referring to example of section (03), find the Z-parameters of its circuit from its Y-parameters. The Y-parameters of the circuit were found to be (see the results of this example (equations))

$$\mathbf{Y}_{11} = \frac{\mathbf{s}+3}{5\mathbf{s}+6} \qquad \mathbf{Y}_{12} = \mathbf{Y}_{21} = \frac{-\mathbf{s}}{5\mathbf{s}+6} \qquad \mathbf{Y}_{22} = \frac{\mathbf{s}+2}{5\mathbf{s}+6}$$

Substituting into (10), where, $\mathbf{D}_{YY} = 1/(5\mathbf{s}+6)$, we obtain

$$\mathbf{Z}_{11} = \mathbf{s}+2$$
$$\mathbf{Z}_{12} = \mathbf{Z}_{21} = \mathbf{s}$$
$$\mathbf{Z}_{22} = \mathbf{s}+3$$

The Z-parameters in the last equation are identical to the Z-parameters of the circuit of the example of the section (02). The two circuits are equivalent as far as the terminals are concerned. This was by design. The figure of the example of the section (02) is the T-equivalent of The figure of the example of the section (03). The equivalence between the figures of these two examples may be verified directly by applying (03) to the Z-parameters given in the last equation to obtain the T-equivalent network.

7. h-Parameters

Some two-port circuits or electronic devices are best characterized by the following terminal equations:

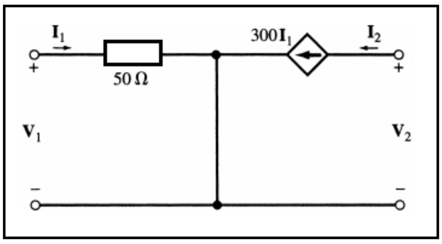
$$\boxed{\begin{aligned} \mathbf{V}_1 &= \mathbf{h}_{11}\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2 \\ \mathbf{I}_2 &= \mathbf{h}_{21}\mathbf{I}_1 + \mathbf{h}_{22}\mathbf{V}_2 \end{aligned}} \qquad (11)$$

where the h_{ij} coefficients are called the hybrid or h-parameters.

Example: Find the h-parameters of the figure below.

This is the simple model of a bipolar junction transistor in its linear region of operation. By inspection, the terminal characteristics of this circuit are:

$$\boxed{\mathbf{V}_1 = 50\mathbf{I}_1 \quad \text{and} \quad \mathbf{I}_2 = 300\mathbf{I}_1}$$



By comparing the last equations and (11) we get

$$\boxed{\mathbf{h}_{11} = 50 \qquad \mathbf{h}_{12} = 0 \qquad \mathbf{h}_{21} = 300 \qquad \mathbf{h}_{22} = 0}$$

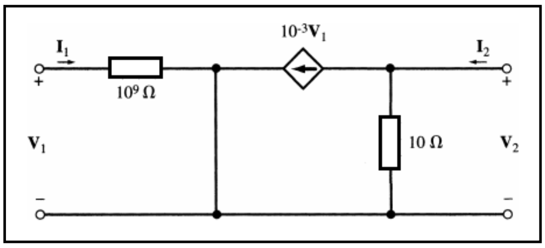
8. g-Parameters

The terminal characteristics of a two-port circuit may also be described by still another set of hybrid parameters as given in (12).

$$\boxed{\begin{aligned} I_1 &= g_{11}V_1 + g_{12}I_2 \\ V_2 &= g_{21}V_1 + g_{22}I_2 \end{aligned}} \qquad (12)$$

where the coefficients g_{ij} are called inverse hybrid or g-parameters.

Example: Find the g-parameters in the circuit shown in figure below.



This is the simple model of a field effect transistor in its linear region of operation. To find the g-parameters, we first derive the terminal equations by applying Kirchhoff's laws at the terminals:

At the input terminal: $V_1 = 10^9\, I_1$

At the output terminal: $V_2 = 10(I_2 - 10^{-3}\, V_1)$

or $I_1 = 10^{-9}\, V_1$ and $V_2 = 10I_2 - 10^{-2}\, V_1$

By comparing (12) and the last equations systems we get

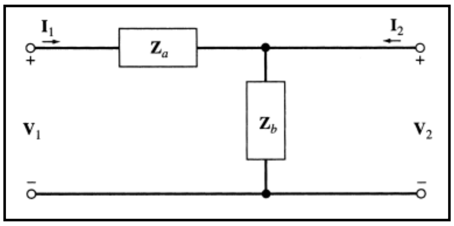
$$\boxed{g_{11} = 10^{-9} \qquad g_{12} = 0 \qquad g_{21} = -10^{-2} \qquad g_{22} = 10}$$

9. Transmission Parameters

The transmission parameters A, B, C, and D express the required source variables V_1 and I_1 in terms of the existing destination variables V_2 and I_2. They are called ABCD or T-parameters and are defined by:

$$\boxed{\begin{aligned} V_1 &= AV_2 - BI_2 \\ I_1 &= CV_2 - DI_2 \end{aligned}} \qquad (13)$$

Example: Find the T-parameters of the figure below where Z_a and Z_b are nonzero.



This is the simple lumped model of an incremental segment of a transmission line. From (13) we have

$$\mathbf{A} = \left.\frac{\mathbf{V}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_2=0} = \frac{\mathbf{Z}_a + \mathbf{Z}_b}{\mathbf{Z}_b} = 1 + \mathbf{Z}_a\mathbf{Y}_b \qquad \mathbf{B} = -\left.\frac{\mathbf{V}_1}{\mathbf{I}_2}\right|_{\mathbf{V}_2=0} = \mathbf{Z}_a \qquad \mathbf{C} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_2=0} = \mathbf{Y}_b \qquad \mathbf{D} = -\left.\frac{\mathbf{I}_1}{\mathbf{I}_2}\right|_{\mathbf{V}_2=0} = 1$$

10. Interconnecting Two-Port Networks

Two-port networks may be interconnected in various configurations, such as series, parallel, or cascade connections, resulting in new two-port networks. For each configuration, a certain set of parameters may be more useful than others to describe the network.

10.1 Series connection

Figure (04) shows a series connection of two two-port networks "a" and "b" with open-circuit impedance parameters Z_a and Z_b, respectively. In this configuration, we use the Z-parameters since they are combined as a series connection of two impedances. The Z-parameters of the series connection are (see example below):

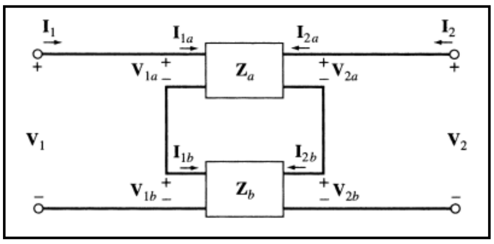


Fig. 04: The series connection of two two-port networks

$$\begin{aligned}
\mathbf{Z}_{11} &= \mathbf{Z}_{11,a} + \mathbf{Z}_{11,b} \\
\mathbf{Z}_{12} &= \mathbf{Z}_{12,a} + \mathbf{Z}_{12,b} \\
\mathbf{Z}_{21} &= \mathbf{Z}_{21,a} + \mathbf{Z}_{21,b} \\
\mathbf{Z}_{22} &= \mathbf{Z}_{22,a} + \mathbf{Z}_{22,b}
\end{aligned} \qquad (14)$$

or, in the matrix form,

$$\boxed{[\mathbf{Z}] = [\mathbf{Z}_a] + [\mathbf{Z}_b]} \tag{15}$$

Example: Two two-port networks "a" and "b", with open-circuit impedances Z_a and Z_b, respectively, are connected in series (see Fig. (04)). Derive the Z-parameter equations (14).

From network "a" we have:

$$\boxed{\begin{aligned} \mathbf{V}_{1a} &= \mathbf{Z}_{11,a}\mathbf{I}_{1a} + \mathbf{Z}_{12,a}\mathbf{I}_{2a} \\ \mathbf{V}_{2a} &= \mathbf{Z}_{21,a}\mathbf{I}_{1a} + \mathbf{Z}_{22,a}\mathbf{I}_{2a} \end{aligned}}$$

From network "b" we have:

$$\boxed{\begin{aligned} \mathbf{V}_{1b} &= \mathbf{Z}_{11,b}\mathbf{I}_{1b} + \mathbf{Z}_{12,b}\mathbf{I}_{2b} \\ \mathbf{V}_{2b} &= \mathbf{Z}_{21,b}\mathbf{I}_{1b} + \mathbf{Z}_{22,b}\mathbf{I}_{2b} \end{aligned}}$$

From the connection between "a" and "b" we have

$$\boxed{\begin{aligned} \mathbf{I}_1 &= \mathbf{I}_{1a} = \mathbf{I}_{1b} \qquad & \mathbf{V}_1 &= \mathbf{V}_{1a} + \mathbf{V}_{1b} \\ \mathbf{I}_2 &= \mathbf{I}_{2a} = \mathbf{I}_{2b} \qquad & \mathbf{V}_2 &= \mathbf{V}_{2a} + \mathbf{V}_{2b} \end{aligned}}$$

Therefore,

$$\boxed{\begin{aligned} \mathbf{V}_1 &= (\mathbf{Z}_{11,a} + \mathbf{Z}_{11,b})\mathbf{I}_1 + (\mathbf{Z}_{12,a} + \mathbf{Z}_{12,b})\mathbf{I}_2 \\ \mathbf{V}_2 &= (\mathbf{Z}_{21,a} + \mathbf{Z}_{21,b})\mathbf{I}_1 + (\mathbf{Z}_{22,a} + \mathbf{Z}_{22,b})\mathbf{I}_2 \end{aligned}}$$

from which the Z-parameters (14) are derived.

10.2 Parallel connection

Figure (05) shows a parallel connection of the two-port networks "a" and "b" with short-circuit admittance parameters Y_a and Y_b, respectively. In this case, the Y-parameters are convenient to work with. The Y-parameters of the parallel connection are (see example below):

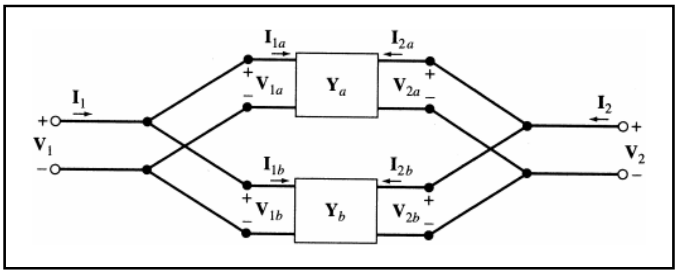


Fig. 05: The parallel connection of two two-port networks

$$\boxed{\begin{aligned} \mathbf{Y}_{11} &= \mathbf{Y}_{11,a} + \mathbf{Y}_{11,b} \\ \mathbf{Y}_{12} &= \mathbf{Y}_{12,a} + \mathbf{Y}_{12,b} \\ \mathbf{Y}_{21} &= \mathbf{Y}_{21,a} + \mathbf{Y}_{21,b} \\ \mathbf{Y}_{22} &= \mathbf{Y}_{22,a} + \mathbf{Y}_{22,b} \end{aligned}} \tag{16}$$

or, in the matrix form,

$$\boxed{[\mathbf{Y}] = [\mathbf{Y}_a] + [\mathbf{Y}_b]} \tag{17}$$

Example: Two two-port networks "a" and "b", with short-circuit admittances Y_a and Y_b, respectively, are connected in parallel (see Fig. (05)). Derive the Y-parameter equations (16).

From network "a" we have:

$$\boxed{\begin{aligned} \mathbf{I}_{1a} &= \mathbf{Y}_{11,a}\mathbf{V}_{1a} + \mathbf{Y}_{12,a}\mathbf{V}_{2a} \\ \mathbf{I}_{2a} &= \mathbf{Y}_{21,a}\mathbf{V}_{1a} + \mathbf{Y}_{22,a}\mathbf{V}_{2a} \end{aligned}}$$

From network "b" we have:

$$\boxed{\begin{aligned} \mathbf{I}_{1b} &= \mathbf{Y}_{11,b}\mathbf{V}_{1b} + \mathbf{Y}_{12,b}\mathbf{V}_{2b} \\ \mathbf{I}_{2b} &= \mathbf{Y}_{21,b}\mathbf{V}_{1b} + \mathbf{Y}_{22,b}\mathbf{V}_{2b} \end{aligned}}$$

From the connection between "a" and "b" we have

$$\boxed{\begin{aligned} \mathbf{V}_1 &= \mathbf{V}_{1a} = \mathbf{V}_{1b} \qquad & \mathbf{I}_1 &= \mathbf{I}_{1a} + \mathbf{I}_{1b} \\ \mathbf{V}_2 &= \mathbf{V}_{2a} = \mathbf{V}_{2b} \qquad & \mathbf{I}_2 &= \mathbf{I}_{2a} + \mathbf{I}_{2b} \end{aligned}}$$

Therefore,

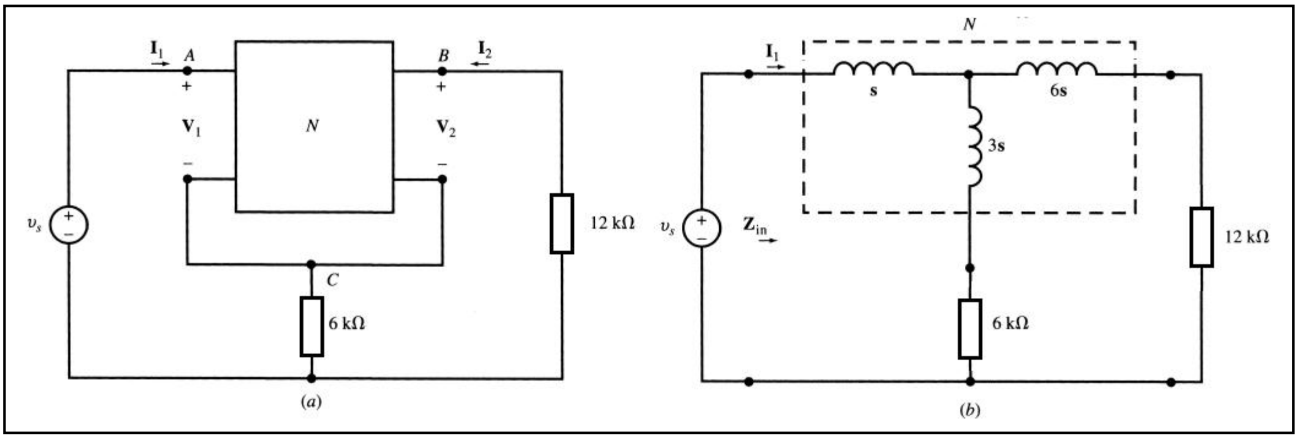
$$\boxed{\begin{aligned} \mathbf{I}_1 &= (\mathbf{Y}_{11,a} + \mathbf{Y}_{11,b})\mathbf{V}_1 + (\mathbf{Y}_{12,a} + \mathbf{Y}_{12,b})\mathbf{V}_2 \\ \mathbf{I}_2 &= (\mathbf{Y}_{21,a} + \mathbf{Y}_{21,b})\mathbf{V}_1 + (\mathbf{Y}_{22,a} + \mathbf{Y}_{22,b})\mathbf{V}_2 \end{aligned}}$$

from which the Z-parameters (16) are derived.

Example: Z-parameters of the two-port network N in Fig. (a) below are $Z_{11} = 4s$, $Z_{12} = Z_{21} = 3s$, and $Z_{22} = 9s$.

(a) Replace N by its T-equivalent. (b) Use part (a) to find input current i_1 for $v_s = \cos 1000t$ (V).

(a) The network is reciprocal. Therefore, its T-equivalent exists. Its elements are found from (03) and shown in the circuit of Fig. (b) below.



$$\mathbf{Z}_a = \mathbf{Z}_{11} - \mathbf{Z}_{12} = 4\mathbf{s} - 3\mathbf{s} = \mathbf{s}$$
$$\mathbf{Z}_b = \mathbf{Z}_{22} - \mathbf{Z}_{21} = 9\mathbf{s} - 3\mathbf{s} = 6\mathbf{s}$$
$$\mathbf{Z}_c = \mathbf{Z}_{12} = \mathbf{Z}_{21} = 3\mathbf{s}$$

(b) We repeatedly combine the series and parallel elements of Fig. (b), with resistors being in kΩ and "s" in "k rad /s", to find Z_{in} in kΩ:

$$\mathbf{Z}_{in}(\mathbf{s}) = \mathbf{V}_s/\mathbf{I}_1 = \mathbf{s} + \frac{(3\mathbf{s}+6)(6\mathbf{s}+12)}{9\mathbf{s}+18} = 3\mathbf{s} + 4 \qquad \text{or} \qquad \mathbf{Z}_{in}(j) = 3j + 4 = 5\angle 36.9^\circ \text{ k}\Omega$$
$$\text{and } i_1 = 0.2\cos(1000t - 36.9^\circ) \text{ (mA)}$$

10.3 Cascade connection

The cascade connection of the two-port networks "a" and "b" is shown in Fig. (06). In this case the T-parameters are particularly convenient. The T-parameters of the cascade combination are

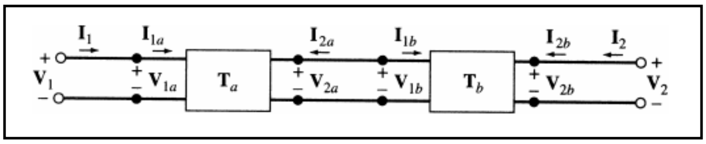


Fig. 06: The cascade connection of two two-port networks

$$\begin{aligned} \mathbf{A} &= \mathbf{A}_a\mathbf{A}_b + \mathbf{B}_a\mathbf{C}_b \\ \mathbf{B} &= \mathbf{A}_a\mathbf{B}_b + \mathbf{B}_a\mathbf{D}_b \\ \mathbf{C} &= \mathbf{C}_a\mathbf{A}_b + \mathbf{D}_a\mathbf{C}_b \\ \mathbf{D} &= \mathbf{C}_a\mathbf{B}_b + \mathbf{D}_a\mathbf{D}_b \end{aligned} \qquad (18)$$

or, in the matrix form,

$$\boxed{[\mathbf{T}] = [\mathbf{T}_a][\mathbf{T}_b]} \tag{19}$$

10.4 Choice of parameter type

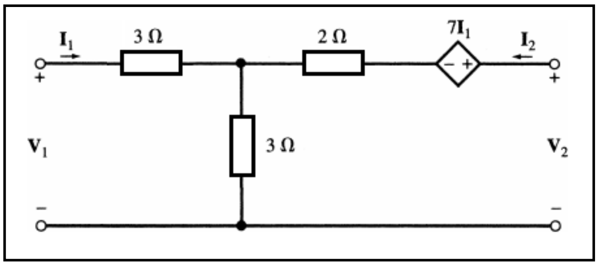
What types of parameters are appropriate for and can best describe a given two-port network or device? Several factors influence the choice of parameters.

(1) It is possible that some types of parameters do not exist as they may not be defined at all (see Example below).

(2) Some parameters are more convenient to work with when the network is connected to other networks, as shown in Section (10). In this regard, by converting the two-port network to its T- and Pi-equivalents and then applying the familiar analysis techniques, such as element reduction and current division, we can greatly reduce and simplify the overall circuit.

(3) For some networks or devices, a certain type of parameter produces better computational accuracy and better sensitivity when used within the interconnected circuit.

Example: Find the Z- and Y-parameters of figure below.



We apply KVL to the input and output loops. Thus,

Input loop: $\mathbf{V}_1 = 3\mathbf{I}_1 + 3(\mathbf{I}_1 + \mathbf{I}_2)$

Output loop: $\mathbf{V}_2 = 7\mathbf{I}_1 + 2\mathbf{I}_2 + 3(\mathbf{I}_1 + \mathbf{I}_2)$

or $\mathbf{V}_1 = 6\mathbf{I}_1 + 3\mathbf{I}_2$ and $\mathbf{V}_2 = 10\mathbf{I}_1 + 5\mathbf{I}_2$

By comparing the last equation and (02) we get

$$\boxed{\mathbf{Z}_{11} = 6 \qquad \mathbf{Z}_{12} = 3 \qquad \mathbf{Z}_{21} = 10 \qquad \mathbf{Z}_{22} = 5}$$

The Y-parameters are, however, not defined, since the application of the direct method of (05) or the conversion from Z-parameters (08) produces $D_{zz} = 6(5) - 3(10) = 0$.

10.5 Summary of terminal parameters and conversion

The various terminal parameters are defined by the following equations:

Z-parameters	Y-parameters	h-parameters	g-parameters
$\mathbf{V_1 = Z_{11}I_1 + Z_{12}I_2}$	$\mathbf{I_1 = Y_{11}V_1 + Y_{12}V_2}$	$\mathbf{V_1 = h_{11}I_1 + h_{12}V_2}$	$\mathbf{I_1 = g_{11}V_1 + g_{12}I_2}$
$\mathbf{V_2 = Z_{21}I_1 + Z_{22}I_2}$	$\mathbf{I_2 = Y_{21}V_1 + Y_{22}V_2}$	$\mathbf{I_2 = h_{21}I_1 + h_{22}V_2}$	$\mathbf{V_2 = g_{21}V_1 + g_{22}I_2}$
$[\mathbf{V}] = [\mathbf{Z}][\mathbf{I}]$	$[\mathbf{I}] = [\mathbf{Y}][\mathbf{V}]$		

Chapter: 03 Semiconductors and semiconductor devices (Diode, Zener diode, varicap diode, LED, photodiode...)

Part I. Semiconductors

This chapter provides essential understanding of semiconductors and semiconductors device characteristics. It explains, in-depth introduction to electronic semiconductor devices and circuits.

1.Introduction

To understand how semiconductors devices (diodes, transistors, and integrated circuits work), you first have to study semiconductors: materials that are neither conductors nor insulators. Semiconductors contain some free electrons, but what makes them unusual is the presence of holes. In this chapter, you will learn about semiconductors, holes, and other related topics.

2. Conductors

Copper is a good conductor. The reason is clear when we look at its atomic structure (Fig. (01)). The nucleus of the atom contains 29 protons (positive charges). When a copper atom has a neutral charge, 29 electrons (negative charges) circle the nucleus. The electrons travel in distinct orbits (also called shells). There are 2 electrons in the first orbit, 8 electrons in the second, 18 in the third, and 1 in the outer orbit.

2.1 Stable orbits

The positive nucleus of Fig. (01) attracts the planetary electrons. The reason why these electrons are not pulled into the nucleus is the centrifugal (outward) force created by their circular motion. This centrifugal force is exactly equal to the inward pull of the nucleus, so the orbit is stable. The larger the orbit of an electron, the smaller the attraction of the nucleus. In a larger orbit, an electron travels more slowly, producing less centrifugal force. The outermost electron in Fig. (01) travels slowly and feels almost no attraction to the nucleus.

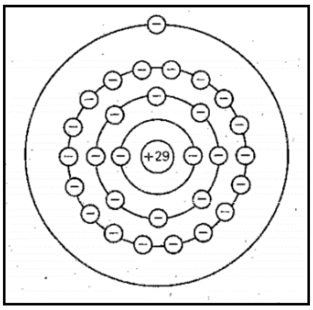


Fig. 01: The Copper atom

2.2 The Core

In electronics, all that matter is the outer orbit. It is called the valence orbit. This orbit controls the electrical properties of the atom. To emphasize the importance of the valence orbit, we define the core of an atom as the nucleus and all the inner orbits. For a copper atom, the core is the nucleus (+ 29), and the first three orbits (- 28).

The core of a copper atom has a net charge of +1 because it contains 29 protons and 28 inner electrons. Figure (02) can help in visualizing the core and the valence orbit. The valence electron is in a large orbit around a core that has a net charge of only +1. Because of this, the inward pull felt by the valence electron is very small.

2.3 Free electron

Since the attraction between the core and the valence electron is weak, an outside force can easily dislodge this electron from the copper atom. This is why we often call the valence electron a free electron. This is also why copper is a good conductor. The slightest energy causes the free electrons to flow from one atom to the next. The best conductors are silver, copper, and gold. All have a core diagram like Fig. (02).

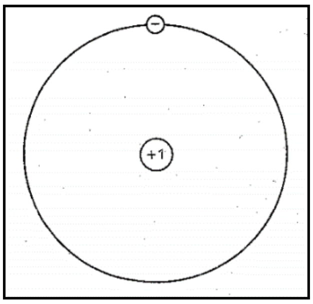


Fig. 02: The core diagram of Copper atom

3. Semiconductors

The best conductors (silver, copper, and gold) have one valence electron, whereas the best insulators have eight valence electrons. A semiconductor is an element with electrical properties between those of a conductor and those of an insulator. As you might expect, the best semiconductors have four valence electrons.

3.1 Germanium

Germanium is an example of a semiconductor. It has four electrons in the valence orbit. Many years ago, germanium was the only material suitable for making semiconductor devices. But these germanium devices had a fatal flaw (their excessive reverse current) that engineers could not overcome. Eventually, another semiconductor named silicon became practical and made germanium obsolete in most electronic applications.

3.2 Silicon

Next to oxygen, silicon is the most abundant element on the earth. But there were certain refining problems that prevented the use of silicon in the early days of semiconductors. Once these problems were solved, the advantages of silicon immediately made it the semiconductor of choice. Without it, modern electronics, communications, and computers would be impossible.

An isolated silicon atom has 14 protons and 14 electrons. As shown in Fig. (03 a), the first orbit contains two electrons and the second orbit contains eight electrons. The four remaining electrons are in the valence orbit. In Fig. (03 a), the core has a net charge of + 4 because it contains 14 protons in the nucleus and 10 electrons in the first two orbits.

Figure (03 b), shows the core diagram of a silicon atom. The four valence electrons tell us that silicon is a semiconductor.

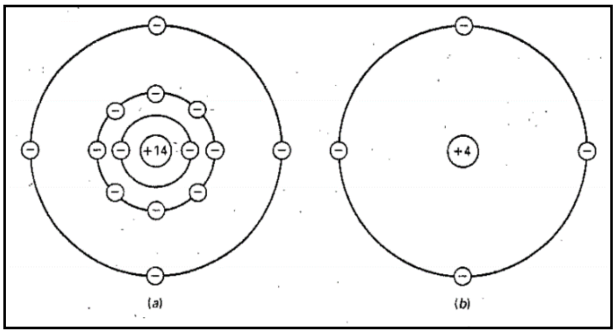


Fig. 03: (a) Silicon atom; (b) core diagram

4. Silicon crystals

When silicon atoms combine to form a solid, they arrange themselves into an orderly pattern called a crystal. Each silicon atom shares its electrons with four neighboring atoms in such a way as to have eight electrons in its valence orbit. For instance, Fig. (04 a) shows a central atom with four neighbors. The shaded circles represent the silicon cores. Although the central atom originally had four electrons in its valence orbit, it now has eight.

4.1 Covalent bonds

Each neighboring atom shares an electron with the central atom. In this way, the central atom has four additional electrons, giving it a total of eight electrons in the valence orbit. The electrons no longer belong to any single atom. Each central atom and its neighbors share the electrons. The same idea is true for all the other silicon atoms. In other words, every atom inside a silicon crystal has four neighbors.

In Fig. (4a), each core has a charge of + 4. Look at the central core and the one to its right. These two cores attract the pair of electrons between them with equal and opposite force. This pulling in opposite directions is what holds the silicon atoms together. The idea is similar to tug-of-war teams pulling on a rope. As long as both teams pull with equal and opposite force, they remain bonded together.

Since each shared electron in Fig. (04 a) is being pulled in opposite directions, the electron becomes a bond between the opposite cores. We call this type of chemical bond a covalent bond. Figure (04 b) is a simpler way to show the concept of the covalent bonds. In a silicon crystal, there are billions of silicon atoms, each with eight valence electrons. These valence electrons are the covalent bonds that hold the crystal together—that give it solidity.

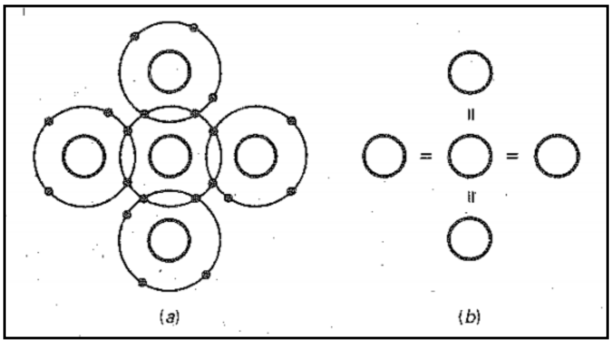


Fig. 04: (a) Atom in crystal has four neighbors; (b) covalent bonds

4.2 Valence saturation

Each atom in a silicon crystal has eight electrons in its valence orbit. These eight electrons produce a chemical stability that results in a solid piece of silicon material. No one is quite sure why the outer orbit of all elements has a predisposition toward having eight electrons. When eight electrons do not exist naturally in an element, there seems to be a tendency for the element to combine and share electrons with other atoms so as to have eight electrons in the outer orbit.

There are advanced equations in physics that partially explain why eight electrons produce chemical stability in different materials, but no one knows the reason why the number eight is so special. It is one of those laws that we observe but cannot fully explain.

When the valence orbit has eight electrons, it is saturated because no more electrons can fit into this orbit. Stated as a law:

$$\boxed{\textbf{Valence saturation: } n = 8} \tag{01}$$

In words, the valence orbit can hold no more than eight electrons. Furthermore, the eight valence electrons are called bound electrons because they are tightly held by the atoms. Because of these bound electrons, a silicon crystal is almost a perfect insulator at room temperature, approximately 25°C.

4.3 The hole

The ambient temperature is the temperature of the surrounding air. When the ambient temperature is above absolute zero (-273°C), the heat energy in this air causes the atoms in a silicon crystal to vibrate. The higher the ambient temperature, the stronger the mechanical vibrations become. When you pick up a warm object, the warmth you feel is the effect of the vibrating atoms.

In a silicon crystal, the vibrations of the atoms can occasionally dislodge an electron from the valence orbit. When this happens, the released electron gains enough energy to go into a larger orbit, as shown in Fig. (05 a). In this larger orbit, the electron is a free electron.

But that's not all. The departure of the electron creates a vacancy in the valence orbit called a hole (see Fig. (05 a)). This hole behaves like a positive charge because the loss of the electron produces a positive ion. The hole will attract and capture any electron in the immediate vicinity. The existence of holes is the critical difference between conductors and semiconductors. Holes enable semiconductors to do all kinds of things that are impossible with conductors.

At room temperature, thermal energy produces only a few holes and free electrons. To increase the number of holes and free electrons, it is necessary to dope the crystal.

4.4 Recombination and lifetime

In a pure silicon crystal, thermal (heat) energy creates an equal number of free electrons and holes. The free electrons move randomly throughout the crystal. Occasionally, a free electron will approach a hole, feel its attraction, and fall into it. Recombination is the merging of a free electron and a hole (see Fig. (05b)).

The amount of time between the creation and disappearance of a free electron is called the lifetime. It varies from a few nanoseconds to several microseconds, depending on how perfect the crystal is and other factors.

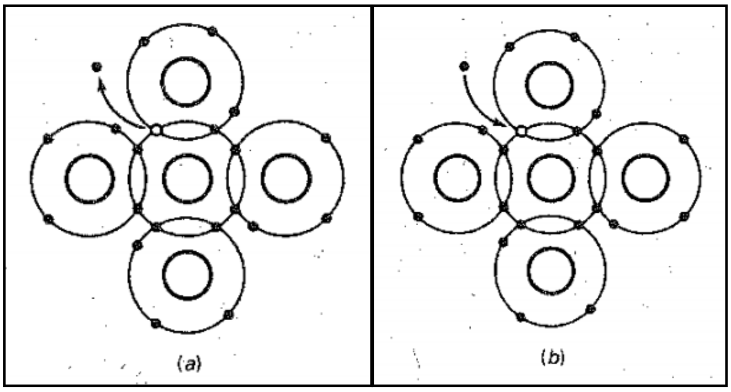


Fig. 05: (a) Thermal energy produces electron and hole;
(b) recombination of free electron and hole

5. Intrinsic semiconductors

An intrinsic semiconductor is a pure semiconductor. A silicon crystal is an intrinsic semiconductor if every atom in the crystal is a silicon atom. At room temperature, a silicon crystal acts like an insulator because it has only a few free electrons and holes produced by thermal energy.

5. 1 Flow of free electrons

Figure (06) shows part of a silicon crystal between charged metallic plates. Assume that thermal energy has produced a free electron and a hole. The free electron is in a large orbit at the right end of the crystal. Because of the negatively charged plate, the free electron is repelled to the left. This free electron can move from one large orbit to the next until it reaches the positive plate.

5. 2 Flow of holes

Notice the hole at the left of Fig. (06). This hole attracts the valence electron at point A. This causes the valence electron to move into the hole.
When the valence electron at point A moves to the left, it creates a new hole at point A. The effect is the same as moving the original hole to the right. The new hole at point A can then attract and capture another valence electron. In this way, valence electrons can travel along the path shown by the arrows. This means the hole can move the opposite way, along path A-B-C-D-E-F, acting the same as a positive charge.

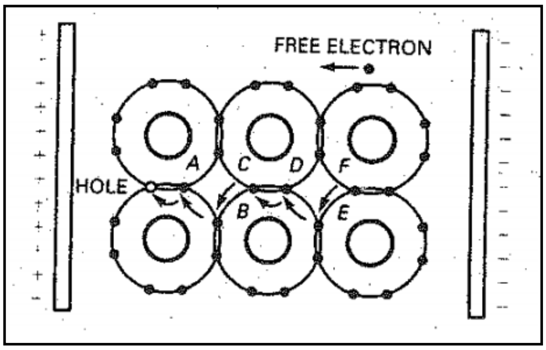


Fig. 06: Hole flow through a semiconductor

5.3 Types of flow

Figure (07) shows an intrinsic semiconductor. It has the same number of free electrons and holes. This is because thermal energy produces free electrons and holes in pairs. The applied voltage will force the free electrons to flow left and the holes to flow right. When the free electrons arrive at the left end of the crystal, they enter the external wire and flow to the positive battery terminal.

On the other hand, the free electrons at the negative battery terminal will flow to the right end of the crystal. At this point, they enter the crystal and recombine with holes that arrive at the right end of the crystal. In this way, a steady flow of free electrons and holes occurs inside the semiconductor. Note that there is no hole flow outside the semiconductor.

In Fig. (07), the free electrons and holes move in opposite directions. From now on, we will visualize the current in a semiconductor as the combined effect of the two types of flow: the flow of free electrons in one direction and the flow of holes in the other direction. Free electrons and holes are often called carriers because they carry a charge from one place to another.

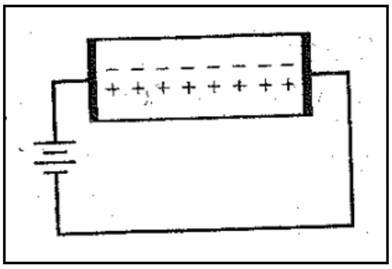


Fig. 07: Intrinsic semiconductor has equal number of free electrons and holes

6. Doping a semiconductor

One way to increase conductivity of a semiconductor is by doping. This means adding impurity atoms to an intrinsic crystal to alter its electrical conductivity. A doped semiconductor is called an extrinsic semiconductor.

6.1 Increasing the free electrons

How does a manufacturer dope a silicon crystal? The first step is to melt a pure silicon crystal. This breaks the covalent bonds and changes the silicon from a solid to a liquid. To increase the number of free electrons, pentavalent atoms are added to the molten silicon. Pentavalent atoms have five electrons in the valence orbit. Examples of pentavalent atoms include arsenic, antimony, and phosphorus. Because these materials will donate an extra electron to the silicon crystal, they are often referred to as donor impurities.

Figure (08 a) shows how the doped silicon crystal appears after it cools down and re-forms its solid crystal structure. A pentavalent atom is in the center, surrounded by four silicon atoms. As before, the neighboring atoms share an electron with the central atom. But this time, there is an extra electron left over. Remember that each pentavalent atom has five valence electrons. Since only eight electrons can fi t into the valence orbit, the extra electron remains in a larger orbit.
In other words, it is a free electron.

Each pentavalent or donor atom in a silicon crystal produces one free electron. This is how a manufacturer controls the conductivity of a doped semiconductor. The more impurity that is added, the greater the conductivity. In this way, a semiconductor may be lightly or heavily doped. A lightly doped semiconductor has a high resistance, whereas a heavily doped semiconductor has a low resistance.

6.2 Increasing the number of holes

How can we dope a pure silicon crystal to get an excess of holes? By using a trivalent impurity, one whose atoms have only three valence electrons. Examples include aluminum, boron, and gallium.

Figure (08 b) shows a trivalent atom in the center. It is surrounded by four silicon atoms, each sharing one of its valence electrons. Since the trivalent atom originally had only three valence electrons and each neighbor shares one electron, only seven electrons are in the valence orbit. This means that a hole exists in the valence orbit of each trivalent atom. A trivalent atom is also called an acceptor atom because each hole it contributes can accept a free electron during recombination.

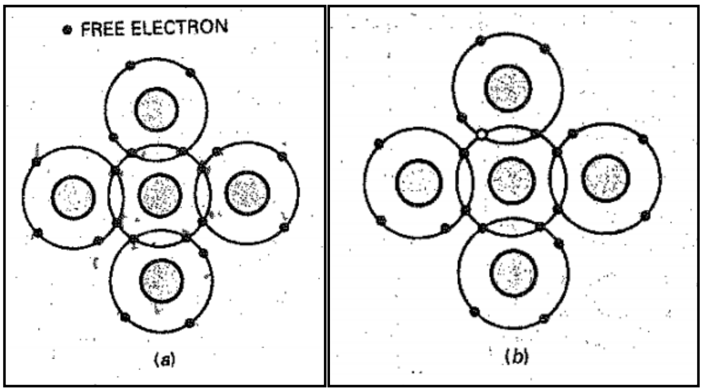


Fig. 08: (a) Doping to get more free electrons;
(b) doping to get more holes

7. Types of extrinsic semiconductors

A semiconductor can be doped to have an excess of free electrons or an excess of holes. Because of this, there are two types of doped semiconductors.

7.1 n-Type semiconductor

Silicon that has been doped with a pentavalent impurity is called an n-type semiconductor, where the n stands for negative. Figure (09) shows an n-type semiconductor. Since the free electrons outnumber the holes in an n-type semiconductor, the free electrons are called the majority carriers and the holes are called the minority carriers.

Because of the applied voltage, the free electrons move to the left and the holes move to the right. When a hole arrives at the right end of the crystal, one of the free electrons from the external circuit enters the semiconductor and recombines with the hole.

The free electrons shown in Fig. (09) flow to the left end of the crystal, where they enter the wire and flow on to the positive terminal of the battery.

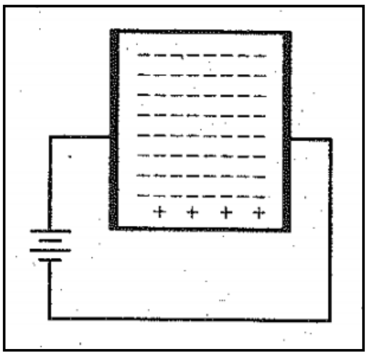


Fig. 09: n-type semiconductor has many free electrons

7.2 p-Type semiconductor

Silicon that has been doped with a trivalent impurity is called a p-type semiconductor, where the p stands for positive. Figure (10) shows a p-type semiconductor. Since holes outnumber free electrons, the holes are referred to as the majority carriers and the free electrons are known as the minority carriers.

Because of the applied voltage, the free electrons move to the left and the holes move to the right. In Fig. (10), the holes arriving at the right end of the crystal will recombine with free electrons from the external circuit.

There is also a flow of minority carriers in Fig. (10). The free electrons inside the semiconductor flow from right to left. Because there are so few minority carriers, they have almost no effect in this circuit.

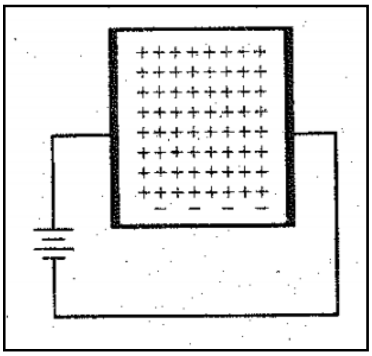


Fig. 10: p-type semiconductor has many holes

8. The diode

By itself, a piece of n-type semiconductor is about as useful as a carbon resistor; the same can be said for a p-type semiconductor. But when a manufacturer dopes a crystal so that one-half of it is p-type and the other half is n-type, something new comes into existence.

The border between p-type and n-type is called the pn junction. The pn junction has led to all kinds of inventions, including diodes, transistors, and integrated circuits. Understanding the pn junction enables you to understand all kinds of semiconductor devices.

8.1 The unbiased diode

As discussed in the preceding section, each trivalent atom in a doped silicon crystal produces one hole. For this reason, we can visualize a piece of p-type semiconductor as shown on the left side of Fig. (11). Each circled minus sign is the trivalent atom, and each plus sign is the hole in its valence orbit.

Similarly, we can visualize the pentavalent atoms and free electrons of an n-type semiconductor as shown on the right side of Fig. (11). Each circled plus sign represents a pentavalent atom, and each minus sign is the free electron it contributes to the semiconductor. Notice that each piece of semiconductor material is electrically neutral because the number of pluses and minuses is equal.

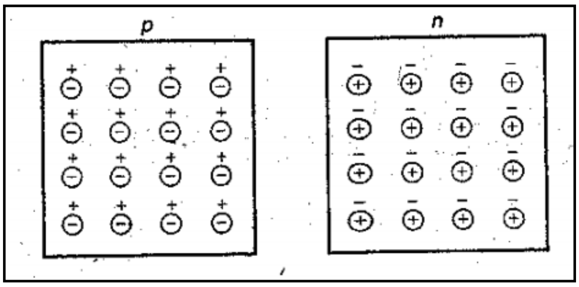


Fig. 11: Types of semiconductor

A manufacturer can produce a single crystal with p-type material on one side and n-type on the other side, as shown in Fig. (12). The junction is the border where the p-type and the n-type regions meet, and junction diode is another name for a pn crystal. The word diode is a contraction of two electrodes, where "di" stands for "two".

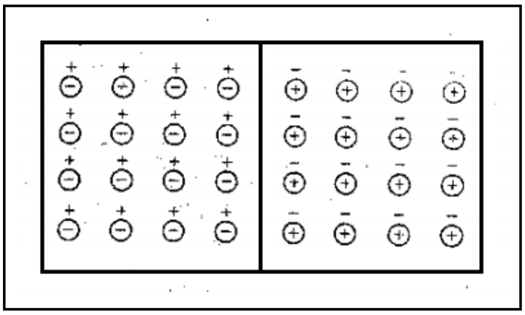


Fig. 12: The pn junction

8.2 The depletion layer

Because of their repulsion for each other, the free electrons on the n side of Fig. (12) tend to diffuse (spread) in all directions. Some of the free electrons diffuse across the junction. When a free electron enters the p region, it becomes a minority carrier. With so many holes around it, this minority carrier has a short lifetime. Soon after entering the p region, the free electron recombines with a hole. When this happens, the hole disappears and the free electron becomes a valence electron.

Each time an electron diffuses across a junction, it creates a pair of ions. When an electron leaves the n side, it leaves behind a pentavalent atom that is short one negative charge; this pentavalent atom becomes a positive ion. After the migrating electron falls into a hole on the p side, it makes a negative ion out of the trivalent atom that captures it.

Figure (13a) shows these ions on each side of the junction. The circled plus signs are the positive ions, and the circled minus signs are the negative ions. The ions are fixed in the crystal structure because of covalent bonding, and they cannot move around like free electrons and holes.

Each pair of positive and negative ions at the junction is called a dipole. The creation of a dipole means that one free electron and one hole have been taken out of circulation. As the number of dipoles builds up, the region near the junction is emptied of carriers. We call this charge-empty region the depletion layer (see Fig. (13b)).

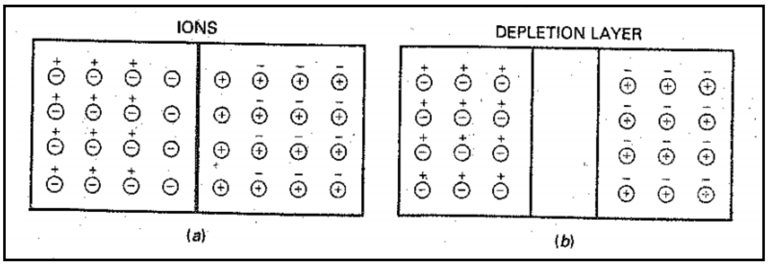


Fig. 13: (a) Creation of ions at junction; (b) depletion layer

8.3 Barrier potential

Each dipole has an electric field between the positive and negative ions. Therefore, if additional free electrons enter the depletion layer, the electric field tries to push these electrons back into the n region. The strength of the electric field increases with each crossing electron until equilibrium is reached. To a first approximation, this means that the electric field eventually stops the diffusion of electrons across the junction.

In Fig. (13 a), the electric field between the ions is equivalent to a difference of potential called the barrier potential. At 25°C, the barrier potential equals approximately 0.3 V for germanium diodes and 0.7 V for silicon diodes.

9. Forward bias

Figure (14) shows a dc source across a diode. The negative source terminal is connected to the n-type material, and the positive terminal is connected to the p-type material. This connection produces what is called forward bias.

9.1 Flow of free electrons

In Fig. (14), the battery pushes holes and free electrons toward the junction. If the battery voltage is less than the barrier potential, the free electrons do not have enough energy to get through the depletion layer. When they enter the depletion layer, the ions will push them back into the n region. Because of this, there is no current through the diode.

When the dc voltage source is greater than the barrier potential, the battery again pushes holes and free electrons toward the junction. This time, the free electrons have enough energy to pass through the depletion layer and recombine with the holes. If you visualize all the holes in the p region moving to the right and all the free electrons moving to the left, you will have the basic idea. Somewhere in the vicinity of the junction, these opposite charges recombine. Since free electrons continuously enter the right end of the diode and holes are being continuously created at the left end, there is a continuous current through the diode.

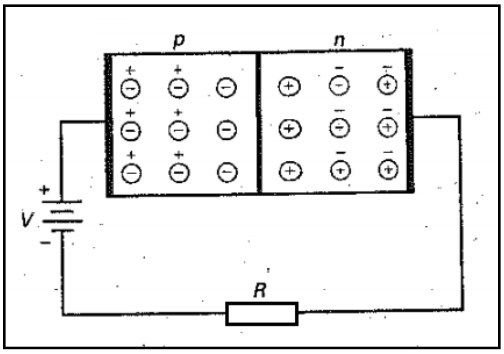


Fig. 14: The forward bias

9.2 Flow of one electron

Let us follow a single electron through the entire circuit. After the free electron leaves the negative terminal of the battery, it enters the right end of the diode. It travels through the n region until it reaches the junction. When the battery voltage is greater than 0.7 V, the free electron has enough energy to get across the depletion layer. Soon after the free electron has entered the p region, it recombines with a hole.

In other words, the free electron becomes a valence electron. As a valence electron, it continues to travel to the left, passing from one hole to the next until it reaches the left end of the diode. When it leaves the left end of the diode, a new hole appears and the process begins again. Since there are billions of electrons taking the same journey, we get a continuous current through the diode. A series resistor is used to limit the amount of forward current.

10. Reverse bias

Turn the dc source around and you get Fig. (15). This time, the negative battery terminal is connected to the p side and the positive battery terminal to the n side. This connection produces what is called reverse bias.

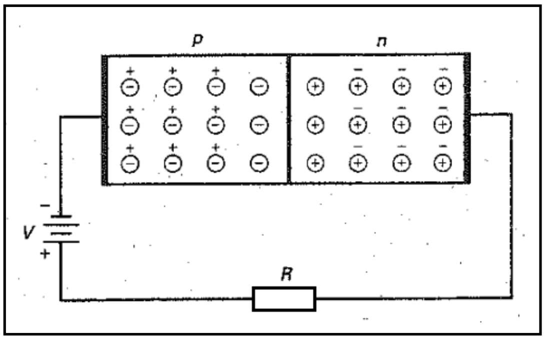


Fig. 15: The reverse bias

10.1 Depletion layer widens

The negative battery terminal attracts the holes, and the positive battery terminal attracts the free electrons. Because of this, holes and free electrons flow away from the junction. Therefore, the depletion layer gets wider.

How wide does the depletion layer get in Fig. (16 a)? When the holes and electrons move away from the junction, the newly created ions increase the difference of potential across the depletion layer. The wider the depletion layer, the greater the difference of potential. The depletion layer stops growing when its difference of potential equals the applied reverse voltage. When this happens, electrons and holes stop moving away from the junction.

Sometimes the depletion layer is shown as a shaded region like that of Fig. (16 b). The width of this shaded region is proportional to the reverse voltage. As the reverse voltage increases, the depletion layer gets wider.

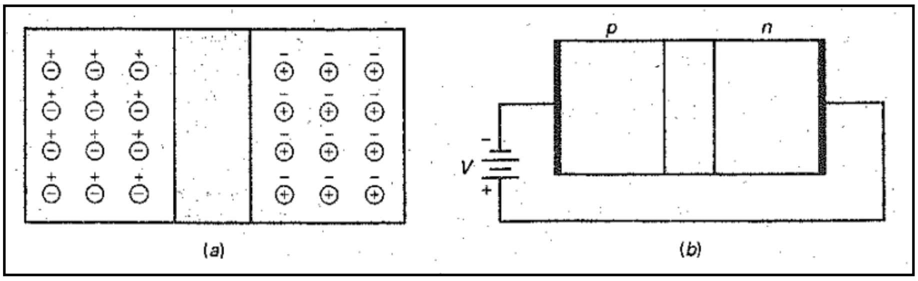


Fig. 16: (a) Depletion layer; (b) increasing reverse bias widens depletion layer

10.2 Minority-carrier current

Is there any current after the depletion layer stabilizes? Yes. A small current exists with reverse bias. Recall that thermal energy continuously creates pairs of free electrons and holes. This means that a few minority carriers exist on both sides of the junction. Most of these recombine with the majority carriers. But those inside the depletion layer may exist long enough to get across the junction. When this happens, a small current flows in the external circuit.

Figure (17) illustrates the idea. Assume that thermal energy has created a free electron and hole near the junction. The depletion layer pushes the free electron to the right, forcing one electron to leave the right end of the crystal. The hole in the depletion layer is pushed to the left. This extra hole on the p side lets one electron enter the left end of the crystal and fall into a hole. Since thermal energy is continuously producing electron-hole pairs inside the depletion layer, a small continuous current flows in the external circuit.

The reverse current caused by the thermally produced minority carriers is called the saturation current. In equations, the saturation current is symbolized by I_S. The name saturation means that we cannot get more minority-carrier current than is produced by the thermal energy. In other words, increasing the reverse voltage will not increase the number of thermally created minority carriers.

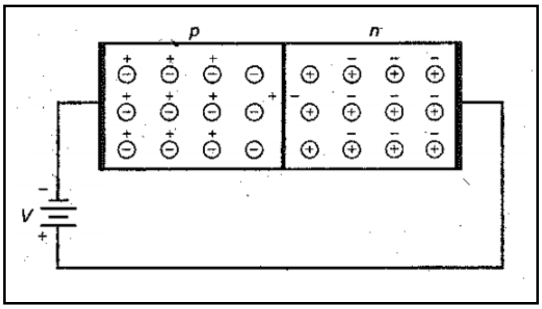


Fig. 17: Thermal production of free electron and hole in depletion layer produces reverse minority-saturation current.

10.3 Surface-leakage current

Besides the thermally produced minority-carrier current, does any other current exist in a reverse-biased diode? Yes. A small current flows on the surface of the crystal. Known as the surface-leakage current, it is caused by surface impurities and imperfections in the crystal structure.

11. Breakdown

Diodes have maximum voltage ratings. There is a limit to how much reverse voltage a diode can withstand before it is destroyed. If you continue increasing the reverse voltage, you will eventually reach the breakdown voltage of the diode. For many diodes, breakdown voltage is at least 50 V. The breakdown voltage is shown on the data sheet for the diode. A data sheet, produced by the manufacturer of the diode, lists important information and typical applications for the device.

Once the breakdown voltage is reached, a large number of the minority carriers suddenly appears in the depletion layer and the diode conducts heavily.

Where do the carriers come from? They are produced by the avalanche effect (see Fig. (18)), which occurs at higher reverse voltages. Here is what happens. As usual, there is a small reverse minority-carrier current. When the reverse voltage increases, it forces the minority carriers to move more quickly. These minority carriers collide with the atoms of the crystal. When these minority carriers have enough energy, they can knock valence electrons loose, producing free electrons. These new minority carriers then join the existing minority carriers to collide with other atoms.

The process is geometric because one free electron liberates one valence electron to get two free electrons. These two free electrons then free two more electrons to get four free electrons. The process continues until the reverse current becomes huge.

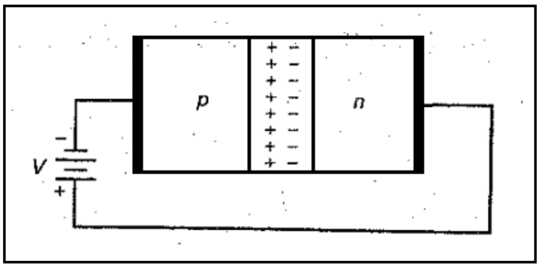


Fig. 18: Avalanche produces many free electrons and holes in depletion layer

Figure (19) shows a magnified view of the depletion layer. The reverse bias forces the free electron to move to the right. As it moves, the electron gains speed. The larger the reverse bias, the faster the electron moves. If the high-speed electron has enough energy, it can bump the valence electron of the first atom into a larger orbit. This results in two free electrons. Both of these then accelerate and go on to dislodge two more electrons. In this way, the number of minority carriers may become quite large and the diode can conduct heavily.

The breakdown voltage of a diode depends on how heavily doped the diode is. With rectifier diodes (the most common type), the breakdown voltage is usually greater than 50 V.

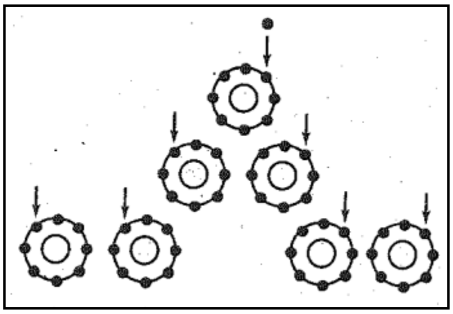
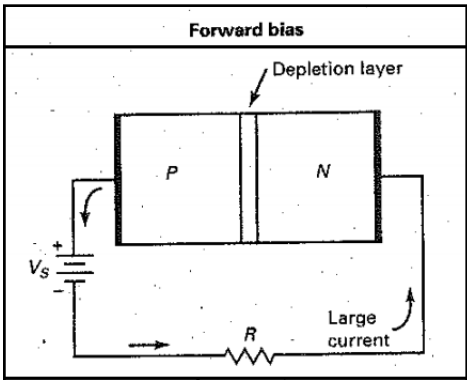
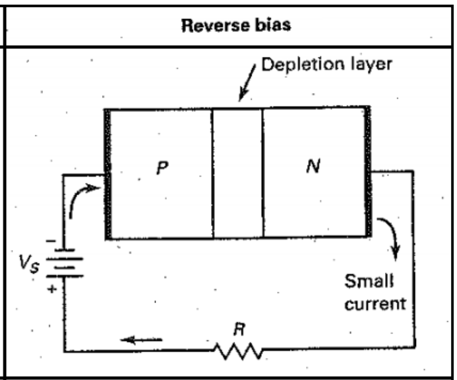


Fig. 19: The process of avalanche is a geometric progression: 1, 2, 4, 8, . . .

Summary Table (01) illustrates the difference between a forward- and reverse-biased diode.

Tab. (01): Summary of the difference between a forward- and reverse-biased diode





	Forward bias	Reverse bias
V_s polarity	(+) to P material (−) to N material	(−) to P materials (+) to N material
Current flow	Large forward current if $V_s > 0.7$ V	Small reverse current (saturation current and surface leakage current) if $V_s <$ breakdown voltage
Depletion layer	Narrow	Wide

12. Energy levels

We can identify the total energy of an electron with the size of its orbit to a good approximation. That is, we can think of each radius of Fig. (20 a) as equivalent to an energy level in Fig. (20 b). Electrons in the smallest orbit are on the first energy level; electrons in the second orbit are on the second energy level; and so on.

12.1 Higher energy in larger orbit

Since an electron is attracted by the nucleus, extra energy is needed to lift an electron into a larger orbit. When an electron is moved from the first to the second orbit, it gains potential energy with respect to the nucleus. Some of the external forces that can lift an electron to higher energy levels are heat, light, and voltage.

For instance, assume that an outside force lifts the electron from the first orbit to the second in Fig. (20 a). This electron has more potential energy because it is farther from the nucleus (Fig. (20 b)). It is like an object above the earth: The higher the object, the greater its potential energy with respect to the earth. If released, the object falls farther and does more work when it hits the earth.

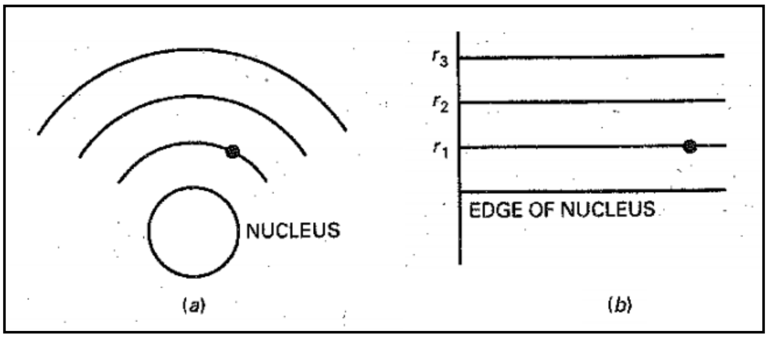


Fig. 20: Energy level is proportional to orbit size. (a) Orbits; (b) energy levels

12.2 Falling electrons radiate light

After an electron has moved into a larger orbit, it may fall back to a lower energy level. If it does, it will give up its extra energy in the form of heat, light, and other radiation.

In a light-emitting diode (LED), the applied voltage lifts the electrons to higher energy levels. When these electrons fall back to lower energy levels, they give off light. Depending on the material used, the radiated light can be a variety of colors, including red, green, orange, or blue. Some LEDs produce infrared radiation (invisible), which is useful in burglar alarm systems.

12.3 Energy bands

When a silicon atom is isolated, the orbit of an electron is influenced only by the charges of the isolated atom. This results in energy levels like the lines of Fig. (20 b). But when silicon atoms are in a crystal, the orbit of each electron is also influenced by the charges of many other silicon atoms. Since each electron has a unique position inside the crystal, no two electrons see exactly the same pattern of surrounding charges. Because of this, the orbit of each electron is different;
or, to put it another way, the energy level of each electron is different.

Figure (21) shows what happens to the energy levels. All electrons in the first orbit have slightly different energy levels because no two electrons see exactly the same charge environment. Since there are billions of first-orbit electrons, the slightly different energy levels form a cluster, or band, of energy. Similarly, the billions of second-orbit electrons, all with slightly different energy levels, form the second energy band—and so on for remaining bands.

Another point: As you know, thermal energy produces a few free electrons and holes. The holes remain in the valence band, but the free electrons go to the next-higher energy band, which is called the conduction band. This is why Fig. (21) shows a conduction band with some free electrons and a valence band with some holes. When the switch is closed, a small current exists in the pure semiconductor. The free electrons move through the conduction band, and holes move through the valence band.

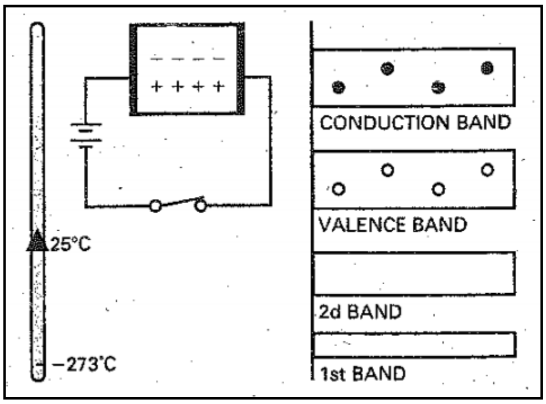


Fig. 21: Intrinsic semiconductor and its energy bands

12.4 n-Type energy bands

Figure (22) shows the energy bands for an n-type semiconductor. As you would expect, the majority carriers are the free electrons in the conduction band, and the minority carriers are the holes in the valence band. Since the switch is closed in Fig. (22), the majority carriers flow to the left, and the minority carriers flow to the right.

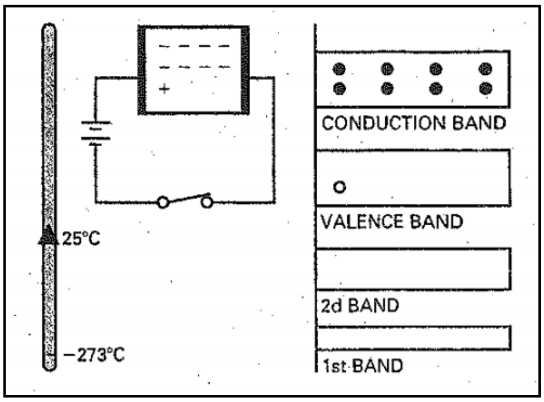


Fig. 22: n-type semiconductor and its energy bands

12.5 p-Type energy bands

Figure (23) shows the energy bands for a p-type semiconductor. Here you see a reversal of the carrier roles. Now, the majority carriers are the holes in the valence band, and the minority carriers are the free electrons in the conduction band. Since the switch is closed in Fig. (23), the majority carriers flow to the right, and the minority carriers flow to the left.

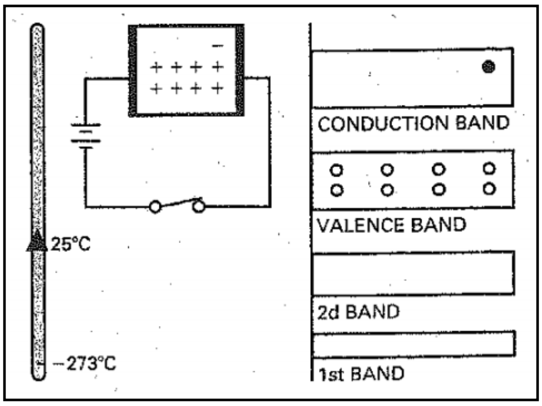


Fig. 23: p-type semiconductor and its energy bands

13. Barrier potential and temperature

The junction temperature is the temperature inside a diode, right at the pn junction. The ambient temperature is different. It is the temperature of the air outside the diode, the air that surrounds the diode. When the diode is conducting, the junction temperature is higher than the ambient temperature because of the heat created by recombination.

The barrier potential depends on the junction temperature. An increase in junction temperature creates more free electrons and holes in the doped regions. As these charges diffuse into the depletion layer, it becomes narrower. This means that there is less barrier potential at higher junction temperatures. Before continuing, we need to define a symbol:

$$\boxed{\Delta = \text{the change in}} \qquad (02)$$

The Greek letter Δ (delta) stands for "the change in". For instance, ΔV means the change in voltage, and ΔT means the change in temperature. The ratio ΔV / ΔT stands for the change in voltage divided by the change in temperature.

Now we can state a rule for estimating the change in barrier potential: The barrier potential of a silicon diode decreases by 2 mV for each degree Celsius rise.

As a derivation:

$$\boxed{\frac{\Delta V}{\Delta T} = -2\ \mathrm{mV/°C}} \qquad (03)$$

By rearranging:

$$\boxed{\Delta V = (-2\ \mathrm{mV/°C})\ \Delta T} \qquad (04)$$

With this, we can calculate the barrier potential at any junction temperature.

Example: Assuming a barrier potential of 0.7 V at an ambient temperature of 25°C, what is the barrier potential of a silicon diode when the junction temperature is 100°C? At 0°C?

When the junction temperature is 100°C, the change in barrier potential is:

$$\boxed{\Delta V = (-2\ \text{mV/°C})\,\Delta T = (-2\ \text{mV/°C})(100\text{°C} - 25\text{°C}) = -150\ \text{mV}}$$

This tells us that the barrier potential decreases 150 mV from its room temperature value, so it equals:

$$\boxed{V_B = 0.7\ \text{V} - 0.15\ \text{V} = 0.55\ \text{V}}$$

When the junction temperature is 0°C, the change in barrier potential is:

$$\boxed{\Delta V = (-2\ \text{mV/°C})\,\Delta T = (-2\ \text{mV/°C})(0\text{°C} - 25\text{°C}) = 50\ \text{mV}}$$

This tells us that the barrier potential increases 50 mV from its room temperature value, so it equals:

$$\boxed{V_B = 0.7\ \text{V} + 0.05\ \text{V} = 0.75\ \text{V}}$$

14. Reverse-biased diode

Let's discuss a few advanced ideas about a reverse-biased diode. To begin with, the depletion layer changes in width when the reverse voltage changes. Let us see what this implies.

14.1 Transient current

When the reverse voltage increases, holes and electrons move away from the junction. As the free electrons and holes move away from the junction, they leave positive and negative ions behind. Therefore, the depletion layer gets wider. The greater the reverse bias, the wider the depletion layer becomes. While the depletion layer is adjusting to its new width, a current flows in the external circuit. This transient current drops to zero after the depletion layer stops growing.

The amount of time the transient current flows depends on the RC time constant of the external circuit. It typically happens in a matter of nanoseconds. Because of this, you can ignore the effects of the transient current below approximately 10 MHz.

14.2 Reverse saturation current

As discussed earlier, forward-biasing a diode decreases the width of the depletion layer and allows free electrons to cross the junction. Reverse bias has the opposite effect: It widens the depletion layer by moving holes and free electrons away from the junction.

Suppose that thermal energy creates a hole and free electron inside the depletion layer of a reverse-biased diode, as shown in Fig. (24). The free electron at A and the hole at B can now contribute to reverse current. Because of the reverse bias, the free electron will move to the right, effectively pushing an electron out of the right end of the diode. Similarly, the hole will move to the left. This extra hole on the p side lets an electron enter the left end of the crystal.

The higher the junction temperature, the greater the saturation current. A useful approximation to remember is this: I_S doubles for each 10°C rise. As a derivation,

$$\boxed{\textbf{Percent } \Delta I_S = \textbf{100\% for a 10°C increase}} \tag{05}$$

In words, the change in saturation current is 100 percent for each 10°C rise in temperature. If the changes in temperature are less than 10°C, you can use this equivalent rule:

$$\boxed{\textbf{Percent } \Delta I_S = \textbf{7\% per °C}} \tag{06}$$

In words, the change in saturation current is 7 percent for each Celsius degree rise. This 7 percent solution is a close approximation of the 10° rule.

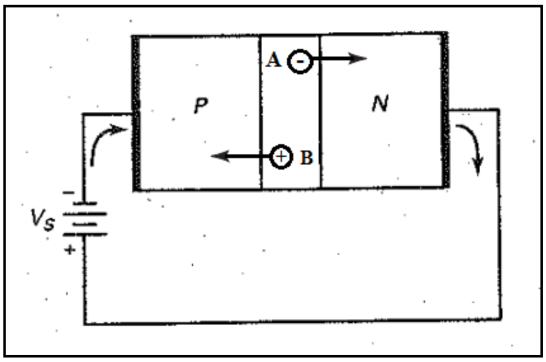


Fig. 24: Thermal energy produces free electron and hole inside depletion layer

14.3 Silicon versus Germanium

In a silicon atom, the distance between the valence band and the conduction band is called the energy gap. When thermal energy produces free electrons and holes, it has to give the valence electrons enough energy to jump into the conduction band. The larger the energy gap, the more difficult it is for thermal energy to produce electron-hole pairs. Fortunately, silicon has a large energy gap; this means that thermal energy does not produce many electron-hole pairs at normal temperatures.

In a germanium atom, the valence band is much closer to the conduction band. In other words, germanium has a much smaller energy gap than silicon has. For this reason, thermal energy produces many more electron-hole pairs in germanium devices. This is the fatal flaw mentioned earlier. The excessive reverse current of germanium devices precludes their widespread use in modern computers, consumer electronics, and communications circuits.

14.4 Surface-leakage current

We discussed surface-leakage current briefly in Sec. (10). Recall that it is a reverse current on the surface of the crystal. Here is an explanation of why surface-leakage current exists. Suppose that the atoms at the top and bottom of Fig. (25 a) are on the surface of the crystal. Since these atoms have no neighbors, they have only six electrons in the valence orbit, implying two holes in each surface atom. Visualize these holes along the surface of the crystal shown in Fig. (25 b). Then you can see that the skin of a crystal is like a p-type semiconductor. Because of this, electrons can enter the left end of the crystal, travel through the surface holes, and leave the right end of the crystal. In this way, we get a small reverse current along the surface.

The surface-leakage current is directly proportional to the reverse voltage. For instance, if you double the reverse voltage, the surface-leakage current ISL doubles. We can define the surface-leakage resistance as follows:

$$\boxed{R_{SL} = \frac{V_R}{I_{SL}}} \qquad (07)$$

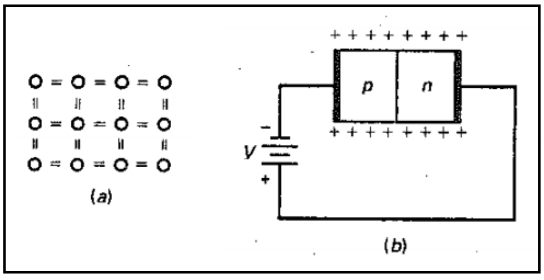


Fig. 25: (a) Atoms on the surface of a crystal have no neighbors;

(b) surface of crystal has holes

Example: A silicon diode has a saturation current of 5 nA at 25°C. What is the saturation current at 100°C?

The change in temperature is:

$$\boxed{\Delta T = 100°\text{C} - 25°\text{C} = 75°\text{C}}$$

With Eq. (05), there are seven doublings between 25°C and 95°C:

$$\boxed{I_S = (2^7)(5 \text{ nA}) = 640 \text{ nA}}$$

With Eq. (06), there are an additional 5° between 95°C and 100°C:

$$\boxed{I_S = (1.07^5)(640 \text{ nA}) = 898 \text{ nA}}$$

Part II. Semiconductor devices (Diode)

1. Introduction

An ordinary resistor is a linear device because the graph of its current versus voltage is a straight line. A diode is different. It is a nonlinear device because the graph of its current versus voltage is not a straight line. The reason is the barrier potential. When the diode voltage is less than the barrier potential, the diode current is small. When the diode voltage exceeds the barrier potential, the diode current increases rapidly.

2. The schematic symbol and case styles

Figure (01 a) shows the pn structure and schematic symbol of a diode. The p side is called the anode, and the n side the cathode. The diode symbol looks like an arrow that points from the p side to the n side, from the anode to the cathode.

3. Basic diode circuit

Figure (01b) shows a diode circuit. In this circuit, the diode is forward biased. How do we know? Because the positive battery terminal drives the p side through a resistor, and the negative battery terminal is connected to the n side. With this connection, the circuit is trying to push holes and free electrons toward the junction.

In more complicated circuits, it may be difficult to decide whether the diode is forward biased. Here is a guideline. Ask yourself this question: Is the external circuit pushing current in the easy direction of flow? If the answer is yes, the diode is forward biased.

What is the easy direction of flow? If you use conventional current, the easy direction is the same direction as the diode arrow. If you prefer electron flow, the easy direction is the other way. When the diode is part of a complicated circuit, we also can use Thevenin's theorem to determine whether it is forward biased. For instance, assume that we have reduced a complicated circuit with Thevenin's theorem to get Fig. (01 b). We would know that the diode is forward biased.

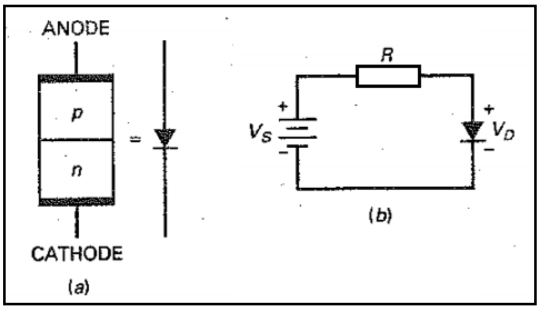


Fig. 01: Diode, (a) Schematic symbol; (b) forward bias

4. The forward region

Figure (01 b) is a circuit that you can set up in the laboratory. After you connect this circuit, you can measure the diode current and voltage. You can also reverse the polarity of the dc source and measure diode current and voltage for reverse bias. If you plot the diode current versus the diode voltage, you will get a graph that looks like Fig. (02).

This is a visual summary of the ideas discussed in the preceding chapter. For instance, when the diode is forward biased, there is no significant current until the diode voltage is greater than the barrier potential. On the other hand, when the diode is reverse biased, there is almost no reverse current until the diode voltage reaches the breakdown voltage. Then, avalanche produces a large reverse current, destroying the diode.

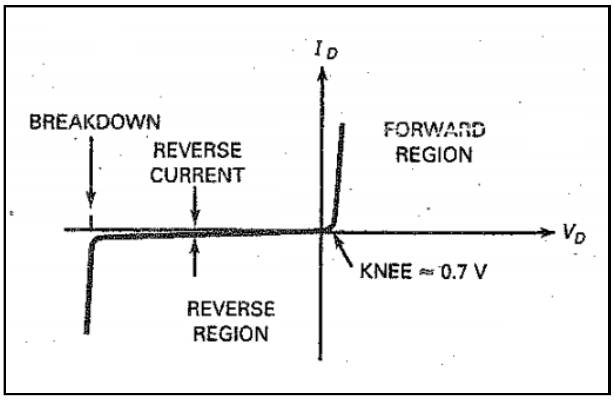


Fig. 02: Diode curve

5. Knee voltage

In the forward region, the voltage at which the current starts to increase rapidly is called the knee voltage of the diode. The knee voltage equals the barrier potential. Analysis of diode circuits usually comes down to determining whether the diode voltage is more or less than the knee voltage. If it's more, the diode conducts easily. If it's less, the diode conducts poorly. We define the knee voltage of a silicon diode as:

$$\boxed{V_K \approx 0.7\ \mathrm{V}} \qquad (01)$$

Even though germanium diodes are rarely used in new designs, you may still encounter germanium diodes in special circuits or in older equipment. For this reason, remember that the knee voltage of a germanium diode is approximately 0.3 V. This lower knee voltage is an advantage and accounts for the use of a germanium diode in certain applications.

6. Bulk resistance

Above the knee voltage, the diode current increases rapidly. This means that small increases in the diode voltage cause large increases in diode current. After the barrier potential is overcome, all that impedes the current is the ohmic resistance of the p and n regions. In other words, if the p and n regions were two separate pieces of semiconductor, each would have a resistance that you could measure with an ohmmeter, the same as an ordinary resistor.

The sum of the ohmic resistances is called the bulk resistance of the diode. It is defined as:

$$\boxed{R_B = R_P + R_N} \qquad (02)$$

The bulk resistance depends on the size of the p and n regions and how heavily doped they are. Often, the bulk resistance is less than 1 Ω.

7. Maximum DC forward current

If the current in a diode is too large, the excessive heat can destroy the diode. For this reason, a manufacturer's data sheet specifies the maximum current a diode can safely handle without shortening its life or degrading its characteristics.

The maximum forward current is one of the maximum ratings given on a data sheet. This current may be listed as I_{max}, $I_{F(max)}$, I_0, etc., depending on the manufacturer. This means that the diode can safely handle a continuous forward current of I_{max} value.

8. Power dissipation

You can calculate the power dissipation of a diode the same way as you do for a resistor. It equals the product of diode voltage and current. As a formula:

$$\boxed{P_D = V_D I_D} \qquad (03)$$

The power rating is the maximum power the diode can safely dissipate without shortening its life or degrading its properties. In symbols, the definition is:

$$\boxed{P_{max} = V_{max} I_{max}} \qquad (04)$$

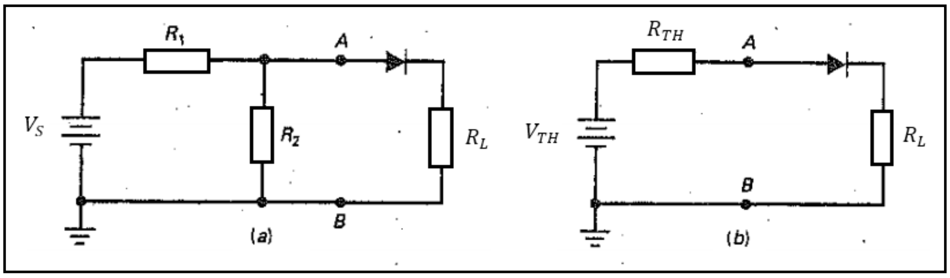
where Vmax is the voltage corresponding to Imax. For instance, if a diode has a maximum voltage and current of 1 V and 2 A, its power rating is 2 W.

<u>Example:</u> Is the diode of the figure (a) below, forward biased or reverse biased?

The voltage across R_2 is positive; therefore, the circuit is trying to push current in the easy direction of flow. If this is not clear, visualize the Thevenin circuit facing the diode as shown in the figure (b) below. In order to determine the Thevenin equivalent, remember that:

$$\boxed{V_{TH} = \frac{R_2}{R_1+R_2}\,(V_S) \quad \text{and} \quad R_{TH} = \frac{R_1}{R_2}}$$

In this series circuit, you can see that the dc source is trying to push current in the easy direction of flow. Therefore, the diode is forward biased. Whenever in doubt, reduce the circuit to a series circuit. Then, it will be clear whether the dc source is trying to push current in the easy direction or not.



Example: A diode has a power rating of 5 W. If the diode voltage is 1.2 V and the diode current is 1.75 A, what is the power dissipation? Will the diode be destroyed?

$$\boxed{P_D = (1.2\text{ V})(1.75\text{ A}) = 2.1\text{ W}}$$

This is less than the power rating, so the diode will not be destroyed.

9. The ideal diode

9.1 The first approximation

Figure (03) shows a detailed graph of the forward region of a diode. Here you see the diode current I_D versus diode voltage V_D. Notice how the current is approximately zero until the diode voltage approaches the barrier potential. Somewhere in the vicinity of 0.6 to 0.7 V, the diode current increases. When the diode voltage is greater than 0.8 V, the diode current is significant and the graph is almost linear.

Depending on how a diode is doped and its physical size, it may differ from other diodes in its maximum forward current, power rating, and other characteristics. If we need an exact solution, we have to use the graph of the particular diode. Although the exact current and voltage points will differ from one diode to the next, the graph of any diode is similar to Fig. (03). All silicon diodes have a knee voltage of approximately 0.7 V.

Most of the time, we do not need an exact solution. This is why we can and should use approximations for a diode. We will begin with the simplest approximation, called an ideal diode. In the most basic terms, what does a diode do? It conducts well in the forward direction and poorly in the reverse direction. Ideally, a diode acts like a perfect conductor (zero resistance) when forward biased and like a perfect insulator (infinite resistance) when reverse biased.

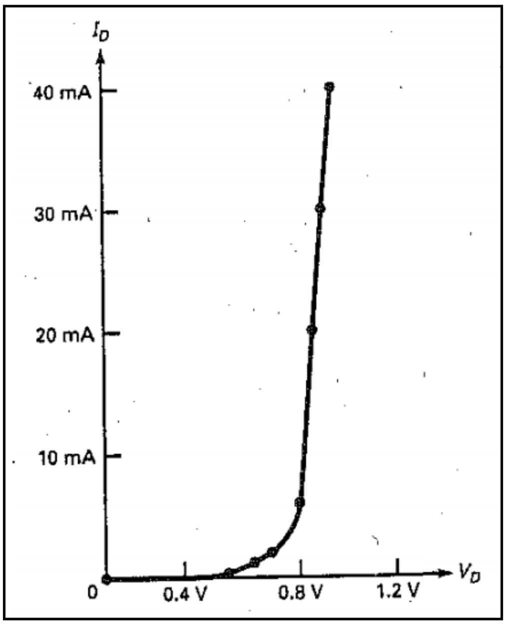


Fig. 03: Graph of forward current

Figure (04 a) shows the current-voltage graph of an ideal diode. It echoes what we just said: zero resistance when forward biased and infinite resistance when reverse biased. It is impossible to build such a device, but this is what manufacturers would produce if they could.

Is there any device that acts like an ideal diode? Yes. An ordinary switch has zero resistance when closed and infinite resistance when open. Therefore, an ideal diode acts like a switch that closes when forward biased and opens when reverse biased. Figure (04 b) summarizes the switch idea.

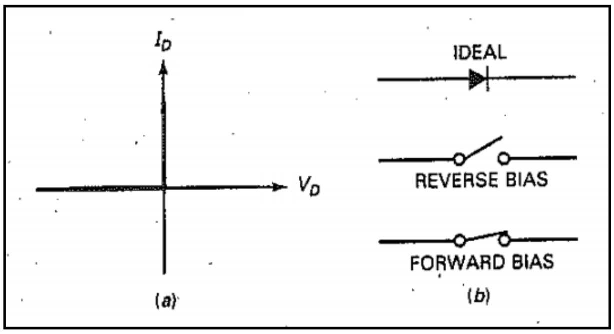


Fig. 04: (a) Ideal diode curve; (b) ideal diode acts like a switch.

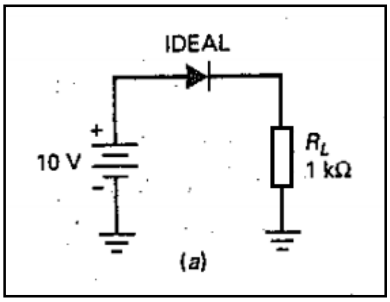
Example: Use the ideal diode (the first approximation) to calculate the load voltage and load current in the figure (a) below.

Since the diode is forward biased, it is equivalent to a closed switch. Visualize the diode as a closed switch. Then, you can see that all of the source voltage appears across the load resistor:

$$\boxed{V_L = 10\ \text{V}}$$

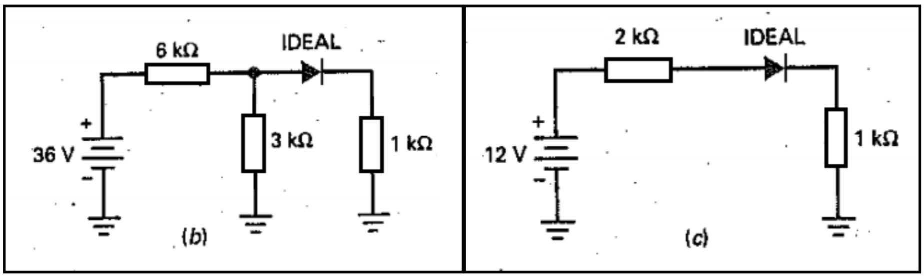
With Ohm's law, the load current is:

$$\boxed{I_L = \frac{10\ \text{V}}{1\ \text{k}\Omega} = 10\ \text{mA}}$$



Calculate the load voltage and load current in the figure (b) below using an ideal diode.

One way to solve this problem is to Thevenize the circuit to the left of the diode. Looking from the diode back toward the source, we see a voltage divider with 6 kΩ and 3 kΩ. The Thevenin voltage is 12 V, and the Thevenin resistance is 2 kΩ. Figure (c) shows the Thevenin circuit driving the diode.



Now that we have a series circuit, we can see that the diode is forward biased. Visualize the diode as a closed switch. Then, the remaining calculations are:

$$\boxed{I_L = \frac{12\ \text{V}}{3\ \text{k}\Omega} = 4\ \text{mA}}$$

And

$$\boxed{V_L = (4\ \text{mA})(1\ \text{k}\Omega) = 4\ \text{V}}$$

You don't have to use Thevenin's theorem. You can analyze Fig. (b) by visualizing the diode as a closed switch. Then, you have 3 kΩ in parallel with 1 kΩ, equivalent to 750 Ω. Using Ohm's law, you can calculate a voltage drop of 32 V across the 6 kΩ. The rest of the analysis produces the same load voltage and load current.

9.2 The second approximation

The ideal approximation is all right in most troubleshooting situations. But we are not always troubleshooting. Sometimes, we want a more accurate value for load current and load voltage. This is where the second approximation comes in.

Figure (05 a) shows the graph of current versus voltage for the second approximation. The graph says that no current exists until 0.7 V appears across the diode. At this point, the diode turns on. Thereafter, only 0.7 V can appear across the diode, no matter what the current.

Figure (05 b) shows the equivalent circuit for the second approximation of a silicon diode. We think of the diode as a switch in series with a barrier potential of 0.7 V. If the Thevenin voltage facing the diode is greater than 0.7 V, the switch will close. When conducting, then the diode voltage is 0.7 V for any forward current.

On the other hand, if the Thevenin voltage is less than 0.7 V, the switch will open. In this case, there is no current through the diode.

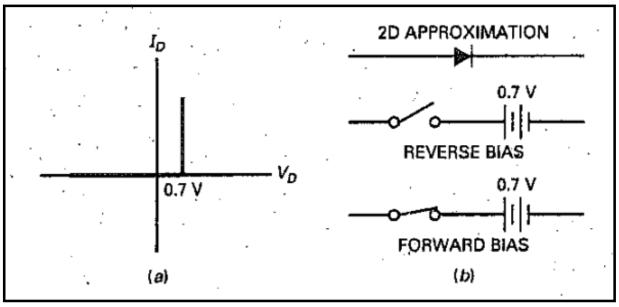
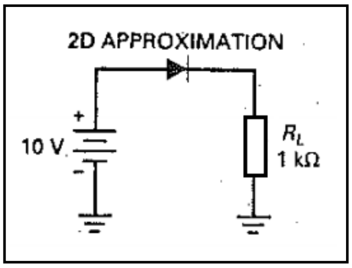


Fig. 05: (a) Diode curve for second approximation; (b) equivalent circuit for second approximation

<u>Example:</u> Use the second approximation to calculate the load voltage, load current, and diode power in the figure below.



Since the diode is forward biased, it is equivalent to a battery of 0.7 V. This means that the load voltage equals the source voltage minus the diode drop:

$$\boxed{V_L = 10\text{ V} - 0.7\text{ V} = 9.3\text{ V}}$$

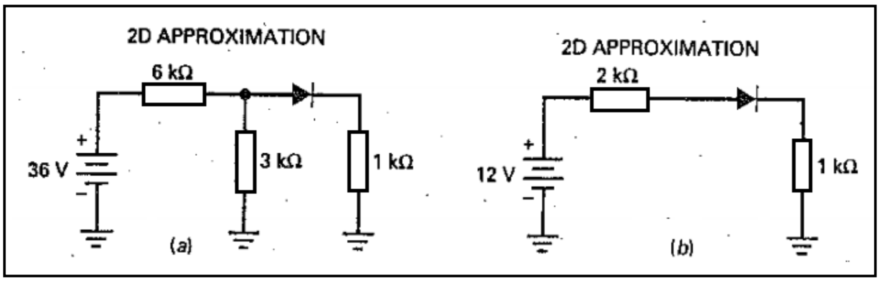
With Ohm's law, the load current is:

$$\boxed{I_L = \frac{9.3\text{ V}}{1\text{ k}\Omega} = 9.3\text{ mA}}$$

The diode power is:

$$\boxed{P_D = (0.7\text{ V})(9.3\text{ mA}) = 6.51\text{ mW}}$$

Example: Calculate the load voltage, load current, and diode power in the figure (a) below using the second approximation.



Again, we will Thevenize the circuit to the left of the diode. As before, the Thevenin voltage is 12 V and the Thevenin resistance is 2 kΩ. Figure (b) shows the simplified circuit.
Since the diode voltage is 0.7 V, the load current is:

$$\boxed{I_L = \frac{12\text{ V} - 0.7\text{ V}}{3\text{ k}\Omega} = 3.77\text{ mA}}$$

The load voltage is:

$$\boxed{V_L = (3.77\text{ mA})(1\text{ k}\Omega) = 3.77\text{ V}}$$

and the diode power is:

$$\boxed{P_D = (0.7\text{ V})(3.77\text{ mA}) = 2.64\text{ mW}}$$

9.3 The third approximation

In the third approximation of a diode, we include the bulk resistance R_B. Figure (06 a) shows the effect that R_B has on the diode curve. After the silicon diode turns on, the voltage increases linearly with an increase in current. The greater the current, the larger the diode voltage because of the voltage drop across the bulk resistance.

The equivalent circuit for the third approximation is a switch in series with a barrier potential of 0.7 V and a resistance of R_B (see Fig. (06 b)). When the diode voltage is larger than 0.7 V, the diode conducts. During conduction, the total voltage across the diode is:

$$\boxed{V_D = 0.7\ \text{V} + I_D R_B} \tag{05}$$

Often, the bulk resistance is less than 1 Ω, and we can safely ignore it in our calculations. A useful guideline for ignoring bulk resistance is this definition:

$$\boxed{\textbf{Ignore bulk: } R_B < 0.01 R_{TH}} \tag{06}$$

This says to ignore the bulk resistance when it is less than 1/100 of the Thevenin resistance facing the diode. When this condition is satisfied, the error is less than 1 percent. The third approximation is rarely used by technicians because circuit designers usually satisfy Eq. (06).

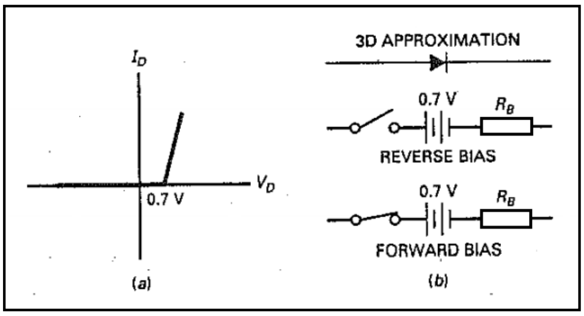
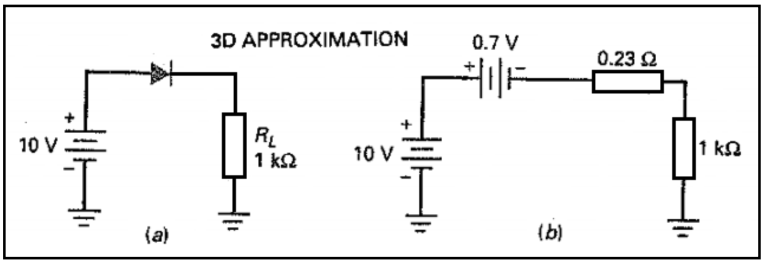


Fig. 06: (a) Diode curve for third approximation;
(b) equivalent circuit for third approximation

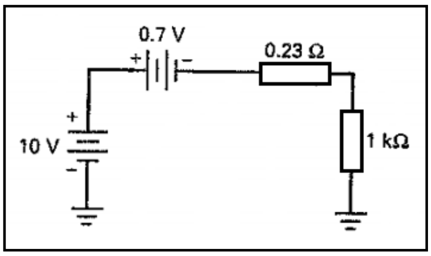
Example: The diode of figure (a) below has a bulk resistance of 0.23 Ω. What is the load voltage, load current, and diode power?

Replacing the diode with its third approximation, we get figure (b) below. The bulk resistance is small enough to ignore because it is less than 1/100 of the load resistance. In this case, we can use the second approximation to solve the problem. We already did this in Example 3-6, where we found a load voltage, load current, and diode power of 9.3 V, 9.3 mA, and 6.51 mW.



Example: Repeat the preceding example for a load resistance of 10 V.

Figure below shows the equivalent circuit.



The total resistance is:

$$\boxed{R_T = 0.23\ \Omega + 10\ \Omega = 10.23\ \Omega}$$

The total voltage across R_T is:

$$\boxed{V_T = 10\ \text{V} - 0.7\ \text{V} = 9.3\ \text{V}}$$

Therefore, the load current is:

$$\boxed{I_L = \frac{9.3\ \text{V}}{10.23\ \Omega} = 0.909\ \text{A}}$$

The load voltage is:

$$\boxed{V_L = (0.909\ \text{A})(10\ \Omega) = 9.09\ \text{V}}$$

To calculate the diode power, we need to know the diode voltage. We can get this in either of two ways. We can subtract the load voltage from the source voltage:

$$\boxed{V_D = 10\ \text{V} - 9.09\ \text{V} = 0.91\ \text{V}}$$

or we can use Eq. (05):

$$\boxed{V_D = 0.7\ \text{V} + (0.909\ \text{A})(0.23\ \Omega) = 0.909\ \text{V}}$$

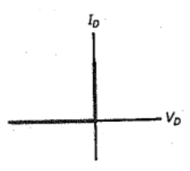
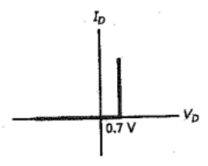
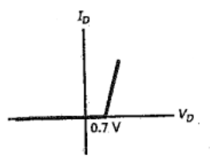
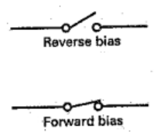
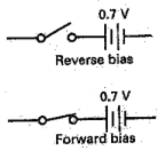
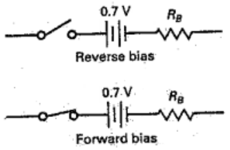
The slight difference in the last two answers is caused by rounding. The diode power is:

$$\boxed{P_D = (0.909\ \text{V})(0.909\ \text{A}) = 0.826\ \text{W}}$$

The used diode has a maximum forward current of 1 A and a power rating of 1 W, so the diode is being pushed to its limits with a load resistance of 10 Ω, and the load voltage calculated with the third approximation is 9.09 V.

Summary Table (01) illustrates the differences between the three diode approximations.

Tab. 01: Differences between the three diode approximations

Diode approximations			
	First or ideal approximation	Second or practical approximation	Third approximation
Diode curve			
Equivalent circuit			

10. How to calculate bulk resistance

When you are trying to analyze a diode circuit accurately, you will need to know the bulk resistance of the diode. Here is the derivation for bulk resistance:

$$\boxed{R_B = \frac{V_2 - V_1}{I_2 - I_1}} \tag{07}$$

where V_1 and I_1 are the voltage and current at some point at or above the knee voltage; V_2 and I_2 are the voltage and current at some higher point on the diode curve.

Example: For instance, the data sheet of a diode gives a forward voltage of 0.93 V for a current of 1 A. Since this is a silicon diode, it has a knee voltage of approximately 0.7 V and a current of approximately zero. Therefore, the values to use are $V_2 = 0.93$ V, $I_2 = 1$ A, $V_1 = 0.7$ V, and $I_1 = 0$. Substituting these values into an equation, we get a bulk resistance of:

$$\boxed{R_B = \frac{V_2 - V_1}{I_2 - I_1} = \frac{0.93\text{ V} - 0.7\text{ V}}{1\text{ A} - 0\text{ A}} = \frac{0.23\text{ V}}{1\text{ A}} = 0.23\ \Omega}$$

Incidentally, the diode curve is a graph of current versus voltage. The bulk resistance equals the inverse of the slope above the knee. The greater the slope of the diode curve, the smaller the bulk resistance. In other words, the more vertical the diode curve is above the knee, the lower the bulk resistance.

11. DC Resistance of a diode

If you take the ratio of total diode voltage to total diode current, you get the DC resistance of the diode. In the forward direction, this dc resistance is symbolized by R_F; in the reverse direction, it is designated R_R.

11.1 Forward resistance

Because the diode is a nonlinear device, its DC resistance varies with the current through it.

<u>Example:</u> For example, here are some pairs of forward current and voltage for a diode: 10 mA at 0.65 V, 30 mA at 0.75 V, and 50 mA at 0.85 V. At the first point, the dc resistance is:

$$\boxed{R_F = \frac{0.65\text{ V}}{10\text{ mA}} = 65\ \Omega}$$

At the second point:

$$\boxed{R_F = \frac{0.75\text{ V}}{30\text{ mA}} = 25\ \Omega}$$

And at the third point:

$$\boxed{R_F = \frac{0.85\text{ mV}}{50\text{ mA}} = 17\ \Omega}$$

Notice how the DC resistance decreases as the current increases. In any case, the forward resistance is low compared to the reverse resistance.

11.2 Reverse resistance

Example: Similarly, here are two sets of reverse current and voltage for a diode: 25 nA at 20 V; 5µ A at 75 V. At the first point, the DC resistance is:

$$\boxed{R_R = \frac{20\ \text{V}}{25\ \text{nA}} = 800\ \text{M}\Omega}$$

At the second point:

$$\boxed{R_R = \frac{75\ \text{V}}{5\ \mu\text{A}} = 15\ \text{M}\Omega}$$

Notice how the DC resistance decreases as we approach the breakdown voltage (75 V).

11.3 DC Resistance versus bulk resistance

The DC resistance of a diode is different from the bulk resistance. The DC resistance of a diode equals the bulk resistance plus the effect of the barrier potential. In other words, the DC resistance of a diode is its total resistance, whereas the bulk resistance is the resistance of only the p and n regions. For this reason, the DC resistance of a diode is always greater than the bulk resistance.

12. Load lines

This section is about the load line, a tool used to find the exact value of diode current and voltage.

12.1 Equation for the load line

How can we find the exact diode current and voltage in Fig. (07)? The current through the resistor is:

$$\boxed{I_D = \frac{V_S - V_D}{R_s}} \qquad (08)$$

Because of the series circuit, this current is the same through the diode.

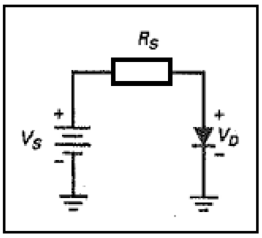
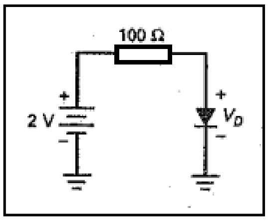


Fig. 07: Load-line analysis

Example: If the source voltage is 2 V and the resistance is 100 V as shown in figure below,



then Eq. (08) becomes:

$$\boxed{I_D = \frac{2 - V_D}{100}}$$

This equation is a linear relationship between current and voltage. If we plot this equation, we will get a straight line. For instance, let V_D equal zero. Then:

$$\boxed{I_D = \frac{2\text{ V} - 0\text{ V}}{100\ \Omega} = 20\text{ mA}}$$

Plotting this point (I_D = 20 mA, V_D = 0) gives the point on the vertical axis of the figure below. This point is called saturation because it represents maximum current with 2 V across 100 V.

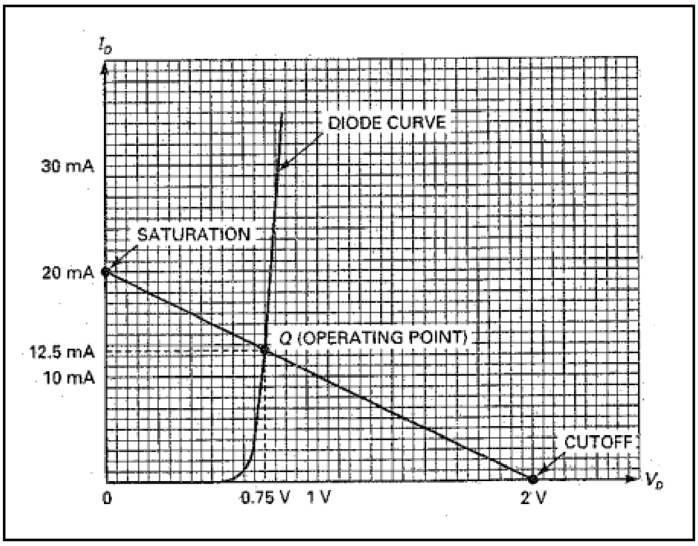


Fig. 08: Load Q point is the intersection of the diode curve and the load line

Here's how to get another point. Let V_D equal 2 V. Then equation of "I_D" gives:

$$\boxed{I_D = \frac{2\text{ V} - 2\text{ V}}{100\ \Omega} = 0}$$

When we plot this point (I_D = 0, V_D =2 V), we get the point shown on the horizontal axis (Fig. (08)). This point is called cutoff because it represents minimum current.

By selecting other voltages, we can calculate and plot additional points. Because equation of "I_D" is linear, all points will lie on the straight line shown in Fig. (08). The straight line is called the load line.

12.2 The Q point

Figure (08) shows the load line and a diode curve. The point of intersection, known as the Q point, represents a simultaneous solution between the diode curve and the load line. In other words, the Q point is the only point on the graph that works for both the diode and the circuit. By reading the coordinates of the Q point, we get a current of 12.5 mA and a diode voltage of 0.75 V.

Incidentally, the Q point has no relationship to the figure of merit of a coil. In the present discussion, Q is an abbreviation for quiescent, which means "at rest."

III. Diode circuits

1. Introduction to electronic systems

In your study of Electronic Principles, you will be introduced to a variety of electronic semiconductor devices. Each of these devices will have unique properties and characteristics. Your knowledge of how these individual components function is very important. But this is just the beginning.

These electronic devices normally do not function on their own. Instead, with the addition of other electronic components, such as resistors, capacitors, inductors, and other semiconductor devices, they are interconnected to form electronic circuits. These electronic circuits are often categorized into subsets, such as analog circuits and digital circuits, or application specific circuits as amplifiers, converters, rectifiers, and so on. While analog circuits operate with infinitely varying quantities, often referred to as linear electronics, digital circuits generally operate with signal levels found in two distinct states representing logical or numeric values.

2. The Half-Wave Rectifier

Figure (01a) shows a half-wave rectifier circuit. The AC source produces a sinusoidal voltage. Assuming an ideal diode, the positive half-cycle of source voltage will forward-bias the diode. Since the switch is closed, as shown in Fig. (01b), the positive half-cycle of source voltage will appear across the load resistor. On the negative half-cycle, the diode is reverse biased. In this case, the ideal diode will appear as an open switch, as shown in Fig. (01c), and no voltage appears across the load resistor.

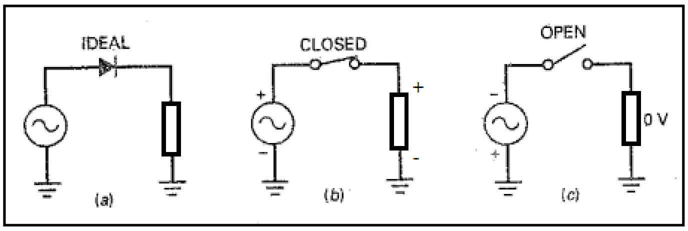


Fig. 01: (a) Ideal half-wave rectifier; (b) on positive half-cycle; (c) on negative half-cycle

2.1 Ideal waveforms

Figure (02 a) shows a graphical representation of the input voltage waveform. It is a sine wave with an instantaneous value of v_{in} and a peak value of Vp(in). A pure sinusoid like this has an average value of zero over one cycle because each instantaneous voltage has an equal and opposite voltage half a cycle later. If you measure this voltage with a DC voltmeter, you will get a reading of zero because a DC voltmeter indicates the average value.

In the half-wave rectifier of Fig. (02 b), the diode is conducting during the positive half-cycles but is non-conducting during the negative half-cycles. Because of this, the circuit clips off the negative half-cycles, as shown in Fig. (02 c). We call a waveform like this a half-wave signal. This half-wave voltage produces a unidirectional load current. This means that it flows in only one direction. If the diode were reversed (Fig. (02 d)), it would become forward biased when the input voltage was negative. As a result, the output pulses would be negative. This is shown in Fig. (02 e). Notice how the negative peaks are offset from the positive peaks and follow the negative alternations of the input voltage.

A half-wave signal like the one in Fig. (02 c) is a pulsating DC voltage that increases to a maximum, decreases to zero, and then remains at zero during the negative half-cycle. This is not the kind of DC voltage we need for electronics equipment. What we need is a constant voltage, the same as you get from a battery. To get this kind of voltage, we need to filter the half-wave signal.

When you are troubleshooting, you can use the ideal diode to analyze a half-wave rectifier. It's useful to remember that the peak output voltage equals the peak input voltage:

$$\boxed{\textbf{Ideal half wave: } V_{p(\text{out})} = V_{p(\text{in})}} \qquad (01)$$

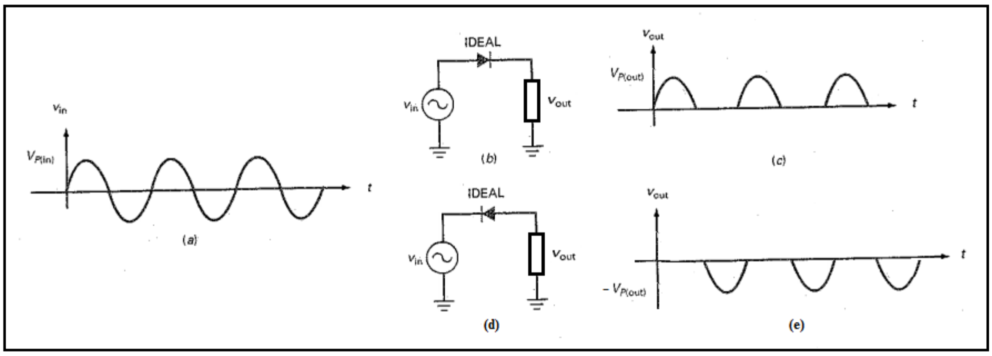


Fig. 02: (a) Input to half-wave rectifier; (b) circuit during the positive half-cycles; (c) output of positive half-wave rectifier; (d) circuit the negative half-cycles; (e) output of negative half-wave rectifier.

2.2 DC value of half-wave signal

The dc value of a signal is the same as the average value. If you measure a signal with a DC voltmeter, the reading will equal the average value. In basic courses, the DC value of a half-wave signal is derived. The formula is:

$$\boxed{\textbf{Half wave: } V_{\text{dc}} = \frac{V_p}{\pi}} \qquad (02)$$

The proof of this derivation requires calculus because we have to work out the average value over one cycle.

Since $1/\pi \approx 0.318$, you may see Eq. (02) written as:

$$\boxed{V_{dc} \approx 0.318V_p}$$

When the equation is written in this form, you can see that the dc or average value equals 31.8 percent of the peak value. For instance, if the peak voltage of the half-wave signal is 100 V, the dc voltage or average value is 31.8 V.

2.3 Output frequency

The output frequency is the same as the input frequency. This makes sense when you compare Fig. 2c with Fig. 2a. Each cycle of input voltage produces one cycle of output voltage. Therefore, we can write:

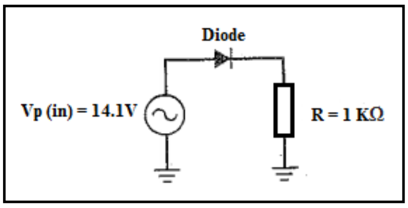
$$\boxed{\textbf{Half wave: } f_{out} = f_{in}} \qquad (03)$$

2.4 Second approximation

We don't get a perfect half-wave voltage across the load resistor. Because of the barrier potential, the diode does not turn on until the ac source voltage reaches approximately 0.7 V. When the peak source voltage is much greater than 0.7 V, the load voltage will resemble a half-wave signal. For instance, if the peak source voltage is 100 V, the load voltage will be close to a perfect half-wave voltage. If the peak source voltage is only 5 V, the load voltage will have a peak of only 4.3 V. When you need to get a better answer, use this derivation:

$$\boxed{\textbf{2d half wave: } V_{p(out)} = V_{p(in)} - 0.7\text{ V}} \qquad (04)$$

Example: Calculate the values of the peak load voltage and the DC load voltage for the figure below.



The peak source voltage in the figure above is: $V_p = 14.1$ V

With an ideal diode, the peak load voltage is: $V_{p\,(out)} = V_{p\,(in)} = 14.1$ V

The DC load voltage is: $V_{DC} = V_p / \pi = 14.1 \text{ V} / \pi = 4.49$ V

With the second approximation, we get a peak load voltage of:
$V_{p\,(out)} = V_{p\,(in)} - 0.7 \text{ V} = 14.1 \text{ V} - 0.7 \text{ V} = 13.4$ V
and a DC load voltage of: $V_{DC} = V_p / \pi = 13.4 \text{ V} / \pi = 4.27$ V

3. The Full-wave rectifier

Figure (03 a) shows a full-wave rectifier circuit. Notice the grounded center tap on the secondary winding. The full-wave rectifier is equivalent to two half-wave rectifiers. Because of the center tap, each of these rectifiers has an input voltage equal to half the secondary voltage. Diode D_1 conducts on the positive half-cycle, and diode D_2 conducts on the negative half-cycle. As a result, the rectified load current flows during both half-cycles. The full-wave rectifier acts the same as two back-to-back half-wave rectifiers.

Figure (03 b) shows the equivalent circuit for the positive half-cycle. As you see, D_1 is forward biased. This produces a positive load voltage as indicated by the plus-minus polarity across the load resistor. Figure (03 c) shows the equivalent circuit for the negative half-cycle. This time, D_2 is forward biased. As you can see, this also produces a positive load voltage.

During both half-cycles, the load voltage has the same polarity and the load current is in the same direction. The circuit is called a full-wave rectifier because it has changed the ac input voltage to the pulsating dc output voltage shown in Fig. (03 d). This waveform has some interesting properties that we will now discuss.

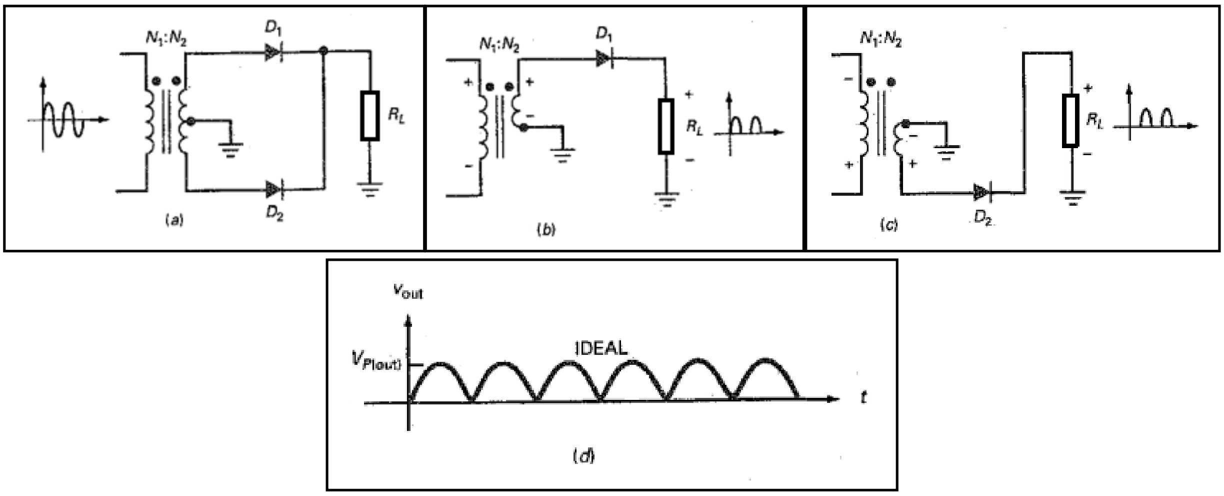


Fig. 03: (a) Full-wave rectifier; (b) equivalent circuit for positive half-cycle; (c) equivalent circuit for negative half-cycle; (d) full-wave output

3.1 DC or average value

Since the full-wave signal has twice as many positive cycles as the half-wave signal, the DC or average value is twice as much, given by:

$$\boxed{\textbf{Full wave: } V_{dc} = \frac{2V_p}{\pi}} \qquad (05)$$

Since 2/ π = 0.636, you may see Eq. (05) written as: $V_{dc} \approx 0.636V_p$. In this form, you can see that the dc or average value equals 63.6 percent of the peak value. For instance, if the peak voltage of the full-wave signal is 100 V, the DC voltage or average value is 63.6 V.

3.2 Output frequency

With a half-wave rectifier, the output frequency equals the input frequency. But with a full-wave rectifier, something unusual happens to the output frequency. The ac line voltage has a frequency of 60 Hz. Therefore, the input period equals:

$$\boxed{T_{in} = \frac{1}{f} = \frac{1}{60\text{ Hz}} = 16.7\text{ ms}}$$

Because of the full-wave rectification, the period of the full-wave signal is half the input period:

$$\boxed{T_{out} = 0.5(16.7\text{ ms}) = 8.33\text{ ms}}$$

The output frequency:

$$\boxed{f_{out} = \frac{1}{T_{out}} = \frac{1}{8.33\text{ ms}} = 120\text{ Hz}}$$

The frequency of the full-wave signal is double the input frequency. This makes sense. A full-wave output has twice as many cycles as the sine-wave input has. The full-wave rectifier inverts each negative half-cycle so that we get double the number of positive half-cycles. The effect is to double the frequency. As a derivation:

$$\boxed{\textbf{Full wave: } f_{out} = 2f_{in}} \qquad (06)$$

3.3 Second approximation

Since the full-wave rectifier is like two back-to-back half-wave rectifiers, we can use the second approximation given earlier. The idea is to subtract 0.7 V from the ideal peak output voltage. The following example will illustrate the idea.

4. The bridge rectifier

Figure (04 a) shows a bridge rectifier circuit. The bridge rectifier is similar to a full-wave rectifier because it produces a full-wave output voltage. Diodes D_1 and D_2 conduct on the positive half-cycle, and D_3 and D_4 conduct on the negative half-cycle. As a result, the rectified load current flows during both half-cycles.

Figure (04 b) shows the equivalent circuit for the positive half-cycle. As you can see, D_1 and D_2 are forward biased. This produces a positive load voltage as indicated by the plus-minus polarity across the load resistor. As a memory aid, visualize D_2 shorted. Then, the circuit that remains is a half-wave rectifier, which we are already familiar with.

Figure (04 c) shows the equivalent circuit for the negative half-cycle. This time, D_3 and D_4 are forward biased. This also produces a positive load voltage. If you visualize D_3 shorted, the circuit looks like a half-wave rectifier. So the bridge rectifier acts like two back-to-back half-wave rectifiers.

During both half-cycles, the load voltage has the same polarity and the load current is in the same direction. The circuit has changed the ac input voltage to the pulsating dc output voltage shown in Fig. (04 d). Note the advantage of this type of full-wave rectification over the center-tapped version in the previous section: The entire secondary voltage can be used.

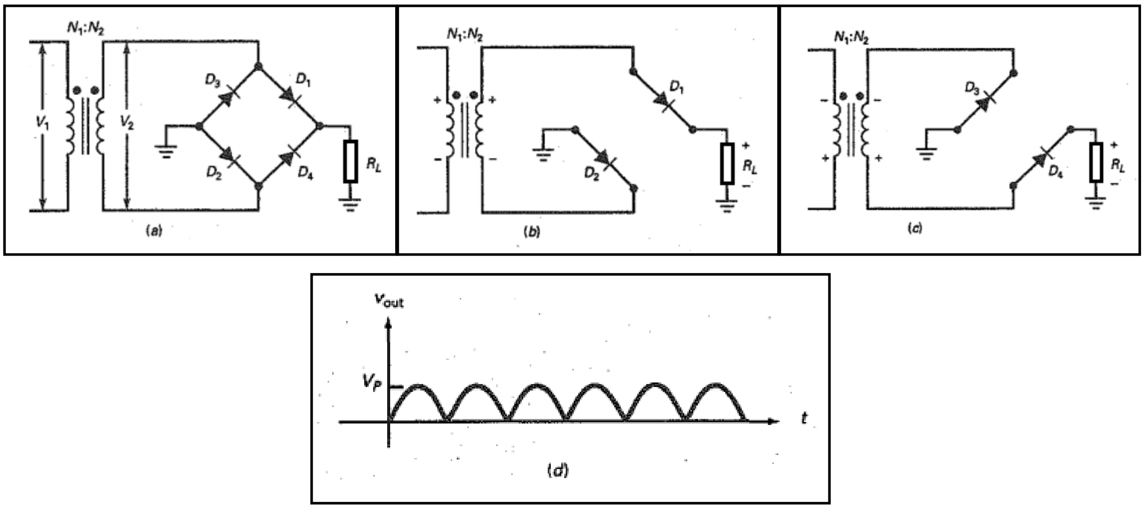


Fig. 04: (a) Bridge rectifier; (b) equivalent circuit for positive half-cycle; (c) equivalent circuit for negative half-cycle; (d) full-wave output; (e) bridge rectifier packages

4.1 Average value and output frequency

Because a bridge rectifier produces a full-wave output, the equations for average value and output frequency are the same as those given for a full-wave rectifier:

$$\boxed{V_{dc} = \frac{2V_p}{\pi}} \text{ and } \boxed{f_{out} = 2f_{in}}$$

The average value is 63.6 percent of the peak value, and the output frequency is 120 Hz, given a line frequency of 60 Hz.

One advantage of a bridge rectifier is that all the secondary voltage is used as the input to the rectifier. Given the same transformer, we get twice as much peak voltage and twice as much dc voltage with a bridge rectifier as with a full-wave rectifier. Doubling the dc output voltage compensates for having to use two extra diodes. As a rule, you will see the bridge rectifier used a lot more than the full-wave rectifier.

Incidentally, the full-wave rectifier was in use for many years before the bridge rectifier was used. For this reason, it has retained the name full-wave rectifier even though a bridge rectifier also has a full-wave output. To distinguish the full-wave rectifier from the bridge rectifier, some literature may refer to a full-wave rectifier as a conventional full-wave rectifier, a two-diode full-wave rectifier, or a center-tapped full-wave rectifier.

4.2 Second approximation and other losses

Since the bridge rectifier has two diodes in the conducting path, the peak output voltage is given by: (2d bridge)

$$\boxed{V_{p(\mathrm{out})} = V_{p(\mathrm{in})} - 1.4\ \mathrm{V}} \tag{07}$$

As you can see, we have to subtract two diode drops from the peak to get a more accurate value of peak load voltage. Summary Table (01) compares the three rectifiers and their properties.

Unfiltered rectifiers			
	Half-wave	**Full-wave**	**Bridge**
Number of diodes	1	2	4
Rectifier input	$V_{p(2)}$	$0.5V_{p(2)}$	$V_{p(2)}$
Peak output (ideal)	$V_{p(2)}$	$0.5V_{p(2)}$	$V_{p(2)}$
Peak output (2d)	$V_{p(2)} - 0.7\ \mathrm{V}$	$0.5V_{p(2)} - 0.7\ \mathrm{V}$	$V_{p(2)} - 1.4\ \mathrm{V}$
DC output	$V_{p(\mathrm{out})}/\pi$	$2V_{p(\mathrm{out})}/\pi$	$2V_{p(\mathrm{out})}/\pi$
Ripple frequency	f_{in}	$2f_{\mathrm{in}}$	$2f_{\mathrm{in}}$

*$V_{p(2)}$ = peak secondary voltage; $V_{p(\mathrm{out})}$ = peak output voltage.

5. The Choke-input filter

At one time, the choke-input filter was widely used to filter the output of a rectifier. Although not used much anymore because of its cost, bulk, and weight, this type of filter has instructional value and helps make it easier to understand other filters.

5.1 Basic idea

Look at Fig. (05 a). This type of filter is called a choke-input filter. The ac source produces a current in the inductor, capacitor, and resistor. The ac current in each component depends on the inductive reactance, capacitive reactance, and the resistance. The inductor has a reactance given by:

$$\boxed{X_L = 2\pi fL}$$

The capacitor has a reactance given by:

$$\boxed{X_C = \frac{1}{2\pi fC}}$$

As you learned in previous courses, the choke (or inductor) has the primary characteristic of opposing a change in current. Because of this, a choke-input filter ideally reduces the ac current in the load resistor to zero. To a second approximation, it reduces the ac load current to a very small value. Let us find out why.

The first requirement of a well-designed choke-input filter is to have X_C at the input frequency be much smaller than R_L. When this condition is satisfied, we can ignore the load resistance and use the equivalent circuit of Fig. (05 b). The second requirement of a well-designed choke-input filter is to have X_L be much greater than X_C at the input frequency. When this condition is satisfied, the AC output voltage approaches zero. On the other hand, since the choke approximates a short circuit at 0 Hz and the capacitor approximates an open at 0 Hz, the DC current can be passed to the load resistance with minimum loss.

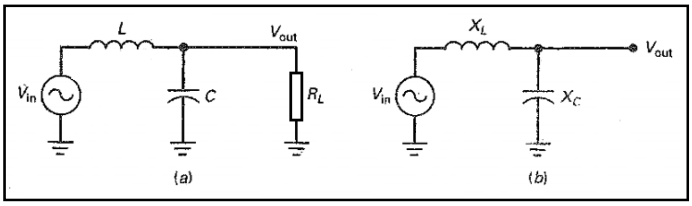


Fig. 05: (a) Choke-input filter; (b) AC-equivalent circuit

In Fig. (05 b), the circuit acts like a reactive voltage divider. When X_L is much greater than X_C, almost all the ac voltage is dropped across the choke. In this case, the ac output voltage equals:

$$\boxed{V_{out} \approx \frac{X_C}{X_L} V_{in}} \qquad (08)$$

5.2 Filtering the output of a rectifier

Figure (06 a) shows a choke-input filter between a rectifier and a load. The rectifier can be a half-wave, full-wave, or bridge type. What effect does the choke-input filter have on the load voltage? The easiest way to solve this problem is to use the superposition theorem. Recall what this theorem says: If you have two or more sources, you can analyze the circuit for each source separately and then add the individual voltages to get the total voltage.

The rectifier output has two different components: a dc voltage (the average value) and an AC voltage (the fluctuating part), as shown in Fig. (06 b). Each of these voltages acts like a separate source. As far as the ac voltage is concerned, X_L is much greater than X_C, and this results in very little ac voltage across the load resistor. Even though the ac component is not a pure sine wave, Eq. (08) is still a close approximation for the ac load voltage.

The circuit acts like Fig. (06 c) as far as dc voltage is concerned. At 0 Hz, the inductive reactance is zero and the capacitive reactance is infinite. Only the series resistance of the inductor windings remains. Making R_S much smaller than R_L causes most of the dc component to appear across the load resistor.

That's how a choke-input filter works: Almost all of the dc component is passed on to the load resistor, and almost all of the ac component is blocked. In this way, we get an almost perfect dc voltage, one that is almost constant, like the voltage out of a battery. Figure (06 d) shows the filtered output for a full-wave signal. The only deviation from a perfect dc voltage is the small ac load voltage shown in Fig. (06 d). This small ac load voltage is called ripple. With an oscilloscope, we can measure its peak-to-peak value. To measure the ripple value, set the oscilloscope's vertical input coupling switch or setting to ac instead of dc. This will allow you to see the ac component of the waveform while blocking the dc or average value.

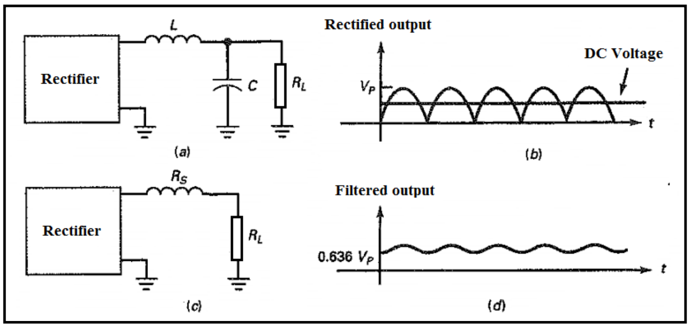


Fig. 06: (a) Rectifier with choke-input filter; (b) rectifier output has DC and AC components; (c) dc-equivalent circuit; (d) filter output is a DC voltage with small ripple

6. The capacitor-input filter

The choke-input filter produces a DC output voltage equal to the average value of the rectified voltage. The capacitor-input filter produces a dc output voltage equal to the peak value of the rectified voltage. This type of filter is the most widely used in power supplies.

6.1 Basic idea

Figure (07 a) shows an AC source, a diode, and a capacitor. The key to understanding a capacitor-input filter is understanding what this simple circuit does during the first quarter-cycle.

Initially, the capacitor is uncharged. During the first quarter-cycle of Fig. (07 b), the diode is forward biased. Since it ideally acts like a closed switch, the capacitor charges, and its voltage equals the source voltage at each instant of the first quarter-cycle. The charging continues until the input reaches its maximum value. At this point, the capacitor voltage equals V_p.

After the input voltage reaches the peak, it starts to decrease. As soon as the input voltage is less than V_p, the diode turns off. In this case, it acts like the open switch of Fig. (07 c). During the remaining cycles, the capacitor stays fully charged and the diode remains open. This is why the output voltage of Fig. (07 b) is constant and equal to V_p.

Ideally, all that the capacitor-input filter does is charge the capacitor to the peak voltage during the first quarter-cycle. This peak voltage is constant; the perfect dc voltage we need for electronics equipment. There's only one problem: there is no load resistor.

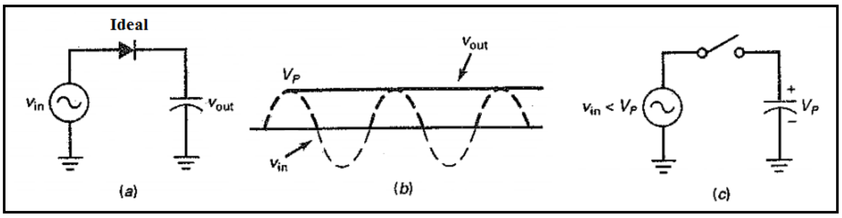


Fig. 07: (a) Unloaded capacitor-input filter; (b) output is pure dc voltage; (c) capacitor remains charged when diode is off.

6.2 Effect of load resistor

For the capacitor-input filter to be useful, we need to connect a load resistor across the capacitor, as shown in Fig. (08 a). As long as the R_LC time constant is much greater than the period, the capacitor remains almost fully charged and the load voltage is approximately V_P. The only deviation from a perfect dc voltage is the small ripple seen in Fig. (08 b). The smaller the peak-to-peak value of this ripple, the more closely the output approaches a perfect dc voltage.

Between peaks, the diode is off and the capacitor discharges through the load resistor. In other words, the capacitor supplies the load current. Since the capacitor discharges only slightly between peaks, the peak-to-peak ripple is small. When the next peak arrives, the diode conducts briefly and recharges the capacitor to the peak value. A key question is: What size should the capacitor be for proper operation? Before discussing capacitor size, consider what happens with the other rectifier circuits.

6.3 Full-wave filtering

If we connect a full-wave or bridge rectifier to a capacitor-input filter, the peak-to-peak ripple is cut in half. Figure (08 c) shows why. When a full-wave voltage is applied to the RC circuit, the capacitor discharges for only half as long. Therefore, the peak-to-peak ripple is half the size it would be with a half-wave rectifier.

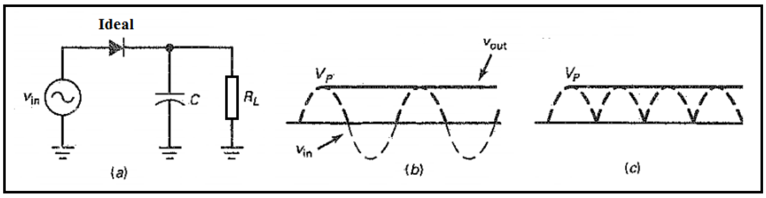


Fig. 08: (a) Loaded capacitor-input filter; (b) output is a dc voltage with small ripple; (c) full-wave output has less ripple

6.4 The ripple formula

Here is a derivation we will use to estimate the peak-to-peak ripple out of any capacitor-input filter:

$$\boxed{V_R = \frac{I}{fC}} \tag{10}$$

Where:

- V_R: peak-to-peak ripple voltage,
- I: DC load current,
- f: ripple frequency,
- C: capacitance.

This is an approximation, not an exact derivation. We can use this formula to estimate the peak-to-peak ripple.

IV. Special-purpose diodes

1. Introduction

Rectifier diodes are the most common type of diode. They are used in power supplies to convert AC voltage to DC voltage. But rectification is not all that a diode can do. Now we will discuss diodes used in other applications. The chapter begins with the Zener diode, which is optimized for its breakdown properties.

Zener diodes are very important because they are the key to voltage regulation. The chapter also covers optoelectronic diodes, including light-emitting diodes (LEDs), and other diodes.

2. The Zener diode

Small-signal and rectifier diodes are never intentionally operated in the breakdown region because this may damage them. A Zener diode is different; it is a silicon diode that the manufacturer has optimized for operation in the breakdown region. The Zener diode is the backbone of voltage regulators, circuits that hold the load voltage almost constant despite large changes in line voltage and load resistance.

2.1 I-V Graph

Figure (01 a) shows the schematic symbol of a Zener diode; Fig. (01 b) is an alternative symbol. In either symbol, the lines resemble a z, which stands for “Zener”. By varying the doping level of silicon diodes, a manufacturer can produce Zener diodes with breakdown voltages from about 2 to over 1000 V. These diodes can operate in any of three regions: forward, leakage, and breakdown.

Figure (01 c) shows the I-V graph of a Zener diode. In the forward region, it starts conducting around 0.7 V, just like an ordinary silicon diode. In the leakage region (between zero and breakdown), it has only a small reverse current. In a Zener diode, the breakdown has a very sharp knee, followed by an almost vertical increase in current. Note that the voltage is almost constant, approximately equal to V_Z over most of the breakdown region. Data sheets usually specify the value of V_Z at a particular test current I_{ZT}.

Figure (01 c) also shows the maximum reverse current I_{ZM}. As long as the reverse current is less than I_{ZM}, the diode is operating within its safe range. If the current is greater than I_{ZM}, the diode will be destroyed. To prevent excessive reverse current, a current-limiting resistor must be used.

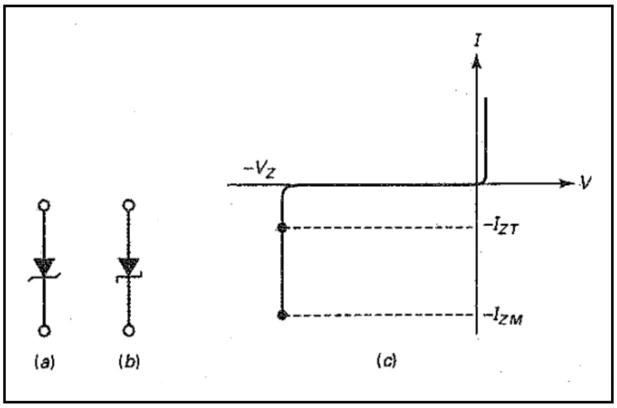


Fig. 01: Zener diode. (a) Schematic symbol;
(b) alternative symbol; (c) graph of current versus voltage;

2.2 Zener resistance

In the third approximation of a silicon diode, the forward voltage across a diode equals the knee voltage plus the additional voltage across the bulk resistance.

Similarly, in the breakdown region, the reverse voltage across a diode equals the breakdown voltage plus the additional voltage across the bulk resistance. In the reverse region, the bulk resistance is referred to as the Zener resistance. This resistance equals the inverse of the slope in the breakdown region. In other words, the more vertical the breakdown region, the smaller the Zener resistance.

In Fig. (01 c), the Zener resistance means that an increase in reverse current produces a slight increase in reverse voltage. The increase in voltage is very small, typically only a few tenths of a volt. This slight increase may be important in design work, but not in troubleshooting and preliminary analysis. Unless otherwise indicated, our discussions will ignore the Zener resistance.

2.3 Zener regulator

A Zener diode is sometimes called a voltage-regulator diode because it maintains a constant output voltage even though the current through it changes. For normal operation, you have to reverse-bias the Zener diode, as shown in Fig. (02 a). Furthermore, to get breakdown operation, the source voltage VS must be greater than the Zener breakdown voltage V_Z. A series resistor R_S is always used to limit the Zener current to less than its maximum current rating. Otherwise, the Zener diode will burn out, like any device with too much power dissipation.

Figure (02 b) shows an alternative way to draw the circuit with grounds. Whenever a circuit has grounds, you can measure voltages with respect to ground. For instance, suppose you want to know the voltage across the series resistor of Fig. (02 b). Here is the one way to find it when you have a built-up circuit. First, measure the voltage from the left end of R_S to ground. Second, measure the voltage from the right end of R_S to ground. Third, subtract the two voltages to get the voltage across R_S. If you have a floating VOM or DMM, you can connect directly across the series resistor.

Figure (02 c) shows the output of a power supply connected to a series resistor and a Zener diode. This circuit is used when you want a dc output voltage that is less than the output of the power supply. A circuit like this is called a Zener voltage regulator, or simply a Zener regulator.

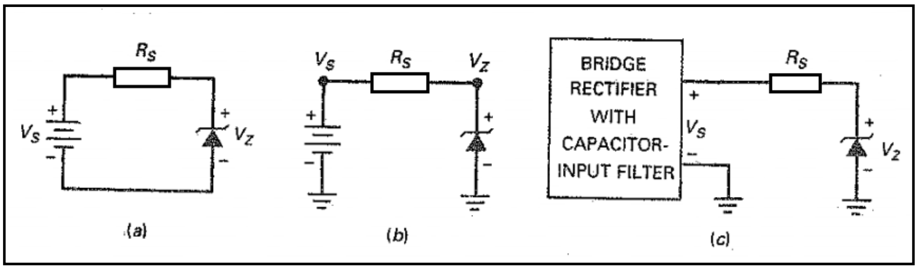


Fig. 02: Zener regulator. (a) Basic circuit; (b) same circuit with grounds; (c) power supply drives regulator

2.4 Ohm's law again

In Fig. (02), the voltage across the series or current-limiting resistor equals the difference between the source voltage and the Zener voltage. Therefore, the current through the resistor is:

$$\boxed{I_S = \frac{V_S - V_Z}{R_S}} \tag{01}$$

Once you have the value of series current, you also have the value of Zener current. This is because Fig. (02) is a series circuit. Note that I_S must be less than I_{ZM}.

2.5 Ideal Zener diode

For troubleshooting and preliminary analysis, we can approximate the breakdown region as vertical. Therefore, the voltage is constant even though the current changes, which is equivalent to ignoring the Zener resistance. Figure (03) shows the ideal approximation of a Zener diode. This means that a Zener diode operating in the breakdown region ideally acts like a battery. In a circuit, it means that you can mentally replace a Zener diode by a voltage source of VZ, provided the Zener diode is operating in the breakdown region.

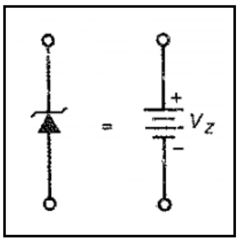
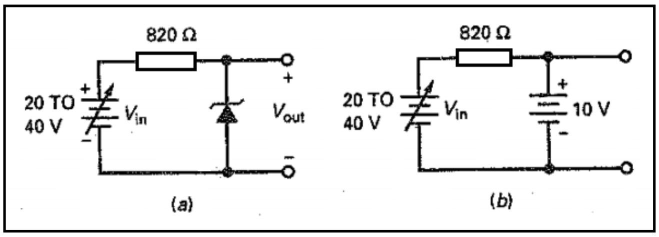


Fig. 03: Ideal approximation of a Zener diode

Example: Suppose the Zener diode of the figure (a) below has a breakdown voltage of 10 V. What are the minimum and maximum Zener currents?



The applied voltage may vary from 20 to 40 V. Ideally; a Zener diode acts like the battery shown in Fig. (b). Therefore, the output voltage is 10 V for any source voltage between 20 and 40 V.

The minimum current occurs when the source voltage is minimum. Visualize 20 V on the left end of the resistor and 10 V on the right end. Then you can see that the voltage across the resistor is 20 V - 10 V, or 10 V. The rest is Ohm's law:

$$\boxed{I_S = \frac{10\ \text{V}}{820\ \Omega} = 12.2\ \text{mA}}$$

The maximum current occurs when the source voltage is 40 V. In this case, the voltage across the resistor is 30 V, which gives a current of:

$$\boxed{I_S = \frac{30\ \text{V}}{820\ \Omega} = 36.6\ \text{mA}}$$

In a voltage regulator like Fig. (a), the output voltage is held constant at 10 V, despite the change in source voltage from 20 to 40 V. The larger source voltage produces more Zener current, but the output voltage holds rock-solid at 10 V. (If the Zener resistance is included, the output voltage increases slightly when the source voltage increases.)

2.6 The loaded Zener regulator

Figure (04 a) shows a loaded Zener regulator, and Fig. (04 b) shows the same circuit with grounds. The Zener diode operates in the breakdown region and holds the load voltage constant. Even if the source voltage changes or the load resistance varies, the load voltage will remain fixed and equal to the Zener voltage.

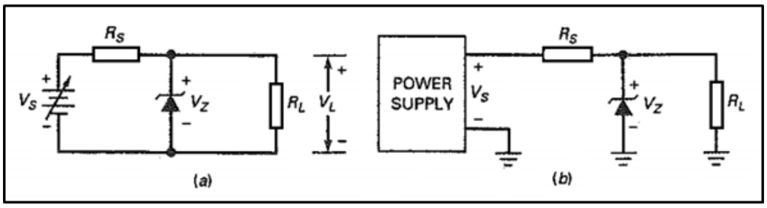


Fig. 04: Loaded Zener regulator. (a) Basic circuit; (b) practical circuit

2.6.1 Breakdown operation

How can you tell whether the Zener diode of Fig. (04) is operating in the breakdown region? Because of the voltage divider, the Thevenin voltage facing the diode is:

$$\boxed{V_{TH} = \frac{R_L}{R_S + R_L} V_S} \qquad (02)$$

This is the voltage that exists when the Zener diode is disconnected from the circuit. This Thevenin voltage has to be greater than the Zener voltage; otherwise, breakdown cannot occur.

2.6.2 Series current

Unless otherwise indicated, in all subsequent discussions we assume that the Zener diode is operating in the breakdown region. In Fig. (04), the current through the series resistor is given by:

$$\boxed{I_S = \frac{V_S - V_Z}{R_S}} \qquad (03)$$

This is Ohm's law applied to the current-limiting resistor. It is the same whether or not there is a load resistor. In other words, if you disconnect the load resistor, the current through the series resistor still equals the voltage across the resistor divided by the resistance.

2.6.3 Load current

Ideally, the load voltage equals the Zener voltage because the load resistor is in parallel with the Zener diode. As an equation:

$$\boxed{V_L = V_Z} \qquad (04)$$

This allows us to use Ohm's law to calculate the load current:

$$\boxed{I_L = \frac{V_L}{R_L}}$$

(05)

2.6.4 Zener current

With Kirchhoff's current law:

$$\boxed{I_S = I_Z + I_L}$$

The Zener diode and the load resistor are in parallel. The sum of their currents has to equal the total current, which is the same as the current through the series resistor.
We can rearrange the foregoing equation to get this important formula:

$$\boxed{I_Z = I_S - I_L}$$

This tells you that the Zener current no longer equals the series current, as it does in an unloaded Zener regulator. Because of the load resistor, the Zener current now equals the series current minus the load current.

2.6.5 Zener effect

When the breakdown voltage is greater than 6 V, the cause of the breakdown is the avalanche effect. The basic idea is that minority carriers are accelerated to high enough speeds to dislodge other minority carriers, producing a chain or avalanche effect that results in a large reverse current.

The Zener effect is different. When a diode is heavily doped, the depletion layer becomes very narrow. Because of this, the electric field across the depletion layer (voltage divided by distance) is very intense. When the field strength reaches approximately 300,000 V/cm, the field is intense enough to pull electrons out of their valence orbits. The creation of free electrons in this way is called the Zener effect (also known as high-field emission). This is distinctly different from the avalanche effect, which depends on high-speed minority carriers dislodging valence electrons.

When the breakdown voltage is less than 4 V, only the Zener effect occurs. When the breakdown voltage is greater than 6 V, only the avalanche effect occurs. When the breakdown voltage is between 4 and 6 V, both effects are present.

The Zener effect was discovered before the avalanche effect, so all diodes used in the breakdown region came to be known as Zener diodes. Although you may occasionally hear the term avalanche diode, the name Zener diode is in general use for all breakdown diodes.

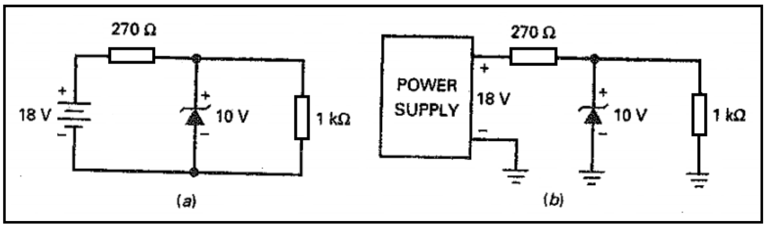
2.6.6 Temperature coefficients

When the ambient temperature changes, the Zener voltage will change slightly. On data sheets, the effect of temperature is listed under the temperature coefficient, which is defined as the change in breakdown voltage per degree of increase. The temperature coefficient is negative for breakdown voltages less than 4 V (Zener effect). For instance, a Zener diode with a breakdown voltage of 3.9 V may have a temperature coefficient of 21.4 mV/°C. If temperature increases by 1°, the breakdown voltage decreases by 1.4 mV.

On the other hand, the temperature coefficient is positive for breakdown voltages greater than 6 V (avalanche effect). For instance, a Zener diode with a breakdown voltage of 6.2 V may have a temperature coefficient of 2 mV/°C. If the temperature increases by 1°, the breakdown voltage increases by 2 mV.

Between 4 and 6 V, the temperature coefficient changes from negative to positive. In other words, there are Zener diodes with breakdown voltages between 4 and 6 V in which the temperature coefficient equals zero. This is important in some applications when a solid Zener voltage is needed over a large temperature range.

<u>Example:</u> Is the Zener diode of the figure (a) below operating in the breakdown region?



With Eq. (02):

$$\boxed{V_{TH} = \frac{1\text{ k}\Omega}{270\ \Omega + 1\text{ k}\Omega}(18\text{ V}) = 14.2\text{ V}}$$

Since this Thevenin voltage is greater than the Zener voltage, the Zener diode is operating in the breakdown region.

What does the Zener current equal in the figure (b) below?

You are given the voltage on both ends of the series resistor. Subtract the voltages, and you can see that 8 V is across the series resistor. Then Ohm's law gives:

$$\boxed{I_S = \frac{8\text{ V}}{270\ \Omega} = 29.6\text{ mA}}$$

Since the load voltage is 10 V, the load current is:

$$I_L = \frac{10\text{ V}}{1\text{ k}\Omega} = 10\text{ mA}$$

The Zener current is the difference between the two currents:

$$I_Z = 29.6\text{ mA} - 10\text{ mA} = 19.6\text{ mA}$$

2.7 Second approximation of a Zener diode

Figure (05 a) shows the second approximation of a Zener diode. A Zener resistance is in series with an ideal battery. The total voltage across the Zener diode equals the breakdown voltage plus the voltage drop across the Zener resistance. Since R_Z is relatively small in a Zener diode, it has only a minor effect on the total voltage across the Zener diode.

2.7.1 Effect on load voltage

How can we calculate the effect of the Zener resistance on the load voltage? Figure (05 b) shows a power supply driving a loaded Zener regulator. Ideally, the load voltage equals the breakdown voltage V_Z. But in the second approximation, we include the Zener resistance as shown in Fig. (05 c). The additional voltage drop across R_Z will slightly increase the load voltage.

Since the Zener current flows through the Zener resistance in Fig. (05 c), the load voltage is given by:

$$V_L = V_Z + I_Z R_Z$$

As you can see, the change in the load voltage from the ideal case is:

$$\Delta V_L = I_Z R_Z \tag{06}$$

Usually, R_Z is small, so the voltage change is small, typically in tenths of a volt.
For instance, if I_Z = 10 mA and R_Z = 10 V, then ΔV_L = 0.1 V.

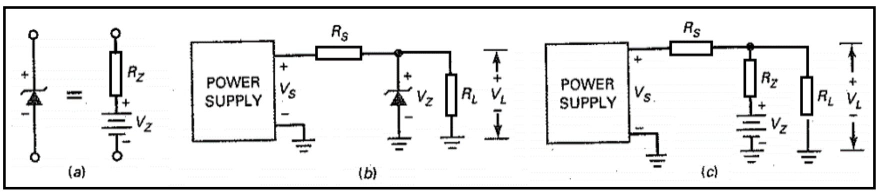


Fig. 05: Second approximation of a Zener diode. (a) Equivalent circuit;
(b) power supply drives Zener regulator; (c) Zener resistance included in analysis

2.7.2 Effect on ripple

As far as ripple is concerned, we can use the equivalent circuit shown in Fig. (06 a). In other words, the only components that affect the ripple are the three resistances shown. We can simplify this even further. In a typical design, R_Z is much smaller than R_L. Therefore, the only two components that have a significant effect on ripple are the series resistance and Zener resistance shown in Fig. (06 b). Since Fig. (06 b) is a voltage divider, we can write the following equation for the output ripple:

$$\boxed{V_{R(\text{out})} = \frac{R_Z}{R_S + R_Z} V_{R(\text{in})}}$$

Ripple calculations are not critical; that is, they don't have to be exact. Since R_S is always much greater than R_Z in a typical design, we can use this approximation for all troubleshooting and preliminary analysis:

$$\boxed{V_{R(\text{out})} \approx \frac{R_Z}{R_S} V_{R(\text{in})}} \qquad (07)$$

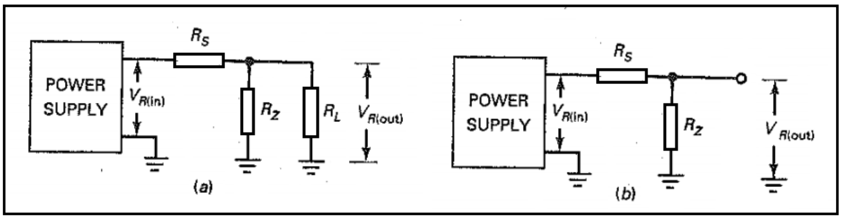


Fig. 06: Zener regulator reduces ripple. (a) Complete AC-equivalent circuit; (b) simplified AC-equivalent circuit

Example: The Zener diode of the figure below has a breakdown voltage of 10 V and a Zener resistance of 8.5 V. Use the second approximation to calculate the load voltage when the Zener current is 20 mA.

The change in load voltage equals the Zener current times the Zener resistance:

$$\boxed{\Delta V_L = I_Z R_Z = (20\text{ mA})(8.5\ \Omega) = 0.17\text{ V}}$$

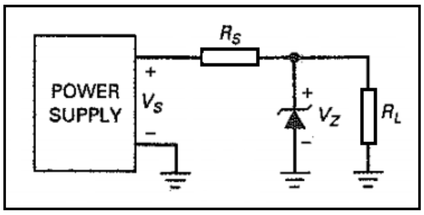
To a second approximation, the load voltage is:

$$\boxed{V_L = 10\text{ V} + 0.17\text{ V} = 10.17\text{ V}}$$

$R_S = 270\ \Omega$, $R_Z = 8.5\ \Omega$, and $V_R(\text{in}) = 2$ V. What is the approximate ripple voltage across the load?

The load ripple approximately equals the ratio of R_Z to R_S multiplied by the input ripple:

$$V_{R(out)} \approx \frac{8.5\ \Omega}{270\ \Omega} 2\ \text{V} = 63\ \text{mV}$$



Loaded Zener regulator

2.8 Zener drop-out point

For a Zener regulator to hold the output voltage constant, the Zener diode must remain in the breakdown region under all operating conditions. This is equivalent to saying that there must be Zener current for all source voltages and load currents.

2.8.1 Worst-case conditions

Figure (07 a) shows a Zener regulator. It has the following currents:

$$I_S = \frac{V_S - V_Z}{R_S} = \frac{20\ \text{V} - 10\ \text{V}}{200\ \Omega} = 50\ \text{mA}$$

$$I_L = \frac{V_L}{R_L} = \frac{10\ \text{V}}{1\ \text{k}\Omega} = 10\ \text{mA}$$

$$I_Z = I_S - I_L = 50\ \text{mA} - 10\ \text{mA} = 40\ \text{mA}$$

Now, consider what happens when the source voltage decreases from 20 to 12 V. In the foregoing calculations, you can see that I_S will decrease, I_L will remain the same, and I_Z will decrease. When V_S equals 12 V, I_S will equal 10 mA, and I_Z 5 0. At this low source voltage, the Zener diode is about to come out of the breakdown region. If the source decreases any further, regulation will be lost. In other words, the load voltage will become less than 10 V. Therefore, a low source voltage can cause the Zener circuit to fail to regulate.

Another way to get a loss of regulation is by having too much load current. In Fig. (07 a), consider what happens when the load resistance decreases from 1 kΩ to 200 Ω. When the load resistance is 200 Ω, the load current increases to 50 mA, which is equal to the current through R_S, and the Zener current decreases to zero. Again, the Zener diode is about to come out of breakdown. Therefore, a Zener circuit will fail to regulate if the load resistance is too low.

Finally, consider what happens when R_S increases from 200 **Ω** to 1 k**Ω**. In this case, the series current decreases from 50 to 10 mA. Therefore, a high series resistance can bring the circuit out of regulation.

Figure (07 b) summarizes the foregoing ideas by showing the worst-case conditions. When the Zener current is near zero, the Zener regulation is approaching a drop-out or failure condition. By analyzing the circuit for these worst-case conditions, it is possible to derive the following equation:

$$\boxed{R_{S(\max)} = \left(\frac{V_{S(\min)}}{V_Z} - 1\right) R_{L(\min)}} \qquad (08)$$

An alternative form of this equation is also useful:

$$\boxed{R_{S(\max)} = \frac{V_{S(\min)} - V_Z}{I_{L(\max)}}} \qquad (09)$$

These two equations are useful because you can check a Zener regulator to see whether it will fail under any operating conditions.

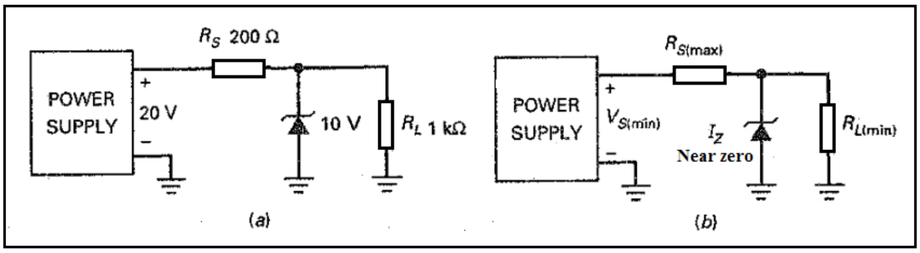


Fig. 07: Zener regulator. (a) Normal operation;
(b) worst-case conditions at drop-out point.

<u>Example:</u> A Zener regulator has an input voltage that may vary from 22 to 30 V. If the regulated output voltage is 12 V and the load resistance varies from 140 **Ω** to 10 k**Ω**, what is the maximum allowable series resistance?

Use Eq. (08) to calculate the maximum series resistance as follows:

$$\boxed{R_{S(\max)} = \left(\frac{22\text{ V}}{12\text{ V}} - 1\right) 140\ \Omega = 117\ \Omega}$$

As long as the series resistance is less than 117 Ω, the Zener regulator will work properly under all operating conditions.

2.9 Zener diode data sheet

2.9.1 Maximum power

The power dissipation of a Zener diode equals the product of its voltage and current:

$$\boxed{P_Z = V_Z I_Z} \tag{10}$$

For instance, if V_Z = 12 V and I_Z = 10 mA, then P_Z = (12 V) (10 mA) = 120 mW

As long as P_Z is less than the power rating, the Zener diode can operate in the breakdown region without being destroyed.

2.9.2 Maximum current

Data sheets often include the maximum current I_{ZM} a Zener diode can handle without exceeding its power rating. If this value is not listed, the maximum current can be found as follows:

$$\boxed{I_{ZM} = \frac{P_{ZM}}{V_Z}} \tag{11}$$

Where:

- I_{ZM} = maximum rated Zener current,
- P_{ZM} = power rating,
- V_Z = Zener voltage,

For example, the Zener diode has a Zener voltage of 12 V and a 1 W power rating.

Therefore, it has a maximum current rating of:

$$\boxed{I_{ZM} = \frac{1\ \text{W}}{12\ \text{V}} = 83.3\ \text{mA}}$$

If you satisfy the current rating, you automatically satisfy the power rating.

2.9.3 Zener resistance

The Zener resistance (also called Zener impedance) may be designated R_{ZT} or Z_{ZT}.

For instance, the Zener diode has a Zener resistance of 8.0 V measured at a test current of 20.0 mA. As long as the Zener current is beyond the knee of the curve, you can use 8.0 V as the approximate value of the Zener resistance. But note how the Zener resistance increases at the knee of the curve (1000 V). The point is this: Operation should be at or near the test current, if at all possible. Then you know that the Zener resistance is relatively small.

2.10 Load lines

The current through the Zener diode of Fig. (08 a) is given by:

$$\boxed{I_Z = \frac{V_S - V_Z}{R_S}}$$

Suppose V_S = 20 V and R_S = 1 kΩ. Then, the foregoing equation reduces to:

$$\boxed{I_Z = \frac{20 - V_Z}{1000}}$$

We get the saturation point (vertical intercept) by setting V_Z equal to zero and solving for I_Z to get 20 mA. Similarly, to get the cutoff point (horizontal intercept), we set I_Z equal to zero and solve for V_Z to get 20 V.

Alternatively, you can get the ends of the load line as follows. Visualize Fig. (08 a) with V_S = 20 V and R_S = 1 kΩ. With the Zener diode shorted, the maximum diode current is 20 mA. With the diode open, the maximum diode voltage is 20 V.

Suppose the Zener diode has a breakdown voltage of 12 V. Then its graph appears as shown in Fig. (08 b). When we plot the load line for V_S = 20 V and R_S = 1 kΩ, we get the upper load line with an intersection point of Q_1. The voltage across the Zener diode will be slightly more than the knee voltage at breakdown because the curve slopes slightly.

To understand how voltage regulation works, assume that the source voltage changes to 30 V. Then, the Zener current changes to:

$$\boxed{I_Z = \frac{30 - V_Z}{1000}}$$

This implies that the ends of the load line are 30 mA and 30 V, as shown in Fig. 5-18b. The new intersection is at Q2. Compare Q2 with Q1, and you can see that there is more current through the Zener diode, but approximately the same Zener voltage. Therefore, even though the source voltage has changed from 20 to 30 V, the Zener voltage is still approximately equal to 12 V. This is the basic idea of voltage regulation; the output voltage has remained almost constant even though the input voltage has changed by a large amount.

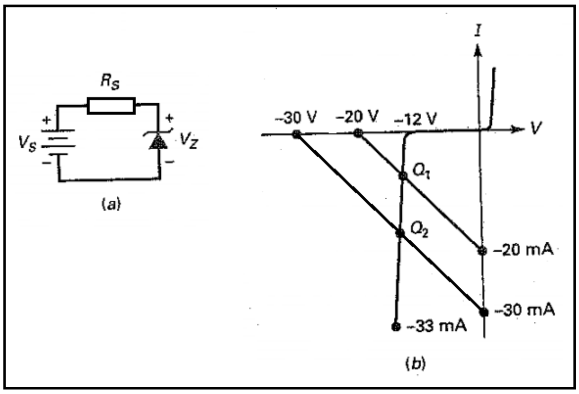


Fig. 08: (a) Zener regulator circuit; (b) load lines.

3. Light-emitting diodes (LEDs)

3.1 Introduction

Optoelectronics is the technology that combines optics and electronics. This field includes many devices based on the action of a pn junction. Examples of optoelectronic devices are light-emitting diodes (LEDs), photodiodes, optocouplers, and laser diodes. Our discussion begins with the LED.

3.2 Light-emitting diode

LEDs have replaced incandescent lamps in many applications because of the LED’s lower energy consumption, smaller size, faster switching and longer lifetime. Just as in an ordinary diode, the LED has an anode and a cathode that must be properly biased. The outside of the plastic case typically has a flat spot on one side which indicates the cathode side of the LED. The material used for the semiconductor die will determine the LED’s characteristics.

Figure (09 a) shows a source connected to a resistor and an LED. The outward arrows symbolize the radiated light. In a forward-biased LED, free electrons cross the pn junction and fall into holes. As these electrons fall from a higher to a lower energy level, they radiate energy in the form of photons. In ordinary diodes, this energy is radiated in the form of heat. But in an LED, the energy is radiated as light. This effect is referred to as electroluminescence.

The color of the light, which corresponds to the wavelength energy of the photons, is primarily determined by the energy band gap of the semiconductor materials that are used. By using elements like gallium, arsenic, and phosphorus, a manufacturer can produce LEDs that radiate red, green, yellow, blue, orange, white or infrared (invisible) light. LEDs that produce visible radiation are useful as indicators in applications such as instrumentation panels, internet routers, and so on. The infrared LED finds applications in security systems, remote controls, industrial control systems, and other areas requiring invisible radiation.

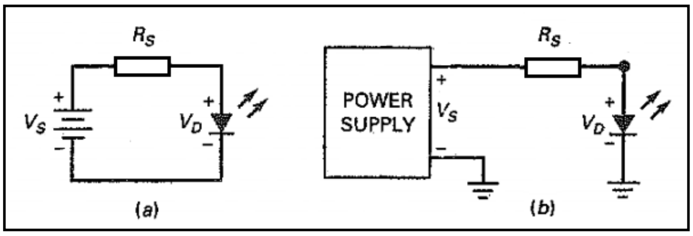


Fig. 09: LED indicator. (a) Basic circuit; (b) practical circuit

3.2.1 LED voltage and current

The resistor of Fig. (09 b) is the usual current-limiting resistor that prevents the current from exceeding the maximum current rating of the diode. Since the resistor has a node voltage of V_S on the left and a node voltage of V_D on the right, the voltage across the resistor is the difference between the two voltages. With Ohm's law, the series current is:

$$\boxed{I_S = \frac{V_S - V_D}{R_S}} \tag{12}$$

For most commercially available low-power LEDs, the typical voltage drop is from 1.5 to 2.5 V for currents between 10 and 50 mA. The exact voltage drop depends on the LED current, color, tolerance, along with other factors.

3.2.2 LED brightness

The brightness of an LED depends on the current. The amount of light emitted is often specified as its luminous intensity I_V and is rated in candelas (cd). Low-power LEDs generally have their ratings given in millicandelas (mcd). For instance, a red LED with a forward voltage drop of 1.8 V and an I_V rating of 70 mcd at 20 mA. The luminous intensity drops to 3 mcd at a current of 1 mA. When V_S is much greater than V_D in Eq. (12), the brightness of the LED is approximately constant. If a circuit like Fig. (09 b) is mass-produced using this LED, the brightness of the LED will be almost constant if V_S is much greater than V_D. If V_S is only slightly more than V_D, the LED brightness will vary noticeably from one circuit to the next.

The best way to control the brightness is by driving the LED with a current source. This way, the brightness is constant because the current is constant.

3.2.3 High-power LEDs

Typical power dissipation levels of the LEDs discussed up to this point are in the low milli-watt range. As an example, a LED has a maximum power rating of 100 mW and generally operates at approximately 20 mA with a typical forward voltage drop of 1.8 V. This results in a power dissipation of 36 mW. High-power LEDs are now available with continuous power ratings of 1 W and above. These power LEDs can operate in the hundreds of mAs to over 1 A of current.

Efficiency of a light source is an essential factor in most applications. Because an LED produces both light and heat, it is important to understand how much electrical power is used to produce the light output. A term used to describe this is called luminous efficacy. Luminous efficacy of a source is the ratio of output luminous flux (lm) to electrical power (W) given in lm/W. With a test current of 700 mA, the emitter has a typical luminous flux output of 245 lm. At this forward current level, the typical forward voltage drop is 2.80 V. Therefore, the amount of power dissipated is $P_D = I_F \times V_F$ = 700 mA x 2.80 V = 1.96 W.

The efficacy value for this emitter would be found by:

$$\boxed{\text{Efficacy} = \frac{\text{lm}}{\text{W}} = \frac{245\ \text{lm}}{1.96\ \text{W}} = 125\ \text{lm/W}}$$

As a comparison, the luminous efficacy of a typical incandescent bulb is 16 lm/W and a compact fluorescent bulb has a typical rating of 60 lm/W. When looking at the overall efficiency of these types of LEDs, it is important to note that electronic circuits, called drivers, are required to control the LED's current and light output. Since these drivers also use electrical power, the overall system efficiency is reduced.

4. Other optoelectronic devices

Besides standard low-power through high-power LEDs, there are many other optoelectronic devices which are based on the photonic action of a pn junction. These devices are used to source, detect and control light in an enormous variety of electronic applications.

4.1 Photodiode

As previously discussed, one component of reverse current in a diode is the flow of minority carriers. These carriers exist because thermal energy keeps dislodging valence electrons from their orbits, producing free electrons and holes in the process. The lifetime of the minority carriers is short, but while they exist, they can contribute to the reverse current.

When light energy bombards a pn junction, it can dislodge valence electrons. The more light striking the junction, the larger the reverse current in a diode. A photodiode has been optimized for its sensitivity to light. In this diode, a window lets light pass through the package to the junction. The incoming light produces free electrons and holes. The stronger the light, the greater the number of minority carriers and the larger the reverse current.

Figure (10) shows the schematic symbol of a photodiode. The arrows represent the incoming light. Especially important, the source and the series resistor reverse-bias the photodiode. As the light becomes brighter, the reverse current increases. With typical photodiodes, the reverse current is in the tens of microamperes.

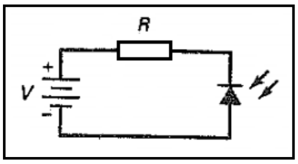


Fig. 10: Incoming light increases reverse current in photodiode

4.2 Optocouplers

An optocoupler combines an LED and a photodiode in a single package. Figure (11) shows an optocoupler. It has an LED on the input side and a photodiode on the output side. The left source voltage and the series resistor set up a current through the LED. Then the light from the LED hits the photodiode, and this sets up a reverse current in the output circuit. This reverse current produces a voltage across the output resistor. The output voltage then equals the output supply voltage minus the voltage across the resistor.

When the input voltage is varying, the amount of light is fluctuating. This means that the output voltage is varying in step with the input voltage. This is why the combination of an LED and a photodiode is called an optocoupler. The device can couple an input signal to the output circuit. Other types of optocouplers use phototransistors, photothyristors, and other photo devices in their output circuit side. These devices will be discussed in later chapters.

The key advantage of an optocoupler is the electrical isolation between the input and output circuits. With an optocoupler, the only contact between the input and the output is a beam of light. Because of this, it is possible to have an insulation resistance between the two circuits in the thousands of megohms. Isolation like this is useful in high-voltage applications in which the potentials of the two circuits may differ by several thousand volts.

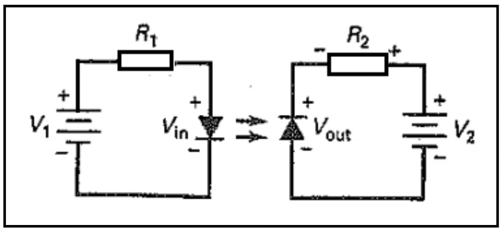


Fig. 11: Optocoupler combines an LED and a photodiode

4.3 Laser diode

In an LED, free electrons radiate light when falling from higher energy levels to lower ones. The free electrons fall randomly and continuously, resulting in light waves that have every phase between 0 and 360°. Light that has many different phases is called noncoherent light. An LED produces noncoherent light.

A laser diode is different. It produces a coherent light. This means that all the light waves are in phase with each other. The basic idea of a laser diode is to use a mirrored resonant chamber that reinforces the emission of light waves at a single frequency of the same phase. Because of the resonance, a laser diode produces a narrow beam of light that is very intense, focused, and pure.

Laser diodes are also known as semiconductor lasers. These diodes can produce visible light (red, green, or blue) and invisible light (infrared). Laser diodes are used in a large variety of applications. They are used in telecommunications, data communications, broadband access, industrial, aerospace, test and measurement, and medical and defense industries.

5. The Schottky diode

As frequency increases, the action of small-signal rectifier diodes begins to deteriorate. They are no longer able to switch off fast enough to produce a well- defined half-wave signal. The solution to this problem is the Schottky diode. Before describing this special-purpose diode, let us look at the problem that arises with ordinary small-signal diodes.

5.1 Charge storage

Figure (12 a) shows a small-signal diode, and Fig. (12 b) illustrates its energy bands. As you can see, conduction-band electrons have diffused across the junction and traveled into the p region before recombining (path A). Similarly, holes have crossed the junction and traveled into the n region before recombination occurs (path B). The greater the lifetime, the farther the charges can travel before recombination occurs.

This allows the free electrons to penetrate deeply into the p region, where they remain temporarily stored at the higher energy band. Similarly, the holes penetrate deeply into the n region, where they are temporarily stored in the lower energy band.

The greater the forward current, the larger the number of charges that have crossed the junction. The greater the lifetime, the deeper the penetration of these charges and the longer the charges remain in the high and low energy bands. The temporary storage of free electrons in the upper energy band and holes in the lower energy band is referred to as charge storage.

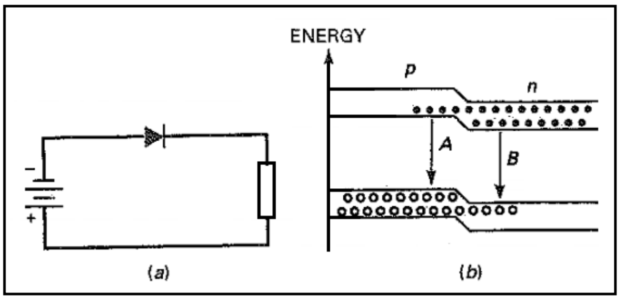


Fig. 12: Charge storage. (a) Forward bias creates stored charges; (b) stored charges in high and low energy bands.

5.2 Charge storage produces reverse current

When you try to switch a diode from on to off, charge storage creates a problem. Why? Because if you suddenly reverse-bias a diode, the stored charges will flow in the reverse direction for a while. The greater the lifetime, the longer these charges can contribute to reverse current. For example, suppose a forward-biased diode is suddenly reverse biased, as shown in Fig. (13 a). Then a large reverse current can exist for a while because of the flow of stored charges in Fig. (13 b). Until the stored charges either cross the junction or recombine, the reverse current will continue.

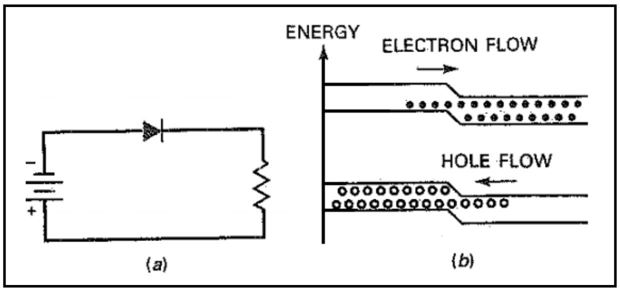


Fig. 13: Charge Stored charges allow a brief reverse current. (a) Sudden reversal of source voltage; (b) flow of stored charges in reverse direction.

5.3 Eliminating charge storage

The solution to the problem of tails is a special-purpose device called a Schottky diode. This kind of diode uses a metal such as gold, silver, or platinum on one side of the junction and doped silicon (typically n-type) on the other side. Because of the metal on one side of the junction, the Schottky diode has no depletion layer. The lack of a depletion layer means that there are no stored charges at the junction.

When a Schottky diode is unbiased, free electrons on the n side are in smaller orbits than are the free electrons on the metal side. This difference in orbit size is called the Schottky barrier, approximately 0.25 V. When the diode is forward biased, free electrons on the n side can gain enough energy to travel in larger orbits. Because of this, free electrons can cross the junction and enter the metal, producing a large forward current. Since the metal has no holes, there is no charge storage and no reverse recovery time.

5.4 High-speed turnoff

The lack of charge storage means that the Schottky diode can switch off faster than an ordinary diode can. In fact, a Schottky diode can easily rectify frequencies above 300 MHz. When it is used in a circuit like Fig. (14 a), the Schottky diode produces a perfect half-wave signal like Fig. (14 b) even at frequencies above 300 MHz.

Figure (14 a) shows the schematic symbol of a Schottky diode. Notice the cathode side. The lines look like a rectangular S, which stands for Schottky. This is how you can remember the schematic symbol.

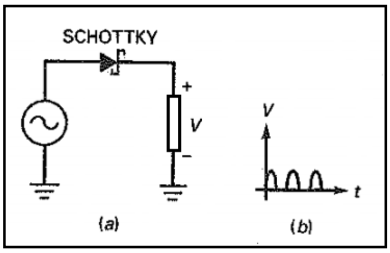


Fig. 14: Schottky diodes eliminate tails at high frequencies. (a) Circuit with Schottky diode; (b) half-wave signal at 300 MHz

The most important application of Schottky diodes is in digital computers. The speed of computers depends on how fast their diodes and transistors can turn on and off. This is where the Schottky diode comes in. Because it has no charge storage, the Schottky diode has become the backbone of low-power Schottky TTLs, a group of widely used digital devices.

A final point. Since a Schottky diode has a barrier potential of only 0.25 V, you may occasionally see it used in low-voltage bridge rectifiers because you subtract only 0.25 V instead of the usual 0.7 V for each diode when using the second approximation. In a low-voltage supply, this lower diode voltage drop is an advantage.

6. The Varactor

The varactor (also called the voltage-variable capacitance, varicap, epicap, and tuning diode) is widely used in television receivers, FM receivers, and other communications equipment because it can be used for electronic tuning.

6.1 Basic idea

In Fig. (15 a), the depletion layer is between the p region and the n region. The p and n regions are like the plates of a capacitor, and the depletion layer is like the dielectric. When a diode is reverse biased, the width of the depletion layer increases with the reverse voltage. Since the depletion layer gets wider with more reverse voltage, the capacitance becomes smaller. It's as though you moved apart the plates of a capacitor. The key idea is that capacitance is controlled by reverse voltage.

6.2 Equivalent circuit and symbol

Figure (15 b) shows the ac-equivalent circuit for a reverse-biased diode. In other words, as far as an AC signal is concerned, the varactor acts the same as a variable capacitance. Figure (15 c) shows the schematic symbol for a varactor. The inclusion of a capacitor in series with the diode is a reminder that a varactor is a device that has been optimized for its variable-capacitance properties.

6.3 Capacitance decreases at higher reverse voltages

Figure (15 d) shows how the capacitance varies with reverse voltage. This graph shows that the capacitance gets smaller when the reverse voltage gets larger. The really important idea here is that reverse dc voltage controls capacitance.

How is a varactor used? It is connected in parallel with an inductor to form a parallel resonant circuit. This circuit has only one frequency at which maximum impedance occurs. This frequency is called the resonant frequency. If the DC reverse voltage to the varactor is changed, the resonant frequency is also changed. This is the principle behind electronic tuning of a radio station, a TV channel, and so on.

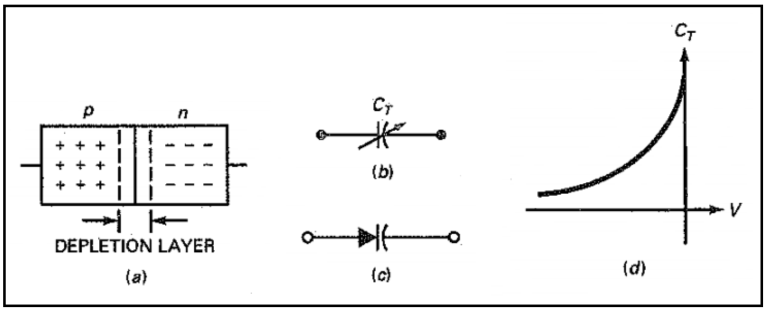


Fig. 15: Varactor. (a) Doped regions are like capacitor plates separated by a dielectric; (b) ac-equivalent circuit; (c) schematic symbol; (d) graph of capacitance versus reverse voltage.

Because the capacitance is voltage controlled, varactors have replaced mechanically tuned capacitors in many applications.

The tuning range of a varactor depends on the doping level. For instance, Fig. (16 a) shows the doping profile for an abrupt-junction diode (the ordinary type of diode).

To get larger tuning ranges, some varactors have a hyper abrupt junction, one whose doping profile looks like Fig. (16 b). This profile tells us that the doping level increases as we approach the junction. The heavier doping produces a narrower depletion layer and a larger capacitance. Furthermore, changes in reverse voltage have more pronounced effects on capacitance.

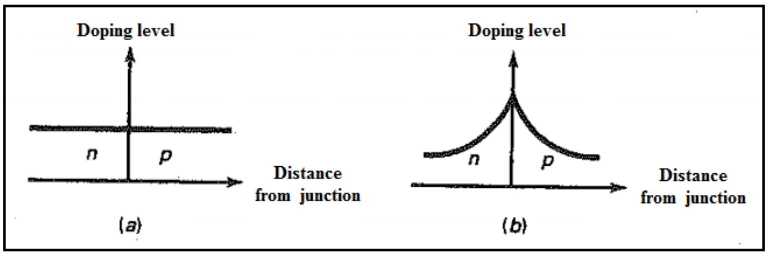


Fig. 16: Doping profiles. (a) Abrupt junction; (b) hyper abrupt junction

7. Tunnel diodes

By increasing the doping level of a back diode, we can get breakdown to occur at 0 V. Furthermore, the heavier doping distorts the forward curve, as shown in Fig. (17 a). A diode with this graph is called a tunnel diode.

Figure (17 b) shows the schematic symbol for a tunnel diode. This type of diode exhibits a phenomenon known as negative resistance. This means that an increase in forward voltage produces a decrease in forward current, at least over the part of the graph between V_P and V_V. The negative resistance of tunnel diodes is useful in high-frequency circuits called oscillators. These circuits are able to generate a sinusoidal signal, similar to that produced by an AC generator. But unlike the ac generator that converts mechanical energy to a sinusoidal signal, an oscillator converts dc energy to a sinusoidal signal.

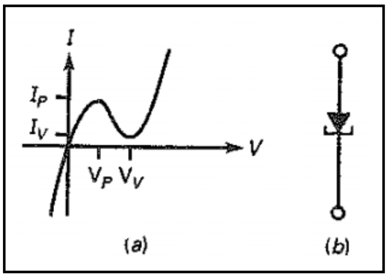


Fig. 17: Tunnel diode. (a) Breakdown occurs at 0 V;
(b) schematic symbol

8. PIN diodes

A PIN diode is a semiconductor device that operates as a variable resistor at RF and microwave frequencies. Figure (18 a) shows its construction. It consists of an intrinsic (pure) semiconductor material sandwiched between p-type and n-type materials. Figure (18 b) shows the schematic symbol for the PIN diode.

When the diode is forward biased, it acts like a current-controlled resistance. Figure (18 c) shows how the PIN diode's series resistance R_S decreases as its forward current increases. When reverse biased, the PIN diode acts like a fixed capacitor. The PIN diode is widely used in modulator circuits for RF and microwave applications.

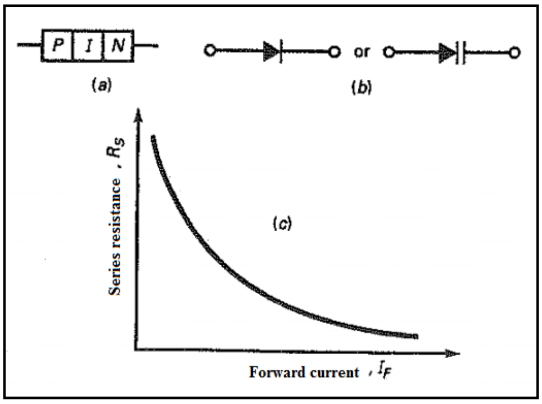


Fig. 18: PIN diode. (a) Construction; (b) schematic symbol;
(c) series resistance

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Abstract

This course brings together all the essential elements of electro kinetics, passive quadrupoles, and diodes. It is structured into three chapters which deal with the fundamental notions of electrical circuits in continuous, sinusoidal and transient regimes then review the passive quadrupoles and ending with the study of diodes.

The student will first discover, in a few pages, the essentials of the course where the essential knowledges are presented in a clear and precise manner. He will then be confronted with numerous exercises of varying difficulties. From simple applications of the course to more original cases, through very classic themes, the exercises and problems will allow the student to familiarize themselves with the basics of electricity, quadrupoles and diodes, then, by tackling more complex subjects, to acquire sufficient know-how to successfully solve any electronics problems.

Keywords: electro kinetics, passive quadrupoles, diodes, electrical networks, continuous regime, variable regime, semi-conductors.

الملخص

يجمع هذا المقرر بين جميع العناصر الأساسية للكهرباء الحركية، ورباعيات الاقطاب الخاملة ، وثنائيات الوصلة.
وهو مقسم إلى ثلاثة فصول تتناول المفاهيم الأساسية للداراة الكهربائية في الأنظمة المستمرة والجيبية والمؤقتة ثم تستعرض ورباعيات الاقطاب الخاملة وتنتهي بدراسة ثنائيات الوصلة .
سيكتشف الطالب أولاً، في بضع الصفحات، أساسيات المقرر حيث يتم تقديم المعارف الأساسية بطريقة واضحة ودقيقة ، ثم سيواجه العديد من التمارين ذات الصعوبات المتفاوتة ، من التطبيقات البسيطة للمقرر إلى الحالات الأكثر صعوبة، من خلال مواضيع كلاسيكية للغاية، ستسمح التمارين والمشكلات للطالب بالتعرف على أساسيات الكهرباء ورباعيات الاقطاب وثنائيات الوصلة ، ثم من خلال معالجة مواضيع أكثر تعقيدًا، لاكتساب المعرفة الكافية لحل معظم المشاكل إلكترونية بنجاح.

الكلمات المفتاحية: الكهرباء الحركية، رباعيات الاقطاب الخاملة ، ثنائيات الوصلة ، الداراة الكهربائية، النمط المستمر، النمط المتغير، أشباه الموصلات.