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Solvability to a Second Order Multipoint Boundary Value Problem on a Half-Line

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Notations

We will use the following notations throughout this Memory:

General notations

L	A continuous linear operator
$D(L)$	domain of the operator L
L^{-1}	the inverse application of L
$\mathbb{R}$	the set of real numbers
$[a,\, b]$	the closed interval $a \leq x \leq b$
$(a,\, b)$	the open interval $a < x < b$
$(.)' = \frac{d(.)}{dt}$	the ordinary derivative with respect to t
$x(+\infty)$	$\lim\limits_{t \to +\infty} x(+\infty)$
a.e.	almost everywhere
i.e.	that is to say.

Spaces

$\Omega \subset \mathbb{R}^n$: open set in $\mathbb{R}^n$

$\overline{\Omega}$: closure of Ω

$\partial\Omega$: boundary of Ω

$L^p(\Omega)$ $= \{$u measurable on Ω and $\int_\Omega |u|^p dx < \infty\}$, $1 \leq p < \infty$

$C(\overline{\Omega})$: Space of continuous functions on Ω

$C^n(\overline{\Omega})$: Space of n times continuously differentiable functions on Ω

In this Memory, we will specifically use the following spaces:

• $C([0,1],\mathbb{R})$: Banach space of continuous functions $u : [0,1] \to \mathbb{R}$, equipped with the norm

$$\|u\|_\infty = \sup\{|u(t)|,\, t \in [0,1]\}.$$

• $C^1([0,1],\mathbb{R})$: Space of functions $u : [0,1] \to \mathbb{R}$ such that u, u' are continuous functions on [0,1], equipped with the norm

$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}.$$

• $L^1([0,1],\mathbb{R})$: Space of Lebesgue integrable functions $u : [0,1] \to \mathbb{R}$ equipped with the norm

$$\|u\|_1 := \|u\|_{L^1} = \int_0^1 |u(t)|dt.$$

Introduction

Boundary value problems involving ordinary differential equations arise in physical sciences and applied mathematics. In some of these problems, additional conditions such that multipoint conditions are imposed the two boundary. Multipoint conditions are important in practice since the measurements needed by a multipoint conditions can be more precise than measurement given by a the two boundary conditions. For example, -the classical Robin problem with two boundary conditions is given by

$$\begin{cases} u''(t) + f(t, u(t), u'(t)) = 0, & t \in (0,1) \\ u(0) = 0, \qquad u'(1) = 0. \end{cases}$$

If the local condition $u'(1) = 0$ is replaced by the three point conditions $u(1) = u(\eta)$ (where $\eta \in (0,1)$), then the problem

$$\begin{cases} u''(t) + f(t, u(t), u'(t)) = 0, & t \in (0,1) \\ u(0) = 0, \qquad u(1) = u(\eta), \end{cases}$$

is a multipoint problem. On the interval $(0,1)$, by the Rolle theorem, the two boundary problems can be thought as the limiting case of multipoint problem as $\eta \to 1^-$. In the process of scientific experiment and numerical computation, it is more difficult to determine the value of $u'(1)$ than that of $\frac{u(\eta)-u(1)}{\eta-1}$. So the multipoint problem gives better effect than the two boundary problem. Therefore, multipoint problem may be regarded as boundary value problem involving continuous equations.

Three-point boundary value problems and more generally m-point boundary value problems were initiated in the early 1960s by Bitsadze [2] and later by Bitsadze and Samarskii [3] and Il'in and Moiseev [15]. In the simplest case of a three-point boundary value problem, Gupta [8], [9] has first applied functional analytical methods to prove existence of solutions; he was then followed by Ma [16], Marano [19], and others. Since then, the existence of solutions for nonlinear three-point, m-point and integral boundary value problems has been studied by several authors by using the nonlinear alternatives theorem of Leray-Schauder, coincidence degree theory, and the fixed-point theorem in cones...We refer the reader to [8, 9, 10, 11, 14, 16, 17, 18, 19, 21, 22] and the references therein. However, most of those papers have studied the existence and multiplicity of solutions (or positive solutions) to boundary value problems with boundary conditions specified as solution values or the slope of solution values at given points.

Multipoint boundary value problems have been well studied especially on compact intervals; in this context, we refer to [18] for a survey of some of the related research works and for references to many further contributions. The discussion about boundary value problems on the half-line is more complicated, in particular, for boundary value problems with integral boundary conditions on infinite intervals. The main difficulty in dealing with this class of boundary value problems is the possible lack of compactness due to the infinite interval; see, e g., [12, 13, 20, 24] and references therein. In order to overcome these difficulties, the authors usually consider a special Banach space in order to establish similar inequalities as finite interval.

This memory is organized as follows.

The first chapter is devoted to presenting some preliminaries and general notions used throughout. Some basic tools from functional analysis, compact operators and its properties and Leray-Schauder's nonlinear alternative fixed point.

In **the second chapter**, we discuss the boundary value problem:

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0, & t \in (0, +\infty) \\ x(0) = 0, & \lim\limits_{t \to +\infty} x'(t) = \alpha x(\eta) \end{cases}$$

where $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a function L^1-Caratheodory and $\eta \in (0, +\infty), \alpha \in \mathbb{R}$ such that $\alpha \neq \frac{1}{\eta}$ with the constant selection $\rho > 0$ which satisfies

$$\frac{\rho(1 - \alpha\eta)}{\psi(2\rho) \int_0^\infty \varphi(s) ds} > 1$$

there may be a solution to the given problem.

In the second section, we will consider the boundary value problem:

$$\begin{cases} x''(t) + f(t, x(t)) = 0, & t \in (0, +\infty), \\ x(0) = 0, & \lim\limits_{t \to +\infty} x'(t) = \alpha \int_0^\eta x(s)ds, \end{cases}$$

where $\eta \in (0, +\infty)$, $\alpha \in \mathbb{R}$ such that $0 < \alpha < \frac{2}{\eta^2}$. In this case we choose the constant $\rho > 0$ which satisfies

$$\frac{\rho(2 - \alpha\eta^2)}{\psi(\rho) \int_0^\infty \varphi(s)ds} > 1$$

there may be a solution to the given problem.

In **the third chapter**, we studied the existence solutions of the second order differential equation with multi-point boundary conditions in general case which represented in the following problem:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \in (0, +\infty) \\ x(0) = \sum_{i=1}^{m-2} c_i x(\xi_i), & x(+\infty) = \sum_{j=1}^{n-2} a_j x(\tau_j), \end{cases}$$

such that $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and c_i, $a_j \in \mathbb{R}$, ξ_i, $\tau_j \in (0, +\infty)$, $i = 1, 2, ..., m-2$, $j = 1, 2, ..., n-2$,
$0 < \xi_1 < \xi_2 < ... < \xi_{m-2}$, $0 < \tau_1 < \tau_2 < ... < \tau_{n-2}$.
Every time when studying any problem, we transform each given differential equation into an equation with the operator and by means of the Leray-Schauder nonlinear alternative theorem, sufficient conditions are obtained that guarantee the existence of solutions to the above problems. As applications, some examples are included to illustrate the existence results.

Chapter 1

Preliminaries

1.1 Basic Tools

1.1.1 Caratheodory Functions

Definition 1.1. *We say that* $f : [0,\ 1] \times \mathbb{R}^n \to \mathbb{R}^n$ *is a* L^1***-Caratheodory function*** *if*

(i) $t \mapsto f(t,\, u)$ *is measurable on* $[0,\ 1]$ *for each* $u \in \mathbb{R}^n$,

(ii) $u \mapsto f(t,\, u)$ *is continuous on* $\mathbb{R}^n$ *for a.e.* $t \in [0, 1]$,

(iii) *for each* $r \in (0,\, +\infty)$, *there exists an* $g_r \in L^1[0, 1]$ *such that*

$$|f(t,\, u)| \leq g_r(t) \text{ for a.e. } t \in [0,1] \text{ and all } u \in \mathbb{R}^n \text{ with } \|u\| \leq r.$$

1.1.2 Compact Operator

Definition 1.2. *Let* X *and* Y *be Banach spaces, and let* $T : X \to Y$ *be a (not necessarily linear) operator. Then* T *is called a* ***compact operator*** *if it maps bounded subsets of* X *into relatively compact subsets of* Y.
that is, for every bounded sequence $(x_n) \subset X$, *the sequence* $(Tx_n) \subset Y$ *has a convergent subsequence in* Y.

1.1.3 Completely Continuous Operator

Definition 1.3. *Let X and Y be Banach spaces, and let $T : X \to Y$ be an operator (not necessarily linear). The operator T is called **completely continuous** if for every bounded sequence $(x_n) \subset X$ such that $x_n \rightharpoonup x$ weakly in X, the image sequence (Tx_n) converges strongly to Tx in Y, i.e.,*

$$x_n \rightharpoonup x \quad \text{in } X \qquad \Longrightarrow \qquad Tx_n \to Tx \quad \text{in } Y.$$

Remark 1.1. *In the linear case, the notions of compact operator and completely continuous operator coincide.*

1.1.4 Green Function

Let $p,\ q,\ f \in C([a,b])$ where $p \in C^1([a,b])$, $(a < b)$ and $(\alpha_i, \beta_i) \in \mathbb{R}^2$ such that $\forall\, i = 1, 2$, $|\alpha_1| + |\alpha_2|$, $|\beta_1| + |\beta_2| \neq 0$. Consider the ordinary differential equations:...

$$\begin{aligned} (H) \quad & (py')' + qy = 0 \\ (NH) \quad & (py')' + qy = f \end{aligned}$$

and the associated edge conditions:

$$(CB)_h \quad \begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \end{cases}$$

$$(CB)_{nh} \begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) = \gamma \\ \beta_1 y(b) + \beta_2 y'(b) = \delta. \end{cases}$$

Definition 1.4. *We call the **Green's function** associated with the homogeneous problem $(H) - (CB)_h$ a function $G\colon\ [a,b] \times [a,b] \to \mathbb{R}$ satisfied The following properties:*

(a) *G is continuous on $[a,b] \times [a,b]$;*

(b) *G is symmetric: $G(x,y) = G(y,x)$, $\forall\,(x,y) \in [a,b]^2$;*

(c) *$\dfrac{\partial G}{\partial x}(x,y)$ is continuous for all $x \neq y$;*

(d) $\dfrac{\partial G}{\partial x}(y^+, y) - \dfrac{\partial G}{\partial x}(y^-, y) = \dfrac{1}{p(y)}$ *for all* $y \in [a, b]$;

(e) *Partial Function* $x \mapsto G(x, y)$ *is the solution of the equation* (H) *for all* $x \neq y$;

(f) *Partial Function* $x \mapsto G(x, y)$ *satisfied the conditions* $(CB)_h$ *for all* $y \in [a, b]$.

Theorem 1.1. *(***(Existence and uniqueness of the Green function)***) Suppose that the homogeneous problem* $(H) - (CB)_h$ *does not admit of a non-trivial solution. Then there exists a (and only one) function* G *that does not depend on* f*, and is called Green's function, such that, for any function* f*, the solution* y *of the non-homogeneous problem* $(NH) - (CB)_h$ *is uniquely written as::*

$$y(x) = \int_a^b G(x, s) f(s)\, ds.$$

Proof:

(i) Existence of the function G**:** Let ϕ_1 and ϕ_2 the respective solutions of the problems with initial conditions.

$$(H) + \begin{cases} \phi_1(a) = \alpha_2 \\ \phi_1'(a) = -\alpha_1, \end{cases} \quad \text{and} \quad (H) + \begin{cases} \phi_2(b) = \beta_2 \\ \phi_2'(b) = -\beta_1. \end{cases}$$

Then, ϕ_1, $\varphi_2 \not\equiv 0$ are linearly independent because otherwise ϕ_1 (and also ϕ_2)would be the solution to the problem $(P_0) : (H) + (CB)_h$ a contradiction hypotheses . So be it $W = \phi_1 \phi_2' - \phi_1' \phi_2 \neq 0$ their Vronskian and G the Green function defined by

$$G(x, s) = \begin{cases} \dfrac{\phi_1(s)\phi_2(x)}{p(s)W(s)}, & a \leqslant s \leqslant x, \\ \dfrac{\phi_1(x)\phi_2(s)}{p(s)W(s)}, & x \leqslant s \leqslant b. \end{cases}$$

Because, for any solution y of the non-homogeneous problem $(NH) - (CB)_h$ is written in the form :

$$\begin{aligned} y(x) &= \int_a^x \frac{\phi_1(s)\phi_2(x)}{p(s)W(s)} f(s)\, ds + \int_x^b \frac{\phi_1(x)\phi_2(s)}{p(s)W(s)} f(s)\, ds \\ &= \int_a^b G(x, s) f(s)\, ds. \end{aligned}$$

Note that the product pW is constant because $p(x)W(x) = p(s)W(s) = p(a)W(a) = p(b)W(b) \neq 0,\ \forall\, x, s \in [a, b]$. Indeed, according to Lagrange's formula, we have

$$\begin{aligned} \phi_2(p\phi_1')' - \phi_1(p\phi_1')' = 0 \quad &\Leftrightarrow \quad (p\phi_2\phi_2')' - (p\phi_1\phi_2')' = 0 \\ &\Leftrightarrow \quad [p(\phi_2\phi_1' - \phi_1\phi_2')]' = 0 \\ &\Leftrightarrow \quad pW = cte. \end{aligned}$$

Finally, G satisfies the hypotheses of the Green function.

(ii) Uniqueness of the function G Let G, H two Green functions, then $\int_a^b [G(x,s) - H(x,s)]f(s)ds = 0,\ \forall\, x \in [a, b]$ and $\forall\, f\colon\ [a, b] \to \mathbb{R}$ continuous function. Four x fixed, let's put down $f(s) = G(x,s) - H(x,s)$; We have $\int_a^b [G(x,s) - H(x,s)]^2 ds = 0,\ \forall\, x \in [a, b]$. As G and H are continuous , $G \equiv H$, that is $(G(x,.) = H(x,.), \forall\, x \in [a, b])$.

(iii) Existence and uniqueness of a solution: The function F defined by,

$$F(x) = \int_a^b G(x,s)f(s)ds = \frac{\phi_2(x)}{pW}\int_a^x \phi_1(s)f(s)ds + \frac{\phi_1(x)}{pW}\int_x^b \phi_2(s)f(s)ds$$

is the solution to the problem $(NH) + (CB)_h$.Indeed ,

$$F'(x) = \int_a^b \frac{\partial G}{\partial x}(x,s)f(s)\,ds$$

$$\text{and}\quad F''(x) = \int_a^b \frac{\partial^2 G}{\partial x^2}(x,s)f(s)\,ds + f(x)[\frac{\partial G}{\partial x}(x,x^-) - \frac{\partial G}{\partial x}(x,x^+)].$$

Or,

$$\begin{aligned} \frac{\partial G}{\partial x}(x^+,x) &= \frac{\partial G}{\partial x}(x,x^-) \\ \frac{\partial G}{\partial x}(x^-,x) &= \frac{\partial G}{\partial x}(x,x^+) \\ \frac{\partial G}{\partial x}(x^+,x) &- \frac{\partial G}{\partial x}(x^-,x) = \frac{1}{p}. \end{aligned}$$

From this we deduce the expression:

$$F''(x) = \int_a^b \frac{\partial^2 G}{\partial x^2}(x,s)f(s)\,ds + \frac{f(x)}{p(x)}$$

as well as:

$$(pF')' = \int_a^b \left(p\frac{\partial G}{\partial x}\right)'(x,s)f(s)\,ds + f(x).$$

As a result,

$$\begin{aligned}(pF')' + qF &= \int_a^b \left[\left(p\frac{\partial G}{\partial x}\right)' + qG\right] f(s)\,ds + f(x)\\ \text{(by definition of } G) &= -\int_a^b 0 f(s)\,ds + f(x)\\ &= f(x).\end{aligned}$$

The uniqueness of the solution y results from the hypothesis on the homogeneous problem as well as from Fredholm's Alternative.

Example 1.1. *Consider the two-point problem posed over an interval* $[a, b]$.

$$\begin{cases} y'' = f(x), & a < x < b \\ y(a) = 0, & y(b) = 0. \end{cases} \tag{1.1}$$

Let's build the functions φ_1 *and* φ_2 *solution of Cauchy problems :*

$$\begin{cases} \varphi_1'' = 0 \\ \varphi_1(a) = 0 \\ \varphi_1'(a) = -1 \end{cases} \qquad \begin{cases} \varphi_2'' = 0 \\ \varphi_2(b) = 0 \\ \varphi_2'(b) = -1. \end{cases}$$

Then, $\varphi_1(x) = (a - x)$, $\varphi_2(x) = (b - x)$ *and* $W(\varphi_1, \varphi_2) = b - a$.
Hence the function of Green:

$$G(x, s) = \begin{cases} \dfrac{(x-b)(s-a)}{(b-a)}, & si\ a \leqslant s \leqslant x; \\ \dfrac{(x-a)(s-b)}{(b-a)}, & si\ x \leqslant s \leqslant b. \end{cases}$$

The unique solution of the problem (1.1) *is therefore given by*

$$y(x) = \int_a^b G(x, s) f(s) ds = \frac{x-b}{b-a}\int_a^x (s-a) f(s)\,ds + \frac{x-a}{b-a}\int_x^b (s-b) f(s)\,ds.$$

1.2 Important theorems

1.2.1 Ascoli-Arzela theorem

Let (X, d_X) be a compact metric space and (Y, d_Y) be a complete metric space. By $C(X, Y)$ we denote the vector space consisting of all continuous function $f : X \to Y$.

Definition 1.5. *The family $F \subset C(X,Y)$ is called* ***equicontinuous*** *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(t), f(s)) < \varepsilon$ for all $t, s \in X$ satisfying $d_X(t,s) < \delta$ and all $f \in F$.*

Theorem 1.2. *([5]) The family $F \subset C(X,Y)$ is relatively compact if and only if:*
1. F is equicontinuous and
2. for all $t \in X$, the set

$$M(t) = \{x(t);\, x(.) \in F\}$$

is relatively compact in Y.

Remark 1.2. *1. If Y is a finite dimensional Banach space, the second condition of Ascoli-Arzela theorem is equivalent to F is uniformly bounded i.e., there exists $M > 0$ such that for all $x \in F$*

$$\|x\|_\infty = \sup_{t \in X} \|x(t)\| \leq M.$$

2. If X is a compact metric space, then Ascoli-Arzela theorem can be expressed as follows: the family $F \subset C(X,Y)$ is relatively compact if and only if F is equicontinuous.

When the space X fails to be compact, Ascoli-Arzela theorem does not hold. The following compactness criterion is:

Let

$$C_l(\mathbb{R}^+, \mathbb{R}) = \{x \in C(\mathbb{R}^+, \mathbb{R}),\ \lim_{t \to +\infty} x(t) \text{ exists}\}$$

with the norm $\|x\|_l = \sup\limits_{t \geq 0} |x(t)|$.

Lemma 1.1. *([13]) A family $M \subset C_l(\mathbb{R}^+, \mathbb{R})$ is relatively compact if and only if*

1. *M is bounded,*
2. *M is equicontinuous on every compact interval of $\mathbb{R}^+$,*
3. *M is equiconvergent.*

Here

Definition 1.6. *$M \subset C_l(\mathbb{R}^+, \mathbb{R})$ is called* ***equiconvergent*** *if it satisfies*

$$\forall \varepsilon > 0,\ \exists T = T(\varepsilon) > 0, \forall t_1, t_2 \in \mathbb{R}^+ :\ t_1, t_2 > T \Rightarrow |u(t_1) - u(t_2)| < \varepsilon, \textit{for any } u \in M.$$

In this thesis, we will use some modified versions of this compactness theorem.

1.2.2 Lebesgue dominated convergence theorem

Theorem 1.3. *([4]) Let (f_n) be a sequence of functions in $L^1(\Omega)$ that satisfies*
(i) $f_n(t) \to f(t)$ for almost every $t \in \Omega$,
(ii) there is a function $g \in L^1(\Omega)$ such that $|f_n(t)| \leq g(t)$ for all $n \in \mathbb{N}$ and for almost every $t \in \Omega$.
Then $f \in L^1(\Omega)$ and $\|f_n - f\|_1 \to 0$.

1.3 Some fixed point theorems

We recall the following fixed point theorems: nonlinear alternative for Leray-Schauder, fixed point.

Definition 1.7. *Let E be a Banach space equipped with the norm $\|.\|$ and $\mathrm{T} : E \longrightarrow E$ a mapping. An element x of E is said to be a* ***fixed point*** *of T if $\mathrm{T}x = x$.*

Theorem 1.4. *([1])(Leray-Schauder nonlinear alternative theorem). Let C be a convex subset of a Banach space and Ω an open subset of C with $0 \in \Omega$. Then every completely continuous map $T : \overline{\Omega} \to C$ satisfies at least one of the following two properties:*
$(A1)$ T has a fixed point in $\overline{\Omega}$, or
$(A2)$ There is an $x \in \partial\Omega$ and $\lambda \in (0,\ 1)$ with $x = \lambda T x$.

Remark 1.3. *Hypothesis $(A2)$ indicates that the set $S = \{x \in \Omega : x = \lambda \mathrm{T}x; \lambda \in [0,1]\}$ is not bounded. So to apply the nonlinear alternative of Leray-Schauder, one can just show that T is completely continuous by checking the following a priori estimate:*

$$\exists R > 0, \forall x \in E, \forall \lambda \in [0,1] : (x = \lambda \mathrm{T}x \Longrightarrow \|x\| < R)$$

to assert that T admits at least one fixed point in $B(0,\ R)$.

Chapter 2

Some existence results for second order three-point boundary value problems on half -line

2.1 Solvability of a Second-Order Boundary Value Problem on an Unbounded Interval

In this section, we will consider the boundary value problem:

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0, & t \in (0, +\infty), \\ x(0) = 0, & \lim_{t \to +\infty} x'(t) = \alpha x(\eta), \end{cases} \tag{2.1}$$

where $\eta \in (0, +\infty), \alpha \in \mathbb{R}$ such that $\alpha \neq \frac{1}{\eta}$.

2.1.1 Lemma auxiliaries

In the following, we collect some auxiliary facts which will be needed throughout this study.

Consider the spaces E, X defined by:

$$E = \left\{ x \in C([0, +\infty], \mathbb{R}), \lim_{t \to +\infty} x(t) \text{ exists} \right\},$$

with the norm:

$$\|x\|_E = \sup_{t\geq 0} \mid x(t) \mid .$$

$$X = \left\{ x \in C^1([0,+\infty], \mathbb{R}), \lim_{t\to+\infty} x(t) \text{ and } \lim_{t\to+\infty} x'(t) \text{ exists} \right\},$$

with the norm:

$$\|x\|_X = \max\left\{ \sup_{t\geq 0} \frac{\mid x(t) \mid}{1+t} \ , \ \sup_{t\geq 0} \mid x'(t) \mid \right\}.$$

Lemma 2.1. *$(E, \|.\|_E)$ and $(X, \|.\|_X)$ are Banach spaces.*

Proof. (1) $(E, \|.\|_E)$ is a Banach space:

Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in the space $(E, \|.\|_E)$. Then,

$$\forall \varepsilon > 0, \exists N > 0 \text{ such that } \|x_n - x_m\|_E \leq \varepsilon \text{ for any } n, m > N.$$

As a consequence, the sequence $(x_n(t))_{n\in\mathbb{N}}$, for $t \in [0, +\infty)$, is a Cauchy sequence in $\mathbb{R}$ too. Hence there exists $x(t) \in \mathbb{R}$ such that $\lim_{t\to+\infty} \mid x_n(t) - x(t) \mid= 0$, $(t \geq 0)$ this implies $\lim_{t\to+\infty} \|x_n - x\|_E = 0$, proving that $(E, \|.\|_E)$ is a Banach space.

(2) $(X, \|x\|_X)$ is a Banach space:

If $(x_n)_{n\in\mathbb{N}} \subseteq X$ is a Cauchy sequence, then $(y_n)_{n\in\mathbb{N}} \subseteq E$ and $(z_n)_{n\in\mathbb{N}} \subseteq E$, with: $y_n(t) = \frac{x_n(t)}{1+t}$ and $z_n(t) = x'_n(t)$, are Cauchy sequences too, Thus there exists $y_0 \in E$ and $z_0 \in E$ such that:

$$\lim_{n\to+\infty} \|y_n - y_0\|_E = 0 \text{ and } \lim_{n\to+\infty} \|z_n - z_0\|_E = 0.$$

Let $x_0(t) = (1+t)y_0(t),\ t \geq 0$. We conclude $(x_n(t))$ converges to $x_0(t)$ and $(x'_n(t))$ converges to $z_0(t)$ with $z_0(t) = x'_0(t)$.

Then:

$$\lim_{n\to+\infty} \|x_n - x_0\|_X = \lim_{n\to+\infty} \max\left\{\|y_n - y_0\|_E, \|z_n - z_0\|_E\right\} = 0.$$

proving that $(X, \|.\|_X)$ is a Banach space.

$\square$

Lemma 2.2. *If $\alpha \neq \frac{1}{\eta}$ and $h \in L^1[0,+\infty)$, then the problem:*

$$\begin{cases} x''(t) + h(t) = 0, & t \in (0,+\infty), \\ x(0) = 0, & \lim\limits_{t\to+\infty} x'(t) = \alpha x(\eta), \end{cases} \tag{2.2}$$

has a unique solution given by

$$x(t) = -\int_0^t (t-s)h(s)ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)h(s)ds + \frac{t}{1-\alpha\eta}\int_0^{+\infty} h(s)ds.$$

Proof. From $x''(t) = -h(t)$, by integrating between 0 and t we get

$$x'(t) = x'(0) - \int_0^t h(s)ds.$$

Also, by integrating once again between 0 and t we get

$$x(t) = x'(0)t - \int_0^t (t-s)h(s)ds,$$

So, we have

$$\lim_{t\to+\infty} x'(t) = x'(0) - \int_0^{+\infty} h(s)ds.$$

and

$$\alpha x(\eta) = \alpha x'(0)\eta - \alpha\int_0^\eta (\eta-s)h(s)ds.$$

According to the condition $\lim\limits_{t\to+\infty} x'(t) = \alpha x(\eta)$, we get

$$x'(0) = -\frac{\alpha}{1-\alpha\eta}\int_0^\eta (\eta-s)h(s)ds + \frac{1}{1-\alpha\eta}\int_0^{+\infty} h(s)ds.$$

Then

$$x(t) = -\int_0^t (t-s)h(s)ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)h(s)ds + \frac{t}{1-\alpha\eta}\int_0^{+\infty} h(s)ds.$$

□

Lemma 2.3. *Inder the assumptions of Lemma (2.2), the unique solution of the problem (2.2) can be written as:*

$$x(t) = \int_0^{+\infty} G(t,s)h(s)ds$$

where $G(t,s)$ is the ***Green's function*** *defined by :*

$$G(t,s) = \frac{1}{1-\alpha\eta}\begin{cases} s-\alpha\eta s+\alpha ts, & s \le \min\{t,\eta\} \\ -\alpha t\eta+\alpha ts+t, & t \le s \le \eta \\ \alpha t\eta-\alpha\eta s+s, & \eta \le s \le t \\ t, & \max\{t,\eta\} \le s. \end{cases}$$

Proof. We divide the proof into two cases:

<u>case 1</u>: If $t \le \eta$, the solution of the problem (2.2) can be expressed as:

$$\begin{aligned} x(t) &= -\int_0^t (t-s)h(s)ds \\ &\quad -\frac{\alpha t}{1-\alpha\eta}\left[\int_0^t (\eta-s)h(s)ds+\int_t^\eta (\eta-s)h(s)ds\right] \\ &\quad +\frac{t}{1-\alpha\eta}\left[\int_0^t h(s)ds+\int_t^\eta h(s)ds+\int_\eta^{+\infty} h(s)ds\right] \\ &\quad = \int_0^t \frac{-(t-s)(1-\alpha\eta)-\alpha t(\eta-s)+t}{1-\alpha\eta}h(s)ds \\ &\quad +\int_t^\eta \frac{-\alpha t(\eta-s)+t}{1-\alpha\eta}h(s)ds+\int_\eta^{+\infty}\frac{t}{1-\alpha\eta}h(s)ds \\ &\quad = \int_0^t \frac{s-\alpha\eta s+\alpha ts}{1-\alpha\eta}h(s)ds \\ &\quad +\int_t^\eta \frac{-\alpha t\eta+\alpha ts+t}{1-\alpha\eta}h(s)ds+\int_\eta^{+\infty}\frac{t}{1-\alpha\eta}h(s)ds \\ &= \int_0^\infty G(t,s)h(s)ds. \end{aligned}$$

<u>case 2</u>: If $t \ge \eta$, the solution of the problem (2.2) can be expressed as:

$$\begin{aligned} x(t) &= -\int_0^\eta (t-s)h(s)ds-\int_\eta^t (t-s)h(s)ds-\frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)h(s)ds \\ &\quad +\frac{t}{1-\alpha\eta}\left[\int_0^\eta h(s)ds+\int_\eta^t h(s)ds+\int_t^{+\infty} h(s)ds\right] \\ &\quad = \int_0^\eta \frac{s-\alpha\eta s+\alpha ts}{1-\alpha\eta}h(s)ds+\int_\eta^t \frac{\alpha t\eta-\alpha\eta s+s}{1-\alpha\eta}h(s)ds \\ &\quad +\int_t^{+\infty}\frac{t}{1-\alpha\eta}h(s)ds \\ &\quad = \int_0^{+\infty} G(t,s)h(s)ds \end{aligned}$$

$\square$

Lemma 2.4. *If* $0 < \alpha < \frac{1}{\eta}$, *then the Green function of problem (2.2) satisfies:* $0 \leq G(t,s) \leq \frac{t}{1-\alpha\eta}$ *, for* $t, s \in [0, +\infty)$.

Proof. **<u>Case 1</u>:**
We show that $G(t,s) \geq 0$.

- For $s \leq \min\{t, \eta\}$,

$$\begin{aligned} G(t,s) &= \frac{-(t-s)(1-\alpha\eta) + t - \alpha t(\eta - s)}{1-\alpha\eta} \\ &\geq \frac{-t(1-\alpha\eta) + t - \alpha t(\eta - s)}{1-\alpha\eta} \\ &= \frac{\alpha t s}{1-\alpha\eta} \geq 0 \end{aligned}$$

- For $0 \leq t \leq s \leq \eta$,

$$\begin{aligned} G(t,s) &= \frac{t - \alpha t(\eta - s)}{1-\alpha\eta} \\ &\geq \frac{t - \alpha t\eta}{1-\alpha\eta} = t \geq 0 \end{aligned}$$

- For $0 \leq \eta \leq s \leq t$,

$$\begin{aligned} G(t,s) &= \frac{s + \alpha t\eta - \alpha\eta s}{1-\alpha\eta} \\ &= \frac{s + \alpha\eta(t - s)}{1-\alpha\eta} \geq 0. \end{aligned}$$

- For $s \geq \max\{t, \eta\}$,

$$G(t,s) = \frac{t}{1-\alpha\eta} \geq 0.$$

<u>Case 2</u>: We show that $G(t,s) \leq \frac{t}{1-\alpha\eta}$.

- For $s \leq \min\{t, \eta\}$,

$$\begin{aligned} G(t,s) &= \frac{s-\alpha\eta s+\alpha st}{1-\alpha\eta} \\ &\leq \frac{s(1-\alpha\eta)+\alpha\eta t}{1-\alpha\eta} \\ &\leq \frac{s(1-\alpha\eta)+\alpha\eta t+t-t}{1-\alpha\eta} \\ &\leq \frac{s(1-\alpha\eta)-t(1-\alpha\eta)+t}{1-\alpha\eta} \\ &\leq s-t+\frac{t}{1-\alpha\eta} \\ &\leq \frac{t}{1-\alpha\eta}. \end{aligned}$$

- For $0 \leq t \leq s \leq \eta$,

$$\begin{aligned} G(t,s) &= \frac{t-\alpha\eta t+\alpha st}{1-\alpha\eta} \\ &\leq \frac{t-\alpha\eta t+\alpha\eta t}{1-\alpha\eta} \\ &\leq \frac{t}{1-\alpha\eta}. \end{aligned}$$

- For $0 \leq \eta \leq s \leq t$,

$$\begin{aligned} G(t,s) &= \frac{s-\alpha\eta s+\alpha\eta t}{1-\alpha\eta} \\ &= \frac{s(1-\alpha\eta)+\alpha\eta t+t-t}{1-\alpha\eta} \\ &= \frac{s(1-\alpha\eta)-t(1-\alpha\eta)+t}{1-\alpha\eta} \\ &= s-t+\frac{t}{1-\alpha\eta} \\ &\leq \frac{t}{1-\alpha\eta}. \end{aligned}$$

- For $s \geq \max\{t, \eta\}$,

$$G(t,s) = \frac{t}{1-\alpha\eta}.$$

We conclude,

$0 \leq G(t,s) \leq \frac{t}{1-\alpha\eta}$ for $t, s \in [0, +\infty)$. □

Lemma 2.5. *If* $0 \leq \alpha \leq \frac{1}{\eta}$, *then the Green function of problem (2.2) satisfies* $0 \leq \frac{\partial G}{\partial t}(t,s) \leq \frac{1}{1-\alpha\eta}$ *for* $t, s \in [0, +\infty)$.

Proof.

$$\frac{\partial G}{\partial t}(t,s) = \frac{1}{1-\alpha\eta}\begin{cases} \alpha s, & s \leq \min\{t,\eta\} \\ -\alpha\eta + \alpha s + 1, & t \leq s \leq \eta \\ \alpha\eta, & \eta \leq s \leq t \\ 1, & \max\{t,\eta\} \leq s \end{cases}$$

Case 1: Evident

$\frac{\partial G}{\partial t} \geq 0$ for $t, s \in [0, +\infty[$

Case 2: We show that $\frac{\partial G}{\partial t}(t,s) \leq \frac{1}{1-\alpha\eta}$

- For $s \leq \min\{t,\eta\}$,

$$\begin{aligned} \frac{\partial G}{\partial t}(t,s) &= \frac{\alpha s}{1-\alpha\eta} \\ &\leq \frac{\alpha\eta}{1-\alpha\eta} \\ &\leq \frac{-(1-\alpha\eta)+1}{1-\alpha\eta} \\ &\leq \frac{1}{1-\alpha\eta}. \end{aligned}$$

- For $t \leq s \leq \eta$,

$$\begin{aligned} \frac{\partial G}{\partial t}(t,s) &= \frac{-\alpha\eta + \alpha s + 1}{1-\alpha\eta} \\ &\leq \frac{-\alpha\eta + \alpha\eta + 1}{1-\alpha\eta} \\ &\leq \frac{1}{1-\alpha\eta}. \end{aligned}$$

- For $\eta \leq s \leq t$,

$$\begin{aligned} \frac{\partial G}{\partial t}(t,s) &= \frac{\alpha\eta}{1-\alpha\eta} \\ &\leq \frac{-(1-\alpha\eta)+1}{1-\alpha\eta} \\ &\leq \frac{1}{1-\alpha\eta}. \end{aligned}$$

- For $\max\{t,\eta\} \leq s$,

$$\frac{\partial G}{\partial t}(t,s) = \frac{1}{1-\alpha\eta}.$$

□

Now, define the operator N by

$$(Nx)(t) = \int_0^{+\infty} G(t,s)f(s,x(s),x'(s))ds, \quad for \;\; x \in Y.$$

We know that, x is a solution of problem (2.1) if and only if x is a fixed point of operator N, to do this we will check that the operator N satisfies the Leray-Schauder's nonlinear alternative fixed point.
Note that the Ascoli-Arzela theorem is not applicable to work in X, we need the following compactness criterion.

Lemma 2.6. *[23] Let $Z \subseteq X$ be a bounded set, then Z is relatively compact in X if the following conditions hold:*

i For any $x \in Z$, $\frac{x(t)}{1+t}$ and $x'(t)$ are equicontinuous on any compact interval of $[0,+\infty)$.

ii For any $x \in Z$, $\frac{x(t)}{1+t}$ and $x'(t)$ are equiconvergent at infinity , i.e : Given $\varepsilon > 0$, there exists a constant $\delta = \delta(\epsilon) > 0$ suth that $\mid \frac{x(t_1)}{1+t_1} - \frac{x(t_2)}{1+t_2} \mid < \varepsilon$ and $\mid x'(t_1) - x'(t_2) \mid < \varepsilon$ for any $t_1, t_2 \geq \delta$ and $x(t) \in Z$.

Before starting the result , we will state the following conditions:

(C1) $0 < \alpha < \frac{1}{\eta}$,

(C2) $f : [0, +\infty) \times \mathbb{R}^2 \to [0, +\infty)$ *is continuous*

(C3) $\mid f(t, (1+t)u, v) \mid \leq \varphi(t)\psi \mid u \mid + \mid v \mid)$ *on* $[0, +\infty) \times \mathbb{R}^2$ *with* $\varphi \in L^1[0, +\infty)$ *and*

$\psi \in C([0, +\infty), [0, +\infty))$ *nondecreasing.*

Lemma 2.7. *If* $(C1)-(C3)$ *holds, then the operator* $N : X \to X$ *is completely continuous.*

Proof. **<u>Case 1</u>:** We show that the operator N is continuous.
Let $x_n \to x$ as $n \to +\infty$. From $(C1) - (C3)$ and by the Lebesgue convergence Theorem we have,

$$\begin{aligned}\frac{|(Nx_n)(t) - (Nx)(t)|}{1+t} &\leq \int_0^{+\infty} \frac{G(t,s)}{1+t} \left| f(s, x_n(s), x_n'(s)) - f(s, x(s), x'(s)) \right| ds \\ &\longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty, \quad \text{for} \quad t \geq 0.\end{aligned}$$

Then

$$\sup_{t \leq 0} \frac{|(Nx_n)(t) - (Nx)(t)|}{1+t} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty$$

and

$$\begin{aligned}|(N'x_n)(t) - (N'x)(t)| &\leq \int_0^{+\infty} \frac{\partial G}{\partial t}(t,s) \left| f(s, x_n(s), x_n'(s)) - f(s, x(s), x'(s)) \right| ds \\ &\longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty, \quad \text{for} \quad t \geq 0.\end{aligned}$$

Then

$$\sup_{t \leq 0} |(N'x_n)(t) - (N'x)(t)| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty$$

this implies

$$\|Nx_n - Nx\|_X \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty$$

thus N is continuous.

<u>Case 2</u>: We show that the operator N is relatively compact.

- Let $\omega = \{x \in [0,+\infty), \quad \|x\|_X \leqslant k\},\ (k>0)$, be a bounded subset of X.

  From Lemma 2.7 and $(C3)$, we have

$$\begin{aligned}
\sup_{t\in[0,+\infty)} \frac{(Nx)(t)}{1+t} &= \sup_{t\in[0,+\infty)} \int_0^{+\infty} \frac{G(t,s)}{1+t} \left| f\left(s,x(s),x'(s)\right)\right| ds \\
&\leqslant \frac{1}{1-\alpha\eta}\int_0^{+\infty} \left| f\left(s,x(s),x'(s)\right)\right| ds \\
&\leqslant \frac{1}{1-\alpha\eta}\int_0^{+\infty} \left| f\left(s,\frac{(1+s)x(s)}{1+s},x'(s)\right)\right| ds \\
&\leqslant \frac{1}{1-\alpha\eta}\int_0^{+\infty} \varphi(s)\psi\left(\frac{|\ x(s)\ |}{1+s}+x'(s)\right) ds \\
&\leqslant \frac{1}{1-\alpha\eta}\int_0^{+\infty} \varphi(s)\psi\left(2\|x\|_X\right) ds \\
&\leqslant \frac{\psi(2k)}{1-\alpha\eta}\int_0^{+\infty} \varphi(s) ds
\end{aligned}$$

  and

$$\begin{aligned}
\sup_{t\in[0,+\infty)} |\ (Nx)'(t)\ | &\leqslant \frac{1}{1-\alpha\eta}\int_0^{+\infty} \left| f\left(s,x(s),x'(s)\right)\right| ds \\
&\leqslant \frac{\psi(2k)}{1-\alpha\eta}\int_0^{+\infty} \varphi(s) ds.
\end{aligned}$$

  Since $\varphi \in L^1[0,+\infty)$, then $\|Nx\|_X$ is bounded

- We show that $\left\{x\in\omega,\ \sup\limits_{t\in[0,+\infty)}\dfrac{(Nx)(t)}{1+t}\right\}$ and $\{x\in\omega,(Nx)'(t)\}$ are equicontinuous on any compact interval of $[0,+\infty)$.

  Let $\beta>0, t_1,t_2\in[0,\beta]$ with $(t_1<t_2)$ and $x\in\omega$. Then

$$\begin{aligned}
\left|\frac{(Nx)(t_2)}{1+t_2}-\frac{(Nx)(t_1)}{1+t_1}\right| &= \left|\int_0^{+\infty}\frac{G(t_2,s)}{1+t_2}f(s,x(s),x'(s))ds-\int_0^{+\infty}\frac{G(t_1,s)}{1+t_1}f(s,x(s),x'(s))ds\right| \\
&\leqslant \int_0^{+\infty}\left|\frac{G(t_2,s)}{1+t_2}-\frac{G(t_1,s)}{1+t_2}+\frac{G(t_1,s)}{1+t_2}-\frac{G(t_1,s)}{1+t_1}\right| |f(s,x(s),x'(s))|ds \\
&\leqslant \int_0^{+\infty}\left(\frac{|G(t_2,s)-G(t_1,s)|}{1+t_2}+\frac{|t_1-t_2|G(t_1,s)}{(1+t_2)(1+t_1)}\right) \\
&\qquad \left|f\left(s,\frac{(1+s)x(s)}{1+s},x'(s)\right)\right|ds \\
&\leqslant \psi(2k)\int_0^{+\infty}\left(\frac{|G(t_2,s)-G(t_1,s)|}{1+t_2}+\frac{|t_1-t_2|G(t_1,s)}{(1+t_2)(1+t_1)}\right)\varphi(s)ds.
\end{aligned}$$

From $t \mapsto G(t,s)$ is continuous on $[0,\beta]$, then

$$|G(t_2,s)-G(t_1,s)| \longrightarrow 0, \quad \text{as} \quad |t_1-t_2| \longrightarrow 0.$$

So

$$\left|\frac{(Nx)(t_2)}{1+t_2}-\frac{(Nx)(t_1)}{1+t_1}\right| \longrightarrow 0, \quad \text{as} \quad |t_1-t_2| \longrightarrow 0,$$

and

$$\begin{aligned}|(Nx)'(t_2)-(Nx)'(t_1)| &= \left|\int_0^{+\infty}\left(\frac{\partial G}{\partial t}(t_2,s)-\frac{\partial G}{\partial t}(t_1,s)\right)f(s,x(s),x'(s))ds\right| \\ &\longrightarrow 0, \quad \text{as} \quad |t_1-t_2| \longrightarrow 0\end{aligned}$$

then $\left\{x \in \omega, \sup\limits_{t\in[0,+\infty)} \dfrac{(Nx)(t)}{1+t}\right\}$ and $\{x \in \omega, (Nx)'(t)\}$ are equicontinuous on $[0,+\infty)$.

<u>**Case 3**</u>**:** We show that $\left\{x \in \omega, \sup\limits_{t\in[0,+\infty)} \dfrac{(Nx)(t)}{1+t}\right\}$ and $\{x \in \omega, (Nx)'(t)\}$ are equiconvergente at infinity.

From

$$\int_0^{+\infty}|f(s,x(s),x'(s))|ds \leqslant \psi(2k)\int_0^{+\infty}\varphi(s)ds.$$

we get $f \in L^1[0,+\infty)$.

Then, we have

$$\begin{aligned}\lim_{t\to+\infty}\left(\frac{(Nx)(t)}{1+t}\right) &= \lim_{t\to+\infty}\left(-\int_0^t \frac{t-s}{1+t}f(s,x(s),x'(s))\,ds\right.\\ &\quad -\frac{\alpha t}{(1+t)(1-\alpha\eta)}\int_0^\eta(\eta-s)f(s,x(s),x'(s))\,ds\\ &\quad \left.+\frac{t}{(1+t)(1-\alpha\eta)}\int_0^{+\infty}f(s,x(s),x'(s))\,ds\right)\\ &= \frac{\alpha\eta}{1-\alpha\eta}\int_0^{+\infty}f(s,x(s),x'(s))ds-\frac{\alpha}{1-\alpha\eta}\int_0^\eta(\eta-s)f(s,x(s),x'(s))ds\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to+\infty}((Nx)'(t)) &= \lim_{t\to+\infty}\int_0^{+\infty}\frac{\partial G}{\partial t}(t,s)f(s,x(s),x'(s))ds\\
&= \lim_{t\to+\infty}-\int_0^{t}(1-s)f(s,x(s),x'(s))ds\\
&- \frac{\alpha}{1-\alpha\eta}\int_0^{\eta}(\eta-s)f(s,x(s),x'(s))ds\\
&+ \frac{1}{1-\alpha\eta}\int_0^{+\infty} f(s,x(s),x'(s))ds
\end{aligned}$$

$$\begin{aligned}
\lim_{t\to+\infty}((Nx)'(t)) &= -\int_0^{+\infty}(1-s)f(s,x(s),x'(s))ds\\
&- \frac{\alpha}{1-\alpha\eta}\int_0^{\eta}(\eta-s)f(s,x(s),x'(s))ds\\
&+ \frac{1}{1-\alpha\eta}\int_0^{+\infty} f(s,x(s),x'(s))ds.
\end{aligned}$$

Then $\left\{x\in\omega, \sup\limits_{t\in[0,+\infty)}\dfrac{(Nx)(t)}{1+t}\right\}$ and $\{x\in\omega,(Nx)'(t)\}$ are equiconvergente at infinity.
We conclude that N is completely continuous. $\square$

2.1.2 Main existence result

Theorem 2.1. *Assume that* $(C1)-(C3)$ *and the following condition holds:*
$(C4)$ *There exists* $\rho>0$ *such that*

$$\frac{\rho(1-\alpha\eta)}{\psi(2\rho)\int_0^{\infty}\varphi(s)ds}>1.$$

Then problem (2.1) *has an solution* $x=x(t)$ *such that*

$$0\leqslant\frac{|x(t)|}{1+t}\leqslant\rho \quad \textit{and} \quad 0\leqslant|x'(t)|\leqslant\rho \quad \textit{for } t\in[0,+\infty).$$

Proof. we have
($x\in X$ is solution of problem(2.1))$\Leftrightarrow$ (x is solution of the operational equation $x+Nx=0$) i.e. x is the fixed point of the operator N.

So it's necessary to find this fixed point, to do this we apply the theorem of Leray-Schauder. For that reason, it's sufficient to check that the set of all possible solutions of the family of equations

$$(1_\lambda)\begin{cases} x''(t)+\lambda f(t,x(t),x'(t))=0, & t\in(0,+\infty) \\ x(0)=0, & \lim\limits_{t\to+\infty} x'(t)=\alpha x(\eta) \end{cases}$$

is bounded on X by an independent constant $\lambda\in[0,1]$.
As above, we have shown that N is completely continuous.
Now, let x is solution of problem (1_λ), let $\Omega=\{x\in X, \|x\|_X<\rho\}$.
Use the contrary, if there $x\in\partial\Omega$ with $x+\lambda Nx=0$, then for $\lambda\in[0,1]$, we have

$$\begin{aligned} \sup_{t\in[0,+\infty)}\left|\frac{x(t)}{1+t}\right| &= \sup_{t\in[0,+\infty)}\left|\frac{(\lambda Nx)(t)}{1+t}\right| \\ &\le \sup_{t\in[0,+\infty)}\left|\frac{(Nx)(t)}{1+t}\right| \\ &\le \sup_{t\in[0,+\infty)}\int_0^{+\infty}\frac{G(t,s)}{1+t}\left|f\left(s,\frac{(1+s)x(s)}{1+s},x'(s)\right)\right|ds \\ &\le \int_0^{+\infty}\frac{1}{1-\alpha\eta}\varphi(s)\psi\left(\left|\frac{x(s)}{1+s}\right|+\mid x'(s)\mid\right)ds \\ &\le \frac{1}{1-\alpha\eta}\int_0^{+\infty}\varphi(s)\psi(2\|x\|_X)ds \\ &\le \frac{\psi(2\rho)}{1-\alpha\eta}\int_0^{+\infty}\varphi(s)ds \end{aligned}$$

and

$$\begin{aligned} \sup_{t\in[0,+\infty)}\mid x'(t)\mid &= \sup_{t\in[0,+\infty)}\mid(\lambda Nx)'(t)\mid \\ &\le \sup_{t\in[0,+\infty)}\int_0^{+\infty}\frac{\partial G}{\partial t}(t,s)\left|f\left(s,\frac{(1+s)x(s)}{1+s},x'(s)\right)\right|ds \\ &\le \int_0^{+\infty}\frac{1}{1-\alpha\eta}\varphi(s)\psi\left(\left|\frac{x(s)}{1+s}\right|+x'(s)\right)ds \\ &\le \frac{1}{1-\alpha\eta}\int_0^{+\infty}\varphi(s)\psi(2\|x\|_X)ds \\ &\le \frac{\psi(2\rho)}{1-\alpha\eta}\int_0^{+\infty}\varphi(s)ds. \end{aligned}$$

Then

$$\|x\|_X \leq \frac{\psi(2\rho)}{1-\alpha\eta}\int_0^{+\infty}\varphi(s)ds$$

and from $x \in \partial\Omega$,

$$\rho \leq \frac{\psi(2\rho)}{1-\alpha\eta}\int_0^{+\infty}\varphi(s)ds.$$

So

$$\frac{\rho(1-\alpha\eta)}{\psi(2\rho)\int_0^{+\infty}\varphi(s)ds} \leq 1.$$

Which contradicts with the condition $(C4)$. Then, by theorem of Leray-Schauder the problem (2.1) has a solution on X. □

Example :

Consider the following boundary value problem on the half-line:

$$\begin{cases} x''(t) + \sqrt{\mid \frac{|x(t)|}{1+t} + x'(t) \mid}\ e^{-t} = 0, \quad 0 < t < +\infty \\ x(0) = 0, \qquad \lim\limits_{t\to+\infty} x'(t) = \alpha x(\eta). \end{cases} \tag{2.3}$$

where $\alpha = \frac{1}{2}$, $\eta = 1$. We will apply Theorem 2.1 to show that problem (2.3) has at least a solution. Let

$$f(t,x,x') = \sqrt{\mid \frac{\mid x \mid}{1+t} + x'(t) \mid}\ e^{-t}.$$

Choose

$$\rho > 2.$$

Then, we have

$(C1)$ Since $\eta = 1$ so $0 < \alpha < \frac{1}{\eta}$ satisfies.

$(C2)$ $f : [0,+\infty) \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous.

$(C3)$ $\mid f(t,(1+t)x,x') \mid \leq \sqrt{\mid x + x' \mid}\ e^{-t}$ then we have $\psi(x) = \sqrt{x}$, $\varphi(t) = e^{-t}$ then $\mid f(t,(1+t)x,x') \mid \leq \sqrt{\mid x + x' \mid}\ e^{-t} = \psi(\mid x + x' \mid)\varphi(t)$ on $[0,+\infty) \times \mathbb{R}^2$ with $\varphi \in L^1[0,+\infty)$ and $\psi \in C([0,+\infty),[0,+\infty))$ is nondecreasing.

$(C4)$
$$\frac{\rho(1-\alpha\eta)}{\psi(\rho)\int_0^{+\infty}\varphi(s)ds}=\frac{\rho(1-\alpha)}{\sqrt{\rho}\times 1}=\sqrt{\rho}(\frac{1}{2})>1.$$

Hence all conditions of Theorem 2.1 hold. As a consequence, problem (2.3) has at least a solution x such that

$0 \leq \frac{|x(t)|}{1+t} \leq \rho$ and $0 \leq \mid x'(t) \mid \leq \rho$, for $t \in [0,+\infty)$.

2.2 Existence of Solutions for Integral Boundary Value Problem of Second-Order on an Infinite Interval

In this section, we will consider the boundary value problem

$$\begin{cases} x''(t)+f(t,x(t))=0, & t\in(0,+\infty), \\ x(0)=0, & \lim\limits_{t\to+\infty} x'(t)=\alpha\int_0^\eta x(s)ds, \end{cases} \tag{2.4}$$

where $\eta \in (0,+\infty)$, $\alpha \in \mathbb{R}$ such that $0<\alpha<\frac{2}{\eta^2}$.

Concerning Boundary value problems posed on unbounded intervals, we refer the reader, e.g., to ([8, 9, 10, 11, 14, 15, 16, 17, 19, 21, 22]) and references therein. For instance, Tariboon and Sitthiwirattham in [21] proved the existence of positive solutions for the three-points Boundary value problem with an integral condition:

$$\begin{cases} x''+a(t)f(x)=0, & t\in(0,1), \\ x(0)=0, & x(1)=\alpha\int_0^\eta x(s)ds, \end{cases}$$

where $0<\eta<1$ and $0<\alpha<\frac{2}{\eta^2}$. They have employed the Krasnosel'skii fixed point theorem in cones. In their analysis they assume that the function f is either superlinear or sublinear.

Galvis, Rojas, and Sinitsyn in [6] have considered the same problem on the interval $[0,\gamma]$. Using Schauder's fixed point theorem, they have also proved existence of positive solutions. They do not impose any extra condition on the function f, they only need to prove a global condition (instead of using local arguments): a compactness condition on the involved operators associated to the equation.

We however notice that less papers have dealt with Boundary value problems posed on infinite domains. We cite For instance, [12] where the authors have studied the existence of multiple positive solutions of the following Boundary value problem:

$$\begin{cases} (\varphi(x'(t))' + a(t)f(t, x(t)) = 0, & t \in (0, +\infty), \\ x(0) = \sum\limits_{i=1}^{m-2} \alpha_i x(\xi_i), & x'(\infty) = 0, \end{cases}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing positive homeomorphism with $\varphi(0) = 0$ and $\xi_i \in (0, +\infty)$ with $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < +\infty$. The coefficients α_i satisfy $\alpha_i \in [0, +\infty)$, $0 < \sum\limits_{i=1}^{m-1} \alpha_i < 1$.
O'Regan, Yan, and Agarwal in [20] established the existence of an unbounded solutions for the second order Boundary value problem on the half line:

$$\begin{cases} x''(t) + \Phi(t)f(t, x(t)) = 0, & t \in (a, +\infty), \\ x(a) = 0, & \lim\limits_{t \to +\infty} x'(t) = 0, \end{cases}$$

where $f \in C([a, +\infty) \times (0, +\infty), [0, +\infty))$ with $\lim\limits_{x \to 0^+} f(t, x) = +\infty$ for each $t \in (a, +\infty)$ and $\Phi \in C((a, +\infty), (0, +\infty))$. Also the existence of multiple unbounded positive solutions is discussed by means of the theory of the fixed point index.

In [24], Zima studied the existence of at least one positive solution to the following Boundary value problem for the second-order differential equation posed on the half-line:

$$\begin{cases} x''(t) - k^2 x(t) + f(t, x(t)) = 0, & t \in (0, +\infty), \\ x(0) = 0, & \lim\limits_{t \to +\infty} x(t) = 0, \end{cases}$$

where $k > 0$ and f is a continuous nonnegative function. The existence results are based on an application of the Krasnosel'skii fixed point theorem of cone compression and expansion.

2.2.1 Related lemmas

In the following, we collect some Lemmas which will be needed throughout this work. Consider the space Y defined by

$$Y = \left\{ x \in C([0,\ +\infty), \mathbb{R}), \quad \lim_{t \to +\infty} \frac{x(t)}{1 + 2t} \text{ exists} \right\},$$

with the norm

$$\|x\|_Y = \sup_{t\in[0,\infty)} \left|\frac{x(t)}{1+2t}\right|.$$

Lemma 2.8. $(Y,\ \|.\|_Y)$ *is a Banach space.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in the space $(Y,\ \|.\|_Y)$. Then

$$\forall \epsilon > 0, \exists N > 0 \text{ such that } \|u_n - u_m\|_Y < \epsilon \text{ for any } n, m > N.$$

As a consequence, the sequence $\{u_n(t)\}_{n\in\mathbb{N}}$ for $t \in [0, +\infty)$, is a Cauchy sequence in $\mathbb{R}$, too. Hence there exists $u(t) \in \mathbb{R}$ such that $\lim\limits_{n\to+\infty} |u_n(t) - u(t)| = 0$, $(t \geq 0)$. Then $\lim\limits_{n\to+\infty} |\frac{u_n(t)-u(t)}{1+2t}| = 0$. This implies $\lim\limits_{n\to+\infty} \|u_n - u\|_Y = 0$, proving that $(Y,\ \|.\|_Y)$ is a Banach space. □

The main tool used to prove the main existence result is the Leray-Schauder nonlinear alternative theorem 1.4.

First, we list some hypotheses on the linearity f:

$(H1)$ $0 < \alpha < \frac{2}{\eta^2}$.

$(H2)$ $f : [0, +\infty) \times \mathbb{R} \to [0, +\infty)$ is continuous.

$(H3)$ $|f(t,(1 + 2t)x)| \leqslant \varphi(t)\psi(|x|)$ on $[0, +\infty) \times \mathbb{R}$ with $\varphi \in L^1[0, +\infty)$ and $\psi \in C([0, +\infty), [0, +\infty))$ is nondecreasing.

Lemma 2.9. *If* $\alpha\eta^2 \neq 2$ *and* $e \in L^1[0, +\infty)$, *then the Boundary value problem*

$$\begin{cases} x''(t) + e(t) = 0, & t \in (0, +\infty), \\ x(0) = 0, & \lim\limits_{t\to+\infty} x'(t) = \alpha \int_0^\eta x(s)ds, \end{cases} \tag{2.5}$$

has a unique solution given by

$$x(t) = -\int_0^t (t-s)e(s)ds - \frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta (\eta - s)^2 e(s)ds + \frac{2t}{2-\alpha\eta^2}\int_0^\infty e(s)ds.$$

Proof. Integrating the equation in (2.5) twice from 0 to t yields

$$x(t) = x'(0)t - \int_0^t (t-s)e(s)ds. \tag{2.6}$$

So
$$\lim_{t\to+\infty} x'(t) = x'(0) - \int_0^\infty e(s)ds,$$
Integrating once again (2.6) from 0 to η ($\eta \in (0, +\infty)$) gives
$$\begin{aligned}\int_0^\eta x(s)ds &= x'(0)\frac{\eta^2}{2} - \int_0^\eta \left(\int_0^\tau (\tau - s)e(s)ds\right) d\tau \\ &= x'(0)\frac{\eta^2}{2} - \frac{1}{2}\int_0^\eta (\eta - s)^2 e(s)ds.\end{aligned}$$
From $\lim\limits_{t\to+\infty} x'(t) = \alpha \int_0^\eta x(s)ds$, we obtain
$$x'(0) - \int_0^\infty e(s)ds = x'(0)\frac{\alpha\eta^2}{2} - \frac{\alpha}{2}\int_0^\eta (\eta - s)^2 e(s)ds.$$
Thus
$$x'(0) = -\frac{\alpha}{2-\alpha\eta^2}\int_0^\eta (\eta - s)^2 e(s)ds + \frac{2}{2-\alpha\eta^2}\int_0^\infty e(s)ds.$$
Then
$$x(t) = -\int_0^t (t-s)e(s)ds - \frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta (\eta - s)^2 e(s)ds + \frac{2t}{2-\alpha\eta^2}\int_0^\infty e(s)ds.$$
□

Lemma 2.10. *Under the assumptions of Lemma 2.9, the unique solution to problem (2.5) can be written as*
$$x(t) = \int_0^\infty G(t,s)e(s)ds,$$
where $G(t,s)$ is the Green's function defined by
$$G(t,s) = \frac{1}{2-\alpha\eta^2}\begin{cases} 2s - \alpha t s^2 - \alpha\eta^2 s + 2\alpha t\eta s, & s \leqslant \min\{t,\, \eta\} \\ 2t - \alpha t(\eta - s)^2, & t \leqslant s \leqslant \eta \\ 2s + \alpha\eta^2 t - \alpha\eta^2 s, & \eta \leqslant s \leqslant t \\ 2t, & \max\{t,\, \eta\} \leqslant s. \end{cases}$$

Proof. We distinguish between two cases:

(a) If $t \leqslant \eta$, the solution of problem (2.5) can be expressed as

$$\begin{aligned}
x(t) &= -\int_0^t (t-s)e(s)ds \\
&\quad -\frac{\alpha t}{2-\alpha\eta^2}\left[\int_0^t (\eta-s)^2 e(s)ds + \int_t^\eta (\eta-s)^2 e(s)ds\right] \\
&\quad +\frac{2t}{2-\alpha\eta^2}\left[\int_0^t e(s)ds + \int_t^\eta e(s)ds + \int_\eta^\infty e(s)ds\right] \\
&= \int_0^t \frac{2s-\alpha ts^2-\alpha\eta^2 s+2\alpha t\eta s}{2-\alpha\eta^2}e(s)ds \\
&\quad +\int_t^\eta \frac{2t-\alpha t(\eta-s)^2}{2-\alpha\eta^2}e(s)ds + \int_\eta^\infty \frac{2t}{2-\alpha\eta^2}e(s)ds \\
&= \int_0^\infty G(t,s)e(s)ds.
\end{aligned}$$

(b) If $t \geqslant \eta$, then a solution of problem (2.5) can be expressed as

$$\begin{aligned}
x(t) &= -\int_0^\eta (t-s)e(s)ds - \int_\eta^t (t-s)e(s)ds \\
&\quad -\frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 e(s)ds \\
&\quad +\frac{2t}{2-\alpha\eta^2}\left[\int_0^\eta e(s)ds + \int_\eta^t e(s)ds + \int_t^\infty e(s)ds\right] \\
&= \int_0^\eta \frac{2s-\alpha ts^2-\alpha\eta^2 s+2\alpha t\eta s}{2-\alpha\eta^2}e(s)ds \\
&\quad +\int_\eta^t \frac{2s+\alpha\eta^2 t-\alpha\eta^2 s}{2-\alpha\eta^2}e(s)ds + \int_t^\infty \frac{2t}{2-\alpha\eta^2}e(s)ds \\
&= \int_0^\infty G(t,s)e(s)ds.
\end{aligned}$$

□

Lemma 2.11. *If $(H1)$ holds, then the Green's function to problem (2.5) satisfies*

$$0 \leqslant G(t,s) \leqslant \frac{2t}{2-\alpha\eta^2}, \ \textit{for } t,s \in [0,+\infty).$$

Proof. **Claim 1.** We show that $G(t,s) \geqslant 0$.

For $0 \leqslant s \leqslant \min\{t,\eta\}$, we have

$$\begin{aligned}
G(t,s) &= \frac{2s-\alpha ts^2-\alpha\eta^2 s+2\alpha t\eta s}{2-\alpha\eta^2} \\
&= \frac{s(2-\alpha\eta^2)+\alpha ts(2\eta-s)}{2-\alpha\eta^2} \geqslant 0.
\end{aligned}$$

For $0 \leqslant t \leqslant s \leqslant \eta$,

$$\begin{aligned} G(t,s) &= \frac{2t - \alpha t(\eta - s)^2}{2 - \alpha\eta^2} \\ &= \frac{t(2 - \alpha\eta^2) + \alpha ts(2\eta - s)}{2 - \alpha\eta^2} \geqslant 0. \end{aligned}$$

For $0 \leqslant \eta \leqslant s \leqslant t$,

$$\begin{aligned} G(t,s) &= \frac{2s + \alpha\eta^2 t - \alpha\eta^2 s}{2 - \alpha\eta^2} \\ &= \frac{2s + \alpha\eta^2(t - s)}{2 - \alpha\eta^2} \geqslant 0. \end{aligned}$$

For $s \geqslant \max\{t, \eta\} \geqslant 0$,

$$G(t,s) = \frac{2t}{2 - \alpha\eta^2} \geqslant 0.$$

Claim 2. From $0 < \alpha < \frac{2}{\eta^2}$ we will show that $G(t,s) \leqslant \frac{2t}{2-\alpha\eta^2}$.
For $0 \leqslant s \leqslant \min\{t, \eta\}$, we have

$$\begin{aligned} G(t,s) &= \frac{2s - \alpha ts^2 - \alpha\eta^2 s + 2\alpha t\eta s}{2 - \alpha\eta^2} \\ &= \frac{s(2 - \alpha\eta^2) + \alpha ts(2\eta - s)}{2 - \alpha\eta^2} \\ &\leqslant \frac{t(2 - \alpha\eta^2) + \alpha ts(2\eta - s)}{2 - \alpha\eta^2} \\ &\leqslant \frac{2t - \alpha t(\eta - s)^2}{2 - \alpha\eta^2} \\ &\leqslant \frac{2t}{2 - \alpha\eta^2}. \end{aligned}$$

For $0 \leqslant t \leqslant s \leqslant \eta$,

$$\begin{aligned} G(t,s) &= \frac{2t - \alpha t(\eta - s)^2}{2 - \alpha\eta^2} \\ &\leqslant \frac{2t}{2 - \alpha\eta^2}. \end{aligned}$$

For $0 \leqslant \eta \leqslant s \leqslant t$,

$$\begin{aligned} G(t,s) &= \frac{2s + \alpha\eta^2 t - \alpha\eta^2 s}{2 - \alpha\eta^2} \\ &= \frac{(2 - \alpha\eta^2)s + \alpha\eta^2 t}{2 - \alpha\eta^2} \\ &\leqslant \frac{(2 - \alpha\eta^2)t + \alpha\eta^2 t}{2 - \alpha\eta^2} \\ &\leqslant \frac{2t}{2 - \alpha\eta^2}. \end{aligned}$$

For $s \geqslant \max\{t, \eta\} \geqslant 0$,

$$G(t,s) = \frac{2t}{2 - \alpha\eta^2}.$$

Thus

$$0 \leqslant G(t,s) \leqslant \frac{2t}{2 - \alpha\eta^2}, \text{ for } t, s \in [0, +\infty).$$

□

Now, define the operator T by

$$(Tx)(t) = \int_0^\infty G(t,s) f(s, x(s)) ds, \quad \text{for } x \in Y.$$

Then boundary value problem (2.4) has a solution x if and only if x solves the operator equation $x = Tx$. We will study the existence of a fixed point of T. For this, we verify that the operator T satisfies all conditions of Theorem 1.4. Since the Arzela-Ascoli theorem fails to work in the space Y, we need a modified compactness criterion to prove T is compact. The proof of the following compactness criterion can be found in [13].

Lemma 2.12. *([13]) Let $B = \{x \in Y,\ \|x\|_Y < l\}$ such that $l > 0$,*
$B_1 = \{\frac{x(t)}{1+2t},\ x \in B\}$.
If B_1 is equicontinuous on any compact intervals of $[0, +\infty)$ and equiconvergent at infinity, then B is relatively compact on Y.

Remark 2.1. *Here B_1 is called equiconvergent at infinity if for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that*

$$\left| \frac{x(t)}{1+2t} - \lim_{t \to +\infty} \frac{x(t)}{1+2t} \right| < \epsilon \quad \textit{for any } x \in B \textit{ and } t > \delta.$$

Lemma 2.13. *Under Assumptions* $(H1)-(H3)$*, the operator* $T: Y \to Y$ *is completely continuous.*

Proof. **Claim 1.** We show that the operator $T: Y \to Y$ is continuous.
Let $x_n \to x$ as $n \to +\infty$, then there exists $r_0 > 0$ such that

$$\max\left\{\|x\|_Y,\ \sup_{n\in\mathbb{N}}\|x_n\|_Y\right\} < r_0.$$

From Lemma 2.11 and $(H3)$, we have

$$\begin{aligned}
\int_0^\infty \frac{G(t,s)}{1+2t}|f(s,x_n(s))|ds &\leqslant \frac{1}{2-\alpha\eta^2}\int_0^\infty |f(s,x_n(s))|ds \\
&\leqslant \frac{1}{2-\alpha\eta^2}\int_0^\infty \left|f\left(s,\frac{x_n(s)(1+2s)}{1+2s}\right)\right|ds \\
&\leqslant \frac{1}{2-\alpha\eta^2}\int_0^\infty \varphi(s)\psi\left|\frac{x_n(s)}{1+2s}\right|ds \\
&\leqslant \frac{1}{2-\alpha\eta^2}\int_0^\infty \varphi(s)\psi(\|x_n\|_Y)ds \\
&\leqslant \frac{\psi(r_0)}{2-\alpha\eta^2}\int_0^\infty \varphi(s)ds.
\end{aligned}$$

From $(H3)$ and using the continuity of f, we obtain that
$f(s,x_n(s)) \longrightarrow f(s,x(s))$ and $\frac{G(t,s)}{1+2t}|f(s,x_n(s))| \leq \frac{\psi(r_0)}{2-\alpha\eta^2}\varphi(s)$ with $\varphi \in L^1[0,+\infty)$.
According to the Lebesgue's dominated convergence theorem, we deduce that
$\int_0^\infty \frac{G(t,s)}{1+2t}|f(s,x_n(s)) - f(s,x(s))|ds \to 0, \quad \text{as} \quad n \to +\infty.$
Hence

$$\|Tx_n - Tx\|_Y = \sup_{t\in[0,\infty)}\int_0^\infty \frac{G(t,s)}{1+2t}|f(s,x_n(s)) - f(s,x(s))|ds \to 0, \quad \text{as} \quad n \to +\infty,$$

proving our claim.
Claim 2. We show that the operator $T: Y \to Y$ is relatively compact.
Let $\Omega = \{x \in Y, \quad \|x\|_Y \leqslant k\},\ (k > 0)$, be any bounded subset of Y.

From Lemma 2.11 and $(H3)$, we have

$$\begin{aligned}
\|Tx\|_Y &= \sup_{t\in[0,\infty)} \int_0^\infty \frac{G(t,s)}{1+2t}|f(s,x(s))|ds \\
&\leqslant \frac{1}{2-\alpha\eta^2}\int_0^\infty |f(s,x(s))|ds \\
&\leqslant \frac{1}{2-\alpha\eta^2}\int_0^\infty \left|f\left(s,\frac{x(s)(1+2s)}{1+2s}\right)\right|ds \\
&\leqslant \frac{1}{2-\alpha\eta^2}\int_0^\infty \varphi(s)\psi\left(\left|\frac{x(s)}{1+2s}\right|\right)ds \\
&\leqslant \frac{1}{2-\alpha\eta^2}\int_0^\infty \varphi(s)\psi(\|x\|_Y)ds \\
&\leqslant \frac{\psi(k)}{2-\alpha\eta^2}\int_0^\infty \varphi(s)ds, \text{ for } x\in\Omega.
\end{aligned}$$

Since $\varphi \in L^1[0,+\infty)$, then $\|Tx\|_Y$ is bounded for $x \in \Omega$. Hence, $T\Omega$ is uniformly bounded. To show that $T\Omega$ is equicontinuous on any compact interval of $[0,+\infty)$, let $\beta > 0$, $t_1, t_2 \in [0,\beta]$ $(t_2 > t_1)$ and $x \in \Omega$. Then

$$\begin{aligned}
&\left|\frac{(Tx)(t_2)}{1+2t_2} - \frac{(Tx)(t_1)}{1+2t_1}\right| \\
&\quad= \left|\int_0^\infty \frac{G(t_2,s)}{1+2t_2}f(s,x(s))ds - \int_0^\infty \frac{G(t_1,s)}{1+2t_1}f(s,x(s))ds\right| \\
&\quad\leqslant \int_0^\infty \left|\frac{G(t_2,s)}{1+2t_2} - \frac{G(t_1,s)}{1+2t_2} + \frac{G(t_1,s)}{1+2t_2} - \frac{G(t_1,s)}{1+2t_1}\right| |f(s,x(s))|ds \\
&\quad\leqslant \int_0^\infty \left[\frac{|G(t_2,s)-G(t_1,s)|}{1+2t_2} + \frac{2|t_1-t_2|G(t_1,s)}{(1+2t_2)(1+2t_1)}\right] \\
&\qquad \left|f\left(s,\frac{(1+2s)x(s)}{1+2s}\right)\right|ds \\
&\quad\leqslant \psi(k)\int_0^\infty \left[\frac{|G(t_2,s)-G(t_1,s)|}{1+2t_2} + \frac{2|t_1-t_2|G(t_1,s)}{(1+2t_2)(1+2t_1)}\right]\varphi(s)ds.
\end{aligned}$$

Since for $s \in [0,+\infty)$ the function $t \mapsto G(t,s)$ is continuous on the compact interval $[0,\beta]$, then it is uniformly continuous on $[0,\beta]$. Hence

$$|G(t_2,s) - G(t_1,s)| \to 0, \text{ uniformly as } |t_1 - t_2| \to 0.$$

So

$$\left|\frac{(Tx)(t_2)}{1+2t_2} - \frac{(Tx)(t_1)}{1+2t_1}\right| \to 0, \text{ uniformly as } |t_1 - t_2| \to 0,$$

for all $x \in \Omega$. Thus $T\Omega$ is locally equicontinuous on $[0, +\infty)$.

We show that $T\Omega$ is equiconvergent at infinity. For any $x \in \Omega$,

$$\int_0^\infty |f(s, x(s))| ds \leqslant \psi(k) \int_0^\infty \varphi(s) ds.$$

Since $\varphi \in L^1[0, +\infty)$, we get $f \in L^1[0, +\infty)$, for $x \in \Omega$. Moreover

$$\begin{aligned}
\lim_{t\to\infty} \left(\frac{(Tx)(t)}{1+2t} \right) &= \lim_{t\to\infty} \Bigg(- \int_0^t \frac{t-s}{1+2t} f(s, x(s)) ds \\
&\quad - \frac{\alpha t}{(1+2t)(2-\alpha\eta^2)} \int_0^\eta (\eta - s)^2 f(s, x(s)) ds \\
&\quad + \frac{2t}{(1+2t)(2-\alpha\eta^2)} \int_0^\infty f(s, x(s)) ds \Bigg) \\
&= -\frac{1}{2} \int_0^\infty f(s, x(s)) ds - \frac{\alpha}{2(2-\alpha\eta^2)} \int_0^\eta (\eta - s)^2 f(s, x(s)) ds \\
&\quad + \frac{1}{2-\alpha\eta^2} \int_0^\infty f(s, x(s)) ds \\
&= \frac{\alpha\eta^2}{2(2-\alpha\eta^2)} \int_0^\infty f(s, x(s)) ds - \frac{\alpha}{2(2-\alpha\eta^2)} \int_0^\eta (\eta - s)^2 f(s, x(s)) ds.
\end{aligned}$$

So

$$\begin{aligned}
\left| \frac{(Tx)(t)}{1+2t} - \lim_{t\to\infty} \frac{(Tx)(t)}{1+2t} \right| &\leq \psi(k) \Bigg| - \int_0^t \frac{t-s}{1+2t} \varphi(s) ds \\
&\quad + \left(- \frac{\alpha t}{(1+2t)(2-\alpha\eta^2)} + \frac{\alpha}{2(2-\alpha\eta^2)} \right) \int_0^\eta (\eta - s)^2 \varphi(s) ds \\
&\quad + \left(\frac{2t}{(1+2t)(2-\alpha\eta^2)} - \frac{\alpha\eta^2}{2(2-\alpha\eta^2)} \right) \int_0^\infty \varphi(s) ds \Bigg|.
\end{aligned}$$

Hence

$$\left| \frac{(Tx)(t)}{1+2t} - \lim_{t\to\infty} \frac{(Tx)(t)}{1+2t} \right| \to 0 \quad \text{as} \quad t \to +\infty.$$

Then $T\Omega$ is equiconvergent at infinity. By Lemma 2.12, we conclude that $T : Y \to Y$ is completely continuous. □

2.2.2 Main existence result

Theorem 2.2. *Assume that $(H1) - (H3)$ and the following condition holds:*

$(H4)$ *There exists $\rho > 0$ such that*

$$\frac{\rho(2-\alpha\eta^2)}{\psi(\rho) \int_0^\infty \varphi(s) ds} > 1.$$

Then problem (2.4) has an unbounded solution $x = x(t)$ such that

$$0 \leqslant \frac{|x(t)|}{1+2t} \leqslant \rho, \quad \textit{for } t \in [0, +\infty).$$

Proof. Consider the family of parameterized Boundary value problems :

$$\begin{cases} x''(t) + \lambda f(t, x(t)) = 0, & t \in (0, +\infty), \\ x(0) = 0, & \lim\limits_{t \to +\infty} x'(t) = \alpha \int_0^\eta x(s)ds, \end{cases} \tag{2.7}$$

for $\lambda \in (0\,,\,1)$. Let

$$U = \{x \in Y, \quad \|x\|_Y < \rho\}.$$

Solving problem (2.4) is equivalent to searching for fixed point for operator T in $\overline{U}$.

We will prove that $x \neq \lambda Tx$, for $x \in \partial U$ and $\lambda \in (0\,,\,1)$. On the contrary, assume there exists $x \in \partial U$ with $x = \lambda Tx$, then for $\lambda \in (0\,,\,1)$ we have

$$\begin{aligned} \|x\|_Y &= \sup_{t \in [0,\infty)} \left| \frac{(\lambda Tx)(t)}{1+2t} \right| \\ &\leqslant \sup_{t \in [0,\infty)} \left| \frac{(Tx)(t)}{1+2t} \right| \\ &\leqslant \sup_{t \in [0,\infty)} \int_0^\infty \frac{G(t,s)}{1+2t} \left| f\left(s, \frac{(1+2s)x(s)}{1+2s}\right) \right| ds \\ &\leqslant \int_0^\infty \frac{1}{2-\alpha\eta^2} \varphi(s) \psi\left(\left| \frac{x(s)}{1+2s} \right| \right) ds \\ &\leqslant \frac{1}{2-\alpha\eta^2} \int_0^\infty \varphi(s)\psi(\|x\|_Y) ds \\ &\leqslant \frac{\psi(\rho)}{2-\alpha\eta^2} \int_0^\infty \varphi(s) ds. \end{aligned}$$

So

$$\rho \leqslant \frac{\psi(\rho)}{2-\alpha\eta^2} \int_0^\infty \varphi(s)ds.$$

Hence

$$\frac{\rho(2-\alpha\eta^2)}{\psi(\rho)\int_0^\infty \varphi(s)ds} \leqslant 1,$$

which contradicts condition $(H4)$.

By Theorem 1.4 and Lemma 2.13, we deduce that Boundary value problem (2.4) has a solution x and since it is in $\overline{U}$ then x an unbounded solution satisfying

$$0 \leqslant \frac{|x(t)|}{1+2t} \leqslant \rho, \quad \text{for } t \in [0, +\infty).$$

□

Example : Consider the boundary value problem set on the half-line:

$$\begin{cases} x''(t) + \frac{x^2(t)}{(1+2t)^4} = 0, \qquad 0 < t < +\infty \\ x(0) = 0, \qquad \lim\limits_{t \to +\infty} x'(t) = \frac{1}{2}\int_0^1 x(s)ds. \end{cases} \tag{2.8}$$

We will apply Theorem 2.2 to show that problem (2.8) has at least one solution.
Let

$$f(t,x) = \frac{x^2}{(1+2t)^4}$$

and

$$\psi(x) = x^2, \quad \varphi(t) = \frac{1}{(1+2t)^2}, \quad \rho < 3.$$

Then, we check the hypotheses
$(H1)$ Since $\eta = 1$, $\alpha = \frac{1}{2}$ so $0 < \alpha < \frac{2}{\eta^2}$ satisfies.
$(H2)$ $f : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous.
$(H3)$ $|f(t,(1+2t)x)| = \frac{x^2}{(1+2t)^2} = \psi(|x|)\varphi(t)$ on $[0,+\infty) \times \mathbb{R}$ with $\varphi \in L^1[0,+\infty)$ and $\psi \in C([0,+\infty),[0,+\infty))$ is nondecreasing.
$(H4)$ $\frac{\rho(2-\alpha\eta^2)}{\psi(\rho)\int_0^\infty \varphi(s)ds} = \frac{\rho(\frac{3}{2})}{\rho^2 \times \frac{1}{2}} = \frac{3}{\rho} > 1.$
Hence all conditions of Theorem 2.2 hold. Then problem (2.8) has at least a solution x such that

$$0 \leqslant \frac{|x(t)|}{1+2t} \leqslant \rho, \quad \text{for } t \in [0,+\infty).$$

Chapter 3

Study of existence solutions of the second order differential equation with multi-point boundary conditions

3.1 Introduction

This chapter presents results on the existence of a solution for n-point boundary value problems. For example, consider the multi-point boundary value problem:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \in (0, +\infty) \\ x(0) = \sum_{i=1}^{m-2} c_i x(\xi_i), & x(+\infty) = \sum_{j=1}^{n-2} a_j x(\tau_j), \end{cases}$$

such that $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and c_i, $a_j \in \mathbb{R}$, ξ_i, $\tau_j \in (0, +\infty)$, $i = 1, 2, ..., m-2$, $j = 1, 2, ..., n-2$,
$0 < \xi_1 < \xi_2 < ... < \xi_{m-2}$, $0 < \tau_1 < \tau_2 < ... < \tau_{n-2}$.
In one case, we assume that all the coefficients c_i and a_j have the same sign; in another case, we assume that the c_i and a_j do not all share the same sign. In certain cases or in the general case, we study the existence of solutions to this problem.In many cases, to study the existence of solutions to multipoint boundary value problems as above,we can

use a priori estimates obtained for the corresponding three-point boundary value problem given below

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \in (0, +\infty) \\ x(0) = \alpha x(\eta), & x(+\infty) = 0, \end{cases}$$

with $\gamma = \sum_{i=1}^{m-2} c_i$, $\sum_{j=1}^{n-2} a_j = 0$, and $\eta \in [\xi_1, \xi_{m-2}]$, .
The pointwise study of n-point boundary value problems was initiated by Il'in and Moiseev [14]. Motivated by the work of Il'in and Moiseev [14], Gupta [9] studied nonlinear three-point boundary value problems for nonlinear ordinary differential equations. Since then, more general nonlinear n-point boundary value problems have been studied by several authors, using the nonlinear alternative of the Leray-Schauder theorem. Some recent results concern nonlinear n-point boundary value problems of ([8, 10, 11]).

3.2 Multipoint boundary value problems: reduction to three-point problems

In the case where $\gamma = \sum_{i=1}^{m-2} c_i = 0$, the first problem is called in this case as an n-point boundary value problem, which can be studied directly or reduced to a three-point problem. Let $f : [0, +\infty)$»$\mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $a_i \in \mathbb{R}$, with all the $a_i s$ have the same sign,
$\xi_i \in (0, 1), i = 1, 2, ..., m-2; \quad 0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1, \alpha = \sum_{j=1}^{n-2} a_i$
we consider the differential equation of second order below:

$$x''(t) = f(t, x(t), x'(t)), \qquad t \in (0, +\infty) \tag{3.1}$$

subject to one of the following boundary conditions:

$$x(0) = 0, \qquad x(+\infty) = \sum_{i=1}^{m-2} a_i x(\xi_i), \tag{3.2}$$

$$x'(0) = 0, \qquad x(+\infty) = \sum_{i=1}^{m-2} a_i x(\xi_i), \tag{3.3}$$

$$x(0) = 0, \qquad x'(+\infty) = \sum_{i=1}^{m-2} a_i x'(\xi_i). \tag{3.4}$$

It is well known that the existence of a solution to these problems can be studied by studying the existence of a solution to equation (3.1), subject to one of the following three-point boundary conditions:

$$\begin{aligned} x(0) &= 0, \qquad x(+\infty) = \alpha x(\eta), && (3.5)\\ x'(0) &= 0, \qquad x(+\infty) = \alpha x(\eta), && (3.6)\\ x(0) &= 0, \qquad x'(+\infty) = \alpha x'(\eta) && (3.7)\end{aligned}$$

In fact, let

$$m = \min_{t\in[\xi_1,\xi_{m-2}]} x(t), \quad M = \max_{t\in[\xi_1,\xi_{m-2}]} x(t)$$

then, we have

$$m \le x(\xi_i) \le M, \qquad\qquad \xi_i \in [\xi_1, \xi_{m-2}]$$

<u>Case 1</u>: $a_i > 0$, $i \in \{1, \ldots, m-2\}$ then

$$a_i m \le a_i x(\xi_i) \le a_i M$$

so

$$\sum_{i=1}^{m-2} a_i m \le \sum_{i=1}^{m-2} a_i x(\xi_i) \le \sum_{i=1}^{m-2} a_i M$$

thus

$$m \le \frac{\sum_{i=1}^{m-2} a_i x(\xi_i)}{\sum_{i=1}^{m-2} a_i} \le M$$

Therefore, there exists $\eta \in \{\xi_1, \ldots, \xi_{m-1}\}$ such that

$$x(\eta) = \frac{\sum_{i=1}^{m-2} a_i x(\xi_i)}{\sum_{i=1}^{m-2} a_i} = \frac{x(+\infty)}{\alpha}$$

i.e.

$$x(+\infty) = \alpha x(\eta)$$

<u>Case 2</u>: $a_i < 0$, $i \in \{1, \ldots, m-2\}$ then

$$a_i m \ge a_i x(\xi_i) \ge a_i M$$

so

$$\sum_{i=1}^{m-2} a_i m \geq \sum_{i=1}^{m-2} a_i x(\xi_i) \geq \sum_{i=1}^{m-2} a_i M$$

thus

$$m \leq \frac{\sum_{i=1}^{m-2} a_i x(\xi_i)}{\sum_{i=1}^{m-2} a_i} \leq M$$

Therefore, there exists $\eta \in \{\xi_1, \ldots, \xi_{m-2}\}$ such that

$$x(\eta) = \frac{\sum_{i=1}^{m-2} a_i x(\xi_i)}{\sum_{i=1}^{m-2} a_i} = \frac{x(+\infty)}{\alpha}$$

i.e.

$$x(+\infty) = \alpha x(\eta).$$

With note we have examined these forms of boundary conditions in the second chapter.

3.3 Boundary value problems at multi-points : reduction to four points

Let $f : [0,1]»\mathbb{R}^2 \to \mathbb{R}$ a continuous function or L^1-Caratheodory. Suppose that ξ_i, τ_j, c_i, a_j, $i = 1, 2, \ldots, m-2$, $j = 1, 2, \ldots, n-2$, $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \ldots < \tau_{n-2} < 1$ are given. Consider the second-order ordinary differential equation (3.1) subject to one of the following boundary conditions:

$$x(0) = \sum_{i=1}^{m-2} c_i x(\xi_i), \qquad x(+\infty) = \sum_{j=1}^{n-2} a_j x(\tau_j), \tag{3.8}$$

$$x(0) = \sum_{i=1}^{m-2} c_i x(\xi_i), \qquad x'(+\infty) = \sum_{j=1}^{n-2} a_j x'(\tau_j), \tag{3.9}$$

$$x'(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x(+\infty) = \sum_{j=1}^{n-2} a_j x(\tau_j), \tag{3.10}$$

$$x'(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x'(+\infty) = \sum_{j=1}^{n-2} a_j x'(\tau_j), \tag{3.11}$$

$$x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x(+\infty) = \sum_{j=1}^{n-2} a_j x(\tau_j), \tag{3.12}$$

$$x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x'(+\infty) = \sum_{j=1}^{n-2} a_j x'(\tau_j), \tag{3.13}$$

$$x(0) = \sum_{i=1}^{m-2} c_i x(\xi_i), \qquad x'(+\infty) = \sum_{j=1}^{n-2} a_j x'(\tau_j). \tag{3.14}$$

When all c_i and all a_j have the same sign, it is well known that the existence of solution for (3.1) with each one of the boundary conditions (3.8)-(3.14) can be obtained when subject to the boundary conditions of four points:

$$x(0) = \gamma x(\zeta), \qquad x(+\infty) = \alpha x(\eta), \tag{3.15}$$

$$x(0) = \gamma x(\zeta), \qquad x'(+\infty) = \alpha x'(\eta), \tag{3.16}$$

$$x'(0) = \gamma x'(\zeta), \qquad x(+\infty) = \alpha x(\eta), \tag{3.17}$$

$$x'(0) = \gamma x'(\zeta), \qquad x'(+\infty) = \alpha x'(\eta), \tag{3.18}$$

$$x(0) = \gamma x'(\zeta), \qquad x(+\infty) = \alpha x(\eta), \tag{3.19}$$

$$x(0) = \gamma x'(\zeta), \qquad x'(+\infty) = \alpha x'(\eta), \tag{3.20}$$

$$x(0) = \gamma x(\zeta), \qquad x'(+\infty) = \alpha x'(\eta), \tag{3.21}$$

where $\zeta \in [\xi_1,\, \xi_{m-2}]$, $\eta \in [\tau_1,\, \tau_{n-2}]$, $\gamma = \sum_{i=1}^{m-2} c_i$, $\alpha = \sum_{j=1}^{n-2} a_j$.

When all c_i have the same sign and all a_j have not same sign, it is known that the existence of a solution for (3.1) with each one of the boundary conditions (3.8)-(3.14) can be obtained when subject to the boundary conditions:

$$\begin{aligned}
x(0) &= \gamma x(\zeta), \qquad x(+\infty) = \sum_{j=1}^{n-2} a_j x(\tau_j), && (3.22)\\
x(0) &= \gamma x(\zeta), \qquad x'(+\infty) = \sum_{j=1}^{n-2} a_j x'(\tau_j), && (3.23)\\
x'(0) &= \gamma x'(\zeta), \qquad x(+\infty) = \sum_{j=1}^{n-2} a_j x(\tau_j), && (3.24)\\
x'(0) &= \gamma x'(\zeta), \qquad x'(+\infty) = \sum_{j=1}^{n-2} a_j x'(\tau_j), && (3.25)\\
x(0) &= \gamma x'(\zeta), \qquad x(+\infty) = \sum_{j=1}^{n-2} a_j x(\tau_j), && (3.26)\\
x(0) &= \gamma x'(\zeta), \qquad x'(+\infty) = \sum_{j=1}^{n-2} a_j x'(\tau_j), && (3.27)\\
x(0) &= \gamma x(\zeta), \qquad x'(+\infty) = \sum_{j=1}^{n-2} a_j x'(\tau_j), && (3.28)
\end{aligned}$$

where $\zeta \in [\xi_1, \xi_{m-2}]$, $\gamma = \sum_{i=1}^{m-2} c_i$.

When all c_i do not have the same sign and all a_j have the same sign, it is known that the existence of a solution for (3.1) with each one of the boundary conditions (3.8)-(3.14) can be obtained when subject to the boundary conditions:

$$\begin{aligned}
x(0) &= \sum_{i=1}^{m-2} c_i x(\xi_i), \qquad x(+\infty) = \alpha x(\eta), && (3.29)\\
x(0) &= \sum_{i=1}^{m-2} c_i x(\xi_i), \qquad x'(+\infty) = \alpha x'(\eta), && (3.30)\\
x'(0) &= \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x(+\infty) = \alpha x(\eta), && (3.31)\\
x'(0) &= \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x'(+\infty) = \alpha x'(\eta), && (3.32)\\
x(0) &= \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x(+\infty) = \alpha x(\eta), && (3.33)\\
x(0) &= \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x'(+\infty) = \alpha x'(\eta), && (3.34)\\
x(0) &= \sum_{i=1}^{m-2} c_i x(\xi_i), \qquad x'(+\infty) = \alpha x'(\eta), && (3.35)
\end{aligned}$$

where $\eta \in [\tau_1,\, \tau_{n-2}]$, $\alpha = \sum_{j=1}^{n-2} a_j$.

3.4 Existence results for second order differential equation with n-point boundary conditions

We consider the following boundary value problem:

$$(2)\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \in (0, +\infty) \\ x(0) = \sum_{i=1}^{n-2} c_i x(\xi_i), & \lim\limits_{t\to+\infty} x'(t) = 0 \end{cases}$$

Lemma 3.1. *Let* $h \in L^1\,[0, +\infty)$,$c_i \in \mathbb{R}$, $\xi_i \in \mathbb{R}^+$ *with* $i = 1, 2, ..., n-2$,
$0 < \xi_1 < \xi_2 < ... < \xi_{n-2}$ *and suppose that* $0 < \sum\limits_{i=1}^{n-2} c_i < 1$.

If $\begin{cases} x''(t) = h(t) \\ x(0) = \sum\limits_{i=1}^{n-2} c_i x(\xi_i), & \lim\limits_{t\to+\infty} x'(t) = 0 \end{cases}$

then

$$x(t) = \int_0^t (t-s)h(s)ds - t\int_0^{+\infty} h(s)ds - \frac{\sum_{i=1}^{n-2} c_i\xi_i}{1 - \sum_{i=1}^{n-2} c_i}\int_0^{+\infty} h(s)ds + \frac{\sum_{i=1}^{n-2} c_i}{1 - \sum_{i=1}^{n-2} c_i}\int_0^{\xi_i} (\xi_i - s)h(s)ds$$

Proof. From $x''(t) = h(t)$, we get $x'(t) = x'(0) + \int_0^t h(s)ds$

and $x(t) = x'(0) + x(0)\int_0^t (t-s)h(s)ds$.

Then, we have $x'(0) = -\int_0^{+\infty} h(s)ds$ (because $\lim\limits_{t\to+\infty} x'(t) = 0$)
and from
$x(\xi_i) = x'(0)\xi_i + x(0) + \int_0^{\xi_i} (\xi_i - s)h(s)ds$

so $\sum\limits_{i=1}^{n-2} c_i x(\xi_i) = x'(0)\sum\limits_{i=1}^{n-2} c_i\xi_i + x(0)\sum\limits_{i=1}^{n-2} c_i + \sum\limits_{i=1}^{n-2} c_i \int_0^{\xi_i} (\xi_i - s)h(s)ds$

i.e. $x(0) = x'(0)\sum\limits_{i=1}^{n-2} c_i(\xi_i) + x(0)\sum\limits_{i=1}^{n-2} c_i + \sum\limits_{i=1}^{n-2} c_i \int_0^{\xi_i} (\xi_i - s)h(s)ds$

(because $x(0)=\sum_{i=1}^{n-2} c_i x(\xi_i)$)

Wich implies

$$\left(1-\sum_{i=1}^{n-2} c_i\right) x(0)=\sum_{i=1}^{n-2} c_i(\xi_i)\int_0^{+\infty} h(s)ds+\sum_{i=1}^{n-2} c_i\int_0^{\xi_i}(\xi_i-s)h(s)ds$$

i.e.

$$x(0)=-\frac{\sum_{i=1}^{n-2} c_i(\xi_i)}{1-\sum_{i=1}^{n-2} c_i}\int_0^{+\infty} h(s)ds+\frac{\sum_{i=1}^{n-2} c_i}{1-\sum_{i=1}^{n-2} c_i}\int_0^{\xi_i}(\xi_i-s)h(s)ds$$

we conclude

$$x(t)=\int_0^t (t-s)h(s)ds-t\int_0^{+\infty} h(s)ds-\frac{\sum_{i=1}^{n-2} c_i(\xi_i)}{1-\sum_{i=1}^{n-2} c_i}\int_0^{+\infty} h(s)ds+\frac{\sum_{i=1}^{n-2} c_i}{1-\sum_{i=1}^{n-2} c_i}\int_0^{\xi_i}(\xi_i-s)h(s)ds$$

□

Theorem 3.1. *We suppose*

(H1) $0<\sum_{i=1}^{n-2} c_i<1$.

(H2) $f:[0,+\infty)\times\mathbb{R}^2\longrightarrow[0,+\infty)$ *is continuous.*

(H3) $\mid f(t,(1+t)u,v)\mid\leq\varphi(t)\psi(\mid u\mid+\mid v\mid)$ *on* $[0,+\infty)\times\mathbb{R}^2$ *with* $\varphi\in L^1([0,+\infty))$ *and* $\psi\in C([0,+\infty),[0,+\infty))$ *nondeceasing.*

(H4) There exists $R>0$ *such that* $R>2\left(1+\frac{\sum_{i=1}^{n-2} c_i(\xi_i)}{1-\sum_{i=1}^{n-2} c_i}\right)\psi(2R)\int_0^{+\infty}\varphi(s)ds$.
Then the problem (2) *has a solution on* X.

Proof. Define the operator N by

$$\begin{aligned}(Nx)(t) &= \int_0^t (t-s)f\left(s,x(s),x'(s)\right)ds-t\int_0^{+\infty} f\left(s,x(s),x'(s)\right)ds\\ &\quad -\frac{\sum_{i=1}^{n-2} c_i(\xi_i)}{1-\sum_{i=1}^{n-2} c_i}\int_0^{+\infty} f\left(s,x(s),x'(s)\right)ds+\frac{\sum_{i=1}^{n-2} c_i}{1-\sum_{i=1}^{n-2} c_i}\int_0^{\xi_i}(\xi_i-s)f\left(s,x(s),x'(s)\right)ds\end{aligned}$$

Problem (2) has a solution x if and only if x solves the operator equation $x = Nx$.
We will prove the existence of the fixed point of N. To do this, we will verify that the operator N satisfies fixed point theorem of Leray-Schauder. We show that

$$N : \overline{\Omega} \to X \ \ is \ \ completely \ \ continuous$$

such that $\overline{\Omega} = \{x \in X, \|x\|_X < R\}, \quad R > 0$

(a) N is continuous:
let $x_n \longrightarrow x$ as $n \longrightarrow +\infty$ on $\overline{\Omega}$. We have from condition (H3),

$$\begin{aligned}
\frac{|\,(Nx_n)(t) - (Nx)(t)\,|}{1+t} \ \leq\ & \int_0^t |\, f(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s)\,|\, ds \\
&+\int_0^{+\infty} |\, f(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s)\,|\, ds \\
&+\frac{\sum_{i=1}^{n-2} c_i\xi_i}{1-\sum_{i=1}^{n-2} c_i}\int_0^{+\infty} |\, f\,(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s))\,|\, ds \\
&+\frac{\sum_{i=1}^{n-2} c_i\xi_i}{1-\sum_{i=1}^{n-2} c_i}\int_0^{\xi_i} |\, f\,(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s))\,|\, ds \\
&\leq 2\int_0^{+\infty} |\, f\,(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s))\,|\, ds \\
&+2\frac{\sum_{i=1}^{n-2} c_i\xi_i}{1-\sum_{i=1}^{n-2} c_i}\int_0^{+\infty} |\, f\,(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s))\,|\, ds
\end{aligned}$$

From (H2), we have $\frac{|(Nx_n)(t)-(Nx)(t)|}{1+t} \longrightarrow 0$ as $n \longrightarrow +\infty$.
On the other hand ,

$$(Nx)'(t) = -\int_0^{+\infty} f\,(s, x(s), x'(s))\, ds + \int_0^t f(s, x(s), x'(s))ds.$$

so

$$|\,(Nx_n)'(t) - (Nx)'(t)\,| \leq 2\int_0^{+\infty} |\, f\,(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s))\,|\, ds.$$

From (H2) , we have

$$|\,(Nx_n)'(t) - (Nx)'(t)\,| \longrightarrow 0 \ \ as \ \ n \longrightarrow +\infty.$$

Then $\|Nx_n - Nx\|_X \longrightarrow 0 \ \ as \ \ n \longrightarrow +\infty$.

(b) $N : \overline{\Omega} \longrightarrow X$ is relatively compact .

(b1) $N(\overline{\Omega})$ is bounded:

For $x \in \overline{\Omega}$,

$$\begin{aligned}
\frac{\mid (Nx)(t) \mid}{1+t} &\leq \int_0^t \mid f\left(s, x(s), x'(s)\right) ds + \int_0^{+\infty} \mid f(s, x(s), x'(s)) \mid ds \\
&+ \frac{\sum_{i=1}^{n-2} c_i \xi_i}{1 - \sum_{i=1}^{n-2} c_i} \int_0^{+\infty} \mid f\left(s, x(s), x'(s)\right) \mid ds \\
&+ \frac{\sum_{i=1}^{n-2} c_i \xi_i}{1 - \sum_{i=1}^{n-2} c_i} \int_0^{\xi_i} \mid f\left(s, x(s), x'(s)\right) \mid ds \\
&\leq 2\int_0^{+\infty} \mid f\left(s, x(s), x'(s)\right) \mid ds + 2\frac{\sum_{i=1}^{n-2} c_i \xi_i}{1 - \sum_{i=1}^{n-2} c_i} \int_0^{+\infty} \mid f\left(s, x(s), x'(s)\right) \mid ds \\
&\leq 2\left(1 + \frac{\sum_{i=1}^{n-2} c_i \xi_i}{1 - \sum_{i=1}^{n-2} c_i}\right) \int_0^{+\infty} \mid f\left(s, x(s), x'(s)\right) \mid ds \\
&\leq 2\left(1 + \frac{\sum_{i=1}^{n-2} c_i \xi_i}{1 - \sum_{i=1}^{n-2} c_i}\right) \int_0^{+\infty} \varphi(s) \psi\left(\frac{\mid x(s) \mid}{1+s} + \mid x'(s) \mid\right) ds \\
&\leq 2\left(1 + \frac{\sum_{i=1}^{n-2} c_i \xi_i}{1 - \sum_{i=1}^{n-2} c_i}\right) \psi(2R) \int_0^{+\infty} \varphi(s) ds < +\infty
\end{aligned}$$

On the other hand, for $x \in \overline{\Omega}$

$$\begin{aligned}
\mid (Nx)'(t) \mid &\leq 2\int_0^{+\infty} \mid f\left(s, x(s), x'(s)\right) \mid ds \\
&\leq 2\psi(2R) \int_0^{+\infty} \varphi(s) ds \\
&\leq 2\left(1 + \frac{\sum_{i=1}^{n-2} c_i \xi_i}{1 - \sum_{i=1}^{n-2} c_i}\right) \psi(2R) \int_0^{+\infty} \varphi(s) ds < +\infty.
\end{aligned}$$

Then

$$\|Nx\|_X \leq 2\left(1 + \frac{\sum_{i=1}^{n-2} c_i \xi_i}{1 - \sum_{i=1}^{n-2} c_i}\right) \psi(2R) \int_0^{+\infty} \varphi(s) ds.$$

Hence $N(\overline{\Omega})$ is uniformly bounded .

(b2) $N(\overline{\Omega})$ is equicontinuous on any compact $[0, a]$ on $[0, +\infty)$. For any $t_1, t_2 \in [0, a]$ with $t_1 < t_2$ and for $u \in \overline{\Omega}$, we have

$$\begin{aligned}
\left|\frac{(Nx)(t_2)}{1+t_2} - \frac{(Nx)(t_1)}{1+t_1}\right| &= \Big|\int_0^{t_2}\frac{t_2-s}{1+t_2}f(s,x(s),x'(s))ds + t_2\int_0^{+\infty}f(s,x(s),x'(s))\,ds \\
&- \int_0^{t_1}\frac{t_1-s}{1+t_1}f(s,x(s),x'(s))ds - t_1\int_0^{+\infty}f(s,x(s),x'(s))ds\Big| \\
&= \Big|\int_0^{t_2}\frac{t_2-s}{1+t_2}f(s,x(s),x'(s))ds + \int_0^{t_1}\frac{t_2-s}{1+t_2}f(s,x(s),x'(s))\,ds \\
&- \int_0^{t_1}\frac{t_2-s}{1+t_2}f(s,x(s),x'(s))ds - \int_0^{t_1}\frac{t_1-s}{1+t_1}f(s,x(s),x'(s))ds \\
&+ (t_2-t_1)\int_0^{+\infty}f(s,x(s),x'(s))ds\Big| \\
&= \Big|\int_{t_1}^{t_2}\frac{t_2-s}{1+t_2}f(s,x(s),x'(s))ds \\
&+ \int_0^{t_1}\frac{(t_2-t_1)(1+s)}{(1+t_1)(1+t_2)}f(s,x(s),x'(s))ds \\
&+ (t_2-t_1)\int_0^{+\infty}f(s,x(s),x'(s))ds\Big| \\
&\le \psi(2R)\int_{t_1}^{t_2}\varphi(s)ds + \frac{\psi(2R)\mid t_2-t_1\mid}{(1+t_1)(1+t_2)}\int_0^{t_1}\mid 1+s\mid\varphi(s)ds \\
&+ \mid t_2-t_1\mid\psi(2R)\int_0^{+\infty}\varphi(s)ds. \\
&\longrightarrow 0 \quad as \quad \mid t_2-t_1\mid\longrightarrow 0
\end{aligned}$$

which implies

$$\left|\frac{(Nx)(t_2)}{1+t_2} - \frac{(Nx)(t_1)}{1+t_1}\right| \longrightarrow 0 \quad as \quad \mid t_2-t_1\mid\longrightarrow 0$$

□

On the other hand,

$$\begin{aligned}
|(Nx)'(t_2)-(Nx)'(t_1)| &\le \int_{t_1}^{t_2}|f(s,x(s),x'(s))|\,ds \\
&\le \psi(2R)\int_{t_1}^{t_2}\varphi(s)\,ds
\end{aligned}$$

So

$$|(Nx)'(t_2)-(Nx)'(t_1)| \to 0 \quad \text{as} \quad |t_2-t_1| \to 0.$$

Then

$$\|Nx(t_2) - Nx(t_1)\|_X \to 0 \quad \text{as} \quad |t_2 - t_1| \to 0.$$

(b3) $N(\overline{\Omega})$ is equiconvergent at infinity:
For $x \in \overline{\Omega}$, we have

$$\int_0^{+\infty} f(s, x(s), x'(s))\, ds \leq \psi(2R) \int_0^{+\infty} \varphi(s)\, ds < +\infty.$$

Hence

$$\begin{aligned}
\lim_{t\to+\infty} \frac{(Nx)(t)}{1+t} &= \lim_{t\to+\infty} \Bigg(\int_0^t \frac{t-s}{1+t} f(s, x(s), x'(s))\, ds - \frac{t}{1+t} \int_0^{+\infty} f(s, x(s), x'(s))\, ds \\
&\quad - \frac{\sum_{i=1}^{n-2} c_i \xi_i}{1 - \sum_{i=1}^{n-2} c_i} \cdot \frac{1}{1+t} \int_0^{+\infty} f(s, x(s), x'(s))\, ds \\
&\quad + \frac{1}{1+t} \cdot \frac{\sum_{i=1}^{n-2} c_i}{1 - \sum_{i=1}^{n-2} c_i} \int_0^{\xi_i} (\xi_i - s) f(s, x(s), x'(s))\, ds \Bigg) \\
&= 0
\end{aligned}$$

and

$$\lim_{t\to+\infty} ((Nx)'(t)) = 0.$$

Then $N(\overline{\Omega})$ is equiconvergent at infinity. We conclude $N = \overline{\Omega} \to X$ is relatively compact.

So N is completely continuous.

Notice that $x \in X$ is a solution of problem (2) if and only if x is a solution of the operator equation $x = Nx$. So we apply the Leray-Schauder theorem of fixed point.

For this, we prove the set of all possible solutions of the problem :

$$(2_\lambda) \begin{cases} x''(t) = \lambda f(t, x(t), x'(t)), & t \in (0, +\infty) \\ x(0) = \displaystyle\sum_{i=1}^{n-1} c_i x(\xi_i), & \displaystyle\lim_{t\to+\infty} x'(t) = 0. \end{cases}$$

is bounded for $\lambda \in (0, 1)$.

The existence of a solution of problem (2_λ) is equivalent to that of a fixed point of equation:

$$x = \lambda N x$$

To do this, it suffices to prove that $x \neq \lambda N x$ for all $x \in \partial\Omega$.

Let's show this by contradiction.

If there exists $x \in \partial\Omega$ with $x = \lambda N x$, then, we have

$$\begin{aligned}\left|\frac{x(t)}{1+t}\right| &= \left|\frac{\lambda N x(t)}{1+t}\right| \\ &\leq \left|\frac{N x(t)}{1+t}\right| \\ &\leq 2\left(1+\frac{\sum_{i=1}^{n-2} c_i\xi_i}{1-\sum_{i=1}^{n-2} c_i}\right)\psi(2R)\int_0^{+\infty}\varphi(s)\,ds, \quad \text{for } t\in[0,+\infty), x\in\overline{\Omega}\end{aligned}$$

$$\Rightarrow \sup_{t\in[0,+\infty)}\left|\frac{x(t)}{1+t}\right| \leq 2\left(1+\frac{\sum_{i=1}^{n-2} c_i\xi_i}{1-\sum_{i=1}^{n-2} c_i}\right)\psi(2R)\int_0^{+\infty}\varphi(s)\,ds$$

and

$$\begin{aligned}|x(t)| &\leq 2\psi(2R)\int_0^{+\infty}\varphi(s)\,ds \\ &\leq 2\left(1+\frac{\sum_{i=1}^{n-2} c_i\xi_i}{1-\sum_{i=1}^{n-2} c_i}\right)\psi(2R)\int_0^{+\infty}\varphi(s)\,ds\end{aligned}$$

Then

$$\|x\|_X \leq 2\left(1+\frac{\sum_{i=1}^{n-2} c_i\xi_i}{1-\sum_{i=1}^{n-2} c_i}\right)\varphi(2R)\int_0^{+\infty}\varphi(s)\,ds$$

So

$$R \leq 2\left(1+\frac{\sum_{i=1}^{n-2} c_i\xi_i}{1-\sum_{i=1}^{n-2} c_i}\right)\varphi(2R)\int_0^{+\infty}\varphi(s)\,ds$$

Which contradict condition (H4).

Then by theorem of Leray-Schauder, the problem (2) has a solution on X.

Example :

Consider the following boundary value problem on the half-line:

$$(P)\begin{cases} x''(t) = \dfrac{1}{80}\cdot\dfrac{1}{(1+t)^2}\cdot\dfrac{\dfrac{|x(t)|}{1+t}+|x'(t)|}{1+|x(t)|+|x'(t)|}, & t\in(0,+\infty) \\ x(0) = \sum\limits_{i=1}^{n-2} c_i x(\xi_i), & \lim\limits_{t\to+\infty} x'(t) = 0\end{cases}$$

Let

$$f(t,x,x') = \frac{1}{80}\cdot\frac{1}{(1+t)^2}\cdot\frac{\dfrac{|x(t)|}{1+t}+|x'(t)|}{1+|x(t)|+|x'(t)|}$$

choose :

$$c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{4} \quad \text{and} \quad \xi_1 = 1, \quad \xi_2 = 2$$

Then , we have

(H1) $0 < \sum_{i=1}^{2} c_i < 1.$

(H2) It holds that $f : [0, +\infty) \times \mathbb{R}^2 \longrightarrow [0, +\infty)$ is continuous.

(H3) Moreover

$$\begin{aligned} |\ f(t, (1+t)u, v)\ | &= \frac{1}{80} \cdot \frac{1}{(1+t)^2} \cdot \frac{|u| + |v|}{1 + (1+t)|u| + |v|} \\ &\leq \frac{1}{80} \cdot \frac{1}{(1+t)^2} \cdot (|u| + |v|) \\ &= \varphi(t)\psi(|u| + |v|) \end{aligned}$$

thus,

$$\varphi(t) = \frac{1}{(1+t)^2}, \qquad \psi(t) = \frac{1}{80} t$$

(H4) There exists $R > 0$ such that $R > 2\left(1 + \frac{\sum_{i=1}^{2} c_i \xi_i}{1 - \sum_{i=1}^{2} c_i}\right) \psi(2R) \int_0^{+\infty} \varphi(s) ds.$
In fact,

$$2\left(1 + \frac{\sum_{i=1}^{2} c_i \xi_i}{1 - \sum_{i=1}^{2} c_i}\right) = 2\left(1 + \frac{\frac{1}{2}(1) + \frac{1}{4}(2)}{1 - (\frac{1}{2} + \frac{1}{4})}\right) = 2\left(1 + \frac{1}{\frac{1}{4}}\right) = 2 \times 5 = 10$$

also

$$\psi(2R) = \frac{1}{40} R$$

and

$$\int_0^{+\infty} \varphi(s) ds = \int_0^{+\infty} \frac{1}{(1+s)^2} ds = \left[-\frac{1}{1+s} \right]_0^{+\infty} = 1$$

which gives

$$2\left(1 + \frac{\sum_{i=1}^{2} c_i \xi_i}{1 - \sum_{i=1}^{2} c_i}\right) \psi(2R) \int_0^{+\infty} \varphi(s) ds = \frac{1}{4} R$$

Then the problem (P) has a solution on X.

Conclusion and perspectives

In the present memory, we have considered some boundary value problems of second order differential equations posed on unbounded intervals and associated with multipoint boundary conditions. Most of these problems stem from some physical phenomena.

We have started the memory with a general review on some recent results regarding existence of solutions to Three-point boundary value problems of second-order over unbounded intervals.

In our work was mainly to establish the existence results for some multipoint boundary value problems of second order in the infinite interval of the real line.

The main difficulty in dealing with such types of Boundary value problems was the possible lack of compactness due to the infinite interval together with some difficulties related to the multipoint boundary conditions. In order to overcome these difficulties, we have made use of some special Banach spaces adapted to each situation.

Some methods have been employed to prove existence of solutions. These are methods based on the study of the corresponding Green's function, some appropriate compactness criteria, and fixed point arguments. For this, some sufficient conditions on the nonlinear term are assumed throughout.

All of our existence results are supported by examples of applications.

In the future, we intend to investigate some questions related to the properties of solutions for such classes of problems. Also the extension to more general nonlinear boundary conditions is one of our topics of interest.

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Abstract

In this memory, we have studied the existence of solutions to certain boundary value problems of second-order differential equations posed on unbounded intervals and associated with multi-point boundary conditions. Methods have been used to prove the existence of solutions. These methods are based on the study of the corresponding Green's function and some of its properties. We have transformed the differential equation into an operational equation. Using a fixed point theorem, sufficient conditions are obtained guaranteeing the existence of at least one solution. Some examples illustrate the applicability of our main results.

key words: Differential equation, existence, multipoint, boundary value problem, infinite interval, fixed point theorem.

Résumé

Dans ce mémoire, nous avons étudié l'existence des solutions à certains problèmes aux limites d'équations différentielles du second ordre posées sur des intervalles non bornés et associées à des conditions aux limites à multi points. Des méthodes ont été employées pour prouver l'existence des solutions. Ces méthodes reposent sur l'étude de la fonction de Green correspondante et de certaines de ses propriétés. Nous avons transformé l'équation différentielle en équation opérationnelle. Grâce à un théorème du point fixe, des conditions suffisantes sont obtenues garantissant l'existence d'au moins une solution. Quelques exemples illustrent l'applicabilité de nos principaux résultats.

Mots clés : Équation différentielle, existence, multipoint, problème aux limites, intervalle infini, théorème du point fixe.

ملخص

درسنا في هذا البحث وجود حلول بعض الجمل الحدية لمعادلات تفاضلية من الرتبة الثانية ، المطروحة على مجالات غير محدودة ، و المرتبطة بشروط حدية لعدة نقط . استخدمنا بعض الطرق لاثبات وجود الحلول . تعتمد هذه الطرق على دراسة دالة جرين المرتبطة لها، و بعض خصائصها ، و قد حولنا المعادلة التفاضلية إلى معادلة بالمؤثر، بفضل نظرية النقطة الصامدة، يتم الحصول على شروط كافية تضمن وجود حل واحد على الأقل . قدمنا بعض الأمثلة كتطبيقات للنتائج التي حصلنا عليها .

الكلمات المفتاحية: المعادلات التفاضلية ،الوجود ، متعدد النقط ، الجملة الحدية ، المجال غير المحدود و نظرية النقطة الصامدة .