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SUBJECT OF THE THESIS:

Study and comparison of approximate solutions of certain perturbed differential dynamical systems

Presented by :

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In front of the jury composed of :

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Spécialité : Analyse Fonctionnelles et Appliquées

SUJET DE LA THESE :

Etude et comparaison des solutions approximatives de certains systèmes dynamiques différentiels perturbés

Présentée Par :

LEJDEL ALI Tefaha

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Abstract

Our work is devoted to studying several aspects of the solutions of an important useful model in turbulent dynamical systems, the Van Der Pol equation. At first we introduced all the basic characteristics and concepts used in our study. Then, we prove the approximate analytical solutions of the Van Der Pol equation in its general form by different perturbation methods: simple perturbation method (SPM), Lindstedt-Poincaré method (PLM) and Averaging method (AM) and then compare them with each other and with the exact solution. We studied Van Der Pol systems in fractal order in their general form, we modeled and simulated them using Matlab software, and analyzed our results. Finally, we presented another aspect of studying this type of equations, where we proved the conditions of existence, singularity and asymptotic stability of periodic solutions to an infinite set of nonlinear differential equations, which in turn facilitates the researcher to take and use them on all possible equations directly and quickly.

Key words: Perturbation theory, Nonlinear dynamical systems, Van Der Pol systems in their general form, Van Der Pol systems in the fractal order in their general form, Approximate solutions, Numerical simulations, Approximations of fractional-order operators, asymptotic stability, periodic solutions, Bogolyubov's second theorem.

Résumé

Notre travail est consacré à l'étude de plusieurs aspects des solutions d'un important modèle utile dans les systèmes dynamiques turbulents, l'équation de Van Der Pol. Dans un premier temps, nous avons présenté toutes les caractéristiques et concepts de base utilisés dans notre étude. Ensuite, nous prouvons les solutions analytiques approchées de l'équation de Van Der Pol dans sa forme générale par différentes méthodes de perturbation : méthode de perturbation simple, méthode de Lindstedt-Poincaré et Averaging method puis nous les comparons entre elles et avec la solution exacte.

Nous avons étudié les systèmes de Van Der Pol en ordre fractal dans leur forme générale, nous les avons modélisés et simulés à l'aide du logiciel Matlab, et analysé nos résultats. Enfin, nous avons présenté un autre aspect de l'étude de ce type d'équations, où nous avons prouvé les conditions d'existence, de singularité et de stabilité asymptotique des solutions périodiques à un ensemble infini d'équations différentielles non linéaires, ce qui à son tour facilite au chercheur de les prendre et de les utiliser sur tous équations possibles directement et rapidement.

Mots clés : Théorie des perturbations, Systèmes dynamiques non linéaires, Systèmes de Van Der Pol dans leur forme générale, Systèmes de Van Der Pol dans l'ordre fractionnaire dans leur forme générale, Solutions approchées, Simulations numériques, Approximations d'opérateurs d'ordre fractionnaire, stabilité asymptotique, solutions périodiques, seconde de Bogolyubov théorème.

ملخص

عملنا مكرس لدراسة عدة جوانب لحلول أحد النماذج المفيدة المهمة في الأنظمة الديناميكية المضطربة وهي معادلة فان دير بول. في البداية قدمنا جميع الخصائص والمفاهيم الأساسية المستعملة في دراستنا.

بعد ذلك اثبتنا الحلول التحليلية التقريبية لمعادلة فان دير بول في شكلها العام من خلال طرق الاضطراب المختلفة : طريقة الاضطراب البسيط وطريقة ليندستيد بوانكاريه وطريقة المتوسط ثم مقارنتها مع بعضها البعض ومع الحل الحقيقي.

ودرسنا أنظمة فان دير بول بالترتيب الكسري في شكله العام، حيث قمنا بنمذجتها ومحاكاتها باستخدام برنامج الماتلاب، وقمنا بتحليل نتائجنا.

أخيرا، قدمنا جانبا آخرا من دراسة هذا النوع من المعادلات. حيث اثبتنا شروط الوجود والتفرد والاستقرار المقارب للحلول الدورية لمجموعة لانهائية من المعادلات التفاضلية الغير خطية، والتي بدورها تسهل للباحث أخذها وإستعمالها على جميع المعادلات الممكنة بشكل مباشر وسريع.

الكلمات المفتاحية : نظرية الاضطراب، الأنظمة الديناميكية غير الخطية، أنظمة فان ديربول في شكلها العام، أنظمة فان دير بول بالترتيب الكسري في شكلها العام، الحلول التقريبية، المحاكاة، تقريب المؤثرات بالترتيب الكسري، الاستقرار المقارب، الحلول الدورية، نظرية ثاني بوغوليوبوف.

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Dedication

Thank God Almighty first and foremost for the great grace that has bestowed upon me.
I dedicate this humble thesis as a sign of respect, appreciation and thanks
To my mother ***Fedjra Lejdel*** *and my father* ***Ahmed****, for their great support even when things were really tough, because they kept encouraging me to work harder.*
To my dear sisters: ***Fatma****,* ***Habiba****,* ***Hakima*** *and* ***Bahia****.*
To my dear brothers: ***Djaafar****,* ***Bader****,* ***Noureddine*** *and* ***Saad****.*
To my brothers and sisters- in- law.
To all my nephews and nieces.
I also dedicate this thesis to my friends who made my life a wonderful experience. I cannot list all the names here, but you are always on my mind.
Thank you, Lord, for always being there for me. This thesis is only the beginning of my journey.

General notations

$\phi(t, x)$	A flow.
$f(X)$	A vector field.
$\dot{X}$	The derivative of X with respect to t.
J_f	The Jacobian matrix of the function f.
T	A period.
$N(x_0)$	A neighborhood of x_0.
$\mathbb{R}$	Set of real numbers.
$\mathbb{R}^+$	Set of positive or zero real numbers.
$\mathbb{R}^n$	Real n-dimensional vector space constructed over the field of reals.
$\mathbb{N}$	Set of natural numbers.
$\mathcal{B}$	A tribe.
μ	A measure.
m^*	The external measure.
D^α	Fractional derivative
$C^0 = C(K, F)$	Set of continuous functions from K to F.
C^1	The space of continuous and differentiable functions.
$det(B)$	determinant of matrix B.
$trace(B)$	trace of matrix B.

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General Introduction

Our work concerns part of the domain of perturbation theory, which is one of the theories that can be considered old and new and which is currently gaining great popularity among researchers and mathematicians in the sciences. Perturbation theory was first devised to solve otherwise intractable problems in the calculation of the motions of planets in the solar system. For instance, Newton's law of universal gravitation explained the gravitation between two astronomical bodies, but when a third body is added, the problem was, "How does each body pull on each?" Newton's equation only allowed the mass of two bodies to be analyzed. The gradually increasing accuracy of astronomical observations led to incremental demands in the accuracy of solutions to Newton's gravitational equations, which led several notable 18th and 19th century mathematicians, such as Lagrange and Laplace, to extend and generalize the methods of perturbation theory. These well-developed perturbation methods were adopted and adapted to solve new problems arising during the development of quantum mechanics in 20th century atomic and subatomic physics.

The principle of perturbation theory is the study of the theory of dynamical systems that is used to describe the behavior of complex dynamical systems and considers these systems as small perturbations of "simple" systems. This theory deals with the long-term qualitative behavior of dynamical systems, and studies the nature of, and when possible the solutions of, the equations of motion of systems that are often primarily mechanical or otherwise physical in nature, such as planetary orbits and the behaviour of electronic circuits, as well as systems that arise in biology, economics, and elsewhere. Much of modern research is focused on the study of chaotic systems.

An important useful model in perturbation theory is the Van Der Pol equation. This equation appeared in 1927 by electrical engineer Balthasar Van Der Pol, see [12], where he described oscillations of a triode in electrical circuits. He presented it in its mathematical form as a second-order nonlinear differential equations.

To understand the behavior of this type of equation, many studies have been carried out with different methods to approximate and find the behavior of the best approximate solutions of nonlinear equations, see [44, 30].
Researchers have developed many methods and techniques to study approximate analytic solutions with different perturbation methods, see [15, 19, 37, 4, 5].

In recent years, researchers have developed this type of differential equation by introducing fractional calculus.
Fractional calculus is an old mathematical topic from 17th century. Although it has a long history, applications are only recent focus of interest. Many systems are known to display fractional order dynamics, such as viscoelastic systems [40], dielectric polarization, electrode–electrolyte polarization, and electromagnetic waves. The usual approach for analysing fractional-order systems is the development of continuous and discrete integer-order approximations of fractional-order operators [20, 10, 13]. And through previous studies the researchers found that some of these systems display chaotic movements in the fractional order [41, 53, 47].

The problems of existence and uniqueness of solutions and their qualitative properties is an important field of current scientific research. The stability of certain of those solutions is the first question in dynamical systems theory, i.e., the study of system equilibrium. The stability theory is still of great interest to mathematicians and astronomers, and it attracts the attention of many specialists over its long history. So in modern times it has become widely used in physics, astronomy, chemistry and even in biology.

In mathematics, the second Bogolyubov's theorem study the existence, uniqueness and asymptotic stability of the periodic solutions for the Lipschitz system. This theorem has been extensively developed; for exemple see Fatou 1928 [34], Mandelstam-Papaleksi 1934 [25] an, krylov-Bogolyubov 1937 [32], N. N. Bogolyubov 1945 [33] and also Adriana Buica et all 2009 [2]. Several special cases of the Van Der Pol equations have been considered [2, 52].

This thesis is composed of four chapters as follows

- The first chapter situates the study framework of this thesis, it consists of six sections. In Section 1.1, we list the initial symbols and definitions and gather several tools of the basic concepts of dynamical systems. In section 1.2, we introduce some basic properties of functional analysis in addition to topology. In section 1.3, We present the Brouwer topological degree, its definition and properties. In Section 1.4, we introduce the definition of the fractional order factor and its approximation. In Section 1.5, we introduce the perturbation theory and its tools, which will be use in the following chapters. Finally, In Section 1.6, we introduce the Van Der Pol equation, which is a type of perturbation theory that we will take as an application for our study in the rest of the chapters.

- In the second chapter, we study the approximate solutions to the nonlinear differential equations in its general form. This chapter is divided into two main parts. In the first, we prove the approximate analytic solutions to this equation by different perturbation analytic methods, simple perturbation method (SPM), Lindstedt-Poincaré method (PLM) and Averaging method (AM). In the second part, we compare these approximations with each other and with the exact solution.

- We start the third chapter by presenting some previous studies on the study of the nonlinear differential equation, including the Van Der Pol oscillator with fractional derivatives. In the second part, we address another study, which is to verify the existence of chaos in the generalized Van Der Pol system and its fractional system. Finally, we prove a more general study, which is the study of the Van Der Pol generalized fractional system with the degree of polynomials.

- In the last chapter, we study the existence, uniqueness and asymptotic stability of T periodic solutions of nonlinear dynamical systems using Bogolyubov's second theorem. It is divided into two parts. In the first section , we remind the periodic case of the second Bogolyubov's theorem as well as an application to the Van Der Pol equation. Finally, we prove some theorems using Bogolyubov's second theorem.

Chapter 1

Preliminaries

In this chapter, we present the elementary symbols and definitions and provide many tools on the basic concepts of dynamical systems, functional analysis, topology, Brouwer's degree of topology, and partial derivation that we will use later. We also take up perturbation theory and take as an application of our study the Van Der Pol equation.

1.1 General definitions of dynamical system

We consider a topological space M (i.e. a space of points endowed with a topology). We assume that this space is

- Countable basis : the topology of M has a countable basis of open sets. This property is equivalent to the existence of a dense countable subset (for example $\mathbb{Q}^n$ for $\mathbb{R}^n$),

- Separated : two distinct points have distinct neighborhoods.

Definition 1.1.1. *[50] A map of dimension n on M is a couple (U, φ) formed from*

- *An open $U \subset M$,*

- *A homeomorphism $\varphi : U \longrightarrow \varphi(U) \subset \mathbb{R}^n$ (a homeomorphism is a continuous and invertible map whose inverse is continuous).*

The open U is the domain of the map. For $p \in U$, $\varphi(p) = (x^1(p), ..., x^n(p)) \in \mathbb{R}^n$: φ is what is called a coordinate function.

A point of M can belong to two different domains corresponding to two maps (U, φ) and (V, ψ).

Definition 1.1.2. *[26] A dynamical system on E is a map C^1*

$$\phi : \mathbb{R} \times E \longrightarrow E$$
$$(t, x) \longmapsto \phi(t, x),$$

where E is an open subset of $\mathbb{R}^n$ and if $\phi_t(x) = \phi(t, x)$, then ϕ_t satisfied

1. *$\phi_0(x) = x$ for all $x \in E$.*

2. *$\phi_t \circ \phi_s(x) = \phi_{t+s}(x)$ for all $s, t \in \mathbb{R}$ and $x \in E$.*

Remark 1.1.1. *We say that ϕ_t is a flow if it satisfies properties 1 and 2 of the definition (1.1.2).*

1.1.1 Mathematical representations of dynamical systems

The dynamical system defined by the nonlinear system of ordinary differential equations is given by

$$\dot{X} = f(X), \tag{1.1}$$

such that $f : E \to \mathbb{R}^n$ (E is an open subset of $\mathbb{R}^n$) is a vector function and

$$\dot{X} = \frac{dX}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix},$$

where $f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))$ and

$$\begin{cases} \frac{dx_1}{dt} = f_1(x_1, \ldots, x_n), \\ \vdots \\ \frac{dx_n}{dt} = f_n(x_1, \ldots, x_n). \end{cases}$$

1.1.2 Phase portrait

Definition 1.1.3. *[26] The phase portrait of a dynamical system is a graphical representation of several representative trajectories in phase space. Given a dynamical system* $\dot{X} = f(X, t)$*, without solving the equations one can always at a given time* t *represent graphically (using the arrows) the field of* $\dot{X}$ *(the field of velocities if* X *are coordinates). Reading this graphical representation will be very useful to get an idea of the behavior of the system.*

1.1.3 Critical point

Definition 1.1.4. *[15] The point* $x = a$ *with* $f(a) = 0$ *is called a critical point of equation* $\dot{x} = f(x)$*.*

1.1.4 Periodic solutions

Definition 1.1.5. *[15] Suppose that* $x = \phi(t)$ *is a solution of the equation* $\dot{x} = f(x), x \in D \in \mathbb{R}^n$ *and suppose there exists a positive number* T *such that* $\phi(t+T) = \phi(t)$ *for all* $t \in \mathbb{R}$*. Then* $\phi(t)$ *is called a periodic solution of the equation with period* T*.*

1.1.5 linearization of dynamic system

Consider the nonlinear dynamical system defined by (1.1) and let X_0 be an equilibrium point of this system.

Suppose a small perturbation $y(t)$ is applied near the equilibrium point.

The function f can be expanded into a Taylor series in the neighborhood of point X_0 as follows

$$\dot{y}(t) + \dot{X}_0 = f(X_0 + y(t)) \simeq f(X_0) + J_f(X_0).y(t), \tag{1.2}$$

with $J_f(X_0)$ is the Jacobian matrix of the function f defined by

$$J_f(X_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}_{X=X_0} . \tag{1.3}$$

As $f(X_0) = \dot{X}_0$, then the equation (1.2) becomes

$$\dot{y}(t) = J_f(X_0).y(t). \tag{1.4}$$

The writing (1.4) means that the system (1.1) is linearized. This system can be written in the matrix form

$$\dot{X} = AX, \tag{1.5}$$

where A is a matrix $n \times n$.

The solution of the linear system (1.5) with the initial condition $X(0) = X_0$ is given by

$$X(t) = e^{At}X_0, \tag{1.6}$$

where e^{At} is a matrix function $n \times n$.

Geometrically, the dynamical system describes the motion of points in phase space along solution curves defined by the system of differential equations.

Remark 1.1.2. *For the system (1.1), the function $f : E \to \mathbb{R}^n$ specifies the velocity at each point in phase space E; it is called a vector field.*

Example 1.1.1. *consider the decoupled linear system*

$$\begin{cases} \dot{x}_1 = -x_1, \\ \dot{x}_2 = 2x_2. \end{cases} \tag{1.7}$$

This system can be written in the matrix form

$$\dot{X} = AX, \tag{1.8}$$

where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

The general solution of the above linear system is given by

$$\begin{cases} x_1(t) = c_1 e^{-t}, \\ x_2(t) = c_2 e^{2t}. \end{cases} \tag{1.9}$$

Or equivalently by

$$X(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} C, \tag{1.10}$$

where $C = X(0)$.

The dynamical system defined by the linear system (1.8) in this example is simply the map $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ *defined by the solution* $X(t, C)$ *given by (1.10); that is, the dynamical system for this example is given by*

$$\phi(t, C) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} C. \tag{1.11}$$

1.1.6 Autonomous, non-autonomous systems

Definition 1.1.6. *A differential system is said to be autonomous if the time variable does not appear explicitly in the function* f*. Otherwise, the system is said to be non-autonomous. A non-autonomous system is of the form*

$$\dot{X} = f(X.t).$$

1.2 Concepts in functional analysis and topology

1.2.1 Set and topological preliminaries

Some of the basic concepts of topology are essential in the study of dynamical systems, so we begin our study by recalling some of the symbols and basics from set theory and topology that we will use.

- $\mathbb{R}^n$ is n-dimensional Euclidean space.
- The Euclidean norm is denoted by $\|x\|$.
- A solid ball of radius r around a point x_0 is the closed set

$$B_r(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| \leq r\}.$$

- A topological space is characterized by a collection of open sets. For Euclidean space the basic open sets are the open balls

$$\{x \in \mathbb{R}^n : \|x - x_0\| < r\}.$$

Remark 1.2.1. *By definition, a union of any number of open sets is declared open, as is the intersection of any finite number of open sets. Similarly, the basic closed sets are the closed balls $B_r(x_o)$. By definition, the intersection of any number of closed sets is closed, as well as the union of finitely many closed sets. The word neighborhood is used to denote some arbitrary set that encloses a designated point.*

Definition 1.2.1 (Neighborhood). *[21] N is a neighborhood of a point x if N contains an open set containing x.*

Remark 1.2.2. *Note that a neighborhood can be open or closed, but it must contain some open set. This excludes calling the set $\{x\}$ a neighborhood of x; however, for any $r > 0$, the closed ball $B_r(x)$ is a neighborhood of x. Often, we think of neighborhoods as being "small" sets in some sense, but this is not a requirement.*

1.2.2 Convergence

Sequences are ordered lists; for example, $S = \{s_j \in : j \in \mathbb{N}$. A sequence is convergent if it approaches a fixed value, s^*, i.e., if $\|s_j - s^*\| \to 0$ as $j \to \infty$. Formally, we say that the sequence S converges if for every $\epsilon > 0$ there is an $N(\epsilon)$ such that whenever $n > N(\epsilon)$, then $\|s_j - s^*\| < \epsilon$.

1.2.3 Lipschitz function

Another ingredient that we will need in the existence and uniqueness theorem is a notion that is stronger than continuity but slightly less stringent than differentiability

Definition 1.2.2 (Lipschitz). *[21] Let E be an open subset of $\mathbb{R}^n$. A function $f : E \to \mathbb{R}^n$ is Lipschitz if for all $x, y \in E$, there is a K such that*

$$\|f(x) - f(y)\| \leq K\|x - y\|.$$

The smallest such constant K is called the Lipschitz constant for f on E; it has the geometrical interpretation that the slope of the chord between the two points $(x, f(x))$ and $(y, f(y))$ is at most K in absolute value.
The Lipschitz property implies more than continuity, but less than differentiability.

Definition 1.2.3 (**locally Lipschitz**). *[21] f is locally Lipschitz on an open set E if for every point $x \in E$, there is a neighborhood N such that f is Lipschitz on N.*

The Lipschitz constant can vary with the point and indeed become arbitrarily large.

Remark 1.2.3. *Every differentiable function is locally Lipschitz.*

1.2.4 Measurable in the sense of Lebesgue

Definition 1.2.4. *[16] A tribe over $\mathbb{R}^n (n \geq 1)$ is a family of subsets of $\mathbb{R}^n$ containing $(\varnothing, \mathbb{R}^n)$, stable by passage to the complement and by countable union (and therefore by countable intersection).*
If $\mathcal{B}$ designates a tribe on $\mathbb{R}^n$, the elements of $\mathcal{B}$ are called measurable sets. We say that $(\mathbb{R}^n, \mathcal{B})$ is a measurable space.

Definition 1.2.5 (**Measure**). *[16] Let $\mathcal{B}$ be a tribe of $\mathbb{R}^n$. A positive measure μ on $\mathcal{B}$ is an application from $\mathcal{B}$ to $\overline{\mathbb{R}}_+$ verifying*

1. $\mu(\varnothing) = 0$,
2. *For any countable family (B_i) of two-by-two disjoint elements of B we have*

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i),$$

$(\mathbb{R}^n, \mathcal{B}, \mu)$ is a measured space.

Definition 1.2.6 (**External measure in $\mathbb{R}$**). *[16] Let it be $E in \mathbb{R}$, we call the external measure of E with the Lebesgue concept and denote it by $m^*(E)$ the expression defined by the following*

$$m^*(E) = \inf \mathcal{L}_E = \inf\{\sum_{n=1}^{\infty} |I_n| : \{I_n\}_{n\geqslant 1} \in \mathbb{R} \text{ intervals and } E \subset \bigcup_{n=1}^{\infty} I_n\}.$$

Definition 1.2.7 (**Measurable parts in Lebesgue's concept**)**.** *[16] We say about part $Ein\mathbb{R}$ that it is measurable in the concept of Lebesgue, if*

$$\forall A \subset \mathbb{R} : m^*(A) = m^*(A \cap E) + (A \cap E^c).$$

We denote the set of all measurable parts with the symbol $\mathcal{M}$

$$E \subset \mathcal{M} \Leftrightarrow \forall A \subset \mathbb{R} : m^*(A) = m^*(A \cap E) + (A \cap E^c).$$

1.3 Brouwer topological degree

Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset and $f : \bar{\Omega} \to \mathbb{R}^n$, if f is differentiable at x_0, we call the Jacobian of f at x_0 is $J_f(x_0) = \det f'(x_0)$, if $J_f(x_0) = 0$, then x_0 is said to be a critical point of f and we use $S_f(\Omega) = \{x \in \Omega : J_f(x) = 0\}$ to denote the set of critical points of f in Ω. If $f^{-1}(y) \cap S_f(\Omega) = \emptyset$, then y is said to be a regular value of f. Otherwise, y is said to be a singular value of f.

Definition 1.3.1. *For a regular value $y \notin f(\partial\Omega)$ the C^1-mapping degree is defined by*

$$\deg_B(f, \Omega, y) = \begin{cases} \sum\limits_{x \in f^{-1}(y)\cap\Omega} \operatorname{sgn}(J_f(x)), & f^{-1}(y) \cap \Omega \neq \emptyset. \\ 0, & f^{-1}(y) \cap \bar{\Omega} = \emptyset. \end{cases}$$

Example 1.3.1. *Let $\Omega = B(0, R), Y_0 = (1, 0)$, and $f(x, y) = (x^3 - 3xy^2, -y^3 + 3x^2y)$, then $f(x, y) = (0, 1)$, thus $(x, y) = (1, 0) \vee (-\frac{1}{2}, \frac{\sqrt{3}}{2}) \vee (-\frac{1}{2}, -\frac{\sqrt{3}}{2}) \in \partial\Omega$. Note that at least the point $(1, 0)$ is on the border of the unit ball whatever the standard usual norm we consider the $\mathbb{R}^2$. Therefore, the degree is not defined if $R = 1$.*
If $0 < R < 1$, then $B(0, R) \bigcap f^{-1}(y) = \emptyset$, thus $\deg_B(f, B(0, R), y) = 0$.
Finally, if $R > 1$, then the degree is defined, and we have

$$Df(x, y) = \begin{pmatrix} -3y^2 & -6xy \\ 6xy & -3y^2 + 3x^2 \end{pmatrix}.$$

We conclude that

$$J_f(x, y) = (3x^2 - 3y^2)^2 + 36x^2y^2 = 0 \Leftrightarrow (x, y) = (0, 0).$$

The three points are then regular and as $\operatorname{sgn}(J_f(x,y)) > 0$, $\forall (x,y) \neq (0,0)$ *then*

$$\deg_B(f, B(0,R), y) = 3.$$

Property 1.3.1. *If* $y \notin f(\partial\Omega)$, *then there exists an integer* $\deg_B(f,\Omega,y)$ *satisfying the following properties*

1. ***Normality :***

$$\deg_B(I,\Omega,y) = \begin{cases} 1, & y \in \Omega, \\ 0, & y \notin \bar{\Omega}, \end{cases}$$

where I *denotes the identity mapping.*

2. ***Additivity :*** *For* Ω_1, Ω_2 *are two disjoint open subsets of* Ω, *and* $y \notin f\left(\bar{\Omega} - \Omega_1 \cup \Omega_2\right)$, *it holds that*

$$\deg_B(f,\Omega,y) = \deg_B(f,\Omega_1,y) + \deg_B(f,\Omega_2,y).$$

3. ***Validity :*** *If* $\deg_B(f,\Omega,y) \neq 0$, *then there exists* $x \in \Omega$, *such that* $f(x) = y$.

4. ***Homotopy :*** *Let* $f, g : X \to Y$ *be maps.* f *homotopoic to* g *if there exists a map continuous* $H : X \times I \to Y$ *with* $H(x,0) = f(x)$ *and* $H(x,1) = g(x)$, $\forall x \in X, I = [0,1]$. *If* $f_t(x) : [0,1] \times \bar{\Omega} \to \mathbb{R}^n$ *is continuous and* $y \notin \cup_{t\in[0,1]} f_t(\partial\Omega)$, *then* $\deg_B(f_t,\Omega,y)$ *does not depend on* $t \in [0,1]$.

Example of Homotopy

Let $f, g : [-1,1] \to \mathbb{R}$ *be given by* $f(x) = x^2$ *and* $g(x) = 2$. *As we can see the* map $h : [0,1] \times [-1,1] \to \mathbb{R}$ *given by*

$$h(t,x) = (1-t)f(x) + \operatorname{tg}(x),$$

is a valid homotopy joining f *and* g.

The fractional calculus concerns the study and applications of integrals and derivatives of arbitrary order (real or complex order).

1.4 Fundamentals of fractional calculus

There are different approaches to the fractional calculus, not being all equivalent. The two most commonly used definitions are the Riemann-Liouville and the Grünwald-Letnikov definitions [17, 24, 18, 38, 8].

1.4.1 Definitions of fractional derivatives and integrals

The Riemann-Liouville definition of the fractional-order derivative is ($\alpha > 0$)

$$aD_t^{\alpha} f(t) = \frac{d^n}{dt^n} D^{\alpha-n} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad \alpha > 0.$$

Where $\Gamma(.)$ is a gamma function and n is an integer such that $n-1 \leq \alpha < n$.
In other hand, the Grünwald-Letnikov definition is formulated as ($\alpha \in \mathbb{N}$)

$$aD_t^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} f(t-kh),$$

where h is the time increment and $[x]$ means the integer part of x.

1.4.2 Laplace transform of fractional derivatives

It is necessary to develop approximations to the fractional operators using the standard integer order operators. Fortunately, the Laplace transform which is basic engineering tool for analysing linear systems is still applicable and works

$$L\{\frac{d^{\alpha} f(t)}{dt^{\alpha}}\} = s^{\alpha} L\{f(t)\} - \sum_{k=0}^{n-1} s^k \left[\frac{d^{\alpha-1-k} f(t)}{dt^{\alpha-1-k}}\right]_{t=0} \quad \text{for all } \alpha,$$

where n is an integer such that $n-1 \leq \alpha < n$. Upon considering the initial conditions to be zero, this formula reduces to the more expected form

$$L\{\frac{d^{\alpha} f(t)}{dt^{\alpha}}\} = s^{\alpha} L\{f(t)\}.$$

1.4.3 Fractional nonlinear systems

We consider the general fractional-order nonlinear system represented as follows

$$aD_t^{q_i}x_i(t) = f(x_1(t), x_2(t), ..., x_n(t), t), \qquad x_i(0) = c_i, \quad i = 1, 2, ..., n, \tag{1.12}$$

where c_i are initial conditions. The vector representation of (1.12) is

$$D^q x = f(x), \tag{1.13}$$

where $q = [q_1, q_2, ..., q_n]^T$ for $0 < q_i < 2, (i = 1, 2, ..., n)$ and $x \in \mathbb{R}^n$.
The equilibrium points of system (1.13) are calculated via solving the following equation

$$f(x) = 0$$

and we suppose that $E^* = (x_1^*, x_2^*, ..., x_n^*)$ is an equilibrium point of system (1.13).

1.4.4 Approximations of fractional-order operators

Numerical simulation of the fractal order system has a method based on approximating the behavior of the system in the frequency domain.
In our study we used the Charef's approximation frequency method [3] which is based on linear transfer function. From [47], we get the table of approximating transfer functions for $1/s^\alpha$ a with different fractional orders, $\alpha = 0.1 - 0.9$ steps, giving a maximum error of $2dB$ in the calculations. The basic idea is to aproximate the slope of the magnitude Bode diagram of the transfer function of a single-fractional power pole of the form

$$\frac{1}{s^\alpha} \approx \frac{1}{\left(1 + \frac{s}{P_T}\right)^\alpha}.$$

1.5 Perturbation theory

The principle of perturbation theory is to study dynamical systems that are small perturbations of systems.
Typically perturbation theory explains only part of the dynamics, and in the resulting 'gaps'

the orderly unperturbed motion is replaced by random or chaotic motion.
The goal of perturbation theory is to find an analytic approximation of a given dynamical system whose true solution is difficult (or impossible) to find. So as to study the behavior of the system.

1.5.1 Perturbed differential dynamical systems

The general form of the perturbed differential dynamical systems which we shall study is

$$\dot{x} = f(x, t, \epsilon), \tag{1.14}$$

where x and $f(x, t, \epsilon)$ are vectors, elements of $\mathbb{R}^n$. All quantities used will be real except if explicitly stated otherwise.
Often we shall assume $x \in D \subset \mathbb{R}^n$ with D an open, bounded set, the function $f(t, x, \epsilon)$ is continuous in the variables t, the variable $t \in \mathbb{R}$ is usually identified with time.
We assume $t \geq 0$ or $t \geq t_0$ with t_0 a constant. The parameter ϵ plays the part of a small parameter which characterizes the magnitude of certain perturbations. We usually take ϵ to satisfy either $0 \leq \epsilon \leq \epsilon_0$ or $|\epsilon| \leq \epsilon_0$, but even when $\epsilon = 0$ is not in the domain, we may want to consider limits as $\epsilon \to 0$.
we give example of perturbation problem.

Example 1.5.1. *A simple example of a perturbation arising in a natural way is the following problem. Consider a harmonic oscillation, described by the equation*

$$\ddot{x} + x = 0 \tag{1.15}$$

In deriving this equation the effect of friction has been neglected; in practice however, friction will always be present. If the oscillator is such that the friction is small, an improved model for the oscillations is given by the equation

$$\ddot{x} + \epsilon\dot{x} + x = 0 \tag{1.16}$$

The term $\epsilon\dot{x}$ is called "friction term" or "damping term" and this particular simple form of the friction term has been based on certain assumptions concerning the mechanics of *friction. The parameter ϵ is small*

$$0 < \epsilon << 1$$

If one puts $\epsilon = 0$ in equation (1.16) one recovers the original equation (1.15), we call equation (1.15) the" unperturbed problem". We shall always proceed while assuming that we have sufficient knowledge of the solutions of the unperturbed problem

1.5.2 Basic material

The function f in equation (1.14) has to be expanded with respect to the small parameter ϵ. In the simple case that f has a Taylor expansion with respect to ϵ near $\epsilon = 0$ we have

$$f(t, x, \epsilon) = f(t, x, 0) + \epsilon f_1(t, x) + \epsilon^2 f_2(t, x) + ...\epsilon^n f_n(t, x) + ...,$$

with coefficients $f_1, f_2, ...$ which depend on t and x. The expressions $\epsilon, \epsilon^2, ..., \epsilon^n, ...$ are called order functions.

1.6 Van Der Pol equation

NAYFEH A. H., 1981 has developed several techniques for determining the approximate solution of nonlinear differential equations.
In chapter 6 of this book entitled "Self-excited oscillators" he studied "Self-excited Systems" whose systems are governed by equations of the form

$$m\frac{d^2u^*}{dt^{*^2}} + ku^* = \mu f^*\left(u^*, \tfrac{du^*}{dt^*}\right)\frac{du^*}{dt^*}, \tag{1.17}$$

or

$$\frac{d^2u^*}{dt^{*^2}} + \frac{k}{m}u^* = \frac{\mu}{m}f^*\left(u^*, \tfrac{du^*}{dt^*}\right)\frac{du^*}{dt^*},$$

where m is a mass μ a positive parameter, f^* is positive for u^* small.
We pose

$$f^*\left(u^*, \tfrac{du^*}{dt^*}\right) = \left(1 - \beta u^{*^2} - \alpha(\tfrac{du^*}{dt^*})^2\right), \tag{1.18}$$

where α and β are positive parameters. We replace (1.18) in (1.17) the equation (1.17) becomes

$$m\frac{d^2u^*}{dt^{*^2}} + ku^* = \mu\left(1 - \beta u^{*^2} - \alpha(\tfrac{du^*}{dt^*})^2\right)\frac{du^*}{dt^*}. \tag{1.19}$$

By introducing the following dimensionless variables

$$\begin{cases} u = \frac{u^*}{u_0^*}, \\ t = t^*\sqrt{\frac{k}{m}}, \end{cases}$$

where u_0^* is a characteristic displacement and $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency of the linear oscillator, (1.19) takes the form

$$\ddot{u} + u = \epsilon\left(1 - \beta {u_0^*}^2 u^2 - \frac{\alpha {u_0^*}^2 k}{m}\dot{u}\right)\dot{u}, \tag{1.20}$$

where $\epsilon = \dfrac{\mu}{\sqrt{km}}$. For u_0^* such as $\frac{\alpha {u_0^*}^2 k}{m} = b$ et $\beta {u_0^*}^2 = a$, we obtain the standard form of (1.19) called the equation of the generalized Van Der Pol oscillator

$$\ddot{u} + u = \epsilon\left(1 - au^2 - b\dot{u}^2\right)\dot{u}. \tag{1.21}$$

The generalized Van Der Pol oscillator is a continuous-time differentiable dynamical system with one degree of freedom that models physical phenomena in the biological, electronic, medical, musical fields, etc. For example for $a = 1$ and $b = 0$ we find

$$\ddot{u} + u - \epsilon(1 - u^2)\dot{u} = 0, \tag{1.22}$$

the simplest form of the equation of the oscillator of Van Der Pol.
If we write (1.22) as a first order system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -x + \epsilon(1 - x^2)y. \end{cases} \tag{1.23}$$

We find that there is no exact closed form solution. Numerical integration shows that the limit cycle is a closed curve enclosing the origin in the $x - y$ phase plane. From the fact that eqs.(1.23) are invariant under the transformation $x \to -x, y \to -y$, we may conclude that the curve representing the limit cycle is point symmetric about the origin.

Remark 1.6.1. *The differential equation*

$$\ddot{u} + u - \epsilon(1 - u^2)\dot{u} = \epsilon F \cos\omega t, \tag{1.24}$$

is called the forced Van Der Pol equation. It is a model for situations in which a system which is capable of self-oscillation is acted upon by another oscillator, in this case represented by the $\epsilon F \cos\omega t$ *term.*

Chapter 2

Approximate Solutions of Nonlinear Differential Equation in Their General Form

In this chapter, we dedicate our study on the approximate solutions of Van Der Pol equation in their general form [46]. First, we prove the approximate analytic solutions to this equation by different perturbation methods. Then we compare these approximations with each other and with the exact solution.

2.1 Approximate solutions methods for the nonlinear differential equation

In this section we recall the main lines of different perturbation methods: the simple perturbation method (SPM), the Lindstedt-Poincaré method (PLM), the mean method (AM) and the renormalization group method (RGM). Then we prove the approximate analytical solutions of the generalized Van Der Pol oscillator by these methods

2.1.1 Simple perturbation method (SPM)

Basic idea of the simple perturbation method

Consider the initial value problem

$$\ddot{x} = f\left(t, x, \epsilon\right), \quad \text{with} \;\; x\left(0\right) \;\; \text{given},$$

$t \geq 0$, $x \in D \subset \mathbb{R}^n$. If we can expend $f(t, x, \epsilon)$ in a Taylor series with respect to ϵ. Suppose that the approximate solution is written in the form

$$x\left(t, \epsilon\right) = x_0\left(t\right) + \epsilon x_1\left(t\right) + ... + o(\epsilon^n).$$

Example

We consider the differential equation of the Van Der Pol oscillator in their general form is

$$\ddot{x} + \epsilon\left(ax^2 + b\dot{x}^2 - 1\right)\dot{x} + x = 0, \quad x(0) = A \text{ and } \dot{x}(0) = 0.\, a, b \,\text{and}\, A \in \mathbb{R}. \tag{2.1}$$

Suppose that the approximate solution is

$$y_P(t, \epsilon) = x\left(t, \epsilon\right) = x_0\left(t\right) + \epsilon x_1\left(t\right) + \epsilon^2 x_2(t) + o(\epsilon^2). \tag{2.2}$$

To determine $x_0(t)$, $x_1(t)$ and $x_2(t)$, substituting (2.2) into (2.1), and calculating, we find

$$(\ddot{x_0}+x_0)+\epsilon(\ddot{x_1}+x_1-\dot{x_0}+a\dot{x_0}x_0^2+b\dot{x_0}^3)+\epsilon^2(\ddot{x_2}+x_2-\dot{x_1}+a\dot{x_1}x_0^2+2ax_0\dot{x_0}x_1+3b\dot{x_0}^2\dot{x_1})+o(\epsilon^2) = 0.$$

$$\ddot{x_0} + x_0 = 0, \tag{2.3}$$

$$\ddot{x_1} + x_1 = \dot{x_0} - a\dot{x_0}x_0^2 - b\dot{x_0}^3, \tag{2.4}$$

$$\ddot{x_2} + x_2 = \dot{x_1} - a\dot{x_1}x_0^2 - 2ax_0\dot{x_0}x_1 - 3b\dot{x_0}^2\dot{x_1}. \tag{2.5}$$

Then

$$x_0(t) = A\cos(t) \tag{2.6}$$

Substituting (2.6) into (2.4) we find

$$\ddot{x_1} + x_1 = -A\sin(t) + aA^3\sin(t)\cos^2(t) + bA^3\sin^3(t). \tag{2.7}$$

Then

$$x_0(t) = A\cos(t), \tag{2.8}$$

$$\ddot{x_1} + x_1 = \underbrace{A(\frac{A^2(a+3b)}{4} - 1)\sin(t)}_{f_1} + \underbrace{\frac{A^3(a-b)}{4}\sin(3t)}_{f_1}. \tag{2.9}$$

$$x_{1_{p1}} = \frac{A}{2}(1 - \frac{A^2(a+3b)}{4})t\cos(t), \tag{2.10}$$

$$x_{1_{p2}} = -\frac{A^3(a-b)}{32}\sin(3t). \tag{2.11}$$

Therefore we now have perturbation solution

$$x_1(t) = c_1\cos(t) + c_2\sin(t) + \frac{A}{2}(1 - \frac{A^2(a+3b)}{4})t\cos(t) - \frac{A^3(a-b)}{32}\sin(3t),$$

$$x_1(0) = 0 \quad \Rightarrow c_1 = 0.$$

$$x_1^{'}(t) = -c_1\sin(t) + c_2\cos(t) + \frac{A}{2}(1 - \frac{A^2(a+3b)}{4})\cos(t) - \frac{A}{2}(1 - \frac{A^2(a+3b)}{4})t\sin(t)$$
$$- \frac{3A^3(a-b)}{32}\cos(3t),$$

$$\dot{x}_1(0) = 0 \quad \Rightarrow \quad c_2 = \frac{A}{2}(\frac{A^2(a+3b)}{4} - 1) + \frac{3A^3(a-b)}{32}.$$

So

$$x_1(t) = [\frac{A}{2}(\frac{A^2(a+3b)}{4} - 1) + \frac{3A^3(a-b)}{32}]\sin(t) + \frac{A}{2}(1 - \frac{A^2(a+3b)}{4})t\cos(t) - \frac{A^3(a-b)}{32}\sin(3t).$$

Also substituting (2.6) into (2.5) we find

$$\begin{aligned}
\ddot{x_2} + x_2 &= \dot{x_1} - a\dot{x_1}x_0^2 - 2ax_0\dot{x_0}x_1 - 3b\dot{x_0}^2\dot{x_1}, \\
&= ([\frac{A}{2}(\frac{A^2(a+3b)}{4} - 1) + \frac{3A^3(a-b)}{32}]\sin(t) + \frac{A}{2}(1 - \frac{A^2(a+3b)}{4})t\cos(t) \\
&- \frac{A^3(a-b)}{32}\sin(3t))' - aA^2\cos^2(t)([\frac{A}{2}(\frac{A^2(a+3b)}{4} - 1) + \frac{3A^3(a-b)}{32}]\sin(t) \\
&+ \frac{A}{2}(1 - \frac{A^2(a+3b)}{4})t\cos(t) - \frac{A^3(a-b)}{32}\sin(3t))' + 2aA^2\sin(t)\cos(t)([\frac{A}{2}(\frac{A^2(a+3b)}{4} - 1) \\
&+ \frac{3A^3(a-b)}{32}]\sin(t) + \frac{A}{2}(1 - \frac{A^2(a+3b)}{4})t\cos(t) - \frac{A^3(a-b)}{32}\sin(3t)) \\
&- 3bA^2\sin(t)([\frac{A}{2}(\frac{A^2(a+3b)}{4} - 1) + \frac{3A^3(a-b)}{32}]\sin(t) + \frac{A}{2}(1 - \frac{A^2(a+3b)}{4})t\cos(t) \\
&- \frac{A^3(a-b)}{32}\sin(3t))', \\
&= \underbrace{(\frac{A^5}{128}[(a-9b)(a-b) - 9(a-b)^2 + 8a(a+3b) + 6(a-b)] - \frac{A^3}{32}[5a+3b])\cos(t)}_{f_1} \\
&+ \underbrace{(\frac{A^3}{2}(a+3b) - \frac{3A^5}{32}(a+3b)^2 - \frac{A}{2})t\sin(t)}_{f_2} \\
&+ \underbrace{(\frac{A^3}{32}[5a+3b] + \frac{A^5}{128}[6(a+3b)(a-b) - 3(a-b)(a-3b) - 8a(a+3b) - 6a(a-b)])\cos(3t)}_{f_3} \\
&+ \underbrace{\frac{A^5}{128}(5a-9b)(a-b)\cos(5t)}_{f_4} + \underbrace{(\frac{3A^3}{8}(a-b) - \frac{3A^5}{32}(a-b)(a+3b))t\sin(3t)}_{f_5}.
\end{aligned}$$

$$
\begin{aligned}
x_{2_{p_1}} &= (\frac{A^5}{256}[(a-9b)(a-b)-9(a-b)^2+8a(a+3b)+6(a-b)]-\frac{A^3}{64}[5a+3b])t\sin(t),\\
x_{2_{p_2}} &= (\frac{3A^5}{128}(a+3b)^2+\frac{A}{8}-\frac{A^3}{8}(a+3b))t^2\cos(t)\\
&+(\frac{A^3}{8}(a+3b)-\frac{3A^5}{128}(a+3b)^2-\frac{A}{8})t\sin(t),\\
x_{2_{p_3}} &= (-\frac{A^5}{1024}[6(a+3b)(a-b)-3(a-b)(a-3b)-8a(a+3b)-6a(a-b)]\\
&-\frac{A^3}{256}[5a+3b])\cos(3t),\\
x_{2_{p_4}} &= -\frac{A^5}{3072}(5a-9b)(a-b)\cos(5t),\\
x_{2_{p_5}} &= (\frac{9A^5}{1024}(a-b)(a+3b)-\frac{9A^3}{256}(a-b))\cos(3t)\\
&+(\frac{3A^5}{256}(a-b)(a+3b)-\frac{3A^3}{64}(a-b))t\sin(3t).
\end{aligned}
$$

$$
\begin{aligned}
x_2(t) &= c_1\cos(t)+c_2\sin(t)+(\frac{A^5}{256}[(a-9b)(a-b)-9(a-b)^2+8a(a+3b)+6(a-b)]\\
&-\frac{A^3}{64}[5a+3b])t\sin(t)\\
&+(\frac{3A^5}{128}(a+3b)^2+\frac{A}{8}-\frac{A^3}{8}(a+3b))t^2\cos(t)+(\frac{A^3}{8}(a+3b)-\frac{3A^5}{128}(a+3b)^2-\frac{A}{8})t\sin(t)\\
&+(-\frac{A^5}{1024}[6(a+3b)(a-b)-3(a-b)(a-3b)-8a(a+3b)-6a(a-b)]\\
&-\frac{A^3}{256}[5a+3b])\cos(3t)-\frac{A^5}{3072}(5a-9b)(a-b)\cos(5t)\\
&+(\frac{9A^5}{1024}(a-b)(a+3b)-\frac{9A^3}{256}(a-b))\cos(3t)+(\frac{3A^5}{256}(a-b)(a+3b)-\frac{3A^3}{64}(a-b))t\sin(3t),\\
&= c_1\cos(t)+c_2\sin(t)+(\frac{A^5}{256}[(a-9b)(a-b)-9(a-b)^2+2(a-9b)(a+3b)+6(a-b)]\\
&+\frac{3A^3}{64}[a+7b]-\frac{A}{8})t\sin(t)+(\frac{3A^5}{128}(a+3b)^2+\frac{A}{8}-\frac{A^3}{8}(a+3b))t^2\cos(t)\\
&+(\frac{A^5}{1024}[3(a+3b)(a-b)+3(a-b)(a-3b)+8a(a+3b)+6a(a-b)]\\
&-\frac{2A^3}{256}[(7a-3b)])\cos(3t)-\frac{A^5}{3072}(5a-9b)(a-b)\cos(5t)\\
&+(\frac{3A^5}{256}(a-b)(a+3b)-\frac{3A^3}{64}(a-b))t\sin(3t).
\end{aligned}
$$

Then

$$x_2(0) = c_1 + 0 + 0 + 0 + (\frac{A^5}{1024}[3(a+3b)(a-b) + 3(a-b)(a-3b) + 8a(a+3b) + 6a(a-b)] - \frac{2A^3}{256}[7a-3b]) - \frac{A^5}{3072}(5a-9b)(a-b) + 0 = 0,$$

$$c_1 = -\frac{A^5}{3072}[9(a+3b)(a-b) + 9(a-b)(a-3b) + 24a(a+3b) + 18a(a-b) - (5a-9b)(a-b)] + \frac{2A^3}{256}[7a-3b]).$$

$$\dot{x}_2(0) = 0 + c_2 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0 \;\Rightarrow\; c_2 = 0.$$

$$x_2(t) = (-\frac{A^5}{3072}[9(a+3b)(a-b) + 9(a-b)(a-3b) + 24a(a+3b) + 18a(a-b) - (5a-9b)(a-b)] + \frac{2A^3}{256}[7a-3b])\cos(t) + (\frac{A^5}{256}[(a-9b)(a-b) - 9(a-b)^2 + 2(a-9b)(a+3b) + 6(a-b)] + \frac{3A^3}{64}[7b+a] - \frac{A}{8})t\sin(t) + (\frac{3A^5}{128}(a+3b)^2 + \frac{A}{8} - \frac{A^3}{8}(a+3b))t^2\cos(t) + (-\frac{A^5}{1024}[-3(a+3b)(a-b) - 3(a-b)(a-3b) - 8a(a+3b) - 6a(a-b)] - \frac{A^3}{256}[2(7a-3b)])\cos(3t) - \frac{A^5}{3072}(5a-9b)(a-b)\cos(5t) + (\frac{3A^5}{256}(a-b)(a+3b) - \frac{3A^3}{64}(a-b))t\sin(3t).$$

Therefore we now have the final perturbation solution, which is

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + o(\epsilon^2).$$

$$
\begin{aligned}
&y_P(t,\epsilon) = A\cos(t) \\
&+\epsilon\left[(\frac{A^3}{32}[7a+9b]-\frac{A}{2})\sin(t)+(\frac{A}{2}-\frac{A^3(a+3b)}{8})t\cos(t)-\frac{A^3(a-b)}{32}\sin(3t)\right] \\
&+\epsilon^2((-\frac{A^5}{3072}[9(a+3b)(a-b)+9(a-b)(a-3b)+24a(a+3b)+18a(a-b) \\
&-(5a-9b)(a-b)]+\frac{2A^3}{256}[7a-3b])\cos(t)+(\frac{A^5}{256}[(a-9b)(a-b)-9(a-b)^2 \\
&+2(a-9b)(a+3b)+6(a-b)]+\frac{3A^3}{64}[7b+a]-\frac{A}{8})t\sin(t)+(\frac{3A^5}{128}(a+3b)^2 \\
&+\frac{A}{8}-\frac{A^3}{8}(a+3b))t^2\cos(t)+(\frac{A^5}{1024}[3(a+3b)(a-b)+3(a-b)(a-3b) \\
&+8a(a+3b)+6a(a-b)]-\frac{2A^3}{256}[7a-3b])\cos(3t)-\frac{A^5}{3072}(5a-9b)(a-b)\cos(5t) \\
&+(\frac{3A^5}{256}(a-b)(a+3b)-\frac{3A^3}{64}(a-b))t\sin(3t))+o(\epsilon^2).
\end{aligned}
$$

2.1.2 Lindstedt-Poincaré Method (LPM)

We present an approximation method, based on the expansion of a solution of a differential equation in a series in a small parameter. It is used to construct uniformly valid periodic solutions to second-order nonlinear differential equations in the form

$$\frac{d^2y}{dt^2}(t,\epsilon)+y(t,\epsilon)=\epsilon F\left(y(t,\epsilon),\frac{dy}{dt}(t,\epsilon)\right), \quad 0<\epsilon<<1, \tag{2.12}$$

with $y(0,\epsilon)=A$, $\frac{dy}{dt}(0,\epsilon)=0$, where $0<\epsilon<<1$ is a small positive parameter, and generally F is assumed to be an analytic function with respect to y and dy/dt. The starting point of the perturbation method is the assumption that a periodic solution of equation (2.12) can be written in the form

$$y\left(t,\epsilon\right)=\sum_{m=0}^{n}\epsilon^m y_m\left(t\right)+O\left(\epsilon^{n+1}\right). \tag{2.13}$$

The general procedure of the simple approximation is to substitute (2.13) into (2.12), develop in powers of ϵ, and put all coefficients of the powers of ϵ equal to zero. This gives a system of linear nonhomogeneous differential equations that we can solve recursively.

But the simple approximation takes us on a problem, if we need to calculate an analytical approximations of periodic solutions of nonlinear differential equations in the form given by (1). We illustrate this type of difficulty in the following example.

Example 2.1.1. *We apply the simple approximation to the following equation*

$$\frac{d^2y}{dt^2} + y + \epsilon y^3 = 0, 0 < \epsilon << 1, \tag{2.14}$$

with the initial values $y(0, \epsilon) = A$, $\frac{dy}{dt}(0, \epsilon) = 0$.
The approximate solution of (2.14) is $y(t, \epsilon) = y_0(t, 0) + \epsilon y_1(t, 0) + O(\epsilon^2)$. *After substituting and calculating, we find*

$$y\,(t, \epsilon) = A \cos\, t + \epsilon \left(\frac{A^3}{32}\right) \left[(\cos\, 3t - \cos\, t) - 12t\, \sin\, t\right]. \tag{2.15}$$

We remark that the terms $y_1(t, 0)$ is nonperiodic and unbounded as $t \to +\infty$. This leads to the notion of secular terms.

Secular Terms

The conservation of a finite numbers of terms on the right-side of expansion (2.15) determines a function that is not only nonperiodic, but also unbounded as $t \to +\infty$.

Definition 2.1.1. *Terms such as* $t^m \cos(pt)$ *or* $t^m \sin(nt)$ *where* $m, n \in \mathbb{N}^*, p \in \mathbb{N}$ *are called secular terms.*

These expressions appear because expansion (2.15) is not uniformly valid. The existence of such expressions destroys the periodicity of expansion (2.15) when only a finite number of terms is conserved. Therefore, to obtain a uniformly valid solution, we must look for an approximation that eliminates secular terms. A technique to avoid the presence of secular terms and allows for an approximation that is valid for all time has been developed by Lindstedt-Poincare.

Basic Idea of the Lindstedt-Poincaré method

The substance of this method is to introduce a new independent variable linearly linked to the old independent variable. This transformation completely eliminates the secular terms. The basic idea came from the astronomer Lindstedt, based on the change of variable $\theta = \omega t$ with $\omega_0 = \omega(0) = 1$, $\omega(\epsilon) \neq 1$, and both y and ω are expanded to powers of as follows

$$y(\theta, \epsilon) = y_0(\theta) + \epsilon y_1(\theta) + ... + \epsilon^n y_n(\theta) + ... + o(\epsilon^n), \tag{2.16}$$

$$\omega(\epsilon) = 1 + \epsilon\omega_1 + ... + \epsilon^n \omega_n + ... + o(\epsilon^n), \tag{2.17}$$

and we note that, in this step, ω_j are unknowns, we obtain them by elimination of the secular terms.

we introduce the following notations

$$\dot{y} \equiv \frac{dy}{d\theta}, \qquad \ddot{y} \equiv \frac{d^2y}{d\theta^2}. \tag{2.18}$$

$$F_y(y, \dot{y}) \equiv \frac{\partial F(y, \dot{y})}{\partial y}, \qquad F_{\dot{y}}(y, \dot{y}) \equiv \frac{\partial F(y, \dot{y})}{\partial \dot{y}}, \tag{2.19}$$

and (2.12) becomes

$$\omega^2 \ddot{y} + y = \epsilon F(\epsilon, \omega\dot{y}). \tag{2.20}$$

If the equations (2.16) and (2.17) are substituted in the equation (2.20) and the coefficients of the different powers of ϵ are equal to zero, we get

$$\ddot{y}_0 + y_0 = 0, \tag{2.21}$$

$$\ddot{y}_1 + y_1 = -2\omega_1\ddot{y}_0 + F(y_0, \dot{y}_0), \tag{2.22}$$

$$\ddot{y}_2 + y_2 = -2\omega_1\ddot{y}_1 - \left(\omega_1^2 + 2\omega_2\right)\ddot{y}_0 + F_y(y_0, \ddot{y}_0)\, y_1 + F_{\dot{y}}(y_0, \dot{y}_0)\left(\omega_1\dot{y}_0 + \dot{y}_1\right), \tag{2.23}$$

....

$$\ddot{y}_n + y_n = G_n\left(\dot{y}_0, \dot{y}_1, ..., \dot{y}_{n-1}; \dot{y}_0, \dot{y}_1, ..., \dot{y}_{n-1}\right). \tag{2.24}$$

If F is a polynomial function in y and dy/dt, then G is also a polynomial function with respect to its arguments.

To calculate an approximate periodic solutions of (2.20), we must solve (2.21), (2.22), (2.23) and (2.24).

Note here that the Lindstedt-Poincaré approximations are periodic due to the following proposition.

Proposition 2.1.1. *Let the equation*

$$\ddot{y} + y = G(\theta), \qquad y(0) = 0, \, \dot{y}(0) = 0. \tag{2.25}$$

Where $G(\theta) = -2\omega_1 \ddot{y}_0 + F(y_0, \dot{y}_0)$. *The solution to this problem is*

$$y(\theta) = \int_0^\theta \sin(\theta - \tau) G(\tau) \, d\tau. \tag{2.26}$$

Moreover, the equation (2.25) has a periodic solution $y_1(\theta)$ *if and only if*

$$\begin{cases} \int_0^{2\pi} F(A \cos\theta, -A \sin\theta) \sin\theta \, d\theta = 0, \\ 2\pi\omega_1 A + \int_0^{2\pi} F(A \cos\theta, -A \sin\theta) \cos\theta \, d\theta = 0. \end{cases}$$

Proof. See [44] page 3.

We know that the solution of (2.25) is $y(\theta) = C_1 \cos(\theta) + C_2 \sin(\theta) + y_p(\theta)$, such that $y_p(\theta) = C_1(\theta) \cos\theta + C_2(\theta) \sin\theta$. By variation of constants we find

$$\begin{cases} C_1'(\theta) \cos\theta + C_2'(\theta) \sin\theta = 0, \\ -C_1'(\theta) \sin\theta + C_2'(\theta) \cos\theta = G(\theta). \end{cases}$$

$$\Rightarrow \begin{cases} -C_1'(\theta) = -\sin\theta \, G(\theta) \Rightarrow C_1(\theta) = -\int_0^\theta \sin\tau \, G(\tau) \, d\tau, \quad C_1(0) = 0, \\ -C_2'(\theta) = \cos\theta \, G(\theta) \Rightarrow C_1(\theta) = \int_0^\theta \cos\tau \, G(\tau) \, d\tau, \quad C_2(0) = 0, \end{cases}$$

$$\begin{aligned} y_p(\theta) &= (-\int_0^\theta \sin\tau \, G(\tau) \, d\tau) \cos\theta + (\int_0^\theta \cos\tau \, G(\tau) \, d\tau) \sin\theta \\ &= \int_0^\theta (-\sin\tau \cos\theta + \cos\tau \sin\theta) G(\tau) \, d\tau, \end{aligned}$$

$$\Rightarrow y_p(\theta) = \int_0^\theta \sin(\theta - \tau) G(\tau) \, d\tau \Rightarrow y(\theta) = C_1 \cos\theta + C_2 \sin\theta + \int_0^\theta \sin(\theta - \tau) G(\tau) \, d\tau,$$

with the initial values $y(0) = 0, \, \dot{y}(0) = 0$ owe have $C_1 = C_2 = 0$. so we deduce that problem (2.25) admits (2.26)as a solution.

Moreover, (16) gives

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = -y_1 + G(\tau). \end{cases}$$

On the other hand, the condition of periodicity for the new variable θ can be expressed as

$$y(\theta) = y(\theta + 2\pi). \tag{2.27}$$

So the corresponding conditions for $y_n(\theta)$ are

$$y_n(\theta) = y_n(\theta + 2\pi), \quad n = 1, 2, ... \tag{2.28}$$

$$\Rightarrow \begin{cases} y_1(2\pi) = y_1(0) = 0, \\ y_2(2\pi) = y_2(0) = 0, \end{cases}$$

which yields to the periodicity condition

$$\int_{\theta}^{\theta+2\pi} \sin(\theta - \tau) G(\tau) d\tau = 0, \tag{2.29}$$

$$\Rightarrow \begin{cases} \int_0^{2\pi} \cos\theta G(\theta) d\theta = 0, \\ \int_0^{2\pi} \sin\theta G(\theta) d\theta = 0. \end{cases} \tag{2.30}$$

According to (2.22) we have $G(\theta) = -2\omega_1 \ddot{y_0} + F(y_0, \dot{y_0}), y_0 = A \cos\theta$.
we rewrite (2.30) as

$$\Rightarrow \begin{cases} \int_0^{2\pi} \cos\theta \left[2\omega_1 A \cos\theta + F(A \cos\theta, -A \sin\theta)\right] d\theta = 0, \\ \int_0^{2\pi} \sin\theta \left[2\omega_1 A \cos\theta + F(A \cos\theta, -A \sin\theta)\right] d\theta = 0, \end{cases} \tag{2.31}$$

$$\Rightarrow \begin{cases} 2\omega_1 A \int_0^{2\pi} \cos^2\theta d\theta + \int_0^{2\pi} \cos\theta F(A \cos\theta, -A \sin\theta) d\theta = 0, \\ 2\omega_1 A \int_0^{2\pi} \sin\theta \cos\theta d\theta + \int_0^{2\pi} \sin\theta F(A \cos\theta, -A \sin\theta) d\theta = 0, \end{cases}$$

which is required. □

Example

In order to apply the Lindstedt method we put $\theta = \omega t$, $y(t) = x(\theta)$, $\ddot{y} = \omega^2 \ddot{x}$ and $\dot{y} = w\dot{x}$ so (2.1) become

$$\omega^2 \ddot{x} - \epsilon(1 - ax^2 - b\dot{y}^2)\omega\dot{x} + x = 0 \quad x(0) = 1 \text{ et } \dot{x}(0) = 0. \tag{2.32}$$

We have got $\omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + o(\epsilon^2)$ and $x(\theta) = x_0(\theta) + \epsilon x_1(\theta) + \epsilon^2 x_2(\theta) + o(\epsilon^2)$ we get

$$
\begin{aligned}
&(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + o(\epsilon^2))(\ddot{x}_0 + \epsilon\ddot{x}_1 + \epsilon^2\ddot{x}_2 + o(\epsilon^2)) \\
&+ (x_0 + \epsilon x_1 + \epsilon^2 x_2 + o(\epsilon^2)) \\
&- \epsilon(1 - a(x_0 + \epsilon x_1 + \epsilon^2 x_2 + o(\epsilon^2))^2 - b(\dot{x}_0 + \epsilon\dot{x}_1 + \epsilon^2\dot{x}_2 + o(\epsilon^2))^2) \\
&(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + o(\epsilon^2))(\dot{x}_0 + \epsilon\dot{x}_1 + \epsilon^2\dot{x}_2 + O(\epsilon)) = 0, \\
&= \ddot{x}_0 + x_0 + \epsilon(\ddot{x}_1 + x_1 + 2\omega_1\ddot{x}_0 - \dot{x}_0 + ax_0^2\dot{x}_0 + b\dot{x}_0^3) \\
&+ \epsilon^2(\ddot{x}_2 + x_2 + 2\omega_1\ddot{x}_1 + (\omega_1^2 + 2\omega_2)\ddot{x}_0 + 2ax_0x_1\dot{x}_0 \\
&- \dot{x}_1 - \omega_1\dot{x}_0 + ax_0^2\dot{x}_1 + a\omega_1x_0^2\dot{x}_0 + 3b\dot{x}_0^2\dot{x}_1 + 3b\omega_1\dot{x}_0^3 + o(\epsilon^2) = 0.
\end{aligned}
$$

$$\ddot{x}_0 + x_0 = 0, \tag{2.33}$$

$$\ddot{x}_1 + x_1 = -2\omega_1\ddot{x}_0 + \dot{x}_0 - ax_0^2\dot{x}_0 - b\dot{x}_0^2, \tag{2.34}$$

$$\ddot{x}_2 + x_2 = -2\omega_1\ddot{x}_1 - (\omega_1^2 + 2\omega_2)\ddot{x}_0 - 2(ax_0x_1 + b\dot{x}_0\dot{x}_1)\dot{x}_0 + \dot{x}_1 \tag{2.35}$$

$$+ \omega_1\dot{x}_0 - ax_0^2\dot{x}_1 - a\omega_1x_0^2\dot{x}_0 - b\dot{x}_0^2\dot{x}_1 - 3b\omega_1\dot{x}_0^3. \tag{2.36}$$

The solution to Eq.(2.33) is

$$x_0 = A\cos\theta. \tag{2.37}$$

Substituting Eq.(2.37) into Eq.(2.34) and simplifying the resulting expression gives

$$
\begin{aligned}
\ddot{x}_1 + x_1 &= 2A\omega_1\cos\theta - A(1 - aA^2\cos^2\theta - bA^2\sin^2\theta)\sin\theta, \\
&= 2A\omega_1\cos\theta - A(1 - aA^2(\frac{1}{2} + \frac{1}{2}\cos 2\theta) - bA^2(\frac{1}{2} + \frac{1}{2}\cos 2\theta))\sin\theta, \\
&= 2A\omega_1\cos\theta - A(\sin\theta - aA^2(\frac{1}{2}\sin\theta + \frac{1}{2}\cos 2\theta\sin\theta) - bA^2(\frac{1}{2}\sin\theta - \frac{1}{2}\cos 2\theta\sin\theta)), \\
&= 2A\omega_1\cos\theta - A(\sin\theta - \frac{aA^2}{2}\sin\theta - \frac{aA^2}{2}\cos 2\theta\sin\theta - \frac{bA^2}{2}\sin\theta + \frac{bA^2}{2}\cos 2\theta\sin\theta), \\
&= 2A\omega_1\cos\theta - A(\sin\theta - \frac{aA^2}{2}\sin\theta - \frac{aA^2}{2}(\frac{1}{2}(\sin 3\theta - \sin(\theta)) - \frac{bA^2}{2}\sin\theta \\
&+ \frac{bA^2}{2}(\frac{1}{2}(\sin 3\theta - \sin\theta))), \\
&= 2A\omega_1\cos\theta - A(1 - \frac{aA^2}{4} - 3\frac{bA^2}{4})\sin\theta + \frac{A^3}{4}(a - b)\sin 3\theta, \\
&= 2A\omega_1\cos\theta - A(1 - \frac{A^2}{4}(a + 3b))\sin\theta + \frac{A^3}{4}(a - b)\sin 3\theta.
\end{aligned}
$$

Elimination of the secular terms gives

$$\omega_1 = 0, \qquad A = \frac{2}{\sqrt{a+3b}}, \tag{2.38}$$

and

$$\ddot{x}_1 + x_1 = \frac{A^3}{4}(a-b)\sin 3\theta, \quad x_1(0) = 0, \quad \dot{x}_1(0) = 0.$$

$$\Rightarrow \quad \ddot{x}_1 + x_1 = \underbrace{\frac{1}{a+3b}\frac{2}{\sqrt{a+3b}}(a-b)\sin 3\theta}_{f}, \quad x_1(0) = 0, \quad \dot{x}_1(0) = 0.$$

$$x_{1_p} = -\frac{1}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)\sin 3\theta.$$

Therefore we now have perturbation solution

$$x_1(t) = c_1\cos(\theta) + c_2\sin(\theta) - \frac{1}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)\sin(3\theta),$$

$$x_1(0) = 0 \quad \Rightarrow c_1 = 0.$$

$$x_1^{'}(t) = -c_1\sin(\theta) + c_2\cos(\theta) - \frac{3}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)\cos(3\theta),$$

$$x_1^{'}(0) = 0 \quad \Rightarrow \quad c_2 = \frac{3}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b).$$

The solution to this equation is

$$x_1(\theta) = \frac{3}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)\sin(\theta) - \frac{1}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)\sin(3\theta).$$

$$x_1(\theta) = \frac{1}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)(3\sin(\theta) - \sin(3\theta)). \tag{2.39}$$

Substituting Eq.(2.37), Eq.(2.38) and Eq.(2.39) into Eq.(2.36) gives

$$\begin{aligned}
\ddot{x}_2 + x_2 =& \frac{4}{\sqrt{a+3b}}\omega_2\cos(\theta) + 2(\frac{2a}{\sqrt{a+3b}}\cos(\theta)[\frac{1}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)(3\sin(\theta)-\sin(3\theta))] \\
&- \frac{2b}{\sqrt{a+3b}}\sin(\theta)[\frac{1}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)(3\cos(\theta)-3\cos(3\theta))])\frac{2}{\sqrt{a+3b}}\sin(\theta) \\
&+ \frac{1}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)(3\cos(\theta)-3\cos(3\theta)) \\
&- \frac{4a}{a+3b}\cos^2(\theta)[\frac{1}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)(3\cos(\theta)-3\cos(3\theta))] \\
&- \frac{4a}{a+3b}\sin^2(\theta)[\frac{1}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)(3\cos(\theta)-3\cos(3\theta))], \\
=& \frac{4}{\sqrt{a+3b}}\omega_2\cos(\theta) + \frac{2a(a-b)}{(a+3b)^2\sqrt{a+3b}}[3\sin^2(\theta)\cos(\theta)-\sin(3\theta)\cos(\theta)\sin(\theta)] \\
&- \frac{3b(a-b)}{(a+3b)^2\sqrt{a+3b}}[3\sin^2(\theta)\cos(\theta)-3\cos(3\theta)\sin^2(\theta)] \\
&+ \frac{(a-b)}{4(a+3b)\sqrt{a+3b}}(3\cos(\theta)-3\cos(3\theta)) \\
&- \frac{a(a-b)}{(a+3b)^2\sqrt{a+3b}}[3\cos^2(\theta)\cos(\theta)-3\cos(3\theta)\cos^2(\theta)], \\
=& (\frac{4}{\sqrt{a+3b}}\omega_2 - \frac{a(a-b)}{2(a+3b)^2\sqrt{a+3b}} + \frac{3(a-b)}{4(a+3b)\sqrt{a+3b}} - \frac{9b(a-b)}{2(a+3b)^2\sqrt{a+3b}})\cos(\theta) \\
&+ (\frac{-3a(a-b)}{4(a+3b)^2\sqrt{a+3b}} - \frac{3(a-b)}{4(a+3b)\sqrt{a+3b}} + \frac{9b(a-b)}{(a+3b)^2\sqrt{a+3b}})\cos(3\theta) \\
&+ (\frac{5a(a-b)}{4(a+3b)^2\sqrt{a+3b}} - \frac{9b(a-b)}{2(a+3b)^2\sqrt{a+3b}})\cos(5\theta).
\end{aligned}$$

The requirement of no secular terms gives the following results

$$\omega_2 = \frac{a(a-b)}{8(a+3b)^2} - \frac{3(a-b)}{16(a+3b)} + \frac{9b(a-b)}{8(a+3b)^2}. \tag{2.40}$$

$$\begin{aligned}
\ddot{x}_2 + x_2 =& \underbrace{(\frac{-3a(a-b)}{4(a+3b)^2\sqrt{a+3b}} - \frac{3(a-b)}{4(a+3b)\sqrt{a+3b}} + \frac{9b(a-b)}{(a+3b)^2\sqrt{a+3b}})}_{f_1}\cos(3\theta) \\
&+ \underbrace{(\frac{5a(a-b)}{4(a+3b)^2\sqrt{a+3b}} - \frac{9b(a-b)}{2(a+3b)^2\sqrt{a+3b}})\cos(5\theta)}_{f_2}, \quad x_1(0) = 1 \text{ and } \dot{x_1}(0) = 0.
\end{aligned}$$

$$x_{2_{p_1}} = (\frac{3a(a-b)}{32(a+3b)^2\sqrt{a+3b}} + \frac{3(a-b)}{32(a+3b)\sqrt{a+3b}} - \frac{9b(a-b)}{8(a+3b)^2\sqrt{a+3b}})\cos(3\theta),$$
$$x_{2_{p_2}} = (-\frac{5a(a-b)}{96(a+3b)^2\sqrt{a+3b}} + \frac{9b(a-b)}{48(a+3b)^2\sqrt{a+3b}})\cos(5\theta).$$

$$\begin{aligned} x_2 &= c_1\cos(\theta) + c_2\sin(\theta) \\ &+ (\frac{3a(a-b)}{32(a+3b)^2\sqrt{a+3b}} + \frac{3(a-b)}{32(a+3b)\sqrt{a+3b}} - \frac{9b(a-b)}{8(a+3b)^2\sqrt{a+3b}})\cos(3\theta) \\ &+ (-\frac{5a(a-b)}{96(a+3b)^2\sqrt{a+3b}} + \frac{9b(a-b)}{48(a+3b)^2\sqrt{a+3b}})\cos(5\theta), \end{aligned}$$

$$\begin{aligned} & x_2(0) = 0 \Rightarrow \\ & c_1 = \frac{-3a(a-b)}{32(a+3b)^2\sqrt{a+3b}} - \frac{3(a-b)}{32(a+3b)\sqrt{a+3b}} + \frac{9b(a-b)}{8(a+3b)^2\sqrt{a+3b}} \\ & \quad + \frac{5a(a-b)}{96(a+3b)^2\sqrt{a+3b}} - \frac{9b(a-b)}{48(a+3b)^2\sqrt{a+3b}}, \\ & \quad = (\frac{-4a(a-b)}{96(a+3b)^2\sqrt{a+3b}} - \frac{3(a-b)}{32(a+3b)\sqrt{a+3b}} + \frac{45b(a-b)}{48(a+3b)^2\sqrt{a+3b}}). \end{aligned}$$

$$x_2'(0) = 0 \Rightarrow c_2 = 0.$$

The solution to Eq.(2.36) is

$$\begin{aligned} x_2 &= (\frac{-4a(a-b)}{96(a+3b)^2\sqrt{a+3b}} - \frac{3(a-b)}{32(a+3b)\sqrt{a+3b}} + \frac{45b(a-b)}{48(a+3b)^2\sqrt{a+3b}})\cos(\theta) \\ &+ (\frac{3a(a-b)}{32(a+3b)^2\sqrt{a+3b}} + \frac{3(a-b)}{32(a+3b)\sqrt{a+3b}} - \frac{9b(a-b)}{8(a+3b)^2\sqrt{a+3b}})\cos(3\theta) \\ &+ (-\frac{5a(a-b)}{96(a+3b)^2\sqrt{a+3b}} + \frac{9b(a-b)}{48(a+3b)^2\sqrt{a+3a}})\cos(5\theta). \end{aligned}$$

$$\begin{aligned} x(\theta,\epsilon) &= \frac{2}{\sqrt{a+3b}}\cos(\theta) + \epsilon(\frac{1}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)(3\sin(\theta) - \sin(3\theta))) \\ &+ \epsilon^2((\frac{-4a(a-b)}{96(a+3b)^2\sqrt{a+3b}} - \frac{3(a-b)}{32(a+3b)\sqrt{a+3a}} + \frac{45b(a-b)}{48(a+3b)^2\sqrt{a+3b}})\cos(\theta) \\ &+ (\frac{3a(a-b)}{32(a+3b)^2\sqrt{a+3b}} + \frac{3(a-b)}{32(a+3b)\sqrt{a+3b}} - \frac{9b(a-b)}{8(a+3b)2\sqrt{a+3b}})\cos(3\theta) \\ &+ (-\frac{5a(a-b)}{96(a+3b)^2\sqrt{a+3b}} + \frac{9b(a-b)}{48(a+3b)^2\sqrt{a+3b}})\cos(5\theta)) + o(\epsilon^2). \end{aligned}$$

With $y_L(t,\epsilon) = x(\theta,\epsilon)$

$$\omega = 1 - [\frac{a(a-b)}{8(a+3b)^2} - \frac{3(a-b)}{16(a+3b)} + \frac{9b(a-b)}{8(a+3b)^2}]\epsilon^2 + o(\epsilon^2).$$

The equation $\ddot{y}_1 + y_1 = 2A\omega_1\cos\theta - A(1 - aA^2\cos^2\theta - bA^2\sin^2\theta)\sin\theta$ has a periodic solution $y_1(\theta)$ if and only if

$$\begin{cases} \int_0^{2\pi} -A(1 - aA^2\cos^2\theta - bA^2\sin^2\theta)\sin^2\theta\, d\theta = 0, \\ 2\pi\omega_1 A + \int_0^{2\pi} -A(1 - aA^2\cos^2\theta - bA^2\sin^2\theta)\sin\theta\cos\theta\, d\theta = 0. \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1}{4}a\pi A^3 + \frac{3}{4}b\pi A^3 - A\pi = 0, \\ 2\omega_1 A\pi = 0. \end{cases}$$

$$\Rightarrow \begin{cases} A = \frac{2}{\sqrt{a+3b}}, \\ \omega_1 = 0. \end{cases}$$

2.1.3 Averaging method (AM)

Here we present a third method for determining the perturbation solutions of the differential equation for a nonlinear oscillator.

The main advantage of the method is that it not only allows to determine the periodic motions at steady state, but also allows to determine the transient behavior of the motion at a periodic solution.

Indication

This method applies to equations of the form

$$\ddot{x} + \omega^2 x + \epsilon F(x,\dot{x}) = 0. \tag{2.41}$$

For $\epsilon = 0$ the general solution is

$$x(t) = A\sin(\omega t + \Phi), \text{ ou } A \text{ and } \Phi \text{ are any constants.} \tag{2.42}$$

For $\epsilon \neq 0$ small, Krylov and Boyolinbov posed the solution

$$x(t) = A(t)\,\sin(\omega t + \Phi(t)), \tag{2.43}$$

$$\dot{x}(t) = A(t)\,\omega\,\cos(\omega t + \Phi(t)). \tag{2.44}$$

Let $y = \dot{x}$ in (2.41), we find

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\omega^2 x - \epsilon F(x, y). \end{cases} \tag{2.45}$$

$$\Rightarrow A(t)\,\omega\,\cos(\omega t + \Phi(t)) = \dot{A}(t)\,\sin(\omega t + \Phi(t)) + A(t)\left(\omega + \dot{\Phi}(t)\right)\cos(\omega t + \Phi(t))$$

$$\Rightarrow \dot{A}(t)\,\sin(\omega t + \Phi(t)) + A(t)\,\dot{\Phi}(t)\,\cos(\omega t + \Phi(t)) = 0. \tag{2.46}$$

Similarly we have

$$\dot{A}(t)\,\omega\,\cos(\omega t + \Phi(t)) - A(t)\,\omega\left(\omega + \dot{\Phi}(t)\right)\sin(\omega t + \Phi(t)) =$$

$$-A(t)\,\omega^2\,sin(\omega t + \Phi(t)) - \epsilon f(A(t)\,\sin(\omega t + \Phi(t)), A(t)\,\omega\,\cos(\omega t + \Phi(t))),$$

$$\Rightarrow \dot{A}(t)\,\omega\,\cos(\omega t + \Phi(t)) - A(t)\,\omega\dot{\Phi}(t)\,\sin(\omega t + \Phi(t)) =$$

$$\epsilon f(A(t)\,\sin(\omega t + \Phi(t)), A(t)\,\omega\,\cos(\omega t + \Phi(t))). \tag{2.47}$$

By solving (2.46) and (2.47) with respect to $\dot{A}$ and $\dot{\Phi}$, we obtain (by the method of cramer).

$$\begin{cases} \dot{A}(t) = -\dfrac{\epsilon}{\omega}\,\cos(\omega t + \Phi(t))\, f(A(t)\,\sin(\omega t + \Phi(t)), A(t)\,\omega\,\cos(\omega t + \Phi(t))), \\ \dot{\Phi}(t) = -\dfrac{\epsilon}{A(t)\,\omega}\,\sin(\omega t + \Phi(t))\, f(A(t)\,\sin(\omega t + \Phi(t)), A(t)\,\omega\,\cos(\omega t + \Phi(t))). \end{cases} \tag{2.48}$$

Note that $\dot{A}(t)$ and $\dot{\Phi}(t)$ are propositional to $\epsilon \Rightarrow A(t)$ et $\Phi(t)$ vary slowly with time. The Krylov and Boyolinbov approximation is to replace $A(t)$ et $\Phi(t)$ in (2.48) by their average values over a period $T = \frac{2\pi}{\omega}$, (i.e $\frac{1}{T}\int_0^T f(t)\,dt$).

$A(t)$ et $\Phi(t)$ are considered constants by taking the average. This process is known as the averaging method.

$$\begin{cases} \dot{A} = -\frac{\epsilon}{2\pi} \int_0^{\frac{2\pi}{\omega}} \cos(\omega t + \Phi(t)) f(A(t) \sin(\omega t + \Phi(t)), A(t)\omega \cos(\omega t + \Phi(t))) dt, \\ \dot{\Phi} = \frac{\epsilon}{2\pi A} \int_0^{\frac{2\pi}{\omega}} sin(\omega t + \Phi(t)) f(A(t) \sin(\omega t + \Phi(t)), A(t)\omega \cos(\omega t + \Phi(t))) dt. \end{cases}$$

Let $\theta = \omega t + \Phi$, we find

$$\begin{cases} \dot{A} = -\frac{\epsilon}{2\pi} \int_0^{2\pi} \cos(\theta) f(A \sin\theta, A\omega \cos\theta) d\theta, \\ \dot{\Phi} = \frac{\epsilon}{2\pi A\omega} \int_0^{2\pi} \sin(\theta) f(A\, sin\theta, A\omega \cos\theta) d\theta. \end{cases} \tag{2.49}$$

Once these integrals have been found, we will have to solve differential equations for $A(t)$ and $\Phi(t)$

Remark 2.1.1. *We recall that*

$$I_{m,n} = \int_0^{2\pi} \sin^m x \cos^n x dx = 0, \ if m, n \ are \ odd \ integer.$$

and further

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}, \qquad I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}.$$

We arrive at $I_{0,0} = 2\pi$.

Example

We solving the equation (2.1) by averaging method.
we have $\omega = 1,\ F(x, \dot{x}) = \left(ax^2 + b\dot{x}^2 - 1\right)\dot{x}$.
The system (2.49) is written

$$\begin{cases} \dot{A} = \frac{\epsilon}{2\pi} \int_0^{2\pi} \cos\theta [A \cos\theta \left(1 - aA^2 \sin^2\theta - bA^2 \cos^2(\theta)\right)] d\theta, \\ \dot{\Phi} = -\frac{\epsilon}{2\pi A} \int_0^{2\pi} \sin\theta [A \cos\theta \left(1 - aA^2 \sin^2\theta - bA^2 \cos^2(\theta)\right)] d\theta. \end{cases} \tag{2.50}$$

$$\begin{aligned}
\dot{A} &= \frac{dA}{dt} \\
&= \frac{\epsilon A}{2\pi}\left(A^n I_{0,n+1} - aA^n + 2I_{2,n+1} - bA^n + 2I_{0,n+1}\right) \\
&= \frac{\epsilon A}{2\pi}\left(\frac{1}{2}2\pi - aA^2\frac{1}{4}\frac{1}{2}2\pi - 3bA^2\frac{1}{4}\frac{1}{2}2\pi\right) \\
&= \frac{\epsilon A}{8}\left(4 - A^2(a+3b)\right).
\end{aligned}$$

Then

$$\begin{aligned}
&\int \frac{dA}{A\left(4 - A^2(a+3b)\right)} = \frac{\epsilon}{8}\int dt \\
&\Rightarrow 8\int\left(\frac{1}{4A} + \frac{1}{8\left(2 - A\sqrt{(a+3b)}\right)} - \frac{1}{8\left(2 + A\sqrt{(a+3b)}\right)}\right) dA = \epsilon t + \ln(c) \\
&\Rightarrow 2\ln A - \ln\left(2 - A\sqrt{(a+3b)}\right) - \ln\left(2 + A\sqrt{(a+3b)}\right) = \epsilon t + \ln(c) \\
&\Rightarrow \ln\left(\frac{A^2}{c\left(4 - A^2(a+3b)\right)}\right) = \epsilon t. \text{ Let's pose } A(0) = A_0 \\
&\Rightarrow c = \frac{A_0^2}{4 - A_0^2(a+3b)}, \text{ On the other hand } \ln\left(\frac{A^2}{c\left(4 - A^2(a+3b)\right)}\right) = \epsilon t \\
&\Rightarrow \frac{A^2}{c\left(4 - A^2(a+3b)\right)} = e^{\epsilon t} \\
&\Rightarrow A^2 = c\left(4 - A^2(a+3b)\right)e^{\epsilon t} \\
&\Rightarrow A^2(t) = \frac{\frac{4A_0^2}{4 - A_0^2(a+3b)}e^{\epsilon t}}{1 + \frac{A_0^2}{4 - A_0^2(a+3b)}e^{\epsilon t}} \\
&\Rightarrow A(t) = \frac{2}{\left[\left(\frac{4}{A_0^2} - (a+3b)\right)e^{-\epsilon t} + 1\right]^{\frac{1}{2}}}.
\end{aligned}$$

On the other hand

$$\begin{aligned}\dot{\Phi}(t) &= -\frac{\epsilon}{2\pi A}\int_0^{2\pi}\sin\theta[A\cos\theta\left(1-A^2\sin^2\theta\right)]d\theta \\ &= -\frac{\epsilon}{2\pi}\int_0^{2\pi}[\sin\theta\cos\theta - A^2\cos\theta\sin^3\theta]d\theta \\ &= -\frac{\epsilon}{2\pi}I_{1,1} + \frac{A^2\epsilon}{2\pi}I_{3,1} \\ &= 0+0 \\ &= 0\end{aligned}$$

$$\Rightarrow \quad \dot{\Phi} = \frac{d\Phi}{dt} = 0$$
$$\Phi = \Phi_0$$

The averaging approximate solution is

$$x(t,\epsilon) = \frac{2}{\left[\left(\frac{4}{A_0^2}-(a+3b)\right)e^{-\epsilon t}+1\right]^{\frac{1}{2}}}\sin(t+\Phi_0),$$

where $y_A(t,\epsilon) = x(t,\epsilon)$.

2.1.4 Renormalization group method (RGM)

The renormalization group method is a method for finding the approximate solution of ordinary differential equations in $(\mathbb{R}^n)$ of the form

$$\dot{x} = Fx + g(x,t,\epsilon), \tag{2.51}$$

$$\dot{x} = Fx + \epsilon g_1(x,t) + \epsilon^2 g_2(x,t) + \cdots, x \in \mathbb{R}^n, \tag{2.52}$$

where ϵ is an infinitely small positive parameter. For this system, we assume that

1. F be a square matrix $n * n$, diagonalizable with imaginary eigenvalues,
2. The function $g(x,t,\epsilon)$ is sufficiently differentiable in t, x and ϵ, the power series expansion of ϵ is given by the equation (2.52),
3. Each $g_i(x,t)$ is periodic in $t \in \mathbb{R}$ and polynomial in x.

At first we apply the simple development method. We replace x in (2.52) by

$$x(t) = x_0 + \epsilon x_1 + \epsilon^2 \dot{x_2} + \cdots . \tag{2.53}$$

We have

$\dot{x_0} + \epsilon \dot{x_1} + \epsilon^2 \dot{x_2} + \cdots = F(x_0 + \epsilon x_1 + \epsilon^2 \dot{x_2} + \cdots) + \epsilon g_1(x_0 + \epsilon x_1 + \epsilon^2 \dot{x_2} + \cdots, t) + \epsilon^2 g_2(x_0 + \epsilon x_1 + \epsilon^2 \dot{x_2} + \cdots, t) + \epsilon^3 g_3(x_0 + \epsilon x_1 + \epsilon^2 \dot{x_2} + \cdots, t) \cdots, x \in \mathbb{R}^n$.

Such that

$$F(x_0 + \epsilon x_1 + \epsilon^2 \dot{x_2} + \cdots) = F(x_0) + \epsilon F(x_1) + \epsilon^2 F(x_2),$$

$$g_1(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots, t) = g_1(x_0, t) + \epsilon \frac{\partial g_1}{\partial x}(x_0, t) x_1 + \epsilon^2 \frac{1}{2} \frac{\partial^2 g_1}{\partial x^2}(x_0, t) x_1^2 + \frac{\partial g_1}{\partial x}(x_0, t) x_2 + \cdots,$$

$$g_2(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots, t) = g_2(x_0, t) + \epsilon \frac{\partial g_2}{\partial x}(x_0, t) x_1 + \cdots,$$

$$g_3(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots, t) = g_3(x_0, t) + \cdots .$$

After development and identification of the coefficients of ϵ we have

$$\dot{x}_0 = F x_0, \tag{2.54}$$

$$\dot{x}_1 = F x_1 + G_1(t, x_0), \tag{2.55}$$

$$\dot{x}_i = F x_i + G_i(t, x_0, x_1, .., x_{i-1}), \tag{2.56}$$

where the homogeneous term G_i is a regular function of, x_0, x_{i-1} . For now G_1, G_2, G_3, G_4 are given

$$G_1(t, x_0) = g_1(x_0, t), \tag{2.57}$$

$$G_2(t, x_0, x_1) = \frac{\partial g_1}{\partial x}(x_0, t) x_1 + g_2(x_0, t), \tag{2.58}$$

$$G_3(t, x_0, x_1, x_2) = \frac{1}{2} \frac{\partial^2 g_1}{\partial x^2}(x_0, t) x_1^2 + \frac{\partial g_1}{\partial x}(x_0, t) x_2 + \frac{\partial g_2}{\partial x}(x_0, t) x_1 + g_3(x_0, t). \tag{2.59}$$

We have the following relation

$$\frac{\partial G_i}{\partial x_j} = \frac{\partial g_{i-1}}{\partial x_{j-1}} = \frac{\partial g_{i-j}}{\partial x_0}, i > j \geq 0. \tag{2.60}$$

In the following expressions, we put $e^{Ft} = X(t)$ and we define the functions R_1 and h_t^i on $\mathbb{R}$ by

$$R_1(y) \doteqdot \lim_{t \longrightarrow \infty} \frac{1}{t} \int_{t_0}^{t} [X^{-1}(s) G_1(s, X(s)y)] ds, \tag{2.61}$$

$$h_t^1(y) \doteqdot X(t)\int_{t_0}^{t}[X^{-1}(s)G_1(s,X(s)y)-R_1(y)]ds, \quad (2.62)$$

$$R_i(y) \doteqdot \lim_{t\longrightarrow\infty}\frac{1}{t}\int_{t_0}^{t}[X^{-1}(s)G_1(s,X(s)y,h_s^1(y),...,h_s^{i-1}(y))-X^{-1}(s)\sum_{k=1}^{i-1}(Dh_t^k)_yR_{i-k}(y)]ds, i=2,3.. \quad (2.63)$$

$$h_t^i(y) \doteqdot X(t)\int_{t_0}^{t}[X^{-1}(s)G_1(s,X(s)y,h_s^1(y),...,h_s^{i-1}(y))-X^{-1}(s)\sum_{k=1}^{i-1}(Dh_t^k)_yR_{i-k}(y)-R_i(y)]ds. \quad (2.64)$$

Proposition 2.1.2. *Let $x_0 = X(t)y$ be the solution of the equation (2.54) whose initial condition is $y \in \mathbb{R}$. Then for an arbitrary time $\zeta \in \mathbb{R}$ and $i = 1,2,3,\dots,$ the curve x_i defined by*

$$x_i = x_i(t,\zeta,y) = h_t^i(y) + p_1^i(t,y)(t-\zeta) + p_2^i(t,y)(t-\zeta)^2 + ... + p_i^i(t,y)(t-\zeta)^i, \quad (2.65)$$

is the solution of the equation (2.56) where the functions $p_1^i,...,p_i^i$ are given by

$$p_1^i(t,y) = X(t)R_i(y) + \sum_{k=1}^{i-1}(Dh_t^k)_yR_{i-k}(y), \quad (2.66)$$

$$p_j^i(t,y) = \frac{1}{j}\sum_{k=1(j=2,3,...,i-1);}^{i-1}\frac{\partial p_{j-1}^k}{\partial y}(t,y)R_{i-k}(y), \quad (2.67)$$

$$p_j^i(t,y) = \frac{1}{j}\sum_{k=1}^{i-1}\frac{\partial p_{j-1}^k}{\partial y}(t,y)R_{i-k}(y) = \frac{1}{j}\frac{\partial p_{j-1}^{i-1}}{\partial y}(t,y)R_1(y), \quad (2.68)$$

$$p_j^i(t,y) = 0, (j > i). \quad (2.69)$$

The h_t^i functions are uniformly bounded at t. the solution of the equation (2.52) at 1st order is given by

$$x(t,\zeta,y) = x_0 + x_1 = X(t)y + \epsilon(h_t^1(y) + X(t)R_1(y)(t-\zeta) + O(\epsilon^2). \quad (2.70)$$

It is the solution obtained by simple expansion, it diverges for long times, hence the need for its renormalization. It must not depend on ζ i.e. $(\frac{\partial x}{\partial \zeta} = 0)$, then

$$0 = X(t)\frac{dy(t)}{d\zeta} + \epsilon\frac{\partial h_t^1}{\partial y}\frac{\partial y(t)}{\partial \zeta} - \epsilon X(t)R_1(y). \quad (2.71)$$

We verify that (2.71) admits for solution

$$\frac{dy(t)}{d\zeta} = \epsilon R_1(y) + O(\epsilon^2). \quad (2.72)$$

Let $y(t)$ be a solution of (2.72), then the solution of (2.52) sought by the renormalization group method is given by

$$x(t,t,y) = X(t)y(t) + \epsilon h_t^1(y(t)) + O(\epsilon^2). \tag{2.73}$$

The equation (2.72) is the equation of the renormalization group of (2.52). The calculation for a higher order is done in the same way and we obtain the equation of the renormalization group of order m as follows

$$\frac{dy(t)}{d\zeta} = \epsilon R_1(y) + \epsilon^2 R_2(y) + ... + \epsilon^m R_m(y), y \in \mathbb{R}^n. \tag{2.74}$$

Using $h_t^1, ..., h_t^m$ defined above by (2.61) and (2.62), we define the Transformation of the renormalization group of order m (TGR)
$\alpha_t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $y \longmapsto \alpha_t(y)$ by

$$\alpha_t(y) = X(t)y + \epsilon h_t^1(y) + ... + \epsilon^m h_t^m(y). \tag{2.75}$$

Example

HINVI A. L in [5] solves the equation (2.1) by renormalization group method. for $y = \dot{x} = \frac{dx(t)}{dt}$, the equation (2.1) becomes

$$\dot{y} + x - \epsilon(1 - ax^2 - by^2)y = 0 \tag{2.76}$$

and for $x = (z + \bar{z})$ and $y = i(z - \bar{z})$ the equation (2.76) becomes

$$i(\dot{z} - \dot{\bar{z}}) + (z + \bar{z}) - \epsilon[1 - a(z + \bar{z})^2 - b(i(z - \bar{z}))^2]i(z - \bar{z}) = 0.$$

Then

$$\begin{cases} \dot{z} = iz + \frac{\epsilon}{2}[(z - \bar{z}) - a(z + \bar{z})^2(z - \bar{z}) + b(z - \bar{z})^3] \\ \dot{\bar{z}} = -i\bar{z} - \frac{\epsilon}{2}[(z - \bar{z}) - a(z + \bar{z})^2(z - \bar{z}) + b(z - \bar{z})^3] \end{cases} \tag{2.77}$$

It verifies the hypotheses (1-3) with

$$F = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

The two equations of the system being identical, the problem amounts to solving one of them. With

$$z(t) = z_0 + \epsilon z_1 + \cdots . \tag{2.78}$$

Then

$$\dot{z_0} + \epsilon\dot{z_1} = i(z_0 + \epsilon z_1) + \frac{\epsilon}{2}[(z_0 + \epsilon z_1 - (\bar{z}_0 - \epsilon\bar{z}_1) \\ - a(z_0 + \epsilon z_1 + (\bar{z}_0 - \epsilon\bar{z}_1))^2(z_0 + \epsilon z_1 - (\bar{z}_0 - \epsilon\bar{z}_1)) + b(z_0 + \epsilon z_1 - (\bar{z}_0 - \epsilon\bar{z}_1))^3].$$

$$\dot{z_0} = iz_0, \tag{2.79}$$

$$\dot{z_1} = iz_1 + G_1(t, z_0). \tag{2.80}$$

Where

$$G_1(t, z_0) = \frac{1}{2}[1 - a(z_0 + \bar{z}_0)^2 + b(z_0 - \bar{z}_0)^2](z_0 - \bar{z}_0).$$

To zero order we have

$$z_0 = qe^{it} = qZ(t), \tag{2.81}$$

with q the integration constant of (2.79).

The equations (2.61) and (2.62) respectively give $R_1(q), h_t^1(q)$ as follows

$$
\begin{aligned}
R_1(q) &= \lim_{t\longrightarrow\infty}\frac{1}{t}\int_0^t [Z(s)^{-1}G_1(s,Z(s)q)]ds,\\
&= \frac{1}{2}\lim_{t\longrightarrow\infty}\frac{1}{t}\int_0^t [e^{-is}[(qe^{is}-\bar{q}e^{-is})-a(qe^{is}+\bar{q}e^{-is})^2(qe^{is}-\bar{q}e^{-is})\\
&+b(qe^{is}-\bar{q}e^{-is})^2(qe^{is}-\bar{q}e^{-is})]]ds,\\
&= \frac{1}{2}\lim_{t\longrightarrow\infty}\frac{1}{t}\int_0^t [e^{-is}[(qe^{is}-\bar{q}e^{-is})-a(q^3e^{3is}-q^2\bar{q}e^{is}+q\bar{q}^2e^{-is}-\bar{q}^3e^{-3is}+2q^2\bar{q}e^{is}\\
&-2q\bar{q}^2e^{-is})+b(q^3e^{3is}-q^2\bar{q}e^{is}+q\bar{q}^2e^{-is}-\bar{q}^3e^{-3is}-2q^2\bar{q}e^{is}+2q\bar{q}^2e^{-is})]]ds,\\
&= \frac{1}{2}\lim_{t\longrightarrow\infty}\frac{1}{t}\int_0^t [(q-\bar{q}e^{-2is})-a(q^3e^{2is}-\bar{q}^3e^{-4is}+q^2\bar{q}-q\bar{q}^2e^{-2is})\\
&+b(q^3e^{2is}-\bar{q}^3e^{-4is}-3q^2\bar{q}+3q\bar{q}^2e^{-2is})]]ds,\\
&= \frac{1}{2}\lim_{t\longrightarrow\infty}\frac{1}{t}([qs+\frac{1}{2i}\bar{q}e^{-2is}]_0^t-a([\frac{q^3}{2i}e^{2is}]_0^t+[\frac{\bar{q}^3}{4i}e^{-4is}]_0^t+[q^2\bar{q}s]_0^t+\frac{q\bar{q}^2}{2i}e^{-2is}]_0^t)\\
&+b([\frac{q^3}{2i}e^{2is}]_0^t+[\frac{\bar{q}^3}{4i}e^{-4is}]_0^t-3[q^2\bar{q}s]_0^t-3\frac{q\bar{q}^2}{2i}e^{-2is}]_0^t)),\\
&= \frac{1}{2}\lim_{t\longrightarrow\infty}(q\frac{t}{t}+\frac{1}{2it}\bar{q}e^{-2it}-\frac{1}{2it}\bar{q}-a(\frac{q^3}{2it}e^{2it}-\frac{q^3}{2it}+\frac{\bar{q}^3}{4it}e^{-4it}-\frac{\bar{q}^3}{4it}+q^2\bar{q}\frac{t}{t}+\frac{q\bar{q}^2}{2it}e^{-2it}\\
&-\frac{q\bar{q}^2}{2it})+b(\frac{q^3}{2it}e^{2it}-\frac{q^3}{2it}+\frac{\bar{q}^3}{4it}e^{-4it}-\frac{\bar{q}^3}{4it}-3q^2\bar{q}\frac{t}{t}-3\frac{q\bar{q}^2}{2it}e^{-2it}+3\frac{q\bar{q}^2}{2it}),\\
&= \frac{1}{2}(q-aq^2\bar{q}-a\lim_{t\longrightarrow\infty}\frac{q^3}{2it}e^{2it})+b\lim_{t\longrightarrow\infty}\frac{q^3}{2it}e^{2it}-3bq^2\bar{q}).
\end{aligned}
$$

Then

$$R_1(q) = \frac{1}{2}[q(1-(a+3b)|q|^2).$$

$$
\begin{aligned}
h_1^t(q) &= Z(t)\int_0^t [Z(s)^{-1}G(s,Z(s)q)-R_1(q)]ds,\\
&= e^{it}(\frac{1}{2}qt+\frac{1}{2}\frac{1}{2i}\bar{q}e^{-2it}-\frac{1}{2}\frac{1}{2i}\bar{q}-a\frac{1}{2}\frac{q^3}{2i}e^{2it}+a\frac{1}{2}\frac{q^3}{2i}-a\frac{1}{2}\frac{\bar{q}^3}{4i}e^{-4it}+a\frac{1}{2}\frac{\bar{q}^3}{4i}-a\frac{1}{2}q^2\bar{q}t\\
&-a\frac{1}{2}\frac{q\bar{q}^2}{2i}e^{-2it}+a\frac{1}{2}\frac{q\bar{q}^2}{2i}+b\frac{1}{2}\frac{q^3}{2i}e^{2it}-b\frac{1}{2}\frac{q^3}{2i}+b\frac{1}{2}\frac{\bar{q}^3}{4i}e^{-4it}-b\frac{1}{2}\frac{\bar{q}^3}{4i}\\
&-3b\frac{1}{2}q^2\bar{q}t-3b\frac{1}{2}\frac{q\bar{q}^2}{2i}e^{-2it}+3b\frac{1}{2}\frac{q\bar{q}^2}{2i}-\frac{q}{2}t+a\frac{q^2\bar{q}}{2}t+3b\frac{q^2\bar{q}}{2}t),\\
&= e^{it}(\frac{1}{4i}\bar{q}e^{-2it}-\frac{1}{4i}\bar{q}-a\frac{q^3}{4i}e^{2it}+a\frac{q^3}{4i}-a\frac{\bar{q}^3}{8i}e^{-4it}+a\frac{\bar{q}^3}{8i}\\
&-a\frac{q\bar{q}^2}{4i}e^{-2it}+a\frac{q\bar{q}^2}{4i}+b\frac{q^3}{4i}e^{2it}-b\frac{q^3}{4i}+b\frac{\bar{q}^3}{8i}e^{-4it}-b\frac{\bar{q}^3}{8i}\\
&-3b\frac{q\bar{q}^2}{4i}e^{-2it}+3b\frac{q\bar{q}^2}{4i})
\end{aligned}
$$

$$h_t^1(q) = \frac{i}{4}[(a-b)(q^3e^{3it} + \frac{1}{2}\bar{q}^3e^{-3it}) + ((a+3b)q\bar{q}^2 - \bar{q})e^{-it},$$

with $i^2 = -1$ and C an integration constant which will be taken equal to zero in the following for reasons of simplification.

By the previous proposition and the above results, we have

$$z(t,\zeta,q) = Z(t)q + \epsilon(h_1^t(q) + Z(t)R_1(q)(t-\zeta)) + O(\epsilon^2), \tag{2.82}$$

which diverges for t long because of the last term.

Using the notion of renormalization of the integration constant $\frac{\partial z(t,\zeta,q)}{\partial \zeta} \mid_{\zeta=0} = 0$ mentioned in the previous section we have

$$\begin{cases} z(t,q) = Z(t)q + \epsilon h_t^1(q) + O(\epsilon^2), \\ \frac{dq}{d\zeta} = \epsilon R_1(q) + O(\epsilon^2). \end{cases} \tag{2.83}$$

The first equation of (2.83) is the solution sought by the renormalization group method and the last is the corresponding renormalization group equation.

Putting $q = re^{i\theta(\zeta)}$ with $x = (z + \bar{z})$ and $y = i(z - \bar{z})$ we find

$$\begin{aligned}
z(t,q) &= Z(t)q + \epsilon h_t^1(q) + O(\epsilon^2),\\
&= re^{i\theta(\zeta)}e^{it} + \epsilon(\frac{i}{4}[(a-b)(r^3e^{3i\theta(\zeta)}e^{3it} + \frac{1}{2}\bar{r}^3e^{-3i\theta(\zeta)}e^{-3it})\\
&+ ((a+3b)r^3e^{i\theta(\zeta)}e^{-2i\theta(\zeta)} - re^{-i\theta(\zeta)})e^{-it}]),\\
&= re^{i(\theta(\zeta)+t)} + \epsilon(\frac{i}{4}[(a-b)(r^3e^{3i(\theta(\zeta)+t)} + \frac{1}{2}r^3e^{-3i(\theta(\zeta)+t)})\\
&+ (a+3b)r^3e^{-i(\theta(\zeta)+t)} - re^{-i(\theta(\zeta)+t)}]),\\
&= r\cos(\theta(\zeta)+t) + ir\sin(\theta(\zeta)+t) + \epsilon(\frac{i}{4}[(a-b)((r^3\cos 3(\theta(\zeta)+t)\\
&+ ir^3\sin 3(\theta(\zeta)+t)) + \frac{1}{2}(r^3\cos 3(\theta(\zeta)+t) - ir^3\sin 3(\theta(\zeta)+t)))\\
&+ (a+3b)(r^3\cos(\theta(\zeta)+t) - r^3\sin(\theta(\zeta)+t))\\
&- (r\cos(\theta(\zeta)+t) - ir\sin(\theta(\zeta)+t))]).\\
\bar{z}(t,q) &= \bar{Z}(t)\bar{q} - \epsilon\bar{h}_t^1(q) + O(\epsilon^2),\\
&= r\cos(\theta(\zeta)+t) - ir\sin(\theta(\zeta)+t) - \epsilon(\frac{i}{4}[(a-b)((r^3\cos 3(\theta(\zeta)+t)\\
&- ir^3\sin 3(\theta(\zeta)+t)) - \frac{1}{2}(r^3\cos 3(\theta(\zeta)+t) + ir^3\sin 3(\theta(\zeta)+t)))\\
&- (a+3b)(r^3\cos(\theta(\zeta)+t) + ir^3\sin(\theta(\zeta)+t))\\
&+ (r\cos(\theta(\zeta)+t) + ir\sin(\theta(\zeta)+t))])\\
x &= z + \bar{z},\\
&= r\cos(\theta(\zeta)+t) + ir\sin(\theta(\zeta)+t) + \epsilon(\frac{i}{4}[(a-b)((r^3\cos 3(\theta(\zeta)+t)\\
&+ ir^3\sin 3(\theta(\zeta)+t)) + \frac{1}{2}(r^3\cos 3(\theta(\zeta)+t) - ir^3\sin 3(\theta(\zeta)+t)))\\
&+ (a+3b)(r^3\cos(\theta(\zeta)+t) - ir^3\sin(\theta(\zeta)+t)) - (r\cos(\theta(\zeta)+t) - ir\sin(\theta(\zeta)+t))])\\
&+ r\cos(\theta(\zeta)+t) - ir\sin(\theta(\zeta)+t) - \epsilon(\frac{i}{4}[(a-b)((r^3\cos 3(\theta(\zeta)+t)\\
&- ir^3\sin 3(\theta(\zeta)+t)) - \frac{1}{2}(r^3\cos 3(\theta(\zeta)+t) + ir^3\sin 3(\theta(\zeta)+t)))\\
&- (a+3b)(r^3\cos(\theta(\zeta)+t) + ir^3\sin(\theta(\zeta)+t))\\
&+ (r\cos(\theta(\zeta)+t) + ir\sin(\theta(\zeta)+t))]),
\end{aligned}$$

$$x = 2r\cos(\theta(\zeta)+t) - \frac{r\epsilon}{2}\sin(\theta(\zeta)+t) + \frac{\epsilon}{2}[\left(\tfrac{b-a}{2}\right) r^3\sin 3(\theta(\zeta)+t) + (a+3b)r^3\sin(\theta(\zeta)+t)]. + O(\epsilon^2). \tag{2.84}$$

The equation for the renormalization group transforms into

$$\begin{cases} z(t,q) = Z(t)q + \epsilon h_t^1(q) + O(\epsilon^2), \\ \frac{dq}{d\zeta} = \epsilon R_1(q) + O(\epsilon^2). \end{cases} \Rightarrow \begin{cases} \frac{dq}{d\zeta} = \epsilon R_1(q) + O(\epsilon^2), \\ q = re^{i\theta(\zeta)}. \end{cases}$$

$$\Rightarrow \begin{cases} \frac{dre^{i\theta(\zeta)}}{d\zeta} = \epsilon R_1(re^{i\theta(\zeta)}) + O(\epsilon^2), \\ q = re^{i\theta(\zeta)}. \end{cases}$$

$$\Rightarrow \begin{cases} \frac{dre^{i\theta(\zeta)}}{d\zeta} = \epsilon \frac{1}{2} re^{i\theta(\zeta)}(1-(a+3b)|re^{i\theta(\zeta)}|^2) + O(\epsilon^2), \\ q = re^{i\theta(\zeta)}. \end{cases}$$

$$\Rightarrow \begin{cases} \frac{dr}{d\zeta} e^{i\theta(\zeta)} + ire^{i\theta(\zeta)} \underbrace{\frac{d\theta(\zeta)}{d\zeta}}_{=0} = \frac{\epsilon r}{2} e^{i\theta(\zeta)}(1-(a+3b)r^2) + O(\epsilon^2), \\ \frac{d\theta(\zeta)}{d\zeta} = 0. \end{cases}$$

$$\Rightarrow \begin{cases} \frac{dr}{d\zeta} = \frac{\epsilon r}{2}(1-(a+3b)r^2) \\ \frac{d\theta(\zeta)}{d\zeta} = 0 \end{cases} \tag{2.85}$$

It is easy to prove that EGR has a stable periodic orbit (the limit cycle) of radius $r_s = \sqrt{\frac{1}{a+3b}}$ with $a + 3b > 0$.

2.2 Comparison of approximate solutions

In this section we analyze the approximate solutions of (2.1) are obtained by the four numerical methods (SPM, LPM, AM and RGM) [46].

We compare $y_E(t, 0)$ the exact solution, $y_P(t, \epsilon)$ the approximate solution with simple perturbation method, $y_L(\theta, \epsilon)$ the approximate solution with Lindestedt method and $y_A(t, \epsilon)$ the approximate solution with Averaging method of Ven Der Pol equation.

The figures from 2.1 to 2.4 give the comparison of the approximate solutions in order ϵ, obtained by the three methods (SPM, LM and AM). In order ϵ^2, we have the figures from 2.5 to 2.8 obtained by the four methods (SPM, LM, AM and RGM).

2.2.1 Comparison of approximate solutions to order ϵ^2

In this part, we compare approximate solutions to order ϵ^2. where $g(t)$ is the Taylor series expansion of $y_L((1+[\frac{a(a-b)}{8(a+3b)^2}-\frac{3(a-b)}{16(a+3b)}+\frac{9b(a-b)}{8(a+3b)^2}]\epsilon^2)t,\epsilon)$ in the order ϵ^2 in the neighbourhood of $\epsilon=0$.

$$
\begin{aligned}
g(t) = {} & \frac{2}{\sqrt{a+3b}}\cos(t)+\epsilon(\frac{1}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)(3\sin(t)-\sin(3t))) \\
& +\epsilon^2((\frac{-4a(a-b)}{96(a+3b)^2\sqrt{a+3b}}-\frac{3(a-b)}{32(a+3b)\sqrt{a+3b}} \\
& +\frac{45b(a-b)}{48(a+3b)^2\sqrt{a+3b}})\cos(t)+(\frac{3a(a-b)}{32(a+3b)^2\sqrt{a+3b}} \\
& +\frac{3(a-b)}{32(a+3b)\sqrt{a+3b}}-\frac{9b(a-b)}{8(a+3b)^2\sqrt{a+3b}})\cos(3t) \\
& +(-\frac{5a(a-b)}{96(a+3b)^2\sqrt{a+3b}}+\frac{9b(a-b)}{48(a+3b)^2\sqrt{a+3b}})\cos(5t) \\
& -[\frac{a(a-b)}{2(a+3b)^2\sqrt{a+3b}}+\frac{3(a-b)}{4(a+3b)\sqrt{a+3b}}+\frac{9b(a-b)}{2(a+3b)^2\sqrt{a+3b}}]t\sin(t)).
\end{aligned}
$$

And $h(t)$ is the Taylor series expansion of $y_A(t,\epsilon)$ in the order ϵ^2 in the neighbourhood of $\epsilon=0$.

$$
h(t)=(A+(\frac{A}{2}-\frac{A^3(a+3b)}{8})\epsilon t+(\frac{3A^5}{128}(a+3b)^2+\frac{A}{8}-\frac{A^3}{8}(a+3b))\cos(t).
$$

$y_E(t,0)=A\cos(t)$.

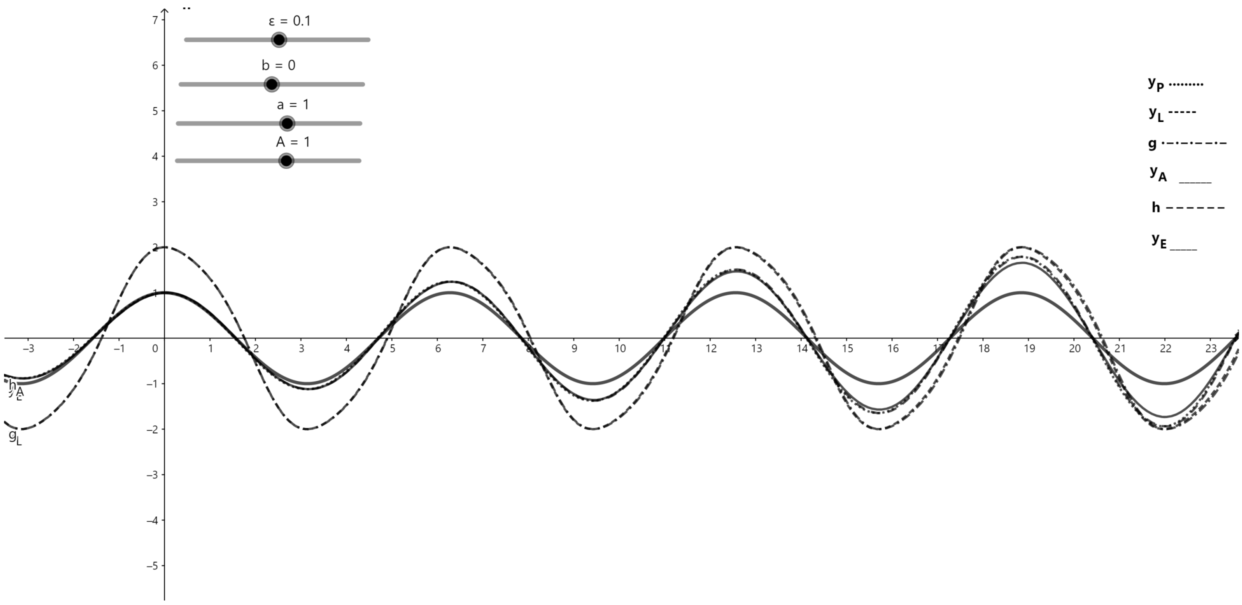


Figure 2.1: Comparison of the SPM solution, LPM solution and AM solution for $\epsilon = 0.1$ and $A = 1$.

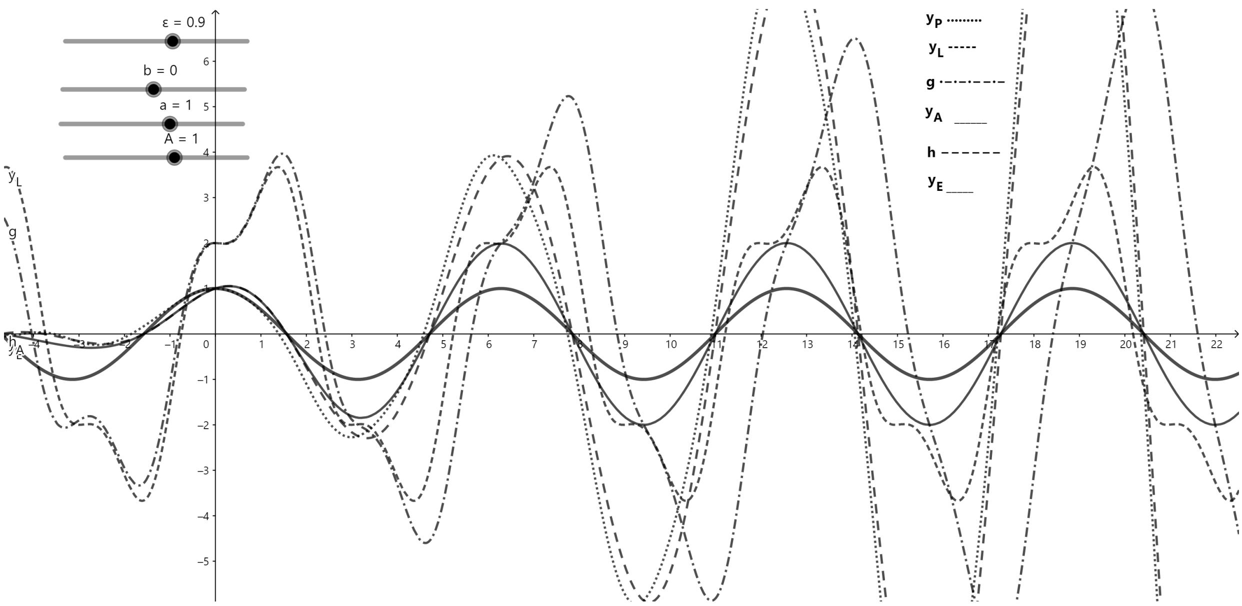


Figure 2.2: Comparison of the SPM solution, LPM solution and AM solution for $\epsilon = 0.9$ and $A = 1$.

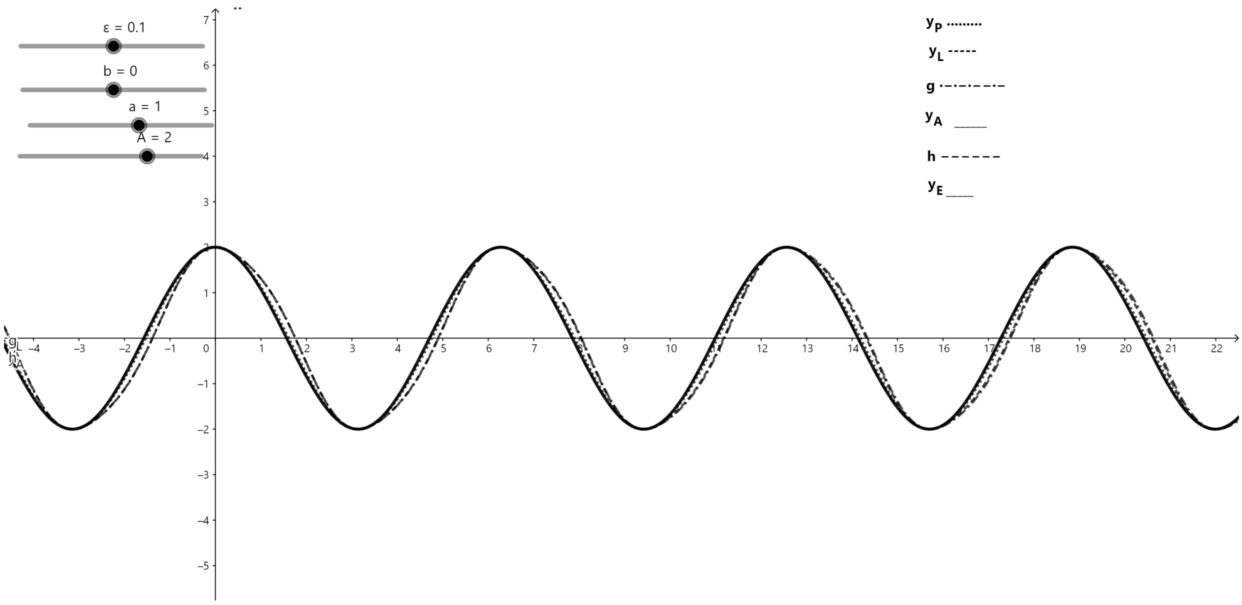


Figure 2.3: Comparison of the SPM solution, LPM solution and AM solution for $\epsilon = 0.1$ and $A = 2$.

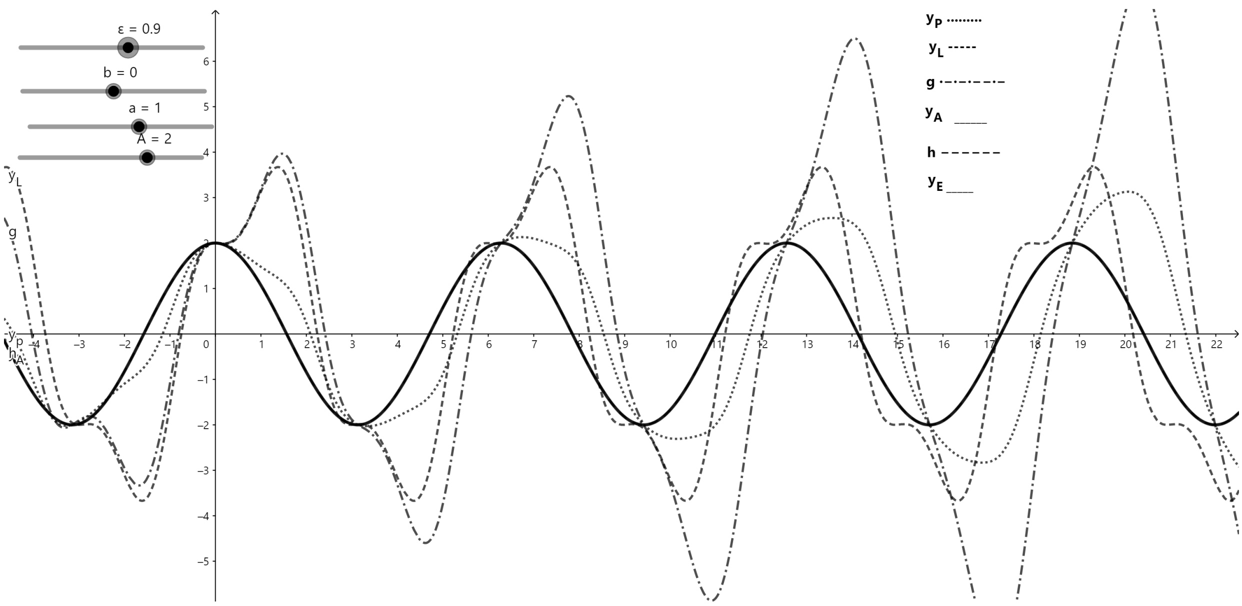


Figure 2.4: Comparison of the SPM solution, LPM solution and AM solution for $\epsilon = 0.9$ and $A = 2$.

2.2.2 Comparison of approximate solutions to order ϵ

To compare the approximate solutions of the first order, we used another approximate solution, which is the solution by the renormalization group method, and denotes it by $y_R(t, \epsilon)$.

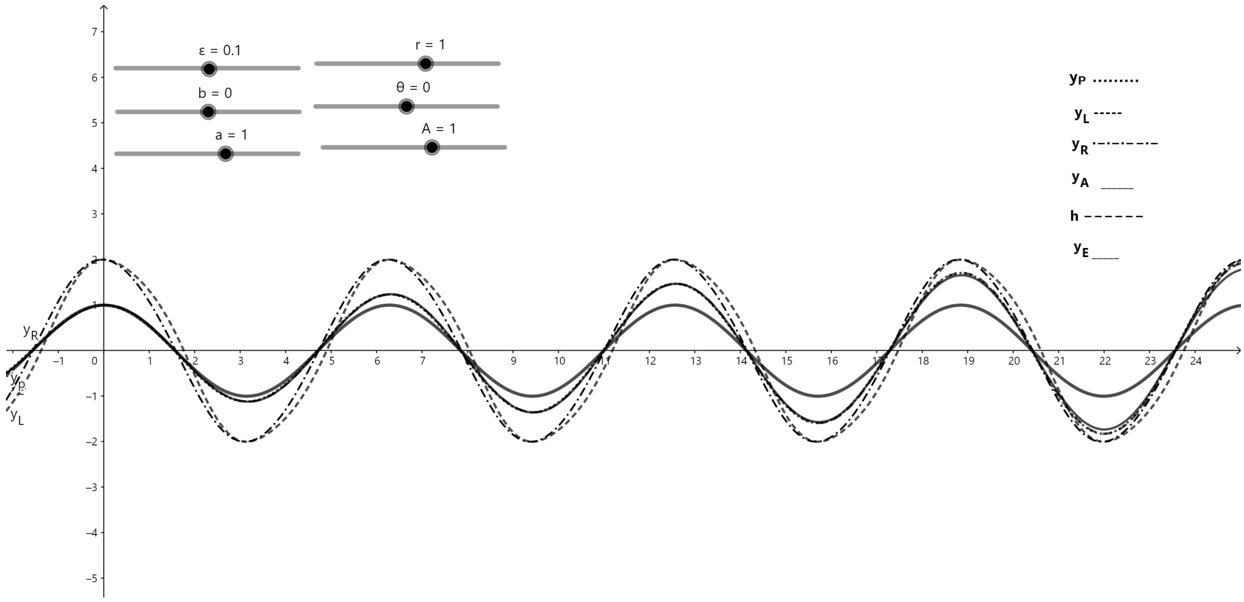


Figure 2.5: Comparison of the SPM solution, LPM solution, AM and solution RGM solution for $\epsilon = 0.1$, $r = 1$ and $A = 1$.

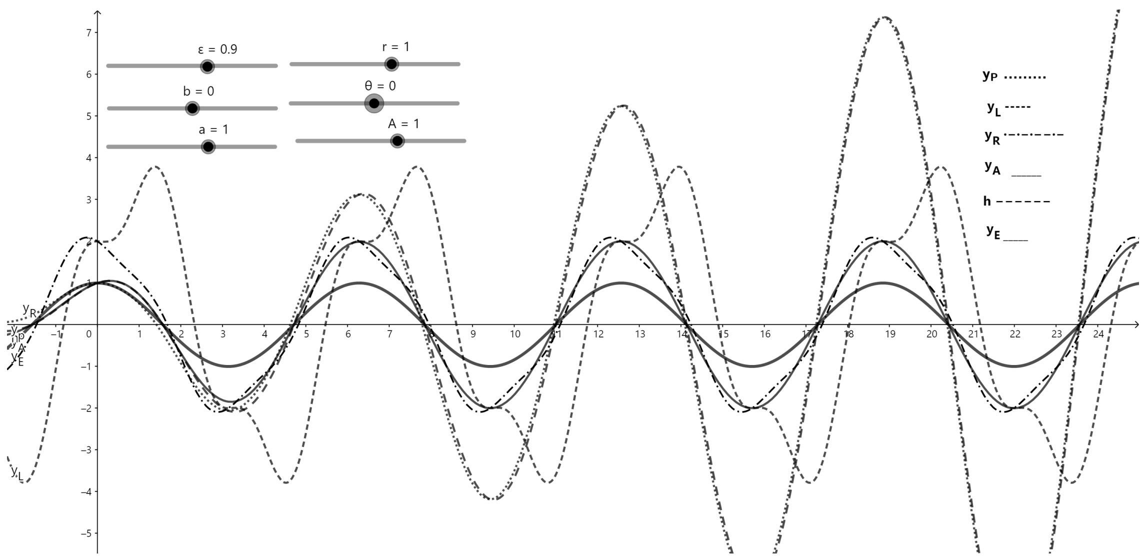


Figure 2.6: Comparison of the SPM solution, LPM solution, AM solution and RGM solution for $\epsilon = 0.9$, $r = 1$ and $A = 1$.

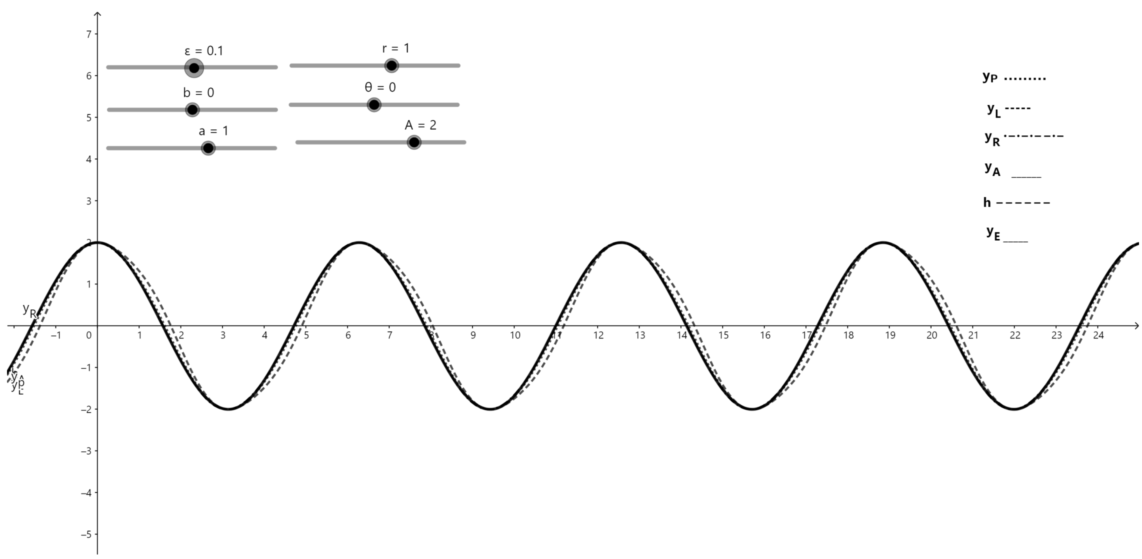


Figure 2.7: Comparison of the SPM solution, LPM solution, AM and solution RGM solution for $\epsilon = 0.1$, $r = 1$ and $A = 2$.

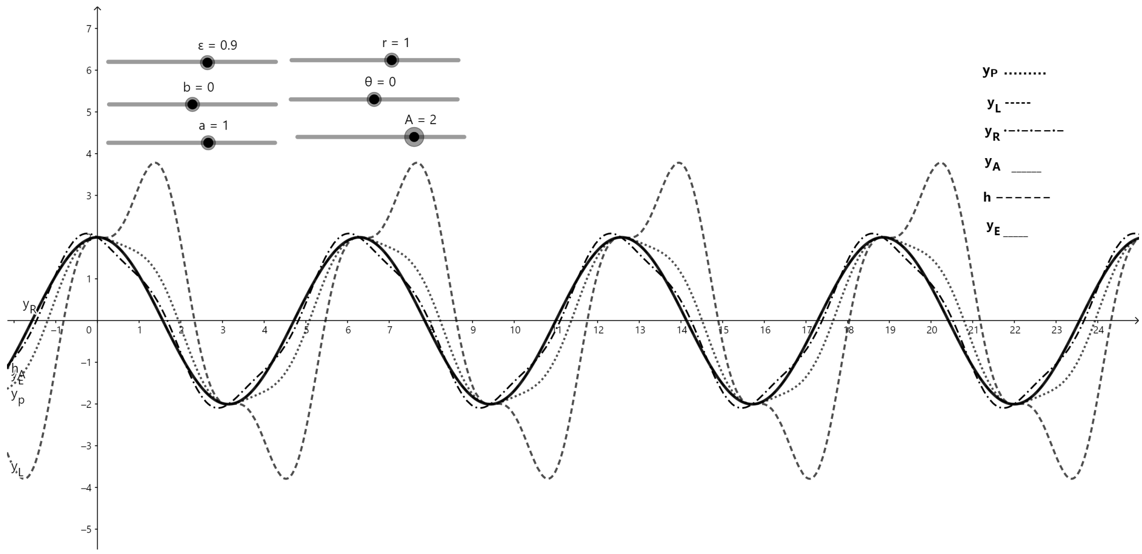


Figure 2.8: Comparison of the SPM solution, LPM solution, AM solution and RGM solution for $\epsilon = 0.9$, $r = 1$ and $A = 2$.

2.3 Discussion of results

Our study consists of two parts

Part 1: $A = 1$, and ϵ equals 0.1 or 0.9.

Part 2: $A = 2$, and ϵ equals 0.1 or 0.9.

In Part 1, when $A = 1$ and $\epsilon = 0.1$, Figure 2.1 shows analytic approximate solutions. According to the obtained results, we found that in the domain $t \in [0, 2]$ the solutions closest to the exact solution y_E are y_P, y_A and h, because the value of ϵ is very close to zero. But the solutions y_L and g move away from it, Because the amplitude of these solutions is equal to the value of 2. We also found that the solutions y_P, y_A, h and $y_E(t, 0)$ are identical and the same for the solutions y_L and g. But when $t \to +\infty$ we found that there is a convergence difference where the solutions closest to the exact solution become y_L, y_A and g while y_P and h are moving away from it. Also, we note that the solutions y_P and h are congruent along the range of t.

And when $\epsilon = 0.9$ we notice from the figure 2.2 that when $t \in [0, 1]$ is the closest solution to the real solution y_E is y_P. Rather, it corresponds to it, as we find that there is a correspondence between y_L with g and y_A with h, because ϵ is very close to 1. But when $t \to +\infty$ behaves differently, the solution becomes the closest to the real solution y_E is y_A and we notice that there is a divergence between y_L and g as well as y_A and h but that h is close very of y_P.

In Part 2, when $A = 2$ and $\epsilon = 0.1$ we see from Figure 2.3 that all solutions are identical at the vertices. As far as the vertices are concerned, we notice a difference, as the solutions y_L and g are far from the other solutions, because the value of ϵ is very close to zero and that the amplitudes of all solutions are equal.
When $\epsilon = 0.9$ from figure 2.4 we notice that there is a match between solutions y_A and h with y_E, Because ϵ is very close to 1.

Figures 2.5, 2.6, 2.7 and 2.8 show the same behavior as the solutions with Figures 2.1, 2.2, 2.3 and 2.4, respectively. But we added another solution to the study, which is y_R. Where in Figure 2.5, when A=1 $\epsilon = 0.1$, we find that y_R is closer to the solution y_L along the t range, because the amplitudes of these solutions are equal and equal to the value of 2.
Figure 2.6 shows when $\epsilon = 0.9$ y_R is divergent from the rest of the solutions, and this is when $t \to 0$, because ϵ is very close to 1. As for $t \to +\infty$, we find that y_R approximates y_A.
Figure 2.7 shows that when $A = 2$ and $\epsilon = 0.1$, y_R behaves the same as the other solutions shown in Figure 2.3.
And when $\epsilon = 0.9$ from Figure 2.8 we find that y_R converges to solutions y_A, h and y_E, because ϵ is very close to 1.

2.4 Concluding remarks

In this chapter, the approximate analytic solutions of the generalized Van Der Pol system are studied. We got these solutions in several ways namely SPM, LPM, AM and RGM. And we studied the behavior of all these solutions in detail. We found that there are several variables that control the behavior of the approximate solutions which are the parameter ϵ, amplitude A and the variable t.

Chapter 3

Study of Nonlinear Differential Equation in Their General Form With Generalized Order

In this chapter we studied the Van Der Pol systems in fractional order in their general form [46], where we modeled and simulated using Matlab software, and analyzed our results. And through previous studies the researchers found that some of these systems display chaotic movements in the fractional order [41, 53, 47].

3.1 Introduction

The Van Der Pol equation is presented in standard form by a second-order nonlinear differential equation of the type

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0, \tag{3.1}$$

where ϵ is the control parameter that reflects the degree of nonlinearity of the system. The equation (3.1) possesses a periodic solution that attracts other solution except the trivial one at the unique equilibrium point $x = \dot{x} = 0$.

The equivalent state space formulation has the form

$$\begin{cases} \frac{dx_1}{dt} = x_2, \\ \frac{dx_2}{dt} = -x_1 - \epsilon(x_1^2 - 1)x_2, \end{cases} \tag{3.2}$$

Figure 3.1 (left) shows the phase portraits of the Van der Pol equation (3.1) for initial

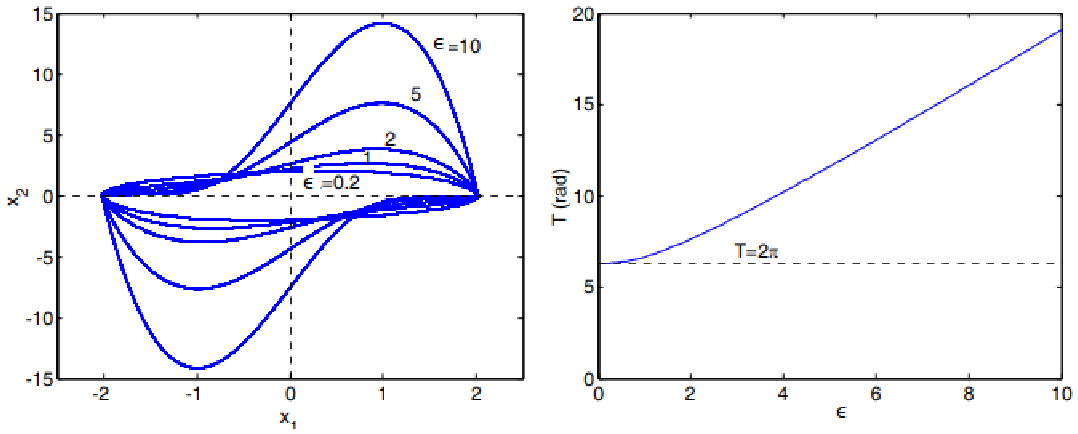


Figure 3.1: The Van der Pol oscillator: left) Phase portraits, right) Period of oscillation $T = 2\pi/\omega$ versus the parameter ϵ.

conditions $x_1(0) = 0, x_2(0) = 2$ as the control parameter ϵ is varied. Clearly, the phase portraits are depending on ϵ, namely: $\epsilon = 0$, harmonic oscillator; $\epsilon > 0$, stable limit cycle; ϵ increasing, nonlinearity increasing. The amplitude of oscillations is nearly constant on the value $A = 2$, but the frequency of oscillation ω (period $T = 2\pi/\omega$) depends on ϵ, as shown in Figure 3.1 (right). For lower values of ϵ the frequency is approximately $\omega = 1(T = 2\pi)$. In this chapter we investigate the influence of a fractional-order time derivative introduced in the Van der Pol equation dynamics (3.2). we propose several versions of the VPO containing fractional derivatives. It is also presented numerical simulations of the fractional Van der Pol system under study. The resulting fractional-order Van der Pol oscillator is analyzed in the time and frequency domains, by using phase portraits, spectral analysis. The standard Van Der Pol equation has undergone extensive modification in which the dependent variable x and/or its derivatives occur to some fractional power. These nonlinear differential equations are called fractional Van Der Pol equations.

3.2 Study of the Van Der Pol oscillator with fractional derivatives

Ramiro S. Barbosa and all in [41] have proposed several versions of the modified Van Der Pol equation. Such modifications consisted on the introduction of a fractional-order time derivative in the sate-space equations of the standard Van Der Pol oscillator. The unforced and forced versions of the resulting fractional-order Van Der Pol oscillators were studied in the time and frequency domains.

3.2.1 Unforced Van Der Pol oscillator with fractional derivatives

Mickens (2002, 2003) in [35, 36] have investigated the following two equations

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x^{1/3} = 0, \tag{3.3}$$

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x}^{1/3} + x = 0. \tag{3.4}$$

More recently, Pereira, and all. (2004) in [14] considered a fractional version of the Van Der Pol equation given by

$$x^{(\alpha)} + \epsilon(x^2 - 1)\dot{x} + x = 0, \qquad 1 < \alpha < 2. \tag{3.5}$$

$$\begin{cases} \dot{x_1} = x_2, \\ \dot{x_2^{\alpha}} = -x_1 - \epsilon(x_1^2 + 1)x_2^{\alpha}, \end{cases} \tag{3.6}$$

which is obtained by substituting the capacitance by a fractance in the nonlinear RLC circuit model. Barbosa, and all. (2004) in [42] has also suggested the introduction of a fractional order time derivative in the state-space equations (3.2) of the standard VPO in the form

$$\begin{cases} \dot{x_1^{\alpha}} = x_2, \\ \dot{x_2} = -x_1 - \epsilon(x_1^2 + 1)x_2^{\alpha} \end{cases} \tag{3.7}$$

where $0 < \alpha < 1$ and $\epsilon > 0$. Note that the system (3.7) reduces to the classical VPO (3.2) when $\alpha = 1$ and that the total system order is changed to $\alpha + 1 < 2$. The differential equation of system (3.7) is given by

$$x^{(1+\alpha)} + \epsilon(x^2 + 1)x^\alpha + x = 0. \tag{3.8}$$

Barbosa presented in [41] numerical simulations of this equation.

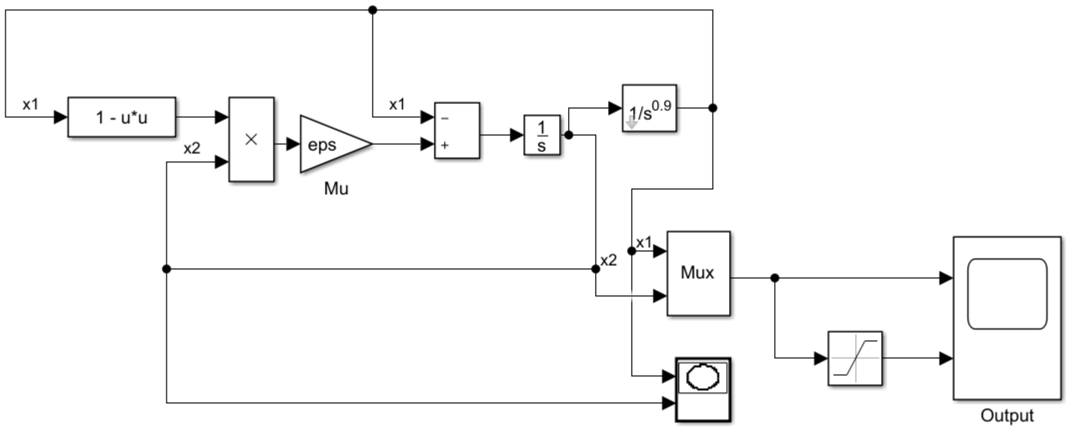


Figure 3.2: Block diagram of the unforced fractional Van Der Pol system under study.

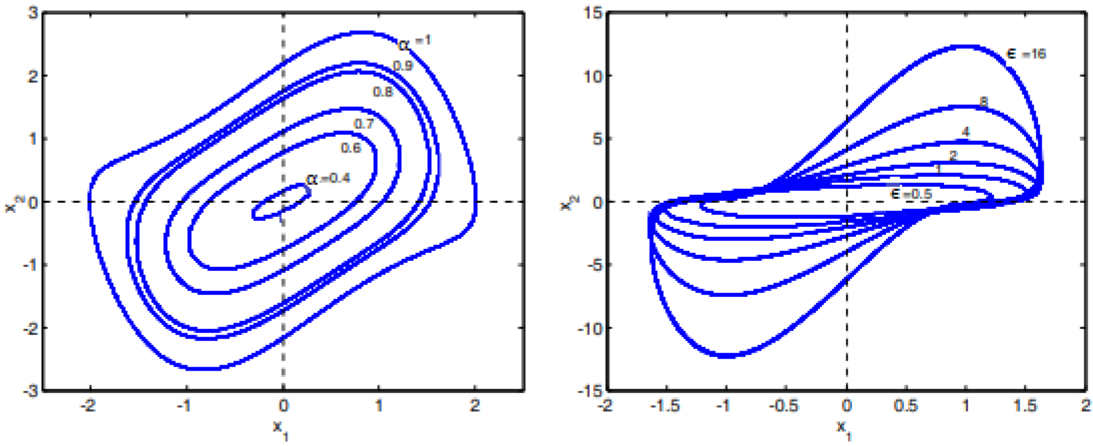


Figure 3.3: Phase portraits: left) $\alpha = 0.4, 0.6, 0.7, 0.8, 0.9, 1.0$ and $\epsilon = 1$, right) $\alpha = 0.8$ and $\epsilon = 0.5, 1, 2, 4, 8, 16$.

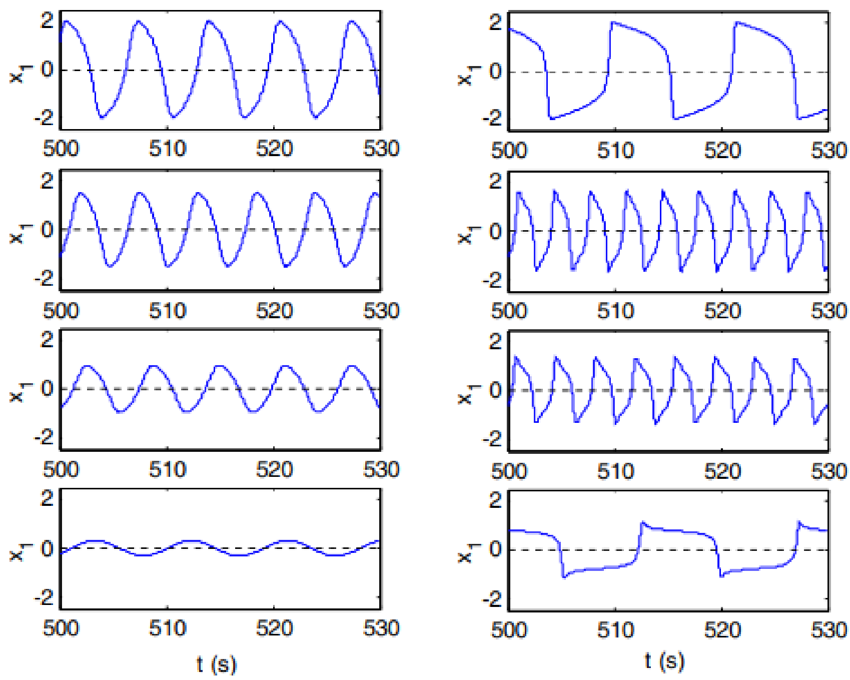


Figure 3.4: Analytical solution of VPO left) with $\alpha = 1, 0.8, 0.6, 0.4$ and $\epsilon = 1$, right) $\alpha = 1, 0.8, 0.6, 0.4$ and $\epsilon = 5$..

The block diagram representation of system (3.8) is illustrated in Figure 3.2. Figure 3.3 shows the phase portraits. In both cases, we verify significant variations of the limit cycle, revealing a large impact of the α-order derivative upon system dynamics. Figure 3.4 show the analytic solutions for several values of $\alpha, \epsilon = 1$ and $\epsilon = 5$, respectively. Once more, we observe the variation of the limit cycle as function of α, noting that the amplitude gets smaller as α is decreased. The system stops oscillating when $\alpha = 0.37(\epsilon = 1)$.

3.2.2 Forced Van Der Pol oscillator with fractional derivatives

Let us now consider the forced version of the fractional VPO system defined inform as

$$\begin{cases} x_1^{(\alpha)} = x_2, \\ \dot{x_1} = -x_1 - \epsilon(x_1^2 - 1)x_2 + f\cos(\omega_f t), \end{cases} \tag{3.9}$$

where f and ω_f are the amplitude and the frequency of the forcing sinusoidal input, respectively.

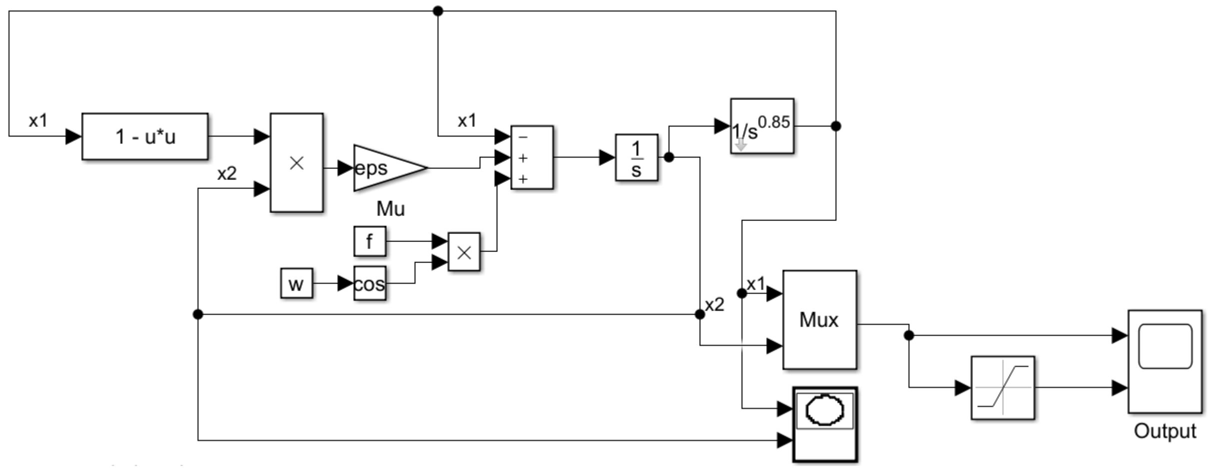


Figure 3.5: Block diagram of the forced fractional Van Der Pol system under study.

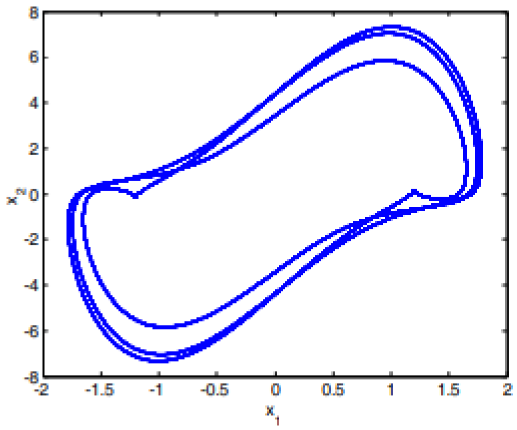


Figure 3.6: Phase plane for $\epsilon = 5, \omega_f = 2.46 rad/s, f = 2.0$ and fractional-order $\alpha = 0.85$: periodic motion.

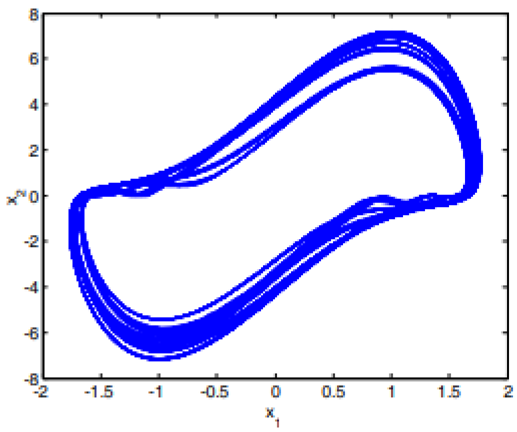


Figure 3.7: Phase plane for $\epsilon = 5, \omega_f = 2.46 rad/s, f = 1.5$ and fractional-order $\alpha = 0.85$: quasiperiodic motion.

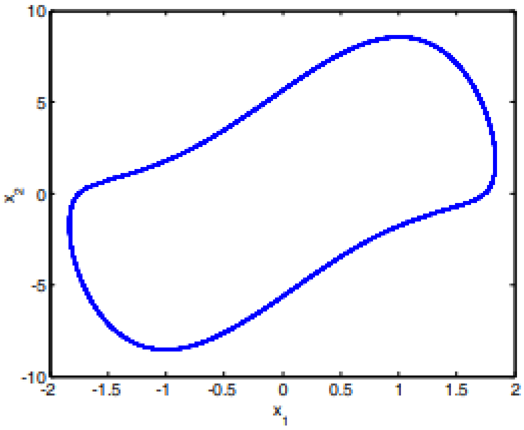


Figure 3.8: Phase plane for $\epsilon = 5, \omega_f = 2.46 rad/s, f = 5.5$ and fractional-order $\alpha = 0.85$: period locked motion.

The block diagram representation of equations (3.9) is depicted in Figure 3.5. Figures 3.6, 3.7 and 3.8 show that the forced Van Der Pol system has different modes, which are periodic motion, quasiperiodic motion and period locked motion. Note that all these modes correspond to a periodic behaviour of the system. The non-periodic behaviour is characterized by the chaos (or sensitivity to initial conditions). It is well-kwon that the classical forced Van Der Pol equation can display chaos for specific set of parameters, even not always easy to find.

3.3 Chaos in a generalized Van Der Pol system and in its fractional order system

In this section, we analyse and present simulation results for the chaotic dynamics of a generalized fractional Van Der Pol system.

Zheng-Ming Ge and Mao-Yuan Hsu in [53] have analysed the nonautonomous and autonomous generalized Van Der Pol system.

3.3.1 Nonautonomous generalized fractional order Van Der Pol system with two states

The generalized Van Der Pol system of (3.2) [9, 6, 48, 43, 35] has the form of a nonautonomous system which is written as

$$\begin{cases} \frac{dx_1}{dt} = x_2, \\ \frac{dx_2}{dt} = -x_1 - \epsilon(1 - x_1^2)(c - ax_1^2)x_2 + b\sin(\omega t), \end{cases} \tag{3.10}$$

where ϵ, a, b, c are parameters, and ω is the circular frequency of the external excitation $b \sin \omega t$. The corresponding nonautonomous fractional order system is

$$\begin{cases} \frac{d^\alpha x_1}{dt} = x_2, \\ \frac{d^\beta x_2}{dt} = -x_1 - \epsilon(1 - x_1^2)(c - ax_1^2)x_2 + b\sin(\omega t), \end{cases} \tag{3.11}$$

where α, β are fractional numbers. According to the results of simulation, it is found that the chaotic motions exist in the nonautonomous system with fractional orders. The lowest total fractional order for chaos existence in this system is 1.4 ($2 \cdot 0.7$). When the total fractional order is 1.2 ($2 \cdot 0.6$), no chaos exists.

3.3.2 Autonomous generalized fractional order Van Der Pol system with three states

A modified version of Eq. (3.11) Suggested by Zheng-Ming Ge and Mao-Yuan Hsu in [53]. The nonautonomous generalized fractional order Van Der Pol system (3.11) with two states is transformed into an autonomous generalized fractional order Van Der Pol system with three states

$$\begin{cases} \frac{d^\alpha x_1}{dt} = x_2, \\ \frac{d^\beta x_2}{dt} = -x_1 - \epsilon(1 - x_1^2)(c - ax_1^2)x_2 + b\sin(\omega x_3), \\ \frac{d^\gamma x_3}{dt} = 1, \end{cases} \tag{3.12}$$

where α, β, γ are fractional numbers, in which the original time t in Eq. (3.11) is changed to a new state x_3. When $\gamma = 1, x_3 = t$, Eq. (3.12) reduces to Eq. (3.11).

In the second part, they studied two cases

Case 1 : α, β, γ take the same fractional numbers. When $\alpha = \beta = \gamma$ and vary from 0.9 to 0.6 in steps of 0.1, no chaos exists.

Case 2 : γ equals 1.1 or 0.9, and α, β take the same fractional numbers. According to the results of simulation in this case, it is found that the chaotic motions exist when γ takes 1.1 and a, β vary from 0.9 to 0.3 in steps of 0.1. When α, β take the fractional number less than 0.3, no chaos is found.

$\gamma = 0.9, \alpha, \beta$ take the same fractional numbers 1.1, 1.2, 1.3 and 1.4, only periodic motions are found, no chaos exists.

3.4 Study of the generalized Van Der Pol oscillator with fractional derivatives

In this section, we present and analyse simulation results of the chaotic dynamics produced from a new generalized fractional Van Der Pol system [46].

3.4.1 Generalized Van Der Pol oscillator with fractional derivatives

The equation (3.1) has undergone several modifications due to the application of some fractional powers to the dependent variable x and/or its derivatives. These nonlinear differential equations are called fractional Van Der Pol equations.

Barbosa and al. [42] suggested the introduction of a fractional-order time derivative in the state-space equations (3.2) of the standard Ven der pol in the form

$$\begin{cases} \frac{d^\alpha x_1}{dt^\alpha} = x_2, \\ \frac{dx_2}{dt} = -x_1 - \epsilon(x_1^2 - 1)x_2, \end{cases} \tag{3.13}$$

where α is fractional number. This system is analyzed by Barbosa and al. [41].

The generalized Van Der Pol system of (3.2) which is written as

$$\begin{cases} \frac{dx_1}{dt} = x_2, \\ \frac{dx_2}{dt} = -x_1 - \epsilon(ax_1^2 + bx_2^2 + cx + d)x_2, \end{cases}$$

where ϵ, a, b, c, d are parameters. The corresponding fractional order system is

$$\begin{cases} \frac{d^\alpha x_1}{dt^\alpha} = x_2, \\ \frac{dx_2}{dt} = -x_1 - \epsilon(ax_1^2 + bx_2^2 + cx + d)x_2, \end{cases} \tag{3.14}$$

where α is fractional number.

A modified version of Eq. (3.14) is now proposed. The generalized fractional order Van Der Pol system (3.14) is transformed into an generalized fractional order Van Der Pol system with the degree of its polynomials.

$$\begin{cases} \frac{d^\alpha x_1}{dt^\alpha} = x_2, \\ \frac{dx_2}{dt} = -x_1 - \epsilon(ax_1^2 + bx_2^2 + cx + d)(x_2)^n, \end{cases} \tag{3.15}$$

where $n \in \mathbb{N}$, $0 < \alpha < 1$ and $\epsilon > 0$.

Note that the system (3.15) reduces to the classical Van Der Pol system (3.13) when $\alpha = 1, n = 1, a = 1, b = 0, c = 0, d = -1$ and that the total system order is changed to $\alpha + 1 < 2$.

The differential equation of system (3.15) is given by

$$x^{(1+\alpha)} + \epsilon(ax^2 + b(x^{(\alpha)})^2 + cx + d)(x^{(\alpha)})^n + x = 0. \tag{3.16}$$

In the next section we investigate the new generalized fractional Van Der Pol system (3.16).

3.4.2 Numerical simulations for the fractional order generalized Van Der Pol systems

Now, we analyse and present simulation results of the new generalized fractional Van Der Pol system (3.16) [46]. Through numerical simulations of the proposed schemes by using approximations to fractional-order operators

Figure 3.9 shows The block diagram representation of system (3.16). Figures 3.10 to 3.19 show phase space, Poincaré maps, and analytic solutions at different values of ϵ, α and n. Where we investigate important differences in the limit cycle, revealing a significant influence of ϵ, α and n on system dynamics.

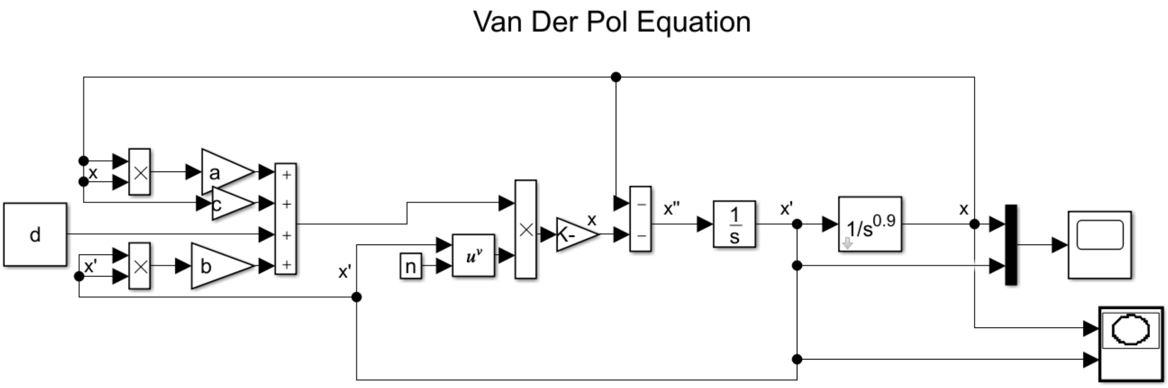
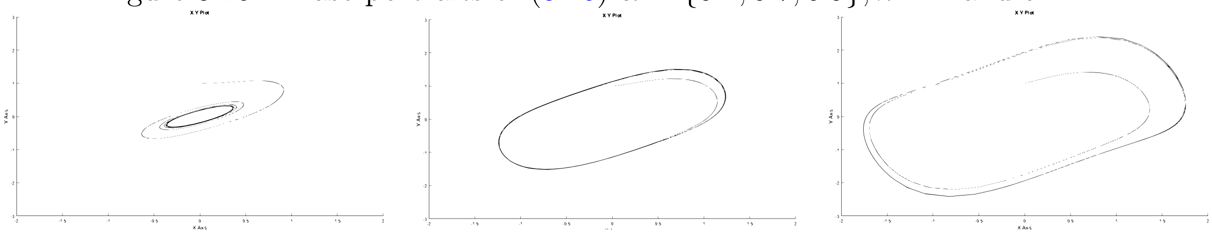


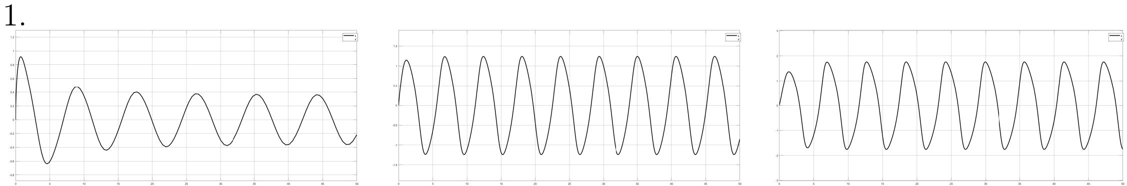
Figure 3.9: Block diagram of the the generalized fractional Van Der Pol system under study.

Figure 3.10: Phase portraits of (3.16) $\alpha = \{0.4, 0.7, 0.9\}, n = 1$ and $\epsilon = 1$.



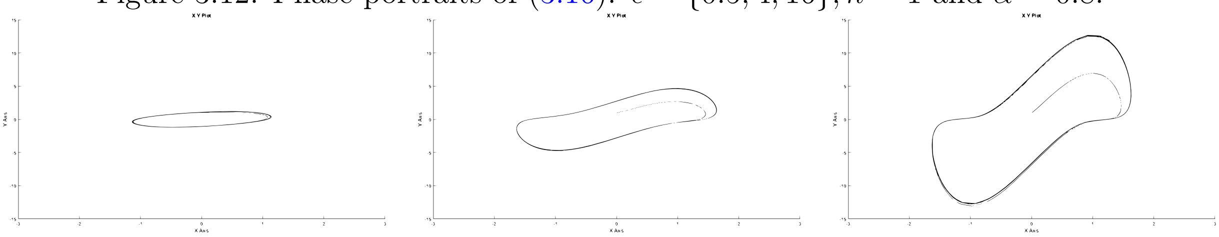
$\alpha = 0.4$ $\alpha = 0.7$ $\alpha = 0.9$

Figure 3.11: Analytical solution of VPO (3.16) such that: $\alpha = \{0.4, 0.7, 0.9\}, n = 1$ and $\epsilon = 1$.



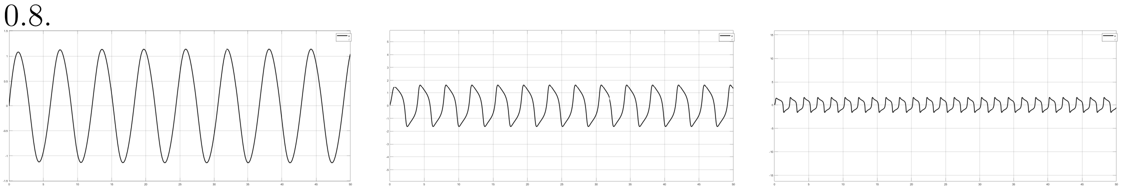
$\alpha = 0.4$ $\alpha = 0.7$ $\alpha = 0.9$

Figure 3.12: Phase portraits of (3.16): $\epsilon = \{0.5, 4, 16\}, n = 1$ and $\alpha = 0.8$.



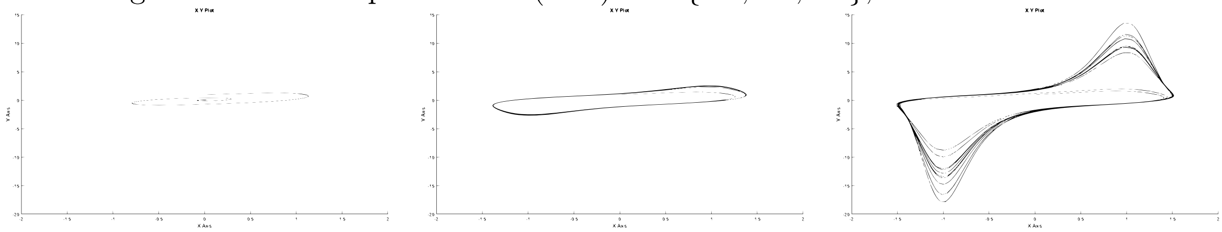
$\epsilon = 0.5$ $\epsilon = 4$ $\epsilon = 16$

Figure 3.13: Analytical solution of VPO (3.16) such that: $\epsilon = \{0.5, 4, 16\}, n = 1$ and $\alpha = 0.8$.



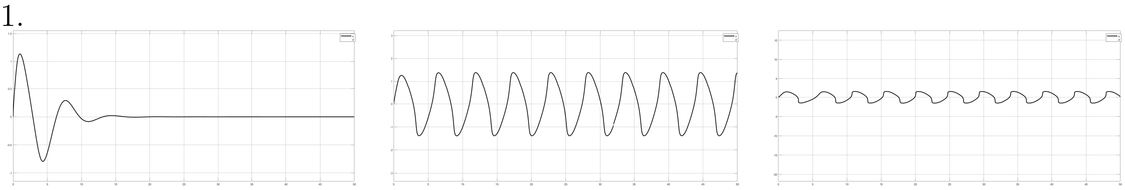
$\epsilon = 0.5$ $\epsilon = 4$ $\epsilon = 16$

Figure 3.14: Phase portraits of (3.16) $\alpha = \{0.6, 0.7, 0.8\}, n = 3$ and $\epsilon = 1$.



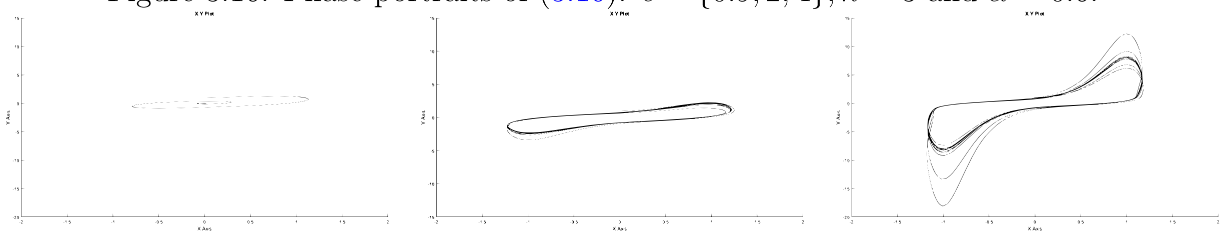
$\alpha = 0.6$ $\alpha = 0.7$ $\alpha = 0.8$

Figure 3.15: Analytical solution of VPO (3.16) such that: $\alpha = \{0.6, 0.7, 0.8\}, n = 3$ and $\epsilon = 1$.



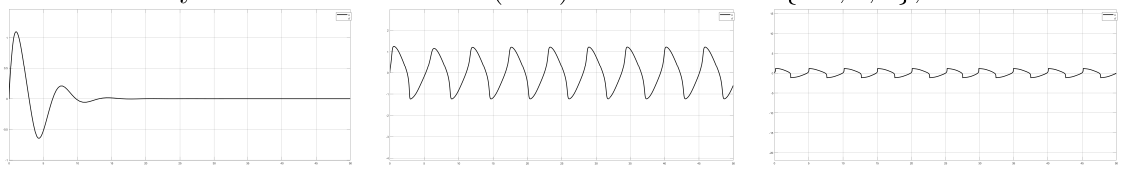
$\alpha = 0.6$ $\alpha = 0.7$ $\alpha = 0.8$

Figure 3.16: Phase portraits of (3.16): $\epsilon = \{0.9, 2, 4\}, n = 3$ and $\alpha = 0.6$.



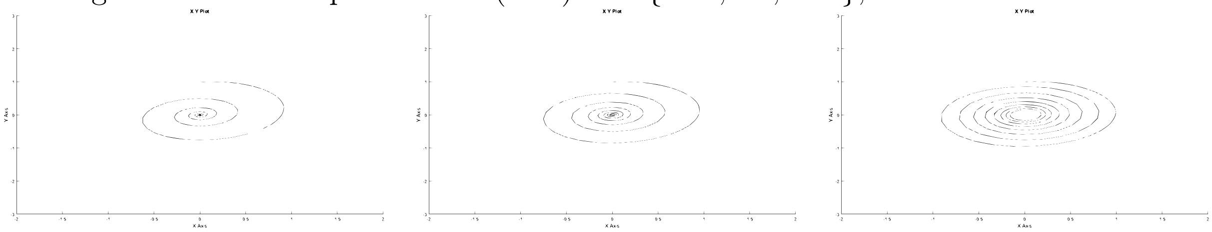
$\epsilon = 0.9$ $\epsilon = 2$ $\epsilon = 4$

Figure 3.17: Analytical solution of VPO (3.16) such that: $\epsilon = \{0.9, 2, 4\}, n = 3$ and $\alpha = 0.6$.



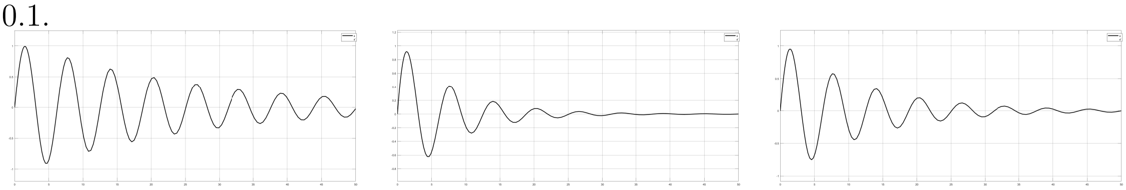
$\epsilon = 0.9$ $\epsilon = 2$ $\epsilon = 4$

Figure 3.18: Phase portraits of (3.16) $\alpha = \{0.85, 0.9, 0.95\}, n = 15$ and $\epsilon = 0.1$.



$\alpha = 0.85$ $\alpha = 0.9$ $\alpha = 0.95$

Figure 3.19: Analytical solution of VPO (3.16) such that: $\alpha = \{0.85, 0.9, 0.95\}, n = 15$ and $\epsilon = 0.1$.



$\alpha = 0.85$ $\alpha = 0.9$ $\alpha = 0.95$

3.5 Conclusions

In this chapter,The Van der Pol system with fractional orders is studied. Where we have proposed several versions of the modified Van Der Pol equation. Such modifications consisted on the introduction of a fractional-order time derivative in the sate-space equations of the standard Van Der Pol oscillator. The unforced and forced versions have been studied. Also, both nonautonomous and autonomous systems were considered. We devoted our study to the new generalized fractional Van Der Pol system. The resulting fractional-order Van Der Pol oscillator is analyzed in the time and frequency domains, by using phase portraits, spectral analysis. The fractional-order dynamics is illustrated through numerical simulations of the proposed schemes by using approximations to fractional-orderoperators. The results reveal that the fractional-order systems can exhibit different behaviour from those obtained with the standard Van Der Pol oscillator depending on order's derivative (or system's order). The fractional order can act as a modulation parameter that may be useful for a better understanding and control of such systems. Also, through numerical analyzes, periodic and chaotic movements were observed. And that chaos exists in the fractional order system with the order both less than and more than the number of the states of the integer order generalized Van Der Pol system.

Chapter 4

Study of Periodic Solutions of Some Nonlinear Dynamical Systems

In this chapter we study the existence, uniqueness and asymptotic stability of the T-periodic solutions for nonlinear dynamical systems using Bogolyubov's second theorem.

4.1 Asymptotic stability of periodic solutions for nonlinear differential equations with applications to the Van Der Pol system

We consider the system

$$\dot{x} = \epsilon g(t, x, \epsilon), \tag{4.1}$$

where $\epsilon > 0$ is a small parameter and the function $g \in C^0(\mathbb{R} \times \mathbb{R}^n \times [0.1], \mathbb{R}^n)$ is T -periodic in the first variable and locally Lipschitz with respect to the second one.
To apply second Bogolyubov's theorem to system (4.1), we use an important notion, which is the averaging function as follows

$$g_0(v) = \int_0^T g(\tau, v, 0)d\tau. \tag{4.2}$$

From this definition we will search for periodic solutions that are near $v_0 \in g_0^{-1}(0)$, if g_0 is of class C^1. In our study, we consider $v_0 = (x_1, x_2) = (A\sin(\phi), A\cos(\phi))$, with $A \in \mathbb{R}$,

$\phi \in [-\pi, \pi]$ and $g_0(v_0) = (g_{0_1}(v_0), g_{0_2}(v_0))$.

We remind the periodic case of the second Bogolyubov's theorem ([2, §2 Theorem 2.11]). It is based on the Jacobian matrix of the averaging function in a neighborhood of v_0, where $det(g_0)'(v_0) > 0$ and $trace(g_0)'(v_0) < 0$ assure the existence and uniqueness, for $\epsilon > 0$ (sufficiently small) of a T-periodic solution of system (4.1), and also its asymptotic stability. If $det(g_0)'(v_0) < 0$, then the system (4.1) has at least one non-asymptotically stable T-periodic solution.

Theorem 4.1.1. *[2] Let $g \in C^0\left(\mathbb{R} \times \Omega \times [0,1], \mathbb{R}^k\right)$ and $v_0 \in \Omega$. Assume the following four conditions.*

(i) For some $L > 0$ we have that $\|g(t, v_1, \varepsilon) - g(t, v_2, \varepsilon)\| \leq L\|v_1 - v_2\|$ for any $t \in [0,T], v_1, v_2 \in \Omega, \varepsilon \in [0,1]$.

(ii) For any $\gamma > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} \| \int_0^T g(\tau, v_1 + u(\tau), \varepsilon)\, d\tau - \int_0^T g(\tau, v_2 + u(\tau), \varepsilon)\, d\tau \\ - \int_0^T g(\tau, v_1, 0)\, d\tau + \int_0^T g(\tau, v_2, 0)\, d\tau \| \leq \gamma \|v_1 - v_2\| \end{aligned}$$

for any $u \in C^0\left([0,T], \mathbb{R}^k\right), \|u\| \leq \delta, v_1, v_2 \in B_\delta(v_0)$ and $\varepsilon \in [0, \delta]$.

(iii) Let g_0 be the averaging function given by (4.2) and consider that $g_0(v_0) = 0$.

(iv) There exist $q \in [0,1), \alpha, \delta_0 > 0$ and a norm $\|\cdot\|_0$ on $\mathbb{R}^k$ such that $\|v_1 + \alpha g_0(v_1) - v_2 - \alpha g_0(v_2)\|_0 \leq q\| v_1 - v_2\|_0$ for any $v_1, v_2 \in B_{\delta_0}(v_0)$.

Then there exists $\delta_1 > 0$ such that for every $\varepsilon \in (0, \delta_1]$ system (4.1) has exactly one T-periodic solution x_ε with $x_\varepsilon(0) \in B_{\delta_1}(v_0)$. Moreover the solution x_ε is asymptotically stable and $x_\varepsilon(0) \to v_0$ as $\varepsilon \to 0$.

When solution $x(\cdot, v, \varepsilon)$ of system (4.1) with initial condition $x(0, v, \varepsilon) = v$ is well defined on $[0,T]$ for any $v \in B_{\delta_0}(v_0)$, the map $v \mapsto x(T, v, \varepsilon)$ is well defined and it is said to be the Poincaré map of system (4.1). The proof of existence, uniqueness and stability of the T-periodic solutions of system (4.1) in Theorem 2.1 reduces to the study of corresponding properties of the fixed points of this map.

In order to prove Theorem 4.1.1 we observe from (4.1) that $x(T,v,\varepsilon)$ can be represented as

$$x(T,v,\varepsilon)=v+\varepsilon g_\varepsilon(v),\ \text{ where } g_\varepsilon(v)=\int_0^T g(\tau,x(\tau,v,\varepsilon),\varepsilon)d\tau,$$

and we use the following result which claims that properties (i) and (ii) are also applied to g_ε in a suitable sense.

Lemma 4.1.1. *Let $g\in C^0\left(\mathbb{R}\times\Omega\times[0,1],\mathbb{R}^k\right)$ and $\delta_0>0$ be such that $B_{\delta_0}(\nu_0)\subset\Omega$. If (ii) is satisfied then there exist $\delta\in[0,\delta_0]$ and $L_1>0$ such that the map $(v,\varepsilon)\mapsto g_\varepsilon(v)$ is well defined and continuous on $B_{\delta_0}(v_0)\times[0,\delta]$ and*

$$\|g_\varepsilon(v_1)-g_\varepsilon(v_2)\|\le L_1\|v_1-v_2\|\ \text{ for any } \varepsilon\in[0,\delta],v_1,v_2\in B_{\delta_0}(v_0)\,.$$

If both (i) and (ii) are satisfied then for any $\gamma>0$ there exists $\delta\in[0,\delta_0]$ such that

$$\|g_\varepsilon(v_1)-g_0(v_1)-g_\varepsilon(v_2)+g_0(v_2)\|\le\gamma\|v_1-v_2\|$$

for any $v_1,v_2,\in B_\delta(v_0)$ and $\varepsilon\in[0,\delta]$.

Proof. Using the continuity of the solution of a differential system with respect to the initial data and the parameter (see [27], Ch. 4, §23, statements G and D), we obtain the existence of $\varepsilon_0>0$ such that $x(t,v,\varepsilon)\in\Omega$ for any $t\in[0,T],v\in B_{\delta_0}(v_0)$ and $\varepsilon\in[0,\varepsilon_0]$. Using the Gronwall–Bellman Lemma ([11, Ch. II, §11]) from the representation $x(t,v,\varepsilon)=v+\varepsilon\int_0^t g(\tau,x(\tau,v,\varepsilon),\varepsilon)d\tau$ and the property (i) we obtain $\|x(t,v_1,\varepsilon)-x(t,v_2,\varepsilon)\|\le e^{\varepsilon LT}\|v_1-v_2\|$ for all $t\in[0,T],v_1,v_2\in B_{\delta_0}(v_0)$ and $\varepsilon\in[0,\varepsilon_0]$. Therefore $y(t,v,\varepsilon)=\int_0^t\int_0^t g(\tau,x(\tau,v,\varepsilon),\varepsilon)d\tau$ satisfies the following property

$$\|y(t,v_1,\varepsilon)-y(t,v_2,\varepsilon)\|\le L_1\|v_1-v_2\| \tag{4.3}$$

for all $t\in[0,T],v_1,v_2\in B_{\delta_0}(v_0)\,,\varepsilon\in[0,\varepsilon_0]$ and $L_1=LTe^{\varepsilon_0 LT}$. Since $g_(v)=y(T,v,\varepsilon)$ the first part of the lemma has been proven.

Taking into account that $x(t,v,\varepsilon)=v+\varepsilon y(t,v,\varepsilon)$ we have

$$y(T,v_1,\varepsilon)-y(T,v_1,0)-y(T,v_2,\varepsilon)+y(T,v_2,0)=I_1(v_1,v_2,\varepsilon)+I_2(v_1,v_2,\varepsilon) \tag{4.4}$$

where

$$I_1(v_1,v_2,\varepsilon) = \int_0^T \left[g(\tau, v_2+\varepsilon y(\tau,v_1,\varepsilon),\varepsilon) - g(\tau, v_2+\varepsilon y(\tau,v_2,\varepsilon),\varepsilon)\right] d\tau,$$

$$I_2(v_1,v_2,\varepsilon) = \int_0^T \left[g(\tau, v_1+\varepsilon y(\tau,v_1,\varepsilon),\varepsilon) - g(\tau, v_2+\varepsilon y(\tau,v_1,\varepsilon),\varepsilon)\right] d\tau - \int_0^T \left(g(\tau,v_1,0) - g(\tau,v_2,0)\right) d\tau.$$

Since $(t,v,\varepsilon) \mapsto y(t,v,z)$ is bounded on $[0,T] \times B_{\delta_0}(v_0) \times [0,\varepsilon_0]$, we have that $\varepsilon y(t,v,\varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly with respect to $t \in [0,T]$ and $v \in B_\delta(v_0)$. Decreasing $\varepsilon_0 > 0$, if necessary, we get that $v_2 + \varepsilon y(t,v_1,\varepsilon) \in \Omega$ for any $t \in [1,T]$, $v_1, v_2 \in B_{\delta_0}(v_0)$, $\quad \varepsilon \in [0, \varepsilon_0 \,|$. By assumption (i) and relation (4.3) we obtain that $\|I_1(v_1,v_2,\varepsilon)\| \le T.\varepsilon L L_1 \|v_1 - v_2\|$ for all $\varepsilon \in [0,\varepsilon_0]$, $v_1, v_2 \in B_{\delta_0}(v_0)$.

We fix $\gamma > 0$ and take $\delta > 0$ given by (ii). Without loss of generality we can consider that $\delta \le \min\{\delta_0, \varepsilon_0, \gamma/(2TLL_1)\}$. Therefore assumption (ii) implies that $\|I_2(v_1,v_2,\varepsilon)\| \le (\gamma/2)\|v_1 - v_2\|$ for any $\varepsilon \in [0,\delta]$, $v_1, v_2 \in B_\delta(v_0)$. Substituting the obtained estimations for I_1 and I_2 into (4.4) we have $\|y(T,v_1,\varepsilon) - y(T,v_1,0) - y(T,v_2,\varepsilon) + y(T,v_2,0)\| \le (\varepsilon TLL_1 + \gamma/2)\|v_1 - v_2\| \le \gamma \|v_1 - v_2\|$ for any $\varepsilon \in [0,\delta]$, $v_1, v_2 \in B_\delta(v_0)$. Hence the proof is complete. □

Lemma 4.1.2. *Let $g_0 : \Omega \to \mathbb{R}^k$ satisfy assumption (iv) with some $q \in (0,1), \alpha, \delta_0 > 0$ and a norm $\|\cdot\|$ on $\mathbb{R}^k$. Then $\|v_1 + \varepsilon g_0(v_1) - v_2 - \varepsilon g_0(v_1)\|_0 \leqslant (1 - \varepsilon(1-q)/\alpha)\|v_1 - v_2\|$ for any $v_1, v_2 \in B_\delta(v_0)$ and any $\varepsilon \in [0,\alpha]$*

Proof. Indeed, the representation $v + \varepsilon g_0(v) = (1-\varepsilon/\alpha)v + \varepsilon/\alpha(v + \alpha g_0(v))$ implies that the Lipschitz constant of the function $I + \varepsilon g_0$ with respect to the norm $\|\cdot\|_0$ is $(1-\varepsilon/\alpha) + \varepsilon/\alpha q = 1 - \varepsilon(1-q)/\alpha$. □

Proof of Theorem 4.1.1. By Lemma 4.1.1 we have that there exists $\delta_1 \in [0,\delta_0]$ such that

$$\|g_\varepsilon(v_1) - g_0(v_1) - g_\varepsilon(v_2) + g_0(v_2)\| \leqslant ((1-q)/(2\alpha))\|v_1 - v_2\|_0 \tag{4.5}$$

for any $\varepsilon \in [0,\delta_1]$, $v_1, v_2 \in B_{\delta_1}(v_0)$. First we prove that there exists $\varepsilon_1 \in [0,\delta_1]$ such that for every $\varepsilon \in [0,\delta_1]$ there exists $v_\varepsilon \in B_{\delta_1}(v_0)$ such that $x(\cdot, v_\varepsilon, \varepsilon)$ is a T − periodic solution (4.1) by showing that there exists v_ε such that $x(T,v_\varepsilon,\varepsilon) = v_\varepsilon$. Using (iii) and (iv) we have

$$\|v + \alpha g_0(v) - v_0\|_0 \le q\|v - v_0\|_0 \text{ for any } v \in B_{\delta_1}(v_0).$$

Therefore we have that the map $I+\alpha_{g_0}$ maps $B_{\delta_1}(v_0)$ into itself. From Lemma 4.1.1 we have that there exists $\varepsilon_0>0$ such that the map $(v,\varepsilon)\to g_\varepsilon(v)$ is well defined and continuous on $B_{\delta_1}(v_0)\times[0,\varepsilon_0]$. We deduce that there exists $\varepsilon_1>0$ sufficiently small such that, for every $\varepsilon\in[0,\varepsilon_1]$, the map $I+\alpha g_0$ maps $B_{\delta_1}(v_0)$ into itself as well. Therefore, by the Brouwer Theorem (see, for example, [29, Theorem 3.1]) we have that $B_{\delta_1}(v_0)$ contains at least one fixed point of the map $I+\alpha g_\varepsilon$ for any $\varepsilon\in[0,\varepsilon_1]$. Denote this fixed point by v_ε. Then we have $g_\varepsilon(v_\varepsilon)=0$ and $x(T,v_\varepsilon,\varepsilon)=v_\varepsilon$ for any $\varepsilon\in[0,\varepsilon_1]$.
Now we prove that $x(\cdot,v_\varepsilon,\varepsilon)$ is the only $T-$periodic solution of (4.1) originating near v_0 and that, moreover, it is asymptotically stable. Knowing that $x(T,v,\varepsilon)=v+\varepsilon g_\varepsilon(v)$ we write the following identity

$$x(T,v,\varepsilon)=v+\varepsilon g_0(v)+\varepsilon(g_\varepsilon(v)-g_0(v)). \tag{4.6}$$

Using Lemma 4.1.2 we have from (4.5) and (4.6) that

$$\begin{aligned}\|x(T,v_1,\varepsilon)-x(T,v_2,\varepsilon)\|_0 &\le (1-\varepsilon(1-q)/\alpha+\varepsilon(1-q)/(2\alpha))\,\|v_1-v_2\|_0\\ &=(1-\varepsilon(1-q)/(2\alpha))\,\|v_1-v_2\|_0\,,\end{aligned}$$

for all $v_1,v_2\in B_{\delta_1}(v_0)$ and $\varepsilon\in[0,\delta_1]$. We proved before that there exists $\varepsilon_1>0$ that, for every $\varepsilon\in[0,\varepsilon_1]$ there exists $v_\varepsilon\in B_{\delta_1}(v_0)$ such that $x(\cdot,v_\varepsilon,\varepsilon)$ is a $T-$periodic solution of (4.1). Since $\varepsilon(1-q)/(2\alpha)>0$ and $\varepsilon_1\le\delta_1$ the lest inequality implies that for each $\varepsilon\in[0,\delta_1]$, the T-periodic solution $x(\cdot,\varepsilon_\varepsilon,\varepsilon)$ is the only T-periodic solution of (4.1) in $B_{\delta_1}(v_0)$ and, moreover (see [29, Lemma 9.2]) it is asymptotically stable. □

Remark 4.1.1. *We note that a similar result close to Theorem 4.1.1 is obtained by Buică and Daniilidis (see [1], Theorem 3.5). But instead of the assumption (iv) with fixed a* $\alpha>0$ *it is assumed to be satisfied for any* $\alpha>0$ *II sufficiently small. Although, Lemma 4.1.2 now implies that it is the same to assume (iv) for only one* $\alpha>0$ *and respectively, for all* $\alpha>0$ *I sufficiently small. The advantage of Theorem 4.1.1 is that it does not require differentiability of* $g(t,\cdot,\varepsilon)$ *at any point, while [1] needs it at* v_0 *See also Remark 4.1.2.*

In general it is not easy to check assumptions (ii) and (iv) in the applications of Theorem 4.1.1. Thus we give also the following theorem based on Theorem 4.1.1 which assumes certain type of piecewise differentiability instead of (ii) and deals with properties of the matrix $(g_0)'(v_0)$ instead of the Lipschitz constant of g_0.

For any sot $M \subset [0,T]$ measurable in the sense of Lebesgue we denote by mes(M) the Lebesgue measure of M (see [7], Ch. V, §3).

Theorem 4.1.2. *[2] Let $g \in C^0(\mathbb{R} \times \Omega \times [0,1], \mathbb{R}^k)$ satisfy (i). Let g_0 be the averaging function given by (4.2) and consider $v_0 \in \Omega$ such that $g_0(v_0) = 0$. Assume that*

(v) given any $\tilde{\gamma} > 0$ there exist $\tilde{\delta} > 0$ and $M \subset [0,T]$ measurable in the sense of Lebesgue with mes$(M) < \tilde{\gamma}$ such that for every $v \in B_{\tilde{\delta}}(v_0)$, $t \in [0,T]\backslash M$ and $\varepsilon \in [0, \tilde{\delta}]$ we have that $g(t, \cdot, \varepsilon)$ is differentiable at v end $\|g'_v(t,v,\varepsilon) - g_v(t, v_0, 0)\| \leqslant \tilde{\gamma}$.
Finally assume that

(vi) g_0 is continuously differentiable in a neighborhood of v_0 and the real parts of all the eigenvalues of $(g_0)'(v_0)$ are negative.

Then there exists $\delta_1 > 0$ such that for every $\varepsilon \in (0, \delta_1]$, system (4.1) has exactly one T -periodic solution x_ε with $x_\varepsilon(0) \in B_{\delta_1}(v_0)$. Moreover the solution x_ε is asymptotically stable and $x_\varepsilon(0) \to v_0$ as $\varepsilon \to 0$.

For proving Theorem 4.1.2 we need two preliminary lemmas.

Lemma 4.1.3. *Let $g \in C_0(\mathbb{R} \times \Omega \times [0,1], \mathbb{R}^k)$ satisfying (i). If (v) holds then (ii) is satisfied.*

Proof. Let $\gamma > 0$ be an arbitrary number. We show that (ii) holds with $\delta = \tilde{\delta}/2$, where $\tilde{\delta}$ is given by (v) applied with $\tilde{\gamma} = min \gamma/(4L), \gamma/(4T)$. We consider also $M \subset [0,T]$ given by (v) applied with the same value of $\tilde{\gamma}$.

Let $u \in C^0([0,T], \mathbb{R}^k)$, $\|u\| \leqslant \delta$ and $F(v) = \int_0^T g(\tau, v + u(\tau), \varepsilon)d\tau - \int_0^T g(\tau, v, 0)d\tau$. Let $v_1, v_2 \in B_\delta(v_0)$ and $\varepsilon \in [0, \delta]$. We have $F(v) = F_1(v) + F_2(v)$, where $F_1(v) = \int_M (g(\tau, v + u(\tau), \varepsilon) - g(\tau, v, 0))d\tau$ and $F_2(v) = \int_{[0,T]\backslash M}(g(\tau, v + u(\tau), \varepsilon) - g(\tau, v, 0))d\tau$. By (i) we have that $\|F_1(v_1) - F_1(v_2)\| \leqslant 2L \cdot mes(M) \backslash \|v_1 - v_2\| < 2L\tilde{\gamma}\|v_1 - v_2\| \leqslant (\tau/2)\|v_1 - v_2\|$. On the other hand, using (v), we will prove that a similar relation holds for F_2. In order to do this, we denote $h(\tau, v) = g(\tau, v + u(\tau), \varepsilon) - g(\tau, v, 0)$. Notice that for each $\tau \in [0,T]$ M we can write $h'_v(\tau, v) = (g'_v(\tau, v + u(\tau), \varepsilon) - g'_v(\tau, v_0, 0)) - (g'v(\tau, v, 0) - g'_v(\tau, v_0, 0))$. As a direct consequence of (v) we deduce that $\|h'_v(\tau, v)\| \leqslant 2\tilde{\gamma}$ for all $v \in B_\delta(v_0)$ and $\tau \in [0,T]\backslash M$. Now applying the mean value theorem for the function $h(\tau, \cdot)$, we have

$\|h(\tau,v_1)-h(\tau,v_2)\| \leqslant 2\tilde{\gamma}\|v_1-v_2\|$ for all $\tau \in [0,T]\backslash M$ and all $v_1,v_2 \in B_\delta(v_0)$. Then $\|F_2(v_1)-F_2(v_2)\| \leqslant \int_{[0,T]\backslash M}\|h(\tau,v_1)-h(\tau,v_2)\|d\| \leqslant 2T\tilde{\gamma}\|v_1-v_2\| \leqslant (\gamma/2)\|v_1-v_2\|$. Therefore, we have proved that $\|F(v_1)-F(v_2)\| \leqslant \gamma\|v_1-v_2\|$, that coincides with (ii). □

Lemma 4.1.4. *Let $g_0 : \Omega \to \mathbb{R}^k$ satisfying assumption (vi) for some $v_0 \in \Omega$. Then there exist $q \in [0,1), \alpha, \delta_0 > 0$ and a norm $\|\cdot\|_0$ on $\mathbb{R}^k$ such that (iv) is satisfied.*

Proof. If λ is an eigenvalue of $\alpha(g_0)'(v_0)$ then $\lambda+1$ is an eigenvalue of $I+(\alpha g_0)'(v_0)$.Since the eigenvalues of $\alpha(g_0)'(v_0)$ tends to 0 as $\alpha \to 0$ and have negative real parts then there exists $\alpha \in [0,1)$ such that the absolute values of all the eigenvalues of $I+\alpha(g_0)'(v_0)$ are less than one. Therefore(see [28, p. 90, Lemma 2.2]) there exist $\tilde{q} \in [0,1)$ and a norm $\|\cdot\|_0$ on $\mathbb{R}^k$ such that $sup_{\|\xi\|_0\leqslant 1}\|\xi+\alpha(g_0)'(v_0)\xi\|_0 \leqslant \tilde{q}$
By continuous differentiability of g_0 in a neighborhood of v_0 we have that $\|g_0(v_1)-g_0(v_2)-(g_0)'(v_0)(v_1-v_2)\|/\|v_1-v_2\| \leqslant \|g_0(v_1)-g_0(v2)-(g_0)'(v_2)(v_1-v_2)\| + \|(g_0)'(v_2)(v_1-v_2)-(g_0)'(v_0)(v_1-v_2)\|/\|v_1-v_2\| \to 0$ as $\max\{\|v_1-v_0\|,\|v_2-v_0\|\} \to 0$. Therefore taking into account that all norms on $\mathbb{R}^k$ are equivalent, there exists $\delta_0 > 0$ such that $\|g_0(v_1)-g_0(v_2)-(g_0)'(v_0)(v_1-v_2)\|_0 \leqslant (1-\tilde{q})/(2\alpha)\|v_1-v_2\|_0$ for all $v_1,v_2 \in B_{\delta_0}(v_0)$. Then

$$\begin{aligned}&\|v_1+\alpha g_0(v_1)-v_2-\alpha g_0(v_2)\|_0\\ &\leqslant \alpha\|g_0(v_1)-g_0(v_2)-(g_0)'(v_0)(v_1-v_2)\|_0 + \|v_1-v_2+\alpha(g_0)'(v_0)(v_1-v_2)\|0\\ &\leqslant (1+\tilde{q})/2\|v_1-v_2\|_0,\end{aligned}$$

for all $v_1,v_2 \in B_{\delta_0}(v_0)$. □

Proof of Theorem 4.1.2. Lemmas 4.1.3 and 4.1.4 imply that assumptions (ii) and (iv) of Theorem 4.1.1 are satisfied. Therefore the conclusion of the theorem follows applying Theorem 4.1.2. □

It was observed by Mitropol'skii in [51] that in spite of the fact that $g(t,\cdot,\varepsilon)$ in (4.1) is only Lipschitz, function g_0 turns out to be differentiable in applications. In particular, one will see in Section 3 that this is the case for the nonsmooth Van Der Pol oscillator.

Clearly if $g \in C^1(\mathbb{R}\times\mathbb{R}^k\times[0,1],\mathbb{R}^k)$ then (i) and (v) hold in any open bounded set $\Omega \subset \mathbb{R}^k$. Therefore Theorem 4.1.2 is a generalization of the periodic case of the second Bogolyubov's theorem formulated in the introduction.

Remark 4.1.2. *Theorem 4.1.2 does not require that the eigenvectors of $(g_0)'(v_0)$ be orthogonal as in the result of Buic ă and Daniilidis [1, Theorem 3.6]. Moreover assumption (H2) of [1] is more restrictive than (v).*

For completeness we give also the following theorem on the existence of non – asymptotically stable T –periodic solutions for (4.1). In the theorem below, $d(F, V)$ denotes the Brouwer topological degree of the vector field $F \in C^0(\mathbb{R}^k, \mathbb{R}^k)$ on the open and bounded set $V \subset \mathbb{R}^k$ (see [29, Ch. 2, §5.2]).

Theorem 4.1.3. *[2] Let $g \in C^0(\mathbb{R} \times \mathbb{R}^k \times [0,1], \mathbb{R}^k)$. Assume that there exists an open bounded set $V \subset \mathbb{R}^k$ such that $g_0(v) \neq 0$ for any $v \in \partial V$ and*

(vii) $d(-g_0, V) < 0$.

Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ system (4.1) has at least one non–asymptotically stable T –periodic solutions x_ε with $x_\varepsilon(0) \in V$.

Proof. Since $g_0(v) \neq 0$ for any $v \in \partial V$ then from Mawhin's Theorem [22] (or [23, Section 5]) we have that there exists $\varepsilon_0 > 0$ such that

$$d(-g_0, V) = d(I - x(T, \cdot, \varepsilon), V) \text{ for any } \varepsilon \in (0, \varepsilon_0]. \tag{4.7}$$

By [29, Theorem 9.6] for any asymptotically stable T –periodic solution x_ε of (4.1) we have that $d(I - x(T, \cdot, \varepsilon), B_\delta(x_\varepsilon(0))) = 1$ for $\delta > 0$ sufficiently small. Therefore if all the possible T –periodic solutions of (4.1) with $\varepsilon \in (0, \varepsilon_0]$ had been asymptotically stable, then the degree $d(I - x(T, \cdot, \varepsilon), V)$ would have been non negative, contradicting (vii) and (4.1). □

Remark 4.1.3. *Assumptions (iii) and (iv) imply that $d(-g_0, V) = 1$ (see [29, Theorem 5.16]).*

Finally thinking in the application to the nonsmooth Van Der Pol oscillator, we formulate the following theorem which combines Mawhin's Theorem (see [22] (or [23, Theorem 3]), Theorem 4.1.2 and Theorem 4.1.3. In this theorem $([g_0]_i)'(j)$ stays for the derivative of the i–th component of the function g_0 with respect to the j–th variable.

Theorem 4.1.4. *[2] Let $g \in C^0(\mathbb{R} \times \Omega \times [0,1], \mathbb{R}^2)$. Let $v_0 \in \Omega$ be such a point that $g_0(v_0) = 0$ and g_0 is continuously differentiable in a neighborhood of v_0.*

(a) If $det(g_0)'(v_0) \neq 0$ then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ system (4.1) has at least one T –periodic solution x_ε such that $x_\varepsilon(0) \to v_0$ as $\varepsilon \to 0$.

(b) If (i) and (v) hold and

$$det(g_0)'(v_0) > 0 \text{ and } ([g_0]_1)'_{(1)}(v0) + ([g_0]_2)'_{(2)}(v_0) < 0, \tag{4.8}$$

then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ system (4.1) has exactly one T –periodic solution x_ε such that $x_\varepsilon(0) \to v_0$ as $\varepsilon \to 0$. Moreover the solution x_ε is asymptotically stable.

(c) If $det(g_0)'(v_0) < 0$, then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ system (4.1) has at least one non–asymptotically stable T –periodic solution x_ε such that $x_\varepsilon(0) \to v_0$ as $\varepsilon \to 0$.

Proof. Statement (a) is added for the completeness of the formulation of Theorem 4.1.4 and it follows from Mawhin's Theorem (see [22] or [23, Theorem 3]).
On the other hand it is a simple calculation to show that (4.8) implies that all the eigenvalues of $(g_0)'(v_0)$ have negative real part. Therefore, assumption (vi) of Theorem 4.1.2 is also satisfied and statement (b) follows from this theorem.
Statement (c) follows from Theorem 4.1.3. Indeed since $det(g_0)'(v_0) < 0$ implies (see [29, Theorem 5.9]) that $d(g_0, B_\rho(v_0))$ is defined for any $\rho > 0$ sufficiently small and that $d(g_0, B_\rho(v_0)) = det(g_0)'(v_0) < 0$. □

Several special cases of the Van Der Pol equations have been considered in the literature. Adriana Buica and others [2] studied it in the form

$$\ddot{u} + \epsilon(u^2 - 1)\dot{u} + (1 + \alpha\epsilon)u = \epsilon\lambda \sin t, \tag{4.9}$$

$$\ddot{u} + \epsilon(|u| - 1)\dot{u} + (1 + \alpha\epsilon)u = \epsilon\lambda \sin t. \tag{4.10}$$

where they proved that it has a unique periodic solution and asymptotic stable by applying a second Bogolyubov's theorem. And they proved that the amplitude for equation (4.9) is $|A| > \sqrt{2}$. The amplitude of the equation (4.10) is $|A| > \frac{\pi}{2}$.

And also Zouhair Diab and Amar Makhlouf [52] studied the Van Der Pol equations in from

$$\ddot{u} + \epsilon(u^2 - 1)\dot{u}^3 + (1 + \alpha\epsilon)u = \epsilon\lambda \sin t, \tag{4.11}$$

$$\ddot{u} + \epsilon(|u| - 1)\dot{u}^3 + (1 + \alpha\epsilon)u = \epsilon\lambda \sin t. \tag{4.12}$$

where they proved that it has a unique periodic solution and asymptotic stable by applying a second Bogolyubov's theorem. And they confirmed that the amplitude for equation (4.11) is $|A| > 2$.The amplitude of the equation (4.12) is $|A| > \frac{3\pi}{4}$.

4.2 Asymptotic stability of periodic solutions for Van Der Pol oscillator in their general form

In this section, we prove some theorems using Bogolyubov's second theorem. We studied the existence, uniqueness and asymptotic stability of the periodic solution for some of the following systems; which represent the oscillatory Van Der Pol equations in their general form

$$\ddot{u} + \epsilon(au^{2h} + b\dot{u}^{2h} + cu + d)\dot{u}^{2k+1} + (1 + \alpha\epsilon)u = \epsilon\lambda \sin t, \;\; (a, \;\; b) \neq (0, 0), \tag{4.13}$$

$$\ddot{u} + \epsilon(f|u| + d)\dot{u}^{2k+1} + (1 + \alpha\epsilon)u = \epsilon\lambda \sin t. \tag{4.14}$$

Where $\alpha, \lambda, a, b, c, f \; and \; d$ are real constants, $k, h \in \mathbb{N}$ and $0 < \epsilon << 1$.
The main idea of our proof is to use a change of variables to convert these equations into Libschitz systems, where we apply the second Bogolyubov's theorem and confirm its conditions, and through it we determine new conditions. We have been able to put general conditions to apply directly in the study of any system of this form in its general case, just by substituting the parameter values in our system.
Through the two results, we will prove the existence, uniqueness and asymptotic stability of the periodic solutions of the systems of differential equations mentioned previously. By applying Theorem 4.1.4, we find that condition (a) proves the existence of the periodic solution, while conditions (b) proves its uniqueness and asymptotic stability.

Theorem 4.2.1. *For all $k, h \in \mathbb{N}$, a, b, c, d, α and $\lambda \in \mathbb{R}$ and $v_0 \in \mathbb{R}^2$, with $(a, b) \neq (0, 0)$ and $0 < \epsilon << 1$, if*

1.
$$A^2(\alpha^2 + (-(a\frac{(2h-1)(2h-3).....53.1}{(2k+2h+2)(2k+2h).....(2k+4)}$$
$$.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4}$$
$$+ b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4})A^{2k+2h}$$
$$- 2d\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}A^{2k})^2) = \lambda^2,$$

2.
$$trace(g_0)'(v_0) = -2(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h)(2k+2h-2).....(2k+4)}$$
$$.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4.2}$$
$$+ b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h)(2k+2h-12).....4.2})A^{2k+2h}\pi$$
$$- 2d(2k+2)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}A^{2k}\pi < 0,$$

3.
$$det(B) = \pi^2(\alpha^2 + (2k+2h+1)(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)}$$
$$.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4}$$
$$+ b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4})^2A^{4k+4h}$$
$$+ (16k+8h+8)(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)}$$
$$.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4.2}$$
$$+ b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4.2}).$$
$$\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}dA^{4k+4h-2}$$
$$+ 4d^2(2k+1)\frac{(2k+1)^2(2k-1)^2.....25.9.1}{(2k+2)^2(2k)^2.....16.4}A^{4k}) > 0.$$

then equation (4.13) has a unique periodic solution and it is asymptotically stable.

In order to prove Theorem 4.2.1, we use Levinson's change of variables (see [31]), which allows us to rewrite equation (4.13) as a system where the second Bogolubov's theorem is applied. For this purpose, we put $(z_1, z_2) = (u, \dot{u})$, so the equation (4.13) becomes

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = -\epsilon(az_1^{2h} + bz_2^{2h} + cz_1 + d)z_2^{2k+1} - (1+\alpha\epsilon)z_1 + \epsilon\lambda\sin t. \end{cases} \tag{4.15}$$

By performing the change of variable

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t), \end{pmatrix}$$

we transform this system into an equation of the form (4.1), and we use the Lipschitz property (in the second variable) of g. Then system (4.15) takes the form

$$\begin{cases} \dot{x}_1 = \epsilon(\sin(-t)(-\alpha(\cos(t)x_1 + \sin(t)x_2) - (a(\cos(t)x_1 + \sin(t)x_2)^{2h} \\ +b(-\sin(t)x_1 + \cos(t)x_2)^{2h} + c(\cos(t)x_1 + \sin(t)x_2) + d)(-\sin(t)x_1 \\ +\cos(t)x_2)^{2k+1} + \lambda\sin(t))), \\ \\ \dot{x}_2 = \epsilon(\cos(-t)(-\alpha(\cos(t)x_1 + \sin(t)x_2) - (a(\cos(t)x_1 + \sin(t)x_2)^{2h} \\ +b(-\sin(t)x_1 + \cos(t)x_2)^{2h} + c(\cos(t)x_1 + \sin(t)x_2) + d)(-\sin(t)x_1 \\ +\cos(t)x_2)^{2k+1} + \lambda\sin(t))). \end{cases}$$

Such that

$$\begin{cases} g_1(t, x_1, x_2) = \alpha(\cos(t)x_1 + \sin(t)x_2)\sin(t) + (a(\cos(t)x_1 + \sin(t)x_2)^{2h} \\ +b(-\sin(t)x_1 + \cos(t)x_2)^{2h} + c(\cos(t)x_1 + \sin(t)x_2) + d)(-\sin(t)x_1 \\ +\cos(t)x_2)^{2k+1}\sin(t) - \lambda\sin^2(t), \\ \\ g_1(t, x_1, x_2) = -\alpha(\cos(t)x_1 + \sin(t)x_2)\cos(t) - (a(\cos(t)x_1 + \sin(t)x_2)^{2h} \\ +b(-\sin(t)x_1 + \cos(t)x_2)^{2h} + c(\cos(t)x_1 + \sin(t)x_2) + d)(-\sin(t)x_1 \\ +\cos(t)x_2)^{2k+1}\cos(t) + \lambda\sin(t)\cos(t). \end{cases} \tag{4.16}$$

with $x_1 = A\sin(\phi)$ and $x_2 = A\cos(\phi)$, so

$$\begin{cases} g_1(t, A\sin(\phi), A\cos(\phi)) = \alpha A\sin(t)\sin(t+\phi) \\ +aA^{2k+2h+1}\sin(t)\sin^{2h}(t+\phi)\cos^{2k+1}(t+\phi) + bA^{2k+2h+1}\sin(t)\cos^{2k+2h+1}(t+\phi) \\ +cA^{2k+2}\sin(t)\sin(t+\phi)\cos^{2k+1}(t+\phi) + dA^{2k+1}\sin(t)\cos^{2k+1}(t+\phi) - \lambda\sin^2(t), \\ \\ g_2(t, A\sin(\phi), A\cos(\phi)) = -\alpha A\cos(t)\sin(t+\phi) \\ -aA^{2k+2h+1}\cos(t)\sin^{2h}(t+\phi)\cos^{2k+1}(t+\phi) - bA^{2k+2h+1}\cos(t)\cos^{2k+2h+1}(t+\phi) \\ -cA^{2k+2}\cos(t)\sin(t+\phi)\cos^{2k+1}(t+\phi) - dA^{2k+1}\cos(t)\cos^{2k+1}(t+\phi) + \lambda\sin(t)\cos(t). \end{cases} \tag{4.17}$$

The corresponding average function g_0, computed according to the formula (4.2), is given by

$$g_0(v_0) = \int_0^{2\pi} g(t, v_0)dt \text{ then } \begin{cases} g_{0_1}(A\sin(\phi), A\cos(\phi)) = g_{0_1}(v_0) = \int_0^{2\pi} g_1(t, v_0)dt, \\ g_{0_2}(A\sin(\phi), A\cos(\phi)) = g_{0_2}(v_0) = \int_0^{2\pi} g_2(t, v_0)dt. \end{cases} \tag{4.18}$$

Such that

$$\begin{aligned} g_{0_1}(A\sin(\phi), A\cos(\phi)) = \int_0^{2\pi} g_1(t, A\sin(\phi), A\cos(\phi))dt &= \int_0^{2\pi} \alpha A\sin(t)\sin(t+\phi)dt \\ &+ aA^{2k+2h+1}\int_0^{2\pi} \sin(t)\sin^{2h}(t+\phi)\cos^{2k+1}(t+\phi)dt \\ &+ bA^{2k+2h+1}\int_0^{2\pi} \sin(t)\cos^{2k+2h+1}(t+\phi)dt \\ &+ cA^{2k+2}\int_0^{2\pi} \sin(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt \\ &+ dA^{2k+1}\int_0^{2\pi} \sin(t)\cos^{2k+1}(t+\phi)dt - \int_0^{2\pi} \lambda\sin^2(t)dt. \end{aligned}$$

and

$$\begin{aligned} g_{0_2}(A\sin(\phi), A\cos(\phi)) = \int_0^{2\pi} g_2(t, A\sin(\phi), A\cos(\phi))dt = -\int_0^{2\pi} \alpha A\cos(t)\sin(t+\phi)dt \\ - aA^{2k+2h+1}\int_0^{2\pi}\cos(t)\sin^{2h}(t+\phi)\cos^{2k+1}(t+\phi)dt \\ - bA^{2k+2h+1}\int_0^{2\pi}\cos(t)\cos^{2k+2h+1}(t+\phi)dt \\ - cA^{2k+2}\int_0^{2\pi}\cos(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt \\ - dA^{2k+1}\int_0^{2\pi}\cos(t)\cos^{2k+1}(t+\phi)dt + \int_0^{2\pi}\lambda\cos(t)\sin(t)dt, \end{aligned}$$

with

$$\int_0^{2\pi}\alpha A\sin(t)\sin(t+\phi)dt = \alpha A\cos(\phi)\pi, \int_0^{2\pi}\lambda\sin^2(t)dt = \lambda\pi,$$

$$\int_0^{2\pi}\alpha A\cos(t)\sin(t+\phi)dt = \alpha A\sin(\phi)\pi, \int_0^{2\pi}\lambda\cos(t)\sin(t)dt = 0,$$

To calculate the average function (4.18) we need the following results. To prove this, we use the following propositions.

Proposition 4.2.1. *[45] For all even integers p, q, we have*

$$\int_0^{2\pi}\sin^p(t)\cos^q(t)dt = \frac{(p-1)(p-3)...5\cdot 3\cdot 1}{(p+q)(p+q-2)...(q+2)}\frac{(q-1)(q-3)...3\cdot 1}{q(q-2)...4\cdot 2}2\pi.$$

Proposition 4.2.2. *[45] If p and q are even integers then*

$$\int_0^{2\pi}\sin^p(t)\cos^q(t)dt = \frac{p-1}{p+q}\int_0^{2\pi}\sin^{p-2}(t)\cos^q(t)dt,$$

and

$$\int_0^{2\pi}\sin^p(t)\cos^q(t)dt = \frac{q-1}{p+q}\int_0^{2\pi}\sin^p(t)\cos^{q-2}(t)dt,$$

and for p or q is an odd integer then

$$\int_0^{2\pi}\sin^p(t)\cos^q(t)dt = 0.$$

Lemma 4.2.1. *For each $h, k \in \mathbb{N}$ and $\phi \in [-\pi, \pi]$, we have*

1. $\int_0^{2\pi}\sin(t)\sin^{2h}(t+\phi)\cos^{2k+1}(t+\phi)dt = -\sin(\phi)\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)}\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4}\pi.$

2. $\int_0^{2\pi} \sin(t)\cos^{2k+1}(t+\phi)dt = -\sin(\phi)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}2\pi.$

3. $\int_0^{2\pi} \sin(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt = 0.$

4. $\int_0^{2\pi} \cos(t)\sin^{2h}(t+\phi)\cos^{2k+1}(t+\phi)dt = \cos(\phi)\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)}\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4}\pi.$

5. $\int_0^{2\pi} \cos(t)\cos^{2k+1}(t+\phi)dt = \cos(\phi)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}2\pi.$

6. $\int_0^{2\pi} \cos(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt = 0.$

Proof. For each $k, h \in \mathbb{N}$ and $\phi \in [-\pi, \pi]$, we calculate the following integral

1.
$$I = \int_0^{2\pi} \sin(t)\sin^{2h}(t+\phi)\cos^{2k+1}(t+\phi)dt.$$

We put $\tau = t + \phi$, so it becomes

$$\begin{aligned}
&\int_0^{2\pi} \sin(t)\sin^{2h}(t+\phi)\cos^{2k+1}(t+\phi)dt,\\
&= \int_0^{2\pi} \sin(\tau-\phi)\sin^{2h}(\tau)\cos^{2k+1}(\tau)d\tau,\\
&= \int_0^{2\pi} (\sin(\tau)\cos(\phi) - \cos(\tau)\sin(\phi))\sin^{2}(\tau)\cos^{2k+1}(\tau)d\tau,\\
&= \cos(\phi)\int_0^{2\pi} \sin^{2h+1}(\tau)\cos^{2k+1}(\tau)d\tau\\
&- \sin(\phi)\int_0^{2\pi} \sin^{2h}(\tau)\cos^{2k+2}(\tau)d\tau.
\end{aligned}$$

Let

$$\begin{aligned}
I_1 &:= \int_0^{2\pi} \sin^{2h+1}(\tau)\cos^{2k+1}(\tau)d\tau,\\
I_2 &:= \int_0^{2\pi} \sin^{2h}(\tau)\cos^{2k+2}(\tau)d\tau.
\end{aligned}$$

$\forall k, h \in \mathbb{N}$, from Proposition 4.2.1, we obtain

$$\begin{aligned}
I_1 &= 0,\\
I_2 &= \frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)}\\
&.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4}\pi.
\end{aligned}$$

Then

$$\begin{aligned} I &= -\sin(\phi)\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)} \\ &\quad .\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4}\pi. \end{aligned}$$

We prove the other integrals in the same way.

2. We put $\tau = t + \phi$ and using the Proposition 4.2.1, we obtain

$$\begin{aligned} \int_0^{2\pi} \sin(t)\cos^{2k+1}(t+\phi)dt &= \int_0^{2\pi} \sin(\tau-\phi)\cos^{2k+1}(\tau)d\tau \\ &= \int_0^{2\pi} (\sin(\tau)\cos(\phi) - \cos(\tau)\sin(\phi))\cos^{2k+1}(\tau)d\tau \\ &= \cos(\phi)\int_0^{2\pi} \sin(\tau)\cos^{2k+1}(\tau) - \sin(\phi)\int_0^{2\pi} \cos^{2k+2}(\tau) \\ &= 0 - \sin(\phi)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}2\pi. \end{aligned}$$

Then

$$\int_0^{2\pi} \sin(t)\cos^{2k+1}(t+\phi)dt = -\sin(\phi)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}2\pi.$$

3.

$$\begin{aligned} &\int_0^{2\pi} \sin(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt = \int_0^{2\pi} \sin(\tau-\phi)\sin(\tau)\cos^{2k+1}(\tau)d\tau, \\ &= \int_0^{2\pi} (\sin(\tau)\cos(\phi) - \cos(\tau)\sin(\phi))\sin(\tau)\cos^{2k+1}(\tau)d\tau, \\ &= \cos(\phi)\int_0^{2\pi} \sin^2(\tau)\cos^{2k+1}(\tau) - \sin(\phi)\int_0^{2\pi} \sin(\tau)\cos^{2k+2}(\tau), \\ &= 0 - 0. \end{aligned}$$

Then

$$\int_0^{2\pi} \sin(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt = 0.$$

4.

$$\int_0^{2\pi} \cos(t)\sin^{2h}(t+\phi)\cos^{2k+1}(t+\phi)dt = \int_0^{2\pi} \cos(\tau-\phi)\sin^{2h}(\tau)\cos^{2k+1}(\tau)d\tau,$$
$$= \int_0^{2\pi} (\cos(\tau)\cos(\phi) - \sin(\tau)\sin(\phi))\sin^{2h}(\tau)\cos^{2k+1}(\tau)d\tau,$$
$$= \cos(\phi)\int_0^{2\pi} \sin^{2h}(\tau)\cos^{2k+2}(\tau)d\tau - \sin(\phi)\int_0^{2\pi} \sin^{2h+1}(\tau)\cos^{2k+1}(\tau)d\tau,$$
$$= \cos(\phi)\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)}\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4}\pi - 0.$$

Then

$$\int_0^{2\pi} \cos(t)\sin^{2h}(t+\phi)\cos^{2k+1}(t+\phi)dt =$$
$$\cos(\phi)\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)}\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4}\pi.$$

5.

$$\int_0^{2\pi} \cos(t)\cos^{2k+1}(t+\phi)dt = \int_0^{2\pi} \cos(\tau-\phi)\cos^{2k+1}(\tau)d\tau,$$
$$= \int_0^{2\pi} (\cos(\tau)\cos(\phi) - \sin(\tau)\sin(\phi))\cos^{2k+1}(\tau)d\tau,$$
$$= \cos(\phi)\int_0^{2\pi} \cos^{2k+2}(\tau) - \sin(\phi)\int_0^{2\pi} \sin(\tau)\cos^{2k+1}(\tau),$$
$$= \cos(\phi)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}2\pi - 0.$$

Then

$$\int_0^{2\pi} \cos(t)\cos^{2k+1}(t+\phi)dt = \cos(\phi)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}2\pi.$$

6.

$$\int_0^{2\pi} \cos(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt = \int_0^{2\pi} \cos(\tau-\phi)\sin(\tau)\cos^{2k+1}(\tau)d\tau,$$
$$= \int_0^{2\pi} (\cos(\tau)\cos(\phi) - \sin(\tau)\sin(\phi))\sin(\tau)\cos^{2k+1}(\tau)d\tau,$$
$$= \cos(\phi)\int_0^{2\pi} \sin(\tau)\cos^{2k+2}(\tau) - \sin(\phi)\int_0^{2\pi} \sin^2(\tau)\cos^{2k+1}(\tau),$$
$$= 0 - 0.$$

Then

$$\int_0^{2\pi} \cos(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt = 0.$$

□

Remark 4.2.1. *For each $k, h \in \mathbb{N}$ and $\phi \in [-\pi, \pi]$, we obtain*

1.
$$\int_0^{2\pi} \sin(t)\cos^{2k+2h+1}(t+\phi)dt = -\sin(\phi)\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4}\pi.$$

2.
$$\int_0^{2\pi} \cos(t)\cos^{2k+2h+1}(t+\phi)dt = \cos(\phi)\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4}\pi.$$

Substituting these results into the average function (4.18), we find

Corollary 4.2.1. $\forall k, h \in \mathbb{N}$, α, λ, a, b, d *and* $A \in \mathbb{R}$ *and* $\phi \in [-\pi, \pi]$. *The average function of function (4.17) is*

$$\begin{cases} g_{0_1}(A\sin(\phi), A\cos(\phi)) = \alpha A\cos(\phi)\pi \\ -A^{2k+2h+1}\sin(\phi)\pi(a\dfrac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)} \\ .\dfrac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4} + b\dfrac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4}) \\ \qquad -dA^{2k+1}\sin(\phi)2\dfrac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k)....4.2}\pi - \lambda\pi, \\ \\ g_{0_2}(A\sin(\phi), A\cos(\phi)) = -\alpha A\sin(\phi)\pi \\ -A^{2k+2h+1}\cos(\phi)\pi(a\dfrac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)} \\ .\dfrac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4} + b\dfrac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4}) \\ \qquad -2dA^{2k+1}\cos(\phi)\dfrac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}\pi, \end{cases}$$

with $x_1 = A\sin(\phi)$ *and* $x_2 = A\cos(\phi)$, *so*

$$\begin{cases} g_{0_1}(x_1,x_2) = \alpha x_2\pi \\ -(a\dfrac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)}\dfrac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4} \\ \qquad +b\dfrac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4})(x_1^2+x_2^2)^{\frac{2k+2h}{2}}x_1\pi \\ -2d\dfrac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{2k}{2}}x_1\pi - \lambda\pi, \\ \\ g_{0_2}(x_1,x_2) = -\alpha x_1\pi \\ -(a\dfrac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)}\dfrac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4} \\ \qquad +b\dfrac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4})(x_1^2+x_2^2)^{\frac{2k+2h}{2}}x_2\pi \\ -2d\dfrac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{2k}{2}}x_2\pi, \end{cases}$$

and it is continuously differentiable in $\mathbb{R}^2 \setminus \{(0,0)\}$.

Remark 4.2.2. *Note here, by statement (1) from theorem 4.1.4, if* $g_0(x_1,x_2) = 0$ *and* $det(g_0)'(x_1,x_2) \neq 0$, $(x_1,x_2) \in \mathbb{R}^2$, *the solution of the unperturbed system*

$$\begin{cases} u_1(t) = x_1\cos(t) + x_2\sin(t), \\ u_2(t) = -x_1\sin(t) + x_2\cos(t). \end{cases} \tag{4.19}$$

is 2π*-periodic solutions of (4.15).*

Let's now prove the Theorem 4.2.1

Proof of Theorem 4.2.1. Obviously, the conditions (i) and (v) of statement (b) in theorem 4.1.4 for $\Omega = \mathbb{R}^2$ satisfies, because the function (4.16) is a polynomial with respect to binary (x_1,x_2). Here, we are interested to the existence of only one periodic solution (one limit cycle) . For this, we must check the two conditions (4.8) of statement (b) in theorem 4.1.4, then the condition $g_0(v_0) = 0$. To prove this we use corollary 4.2.1.

First, we calculate the Jacobian matrix $g_0(x_1,x_2)$.

$$B = J_{g_0}(x_1,x_2) = (g_0)'(x_1,x_2) = \begin{pmatrix} (g_{0_1})'_{x_1}(x_1,x_2) & (g_{0_1})'_{x_2}(x_1,x_2) \\ (g_{0_2})'_{x_1}(x_1,x_2) & (g_{0_2})'_{x_2}(x_1,x_2) \end{pmatrix},$$

with

$$
\begin{aligned}
(g_{0_1})'_{x_1}(x_1,x_2) = &-(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)} \\
&.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4} \\
&+b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4})(x_1^2+x_2^2)^{\frac{2k+2h}{2}}\pi \\
&-(2k+2h)(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)} \\
&.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4} \\
&+b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4})(x_1^2+x_2^2)^{\frac{2k+2h-2}{2}}x_1^2\pi \\
&-2d\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{2k}{2}}\pi \\
&-2d(2k)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{2k-2}{2}}x_1^2\pi,
\end{aligned}
$$

$$
\begin{aligned}
(g_{0_1})'_{x_2}(x_1,x_2) = &\,\alpha\pi \\
&-(2k+2h)(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)} \\
&.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4} \\
&+b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4})((x_1^2+x_2^2))^{\frac{2k+2h-2}{2}}x_1x_2\pi \\
&-2d(2k)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{2k-2}{2}}x_1x_2\pi,
\end{aligned}
$$

$$
\begin{aligned}
(g_{0_2})'_{x_1}(x_1,x_2) = &-\alpha\pi \\
&-(2k+2h)(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)} \\
&.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4} \\
&+b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4})(x_1^2+x_2^2)^{\frac{2k+2h-2}{2}}x_1x_2\pi, \\
&-2d(2k)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{2k-2}{2}}x_1x_2\pi,
\end{aligned}
$$

and

$$\begin{aligned}(g_{0_2})'_{x_2}(x_1,x_2) = &-(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)}\\&.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4}\\&+b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4})(x_1^2+x_2^2)^{\frac{2k+2h}{2}}\pi\\&-(2k+2h)(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)}\\&.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4}\\&+b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4})(x_1^2+x_2^2)^{\frac{2k+2h-2}{2}}x_2^2\pi\\&-2d\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{2k}{2}}\pi\\&-2d(2k)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{2k-2}{2}}x_2^2\pi.\end{aligned}$$

For the calculation of $det(B)$ and $trace(B)$, we assume

$$\begin{aligned}U = &-(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)}\\&.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4}\\&+b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4})\pi,\\V = &-(2k+2h)(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)}\\&.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4}\\&+b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4})\pi,\end{aligned}$$

$$\begin{aligned}X &= -2d\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}\pi,\\Y &= -2d(2k)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}\pi.\end{aligned}$$

Then

$$\begin{aligned}
trace(B) &= trace(J_{g_0}(x_1,x_2)) = (g_{0_1})'_{x_1}(x_1,x_2) + (g_{0_2})'_{x_2}(x_1,x_2),\\
&= (2U+V)(x_1^2+x_2^2)^{\frac{2k+2h}{2}} + (2X+Y)(x_1^2+x_2^2)^{\frac{2k}{2}},\\
trace(B) &= -2(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h)(2k+2h-2).....(2k+4)}\\
&.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4.2}\\
&+ b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h)(2k+2h-12).....4.2})(x_1^2+x_2^2)^{\frac{2k+2h}{2}}\pi\\
&- 2d(2k+2)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{2k}{2}}\pi.
\end{aligned}$$

$$\begin{aligned}
det(B) &= det(g_0'((x_1,x_2)),\\
&= (g_{0_1})'_{x_1}(x_1,x_2)(g_{0_2})'_{x_2}(x_1,x_2) - (g_{0_2})'_{x_1}(x_1,x_2)(g_{0_1})'_{x_2}(x_1,x_2),\\
&= (U^2+UV)(x_1^2+x_2^2)^{2k+2h} + (2UX+UY+VX)(x_1^2+x_2^2)^{2k+2h-1}\\
&+ (X^2+XY)(x_1^2+x_2^2)^{2k+2h-2} + \alpha^2\pi^2,\\
det(B) &= (2k+2h+1)(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)}\\
&.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4}\\
&+ b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4})^2\pi^2(x_1^2+x_2^2)^{2k+2h}\\
&+ (16k+8h+8)(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)}\\
&.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4.2}\\
&+ b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4.2}).\\
&\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}d(x_1^2+x_2^2)^{2k+2h-1}\\
&+ 4d^2(2k+1)\frac{(2k+1)^2(2k-1)^2.....25.9.1}{(2k+2)^2(2k)^2.....16.4}\pi^2(x_1^2+x_2^2)^{2k} + \alpha^2\pi^2.
\end{aligned}$$

with $x_1 = A\sin(\phi)$ and $x_2 = A\cos(\phi)$, so

$$\begin{aligned} trace(B) = -2(a&\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h)(2k+2h-2).....(2k+4)} \\ &.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4.2} \\ &+b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h)(2k+2h-12).....4.2})A^{2k+2h}\pi \\ &-2d(2k+2)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}A^{2k}\pi < 0 \end{aligned}$$

$$\Rightarrow |A| > \left(\frac{-d(2k+2)(2k+2h)(2k+2h-2).....(2k+4)}{a(2h-1)(2h-3).....5.3.1+b(2k+2h+1)(2k+2h-1).....(2k+3)}\right)^{1/(2h)}$$

is defined if

$$\begin{cases} d < 0, \\ a(2h-1)(2h-3).....5.3.1 + b(2k+2h+1)(2k+2h-1).....(2k+3) > 0. \end{cases}$$

$$\Rightarrow \begin{cases} if \quad d<0,\, a>0 \quad \text{and} \quad b>0, \\ \text{or if} \quad d<0,\, a=0 \quad \text{and} \quad b>0, \\ \text{or if} \quad d<0,\, b=0 \quad \text{and} \quad a>0, \\ \text{or if} \quad d<0,\, a>0,\, b<0 \quad \text{and} \quad \frac{b}{a} > \dfrac{-(2h-1)(2h-3).....5.3.1}{(2k+2h+1)(2k+2h-1).....(2k+3)}, \\ \text{and or if} \quad d<0,\, a<0,\, b>0 \quad \text{and} \quad \frac{b}{a} < \dfrac{-(2h-1)(2h-3).....5.3.1}{(2k+2h+1)(2k+2h-1).....(2k+3)}. \end{cases}$$

Or

$$\begin{cases} d > 0, \\ a(2h-1)(2h-3).....5.3.1 + b(2k+2h+1)(2k+2h-1).....(2k+3) < 0. \end{cases}$$

$$\Rightarrow \begin{cases} if \quad d>0,\, a<0 \quad \text{and} \quad b<0, \\ \text{ or if} \quad d>0,\, a=0 \quad \text{and} \quad b<0, \\ \text{ or if} \quad d>0,\, b=0 \quad \text{and} \quad a<0, \\ \text{ or if} \quad d>0,\, a>0,\, b<0 \quad \text{and} \quad \frac{b}{a} < \dfrac{-(2h-1)(2h-3).....5.3.1}{(2k+2h+1)(2k+2h-1).....(2k+3)}, \\ \text{and or if} \quad d>0,\, a<0,\, b>0 \quad \text{and} \quad \frac{b}{a} > \dfrac{-(2h-1)(2h-3).....5.3.1}{(2k+2h+1)(2k+2h-1).....(2k+3)}. \end{cases}$$

And

$$\begin{aligned} det(B) = \pi^2(\alpha^2 + (2k+2h+1)(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)} \\ .\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4} \\ + b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4})^2 A^{4k+4h} \\ + (16k+8h+8)(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)} \\ .\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4.2} \\ + b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4.2}). \\ \frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2} dA^{4k+4h-2} \\ + 4d^2(2k+1)\frac{(2k+1)^2(2k-1)^2.....25.9.1}{(2k+2)^2(2k)^2.....16.4} A^{4k}) > 0. \end{aligned}$$

Now we check condition $g_0(v_0) = 0$ in theorem 4.1.4.

$$\begin{cases} g_{0_1}(A\sin(\phi), A\cos(\phi)) = \alpha A\cos(\phi)\pi \\ -A^{2k+2h+1}\sin(\phi)\pi(a\dfrac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)} \\ .\dfrac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4} \\ \qquad +b\dfrac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4}) \\ \qquad -dA^{2k+1}\sin(\phi)2\dfrac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k)....4.2}\pi - \lambda\pi = 0, \\ g_{0_2}(A\sin(\phi), A\cos(\phi)) = -\alpha A\sin(\phi)\pi \\ -A^{2k+2h+1}\cos(\phi)\pi(a\dfrac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)} \\ .\dfrac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4} \\ \qquad +b\dfrac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4}) \\ \qquad -2dA^{2k+1}\cos(\phi)\dfrac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}\pi = 0. \end{cases}$$

So we have

$$\begin{aligned} \cos(\phi) &= \frac{A\alpha}{\lambda}, \text{where } \lambda \neq 0, \\ \sin(\phi) &= -(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)} \\ &\quad .\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4} \\ &\quad + b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4})\frac{1}{\lambda}A^{2k+2h+1} \\ &\quad - 2d\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}\frac{1}{\lambda}A^{2k+1}. \end{aligned}$$

And also

$$\begin{aligned} &\left(\tfrac{A\alpha}{\lambda}\right)^2 + (-(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)} \\ &\quad .\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4} \\ &\quad + b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4})\frac{1}{\lambda}A^{2k+2h+1} \\ &\quad - 2d\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}\frac{1}{\lambda}A^{2k+1})^2 = 1, \end{aligned}$$

then

$$\begin{aligned}
&A^2(\alpha^2+(-(a\frac{(2h-1)(2h-3).....5.3.1}{(2k+2h+2)(2k+2h).....(2k+4)}\\
&.\frac{(2k+1)(2k-1).....5.3.1}{(2k+2).....4}\\
&+b\frac{(2k+2h+1)(2k+2h-1).....5.3.1}{(2k+2h+2)(2k+2h).....4})A^{2k+2h}\\
&-2d\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}A^{2k})^2)=\lambda^2.
\end{aligned}$$

□

Example 4.2.1. *Buica. A, Llibre. J and Makarenkove. O in [2] studied the existence and asymptotic stability of the periodic solution of the Van Der Pol equation*

$$\ddot{u}+\epsilon(u^2-1)\dot{u}+(1+\alpha\epsilon)u=\epsilon\lambda\sin t.$$

Due to our result in Theorem 4.2.1, this equation has exactly one cycle limit asymptotically stable if the following three equations are verified

$$\begin{cases}
A^2\left(\alpha^2+\left(1-\frac{A^2}{4}\right)^2\right)=\lambda^2,\\
det(B)=1+\alpha^2-A^2+\frac{3}{16}A^4>0,\\
trace(B)=2-A^2<0.
\end{cases}$$

Remark 4.2.3. *In [49], X. Ioakim proves that the equation $\ddot{u}+u+\epsilon(u^{2q}-1)\dot{u}^{p+1}=0$, where $p\in\mathbb{N}_0$ is even, $q\in\mathbb{N}$ and $0<\epsilon\ll 1$ has the unique limit cycle and it is simple and stable for the amplitude*

$$A=\left(\frac{(p+2q+2)(p+2q)...(2q+2)}{(p+2)p...4\cdot 2}\frac{2q(2q-2)...4\cdot 2}{(2q-1)(2q-3)...3\cdot 1}\right)^{\frac{1}{2q}}.$$

Through in theorem 4.2.1, we prove that the equation (4.13) has a unique limit cycle and asymptotically stable for the amplitude

$$|A|>\left(\frac{-d(2k+2)(2k+2h)(2k+2h-2).....(2k+4)}{a(2h-1)(2h-3).....5.3.1+b(2k+2h+1)(2k+2h-1).....(2k+3)}\right)^{1/(2h)}.$$

Theorem 4.2.2. *For all $k\in\mathbb{N}$, $\forall f,d,\alpha$ and $\lambda\in\mathbb{R}$ and $v_0\in\mathbb{R}^2$ and $0<\epsilon<<1$ if*

1. $A^2\left(\alpha^2+\left(-f|A|A^{2k}\frac{4}{(2k+3)\pi}-dA^{2k}\frac{2(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}\right)^2\right)-\lambda^2=0,$

2. $trace(g_0)'(v_0)=-4fA^{2k+1}-d\frac{(2k+2)2(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}A^{2k}\pi<0,$

3. $det(g_0)'(v_0)=f^2\frac{32k+32}{(2k+3)^2}A^{4k+2}-8(4k+1)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}A^{4k+1}fd\pi$
$+d^2\frac{4(2k+1)^3(2k-1).....9.1}{(2k+2)^2(2k)^2.....16.4}A^{4k}\pi^2>0,$

then the equation (4.14) has a unique periodic solution and asymptotically stable.

In order to prove Theorem 4.2.2, we use Levinson's change of variables (see [31]), which allows us to rewrite equation (4.14) as a system where the second Bogolubov's theorem is applied, for this we put $(z_1,z_2)=(u,\dot{u})$, so the equation (4.14) becomes

$$\begin{cases}\dot{z}_1'=z_2,\\ \dot{z}_2=-\epsilon(f|z_1|+d)z_2^{2k+1}-(1+\alpha\epsilon)z_1+\epsilon\lambda\sin t.\end{cases}\tag{4.20}$$

By performing the change of variable

$$\begin{pmatrix}z_1(t)\\ z_2(t)\end{pmatrix}=\begin{pmatrix}\cos(t)&\sin(t)\\ -\sin(t)&\cos(t)\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix},$$

we transform this system into an equation of the form (4.1), and we use the Lipschitz property (in the second variable) of g. Then system (4.20) takes the form

$$\begin{cases}\dot{x}_1=\epsilon(\sin(-t)(-\alpha(\cos(t)x_1+\sin(t)x_2)-(f|\cos(t)x_1+\sin(t)x_2|+d)(-\sin(t)x_1\\ +\cos(t)x_2)^{2k+1}+\lambda\sin(t))),\\ \dot{x}_2=\epsilon(\cos(-t)(-\alpha(\cos(t)x_1+\sin(t)x_2)-(f|\cos(t)x_1+\sin(t)x_2|+d)(-\sin(t)x_1\\ +\cos(t)x_2)^{2k+1}+\lambda\sin(t))).\end{cases}$$

Such that

$$\begin{cases}g_1(t,x_1,x_2)=\alpha(\cos(t)x_1+\sin(t)x_2)\sin(t)+(f|\cos(t)x_1+\sin(t)x_2|+d)\\ \qquad(-\sin(t)x_1+\cos(t)x_2)^{2k+1}\sin(t)-\lambda\sin^2(t),\\ g_1(t,x_1,x_2)=-\alpha(\cos(t)x_1+\sin(t)x_2)\cos(t)-(f|\cos(t)x_1+\sin(t)x_2|+d)\\ \qquad(-\sin(t)x_1+\cos(t)x_2)^{2k+1}\cos(t)+\lambda\sin(t)\cos(t).\end{cases}\tag{4.21}$$

With $x_1 = A\sin(\phi)$ and $x_2 = A\cos(\phi)$, so

$$\begin{cases} g_1(t, A\sin(\phi), A\cos(\phi)) = \alpha A\sin(t)\sin(t+\phi) + fA^{2k+1}|A|\sin(t)|\sin(t+\phi)|\cos^{2k+1}(t+\phi) \\ \qquad\qquad +dA^{2k+1}\sin(t)\cos^{2k+1}(t+\phi) - \lambda\sin^2(t), \\ g_2(t, A\sin(\phi), A\cos(\phi)) = -\alpha A\cos(t)\sin(t+\phi) - fA^{2k+1}|A|\cos(t)|\sin(t+\phi)|\cos^{2k+1}(t+\phi) \\ \qquad\qquad -dA^{2k+1}\cos(t)\cos^{2k+1}(t+\phi) + \lambda\sin(t)\cos(t). \end{cases} \tag{4.22}$$

The corresponding average function g_0 is calculated according to the formula (4.2) and it is given by

$$g_0(v_0) = \int_0^{2\pi} g(t, v_0)dt = \begin{cases} g_{0_1}(A\sin(\phi), A\cos(\phi)) = g_{0_1}(v_0) = \int_0^{2\pi} g_1(t, v_0)dt, \\ g_{0_2}(A\sin(\phi), A\cos(\phi)) = g_{0_2}(v_0) = \int_0^{2\pi} g_2(t, v_0)dt. \end{cases} \tag{4.23}$$

with

$$\begin{aligned} & g_{0_1}(A\sin(\phi), A\cos(\phi)) \\ &= \int_0^{2\pi} g_1(t, A\sin(\phi), A\cos(\phi))dt \\ &= \int_0^{2\pi} \alpha A\sin(t)\sin(t+\phi)dt + f|A|A^{2k+1}\int_0^{2\pi} \sin(t)|\sin(t+\phi)|\cos^{2k+1}(t+\phi)dt \\ &+ \int_0^{2\pi} dA^{2k+1}\sin(t)\cos^{2k+1}(t+\phi)dt - \int_0^{2\pi} \lambda\sin^2(t)dt. \end{aligned}$$

and

$$\begin{aligned} & g_{0_2}(A\sin(\phi), A\cos(\phi)) \\ &= \int_0^{2\pi} g_2(t, A\sin(\phi), A\cos(\phi))dt \\ &= -\int_0^{2\pi} \alpha A\cos(t)\sin(t+\phi)dt - fA^{2k+1}|A|\int_0^{2\pi} \cos(t)|\sin(t+\phi)|\cos^{2k+1}(t+\phi)dt \\ &- \int_0^{2\pi} dA^{2k+1}\cos(t)\cos^{2k+1}(t+\phi)dt + \int_0^{2\pi} \lambda\cos(t)\sin(t)dt. \end{aligned}$$

with

$$\begin{aligned}
&\int_0^{2\pi} \alpha A \sin(t)\sin(t+\phi)dt = \alpha A\cos(\phi)\pi,\\
&\int_0^{2\pi} dA^{2k+1}\sin(t)\cos^{2k+1}(t+\phi)dt\\
&= -dA^{2k+1}\sin(\phi)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}2\pi,\\
&\int_0^{2\pi} \lambda\sin^2(t)dt = \lambda\pi,\\
&\int_0^{2\pi} \alpha A \cos(t)\sin(t+\phi)dt = \alpha A\sin(\phi)\pi,\\
&\int_0^{2\pi} dA^{2k+1}\cos(t)\cos^{2k+1}(t+\phi)dt\\
&= dA^{2k+1}\cos(\phi)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}2\pi,\\
&\int_0^{2\pi} \lambda\cos(t)\sin(t)dt = 0.
\end{aligned}$$

To calculate the average function (4.23) we use the following results.

Lemma 4.2.2. *For each $k \in \mathbb{N}$ and $\phi \in [-\pi, \pi]$*

1. $\int_0^{2\pi} \sin(t)|\sin(t+\phi)|\cos^{2k+1}(t+\phi)dt = \dfrac{-4}{2k+3}\sin(\phi).$
2. $\int_0^{2\pi} \cos(t)|\sin(t+\phi)|\cos^{2k+1}(t+\phi)dt = \dfrac{4}{2k+3}\cos(\phi).$

Proof. 1. $\forall\phi \in [0, \pi]$, $k \in \mathbb{N}$ and

$$|\sin(t+\phi)| = \begin{cases} \sin(t+\phi) \text{ if } \phi+t \in [0,\pi] \\ -\sin(t+\phi) \text{ if } \phi+t \in [-\pi, 0] \end{cases}$$

$$\Rightarrow |\sin(t+\phi)| = \begin{cases} \sin(t+\phi) \text{ if } t \in [-\phi, \pi-\phi] \\ -\sin(t+\phi) \text{ if } t \in [-\pi-\phi, -\phi]. \end{cases}$$

Then

$$\begin{aligned}
&\int_0^{2\pi} \sin(t)|\sin(t+\phi)|\cos^{2k+1}(t+\phi)dt \\
&= \int_0^{\pi-\phi} \sin(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt \\
&- \int_{\pi-\phi}^{-\phi} \sin(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt \\
&+ \int_{-\phi}^{2\pi} \sin(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt \\
&= \frac{-1}{2k+3}\sin(\phi) - \frac{2}{2k+3}\sin(\phi) + \frac{-1}{2k+3}\sin(\phi) = \frac{-4}{2k+3}\sin(\phi).
\end{aligned}$$

If $\phi \in [-\pi, 0]$ then

$$\begin{aligned}
&\int_0^{2\pi} \sin(t)|\sin(t+\phi)|\cos^{2k+1}(t+\phi)dt \\
&= -\int_0^{-\phi} \sin(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt \\
&+ \int_{-\phi}^{\pi-\phi} \sin(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt \\
&- \int_{\pi-\phi}^{2\pi} \sin(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt \\
&= -\frac{1}{2k+3}\sin(\phi) + \frac{-2}{2k+3}\sin(\phi) - \frac{1}{2k+3}\sin(\phi) = \frac{-4}{2k+3}\sin(\phi).
\end{aligned}$$

2. This integral is demonstrated in the same way.

$$\begin{aligned}
&\int_0^{2\pi} \cos(t)|\sin(t+\phi)|\cos^{2k+1}(t+\phi)dt \\
&= \int_0^{\pi-\phi} \cos(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt \\
&- \int_{\pi-\phi}^{-\phi} \cos(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt \\
&+ \int_{-\phi}^{2\pi} \cos(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt \\
&= \frac{1}{2k+3}\cos(\phi) - \frac{-2}{2k+3}\cos(\phi) + \frac{1}{2k+3}\sin(\phi) = \frac{4}{2k+3}\cos(\phi)
\end{aligned}$$

If $\phi \in [-\pi, 0]$ then

$$\begin{aligned}
&\int_0^{2\pi} \cos(t)|\sin(t+\phi)|\cos^{2k+1}(t+\phi)dt \\
&= -\int_0^{-\phi} \cos(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt \\
&+ \int_{-\phi}^{\pi-\phi} \cos(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt \\
&- \int_{\pi-\phi}^{2\pi} \cos(t)\sin(t+\phi)\cos^{2k+1}(t+\phi)dt \\
&= -\frac{-1}{2k+3}\cos(\phi) + \frac{2}{2k+3}\cos(\phi) - \frac{-1}{2k+3}\sin(\phi) = \frac{4}{2k+3}\cos(\phi).
\end{aligned}$$

□

Substituting these results into the average function (4.23), we get

Corollary 4.2.2. *$\forall k \in \mathbb{N}$, α, λ, f, d and $A \in \mathbb{R}$ and $\phi \in [-\pi, \pi]$. The average function of function (4.22) is*

$$\begin{cases}
g_{0_1}(A\sin(\phi), A\cos(\phi)) \\
= \alpha A\cos(\phi)\pi - f|A|A^{2k+1}\frac{4}{2k+3}\sin(\phi) - dA^{2k+1}\sin(\phi)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}2\pi - \lambda\pi, \\
\\
g_{0_2}(A\sin(\phi), A\cos(\phi)) \\
= -\alpha A\sin(\phi)\pi - f|A|A^{2k+1}\frac{4}{2k+3}\cos(\phi) - dA^{2k+1}\cos(\phi)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}2\pi.
\end{cases}$$

With $x_1 = A\sin(\phi)$ and $x_1 = A\cos(\phi)$, so

$$\begin{cases}
g_{0_1}(x_1, x_2) &= \alpha x_2\pi - \frac{4}{2k+3}f(x_1^2+x_2^2)^{\frac{2k+1}{2}}x_1 - d\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}2(x_1^2+x_2^2)^{\frac{2k}{2}}x_1\pi - \lambda\pi, \\
\\
g_{0_2}(x_1, x_2) &= -\alpha x_1\pi - \frac{4}{2k+3}f(x_1^2+x_2^2)^{\frac{2k+1}{2}}x_2 - d\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}2(x_1^2+x_2^2)^{\frac{2k}{2}}x_2\pi.
\end{cases}$$

and it is continuously differentiable in $\mathbb{R}^2 \setminus \{(0,0)\}$.

Remark 4.2.4. *Note here, by statement (1) from theorem 4.1.4, if $g_0(x_1, x_2) = 0$ and $det(g_0)'(x_1, x_2) \neq 0$, $(x_1, x_2) \in \mathbb{R}^2$, the solution of the unperturbed system*

$$\begin{cases}
u_1(t) = x_1\cos(t) + x_2\sin(t), \\
u_2(t) = -x_1\sin(t) + x_2\cos(t).
\end{cases} \tag{4.24}$$

is 2π-periodic solutions of (4.20).

Lemma 4.2.3. *Let $v_0 \in \mathbb{R}^2, v_0 \neq 0$. The function (4.21) satisfies (v) in theorem 4.1.4 for any f, d, α and $\lambda \in \mathbb{R}$ and $k \in \mathbb{N}$.*

Proof. In order to prove that the function (4.21) satisfies condition (v), we use the same as in [2]. So, it is sufficient to demonstrate that $g(t,v) = |[v]_1 \cos t + [v]_2 \sin t|$ satisfies this condition, where $[v]_i$ is the i-th component of the vector $v \in \mathbb{R}^2$ and $g : [0, 2\pi].\mathbb{R}^2 \to \mathbb{R}$.
In the case that $[v_0]_2 \neq 0$, denote $\theta(v) = \arctan\left(-\frac{[v]_1}{[v]_2}\right)$, while when $[v_0]_2 = 0$, denote $\theta(v) = \arctan\left(-\frac{[v]_1}{[v]_2}\right)$ for $[v_0]_1[v]_2 < 0$, $\theta(v) = \frac{\pi}{2}$ and, respectively, $\theta(v) = \arctan\left(-\frac{[v]_1}{[v]_2}\right) + \pi$ for $[v_0]_1[v]_2 > 0$. In any case notice that the function $v \to \theta(v)$ is continuous in every sufficiently small neighborhood of v_0. Fix $\tilde{\gamma} > 0$. Let M be the union of two intervals centered in $\theta(v_0)$ (when $\theta(v_0) < 0$, take $\theta(v_0) + 2\pi$ instead) and, respectively, $\theta(v_0) + \pi$, each of length $\frac{\tilde{\gamma}}{2}$. Denote them M_1 and M_2. Take $\tilde{\delta} > 0$ such that $\theta(v) \in M_1$ for all $v \in B_{\tilde{\delta}}(v_0)$. Of course, also $\theta(v) + \pi \in M_2$ for all $\|v - v_0\| \leq \tilde{\delta}$. This implies that for fixed $t[0, 2\pi] \setminus M$, $[v]_1 \cos t + [v]_2 \sin t$ has constant sign for all $v \in B_{\tilde{\delta}}(v_0)$, that, further, gives that $g(t,.)$ is differentiable and $g\prime_v(t,v) = g\prime_v(t,v_0)$ for all $v \in B_{\tilde{\delta}}(v_0)$. Hence (v) is fulfilled. □

Let's now prove theorem 4.2.2

Proof of Theorem 4.2.2. Obviously, the condition (i) of statement (b) in theorem 4.1.4 for $\Omega = \mathbb{R}^2$ satisfies, because the function (4.21) is absolute value function respect to binary (x_1, x_2).
Here, We are interested by the existence of only one periodic solution (one limit cycle). Before proving the condition $g_0(v_0) = 0$, we must check the two conditions (4.8) in theorem 4.1.4 . To prove this we use the corollary 4.2.2.
Calculating now the Jacobian matrix $g_0(x_1, x_2)$

$$B = J_{g_0}(x_1, x_2) = (g_0)'(x_1, x_2) = \begin{pmatrix} (g_{0_1})'_{x_1}(x_1, x_2) & (g_{0_1})'_{x_2}(x_1, x_2) \\ (g_{0_2})'_{x_1}(x_1, x_2) & (g_{0_2})'_{x_2}(x_1, x_2) \end{pmatrix}.$$

Such that

$$(g_{0_1})'_{x_1}(x_1,x_2) = -\frac{4}{2k+3}f(x_1^2+x_2^2)^{\frac{2k+1}{2}} - \frac{8k+4}{2k+3}f(x_1^2+x_2^2)^{\frac{2k-1}{2}}x_1^2$$
$$- d\frac{2(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{2k}{2}}\pi$$
$$- d\frac{(2k)2(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{2k-2}{2}}x_1^2\pi,$$
$$(g_{0_1})'_{x_2}(x_1,x_2) = \alpha\pi - \frac{8k+4}{2k+3}f(x_1^2+x_2^2)^{\frac{2k-1}{2}}x_1x_2$$
$$- d\frac{(2k)2(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{2k-2}{2}}x_1x_2\pi,$$

and

$$(g_{0_2})'_{x_1}(x_1,x_2) = -\alpha\pi - \frac{8k+4}{2k+3}f(x_1^2+x_2^2)^{\frac{2k-1}{2}}x_1x_2$$
$$- d\frac{(2k)2(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{2k-2}{2}}x_1x_2\pi,$$
$$(g_{0_2})'_{x_2}(x_1,x_2) = -\frac{4}{2k+3}f(x_1^2+x_2^2)^{\frac{2k+1}{2}} - \frac{8k+4}{2k+3}f(x_1^2+x_2^2)^{\frac{2k-1}{2}}x_2^2$$
$$- d\frac{2(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{2k}{2}}\pi$$
$$- d\frac{(2k)2(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{2k-2}{2}}x_2^2\pi.$$

Now we calculate $det(B)$ and $trace(B)$, let

$$u = -f\frac{4}{2k+3}, v = -f\frac{8k+4}{2k+3},$$
$$x = -d\frac{2(2k+1)(2k-1).....5.3.1}{(2k+2)(n-1).....4.2}\pi,$$
$$y = -d\frac{(2k)2(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}\pi.$$

Then

$$
\begin{aligned}
trace(B) &= trace(J_{g_0}(x_1,x_2)) = (g_{0_1})'_{x_2}(x_1,x_2) + (g_{0_2})'_{x_1}(x_1,x_2)\\
&= (2u+v)(x_1^2+x_2^2)^{\frac{2k+1}{2}} + (2x+y)(x_1^2+x_2^2)^{\frac{2k}{2}}\\
&= \frac{-8k-12}{2k+3} f(x_1^2+x_2^2)^{\frac{2k+1}{2}}\\
&- d\frac{(2k+2)2(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{2k}{2}}\pi\\
\Rightarrow trace(B) &= -4f(x_1^2+x_2^2)^{\frac{2k+1}{2}} - d\frac{(2k+2)2(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{2k}{2}}\pi.
\end{aligned}
$$

$$
\begin{aligned}
det(B) &= det(g_0'(x_1,x_2)) = (g_{0_1})'_{x_1}(x_1,x_2)(g_{0_2})'_{x_2}(x_1,x_2) - (g_{0_2})'_{x_1}(x_1,x_2)(g_{0_1})'_{x_2}(x_1,x_2)\\
&= (u^2+uv)(x_1^2+x_2^2)^{2k+1} + (2ux+uy+vx)(x_1^2+x_2^2)^{\frac{4k+1}{2}} + (x^2+xy)(x_1^2+x_2^2)^{2k}\\
\Rightarrow det(B) &= f^2\frac{32k+32}{(2k+3)^2}(x_1^2+x_2^2)^{2k+1}\\
&- 8(4k+1)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}(x_1^2+x_2^2)^{\frac{4k+1}{2}} fd\pi\\
&+ d^2\frac{4(2k+1)^3(2k-1).....9.1}{(2k+2)^2(2k)^2.....16.4}(x_1^2+x_2^2)^{2k}\pi^2.
\end{aligned}
$$

with $x_1 = A\sin(\phi)$ and $x_2 = A\cos(\phi)$, so

$$
trace(B) = -4fA^{2k+1} - d\frac{(2k+2)2(2k+1)(2k-1).....5.3.1}{(2k+2)(2k)...4.2}A^{2k}\pi < 0
$$

$$
\Rightarrow |A| > \frac{-d}{2f}\frac{(2k+2)(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}\pi, \text{ where } f \neq 0.
$$

is defined if

$$
\begin{cases} d < 0, \\ f > 0. \end{cases} \quad \text{or} \quad \begin{cases} d > 0, \\ f < 0. \end{cases}
$$

And if

$$
f = 0, trace(B) = -d\frac{(2k+2)2(2k+1)(2k-1).....5.3.1}{(2k+2)(2k)...4.2}A^{2k}\pi < 0,
$$

then $d > 0$.

$$det(B) = f^2 \frac{32k+32}{(2k+3)^2} A^{4k+2} - 8(4k+1)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2} A^{4k+1} f d\pi$$
$$+ d^2 \frac{4(2k+1)^3(2k-1).....9.1}{(2k+2)^2(2k)^2.....16.4} A^{4k}\pi^2 > 0.$$

Now we check condition $g_0(v_0) = 0$ in theorem 4.1.4.

$$\begin{cases} g_{0_1}(A\sin(\phi), A\cos(\phi)) \\ = \alpha A\cos(\phi)\pi - f|A|A^{2k+1}\frac{4}{2k+3}\sin(\phi) - dA^{2k+1}\sin(\phi)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}2\pi - \lambda\pi = 0, \\ g_{0_2}(A\sin(\phi), A\cos(\phi)) \\ = -\alpha A\sin(\phi)\pi - f|A|A^{2k+1}\frac{4}{2k+3}\cos(\phi) - dA^{2k+1}\cos(\phi)\frac{(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}2\pi = 0. \end{cases}$$

So we have

$$\cos(\phi) = \frac{A\alpha}{\lambda}, \text{ where } \lambda \neq 0,$$
$$\sin(\phi) = -f|A|A^{2k+1}\frac{4}{(2k+3)\lambda\pi} - dA^{2k+1}\frac{2(2k+1)(2k-1).....5.3.1}{\lambda(2k+2)(2k).....4.2},$$

then

$$\left(\frac{A\alpha}{\lambda}\right)^2 + \left(-f|A|A^{2k+1}\frac{4}{(2k+3)\lambda\pi} - dA^{2k+1}\frac{2(2k+1)(2k-1).....5.3.1}{\lambda(2k+2)(2k).....4.2}\right)^2 = 1,$$
$$A^2\left(\alpha^2 + \left(-f|A|A^{2k}\frac{4}{(2k+3)\pi} - dA^{2k}\frac{2(2k+1)(2k-1).....5.3.1}{(2k+2)(2k).....4.2}\right)^2\right) = \lambda^2.$$

□

Example 4.2.2. *Buica. A, Llibre. J and Makarenkove. O in [2] studied the existence and asymptotic stability of the periodic solution of the Van Der Pol equation*

$$\ddot{u} + \epsilon(|u|-1)\dot{u} + (1+\alpha\epsilon)u = \epsilon\lambda\sin t.$$

Due to our result in Theorem 4.2.2, this equation has exactly one cycle limit asymptotically stable if the following three equations are verified

$$\begin{cases} A^2\left(\alpha^2 + \left(1 - \frac{4}{3\pi}|A|\right)^2\right) = \lambda^2, \\ det(B) = \pi^2(1+\alpha^2) + \frac{32}{9}A^2 - 2\pi|A| > 0, \\ trace(B) = 2\pi - 4|A| < 0. \end{cases}$$

4.3 Conclusions

In this chapter, we have proved the conditions of existence, uniqueness and the asymptotic stability of periodic solutions for an infinite set of nonlinear differential equations, which in turn facilitates the researcher to put conditions on all possible equations directly and quickly.

General conclusion

In this thesis, we presented some results related to the study of one of the important models in turbulent dynamical systems, which is the Van der Paul equation. Where we proved the approximate analytical solutions to the Van der Paul equation in its general form through various perturbation methods: the simple perturbation method, the Lindsted Poincaré method, and the average method, then compared them with each other and with the real solution. We studied the behavior of all these solutions in detail. We found that there are several variables that control the behavior of the approximate solutions, which are the parameter ϵ, the amplitude A and the variable t.

We also studied van der Pol systems in fractional order in its general form, where we proposed several versions of the modified Van der Pol equation. These modifications consisted of introducing a fractional time derivative into the Satie space equations of the standard van der Pol oscillator. We have dedicated our study to a new generalized Van Der Pol partial system. The resulting fractional-order Van Der Pol oscillator is analyzed in the time and frequency domains, using phase images, and spectral analysis. The dynamics of the fractal order is illustrated by numerical simulations of the proposed schemes using approximations to the operators of the fractal order. The results reveal that systems of fractional order can exhibit different behavior than those obtained with a standard Van Der Pol oscillator depending on the derivative of the system (or order of the system). where we investigate the significant differences in the limit cycle, revealing a significant influence of ϵ and α and n on the dynamics of the system. Fractional order can act as a modulation parameter that may be useful to better understand and control these systems.

It is well known that the classical Van der Pol equation can display chaos for a finite set of parameters, even it is not always easy to find. The same difficulty can be expected in the case of a new generalized Van Der Pol partial system. This is a topic that will be investigated in future research.

Finally, we presented another aspect of the study of this type of equations. Where we proved the conditions of existence, uniqueness and stability close to periodic solutions for an infinite set of non-linear differential equations, which in turn facilitate the researcher to take and use them on all possible equations directly and quickly. These results were obtained by applying a second Bogolyubov's theorem.

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