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Theme

Behavior of Solutions to Some Systems of Differential Equations of Non-Integer Order: Comparison and Maximum Principle

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Dedication

With profound affection and immense gratitude, I dedicate this work :

*To the man I am proud of and honored to call my father, **"Youssef Ben Othmane "** , the school of my childhood, who was my shadow during all the years of my studies, who instilled in me the strength to become the woman I am today, my eternal example, my moral and material support. Though you may be gone, your spirit lives on in every aspect of my life. With profound pride and gratitude, I dedicate this thesis to you, Dad, as a reflection of the values and lessons you have instilled in me. You were not just my father, but my mentor, my rock, and my greatest source of inspiration. May God have mercy on you and make you dwell in the highest paradise.*

My mother who carried me for nine months, the symbol of tenderness, who sacrificed herself for my happiness and my success. Though you may be gone, your spirit remains a beacon of light in my life.

To my sisters and their husbands, my brothers and their wives and all their children, whose steadfast love and sacrificial support have been guiding lights along my journey to fulfillment.

*To those rare, beautiful friendships, to my best friend, who knows the chapters I left unsaid, who taught me the true meaning of friendship **"Asma Guemoula"** and all her family.*

*To my professor, who taught me the value of thinking and the power of the pen, who saw in me what I failed to see in myself **"nasar ajroud"**.*

To the challenges that made me stronger, the past that shaped me, the present that nurtures me, and the future that awaits me.

To all those who hold a special place in my heart, and to those whom I may have unintentionally overlooked in my acknowledgments.

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This thesis is not only a reflection of my work but also a testament to the support and belief these individuals have invested in me. Thank you.

ملخص

في هذه الأطروحة قمنا بدراسة مسألة جديدة تعرف بمسألة كوشي للمعادلات التفاضلية غير الخطية ذات الرتبة الكسرية التي تحتوي على المشتقة الكسرية لريمان-ليوفيل الموزونة لدالة بالنسبة لدالة أخرى. تعتبر هذه المسألة تعميمًا لمسائل كوشي المعروفة التي تتضمن مشتقات ريمان-ليوفيل الكسرية.

في البداية، أثبتنا تكافؤ هذه المسألة مع معادلة فولتيرا التكاملية غير الخطية من النوع الثاني.

بعد ذلك، قمنا بالتحقق من وجود وتفرد الحل لمسألة كوشي المعطاة، وإثبات ذلك من خلال استخدام نظرية النقطة الثابتة لبناخ.

بالإضافة إلى ذلك، قمنا بتطوير نظريات مقارنة للمتباينات التفاضلية الكسرية، تتضمن هذه المتباينات عوامل تفاضل ريمان-ليوفيل الموزونة لدالة فيما يتعلق بدالة أخرى ، ذات الترتيب الكسري p، $0 < p < 1$ ،والتي تعتمد أساسًا على تقدير جديد لمشتق ريمان-ليوفيل الكسري الموزون للدالة فيما يتعلق بالدالة اخرى عند النقاط القصوى والذي بدوره تم اثباته.

أخيرًا، اختتمنا بملخص يسلط الضوء على مساهماتنا في هذه الأطروحة، بالإضافة إلى بعض المسائل المفتوحة المحتملة التي سيتم التحقيق فيها في المستقبل باعتبارها اتجاهات بحثية جديدة.

الكلمات المفتاحية: التكاملات الكسرية الموزونة ومشتقاتها؛ المعادلات التفاضلية الكسرية؛ معادلة فولتيرا التكاملية؛ عدم المساواة التفاضلية الكسرية؛ مشكلة كوشي؛ نظرية النقطة الثابتة.

Abstract

In this thesis, we study a new problem known as the Cauchy problem for nonlinear differential equations of fractional order containing the weighted Riemann-Liouville fractional derivative of a function with respect to another function. This problem generalizes the well-known Cauchy-type problems involving Riemann-Liouville fractional derivatives.

Initially, we demonstrated the equivalence of this problem with the nonlinear Volterra integral equation of the second kind. Subsequently, we investigated the existence and uniqueness of the solution to the given Cauchy problem, establishing proof through the utilization of Banach's fixed point theorem and the method of successive approximations.

In addition, we have developed comparison theorems for fractional differential inequalities, these inequalities involve weighted Riemann-Liouville differential operators of a function with respect to functions of order ϑ, $0 < \vartheta < 1$. Which is essentially based on a new estimate of the weighted Riemann-Liouville fractional derivative of a function with respect to functions at their extreme points obtained.

Lastly, we concluded with a summary highlighting our contributions to this thesis, as well as, some possible open problems to be investigated in the future as new research directions.

Key words: weighted fractional integrals and derivatives; comparison theorems; fractional differential equations; volterra integral equation; fractional differential inequalities; Cauchy problem; fixed point theorem.

Résumé

Dans cette thèse, nous étudions un nouveau problème connu sous le nom de problème de Cauchy pour les équations différentielles non linéaires d'ordre fractionnaire contenant la dérivée fractionnaire de Riemann-Liouville pondérée d'une fonction par rapport à une autre fonction. Ce problème est considéré comme une généralisation des problèmes de type Cauchy connus impliquant des dérivées fractionnaires de Riemann-Liouville.

Initialement, nous avons démontré l'équivalence de ce problème avec l'équation intégrale de Volterra non linéaire du deuxième type. Ensuite, nous avons étudié l'existence et l'unicité de la solution au problème de Cauchy donné, établissant la preuve grâce à l'utilisation du théorème du point fixe de Banach et de la méthode des approximations successives.

De plus, nous avons développé des théorèmes de comparaison pour les inégalités différentielles fractionnaires, ces inégalités impliquent des opérateurs différentiels de Riemann-Liouville pondérés d'une fonction par rapport à des fonctions d'ordre ϑ, où $0 < \vartheta < 1$. Cela repose essentiellement sur une nouvelle estimation de la dérivée fractionnaire de Riemann-Liouville pondérée d'une fonction par rapport à des fonctions à leurs points extrêmes préalablement obtenue.

Enfin, nous avons conclu par un résumé mettant en évidence nos contributions à cette thèse, ainsi que quelques problèmes ouverts à explorer à l'avenir en tant que nouvelles directions de recherche.

Mots clés : intégrales fractionnaires pondérées et dérivées ; théorèmes de comparaison ; équations différentielles fractionnaires ; équation intégrale de Volterra ; inégalités différentielles fractionnaires ; Problème de Cauchy ; théorème du point fixe...

Contents

INTRODUCTION

Fractional calculus extends the concepts of differentiation and integration to non-integer orders, making it a significant branch of mathematical analysis. Unlike classical calculus, which focuses on derivatives and integrals of functions of integer order, fractional calculus deals with derivatives and integrals of functions with non-integer orders. Tracing its roots back to the contributions of Leibniz, Euler, and Laplace during the 17th and 18th centuries, fractional calculus only began to gain substantial attention in the 20th century with the development of advanced mathematical tools. These tools allowed for the rigorous definition and analysis of fractional derivatives and integrals. Fractional calculus has found extensive applications across diverse fields owing to its efficacy in modeling complex phenomena. Some notable fields where fractional calculus has been widely applied include:

- **Physics:** Fractional calculus has found widespread application in elucidating phenomena such as anomalous diffusion, the behavior of viscoelastic materials, the characterization of fractal structures, and the modeling of non-local interactions.
- **Engineering:** In engineering, fractional calculus has been applied in control theory, signal processing, image processing, and mechanical systems to model and analyze complex dynamic behaviors.
- **Biology and Medicine:** Fractional calculus plays a pivotal role in modeling diverse biological systems, including population dynamics, cell growth, and the spread of diseases. Furthermore, its application extends to critical areas such as medical imaging, biomechanics.

- ▶ **Finance and Economics:** Fractional calculus has been utilized extensively to model the complexities of financial markets, including aspects such as option pricing, risk management, and economic systems characterized by long-range memory and non-local interactions.
- ▶ **Geoscience:** Geophysical processes like seismic wave propagation, groundwater flow, and fractal geology have been studied using fractional calculus to capture non-local and non-Markovian behaviors.
- ▶ **Chemistry:** Fractional calculus has been used in modeling chemical reactions, diffusion processes, and reaction kinetics, especially in systems exhibiting anomalous transport phenomena.
- ▶ **Materials Science:** Fractional calculus has been applied in modeling the properties of complex materials, such as polymers, gels, and porous media, to capture their non-local behaviors and viscoelastic properties.
- ▶ **Electronics and Telecommunications:** Fractional calculus has been employed in designing filters, antennas, communication channels, and electronic circuits to account for memory effects and non-local interactions.

These instances represent only a fraction of the diverse applications of fractional calculus. Its utility in modeling complex phenomena, which classical calculus often fails to adequately address, has led researchers to explore its potential in new and expanding areas.

Fractional differential equations (FDEs) represent an extension of classical differential equations, encompassing derivatives of non-integer orders. Recent years have witnessed significant progress in the theoretical treatment of the existence and uniqueness of solutions for Cauchy-type problems related to fractional differential equations. This advancement is increasingly recognized for its applicability in modeling real-world phenomena across various domains, thereby enhancing the theoretical framework considerably beyond that of conventional ordinary differential equations (see references [1, 2, 3]). Fractional differential equations find applications across numerous scientific disciplines, encompassing but not restricted to engineering [4], chemistry [5, 6] physics[7, 8], and various other fields [9, 10].

The increasing intricacy observed in practical applications has prompted scholars to expand the theoretical underpinnings of fractional derivatives. This progression has led to the emergence of weighted fractional derivatives as a mechanism for extending multiple established fractional operators (see references [11, 12, 13, 14, 15, 16, 17]). Consequently, these advancements enable enhanced analyses and modeling of intricate systems across a multitude of disciplines. Their utility transcends various fields, including signal processing, image analysis, finance, physics, and engineering.

In [18, 19], Agrawal introduced weighted fractional derivatives, a form of generalized fractional derivatives, and elucidated several of their properties. However, the specifics concerning the function spaces wherein these operators operate were not expounded upon. Subsequently, in [20, 21], extensive research was conducted on weighted fractional operators, delving into numerous fundamental properties of these operators. Furthermore, the presentation of the function spaces within which these operators are defined was included. Recent studies have investigated weighted fractional operators, scrutinizing their practical applications and delving into their mathematical properties, as evidenced by the works cited in [9, 17, 18, 19, 22, 23, 24, 25, 26, 27]. Furthermore, an examination of the pertinent differential equations has been undertaken, as detailed in [28, 29, 30]. Nonlocal fractional derivatives can be classified into two distinct categories: classical derivatives, which feature singular kernels like the Riemann-Liouville and Caputo derivatives, and recently developed derivatives with nonsingular kernels, exemplified by the Atangana-Baleanu and Caputo-Fabrizio derivatives (refer to [31, 32]). In recent literature [23, 24], the authors proposed the introduction of weighted Caputo-Fabrizio fractional operators. Additionally, they extended their inquiry to include the investigation of weighted Atangana-Baleanu fractional operators and an analysis of their respective properties.

Comparison theories play a crucial role in fractional calculus, providing indispensable methodologies and tools for analyzing and comprehending fractional differential equations (FDEs) and fractional integral equations (FIEs). Through their application, one gains insight into the intricate behaviors exhibited by these equations, facilitating a deeper understanding of their underlying dynamics. Moreover, comparison theories contribute significantly to the asymptotic analysis and approximation of solutions to fractional equations. By comparing the behavior of solutions across different orders of fractional derivatives or integrals, one can deduce asymptotic expansions and approximate solutions tailored to FDEs and FIEs. Additionally, comparison theories contribute to the analysis of the stability and convergence of numerical schemes designed for fractional calculus problems. In summary, these theories are indispensable for theoretical exploration, numerical computations, and practical applications across various scientific and engineering disciplines.

In previous studies [33, 34, 35, 36], comparison theorems were established concerning fractional differential and integral inequalities that incorporate Riemann-Liouville and Caputo derivatives of order ϑ, where $0 < \vartheta < 1$. The primary contribution of this thesis resides in the extension of the methodology proposed by Chunhai Kou et al. [1] to address fractional Cauchy problems incorporating weighted Riemann-Liouville operators of a function with respect to another function. Our research makes significant contributions by not only confirming the existence and uniqueness of solutions but also extending the comparison result for the problems originally presented by V. Lakshmikantham et al. [33] . Currently, there has been no research initiated on comparison theorems concerning fractional differential equations that incorporate the weighted Riemann-Liouville derivative of a function with respect to another function of order ϑ, $0 < \vartheta < 1$. We hope that the results we have obtained will be a valuable contribution and extension to the existing body of literature. Indeed, the investigations into existence and uniqueness conducted in this study serve as indispensable prerequisites for advancing the proof of the comparison theory for the weighted fractional system under consideration, constituting a fundamental aspect of analyzing the qualitative theory of dynamic systems.

This thesis is organized as follows: In the first chapter, we will present some definitions and characteristics of fractional calculation, such as functional spaces, the Laplace transform, the gamma function, and the Mittag-Leffler function which play an important role in the theory of fractional differential equations. In addition to definitions of derivatives and fractional integrals in the sense of Riemann Liouville, and Caputo and some properties, while expounding on their interrelationships and pertinent properties. In the second chapter, we present new fractional operators, known as weighted fractional operators of a function with respect to another function. These operators generalize various fractional derivatives, such as the Riemann-Liouville and Caputo derivatives, highlighting their crucial properties. They serve as a fundamental framework for deriving the results presented in this thesis. Chapter 3 focuses on establishing both the existence and uniqueness of solutions for Cauchy-type problems that involve weighted R-L rational derivatives of a function in relation to another function. This task will be achieved through the application of Banach's fixed point theorem. Revisiting the Cauchy problem, which incorporates Riemann-Liouville fractional derivatives, is crucial for gaining deeper insights and demonstrating its position as a specific case within the broader scope of our investigation. In Chapter 4, we establish a novel estimate concerning the weighted Riemann-Liouville fractional derivative of a function with respect to function at their extrema. This estimate represents a broader generalization of the estimate pertaining to the Riemann-Liouville fractional derivative of a function at its extrema. Utilizing this estimate, we delve into the examination of comparison theorems for fractional differential inequalities, encompassing both strict and non-strict cases. These inequalities involve weighted Riemann-Liouville differential operators of a function relative to functions of order ϑ, where $0 < \vartheta < 1$. This extension of known theorems involving the Riemann-Liouville derivative adds depth to our exploration. Finally, we encapsulate the findings presented in [37] and this work in a concluding paragraph, discussing potential avenues for future extensions of this research.

Symbols and Notation

$\mathbb{R}$: set of real numbers.

$\mathbb{C}$: set of complex numbers.

$\mathbb{N}$: set of natural numbers.

D : bounded domain in R.

$L^p(D)$: space of measurable functions of power $p \in [0, +\infty[$ integrable on D.

$L^\infty(D)$: space of essentially bounded measurable functions on D.

$C(D)$: space of continuous functions on D.

$C^m(D)$: space of functions $\kappa : D \to \mathbb{R}$ that are m times differentiable and $\kappa^{(m)}$ is continuous.

$AC(D)$: space of absolutely continuous functions on D.

$AC^m(D)$: space of functions κ differentiable to order $n-1$ and such that $\kappa^{n-1} \in AC(D)$.

$C_\varrho(D)$: space of weighted continuous functions.

$\Gamma(.)$: the gamma function.

$B(.,.)$: the beta function.

$E_\vartheta(.)$: the Mittag-Leffler function.

${}^{RL}\Im^\vartheta$: fractional integral in the sense of Riemann-Liouville of order $\vartheta > 0$.

${}^{RL}D^\vartheta\kappa$: fractional derivative in the sense of Riemann-Liouville of order $\vartheta > 0$.

${}^{c}D^\vartheta\kappa$: fractional derivative in the sense of Caputo of order $\vartheta > 0$.

${}^{RL}\Im^{\vartheta,\theta(u)}_{\pi(u)}\kappa$: the π-weighted Riemann-Liouville fractional derivative of order $\vartheta > 0$.

${}^{RL}D^{\vartheta,\theta(u)}_{\pi(u)}\kappa$: the π-weighted Riemann-Liouville fractional derivative of order $\vartheta > 0$.

${}^{c}D^{\vartheta,\theta(u)}_{\pi(u)}\kappa$: the weighted Caputo fractional derivative of order $\vartheta > 0$.

CHAPTER

I

FRACTIONAL CALCULUS

Fractional calculus, a branch of mathematics, focuses on the study of derivatives and integrals with non-integer orders. This field expands the conventional notions of differentiation and integration to encompass non-integer or fractional orders. To gain a deep understanding of fractional calculus, it is crucial to explore its mathematical foundations. This involves examining the definitions of fractional derivatives and integrals, understanding their properties, and studying the theorems that govern their behavior. These operators are defined using fractional orders denoted by δ, where $0 < \delta < 1$ or $\delta > 1$. Such operators serve as potent tools for modeling and comprehending phenomena across disciplines like physics, engineering, biology, and others, where conventional calculus methods may fall short. Such operators serve as potent tools for model disciplines like physics, engineering, biology, and others, where conventional calculus methods may fall short.

I.1 Functional Spaces

In this part, we introduce definitions for spaces of p-integrable, absolutely continuous, and continuous functions, along with their weighted modifications, which will be employed subsequently.

I.1.1 Spaces of Integrable Functions

Definition I.1.1 *[9, 22, 38] Let $D = [\ell, r]$ $(-\infty < \ell < r < +\infty)$ be a finite or infinite interval of the real axis $\mathbb{R}$ and $1 \leq p \leq \infty$.*

1) *The spaces $L^p(D)$, $1 \leq p < \infty$ is defined by*

$$L^p(D) = \left\{ \kappa : D \to \mathbb{R} \ \textit{measurable and} \ \int_D |\kappa(t)|^p < \infty \right\}$$

This space is a Banach space with the norm

$$\|\kappa\|_p = \left(\int_\ell^r |\kappa(t)|^p dt \right)^{\frac{1}{p}}.$$

2) *For $p = \infty$, the spaces $L^\infty(D)$ is defined by*

$$L^\infty(D) = \left\{ \ \kappa : D \to \mathbb{R} \ \textit{measurable}, \exists N > 0 \quad \textit{suth that} \quad |\kappa(t)| \leq N \ \textit{a.e.on} \ D \ \right\}$$

It is a Banach space under the essential sup norm.

$$\|f\|_\infty = \inf \{N; \ |\kappa(t)| \leq N \ \textit{a.e.on} \ D\},$$

or

$$\|\kappa\|_\infty = \operatorname*{ess\,sup}_{\ell \leq t \leq r} |\kappa(t)|.$$

Definition I.1.2 *[9, 22, 38] Let $D = [\ell, r]$ $(-\infty < \ell < r < +\infty)$ be a finite or infinite interval of the real axis $\mathbb{R}$ and $1 \leq p \leq \infty$, $c \in \mathbb{R}$. Denote by $\chi_c^p(\ell, r)$, the weighted L^p-space. For $1 \leq p < \infty$ we define*

$$\chi_c^p(\ell, r) = \left\{ \kappa : (\ell, r) \to \mathbb{R} \ \textit{measurable and} \ \int_\ell^r |t^c \kappa(t)|^p \frac{dt}{t} < \infty \right\},$$

with the norm

$$\|\kappa\|_{\chi_c^p} = \left(\int_\ell^r |t^c \kappa(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}}$$

and

$$\|\kappa\|_{\chi_c^\infty} = \operatorname*{ess\,sup}_{\ell\leq t\leq r}[t^c|\kappa(t)|].$$

In particular, when $c=\frac{1}{p}$, *the space* $\chi_c^p(\ell,r)$ *coincides with the* $L^p(\ell,r)$ *space:*

$$\chi_c^p(\ell,r) = L^p(\ell,r).$$

I.1.2 Spaces of Continuous Functions

Definition I.1.3 *[9] Let* $D=[\ell,r]$ $(-\infty<\ell<r<+\infty)$ *and* $m\in\mathbb{N}_0=\{0,1,...\}$. *We denote by* $C^m(D)$ *a space of function* κ *which are* m *times continuously differentiable on* D *with the norm*

$$\|\kappa\|_{C^m} = \sum_{i=0}^{m}\|f^{(i)}\|_C = \sum_{i=0}^{m}\max_{t\in D}|\kappa^{(i)}(t)|, m\in\mathbb{N}_0.$$

In particular, for $m=0$, $C^0(D)=C(D)$ *is the space of continuous functions* κ *on* D *with the norm*

$$\|\kappa\|_C = \max_{t\in D}|\kappa(t)|.$$

Definition I.1.4 *[9, 22, 38] Let* $D=[\ell,r]$ $(-\infty<\ell<r<+\infty)$ *be a finite interval. we denote by* $AC[\ell,r]$ *the space of primitives of Lebesgue summable function :*

$$AC[\ell,r] = \left\{\kappa\backslash\exists\varphi\in L^1(D):\kappa(t)=c+\int_\ell^t\varphi(t)dt\right\},$$

and we call $AC(D)$ *the space of absolutely continuous functions on* D.

Definition I.1.5 *[9, 22] For* $m\in\mathbb{N}^*$ *we denote by* $AC^m(D)$ *the space of* m *times absolutely continuous differentiable functions, given by*

$$AC^m(D) = \left\{\kappa:(\ell,r]\to\mathbb{R};\quad \kappa^{m-1}\in AC(D)\right\}. \tag{I.1}$$

In particular, $AC^1(D)=AC(D)$.

I.1.3 Spaces of Continuous Functions With Weights

Definition I.1.6 *[9, 39] Let* $D=[\ell,r]$ *be a finite interval and* ϱ *be a parameter such that* $0\leq\varrho<1$.

The weighted space $C_\varrho[\ell,r]$ *of a functions* κ *on* $(\ell,r]$ *is defined by*

$$C_\varrho[\ell,r] = \{\kappa:(\ell,r]\to\mathbb{R};(t-\ell)^\varrho\kappa(t)\in C[\ell,r]\},\quad 0\leq\varrho<1,$$

with the norm

$$\|\kappa\|_{C_\varrho[\ell,r]} = \|(t-\ell)^\varrho \kappa(t)\|_{C[\ell,r]} = \max_{t\in[\ell,r]} |(t-\ell)^\varrho \kappa(t)|.$$

The weighted space $C^n_\varrho[\ell,r]$ *of the functions* κ *on* $(\ell,r]$ *is defined by*

$$C^m_\varrho[\ell,r] = \{\kappa : [\ell,r] \to \mathbb{R}; \kappa \in C^{m-1}([\ell,r]); \quad \kappa^{(m)} \in C_\varrho([\ell,r])\}, \quad 0 \le \varrho < 1,$$

with the norm

$$\|\kappa\|_{C^m_\varrho[\ell,r]} = \sum_{i=0}^{m-1} \|\kappa^{(i)}\|_{C[\ell,r]} + \|\kappa^{(m)}\|_{C_\varrho[\ell,r]}.$$

Clearly, $C^0_\varrho[\ell,r] = C_\varrho[\ell,r]$, *if* $n = 0$.

Definition I.1.7 *[9, 40] Let* $D = [\ell,r]$ $(0 \le \ell < r < \infty)$ *be a finite interval and* ϱ *be a parameter such that* $m-1 \le \varrho < m$ *and let* $\theta : [\ell,r] \to \mathbb{R}$ *be a strictly increasing* C^1 *function, so that* $\theta' > 0$ *everywhere.*

The weighted space $C_{\varrho,\theta}[\ell,r]$ *of functions* κ *on* $(\ell,r]$ *is defined by*

$$C_{\varrho,\theta}[\ell,r] = \{\kappa : (\ell,r] \to \mathbb{R}; (\theta(t)-\theta(\ell))^\varrho \kappa(t) \in C[\ell,r]\},$$

having norm

$$\|\kappa\|_{C_{\varrho,\theta}[\ell,r]} = \|(\theta(t)-\theta(\ell))^\varrho \kappa(t)\|_{C[\ell,r]} = \max_{t\in[\ell,r]} |(\theta(t)-\theta(\ell))^\varrho \kappa(t)|.$$

The weighted space $C^m_{\varrho,\theta}[\ell,r]$ *of functions* κ *on* $(\ell,r]$ *is defined by*

$$C^m_{\varrho,\theta}[\ell,r] = \{\kappa : [\ell,r] \to \mathbb{R}; \kappa \in C^{m-1}[\ell,r]; \quad \kappa^{(m)} \in C_{\varrho,\theta}[\ell,r]\},$$

with the norm

$$\|\kappa\|_{C^m_{\varrho,\theta}[\ell,r]} = \sum_{i=0}^{m-1} \|\kappa^{(i)}\|_{C[\ell,r]} + \|\kappa^{(m)}\|_{C_{\varrho,\theta}[\ell,r]}.$$

The above space satisfies the following properties:

i) $C^0_{\varrho,\theta}[\ell,r] = C_{\varrho,\theta}[\ell,r]$, *for* $m = 0$.

ii) For $m-1 \le \varrho_1 < \varrho_2 < m$, $C_{\varrho_1,\theta}[\ell,r] \subset C_{\varrho_2,\theta}[\ell,r]$.

I.2 Banach Fixed Point Theorem

Definition I.2.1 *[9] Let* (S,σ) *be a metric space. A mapping* $A : S \to S$ *is a contraction mapping, if there exists a constant* λ *with* $0 < \lambda < 1$, *such that*

$$\sigma(Au, Av) \le \lambda \sigma(u,v), \quad \textit{for all} \quad u, v \in S.$$

Theorem I.2.1 *(Banach fixed point theorem [9]) Let (S, σ)be a nonempty complete metric space, let $0 \leq \lambda < 1$, and let $A : S \to S$ be the map such that, for every $u, v \in S$, the relation*

$$\sigma(Au, Av) \leq \lambda\sigma(u, v) \quad (0 < \lambda < 1), \tag{I.2}$$

holds. Then the operator A has a unique fixed point $u^* \in S$.

Furthermore, if $A^i (i \in \mathbb{N})$ is the sequence of operators defined by

$$A^1 = A \quad and \quad A^i = AA^{i-1} \quad (i \in \mathbb{N}/\{1\}), \tag{I.3}$$

then, for any $u_0 \in S$, the sequence $\{A^i u_0\}_{i=1}^{\infty}$ converges to the above fixed point u^*.

Definition I.2.2 *[9] Assume that $\kappa(t, u)$ is defined on the $(\ell, r] \times \Omega (\Omega \subset \mathbb{R})$. $\kappa(t, u)$ is said to satisfy Lipschitzian condition with respect to the second variable, if for all $t \in (\ell, r]$ and for any $u_1, u_2 \in \Omega$ one has*

$$|\kappa(t, u_1) - \kappa(t, u_2)| \leq L|u_1 - u_2|, \tag{I.4}$$

where $L > 0$ does not depend on $t \in [\ell, r]$.

I.3 Special Functions

In this section, we introduce definitions and highlight certain properties of the functions Gamma, Beta, and Mittag-Leffler. Widely employed in mathematical, physical, technical, and statistical sciences, these functions play a pivotal role in the theory of fractional calculus and its applications, which will be used in the other chapters.

I.3.1 Gamma Function

In mathematics, the Gamma function is one of the basic functions of fractional calculus, which generalizes the factorial and allows n to take also non-integer and even complex values. It was first introduced by the swiss Leonhard Euler$(1707-1783)$, later, due to its great importance, it was studied by other eminent mathematicians like, Adrien-Marie Legendre $(1752 - 1833)$, Carl Friedrich Gauss$(1777 - 1855)$, Christoph Gudermann$(1798-1852)$, Joseph Liouville $(1809-1882)$, Karl Weierstrass $(1815-1897)$

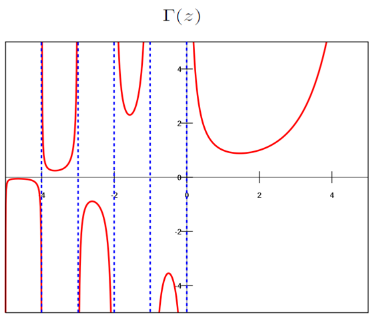

Figure I.1: Gamma Function

Charles Hermite $(1822 - 1901)$ and many others. More detailed information on this function may be found in ([41], [42]), and the book by Erdelyi et al. ([43], $Vol.1, Chapter I$).

Definition I.3.1 *[38] For $z \in \mathbb{C}$ such that $\Re(z) > 0$. The Gamma function $\Gamma(z)$ is defined by the following integral*

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt, \quad (\Re(z) > 0)\,. \tag{I.5}$$

This integral is convergent for all complex $z \in \mathbb{C}$ $(\Re(z) > 0)$.

Example I.3.1 $\Gamma(1) = 1, \Gamma(0^+) = +\infty, \Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Remark I.3.1 *The Gamma function $\Gamma(z)$ is monotonic and strictly decreasing for $0 < z \leq 1$, as shown in Figure I.1.*

Proposition I.3.1 *[9, 38] Let $z \in \mathbb{C}$ $(\Re(z) > 0)$ and $n \in \mathbb{R}$.*

a) *One of the basic properties of the gamma function is that it satisfies the following recurrence relation:*

$$\Gamma(z+1) = z\Gamma(z).$$

b) *Euler's Gamma function generalizes the fractional*

$$\Gamma(n+1) = n!, \quad \forall n \in \mathbb{N}.$$

and

$$\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{4^n n!}, \quad \forall n \in \mathbb{N}.$$

Limit Representation of the Gamma Function

The gamma function can be represented also by the limit

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)...(z+n)},$$

where we initially supposed $\Re(z) > 0$.(prove by I.Podlubny in [38]).

I.3.2 Beta function

The function beta B(x; y) is the name used by Legendre, Whittaker, and Watson in 1990. It is one of the basic functions of fractional calculus. This function plays an important role when combined with the function gamma. In numerous instances, employing the beta function proves to be more practical than dealing with various combinations of Gamma function values.

Definition I.3.2 *[9] The beta function is defined by the Euler integral of the first kind:*

$$B(\mu,\eta) = \int_0^1 t^{\mu-1}(1-t)^{\eta-1}dt, \quad (\Re(\mu) > 0; \Re(\eta) > 0).$$

Proposition I.3.2 *[9] The Beta function is related to the Gamma function by the following relationship:*

$$B(\mu,\eta) = \frac{\Gamma(\mu)\Gamma(\eta)}{\Gamma(\mu+\eta)}, \quad \Re(\mu) > 0, \Re(\eta) > 0.$$

and it is also a symmetric function, i.e,

$$B(\mu,\eta) = B(\eta,\mu), \quad \Re(\mu) > 0, \Re(\eta) > 0. \tag{I.6}$$

Remark I.3.2 *By utilizing the beta function, we can establish two important relationships for the gamma function, as detailed in [38]. The first relationship is expressed as follows:*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{I.7}$$

The second important relationship is the Legendre formula:

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}2^{2z-1}\Gamma(2z), \quad (2z \neq 0, -1, -2, ...). \tag{I.8}$$

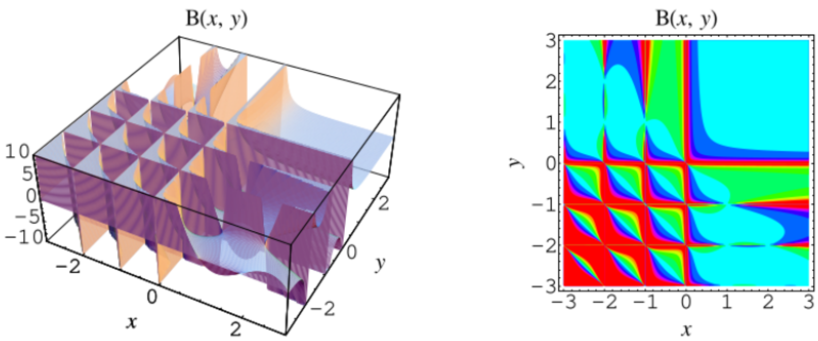

Figure I.2: Beta function

I.3.3 Mittag-Leffler Function

The Mittag-Leffler function plays a very important role in the theory of differential equations of non integer order, and it is also extensively employed in solving fractional-order differential equations, this function was introduced by G.M. Mittag-Leffler [44, 45]. The exponential function $exp(z)$ holds substantial importance in the domain of integer-order differential equations. Mittag-Leffler further extended this by introducing the one-parameter Mittag-Leffler function in [46], which serves as a generalization of the exponential function. Additionally, in [47] Agarwal introduced the two-parameter Mittag-Leffler type function, playing a crucial role in fractional calculus.

Definition I.3.3 *[9, 38] The Mittag-Leffler function $E_\mu(\omega)$ is defined by:*

$$E_\mu(\omega) = \sum_{i=0}^{+\infty} \frac{\omega^i}{\Gamma(\mu i + 1)}, \quad (\omega \in \mathbb{C}, \Re(\mu) > 0). \tag{I.9}$$

The generalized Mittag-Leffler function is defined as follows:

$$E_{\mu,\eta}(\omega) = \sum_{i=0}^{+\infty} \frac{\omega^i}{\Gamma(\mu i + \eta)}, \quad (\omega \in \mathbb{C}, \Re(\mu) > 0, \Re(\eta) > 0). \tag{I.10}$$

In particular, when $\eta = 1$, $E_{\mu,\eta}(\omega)$ coincides with the Mittag-Leffler function $(I.9)$

$$E_{\mu,1}(\omega) = E_\mu(\omega) \quad (\omega \in \mathbb{C}, \Re(\mu) > 0).$$

Example I.3.2 *For special values of μ and η, we have*

1) $E_1(\omega) = e^\omega, \quad E_2(\omega) = \cosh(\sqrt{\omega}).$

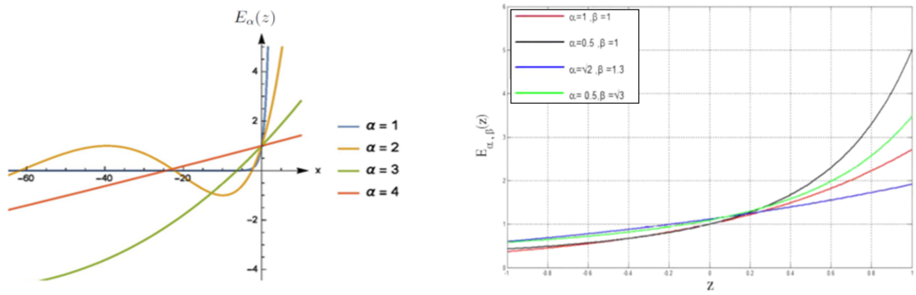

Figure I.3: Mittag Leffler function with one and two parameters.

2) $E_{1,2}(\omega) = \frac{e^{\omega}-1}{\omega}, \quad E_{2,2}(\omega) = \frac{\sinh(\sqrt{\omega})}{\sqrt{\omega}}.$

Remark I.3.3 *The hyperbolic sine and cosine are also particular cases of the Mittag Leffler function* $(I.10)$*:*

$$E_{2,1}(\omega^2) = \sum_{i=0}^{+\infty} \frac{\omega^{2i}}{\Gamma(2i+1)} = \sum_{i=0}^{+\infty} \frac{\omega^{2i}}{\Gamma(2i)!} = \cosh(\omega), \tag{I.11}$$

$$E_{2,2}(\omega^2) = \sum_{i=0}^{+\infty} \frac{\omega^{2i}}{\Gamma(2i+2)} = \frac{1}{\omega}\sum_{i=0}^{+\infty} \frac{\omega^{2i+1}}{\Gamma(2i+1)!} = \frac{\sinh(\omega)}{\omega}. \tag{I.12}$$

I.4 The Laplace Transform

In this section, we will discuss the Laplace transform along with its fundamental properties. The Laplace transform is a wonderful tool for solving ordinary and partial differential equations and systems.

Definition I.4.1 *[48] A function κ has exponential order δ if there exist two positive constants $N > 0$ and u^* such that:*

$$|\kappa(u)| \leq Ne^{\delta u}, \quad u \geq u^*.$$

Definition I.4.2 *[48] Let κ be a function of exponential order δ. We define the Laplace transform of κ as follows:*

$$K(u) = \mathcal{L}(\kappa(u)) = \int_0^{\infty} e^{-\vartheta u}\kappa(u)du = \lim_{w \longmapsto \infty} \int_0^{w} e^{-\vartheta w}\kappa(w)dw, \quad (\Re(\vartheta > 0, \Re(u > 0)). \tag{I.13}$$

Whenever the limit exists as a finite number, the integral $(I.13)$ is considered to converge.

Example I.4.1 *Let $\kappa(u) = e^{\xi u}$, $\xi \in \mathbb{R}$. This function is continuous on $[0, \infty)$ and of exponential order ξ. Then*

$$\begin{aligned}\mathcal{L}(e^{\xi u}) &= \int_0^\infty e^{-\vartheta u} e^{\xi u} du \\ &= \int_0^\infty e^{-(\vartheta-\xi)u} du \\ &= \frac{e^{-(\vartheta-\xi)u}}{-(\vartheta-\xi)}\Big|_0^\infty \\ &= \frac{1}{\vartheta-\xi}, \quad (\Re(\vartheta) > 0).\end{aligned}$$

Example I.4.2 *The Laplace transform of the cosine function, represented as $\cos(\mu u)$, is determined as follows:*

$$\mathcal{L}\{\cos(\mu u)\}(\omega) = \int_0^\infty e^{-\omega u} \cos(\mu u) du$$

To solve this integral, we use the identity $\cos(\mu u) = \frac{e^{i\mu u}+e^{-i\mu u}}{2}$ and properties of exponential functions:

$$\mathcal{L}\{\cos(\mu u)\}(\omega) = \frac{1}{2}\left(\int_0^\infty e^{-(\omega-i\mu)u} du + \int_0^\infty e^{-(\omega+i\mu)u} du\right)$$

Each integral on the right-hand side evaluates to:

$$\int_0^\infty e^{-(\omega\pm i\mu)u} du = \frac{1}{\omega \pm i\mu}$$

Therefore, combining these, we get:

$$\mathcal{L}\{\cos(\mu u)\}(\omega) = \frac{\omega}{\omega^2+\mu^2}$$

Similarly, the sine function $\sin(\mu u)$ has the following Laplace transform:

$$\mathcal{L}\{\sin(\mu u)\}(\omega) = \int_0^\infty e^{-\omega u} \sin(\mu u) du$$

Using the identity $\sin(\mu u) = \frac{e^{i\mu u}-e^{-i\mu u}}{2i}$:

$$\mathcal{L}\{\sin(\mu u)\}(\omega) = \frac{1}{2i}\left(\int_0^\infty e^{-(\omega-i\mu)u} du - \int_0^\infty e^{-(\omega+i\mu)u} du\right)$$

Thus:

$$\mathcal{L}\{\sin(\mu u)\}(\omega) = \frac{\mu}{\omega^2+\mu^2}$$

I.4.1 Properties of the Laplace Transform

Linearity property.[48] One of the fundamental and advantageous characteristics of the Laplace operator $\mathcal{L}$ is its linearity. Specifically, if $\varphi \in S$ for $\Re(\vartheta) > \delta$, $\psi \in S$ for $\Re(\vartheta) > \rho$, then $\varphi + \psi \in S$ for $\Re(\vartheta) > \max\{\delta, \rho\}$, and the following equation holds :

$$\mathcal{L}(\mu\varphi + \eta\psi) = \mu\mathcal{L}(\varphi) + \eta\mathcal{L}(\psi), \tag{I.14}$$

where μ and η are arbitrary constants.

Theorem I.4.1 *[48] If κ is piecewise continuous on $[0,+\infty)$ and has exponential order δ, then*

$$K(\vartheta)=\mathcal{L}(\kappa(u))\to 0, \quad as \quad \Re(\vartheta)\to\infty.$$

I.4.2 Inverse of the Laplace Transform

To address physical problems using the Laplace transform, it is essential to utilize the inverse transform. If $\mathcal{L}(\kappa(u))=K(\vartheta)$, the corresponding inverse Laplace transform is represented as follows:

$$\begin{aligned}\mathcal{L}^{-1}(K(\vartheta)) &= \kappa(u) \quad u\geq 0.\\ &= \int_{N-i\infty}^{N+i\infty} e^{\vartheta u}K(\vartheta)d\vartheta, \quad N=\Re(\vartheta)>N_0,\end{aligned} \tag{I.15}$$

where, N_0 represents the convergence index of the integral, (see in [48]).

Additionally, it should be noted that the inverse Laplace transform $\mathcal{L}^{-1}$, exhibits linearity. In other words, if $\mathcal{L}(m(u))=M(\vartheta)$ and $\mathcal{L}(h(u))=H(\vartheta)$, then $\mathcal{L}^{-1}(\gamma M(\vartheta)+\lambda H(\vartheta))=\gamma m(u)+\lambda h(u)$. This property stems from the linearity of $\mathcal{L}$ and is valid within the shared domain of M and H.

Example I.4.3 *In this example we will present the inverse Laplace transform of some functions.*

⋇ $\mathcal{L}^{-1}\left(\frac{1}{2(\vartheta-1)}+\frac{1}{2(\vartheta+1)}\right)=\frac{1}{2}e^{u}+\frac{1}{2}e^{-u}=coshu, \quad u\geq 0.$

⋇ $\mathcal{L}^{-1}\left(\frac{3\omega+7}{\omega-2\omega-3}\right)=(4e^{3u}-e^{-u})\varepsilon(u).$

⋇ $\mathcal{L}^{-1}\left(\frac{1}{\omega(\omega+2)}\right)=1-\frac{1}{2}e^{-2u}.$

⋇ $\mathcal{L}^{-1}\left(\frac{1}{\omega^2+4\omega+5}\right)=e^{-2}\sin(u).$

I.4.3 Translation Theorems for the Laplace Transform

We introduce two very useful results for calculating Laplace transforms and their inverses. The first result relates to a translation in the s-domain, while the second one involves a translation in the t-domain.

Theorem I.4.2 *[48] If $K(\vartheta) = \mathcal{L}(\kappa(u))$ for $\Re(\vartheta) > 0$, then*

$$K(\vartheta - \xi) = \mathcal{L}(e^{\xi u}\kappa(u)) \quad (\xi \in \mathbb{R}, \Re(\vartheta) > \xi).$$

Theorem I.4.3 *[48] If $K(\vartheta) = \mathcal{L}(\kappa(u))$, then*

$$\mathcal{L}(\varphi_\xi(u)\kappa(u - \xi)) = e^{-\xi\vartheta}K(\vartheta) \quad (\xi > 0).$$

I.4.4 Laplace Transform of Derivatives

In solving differential equations, it is necessary to know the Laplace transform of the derivative $\kappa^{'}$ of a function κ. The advantage of $\mathcal{L}(\kappa^{'})$ is that it can be written in terms of $\mathcal{L}(\kappa)$.

Theorem I.4.4 *[48] If κ is continuous on $(0, \infty)$ with an exponential order of δ, and $\kappa^{'}$ is piecewise continuous on $[0, \infty)$, then the Laplace transform of $\kappa^{'}(u)$ can be expressed as*

$$\mathcal{L}(\kappa^{'}(u)) = \vartheta\mathcal{L}(\kappa(u)) - \kappa(0^+), \quad (\Re(\vartheta) > \delta).$$

Theorem I.4.5 *[48] Suppose that $\kappa(u), \kappa^{'}(u), ..., \kappa^{(m-1)}(u)$ are continuous on $[0, \infty)$. Then*

$$\begin{aligned} \mathcal{L}(\kappa^m(u)) &= \vartheta^m\mathcal{L}(\kappa(u)) - \vartheta^{m-1}\kappa(0^+) - \vartheta^{m-2}\kappa^{'}(0^+) - ... - \kappa^{(m-1)}(0^+) \\ &= \vartheta^m\mathcal{L}(\kappa(u)) - \sum_{i=0}^{m-1}\vartheta^{m-i-1}\kappa^i(0^+). \end{aligned}$$

Example I.4.4 *(First Derivative) Consider the function $\kappa(u) = 3e^{-2u}$ for which we seek the Laplace transform of its derivative. The first derivative $\kappa^{'}(u)$ is calculated as follows:*

$$\kappa^{'}(u) = -6e^{-2u}$$

To compute the Laplace transform of this derivative, we apply the standard transformation formula:

$$\mathcal{L}(\kappa^{'}(u)) = \omega K(\omega) - \kappa(0^+)$$

Given $\kappa(0^+) = 3$, the Laplace transform of $\kappa^{'}(u)$ can be computed as:

$$\mathcal{L}(\kappa^{'}(u)) = -\frac{6}{\omega + 2} - 3$$

Example I.4.5 *(Second Derivative) To compute the Laplace transform of the second derivative of the function $\kappa(u) = \sin(u)$, we first note the expression for the second derivative:*

$$\kappa^{''}(u) = -\sin(u)$$

Using the Laplace transform formula for the second derivative:

$$\mathcal{L}(\kappa^{''}(u)) = \omega^2 K(\omega) - \omega\kappa(0^+) - \kappa^{'}(0^+).$$

Since $\kappa(0^+) = 0$ *and* $\kappa^{'}(0^+) = 1$, *we have:*

$$\mathcal{L}(\kappa^{''}(u)) = \frac{\omega^2}{\omega^2 + 1} - 1.$$

I.4.5 Convolution

The convolution is given by the integral ([48])

$$(\Phi * \Psi)(u) = \int_0^u \Phi(w)\Psi(u - w)dw,$$

which of course exists if Φ and Ψ are, say, piecewise continuous.

By substituting $\tau = u - w$,the convolution can be expressed as:

$$(\Phi * \Psi)(u) = \int_0^u \Psi(\tau)\Phi(u - \tau)d\tau = (\Psi * \Phi)(u),$$

This signifies the commutative property of convolution.

Here are additional fundamental characteristics of convolution:

a) For a constant c, $c(\Phi * \Psi) = (c\Phi) * \Psi = \Phi * (c\Psi)$;

b) The associative property holds: $\Phi * (\Psi * \varphi) = (\Phi * \Psi)\varphi$;

c) The distributive property is satisfied: $\Phi(\Psi + \varphi) = (\Phi * \Psi) + (\Phi * \varphi)$.

Example I.4.6 *The convolution operation between the sine function* $\alpha(u) = \sin(u)$ *and the cosine function* $\beta(u) = \cos(u)$ *can be described as follows:*

$$\alpha(u) * \beta(u) = \int_0^u \alpha(\omega) * \beta(u - \omega)d\omega \tag{I.16}$$

Substituting the given functions:

$$\begin{aligned}
\alpha(u) * \beta(u) &= \int_0^u \sin(\omega)\cos(u-\omega)d\omega \\
&= \frac{1}{2}\int_0^u \left[\sin(u) + \sin(2\omega - u)\right] d\omega \\
&= \frac{1}{2}\int_0^u \sin(u)d\omega + \frac{1}{2}\int_0^u \sin(2\omega - u)d\omega \\
&= \frac{1}{2}\sin(u)\int_0^u d\omega + \frac{1}{4}\int_{-u}^u \sin(x)dx \\
&= \frac{1}{2}\sin(u)u + \frac{1}{4}\int_{-u}^u \sin(x)dx \\
&= \frac{u\sin(u)}{2}.
\end{aligned}$$

A highly notable characteristic of convolution in connection with the Laplace transform is that the Laplace transform of the convolution of two functions equals the product of their individual Laplace transforms.

Theorem I.4.6 *[48] If Φ and Ψ are piecewise continuous on $[0,\infty)$ and of exponential order δ, the Laplace transform of their convolution is given by*

$$\mathcal{L}[(\Phi * \Psi)(u)](\vartheta) = \mathcal{L}(\Phi(u))(\vartheta).\mathcal{L}(\Psi(u))(\vartheta). \quad (\Re(\vartheta) > \delta).$$

I.5 Fractional Integrals and Derivatives

In this section, you will find definitions and properties related to various types of fractional integrals and fractional derivatives. The majority of definitions and properties in this section are derived from ([9, 22]), and we recommend referring to them for a more thorough analysis of the subject.

I.5.1 Riemann-Liouville Fractional Integrals and Derivatives

In this part, we provide definitions for Riemann-Liouville fractional integrals and fractional derivatives on a finite interval and present some of their properties. For a more thorough comprehension, we suggest that readers consult the books ([9, 22, 38]).

The concept of a fractional integral with order $\delta \in \mathbb{C}(\Re(\delta))$, following the Riemann-Liouville approach extends the well-known formula (attributed to Cauchy) of integral repeated n-times.

Let κ be a continuous function over the interval $[r,\ell]$, where $\ell > 0$. An antiderivative of κ is expressed as follows:

$$(\Im_r^1\kappa)(u) = \int_r^u \kappa(v)dv. \tag{I.17}$$

For a second antiderivative and according to Fubini's theorem, we obtain:

$$\begin{aligned}\Im^2\kappa(u) &= \int_0^u \Im^1\kappa(s)ds = \int_0^u (\int_0^s \kappa(v)dv)ds \\ &= \int_0^u (\int_v^u ds)\kappa(v)dv \\ &= \int_0^u (u-v)\kappa(v)dv.\end{aligned}$$

By iterating the process m times, we obtain the nth primitive of the function f expressed in the following form:

$$\begin{aligned}(\Im_{r+}^m\kappa)(u) &= \int_r^u dv_1 \int_r^{v_1} dv_2 ... \int_r^{v_{m-1}} \kappa(v_m)dv_m \\ &= \frac{1}{(m-1)!}\int_r^u (u-v)^{m-1}\kappa(v)dv, \quad (m \in \mathbb{N}).\end{aligned} \tag{I.18}$$

This formula is known as the Cauchy formula. Leveraging the Gamma function's generalization of the factorial, denoted as $\Gamma(n) = (n-1)!$, Riemann discerned that the right-hand side of $(I.18)$ could have meaning even when n takes a non-integer value of n. In response, he introduces the fractional integral through the following definition:

Definition I.5.1 *[9, 22, 38] Let $D = [\ell, r]$ $(-\infty < r < \ell < +\infty)$ be a finite interval on the real axis $\mathbb{R}$, and let $\kappa \in L^1(\ell, r)$. The Riemann-Liouville fractional integrals ${}^{RL}\Im_{\ell+}^{\delta}\kappa$ and ${}^{RL}\Im_{r-}^{\delta}\kappa$ of order $\delta \in \mathbb{R}$ $(\delta > 0)$ are defined as follows:*

$$({}^{RL}\Im_{\ell+}^{\delta}\kappa)(u) = \frac{1}{\Gamma(\delta)}\int_\ell^u (u-v)^{\delta-1}\kappa(v)dv, \quad (u > \ell, \delta > 0) \tag{I.19}$$

and

$$({}^{RL}\Im_{r-}^{\delta}\kappa)(u) = \frac{1}{\Gamma(\delta)}\int_u^r (v-u)^{\delta-1}\kappa(v)dv, \quad (u < r, \Re(\delta) > 0), \tag{I.20}$$

respectively. Here, $\Gamma(\delta)$ represents the Gamma function $(I.4)$.

Example I.5.1 *The Riemann-Liouville integral of the function $\kappa(u) = (u-r)^\eta$.*

Let $\mu > 0$ and $\eta > -1$. By definition, we have:

$${}^{RL}\Im_r^{\mu}(u-r)^\eta = \frac{1}{\Gamma(\mu)}\int_r^u (u-v)^{\eta-1}(v-r)^\eta dv.$$

To evaluate this integral, we perform the variable change $v = r + (u - v)\tau$:

$$
\begin{aligned}
{}^{RL}\Im_r^{\mu}(u-r)^{\eta} &= \frac{(u-r)^{\eta+\mu}}{\Gamma(\mu)}\int_0^1 (1-\tau)^{\mu-1}\tau^{\eta}d\tau \\
&= \frac{(u-r)^{\eta+\mu}}{\Gamma(\mu)}B(\mu,\eta+1) \\
&= \frac{(u-r)^{\eta+\mu}}{\Gamma(\mu)}\frac{\Gamma(\mu)\Gamma(\eta+1)}{\Gamma(\eta+1+\mu)} \\
&= \frac{\Gamma(\eta+1)}{\Gamma(\eta+1+\mu)}(u-r)^{\eta+\mu}.
\end{aligned}
$$

Therefore, the Riemann-Liouville fractional integral of the function $\kappa(u) = (u-r)^{\eta}$ is given by:

$$
{}^{RL}\Im_r^{\mu}(u-r)^{\eta} = \frac{\Gamma(\eta+1)}{\Gamma(\eta+1+\mu)}(u-r)^{\eta+\mu}. \tag{I.21}
$$

Remark I.5.1 *In particular, if $\kappa(u) = C$, the relation $(I.21)$ demonstrates that the Riemann-Liouville fractional integral of order μ of a constant is given by:*

$$
{}^{RL}\Im_r^{\mu}C = \frac{C}{\Gamma(\mu+1)}(u-r)^{\mu}.
$$

Lemma I.5.1 *[9, 22] For $\delta > 0$, the fractional integration operators $\Im_{r+}$ and $\Im_{\ell-}$ are bounded in the space $L_p(r,\ell)$, where $1 \le p \le \infty$. This is expressed by the following inequalities:*

$$
\|I_{r+}^{\delta}\kappa\|_p \le \Lambda\|\kappa\|_p, \quad \|I_{\ell-}^{\delta}\kappa\|_p \le \Lambda\|\kappa\|_p \quad \left(\Lambda = \frac{(\ell-r)^{\delta}}{\delta\Gamma(\delta)}\right). \tag{I.22}
$$

The semigroup property of the fractional integration operators I_{r+}^{δ} and $I_{\ell-}^{\delta}$ is stated in the following result

Lemma I.5.2 *[9, 22] Let $\delta > 0$, $\theta > 0$, and let $\kappa(u) \in L^p(\ell, r)$. The following equations hold for almost every point $u \in [\ell, r]$, with $(1 \le p \le \infty)$*

$$
({}^{RL}\Im_{\ell+}^{\delta}\,{}^{RL}\Im_{\ell+}^{\theta}\kappa)(u) = ({}^{RL}\Im_{\ell+}^{\delta+\theta}\kappa)(u), \quad and \quad ({}^{RL}\Im_{r-}^{\delta}\,{}^{RL}\Im_{r-}^{\theta}\kappa)(u) = ({}^{RL}\Im_{r-}^{\delta+\theta}\kappa)(u). \tag{I.23}
$$

If $\delta + \theta > 1$, then the relations in Eq. $(I.23)$ are satisfied at any point in the interval $[\ell, r]$.

Definition I.5.2 *[9, 38] Let $D = [\ell, r]$ $(-\infty < \ell < r < +\infty)$ be a finite interval on the real axis $\mathbb{R}$, and let $\kappa \in L^1(\ell, r)$. The Riemann-Liouville fractional derivatives ${}^{RL}D_{\ell+}^{\delta}\kappa$ and ${}^{RL}D_{r-}^{\delta}\kappa$ of order $\delta \in \mathbb{R}^+$ $(\delta > 0)$ are defined as follows:*

$$
\begin{aligned}
({}^{RL}D_{\ell+}^{\delta}\kappa)(u) &= \left(\frac{d}{du}\right)^m (\Im_{\ell+}^{m-\delta}\kappa)(u) \\
&= \frac{1}{\Gamma(m-\delta)}\left(\frac{d}{du}\right)^m \int_{\ell}^{u}(u-v)^{m-\delta-1}\kappa(v)dv \quad (m = [\delta]+1, u > \ell)
\end{aligned} \tag{I.24}
$$

and

$$\begin{aligned}({}^{RL}D^{\delta}_{r-}\kappa)(u) &= \left(-\frac{d}{du}\right)^m (\Im^{m-\delta}_{r-}\kappa)(u) \\ &= \frac{1}{\Gamma(m-\delta)}\left(-\frac{d}{du}\right)^m \int_u^r (v-u)^{m-\delta-1}\kappa(v)dv \quad (m=[\delta]+1, u<r),\end{aligned} \tag{I.25}$$

respectively. In particular, when $\delta = m \in \mathbb{N}$*, then*

$$({}^{RL}D^{m}_{\ell+}\kappa)(u) = \kappa^{(m)}(u), and \quad ({}^{RL}D^{m}_{r-}\kappa)(u) = (-1)^m\kappa^m(u).$$

where $\kappa^{(m)}(u)$ *is the usual derivative of* $\kappa(u)$ *of order* m*. If* $0 < \delta < 1$*, then*

$$({}^{RL}D^{\delta}_{\ell+}\kappa)(u) = \frac{1}{\Gamma(1-\delta)}\frac{d}{du}\int_\ell^u (u-v)^{-\delta}\kappa(v)dv \quad (u > \ell) \tag{I.26}$$

and

$$({}^{RL}D^{\delta}_{r-}\kappa)(u) = \frac{1}{\Gamma(1-\delta)} - \frac{d}{du}\int_u^r (v-u)^{-\delta}\kappa(v)dv \quad (u < r), \tag{I.27}$$

respectively.

Example I.5.2 *The Riemann-Liouville derivative of the function* $\kappa(u) = (u-r)^\theta$*.*

Consider $\delta > 0$ *such that* $m-1 < \delta < m$ *and* $\theta > -1$*. According to* $(I.24)$ *and relation* $(I.21)$*,(see ExampleI.5.1), we obtain*

$${}^{RL}D^{\delta}_{\ell}(u-\ell)^\theta = \left(\frac{d}{du}\right)^m \left[\frac{\Gamma(\theta+1)}{\Gamma(\theta+1+m-\delta)}(u-\ell)^{\theta+m-\delta}\right]. \tag{I.28}$$

Taking into account that

$$\begin{aligned}\left(\frac{d}{du}\right)^m (u-\ell)^{\theta+m-\delta} &= (\theta+m-\delta)(\theta+m-\delta-1)...(\theta-\delta+1)(u-\ell)^{\theta-\delta} \\ &= \frac{\Gamma(\theta+m-\delta+1)}{\Gamma(\theta-\delta+1)}(u-\ell)^{\theta-\delta}.\end{aligned} \tag{I.29}$$

We substitute the result $(I.29)$ *into the formula* $(I.28)$ *to yield:*

$${}^{RL}D^{\delta}_{\ell}(u-\ell)^\theta = \frac{\Gamma(\theta+1)}{\Gamma(\theta+1+m-\delta)}\frac{\Gamma(\theta+1+m-\delta)}{\Gamma(\theta+1-\delta)}(u-\ell)^{\theta-\delta} \tag{I.30}$$

$$= \frac{\Gamma(\theta+1)}{\Gamma(\theta+1-\delta)}(u-\ell)^{\theta-\delta}. \tag{I.31}$$

Hence, the Riemann-Liouville fractional derivative of the function $\kappa(u) = (u-r)^\theta$ *is given by:*

$${}^{RL}D^{\delta}_{\ell}(u-\ell)^\theta = \frac{\Gamma(\theta+1)}{\Gamma(\theta+1-\delta)}(u-\ell)^{\theta-\delta}.$$

Remark I.5.2 *In specific cases, when $\theta = 0$ and $\delta > 0$, the Riemann-Liouville fractional derivatives of a constant function are generally non-zero:*

$$({}^{RL}D^{\delta}_{\ell+}1)(u) = \frac{(u-\ell)^{-\delta}}{\Gamma(1-\delta)}, \quad ({}^{RL}D^{\delta}_{r-}1)(u) = \frac{(r-u)^{-\delta}}{\Gamma(1-\delta)} \quad (0<\delta<1). \tag{I.32}$$

Conversely, for $i = 1, 2, ..., [\delta]+1$,

$$({}^{RL}D^{\delta}_{\ell+}(u-\ell)^{\delta-i})(u) = 0, \quad ({}^{RL}D^{\delta}_{r-}(r-u)^{\delta-i})(u) = 0. \tag{I.33}$$

The following Lemma establishes the conditions for the existence of the fractional derivatives ${}^{RL}D^{\delta}_{\ell+}\kappa$ and ${}^{RL}D^{\delta}_{r-}\kappa$ in the space $AC^m[\ell, r]$ defined in $(I.1)$.

Lemma I.5.3 *[9, 22] If $\delta > 0$ and $\kappa(u) \in L^p(\ell, r)$ $(1 \le p \le \infty)$, then the following equalities*

$$(D^{\delta}_{\ell+}\Im^{\delta}_{\ell+}\kappa)(u) = \kappa(u) \quad \textit{and} \quad (D^{\delta}_{r-}\Im^{\delta}_{r-}\kappa)(u) = \kappa(u) \quad (\delta > 0), \tag{I.34}$$

hold almost everywhere on $[\ell, r]$.

Property I.5.1 *[9, 22] If $\mu > \eta > 0$, then, for $\kappa(u) \in L^p(\ell, r)$ $(1 \le p < \infty)$, the relations*

$$(D^{\eta}_{\ell+}\Im^{\mu}_{\ell+}\kappa)(u) = \Im^{\mu-\eta}_{\ell+}\kappa(u) \quad \textit{and} \quad (D^{\eta}_{r-}\Im^{\mu}_{r-}\kappa)(u) = \Im^{\mu-\eta}_{r-}\kappa(u), \tag{I.35}$$

hold almost everywhere on $[\ell, r]$.

In particular, when $\eta = i \in \mathbb{N}$, $\delta > i$ and $\sigma = \frac{d}{du}$, then

$$(\sigma^i\Im^{\delta}_{\ell+}\kappa)(u) = \Im^{\delta-i}_{\ell+}\kappa(u) \quad \textit{and} \quad (\left(\frac{d}{du}\right)^i {}^i\Im^{\delta}_{r-}\kappa)(u) = (-1)^i\Im^{\delta-i}_{r-}\kappa(u). \tag{I.36}$$

Property I.5.2 *[9, 22] Let $\delta \ge 0$, $m \in \mathbb{N}$ and $\sigma = \frac{d}{dt}$.*

(i) If the fractional derivatives $(D^{\delta}_{\ell+}\kappa)(u)$ and $(D^{\delta+m}_{\ell+}\kappa)(u)$ exist, then

$$(\sigma^m D^{\delta}_{\ell+}\kappa)(u) = (D^{\delta+m}_{\ell+}\kappa)(u).$$

(ii) If the fractional derivatives $(D^{\delta}_{r-}\kappa)(u)$ and $(D^{\delta+m}_{r-}\kappa)(u)$ exist, then

$$(\sigma^m D^{\delta}_{r-}\kappa)(u) = (D^{\delta+m}_{r-}\kappa)(u).$$

Lemma I.5.4 *[9, 22] Let $\delta > 0$, $m = [\delta]+1$ and let $\kappa_{m-\delta}(u) = (\Im^{m-\delta}_{\ell+}\kappa)(u)$ be the fractional integral $(I.19)$ of order $m - \delta$.*

(I1) *If* $(1 \leq p \leq \infty)$ *and* $\kappa(u) \in \Im^{\delta}_{\ell+}(L^p)$, *then*

$$(\Im^{\delta}_{\ell+} D^{\delta}_{\ell+} \kappa)(u) = \kappa(u). \tag{I.37}$$

(I2) *If* $\kappa(u) \in L_1(\ell, r)$ *and* $\kappa_{m-\delta}(u) \in AC^m[\ell, r]$, *then the equality*

$$(\Im^{\delta}_{\ell+} D^{\delta}_{\ell+} \kappa)(u) = \kappa(u) - \sum_{i=1}^{m} \frac{\kappa^{(m-i)}_{m-\delta}(\ell)}{\Gamma(\delta - i + 1)} (u - \ell)^{\delta - i}, \tag{I.38}$$

holds almost everywhere on $[\ell, r]$.

In particular, if $0 < \delta < 1$, *then*

$$(\Im^{\delta}_{\ell+} D^{\delta}_{\ell+} \kappa)(u) = \kappa(u) - \frac{\kappa_{1-\delta}(\ell)}{\Gamma(\delta)} (u - \ell)^{\delta - 1}, \tag{I.39}$$

where $\kappa_{1-\delta}(u) = (\Im^{1-\delta}_{\ell+} \kappa)(u)$, *while for* $\delta = m \in \mathbb{N}$, *the following equality holds:*

$$(\Im^{m}_{\ell+} D^{m}_{\ell+} \kappa)(u) = \kappa(u) - \sum_{i=0}^{m-1} \frac{\kappa^{(i)}(\ell)}{i!} (u - \ell)^{i}. \tag{I.40}$$

Property I.5.3 *[9, 22] Let* $\mu > 0$ *and* $\eta > 0$ *be such that* $i - 1 < \delta \leq i$, $j - 1 < \eta \leq j$ $(i, j \in \mathbb{N})$ *and* $\mu + \eta < i$, *and let* $\kappa \in L_1(\ell, r)$ *and* $\kappa_{j-\mu} \in AC^j([\ell, r])$. *Then we have the following index rule:*

$$(D^{\mu} D^{\eta} \kappa)(u) = (D^{\mu+\eta} \kappa)(u) - \sum_{k=1}^{j} (D^{\eta-k}_{\ell+} \kappa)(\ell^{+}) \frac{(u - \ell)^{-k-\mu}}{\Gamma(1 - k - \mu)}.$$

Definition I.5.3 *[9, 22] Let* $\delta > 0$ *and* $1 \leq p \leq \infty$, *the spaces of functions* $\Im^{\delta}_{\ell+}(L^p)$ *and* $\Im^{\delta}_{r-}(L^p)$ *are defined by*

$$\Im^{\delta}_{\ell+}(L^p) = \{\kappa : \kappa = \Im^{\delta}_{\ell+} \psi, \quad \psi \in L^p(\ell, r)\}$$

and

$$\Im^{\delta}_{r-}(L^p) = \{\kappa : \kappa = \Im^{\delta}_{r-} \psi, \quad \psi \in L^p(\ell, r)\},$$

respectively.

Lemma I.5.5 *[9, 22] Let* $\delta > 0$, $m = [\delta] + 1$ *and let* $h_{m-\delta}(u) = (\Im^{m-\delta}_{r-} \kappa)(u)$ *be the fractional integral* $(I.20)$ *of order* $m - \delta$.

(R1) *If* $(1 \leq p \leq \infty)$ *and* $h(u) \in \Im^{\delta}_{r-}(L^p)$, *then*

$$(\Im^{\delta}_{r-} D^{\delta}_{r-} h)(u) = h(u). \tag{I.41}$$

(R2) *If* $h(u) \in L_1(\ell, r)$ *and* $h_{m-\delta}(u) \in AC^m[\ell, r]$, *then the formula*

$$(\Im^{\delta}_{r-} D^{\delta}_{r-} h)(u) = h(u) - \sum_{i=1}^{m} \frac{(-1)^{m-i} h^{(m-i)}_{m-\delta}(r)}{\Gamma(\delta - i + 1)} (r - u)^{\delta - i}, \tag{I.42}$$

holds almost everywhere on $[\ell, r]$.

In particular, if $0 < \delta < 1$, *then*

$$(\Im_{r-}^{\delta} D_{r-}^{\delta} h)(u) = h(u) - \frac{h_{1-\delta}(r)}{\Gamma(\delta)}(r-u)^{\delta-1}, \tag{I.43}$$

where $h_{1-\delta}(u) = (\Im_{r-}^{1-\delta} h)(u)$, *while for* $\delta = m \in \mathbb{N}$, *the following equality holds:*

$$(\Im_{r-}^{m} D_{r-}^{m} h)(u) = h(u) - \sum_{i=0}^{m-1} \frac{(-1)^i h^{(i)}(r)}{i!}(r-u)^i. \tag{I.44}$$

Lemma I.5.6 *[9, 22] Let* $\delta > 0$ *and* $0 \le \varrho < 1$ ($\varrho \in \mathbb{R}$).

- *If* $\varrho > \delta$, *then the fractional integration operators* $\Im_{\ell+}^{\delta}$ *and* $\Im_{r-}^{\delta}$ *are bounded from* $C_{\varrho}[\ell, r]$ *into* $C_{\varrho-\delta}[\ell, r]$:

$$\|\Im_{\ell+}^{\delta}\kappa\|_{C_{\varrho-\delta}} \le \varpi_1 \|\kappa\|_{C_{\varrho}} \quad \textit{and} \quad \|\Im_{r-}^{\delta}\kappa\|_{C_{\varrho-\delta}} \le \varpi \|\kappa\|_{C_{\varrho}}, \quad \varpi_1 = \frac{\Gamma(1-\varrho)}{\Gamma(1+\delta-\varrho)}. \tag{I.45}$$

In particular, $\Im_{\ell_+}^{\delta}$ *and* $\Im_{r_-}^{\delta}$ *are bounded in* $C_{\varrho}[\ell, r]$.

- *If* $\varrho \le \delta$, *then the fractional integration operators* $\Im_{\ell_+}^{\delta}$ *and* $\Im_{r_-}^{\delta}$ *are bounded from* $C_{\varrho}[\ell, r]$ *into* $C[\ell, r]$:

$$\|\Im_{\ell+}^{\delta}\kappa\|_{C} \le \varpi_2 \|\kappa\|_{C_{\varrho}} \quad \textit{and} \quad \|\Im_{r-}^{\delta}\kappa\|_{C} \le \varpi \|\kappa\|_{C_{\varrho}}, \quad \varpi_2 = (r-\ell)^{\delta-\varrho}\frac{\Gamma(1-\varrho)}{\Gamma(1+\delta-\varrho)}. \tag{I.46}$$

In particular, $\Im_{\ell_+}^{\delta}$ *and* $\Im_{r_-}^{\delta}$ *are bounded in* $C_{\varrho}[\ell, r]$.

I.5.2 Caputo Fractional Derivatives

In this section, we introduce the definitions and fundamental properties of Caputo fractional derivatives.

Definition I.5.4 *[9, 38] Let* $D = [\ell, r]$ *be a finite interval on the real axis* $\mathbb{R}$, *and let* $({}^{RL}D_{\ell+}^{\delta}\kappa)(u)$ *and* $({}^{RL}D_{r-}^{\delta}\kappa)(u)$ *represent the Riemann-Liouville fractional derivatives of order* $\delta \in \mathbb{C}$ ($\Re(\delta) \ge 0$). *The Caputo fractional derivatives* $({}^{C}D_{\ell+}^{\delta}\kappa)(u)$ *and* $({}^{C}D_{r-}^{\delta}\kappa)(u)$ *of order* $\delta \in \mathbb{C}$ ($\Re(\delta) \ge 0$) *are defined through the above Riemann-Liouville fractional derivatives as follows:*

$$({}^{C}D_{\ell+}^{\delta}\kappa)(u) = \left({}^{RL}D_{\ell+}^{\delta}\left[\kappa(u) - \sum_{i=0}^{m-1}\frac{\kappa^{(i)}(\ell)}{i!}(u-\ell)^i\right]\right)(u) \tag{I.47}$$

and

$$({}^{C}D_{r-}^{\delta}\kappa)(u) = \left({}^{RL}D_{r-}^{\delta}\left[\kappa(u) - \sum_{i=0}^{m-1}\frac{\kappa^{(i)}(r)}{i!}(r-u)^i\right]\right)(u), \tag{I.48}$$

respectively, where

$$m = [\Re(\delta)] + 1 \quad for \quad \delta \notin \mathbb{N}, \quad m = \delta \quad for \quad \delta \in \mathbb{N}. \tag{I.49}$$

In particular, when $0 < \Re(\delta) < 1$, the relations $(I.47)$ and $(I.48)$ take the following forms:

$$({}^C D^{\delta}_{\ell+}\kappa)(u) = \left({}^{RL} D^{\delta}_{\ell+}\left[\kappa(u) - \kappa(\ell)\right]\right)(u), \quad \text{and} \quad ({}^C D^{\delta}_{r-}\kappa)(u) = \left({}^{RL} D^{\delta}_{r-}\left[\kappa(u) - \kappa(r)\right]\right)(u).$$

Theorem I.5.4 *[9] Let $\Re(\delta) \geq 0$ and let m be given by $(I.49)$. If $\kappa(u) \in AC^m[\ell, r]$, then the Caputo fractional derivatives $({}^C D^{\delta}_{\ell+}\kappa)(u)$ and $({}^C D^{\delta}_{r-}\kappa)(u)$ exist almost everywhere on $[\ell, r]$.*

B1) If $\delta \notin \mathbb{N}$, the Caputo fractional derivatives $({}^C D^{\delta}_{\ell+}\kappa)(u)$ and $({}^C D^{\delta}_{r-}\kappa)(u)$ of order δ are represented by

$$\begin{aligned} ({}^C D^{\delta}_{\ell+}\kappa)(u) &= ({}^{RL}\Im^{m-\delta}_{\ell+}\sigma^m\kappa)(u) \\ &= \frac{1}{\Gamma(m-\delta)}\int_{\ell}^{u}(u-v)^{m-\delta-1}\kappa^m(v)dv \end{aligned} \tag{I.50}$$

and

$$\begin{aligned} ({}^C D^{\delta}_{r}\kappa)(u) &= (-1)^m({}^{RL}\Im^{m-\delta}_{r-}\sigma^m\kappa)(u), \\ &= \frac{(-1)^m}{\Gamma(m-\delta)}\int_{u}^{r}(v-u)^{m-\delta-1}\kappa^m(v)dv. \end{aligned} \tag{I.51}$$

respectively, where $\sigma = \frac{d}{dt}$ and $m = [\Re(\delta)] + 1$.

In particular, when $0 < \Re(\delta) < 1$ and $\kappa(u) \in AC[\ell, r]$,

$$({}^C D^{\delta}_{\ell+}\kappa)(u) = \frac{1}{\Gamma(1-\delta)}\int_{\ell}^{u}(u-v)^{-\delta}\kappa^{'}(v)dv$$

and

$$({}^C D^{\delta}_{r}\kappa)(u) = -\frac{1}{\Gamma(1-\delta)}\int_{u}^{r}(v-u)^{-\delta}\kappa^{'}(v)dv.$$

B2) If $\delta = m \in \mathbb{N}$, then:

$$({}^C D^{\delta}_{\ell+}\kappa)(u) = \kappa^m(v) \quad \text{and} \quad ({}^C D^{\delta}_{r}\kappa)(u) = \kappa^m(v).$$

Example I.5.3 *The Caputo derivative of the function $\kappa(u) = (u-\ell)^{\theta}$.*

Suppose $0 \leq m-1 < \delta < m$, with $\theta > m-1$, then according to $(I.50)$, we obtain

$$\kappa^{(m)}(v) = \frac{\Gamma(\theta+1)}{\Gamma(\theta-m+1)}(v-\ell)^{\theta-m}$$

Hence,

$${}^C D_{\ell}(u-\ell)^{\theta} = \frac{\Gamma(\theta+1)}{\Gamma(m-\delta)\Gamma(\theta-m+1)}\int_{\ell}^{u}(u-v)^{m-\delta-1}(v-\ell)^{\theta-m}dv.$$

Through the change of variables $v = \ell + \omega(u-\ell)$, we derive:

$$
\begin{aligned}
{}^{C}D_{\ell}(u-\ell)^{\theta} &= \frac{\Gamma(\theta+1)}{\Gamma(m-\delta)\Gamma(\theta-m+1)}\int_{\ell}^{u}(u-v)^{m-\delta-1}(v-\ell)^{\theta-m}dv \\
&= \frac{\Gamma(\theta+1)}{\Gamma(m-\delta)\Gamma(\theta-m+1)}(u-\ell)^{\theta-\delta}\int_{0}^{1}(1-\omega)^{m-\delta-1}\omega^{\theta-\delta}d\omega \\
&= \frac{\Gamma(\theta+1)B(m-\delta,\theta-m+1)}{\Gamma(m-\delta)\Gamma(\theta-m+1)}(u-\ell)^{\theta-\delta} \\
&= \frac{\Gamma(\theta+1)\Gamma(m-\delta)\Gamma(\theta-m+1)}{\Gamma(m-\delta)\Gamma(\theta-m+1)\Gamma(\theta-\delta+1)}(u-\ell)^{\theta-\delta} \\
&= \frac{\Gamma(\theta+1)}{\Gamma(\theta-\delta+1)}(u-\ell)^{\theta-\delta}.
\end{aligned}
$$

Remark I.5.3 *In specific cases, when $\theta = 0$ and $\delta > 0$, the Caputo fractional derivatives of a constant function is zero:*

$$({}^{C}D^{\delta}_{\ell+}1)(u) = 0 \quad and \quad ({}^{C}D^{\delta}_{r-}1)(u) = 0$$

For $(i = 0, 1, ..., m-1)$,

$$({}^{C}D^{\delta}_{\ell+}(u-\ell)^{i})(u) = 0 \quad and \quad ({}^{C}D^{\delta}_{r-}(r-u)^{i})(u) = 0.$$

I.5.3 The Relationship Between the Riemann-Liouville and Caputo Derivatives

The following lemma establishes the relationship between Riemann-Liouville and Caputo fractional derivatives.

Lemma I.5.7 *[9] If $\delta \in \mathbb{C}$ $(\Re(\delta) > 0)$ with $m = [\Re(\delta)] + 1$. If $\delta \in \mathbb{N}$ and $\kappa(u)$ is a function for which the Caputo fractional derivatives $({}^{C}D^{\delta}_{\ell+}\kappa)(u)$ and $({}^{C}D^{\delta}_{r-}\kappa)(u)$ of order δ exist together with the Riemann-Liouville fractional derivatives $({}^{RL}D^{\delta}_{\ell+}\kappa)(u)$ and $({}^{RL}D^{\delta}_{r-}\kappa)(u)$, then*

$$({}^{C}D^{\delta}_{\ell+}\kappa)(u) = {}^{RL}D^{\delta}_{\ell+}\kappa(u) - \sum_{i=0}^{m-1}\frac{\kappa(\ell)}{\Gamma(i-\delta+1)}(u-\ell)^{i-\delta}$$

and

$$({}^{C}D^{\delta}_{r-}\kappa)(u) = {}^{RL}D^{\delta}_{r-}\kappa(u) - \sum_{i=0}^{m-1}\frac{\kappa^{(i)}(r)}{\Gamma(i-\delta+1)}(r-u)^{i-\delta}.$$

In particular, when $0 < \Re(\delta) < 1$, we have

$$({}^{C}D^{\delta}_{\ell+}\kappa)(u) = {}^{RL}D^{\delta}_{\ell+}\kappa(u) - \frac{\kappa(\ell)}{\Gamma(1-\delta)}(u-\ell)^{-\delta}, \tag{I.52}$$

and

$$({}^{C}D^{\delta}_{r-}\kappa)(u) = {}^{RL}D^{\delta}_{r-}\kappa(u) - \frac{\kappa(r)}{\Gamma(1-\delta)}(r-u)^{-\delta}.$$

I.5.4 Laplace Transform of the Riemann-Liouville Fractional Integral and Derivative

Lemma I.5.8 *[9, 22, 49] The Laplace transform of the Riemann-Liouville fractional integral of order $\delta > 0$ is given by*

$$\mathcal{L}\{{}^{RL}\Im_{0+}^{\delta}\kappa(u)\}(v) = v^{-\delta}K(v),$$

where $K(v)$ is the Laplace transform of $\kappa(u)$.

Example I.5.4 *Suppose we have a function $\kappa(u) = u^2$, and we want to find the Laplace transform of its Riemann-Liouville fractional integral of order $\delta = \frac{1}{2}$. The Riemann-Liouville fractional integral is given by:*

$$\begin{aligned} {}^{RL}\Im_{0+}^{\delta}\kappa(u) &= \frac{1}{\Gamma(\frac{1}{2})}\int_0^u (u-v)^{-\frac{1}{2}}\kappa(u)du \\ &= \frac{2}{3\sqrt{\pi}}u^{\frac{3}{2}}. \end{aligned}$$

Therefore, the Laplace transform of ${}^{RL}\Im_{0+}^{\frac{1}{2}}\kappa(u)$ is:

$$\mathcal{L}\{{}^{RL}\Im_{0+}^{\frac{1}{2}}\kappa(u)\} = \frac{2}{3\sqrt{\pi}}\mathcal{L}\{u^{\frac{3}{2}}\}$$

Now, we know that the Laplace transform of $u^{\frac{3}{2}}$ is $\frac{\sqrt{\pi}}{v^{\frac{5}{2}}}$, so:

$$\mathcal{L}\{{}^{RL}\Im_{0+}^{\frac{1}{2}}\kappa(u)\} = \frac{2}{3v^{\frac{5}{2}}}.$$

Lemma I.5.9 *[9, 22] The Laplace transform of the Riemann-Liouville fractional derivative of order $\delta > 0$ ($M - 1 < \delta \leq M$), is given by:*

$$\mathcal{L}\{{}^{RL}D_{0+}^{\delta}\kappa(u)\}(v) = v^{\delta}K(v) - \sum_{i=0}^{M-1} v^{M-i-1}\sigma^{i}({}^{RL}\Im_{0+}^{M-\delta}\kappa(0^+)).$$

where $\sigma = \frac{d}{du}$, $K(v)$ is the Laplace transform of $\kappa(u)$.

Example I.5.5 *Suppose we have a function $\varphi(\xi) = e^{\eta\xi}$, and we want to find the Laplace transform of its Riemann-Liouville fractional derivative of order $\delta = \frac{1}{2}$. The Riemann-Liouville fractional derivative is given by:*

$$\begin{aligned} {}^{RL}D_{0+}^{\delta}\varphi(\xi) &= \frac{1}{\Gamma(\frac{1}{2})}\frac{d}{d\xi}\int_0^{\xi} e^{\eta v}(\xi - v)^{-\frac{1}{2}}dv \\ &= \frac{1}{\Gamma(\frac{1}{2})}\frac{d}{d\xi}\left(\frac{2e^{\eta\xi}}{\sqrt{\pi}}\int_0^{\xi} e^{-\eta v^2}dv\right) \end{aligned}$$

Then, the Laplace transform of ${}^{RL}D_{0^+}^{\frac{1}{2}}\varphi(\xi)$ *is:*

$$\mathcal{L}\{{}^{RL}D_{0^+}^{\frac{1}{2}}\varphi(\xi)\}(v) = \frac{\eta}{\sqrt{\pi}}\mathcal{L}\{e^{\eta\xi}\}(v)$$

Now, we know that the Laplace transform of $e^{\eta\xi}$ *is* $\frac{1}{v-\eta}$*, so:*

$$\mathcal{L}\{{}^{RL}D_{0^+}^{\frac{1}{2}}\varphi(\xi)\}(v) = \frac{\eta}{\sqrt{\pi}}\frac{1}{v-\eta}.$$

I.5.5 Laplace Transform of the Caputo Fractional Derivative

Lemma I.5.10 *[9, 22, 49] The Laplace transform of the Caputo fractional integral of order* $\delta > 0$*, is given by:*

$$\mathcal{L}\{{}^{C}D_{0^+}^{\delta}\kappa(u)\}(v) = v^{\delta}K(u) - \sum_{i=0}^{M-1} v^{\delta-i-1}(\sigma^i\kappa)(0).$$

where $\sigma = \frac{d}{du}$*,* $K(v)$ *is the Laplace transform of* $\kappa(u)$*.*

Example I.5.6 *Suppose we have a function* $\omega(\xi) = \xi^2$*, and we want to find the Laplace transform of its Caputo fractional derivative of order* $\delta = \frac{1}{2}$*. The Caputo fractional derivative is given by:*

$$\begin{aligned} {}^{C}D_{0^+}^{\delta}\omega(\xi) &= \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\xi}(\xi - v)^{-\frac{1}{2}}\omega^{'}(v)dv \\ &= \frac{4\xi^{\frac{3}{2}}}{3\Gamma(\frac{3}{2})} \end{aligned}$$

Therefore, the Laplace transform of ${}^{C}D_{0^+}^{\frac{1}{2}}\omega(\xi)$ *is:*

$$\mathcal{L}\{{}^{C}D_{0^+}^{\frac{1}{2}}\omega(\xi)\}(v) = v^{\frac{1}{2}}\mathcal{L}\{\xi^{\frac{3}{2}}\}(v).$$

Now, we can use the Laplace transform of $\xi^{\frac{3}{2}}$ *to find the Laplace transform of* ${}^{C}D_{0^+}^{\frac{1}{2}}\omega(\xi)$*.*

CHAPTER

II

WEIGHTED FRACTIONAL CALCULUS

In this chapter, we present some generalizations of fractional integrals and derivatives and present some of their properties. Specifically, we delve into the concept of weighted fractional calculus, and its extension to the more general class known as weighted fractional calculus with respect to functions. It is worth noting that all the operators mentioned in the first chapter are just special cases of these operators. The use of weighted fractional derivatives and integrals allows for the development of more precise mathematical models. (For further details, refer to [50, 51, 18, 19]).

II.1 Weighted Fractional Operators

Definition II.1.1 *[51] The weighted Riemann–Liouville fractional integral of a function $\kappa \in L^1(\ell, r)$, with a weight function $\pi \in L^\infty(\ell, r)$ and order δ in $\mathbb{R}$ or $\mathbb{C}$, is defined as*

$$({}^{RL}\Im^{\delta}_{\ell,\pi(u)}\kappa)(u) = \frac{1}{\Gamma(\delta)\pi(u)} \int_{\ell}^{u} (u-v)^{\delta-1}\pi(v)\kappa(v)dv, \quad u \in (\ell, r), \tag{II.1}$$

where $\Re(\delta) > 0$, or simply $\delta > 0$ if we consider a real order.

Remark II.1.1 *If we set $\pi(u) = 1$, we obtain the Riemann-Liouville integral as defined in Definition I.5.1.*

Definition II.1.2 *[51] The weighted Riemann–Liouville fractional derivative of a function $\kappa \in AC^m[\ell, r]$, with a weight function $\pi \in AC^m[\ell, r]$ and order δ in $\mathbb{R}$ or $\mathbb{C}$, is defined as*

$$({}^{RL}D^{\delta}_{\ell,\pi(u)}\kappa)(u) = \left(\sigma + \frac{\pi^{'}(u)}{\pi(u)}\right)^{m} {}^{RL}\Im^{m-\delta}_{\ell,\pi}\kappa(u), \quad u \in (\ell, r), \tag{II.2}$$

where $\sigma = \frac{d}{du}$, $\Re(\delta) \geq 0$, or simply $\delta > 0$ if we consider a real order, and $m = [\Re(\delta)] + 1$ so that $m - 1 \leq \Re(\delta) < m$.

Remark II.1.2 *If we consider $\pi(u) = 1$, then it reduces to the Riemann-Liouville derivative as defined in Definition I.5.2.*

Example II.1.1 *The weighted Riemann-Liouville derivative of $\Psi(u) = \frac{(u-\ell)^\theta}{\pi(u)}$ is given by:*

$${}^{RL}D^{\delta}_{\ell,\pi(u)}\left(\frac{(u-\ell)^\theta}{\pi(u)}\right) = \frac{\Gamma(\theta+1)}{\Gamma(\theta-\delta+1)}\frac{(u-\ell)^{\theta-\delta}}{\pi(u)}, \quad \delta \in \mathbb{C}, \Re(\theta) > -1.$$

Definition II.1.3 *[51] The weighted Caputo fractional derivative of a given function $\kappa \in AC^m[\ell, r]$, with a weight function $\pi \in AC^m[\ell, r]$ and order δ in $\mathbb{R}$ or $\mathbb{C}$, is defined as*

$$({}^{C}D^{\delta}_{\ell,\pi(u)}\kappa)(u) = {}^{RL}\Im^{m-\delta}_{\ell,\pi}\left(\sigma + \frac{\pi^{'}(u)}{\pi(u)}\right)^{m} \kappa(u), \quad u \in (\ell, r), \tag{II.3}$$

where $\sigma = \frac{d}{du}$, $\Re(\delta) \geq 0$, or simply $\delta > 0$ if we consider a real order, and $m = [\Re(\delta)] + 1$ so that $m - 1 \leq \Re(\delta) < m$.

Remark II.1.3 *For $\pi(u) = 1$, we get the Caputo derivative as defined in Theorem I.5.4.*

Example II.1.2 *The weighted Caputo derivative of $\Phi(u) = \gamma\frac{\gamma(u-\ell)^\theta}{\pi(u)}$ is given by:*

$${}^{C}D^{\delta}_{\ell,\pi(u)}\left(\frac{E_\delta(\gamma(u-\ell)^\delta)}{\pi(u)}\right) = \gamma\frac{E_\delta(\gamma(u-\ell)^\delta)}{\pi(u)}, \quad \gamma \in \mathbb{C}, \Re(\gamma) > 0.$$

where E_δ is the Mittag-Leffler function.

II.1.1 Conjugation Relations

Proposition II.1.1 *[51] The weighted fractional differintegrals can be viewed as conjugations of the original fractional differintegrals, demonstrated by the following expressions:*

$$\begin{aligned} {}^{RL}\Im^{\delta}_{\ell,\pi(u)} &= G^{-1}_{\pi(u)} \circ^{RL} \Im^{\delta}_{\ell} \circ G_{\pi(u)}, \\ {}^{RL}D^{\delta}_{\ell,\pi(u)} &= G^{-1}_{\pi(u)} \circ^{RL} D^{\delta}_{\ell} \circ G_{\pi(u)}, \\ {}^{C}D^{\delta}_{\ell,\pi(u)} &= G^{-1}_{\pi(u)} \circ^{C} D^{\delta}_{\ell} \circ G_{\pi(u)}, \end{aligned}$$

where the operator $G_{\pi(u)}$ acting on functions is defined as multiplication by the weight function $\pi(u)$:

$$(G_{\pi(u)}\kappa)(u) = \pi(u)\kappa(u). \tag{II.4}$$

Proposition II.1.2 *[51](The semigroup property for weighted fractional integral)*

$$({}^{RL}\Im^{\delta}_{\ell,\pi(u)}\,{}^{RL}\Im^{\theta}_{\ell,\pi(u)}\kappa)(u) = ({}^{RL}\Im^{\delta+\theta}_{\ell,\pi(u)}\kappa)(u), \quad \delta \in \mathbb{C}, \Re(\theta) > 0.$$

Proposition II.1.3 *[51] Let $\delta \in \mathbb{C}$ with $\Re(\theta) > 0$.*

$$\begin{aligned} {}^{RL}\Im^{\delta}_{\ell,\pi(u)}\,{}^{RL}D^{\delta}_{\ell,\pi(u)}\kappa(u) &= \kappa(u) - \sum_{i=1}^{m} \frac{(u-\ell)^{\delta-i}}{\Gamma(\delta-i+1)}.\frac{\pi(\ell^+)}{\pi(u)}.\lim_{u\to\ell^+}({}^{RL}D^{\delta-i}_{\ell,\pi(u)}\kappa)(u), \\ {}^{RL}\Im^{\delta}_{\ell,\pi(u)}\,{}^{C}D^{\delta}_{\ell,\pi(u)}\kappa(u) &= \kappa(u) - \sum_{i=0}^{m-1} \frac{(u-\ell)^{i}}{i!}.\frac{\pi(\ell^+)}{\pi(u)}.\lim_{u\to\ell^+}\left(\vartheta + \frac{\pi^{'}(u)}{\pi(u)}\right)^{i}\kappa(u), \end{aligned}$$

where $\vartheta = \frac{d}{du}$, and $m = [\Re(\delta)] + 1$.

Proposition II.1.4 *[51] The weighted R–L derivative is the analytic continuation of the weighted R–L integral in the complex variable δ, under the convention that integrals of negative order are derivatives of positive order:*

$${}^{RL}D^{\delta}_{\ell,\pi(u)}\kappa(u) = {}^{RL}\Im^{-\delta}_{\ell,\pi(u)}\kappa(u), \quad \Re(\delta) > 0.$$

II.1.2 The Relationship Between the Weighted R-L and Caputo Derivatives:

Lemma II.1.1 *[51] Let $\kappa \in AC^m[\ell, r]$ and Let $\delta \in \mathbb{C}$ with $\Re(\delta) > 0$.*

$$\begin{aligned} {}^{C}D^{\delta}_{\ell,\pi(u)}\kappa(u) &= {}^{RL}D^{\delta}_{\ell,\pi(u)}\kappa(u) - \sum_{i=0}^{m-1} \frac{(u-\ell)^{i-\delta}}{\Gamma(i-\delta+1)}.\frac{\pi(\ell^+)}{\pi(u)}.\lim_{u\to\ell^+}\left(\sigma + \frac{\pi^{'}(u)}{\pi(u)}\right)^{i}\kappa(u) \\ &= {}^{RL}D^{\delta}_{\ell,\pi(u)}\left(\kappa(u) - \sum_{i=0}^{m-1} \frac{(u-\ell)^{i}}{i!}.\frac{\pi(\ell^+)}{\pi(u)}.\lim_{u\to\ell^+}\left(\sigma + \frac{\pi^{'}(u)}{\pi(u)}\right)^{i}\kappa(u)\right), \end{aligned}$$

where $\sigma = \frac{d}{du}$,and $m = [\Re(\delta)] + 1$.

II.1.3 Weighted Laplace Transform and Weighted Convolution

In this section, we introduce the weighted Laplace transform and convolution, essential tools for solving linear fractional differential equations involving weighted Riemann–Liouville and Caputo fractional derivatives.

Definition II.1.4 *[51] Let $\kappa : [0,\infty) \to \mathbb{C}$ be a function, whether real-valued or complex-valued. The weighted Laplace transform of κ with respect to a weight function π is defined as follows:*

$$\mathcal{L}_{\pi(u)}\{\kappa(u)\} = K(v) = \int_0^{\infty} e^{-vu}\pi(u)\kappa(u)du, \tag{II.5}$$

for any $v \in \mathbb{C}$ and any function κ such that this is a convergent integral.

Example II.1.3 *Let $\kappa(u) = e^{-2u}$ with respect to the weight function $\pi(u) = u$, we have*

$$\mathcal{L}_{\pi(u)}\{\kappa(u)\} = \int_0^{\infty} e^{-vu}ue^{-2u}du = \frac{1}{(v+2)^2}$$

For $\kappa(u) = \sin(u)$ and $\pi(u) = e^{-u}$, we obtain

$$\mathcal{L}_{\pi(u)}\{\kappa(u)\} = \int_0^{\infty} e^{-vu}e^{-u}\sin(u)du = \frac{\sin(1)}{1-\cos(1)}$$

Remark II.1.4 *The relationship between the weighted Laplace transform and the classical Laplace transform is defined as follows:*

$$\mathcal{L}_{\pi(u)} = \mathcal{L} \circ G_{\pi(u)}, \tag{II.6}$$

where $G_{\pi(u)}$ is defined as in Equation $(II.4)$.

As consequences of Equation $(II.6)$, we highlight the following results.

Corollary II.1.1 *[51](**Inverse Weighted Laplace Transform**) The inverse weighted Laplace transform exists for any function that possesses a classical inverse Laplace transform, and it can be expressed as follows:*

$$\mathcal{L}^{-1}_{\pi(u)} = G^{-1}_{\pi(u)} \circ \mathcal{L}^{-1},$$

Alternatively,

$$\mathcal{L}^{-1}_{\pi(u)}\{K(u)\} = \frac{1}{2\pi i\pi(u)}\int_{M-i\infty}^{M+i\infty} e^{vu}K(u)dv.$$

II.1.4 Weighted Laplace Transform of the Weighted R-L Fractional Integral and Derivative

Theorem II.1.1 *[51] Let $\delta > 0$ and let κ be a continuous function on $[0,\infty)$ which is of π-weighted exponential order, where π is a continuous weight function. Then, the weighted Laplace transform of the weighted Riemann-Liouville fractional integral of order δ is expressed as:*

L1)

$$\mathcal{L}_{\pi(u)}\{({}^{RL}\Im^{\delta}_{0,\pi(u)}\kappa)(u)\} = v^{-\delta}\mathcal{L}_{\pi(u)}\{\kappa(u)\}.$$

L2) Let $m-1 \leq \Re(\delta) < m$ with $(m \in \mathbb{Z}^+)$, and assume that ${}^{RL}D^{\delta}_{0,\pi(u)}\kappa$ is continuous on $[0,\infty)$ and of π-weighted exponential order. Then, the weighted Laplace transform of the weighted Riemann-Liouville fractional derivative of order δ is defined as:

$$\mathcal{L}_{\pi(u)}\{({}^{RL}D^{\delta}_{0,\pi(u)}\kappa)(u)\} = v^{\delta}\mathcal{L}_{\pi(u)}\{\kappa(u)\} - \pi(0^+)\sum_{j=0}^{m-1} v^{m-j-1}\left({}^{RL}\Im^{m-j-\delta}_{0,\pi(u)}\kappa\right)(0^+).$$

II.1.5 Weighted Laplace Transform of the Weighted Caputo Fractional Derivative

Theorem II.1.2 *[51] Let $m-1 \leq \Re(\delta) < m$ with $(m \in \mathbb{Z}^+)$, and assume that ${}^{C}D^{\delta}_{0,\pi(u)}\kappa$ is continuous on $[0,\infty)$ and of π-weighted exponential order. Then, the weighted Laplace transform of the Caputo fractional derivative of order δ is given by:*

$$\mathcal{L}_{\pi(u)}\{({}^{C}D^{\delta}_{0,\pi(u)}\kappa)(u)\} = v^{\delta}\mathcal{L}_{\pi(u)}\{\kappa(u)\} - \pi(0^+)\sum_{j=0}^{m-1} v^{\delta-j-1}\left[\left(\sigma + \frac{\pi'(u)}{\pi(u)}\right)^{j}\kappa\right](0^+),$$

where $\sigma = \frac{d}{du}$.

Remark II.1.5 *Saying that f is π-exponentially bounded means that the product $\pi \cdot f$ is exponentially bounded.*

II.1.6 Weighted Convolution

Definition II.1.5 *[51] The π-weighted convolution of two functions $\varphi, \psi : [0,\infty) \to \mathbb{C}$ is the function $\varphi *^{\pi} \psi$, given by:*

$$(\varphi *^{\pi} \psi)(u) = \frac{1}{\pi(u)}\int_0^u \pi(u-v)\varphi(u-v)\pi(v)\psi(v)dv.$$

Remark II.1.6 *The weighted convolution is related to the classical convolution as shown by the following formula:*

$$\begin{aligned}(\varphi *^{\pi} \psi)(u) &= \frac{1}{\pi(u)}(\pi\varphi) * (\pi\psi)(u) \\ &= G^{-1}_{\pi(u)}\left((G_{\pi(u)}\varphi) * (G_{\pi(u)}\psi)\right).\end{aligned}$$

Corollary II.1.2 *[51] If $\varphi, \psi : [0, E] \to \mathbb{C}$ are piecewise continuous and their products with π are of exponential order $N > 0$, then*

$$\mathcal{L}_{\pi(u)}\{\varphi *^{\pi} \psi\} = \mathcal{L}_{\pi(u)}\{\varphi\}\mathcal{L}_{\pi(u)}\{\psi\}.$$

II.2 Weighted Fractional Operators of a Function with Respect to Another Function

Definition II.2.1 *[50] Let $D = [\ell, r]$ and $\pi(u) \neq 0$ be a weight function on D, $\theta(u)$ is a differentiable strictly increasing function on D. The space $\chi^p_\pi(\ell, r)$, $1 \leq p \leq \infty$ is defined by*

$$\chi^p_\pi(\ell, r) = \{\kappa : [\ell, r] \to \mathbb{R} \text{ measurable and } \int_\ell^r |\pi(u)\kappa(u)|^p \theta^{'}(u)du < \infty\},$$

having norm

$$\|\kappa\|_{\chi^p_\pi} = \left(\int_\ell^r |\pi(u)\kappa(u)|^p \theta^{'}(u)du\right)^{\frac{1}{p}}, \quad (1 \leq p < \infty)$$

and

$$\|\kappa\|_{\chi^\infty_\pi} = \operatorname*{ess\,sup}_{\ell \leq t \leq r} |\pi(u)\kappa(u)|.$$

Remark II.2.1 *It should be noted that $\kappa \in \chi^p_\pi(\ell, r) \Leftrightarrow \pi(u)\kappa(u)(\theta^{'}(u))^{\frac{1}{p}} \in L^p(\ell, r)$ for $1 \leq p < \infty$ and $\kappa \in \chi^\infty_\pi(\ell, r) \Leftrightarrow \pi(u)\kappa(u) \in L^\infty(\ell, r)$.*

Definition II.2.2 *[50] The space $AC^m_\pi[\ell, r]$ is defined by*

$$AC^m_\pi[\ell, r] = \left\{\kappa : [\ell, r] \to \mathbb{R} \quad \text{such that} \quad \kappa^{(m-1)}_\pi \in AC[\ell, r]\right\}$$

where $AC[\ell, r]$ is the set of absolute continuous functions on the interval $[\ell, r]$, and

$$\kappa^{(m)}_\pi(u) = \frac{1}{\pi(u)}\left(\frac{\sigma}{\theta'(u)}\right)^{(m)}(\pi(u)\kappa(u)), \quad m = 0, 1, 2, ...$$

where $\sigma = \frac{d}{du}$.

Definition II.2.3 *[18, 50, 51] Let $\theta : [\ell, r] \to \mathbb{R}$ be a strictly increasing C^1 function such that $\theta^{'} > 0$ everywhere, and let $\pi \in L^{\infty}(\ell, r)$ be a weight function. Suppose $\kappa \in X^1_{\pi}(\ell, r)$. The π-weighted Riemann-Liouville fractional integral of order $\delta > 0$ of a function $\kappa(u)$ with respect to another function $\theta(u)$ is defined as follows:*

$$(\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)(u) = \frac{1}{\Gamma(\alpha)\pi(u)} \int_{\ell}^{u} (\theta(u) - \theta(v))^{\delta-1}\pi(v)\kappa(v)\theta^{'}(v)dv, \quad u \in (\ell, r). \tag{II.7}$$

Remark II.2.1 *1) If we set $\pi(u) = 1$ and $\theta(u) = u$, we obtain the Riemann-Liouville integral as defined in Definition I.5.1.*

2) Setting $\theta(u) = u$ gives us the Riemann-Liouville integral as defined in Definition II.1.1.

Definition II.2.4 *[18, 50, 51] Let $m \in \mathbb{N}$. The π-weighted derivative of integer order m of a function κ with respect to another function θ is defined as follows:*

$$(D^{m,\theta(u)}_{\pi(u)}\kappa)(u) = \frac{1}{\pi(u)} \left[\left(\frac{\sigma_u}{\theta'(u)} \right)^m (\pi(u)\kappa(u)) \right](u), \tag{II.8}$$

where $\sigma_u = \frac{d}{du}$, and the first-order operator $D^{1,\theta(u)}_{\pi(u)}$ is defined by

$$(D^{1,\theta(u)}_{\pi(u)}\kappa)(u) = \frac{1}{\pi(u)} \left[\left(\frac{\sigma_u}{\theta'(u)} \right) (\pi(u)\kappa(u)) \right](u). \tag{II.9}$$

Definition II.2.5 *[18, 50, 51] Let $\kappa \in AC^m_{\pi}[\ell, r]$, the π-weighted Riemann-Liouville fractional derivative of order $\delta > 0$ of a function κ with respect to another function θ is defined as follows:*

$$\begin{aligned} (D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)(u) &= (D^{m,\theta(u)}_{\pi(u)}\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)(u) \\ &= \frac{D^{m,\theta(u)}_{\pi(u)}}{\Gamma(m-\delta)\pi(u)} \int_{\ell}^{u} (\theta(u) - \theta(v))^{m-\delta-1}\pi(v)\theta^{'}(v)\kappa(v)dv. \end{aligned} \tag{II.10}$$

where $m = [\delta] + 1$ so that $m - 1 < \delta < m$, with $[\delta]$ representing the integer part of δ.

Remark II.2.2 *1) If we set $\pi(u) = 1$ and $\theta(u) = u$, then it simplifies to the Riemann-Liouville derivative as defined in Definition I.5.2.*

2) If we choose $\theta(u) = u$, we obtain the Riemann-Liouville derivative as defined in Definition II.1.2.

Property II.2.1 *[18, 50, 51]*

1) For $\delta > 0$ and $\mu > 0$, we find that

$$\left({}^{RL}\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)} \left[\frac{(\theta(u) - \theta(\ell))^{\mu-1}}{\pi(u)} \right] \right)(u) = \frac{\Gamma(\mu)}{\Gamma(\mu+\delta)\pi(u)} (\theta(u) - \theta(\ell))^{\mu+\delta-1}. \tag{II.11}$$

2) For $\delta < m$ $(m \in \mathbb{N})$ and $\mu > 0$, we obtain

$$\left({}^{RL}D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\left[\frac{(\theta(u)-\theta(\ell))^{\mu-1}}{\pi(u)}\right]\right)(u) = \frac{\Gamma(\mu)}{\Gamma(\mu-\delta)\pi(u)}(\theta(u)-\theta(\ell))^{\mu-\delta-1}. \tag{II.12}$$

Meanwhile, for $i = 1, 2, ..., [\delta]+1$, we have

$$\left({}^{RL}D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\left[\frac{(\theta(u)-\theta(\ell))^{\delta-i}}{\pi(u)}\right]\right)(u) = 0 \tag{II.13}$$

Proof.

$$\begin{aligned}
\left({}^{RL}\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\left[\frac{(\theta(u)-\theta(\ell))^{\mu-1}}{\pi(u)}\right]\right)(u) &= \frac{1}{\Gamma(\delta)\pi(u)}\int_{\ell}^{u}(\theta(u)-\theta(v))^{\delta-1}(\theta(v)-\theta(\ell))^{\delta-1}\theta'(v)dv \\
&= \frac{1}{\Gamma(\delta)\pi(u)}(\theta(u)-\theta(\ell))^{\mu+\delta-1}\int_0^1 \tau^{\delta-1}(1-\tau)^{\mu-1}d\tau \\
&= \frac{1}{\Gamma(\delta)\pi(u)}(\theta(u)-\theta(\ell))^{\mu+\delta-1}\frac{\Gamma(\delta)\Gamma(\mu)}{\Gamma(\delta+\mu)} \\
&= \frac{\Gamma(\mu)}{\Gamma(\delta+\mu)}\frac{(\theta(u)-\theta(\ell))^{\mu+\delta-1}}{\pi(u)},
\end{aligned}$$

where $\tau = \frac{(\theta(v)-\theta(\ell))}{(\theta(u)-\theta(\ell))}$. Now,

$$\begin{aligned}
\left({}^{RL}D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\left[\frac{(\theta(u)-\theta(\ell))^{\mu-1}}{\pi(u)}\right]\right)(u) &= \left(D^{m,\theta(u)}_{\pi(u)}\,{}^{RL}\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)}\left[\frac{(\theta(u)-\theta(\ell))^{\mu-1}}{\pi(u)}\right]\right)(u) \\
&= \frac{\Gamma(\mu)}{\Gamma(\mu+m-\delta)}\left(D^{m,\theta(u)}_{\pi(u)}\left[\frac{(\theta(u)-\theta(\ell))^{\mu+m-\delta-1}}{\pi(u)}\right]\right)(u) \\
&= \frac{\Gamma(\mu)}{\Gamma(\mu+m-\delta)}\frac{\Gamma(\mu+m-\delta)}{\Gamma(\mu-\delta)}\frac{(\theta(u)-\theta(\ell))^{\mu+\delta-1}}{\pi(u)} \\
&= \frac{\Gamma(\mu)}{\Gamma(\mu-\delta)\pi(u)}(\theta(u)-\theta(\ell))^{\mu-\delta-1}.
\end{aligned}$$

This completes the proof. ■

Lemma II.2.1 *[50] For $m \in \mathbb{N}$, we find that*

$$(D^{m,\theta(u)}_{\pi(u)}\,{}^{RL}\Im^{m,\theta(u)}_{\ell^+,\pi(u)}\kappa) = \kappa.$$

Lemma II.2.2 *[50] For $m \in \mathbb{N}$, the following equality holds:*

$$({}^{RL}\Im^{m,\theta(u)}_{\ell^+,\pi(u)} D^{m,\theta(u)}_{\pi(u)}\kappa)(u) = \kappa(u) - \pi^{-1}(u)\sum_{j=1}^{j-1}\frac{(\theta(u)-\theta(\ell))^j}{j!}\kappa_j(\ell),$$

where $\kappa_i(u) = \left(\frac{\sigma_u}{\theta'(u)}\right)^i (\pi(u)\kappa(u))$, $i = 0, 1, 2, ...$

Theorem II.2.2 *[50] Let $\delta > 0$, $1 \le p \le \infty$, and $\kappa \in \chi^p_\pi(\ell, r)$. Then ${}^{RL}\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa$ is bounded in $\chi^P_\pi(\ell, r)$, and we have*

$$\|{}^{RL}\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa\|_{\chi^p_\pi} \le \frac{(\theta(r)-\theta(\ell))^\delta}{\Gamma(\delta+1)}\|\kappa\|_{\chi^p_\pi}.$$

Theorem II.2.3 *[18, 50, 51] Let $\kappa \in \chi^p_\pi(\ell, r)$, $1 \le p \le \infty$, $\delta > 0$, and $\mu > 0$. Then, we have*

$$\left({}^{RL}\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\,{}^{RL}\Im^{\mu,\theta(u)}_{\ell^+,\pi(u)}\kappa\right)(u) = \left({}^{RL}\Im^{\delta+\mu,\theta(u)}_{\ell^+,\pi(u)}\kappa\right)(u). \tag{II.14}$$

Theorem II.2.4 *[18, 50, 51] Let $\delta > k$, where $k \in \mathbb{N}$. Then, we have*

$$D^{k,\theta(u)}_{\pi(u)}\left({}^{RL}\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa\right) = {}^{RL}\,\Im^{\delta-k,\theta(u)}_{\ell^+,\pi(u)}\kappa.$$

Theorem II.2.5 *[50] For $\delta > k$, if $D^{k,\theta(u)}_{\pi(u)}\kappa \in \chi^p_\pi(\ell, r)$, then*

$$({}^{RL}\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)} D^{k,\theta(u)}_{\pi(u)}\kappa)(u) = (D^{k,\theta(u)}_{\pi(u)}\,{}^{RL}\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)(u) - \pi^{-1}(u)\sum_{i=0}^{k-1}\frac{(\theta(u)-\theta(\ell))^{\delta-k+i}}{\Gamma(\delta-k+i+1)}\kappa_i(\ell).$$

Property II.2.6 *[50] Let $\delta > \mu > 0$ and $\kappa \in \chi^p_\pi(\ell, r)$,$1 \le p < \infty$. Then,*

$$\left(D^{\mu,\theta(u)}_{\ell^+,\pi(u)}\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa\right)(u) = \left(\Im^{\delta-\mu,\theta(u)}_{\ell^+,\pi(u)}\kappa\right)(u), \tag{II.15}$$

where $m = [\delta] + 1$.

Theorem II.2.7 *[18, 50, 51] Let $\delta > 0$. Then, we have*

$$\left(D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa\right)(u) = \kappa(u). \tag{II.16}$$

Lemma II.2.3 *[50] Let $\delta > 0$, $m = -[-\delta]$, $\kappa \in \chi^p_\pi(\ell, r)$ and $\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa \in AC^m_\pi[\ell, r]$. Then*

$$\left(\Im^{\delta,\theta(u)}_{\ell^+,\theta(u)} D^{\delta,\theta(u)}_{\pi(u)} k\right)(u) = \kappa(u) - \pi^{-1}(u)\sum_{i=1}^{m}\frac{(\theta(u)-\theta(\ell))^{\delta-i}}{\Gamma(\delta-i+1)}\left(\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa\right)_{m-i}(\ell^+), \tag{II.17}$$

where

$$\left(\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa\right)_i(\ell^+) = \left(\frac{\sigma_u}{\theta'(u)}\right)^i (\pi(u)\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)(\ell^+), \qquad i = 0, 1, ..., m. \tag{II.18}$$

In particular, if $0 < \delta < 1$, then

$$\left(\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)} D^{\delta,\theta(u)}_{\pi(u)}\kappa\right)(u) = \kappa(u) - \frac{\pi(\ell^+)(\Im^{1-\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)(\ell^+)}{\Gamma(\delta)}(\theta(u)-\theta(\ell))^{\delta-1}\pi^{-1}(u). \tag{II.19}$$

Proposition II.2.1 *[51] The w-weighted RL derivative with respect to $\theta(u)$ is the analytic continuation, in the complex variable δ, of the π-weighted RL integral with respect to $\theta(u)$, under the convention that integrals of negative order are derivatives of positive order:*

$$^{RL}D^{\delta,\theta(u)}_{\ell,\pi(u)}\kappa(u) = ^{RL}\Im^{-\delta,\theta(u)}_{\ell,\pi(u)}\kappa(u), \quad \Re(\delta) > 0.$$

Theorem II.2.8 *[50] If $\kappa \in AC^m\pi[\ell, r]$, then the weighted Caputo fractional derivative of order $\delta > 0$, where $m = [\delta] + 1$, can be expressed as*

$$\begin{aligned}({}^{C}D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)(u) &= ({}^{RL}\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)} D^{m,\theta(u)}_{\pi(u)}\kappa)(u) \\ &= \frac{\pi^{-1}(u)}{\Gamma(m-\delta)}\int_{\ell}^{u}(\theta(u)-\theta(v))^{m-\delta-1}\pi(v)D^{m}_{\pi(u)}\kappa(v)\theta'(v)dv\end{aligned}$$

Proposition II.2.2 *[50, 51]*

(1) For $\delta \geq 0$ and $\eta > m$, where $m = [m] + 1$, we have

$$^{C}D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\left(\frac{(\theta(u)-\theta(\ell))^{\eta-1}}{\pi(u)}\right) = \frac{\Gamma(\eta)}{\Gamma(\eta-\delta)}\frac{(\theta(u)-\theta(\ell))^{\eta-\delta-1}}{\pi(u)}.$$

It is noteworthy that

$$^{C}D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\left(\frac{(\theta(u)-\theta(\ell))^{i}}{\pi(u)}\right) = 0, \quad i = 0, 1, ..., m-1.$$

(2) For $\rho \in \mathbb{R}$ and $\delta > 0$, we find

$$^{C}D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\left(\frac{E_{\delta}(\rho(\theta(u)-\theta(\ell))^{\eta})}{\pi(u)}\right) = \rho\frac{E_{\delta}(\rho(\theta(u)-\theta(\ell))^{\eta})}{\pi(u)}.$$

Theorem II.2.9 *[50, 51] Let $\delta \in \mathbb{C}$ with $\Re(\delta) > 0$ and $m = [\Re(\delta)] + 1$. Then*

$$\left({}^{RL}\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\, {}^{C}D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa\right)(u) = \kappa(u) - \pi^{-1}(u)\sum_{i=0}^{m-1}\frac{(\theta(u)-\theta(\ell))^{i})}{i!}\kappa_i(\ell).$$

II.2.1 Conjugation Relations

Theorem II.2.10 *[51] Let $\theta \in C^{\infty}[\ell, r]$, and let $\varnothing_{\theta}$ denote the functional operator of right-composition with $\varnothing_{\theta}h = h \circ \theta$ for any function h defined on the interval $[\theta(\ell), \theta(r)]$. The weighted fractional differintegrals with* *respect to functions can be expressed as conjugations of the original fractional differintegrals, given by:*

$$\begin{aligned}
{}^{RL}\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)} &= G^{-1}_{\pi(u)} \circ \varnothing_\theta \circ^{RL} \Im^{\delta}_{\theta(\ell^+)} \varnothing^{-1}_\theta \circ G_{\pi(u)}, \\
{}^{RL}D^{\delta,\theta(u)}_{\ell^+,\pi(u)} &= G^{-1}_{\pi(u)} \circ \varnothing_\theta \circ^{RL} D^{\delta}_{\theta(\ell^+)} \varnothing^{-1}_\theta \circ G_{\pi(u)}, \\
{}^{C}D^{\delta,\theta(u)}_{\ell^+,\pi(u)} &= G^{-1}_{\pi(u)} \circ \varnothing_\theta \circ^{C} D^{\delta}_{\theta(\ell^+)} \varnothing^{-1}_\theta \circ G_{\pi(u)},
\end{aligned}$$

Here, $G_{\pi(u)}$ is defined as indicated in Equation $(II.4)$.

II.2.2 The Relationship Between the Weighted R-L and Caputo Derivatives with Respect to Functions

Lemma II.2.4 *[50, 51] Let $\kappa \in AC^m_\pi[\ell, r]$ and Let $\delta \in \mathbb{C}$ with $\Re(\delta) > 0$.*

$$\begin{aligned}
{}^{C}D^{\delta,\pi(u)}_{\ell,\pi(u)}\kappa(u) &= {}^{RL}D^{\delta,\theta(u)}_{\ell,\pi(u)}\kappa(u) - \sum_{i=0}^{m-1} \frac{(\theta(u)-\theta(\ell))^{i-\delta}}{\Gamma(i-\delta+1)} . \frac{\pi(\ell^+)}{\pi(u)} . \lim_{u\to\ell^+} \left(D^{\delta,\theta(u)}_{\pi(u)}\right)^i \kappa(u) \\
&= {}^{RL}D^{\delta,\theta(u)}_{\ell,\pi(u)} \left(\kappa(u) - \sum_{i=0}^{m-1} \frac{(\theta(u)-\theta(\ell))^{i}}{i!} . \frac{\pi(\ell^+)}{\pi(u)} . \lim_{u\to\ell^+} \left(D^{\delta,\theta(u)}_{\pi(u)}\right)^i \kappa(u)\right),
\end{aligned}$$

where $m = [\Re(\delta)] + 1$.

II.2.3 The Weighted Laplace Transform of a Function with Respect to Another Function

It is challenging to use the classical Laplace transform for the weighted fractional order. Therefore, in this section, we introduce a modification of the classical Laplace transform known as the w-weighted Laplace transform of a function with respect to another function. Which is ideally suited for studying fractional differential equations which are both weighted and with respect to functions.

Definition II.2.6 *[50, 51] Let $\kappa, \pi : [\ell, \infty) \to\in \mathbb{C}$ $(or\mathbb{R})$, and let θ be a strictly increasing function on $[\ell, \infty)$. Then, the weighted Laplace transform of κ with respect to θ is defined as:*

$$\mathcal{L}^{\theta(u)}_{\pi(u)}\{\kappa\}(v) = \int_\ell^\infty e^{-v(\theta(u)-\theta(\ell))}\pi(u)\kappa(u)\theta'(u)du, \tag{II.20}$$

for any $v \in \mathbb{C}$, and any function κ such that this is a convergent integral.

Example II.2.1 *The weighted Laplace transform of a weighted Mittag-Leffler function is computed as follows:*

$$\begin{aligned}
\mathcal{L}_{\pi(u)}^{\theta(u)}\left\{\frac{(\theta(u)-\theta(\ell))^{\eta-1}}{\pi(u)}E_{\delta,\eta}[\rho(\theta(u)-\theta(\ell))^{\delta}]\right\}(v) &= \mathcal{L}_{\pi(u)}^{\theta(u)}\left\{\sum_{i=0}^{\infty}\frac{\rho^{i}(\theta(u)-\theta(\ell))^{i\delta+\eta-1}}{\Gamma(i\delta+\eta)\pi(u)}\right\}(v) \\
&= \sum_{i=0}^{\infty}\frac{\rho^{i}\mathcal{L}_{\pi(u)}^{\theta(u)}\left\{\frac{(\theta(u)-\theta(\ell))^{i\delta+\eta-1}}{\pi(u)}\right\}(v)}{\Gamma(i\delta+\eta)} \\
&= \sum_{i=0}^{\infty}\frac{\rho^{i}}{\Gamma(i\delta+\eta)}\frac{\Gamma(i\delta+\eta)}{v^{i\delta+\eta}} \\
&= \frac{1}{v^{\delta}}\sum_{i=0}^{\infty}\left(\frac{\rho}{v^{\delta}}\right)^{i} \\
&= \frac{v^{\delta-\eta}}{v^{\delta}-\delta}.
\end{aligned}$$

Theorem II.2.11 *[51] The π-weighted Laplace transform with respect to θ can be expressed as:*

$$\mathcal{L}_{\pi(u)}^{\theta(u)} = \mathcal{L}\circ\varnothing_{\theta(u)-\theta(\ell)}^{-1}\circ G_{\pi(u)},$$

where G and $\varnothing$ are the operators defined in Equation $(II.4)$ and Theorem II.2.10, respectively.

Remark II.2.4 *If $\theta:[0,\infty)\to[0,\infty)$ is an increasing bijection, then the π-weighted Laplace transform with respect to θ can be expressed as:*

$$\mathcal{L}_{\pi(u)}^{\theta(u)} = \mathcal{L}\circ\varnothing_{\theta(u)}^{-1}\circ G_{\pi(u)},$$

Proposition II.2.3 *[50] For $\rho\in\mathbb{R}$ and $v>\rho$, we have*

$$\mathcal{L}_{\pi(u)}^{\theta(u)}\left\{\frac{e^{\rho(\theta(u)-\theta(\ell))}}{\pi(u)}\right\}(v) = \frac{1}{v-\rho}.$$

For $\eta>-1$ and $v>0$, we obtain

$$\mathcal{L}_{\pi(u)}^{\theta(u)}\left\{\frac{(\theta(u)-\theta(\ell))^{\eta-1}}{\pi(u)}\right\}(v) = \frac{\Gamma(\eta)}{v^{\eta}}.$$

Corollary II.2.1 *[51] The inverse π-weighted Laplace transform with respect to θ exists for any function that has a classical inverse Laplace transform denoted by $K(v)$. It can be expressed as follows:*

$$\mathcal{L}_{\theta(u),\pi(u)}^{-1} = G_{\pi(u)}^{-1}\circ\varnothing_{\theta(u)-\theta(\ell)}\circ\mathcal{L}^{-1},$$

Alternatively,

$$\mathcal{L}_{\theta(u),\pi(u)}^{-1}\{K(v)\} = \frac{1}{2\pi i\pi(u)}\int_{N-i\infty}^{N+\infty}e^{v[\theta(u)-\theta(\ell)]}K(v)dv.$$

Next, we will define the weighted convolution of two functions.

Definition II.2.7 *[50] The weighted convolution of functions Φ and Ψ is defined as follows:*

$$\Phi *_{\theta}^{\pi} \Psi(u) = \frac{1}{\pi(u)} \int_{\ell}^{u} \pi(\theta^{-1}(\theta(u) + \theta(\ell) - \theta(v)))\Phi(\theta^{-1}(\theta(u) + \theta(\ell) - \theta(v)))\pi(v)\Psi(v)\theta^{'}(v)dv.$$

Here, θ^{-1} denotes the inverse function of θ.

Remark II.2.5 *The π-weighted θ-convolution is connected to the classical convolution through the following formula:*

$$\Phi *_{\theta}^{\pi} \Psi(u) = G_{\pi(u)}^{-1} \circ \varnothing_{\theta(u)-\theta(\ell)} \left(\varnothing_{\theta(u)-\theta(\ell) \circ G_{\pi(u)}\Phi}^{-1}\right) * \left(\varnothing_{\theta(u)-\theta(\ell) \circ G_{\pi(u)}\Psi}^{-1}\right),$$

where the G and $\varnothing$ operators are as defined in $(II.4)$ and Theorem II.2.10, respectively.

Theorem II.2.12 *[50] Let the weighted Laplace transform of Φ and Ψ exist for $v > N_1$ $v > N_2$, respectively. Then, we have*

$$\mathcal{L}_{\pi(u)}^{\theta(u)}\{\Phi *_{\theta}^{\pi} \Psi(u)\}(v) = \mathcal{L}_{\pi(u)}^{\theta(u)}\{\Phi(u)\}(v)\mathcal{L}_{\pi(u)}^{\theta(u)}\{\Psi(u)\}(v), \quad v > \max\{N_1, N_2\}.$$

Definition II.2.8 *[50] Let $\kappa, \pi : [\ell, \infty) \to \mathbb{R}$. We say that κ is a π-weighted θ-exponential function if there exist constants N, ξ, and U such that*

$$|\pi(u)\kappa(u)| \leq Ne^{\xi\theta(u)} \quad for \quad u > U.$$

The following theorem outlines the sufficient conditions for the existence of the weighted Laplace transform of a function with respect to another function.

Theorem II.2.13 *[50] Let $\kappa, \pi : [\ell, \infty) \to \mathbb{R}$ be functions such that $\pi\kappa$ is piecewise continuous and κ is a π-weighted θ-exponential function. Then, the weighted Laplace transform of κ exists for $v > N$.*

Theorem II.2.14 *[50] Let $\kappa \in AC_{\pi}[\ell, u]$ and is of π-weighted θ-exponential order. Suppose $D_{\pi(u)}^{1,\theta(u)}\kappa$ is piecewise continuous on every interval $[\ell, U)$. Then, the weighted Laplace transform of $D_{\pi}^{1,\theta(u)}\kappa$ exists and*

$$\mathcal{L}_{\pi(u)}^{\theta(u)}\{D_{\pi(u)}^{1,\theta(u)}\kappa\}(v) = v\mathcal{L}_{\pi(u)}^{\theta(u)}\{\kappa\}(v) - \pi(u)\kappa(u).$$

Corollary II.2.2 *[50] Let $\kappa \in AC_{\pi}^{m-1}[\ell, u]$, such that $D_{\pi(u)}^{i,\theta(u)}\kappa$, $i = 0, 1, ..., m-1$, are of π-weighted θ-exponential order. Suppose $D_{\pi(u)}^{m,\theta(u)}\kappa$ is piecewise continuous on every interval $[\ell, U)$. Then, the weighted Laplace transform of $D_{\pi(u)}^{m,\theta(u)}\kappa$ exists and is given by*

$$\mathcal{L}_{\pi(u)}^{\theta(u)}\{D_{\pi(u)}^{m,\theta(u)}\kappa\}(v) = v^m\mathcal{L}_{\pi(u)}^{\theta(u)}\{\kappa\}(v) - \sum_{i=0}^{m-1} v^{m-1-i}\kappa_i(\ell).$$

II.2.4 The Weighted Laplace Transforms of the Weighted Fractional Operators of a Function with Respect to Another Function

In what follows, we will introduce the weighted Laplace transforms of the weighted fractional operators with respect to another function, which plays a crucial role in the analysis and study of fractional differential equations.

Theorem II.2.15 *[50] Let κ be a piecewise continuous function on each interval $[\ell, u]$ and of π-weighted θ-exponential order. Then, the weighted Laplace transform of the weighted fractional integral $(\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)(u)$ is expressed as follows:*

$$\mathcal{L}^{\theta(u)}_{\pi(u)}\{({}^{RL}\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)(u)\}(v) = \frac{\mathcal{L}^{\theta(u)}_{\pi(u)}\{\kappa(u)\}(v)}{v^{\delta}}.$$

Corollary II.2.3 *[50] Let $\delta > 0$, $\kappa \in AC^m_{\pi}[\ell, r]$ for any $r > 0$, $\theta \in C^m[\ell, r]$, $\theta'(u) > 0$ and $(\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)_i$, $i = 0, 1, ..., m-1$ be π-weighted θ-exponential order. Then, the weighted Laplace transform of the weighted fractional derivative $({}^{RL}D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)(u)$ is given by:*

$$\mathcal{L}^{\theta(u)}_{\pi(u)}\{({}^{RL}D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)(u)\}(v) = v^{\delta}\mathcal{L}^{\theta(u)}_{\pi(u)}\{\kappa(u)\}(v) - \sum_{i=0}^{m-1} v^{m-i-1}({}^{RL}\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)_i(\ell^+).$$

Corollary II.2.4 *[50] Let $\delta > 0$, $\kappa \in AC^m_{\pi}[\ell, r]$ for any $r > 0$, $\theta \in C^m[\ell, r]$, $\theta'(u) > 0$ and κ_i, $i = 0, ..., m-1$ be π-weighted θ-exponential order. Then, the expression for the weighted Laplace transform of the weighted fractional derivative $({}^{C}D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)(u)$ is as follows:*

$$\mathcal{L}^{\theta(u)}_{\pi(u)}\{({}^{C}D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)(u)\}(v) = v^{\delta}\left[\mathcal{L}^{\theta(u)}_{\pi(u)}\{\kappa(u)\}(v) - \sum_{i=0}^{m-1} v^{-i-1}\kappa_i(\ell)\right]. \quad \text{(II.22)}$$

Example II.2.2 *Consider the linear Caputo weighted fractional initial value problem:*

$$({}^{C}D^{\delta,\omega(u)}_{\ell^+,\pi(u)}\kappa)(u) - \rho\omega(u) = \kappa(u), \quad \omega(\ell) = \omega_{\ell}, \quad 0 < \delta < 1.$$

Then, $\omega(u)$ is a solution of $(II.2.2)$ if and only if it satisfies the integral equation:

$$\begin{aligned}
\omega(u) &= \pi(\ell)\kappa(\ell)\pi^{-1}(u)E_{\delta,1}(\rho(\theta(u)-\theta(\ell))^{\delta}) \\
&+ \pi^{-1}(u)(\theta(u)-\theta(\ell))^{\delta-1}E_{\delta,\delta}(\rho(\theta(u)-\theta(\ell))^{\delta}) *^{\theta}_{\pi} \kappa(u) \\
&= \pi^{-1}(u)[\pi(\ell)\kappa(\ell)E_{\delta,1}(\rho(\theta(u)-\theta(\ell))^{\delta}) \\
&+ \int_{\ell}^{u}(\theta(u)-\theta(v))^{\delta-1}E_{\delta,\delta}(\rho(\theta(u)-\theta(v))^{\delta})\kappa(v)\theta'(v)dv].
\end{aligned}$$

CHAPTER

III

ON CAUCHY-TYPE PROBLEMS WITH WEIGHTED R-L FRACTIONAL DERIVATIVES

This chapter is dedicated to establishing the existence and uniqueness of solutions to Cauchy-type problems involving weighted R-L fractional derivatives of a function with respect to another function. This will be achieved through the application of the Banach fixed-point theorem.

III.1 Formulation of Cauchy-Type Problems with Weighted R-L Fractional Derivatives

In this section, we delve into the investigation of the Cauchy problem concerning nonlinear differential equations of fractional order, incorporating the weighted Riemann-Liouville fractional derivative of a function with respect to another function. We establish the equivalence between this problem and a nonlinear Volterra-type integral equation of the second kind. Moreover, we provide proofs for the existence and uniqueness of the solution to the addressed Cauchy problem, employing Banach's fixed-point theorem and the method of successive approximations.

Let $D = [\ell, r]$ be a finite interval and ϱ be a parameter such that $m-1 < \varrho \leq m$, then

1) *The weighted space $C^{\pi}_{\varrho,\theta}[a,b]$ of functions κ with respect to θ and weighted π on $[\ell, r]$ is defined by*

$$C^{\pi}_{\varrho,\theta}[\ell, r] = \{\kappa : (\ell, r] \to \mathbb{R}; \quad (\theta(u) - \theta(\ell))^{\varrho}\pi(u)\kappa(u) \in C[\ell, r]\}, \tag{III.1}$$

having norm

$$\|\kappa\|_{C^{\pi}_{\varrho,\theta}[\ell,r]} = \|(\theta(u) - \theta(\ell))^{\varrho}\pi(u)\kappa(u)\|_{C[\ell,r]}.$$

The above space satisfies the following properties:

i) $C^{\pi}_{\varrho,\theta}[\ell, r] = C[\ell, r]$, *for* $\varrho = 0$ *and* $\pi(u) = 1$.

ii) *For* $\pi(u) = 1$, $C^{\pi}_{\varrho,\theta}[\ell, r] = C_{\varrho,\theta}[\ell, r]$.

2) *The weighted space $C^{m,\pi}_{\varrho,\theta}[\ell, r]$ of functions κ with respect to θ and weighted π on $[\ell, r]$ is defined by*

$$C^{m,\pi}_{\varrho,\theta}[\ell, r] = \left\{\kappa : [\ell, r] \to \mathbb{R}; \quad (\pi\kappa)(u) \in C^{m-1}[\ell, r]; \quad (D^{m,\theta(u)}_{\pi(u)}\kappa)(u) \in C^{\pi}_{\varrho,\theta}[\ell, r]\right\}. \tag{III.2}$$

where

$$(D^{m,\theta(u)}_{\pi(u)}\kappa)(u) = \frac{1}{\pi(u)}\left(\frac{\sigma_u}{\theta'(u)}\right)^m (\pi(u)\kappa(u)), \quad \sigma_u = \frac{d}{du}. \tag{III.3}$$

along with the norm

$$\|\kappa\|_{C^{m,\pi}_{\varrho,\theta}[\ell,r]} = \sum_{i=0}^{m-1} \|(\pi\kappa)^{(i)}\|_{C[\ell,r]} + \|D^{m,\theta(u)}_{\pi(u)}\kappa\|_{C^{\pi}_{\varrho,\theta}[\ell,r]}.$$

The above space satisfies the following properties:

p1) $C^{0,\pi}_{\varrho,\theta}[\ell, r] = C^{\pi}_{\varrho,\theta}[\ell, r]$, *for* $m = 0$.

p2) $C^{\pi}_{\varrho,\theta}[\ell,r]=C_{\varrho,\theta}[\ell,r]$ *and* $C^{m,\pi}_{\varrho,\theta}[\ell,r]=C^{m}_{\varrho,\theta}[\ell,r]$, *for* $\pi(u)=1$.

3) For $m-1<\delta\leq m$ $(m\in\mathbb{N})$, *we denote by* $C^{\delta,\pi}_{\varrho,\theta}[\ell,r]$

$$C^{\delta,\pi}_{\varrho,\theta}[\ell,r]=\left\{\kappa(t)\in C^{\pi}_{\varrho,\theta}[\ell,r]:\quad (D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)(u)\in C^{\pi}_{\varrho,\theta}[\ell,r]\right\} \tag{III.4}$$

4) The space $C^{\pi}[\ell,r]$ *of functions* κ *with respect to weighted* π *on* $[\ell,r]$ *is defined by*

$$C^{\pi}[\ell,r]=\{\kappa:(\ell,r]\to\mathbb{R};\quad \pi(u)\kappa(u)\in C[\ell,r]\}. \tag{III.5}$$

We will study the existence and uniqueness of a Cauchy-type problem with a π*-weighted Riemann-Liouville fractional derivative of a function with respect to another function*

$$(D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u)=\kappa(u,\varphi(u))\quad(\delta>0,\quad u>\ell), \tag{III.6}$$

with initial conditions

$$(\pi D^{\delta-k,\theta(u)}_{\ell^+\pi(u)}\varphi)(\ell^+)=r_k,\quad r_k\in\mathbb{R}\quad(k=1,...,m=-[-\delta]). \tag{III.7}$$

From the above initial condition and by definition II.2.5, it is clear that

$$(\pi D^{\delta-k,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u)=(\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)_{m-k}(u)=\left(\frac{\sigma_u}{\theta'(u)}\right)^{m-k}(\pi\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u). \tag{III.8}$$

where, $(\Im^{\delta,\theta}_{\ell^+,\pi(u)}\varphi(u))$ *is the* π*-weighted Riemann-Liouville fractional integration operator of order* δ *defined by* $(II.7)$.

The notation $(D^{\delta-k,\theta(u)}_{\ell^+,\pi(u)}\varphi)(\ell^+)$ *means that the limit is taken at almost all points of the right-sided neighborhood* $(\ell,\ell+\varepsilon)$ $(\varepsilon>0)$ *of a as follows:*

$$(\pi D^{\delta-k,\theta(u)}_{\ell^+,\pi(u)}\varphi)(\ell^+)=\pi(\ell^+)\lim_{u\to\ell^+}(D^{\delta-k,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u)\quad(1\leq k\leq m-1), \tag{III.9}$$

$$(\pi D^{\delta-m,\theta(u)}_{\ell^+,\pi(u)}\varphi)(\ell^+)=\pi(\ell^+)\lim_{u\to\ell^+}(\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u),\quad(\delta\neq m). \tag{III.10}$$

The nonlinear Volterra integral equation of the second kind corresponding to the problem $(III.6)-(III.7)$ *takes the form*

$$\varphi(u)=\frac{1}{\pi(u)}\sum_{j=1}^{m}\frac{r_j}{\Gamma(\delta-j+1)}(\theta(u)-\theta(\ell))^{\delta-j}+\frac{1}{\pi(u)\Gamma(\delta)}\int_{\ell}^{u}(\theta(u)-\theta(s))^{\delta-1}\pi(s)\theta'(s)\kappa(s,\varphi(s))ds. \tag{III.11}$$

In particular, if $0 < \delta < 1$, the problem $(III.6)$-$(III.7)$ takes the form

$$\begin{cases} (D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u) = \kappa(u,\varphi(u)), \quad (0<\delta<1), \\ \\ (\pi\Im^{1-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(\ell^+) = r \quad r\in\mathbb{R}. \end{cases} \tag{III.12}$$

and this problem can be rewritten as a weighted Cauchy type problem

$$\begin{cases} (D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u) = \kappa(u,\varphi(u)), \quad (0<\delta<1), \\ \\ \lim_{u\to\ell^+}(\theta(u)-\theta(\ell))^{1-\delta}\pi(u)\varphi(u)] = C \quad C\in\mathbb{R}. \end{cases} \tag{III.13}$$

The corresponding integral equation to the problem $(III.12)$ has the form:

$$\varphi(u) = \frac{r(\theta(u)-\theta(\ell))^{\delta-1}}{\Gamma(\delta)\pi(u)} + \frac{1}{\pi(u)\Gamma(\delta)}\int_\ell^u (\theta(u)-\theta(s))^{\delta-1}\pi(s)\theta^{'}(s)\kappa(s,\varphi(s))ds. \tag{III.14}$$

In this subsection we give conditions for a unique solution $\varphi(u)$ to the Cauchy type problem $(III.6)-(III.7)$ in the space $C^{\delta,\pi}_{m-\delta,\theta}[\ell,r]$.

III.1.1 Equivalence of Cauchy-Type Problems Involving Weighted R-L Fractional Derivatives and Volterra Integral Equations

First, we prove that the Cauchy type problem $(III.6)-(III.7)$ and the nonlinear Volterra integral equation $(III.11)$ are equivalent in the space $C^{\pi}_{m-\delta,\theta}[\ell,r]$, in the sense that, if $\varphi(u)\in C^{\pi}_{m-\delta,\theta}[\ell,r]$ satisfies one of these relations, then it also satisfies the other one. For that, we need the following lemmas:

Lemma III.1.1 *If $\varrho\in\mathbb{R}(0\le\varrho<1)$, then the π-weighted Riemann-Liouville fractional integral operator $\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}$ with $\delta\in\mathbb{R}$ $(\delta>0)$ is bounded from $C^{\pi}_{\varrho,\theta}[\ell,r]$ into $C^{\pi}_{\varrho,\theta}[\ell,r]$, and*

$$\|\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa\|_{C^{\pi}_{\varrho,\theta}[\ell,r]} \le (\theta(r)-\theta(\ell))^{\delta}\frac{\Gamma(1-\varrho)}{\Gamma(1+\delta-\varrho)}\|\kappa\|_{C^{\pi}_{\varrho,\theta}[\ell,r]}. \tag{III.15}$$

Proof. *Using the definition of weighted fractional integral $(II.7)$ and property II.2.1, then for any $\kappa\in$*

$C^{\pi}_{\varrho,\theta}[\ell,r]$ and $t\in[\ell,r]$, we obtain

$$\begin{aligned}|(\theta(u)-\theta(\ell))^{\varrho}\pi(u)\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa(u)| &= |(\theta(u)-\theta(\ell))^{\varrho}\frac{1}{\Gamma(\delta)}\int_{\ell}^{u}(\theta(u)-\theta(s))^{\delta-1}\theta'(s)\pi(s)\kappa(s)ds|\\ &\le \frac{|(\theta(u)-\theta(\ell))^{\varrho}|}{|\Gamma(\delta)|}\int_{\ell}^{u}|(\theta(u)-\theta(s))^{\delta-1}(\theta(s)-\theta(\ell))^{-\varrho}\theta'(s)|\\ &\times |(\theta(s)-\theta(\ell))^{\varrho}\kappa(s)\pi(s)|ds\\ &\le \pi(u)(\theta(u)-\theta(\ell))^{\varrho}\|\kappa\|_{C^{\pi}_{\varrho,\theta}[\ell,r]}\left(\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}(\pi^{-1}(u)(\theta(u)-\theta(\ell))^{-\varrho})\right)(u)\\ &= (\theta(u)-\theta(\ell))^{\delta}\frac{\Gamma(1-\varrho)}{\Gamma(1+\delta-\varrho)}\|\kappa\|_{C^{\pi}_{\varrho,\theta}[\ell,r]}\end{aligned}$$

Now, by the definition of the weighted space $C^{\pi}_{\varrho,[\ell,r]}$ defined by (III.1), we get

$$\|\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa\|_{C^{\pi}_{\varrho,\theta}[\ell,r]} = \|(\theta(u)-\theta(\ell))^{\varrho}\pi(u)\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa(u)\|_{C[\ell,r]}\le(\theta(r)-\theta(\ell))^{\delta}\frac{\Gamma(1-\varrho)}{\Gamma(1+\delta-\varrho)}\|\kappa\|_{C^{\pi}_{\varrho,\theta}[\ell,r]}.$$

Hence, the proof of this lemma is complete. ■

Lemma III.1.2 *Let $\delta>0$, $\mu>0$ and $0\le\varrho<1$. The following assertions are then true:*

a) If $\kappa(u)\in C^{\pi}_{\varrho,\theta}[\ell,r]$, then the relation $(II.14)$ hold any point $u\in(\ell,r]$.

b) If $\kappa(u)\in C^{\pi}_{\varrho,\theta(u)}[\ell,r]$, then the equality $(II.16)$ hold any point $u\in(\ell,r]$.

c) Let $\delta>\mu>0$. If $\kappa(u)\in C^{\pi}_{\varrho,\theta}[\ell,r]$, then the relation $(II.15)$ hold at any point $t\in(\ell,r]$.

d) Let $m=[\delta]+1$. Also let $(\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)(u)$ the weighted fractional integral $(II.7)$ be, of order $m-\delta$. If $\kappa(u)\in C^{\omega}_{\varrho,\theta}[\ell,r]$ and $(\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)(u)\in C^{m,\pi}_{\varrho,\theta}[\ell,r]$, then the relation $(II.17)$ hold at any point $u\in(\ell,r]$.
In particular, when $0<\delta<1$ and $(\Im^{1-\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa)(u)\in C^{1,\pi}_{\varrho,\theta}[\ell,r]$, the equality $(II.19)$ is valid.

Proof. *As the proof is similar to the proofs in [1], we deleted it.* ■

Lemma III.1.3 *Let $0<\ell<r<\infty$, $\delta>0$ and $m-1\le\varrho<m$ with $m\in\mathbb{N}$ and $\kappa\in C^{\pi}_{\varrho,\theta}[\ell,r]$.*
If $\delta>\varrho$ and $\pi(u)>0$, for all $u\in[\ell,r]$, then $\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa$ is continuous on $[\ell,r]$ and

$$\left(\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa\right)(\ell)=\lim_{u\to\ell^+}\left(\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa\right)(u)=0. \tag{III.16}$$

Proof. *Since $\kappa\in C^{\pi}_{\varrho,\theta}[\ell,r]$, then $(\theta(u)-\theta(\ell))^{\varrho}\pi(u)\kappa(u)$ is continuous on $[\ell,r]$ and hence*

$$|(\theta(u)-\theta(\ell))^{\varrho}\pi(u)\kappa(u)|<C,$$

where $t\in[\ell,r]$ and $C>0$ is a constant.

Therefore,

$$|\left(\Im^{\varrho,\theta(u)}_{\ell^+,\pi(u)}\kappa\right)(u)|<C\left(\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}(\pi^{-1}(u)(\theta(u)-\theta(\ell))^{-\varrho})\right)(u),$$

and by Property II.2.1, we can write

$$|\left(\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa\right)(u)| < C\frac{\Gamma(1-\varrho)}{\Gamma(\delta-\varrho+1)}(\pi^{-1}(u)(\theta(u)-\theta(\ell))^{\delta-\varrho})). \quad \text{(III.17)}$$

As $\delta > \varrho$, the right-hand side of (III.17) goes to zero when $t \to \ell^+$, we obtain the result. ■

Theorem III.1.1 *Let $\delta > 0$, $m = -[-\delta]$. Let B be an open set in $\mathbb{R}$ and let $\kappa : (\ell, r] \times B \to \mathbb{R}$ be a function such that $\kappa(u, \varphi(u)) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ for any $\varphi(u) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$. If $\varphi(u) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$, then $\varphi(u)$ satisfies the relations $(III.6) - (III.7)$ if, and only if, $\varphi(u)$ satisfies the Volterra integral equation $(III.11)$.*

Proof. *First, we prove the necessity. Let $\varphi(u) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ satisfy the relations $(III.6) - (III.7)$. By hypothesis, $\kappa(u, \varphi(u)) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ and it follows from $(III.6)$ that*

$$(D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u) \in C^{\pi}_{m-\delta,\theta}[\ell, r].$$

According to $(II.10)$

$$(D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u) = (D^{m,\theta(u)}_{\pi(u)}\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)((u)), \quad m = -[-\delta], \quad \text{(III.18)}$$

and hence, by Lemma III.1.1, we have

$$(\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u) \in C^{m,\pi(u)}_{m-\delta,\theta(u)}[\ell, r].$$

Thus, we can apply LemmaIII.1.2 (d), and, in accordance with $(II.17)$, we have

$$\left(\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}D^{\delta,\theta(u)}_{\pi(u)}u\right)(u) = \varphi(u) - \sum_{j=1}^{m}\frac{(\theta(u)-\theta(\ell))^{\delta-j}\pi^{-1}(u)}{\Gamma(\delta-j+1)}\left(\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi\right)_{m-j}(\ell^+), \quad \text{(III.19)}$$

where

$$\left(\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi\right)_{m-j}(\ell^+) = \left(\frac{\sigma_u}{\theta'(u)}\right)^{m-j}(\pi(u)\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(\ell^+), \quad \sigma_u = \frac{d}{du}.$$

By $(III.7)$ and $(III.9)$, we rewrite $(III.19)$ in the form

$$\begin{aligned}\left(\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}D^{\delta,\theta(u)}_{\pi(u)}\varphi\right)(u) &= \varphi(u) - \sum_{j=1}^{m}\frac{(\theta(u)-\theta(\ell))^{\delta-j}\pi^{-1}(u)}{\Gamma(\delta-j+1)}\left(\pi D^{\delta-j,\theta(u)}_{\ell^+,\pi(u)}t\right)(\ell^+)\\ &= \varphi(u) - \sum_{j=1}^{m}\frac{r_j}{\Gamma(\delta-j+1)}(\theta(u)-\theta(\ell))^{\delta-j}\pi^{-1}(u) \quad \text{(III.20)}\end{aligned}$$

By Lemma III.1.1, the integral $\left(\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa(u,\varphi(u))\right)(u) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ exists on $[\ell, r]$. Applying the operator $\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}$ to both sides of $(III.6)$ and using $(III.20)$, we obtain the equation $(III.11)$, and hence necessity is proved.

Now, we prove the sufficiency. Let $\varphi(u) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ satisfy the equation $(III.11)$. Applying the operator $D^{\delta,\theta(u)}_{\ell^+\pi(u)}$ to both sides of $(III.11)$, we have

$$\left(D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi\right)(u) = \sum_{j=1}^{m} \frac{r_j}{\Gamma(\delta-j+1)} \left(D^{\delta,\theta}_{\ell^+,\pi(u)}(\pi^{-1}(u)(\theta(u)-\theta(\ell))^{\delta-j})\right)(u) + \left(D^{\delta,\theta(u)}_{\ell^+,\pi(u)} \Im^{\delta,\theta(u)}_{\ell^+,\pi(u)} \kappa(u,\varphi(u))\right)(u).$$

From here, in accordance with the formula $(II.13)$ and Lemma III.1.2 (b), we arrive at the equation $(III.6)$. Now we show that the relation in $(III.7)$ also holds. For this, we apply the operators $D^{\delta-k}_{\ell^+,\pi(u)}$ $(k = 1, ..., m)$ to both sides of $(III.11)$.

If $1 \le k \le m-1$, then, in accordance with $(II.12)$ and Lemma III.1.2(c), we have

$$\begin{aligned}\left(D^{\delta-k,\theta(u)}_{\ell^+,\pi(u)}\varphi\right)(u) &= \sum_{j=1}^{m} \frac{r_j}{\Gamma(\delta-j+1)} \left(D^{\delta-k,\theta(u)}_{\ell^+,\pi(u)}((\theta(u)-\theta(\ell))^{\delta-j}\pi^{-1}(u))\right)(u) + \left(D^{\delta-k,\theta(u)}_{\ell^+,\pi(u)} \Im^{\delta,\theta(u)}_{\ell^+,\pi(u)} \kappa(u,\varphi(u))\right)(u) \\ &= \sum_{j=1}^{m} \frac{r_j}{\Gamma(k-j+1)} (\theta(u)-\theta(\ell))^{k-j}\pi^{-1}(u) + \left(\Im^{k,\theta(u)}_{\ell^+,\pi(u)} \kappa(u,\varphi(u))\right)(u),\end{aligned}$$

Hence,

$$\left(D^{\delta-k,\theta(u)}_{\ell^+,\pi(u)}\varphi\right)(u) = \sum_{j=1}^{m} \frac{r_j}{(k-j)!} (\theta(u)-\theta(\ell))^{k-j}\pi^{-1}(u) + \frac{\pi^{-1}(u)}{(k-1)!} \int_{\ell}^{u} (\theta(u)-\theta(s))^{k-1}\pi(s)\kappa(s,\varphi(s))\theta'(s)ds. \tag{III.21}$$

If $k = m$, then, in accordance with $(III.10)$, $(II.11)$ and Lemma III.1.2 (a) and using Lemma III.1.3, similarly to $(III.21)$, we obtain

$$\begin{aligned}\left(D^{\delta-m,\theta(u)}_{\ell^+,\pi(u)}\varphi\right)(u) &= \sum_{j=1}^{m} \frac{r_j}{\Gamma(\delta-j+1)} \left(\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(t)}((\theta(u)-\theta(\ell))^{\delta-j}\pi^{-1}(u))\right)(u) + \left(\Im^{m-\delta,\theta(u)}_{\ell^+,\pi(u)} \Im^{\delta,\theta(u)}_{\ell^+\pi} \kappa(u,\varphi(u))\right)(u) \\ &= \sum_{j=1}^{m} \frac{r_j}{\Gamma(m-j+1)} (\theta(u)-\theta(\ell))^{m-j}\pi^{-1}(u) + \left(\Im^{m,\theta(u)}_{\ell^+\pi(u)} \kappa(u,\varphi(u))\right)(u),\end{aligned}$$

Therefore,

$$\left(D^{\delta-m,\theta(u)}_{\ell^+,\pi(u)}(u)\right)(u) = \sum_{j=1}^{m} \frac{r_j}{(m-j)!} (\theta(u)-\theta(\ell))^{m-j}\pi^{-1}(u) + \frac{\pi^{-1}(u)}{(m-1)!} \int_{\ell}^{u} (\theta(u)-\theta(s))^{m-1}\pi(s)\kappa(s,\varphi(s))\psi'(s)ds. \tag{III.22}$$

Multiplying $(III.21)$, $(III.22)$ by $\pi(u)$ and taking the limit as $u \to 0^+$ above, we obtain $(III.7)$. Thus, sufficiency is proven, and the proof is completed. ■

Corollary III.1.1 *Let $0 < \delta < 1$, let B be an open set in $\mathbb{R}$ and let $\kappa : (\ell, r] \times B \to \mathbb{R}$ be a function such that $\kappa(u, \varphi(u)) \in C^{\pi}_{1-\delta,\theta}[\ell, r]$ for any $\varphi(u) \in C^{\pi}_{1-\delta,\theta}[\ell, r]$. If $\varphi(u) \in C^{\pi}_{1-\delta,\theta}[\ell, r]$, then $\varphi(u)$ satisfies the relations in $(III.12)$ if, and only if, $\varphi(u)$ satisfies the integral equation $(III.14)$.*

III.1.2 Existence and Uniqueness of Solutions to Cauchy-Type Problems Involving Weighted R-L Fractional Differential Equations

Next, we will establish the existence and uniqueness of a solution for the Cauchy-type problem $(III.6) - (III.7)$ *in the space* $C^{\delta,\omega}_{\varrho,\theta}[\ell, r]$*, as defined in* $(III.4)$*, utilizing the Banach fixed point theorem. This requires the following lemmas:*

Lemma III.1.4 *Let* $\lambda \in [0, \infty)$*,* $\ell < n < r$*,* $\kappa \in C^{\pi}_{\lambda,\theta}[\ell, n]$ *and* $\kappa \in C^{\pi}[n, r]$*. Then* $\kappa \in C^{\pi}_{\lambda,\theta}[\ell, r]$ *and*

$$\|\kappa\|_{C^{\pi}_{\lambda,\theta}[\ell,r]} \leq \max\{\|\kappa\|_{C^{\pi}_{\lambda,\theta}[\ell,n]}, (\theta(r) - \psi(\ell))^{\lambda}\|\kappa\|_{C^{\pi}[n,r]}\}, \tag{III.23}$$

where the space $C^{\pi}[\ell, r]$ *is the same as defined in* $(III.5)$*.*

Proof. *Since* $\kappa \in C^{\pi}_{\lambda,\theta}[\ell, n]$ *and* $\kappa \in C^{\pi}[n, r]$*, then we obtain*

$$\kappa \in C^{\pi}(\ell, r] \quad and \quad \kappa \in C^{\pi}_{\lambda,\theta}[\ell, r].$$

Now, we prove the estimate. Because $\kappa \in C^{\pi}_{\lambda,\theta}[\ell, r]$*, there exists* $u^* \in [\ell, r]$ *such that*

$$\|\kappa\|_{C^{\pi}_{\lambda,\theta}[\ell,r]} = |(\theta(u^*) - \theta(\ell))^{\lambda}\pi(u^*)\kappa(u^*)|. \tag{III.24}$$

Assume that $u^* \in [\ell, n]$*, then we have*

$$\|\kappa\|_{C^{\pi}_{\lambda,\theta}[\ell,r]} \leq \|\kappa\|_{C^{\pi}_{\lambda,\theta}[\ell,n]}.$$

Similarly, if we suppose that $u^* \in [n, r]$*, then we have*

$$\|\kappa\|_{C^{\pi}_{\lambda,\theta}[\ell,r]} \leq (\theta(r) - \theta(\ell))^{\lambda}\|\kappa\|_{C^{\pi}[n,r]}.$$

Now, we can write this result

$$\|\kappa\|_{C^{\pi}_{\lambda,\theta}[\ell,r]} \leq \max\{\|\kappa\|_{C^{\pi}_{\lambda,\theta}[\ell,n]}, (\theta(r) - \theta(\ell))^{\lambda}\|\kappa\|_{C^{\pi}[n,r]}\}.$$

The proof of this lemma is complete. ■

Lemma III.1.5 *The weighted fractional integration operator* $\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}$ *of order* δ $(\delta > 0)$ *is a mapping from* $C^{\pi}[\ell, r]$ *to* $C^{\pi}[\ell, r]$*, and*

$$\|\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa\|_{C^{\pi}[\ell,r]} \leq \frac{(\theta(r) - \psi(\ell))^{\delta}}{\delta\Gamma(\delta)}\|\kappa\|_{C^{\pi}[\ell,r]}, \tag{III.25}$$

where $\kappa \in C^{\pi}[\ell, r]$*.*

Proof. *we prove the estimate in $(III.25)$ as follows:*

$$\begin{aligned}|\pi(u)\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa(u)| &= |\frac{1}{\Gamma(\delta)}\int_\ell^u(\theta(u)-\theta(s))^{\delta-1}\theta^{'}(s)\pi(s)\kappa(s)ds| \\ &\leq \frac{\|\pi\kappa\|_{C[\ell,r]}}{\Gamma(\delta)}\int_\ell^u(\theta(u)-\theta(s))^{\delta-1}\theta^{'}(s)ds \\ &= \frac{(\theta(u)-\theta(\ell))^\delta}{\delta\Gamma(\delta)}\|\kappa\|_{C^\pi[\ell,r]}.\end{aligned}$$

Therefore,

$$\|\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}\kappa\|_{C^\pi[\ell,r]} \leq \frac{(\theta(r)-\theta(\ell))^\delta}{\delta\Gamma(\delta)}\|\kappa\|_{C^\pi[\ell,r]}.$$

Hence, the proof of this lemma is complete. ■

Theorem III.1.2 *Let $\delta > 0$ and $m = -[-\delta]$. Let B be an open set in $\mathbb{R}$ and let $\kappa : (\ell, r] \times B \to \mathbb{R}$ be a function such that $\kappa(u, \varphi(u)) \in C^\pi_{m-\delta,\theta}[\ell, r]$ for any $\varphi(u) \in C^\pi_{m-\delta,\theta}[\ell, r]$ and the Lipschitzian condition $(I.4)$ holds. Then there exists a unique solution $\varphi(u)$ to the Cauchy type problem $(III.6) - (III.7)$ in the space $C^\pi_{m-\delta,\theta}[\ell, r]$.*

Proof. Step1. *First we prove the existence of a unique solution $\varphi(u) \in C^\pi_{m-\delta,\theta}[\ell, r]$. According to the previous Theorem III.1.1, it is sufficient to prove the existence of a unique solution $\varphi(u) \in C^\pi_{m-\delta,\theta}[\ell, r]$ to the nonlinear Volterra integral equation $(III.11)$, which, mainly based on Theorem I.2.1(Banach fixed point theorem).*

Divide the interval $[\ell, r]$ into M subdivisions $[\ell, u_1], [u_1, u_2], ..., [u_{M-1}, r]$ such that $\ell < u_1 < u_2 < ... < u_{M-1} < r$.

I) Choose $u_1 \in (\ell, r]$ such that the following estimate holds

$$\xi_1 = L(\theta(u_1)-\theta(\ell))^\delta\frac{\Gamma(\delta-m+1)}{\Gamma(2\delta-m+1)} < 1, \tag{III.26}$$

where L is the Lipschitzian constant. Now we prove that there exists a unique solution $\varphi(u) \in C^\omega_{m-\delta,\theta}[\ell, u_1]$ to $(III.11)$ on the interval $(\ell, u_1]$. To do this, we apply the Banach fixed point theorem (Theorem I.2.1) for the space $C^\pi_{m-\delta,\theta}[\ell, r]$, which is the complete metric space equipped with the distance given by

$$\sigma(\varphi_1,\varphi_2) = \|\varphi_1-\varphi_2\|_{C^\pi_{m-\delta,\theta}[\ell,u_1]} = \max_{u\in[\ell,u_1]}|(\theta(u)-\theta(\ell))^{m-\delta}\pi(u)[\varphi_1(u)-\varphi_2(u)]|.$$

For any $\varphi(u) \in C^\pi_{m-\delta,\theta}[\ell, u_1]$, we define the operator A by expressing the integral equation $(III.11)$ as follows:

$$\varphi(u) = (A\varphi)(u),$$

$$(A\varphi)(u) = \varphi_0(u) + \frac{1}{\Gamma(\delta)\pi(u)}\int_\ell^u(\theta(u)-\theta(s))^{\delta-1}\theta^{'}(s)\omega(s)\kappa(s,\varphi(s))ds, \tag{III.27}$$

with

$$\varphi_0(u) = \frac{1}{\pi(u)} \sum_{j=1}^{m} \frac{r_j}{\Gamma(\delta - j + 1)} (\theta(u) - \theta(\ell))^{\delta - j}. \tag{III.28}$$

Applying the Banach contraction mapping, we shall prove that A has a unique fixed point.

Firstly, we have to show that:

I1) If $\varphi(u) \in C^{\pi}_{m-\delta,\theta}[\ell, u_1]$, then $(A\varphi)(u) \in C^{\pi}_{m-\delta,\theta}[\ell, u_1]$.

I2) For any $\varphi_1, \varphi_2 \in C^{\pi}_{m-\delta,\theta}[\ell, u_1]$ the following estimate holds:

$$\|A\varphi_1 - A\varphi_2\|_{C^{\pi}_{m-\delta,\theta}[\ell,u_1]} \leq \xi_1 \|\varphi_1 - \varphi_2\|_{C^{\pi}_{m-\delta,\theta}[\ell,u_1]}, \quad \xi_1 = L(\theta(u_1) - \theta(\ell))^{\delta} \frac{\Gamma(\delta - m + 1)}{\Gamma(2\delta - m + 1)}. \tag{III.29}$$

It is evident from equation (III.28) that $\varphi_0(u) \in C^{\pi}_{m-\delta,\theta}[\ell, u_1]$. Since $\kappa(u, \varphi(u)) \in C^{\pi}_{m-\delta,\theta}[\ell, u_1]$ for any $\varphi(u) \in C^{\pi}_{m-\delta,\theta}[\ell, u_1]$, then, by Lemma III.1.1, the integral in the right-hand side of $(III.27)$ also belongs to $C^{\pi}_{m-\delta,\theta}[\ell, u_1]$. The above implies that $(A\varphi)(u) \in C^{\pi}_{m-\delta,\theta}[\ell, u_1]$.

Let $\varphi_1, \varphi_2 \in C^{\pi}_{m-\delta,\theta}[\ell, u_1]$. Using $(III.27)$, $(I.4)$ and hance by Lemma III.1.1, we obtain

$$\begin{aligned}
\|A\varphi_1 - A\varphi_2\|_{C^{\pi}_{m-\delta,\theta}[\ell,u_1]} &= \|\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)}(\kappa(u, \varphi_1(u)) - \kappa(u, \varphi_2(u)))\|_{C^{\pi}_{m-\delta,\theta}[\ell,u_1]} \\
&\leq (\theta(u_1) - \theta(\ell))^{\delta} \frac{\Gamma(\delta - m + 1)}{\Gamma(2\delta - m + 1)} \|\kappa(u, \varphi_1(u)) - \kappa(u, \varphi_2(u))\|_{C^{\pi}_{m-\delta,\theta}}[\ell, u_1] \\
&\leq L(\theta(u_1) - \theta(\ell))^{\delta} \frac{\Gamma(\delta - m + 1)}{\Gamma(2\delta - m + 1)} \|\varphi_1 - \varphi_2\|_{C^{\pi}_{m-\delta,\theta}}[\ell, u_1] \\
&= \xi_1 \|\varphi_1 - \varphi_2\|_{C^{\pi}_{m-\delta,\theta}}[\ell, u_1].
\end{aligned}$$

This results in obtaining the estimate $(III.29)$. By $(III.26)$, $0 < \xi_1 < 1$, and therefore by using the Banach fixed point theorem, there exists a unique solution $\varphi^*(u) \in C^{\pi}_{m-\delta,\theta}[\ell, u_1]$ to $(III.11)$ on the interval $[\ell, u_1]$. This solution $\varphi^*(u)$ is a limit of a convergent sequence $(A^i\varphi_0^*)(u)$:

$$\lim_{i\to\infty} \|A^i\varphi_0^* - \varphi^*\|_{C^{\pi}_{m-\delta,\theta}[\ell,u_1]} = 0, \tag{III.30}$$

where $\varphi_0^*(u)$ is any function in $C^{\pi}_{m-\delta,\theta}[\ell, u_1]$, and

$$(A^i\varphi_0^*)(u) = \varphi_0(u) + \frac{1}{\Gamma(\delta)\pi(u)} \int_{\ell}^{u} (\theta(u) - \theta(s))^{\delta-1}\theta'(s)\pi(s)\kappa(s, (A^{i-1}\varphi_0^*)(s))ds, \quad (i \in \mathbb{N}).$$

If we denote $\varphi_i(u) = (A^i\varphi_0^*)(u)$, $(i \in \mathbb{N})$, then it is clear that

$$\lim_{m\to\infty} \|\varphi_i - \varphi^*\|_{C^{\pi}_{m-\delta,\theta}[\ell,u_1]} = 0. \tag{III.31}$$

If there exists at least one $r_k \neq 0$ in the initial condition $(III.7)$, then we can choose $\varphi_0^*(u) = \varphi_0(u)$, where $\varphi_0(u)$ is defined by $(III.28)$.

E) *Next, we prove the existence of a unique solution $u \in C^{\pi}[u_1, r]$ to $(III.11)$ on the interval $[u_1, r]$. Moreover, if we consider the interval $[u_1, r]$, we can express Eq $(III.11)$ in the following manner:*

$$\varphi(u) = \varphi_{01}(u) + \frac{1}{\pi(u)\Gamma(\delta)} \int_{u_1}^{u} (\theta(u) - \theta(s))^{\delta-1} \pi(s)\theta^{'}(s)\kappa(s, \varphi(s))ds, \tag{III.32}$$

where $\varphi_{01}(u)$ is defined by

$$\varphi_{01}(u) = \sum_{j=1}^{m} \frac{r_j}{\Gamma(\delta - j + 1)} (\theta(u) - \theta(\ell))^{\delta-j} \pi^{-1}(u) + \frac{1}{\pi(u)\Gamma(\delta)} \int_{\ell}^{u_1} (\theta(u) - \theta(s))^{\delta-1} \pi(s)\theta^{'}(s)\kappa(s, \varphi(s))ds,$$

is a known function. We note that $\varphi_{01}(u) \in C^{\pi}[u_1, r]$. Because, $\varphi_0(u) \in C^{\pi}[u_1, r]$. Also, by hypothesis $\kappa(u, \varphi(u)) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ for any $\varphi(u) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$, then, $\kappa(u, \varphi(u)) \in C^{\pi}[u_1, r]$, therefore, we can apply Lemma III.1.5, we have $\Im^{\delta,\theta(u)}_{\ell^+,\pi(u)} \kappa(u, \varphi(u)) \in C^{\pi}[u_1, r]$. Thus, $\varphi_{01}(u) \in C^{\pi}[u_1, r]$. We consider the interval $[u_1, u_2]$, where $u_2 = u_1 + \varepsilon_1$ and $\varepsilon_1 > 0$ are such that $u_2 \in (u_1, r]$. We also use Banach fixed point theorem for the space $C^{\pi}[u_1, u_2]$, where u_2 satisfies

$$\xi_2 = \frac{L(\theta(u_2) - \theta(u_1))^{\delta}}{\delta\Gamma(\delta)} < 1. \tag{III.33}$$

The space $C^{\pi}[u_1, u_2]$ is a complete metric space, with the distance given by

$$\delta(\varphi_1, \varphi_2) = \|\varphi_1 - \varphi_2\|_{C^{\pi}[u_1,u_2]} = \max_{u\in[u_1,u_2]} |\pi(u)[\varphi_1(u) - \varphi_2(u)]|.$$

Also, we can rewrite the integral equation $(III.32)$ in the form:

$$\varphi(u) = (A\varphi)(u),$$

where A is the operator given by

$$(A\varphi)(u) = \varphi_{01}(u) + \frac{1}{\Gamma(\delta)\pi(u)} \int_{u_1}^{u} (\theta(u) - \theta(s))^{\delta-1} \theta^{'}(s)\pi(s)\kappa(s, \varphi(s))ds. \tag{III.34}$$

To apply Theorem I.2.1, we have to prove the following:

E1) *If $\varphi(u) \in C^{\pi}[u_1, u_2]$, then $(A\varphi)(u) \in C^{\pi}[u_1, u_2]$.*

E2) *For any $\varphi_1, \varphi_2 \in C^{\pi}[u_1, u_2]$ the following estimate holds:*

$$\|A\varphi_1 - A\varphi_2\|_{C^{\pi}[u_1,u_2]} \leq \xi_2 \|\varphi_1 - \varphi_2\|_{C^{\pi}[u_1,u_2]}, \quad \xi_2 = \frac{L(\theta(u_2) - \theta(u_1))^{\delta}}{\delta\Gamma(\delta)}. \tag{III.35}$$

Similarly, by hypothesis $\kappa(u, \varphi(u)) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ for any $\varphi(u) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$, then, by Lemma III.1.5, the integral in the right-hand side of $(III.34)$ also belongs to $C^{\pi}[u_1, u_2]$, and hence $(A\varphi)(u) \in C^{\pi}[u_1, u_2]$. Now, we *prove the estimate in $(III.35)$, using the Lipschitz condition and applying Lemma III.1.5, we find*

$$\begin{aligned}\|A\varphi_1 - A\varphi_2\|_{C^\pi[u_1,u_2]} &= \|\Im^{\delta,\theta(u)}_{\ell+,\pi(u)}(\kappa(u,\varphi_1(u)) - \kappa(u,\varphi_2(u)))\|_{C^\pi[u_1,u_2]} \\ &\le \frac{(\psi(u_2)-\psi(u_1))^\delta}{\delta\Gamma(\delta)}\|\kappa(u,\varphi_1(u)) - \kappa(u,\varphi_2(u))\|_{C^\pi[u_1,u_2]} \\ &\le \frac{L(\theta(u_2)-\theta(u_1))^\delta}{\delta\Gamma(\delta)}\|\varphi_1-\varphi_2\|_{C^\pi[u_1,u_2]} \\ &= \xi_2\|\varphi_1-\varphi_2\|_{C^\pi[u_1,u_2]},\end{aligned}$$

which yields the estimate $(III.35)$. This, together with our assumption $0 < \xi_2 < 1$, shows that A is a contraction and therefore from Theorem I.2.1, there exists a unique solution $\varphi_1^*(u) \in C^\pi[u_1,u_2]$ to $(III.11)$ on the interval $[t_1,t_2]$. Further, Theorem I.2.1 guarantees that this solution $\varphi_1^*(u)$ is the limit of the convergent sequence $(A^i\varphi_{01}^*)(u)$:

$$\lim_{i\to\infty}\|A^i\varphi_{01}^* - \varphi_1^*\|_{C^\pi[u_1,u_2]} = 0, \tag{III.36}$$

where $\varphi_{01}^*(u)$ is any function in $C^\pi[u_1,u_2]$. If $\varphi_0(u) \neq 0$ on $[u_1,u_2]$, then we can take $\varphi_{01}^*(u) = \varphi_0(u)$ with $\varphi_0(u)$ defined by $(III.28)$. The last relation can be rewritten in the form

$$\lim_{i\to\infty}\|\varphi_i - \varphi_1^*\|_{C^\pi[u_1,u_2]} = 0, \tag{III.37}$$

where

$$\varphi_i(u) = (A^i\varphi_{01}^*)(u) = \varphi_{01}(u) + \frac{1}{\pi(u)\Gamma(\delta)}\int_{u_1}^{u}(\theta(u)-\theta(s))^{\delta-1}\pi(s)\theta'(s)\kappa(s,(A^{i-1}\varphi_{01}^*)(s))ds, \quad (i\in\mathbb{N}). \tag{III.38}$$

E3) *Next, if $u_2 \neq r$, we consider the interval $[u_2,u_3]$, where $u_3 = u_2+\varepsilon_2, \varepsilon_2 > 0$, such that $u_3 \le r$ and*

$$\xi_3 = \frac{L(\theta(u_3)-\theta(u_2))^\delta}{\delta\Gamma(\delta)} < 1.$$

By using the same arguments as above, we conclude that there exists a unique solution $\varphi_2^*(u) \in C^\pi[u_2,u_3]$ to $(III.11)$ on the interval $[u_2,u_3]$. If $u_3 \neq r$, repeating the above process, then we find that there exists a unique solution $\varphi(u)$ to $(III.11)$, $\varphi(u) = \varphi_k^*(u)$, and $\varphi_k^*(u) \in C^\pi[u_{k-1},u_k]$ for $k = 1,...,M$, where $a = u_0 < u_1 < ... < tuM = r$ and

$$\xi_k = \frac{L(\theta(u_k)-\theta(u_{k-1}))^\delta}{\delta\Gamma(\delta)} < 1.$$

Consequently, there exists a unique solution $\varphi(u) \in C^\pi[u_1,r]$ to $(III.11)$ on the interval $[u_1,r]$. Using Lemma III.1.4, we can conclude that there exists a unique solution $u(t) \in C^\pi_{m-\delta,\theta}[\ell,r]$ to the Volterra integral equation

$(III.11)$ on the whole interval $[\ell, r]$. Therefore, $\varphi(u) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ is the unique solution to the Cauchy-type problem $(III.6) - (III.7)$.

Step 2. *Finally, it remains to show that such a unique solution is actually in $C^{\delta,\pi}_{m-\delta,\theta}[\ell, r]$. By $(III.4)$, it is sufficient to prove that $(D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$. By the above proof, the solution $\varphi(u) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ is a limit of the sequence $\varphi_i(u)$, where $\varphi_i(u) = (A^i\varphi^*_0)(u) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$:*

$$\lim_{i\to\infty} \|\varphi_i - \varphi\|_{C^{\pi}_{m-\delta,\theta}[\ell,r]} = 0, \tag{III.39}$$

with the choice of certain φ^*_0 on each $[\ell, u_1], ..., [u_{M-1}, r]$. If $\varphi_0(u) \neq 0$, then we can take $\varphi^*_0(u) = \varphi_0(u)$. Hence, by using $(III.6)$ and $(I.4)$, we have

$$\begin{aligned} \|D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi_i - D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi\|_{C^{\pi}_{m-\delta,\theta}[\ell,r]} &= \|\kappa(u, \varphi_i(u)) - \kappa(u, \varphi(u))\|_{C^{\pi}_{m-\delta,\theta}[\ell,r]} \\ &\leq L\|\varphi_i - \varphi(u)\|_{C^{\pi}_{m-\delta,\theta}[\ell,r]}. \end{aligned} \tag{III.40}$$

In virtue of $(III.39)$ and $(III.40)$, it can be said that

$$\lim_{i\to\infty} \|D^{\delta,\theta}_{\ell^+,\pi(u)}\varphi_i - D^{\delta,\theta}_{\ell^+,\pi(u)}\varphi\|_{C^{\pi}_{m-\delta,\theta}[\ell,r]} = 0.$$

By hypothesis, $(D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi_i)(u) = \kappa(u, \varphi_{i-1}(u))$ and $\kappa(u, \varphi(u)) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ for any $\varphi(u) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$, we have $(D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi_i)(u) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$. Hence $(D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$. Consequently, $\varphi(u) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ is the unique solution to the problem $(III.6) - (III.7)$. The proof is complete. ■

Corollary III.1.2 *Let $0 < \delta < 1$, let B be an open set in $\mathbb{R}$ and let $\kappa : (\ell, r] \times B \to \mathbb{R}$ be a function such that $\kappa(u, \varphi(u)) \in C^{\pi}_{1-\delta,\theta}[\ell, r]$ for any $\varphi(u) \in C^{\pi}_{1-\delta,\theta}[\ell, r]$ and $(I.4)$ holds. Then there exists a unique solution $u(t)$ to the Cauchy type problem $(III.12)$ in the space $C^{\delta,\pi}_{1-\delta,\theta}[\ell, r]$.*

III.2 The Weighted Cauchy Type Problem with Weighted R-L Fractional Derivatives of a Function with Respect to Another Function

When $0 < \delta < 1$, the result of Corollary III.1.2 remains true for the following weighted Cauchy type problem $(III.13)$ with $C \in \mathbb{R}$:

$$(D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u) = \kappa(u, \varphi(u)); \quad \lim_{t\to\ell^+} [(\theta(u) - \theta(\ell))^{1-\delta}\pi(u)\varphi(u)] = C, \quad (0 < \delta < 1). \tag{III.41}$$

Its proof is based on the following lemma assertion:

Lemma III.2.1 *Let $0 < \delta < 1$ and let $\varphi(u) \in C^{\pi}_{1-\delta,\theta}[\ell, r]$.*

S1) If there exists a limit

$$\lim_{u\to\ell^+} \left[(\theta(u) - \theta(\ell))^{1-\delta}\pi(u)\varphi(u)\right] = C, \quad C \in \mathbb{R}, \tag{III.42}$$

then there also exists a limit

$$(\pi\Im^{1-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(\ell^+) = \lim_{u\to\ell^+}(\pi\Im^{1-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u) = C\Gamma(\delta). \tag{III.43}$$

S2) If there exists a limit

$$\lim_{u\to\ell^+}(\pi\Im^{1-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u) = r, \quad r \in \mathbb{R} \tag{III.44}$$

and if there exists the limit $\lim_{t\to\ell^+}\left[(\theta(u) - \theta(\ell))^{1-\delta}\pi(u)\varphi(u)\right]$, then

$$\lim_{u\to\ell^+} \left[(\theta(u) - \theta(\ell))^{1-\delta}\pi(u)\varphi(u)\right] = \frac{r}{\Gamma(\delta)}. \tag{III.45}$$

Proof. *Choose an arbitrary $\varepsilon > 0$. According to $(III.42)$, there exists $\eta = \eta(\varepsilon) > 0$ such that*

$$|(\theta(u) - \theta(\ell))^{1-\delta}\pi(u)\varphi(u) - C| < \frac{\varepsilon}{\Gamma(\delta)}. \tag{III.46}$$

For $\ell < t < \ell + \eta$.By using $(II.11)$, we have

$$\Gamma(\delta) = (\pi\Im^{1-\delta,\theta(u)}_{\ell^+,\pi(u)}(\pi^{-1}(u)(\theta(u) - \theta(\ell))^{\delta-1}))(u), \quad 0 < \delta < 1. \tag{III.47}$$

Using this equality and $(II.7)$, we obtain

$$\begin{aligned}
|(\omega\Im^{1-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u) - C\Gamma(\delta)| &= |(\pi\Im^{1-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u) - C(\pi\Im^{1-\delta,\theta(u)}_{\ell^+,\pi(u)}(\pi^{-1}(u)(\theta(u) - \theta(\ell))^{\delta-1}))(u)| \\
&\leq \frac{1}{\Gamma(1-\delta)}\int_{\ell}^{u}(\theta(u) - \theta(s))^{-\delta}\theta'(s)|\pi(s)\varphi(s) - C(\theta(s) - \theta(\ell))^{\delta-1}|ds \\
&\leq \frac{1}{\Gamma(1-\delta)}\int_{\ell}^{u}(\theta(u) - \theta(s))^{-\delta}\theta'(s)(\theta(s) - \theta(\ell))^{\delta-1}|(\theta(s) - \theta(\ell))^{1-\delta}\pi(s)\varphi(s) - C|ds.
\end{aligned}$$

Now, by making use of $(III.46)$ and the formula $(III.47)$, we have

$$|(\pi\Im^{1-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u) - C\Gamma(\delta)| \leq \frac{\varepsilon\pi(u)}{\Gamma(\delta)}(\Im^{1-\delta,\theta(u)}_{\ell^+,\pi(u)}(\pi^{-1}(u)(\theta(u) - \theta(\ell))^{\delta-1}))(t) = \varepsilon, \tag{III.48}$$

which proves the assertion $(S1)$ of Lemma III.2.1. Assume that the limit in $(III.45)$ is equal to C :

$$\lim_{u\to\ell^+} \left[(\theta(u) - \theta(\ell))^{1-\delta}\pi(u)\varphi(u)\right] = C.$$

Consequently, based on $(S1)$*, we have*

$$(\pi\Im^{1-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(\ell^+) = \lim_{u\to\ell^+}(\pi\Im^{1-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u) = C\Gamma(\delta),$$

and hence, by $(III.44)$*,* $C = \frac{r}{\Gamma(\delta)}$*, which proves* $(III.45)$*.* ■

Now, by Corollary III.1.2 and Lemma III.2.1, we deduce the existence and uniqueness result for the weighted Cauchy type problem $(III.41)$*.*

Theorem III.2.1 *Let* $0 < \delta < 1$*, let* B *be an open set in* $\mathbb{R}$ *and let* $\kappa : (\ell, r] \times B \to \mathbb{R}$ *be a function such that* $\kappa(u, \varphi(u)) \in C^{\pi}_{1-\delta,\theta}[\ell, r]$ *for any* $\varphi(u) \in C^{\pi}_{1-\delta,\theta}[\ell, r]$ *and the Lipschitzian condition* $(I.4)$ *holds. Then there exists a unique solution* $\varphi(u)$ *to the weighted Cauchy type problem* $(III.41)$ *in the space* $C^{\delta,\pi}_{1-\delta,\theta}[\ell, r]$*.*

Proof. *If* $\varphi(u)$ *fulfills the conditions* $(III.41)$*, then, according to Lemma III.2.1* $(S1)$*,* $\varphi(u)$ *also satisfies the conditions* $(III.12)$ *with* $r = C\Gamma(\delta)$*:*

$$(D^{\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(u) = \kappa(u, \varphi(u)) \quad (0 < \delta < 1), \quad (\pi\Im^{1-\delta,\theta(u)}_{\ell^+,\pi(u)}\varphi)(\ell^+) = C\Gamma(\delta) \in \mathbb{R}. \tag{III.49}$$

By Corollary III.1.2, there exists a unique solution $\varphi(u) \in C^{\delta,\pi}_{1-\delta,\theta}[\ell, r]$ *to this problem. Furthermore, by Lemma III.2.1* $(S2)$*,* $u(t)$ *is also a unique solution to the weighted Cauchy problem* $(III.41)$*.* ■

III.3 On Cauchy-Type Problems with Riemann-Liouville Fractional Derivatives

In this section, we establish the existence and uniqueness results within the space $C^{\vartheta}_{m-\vartheta}[\ell, r]$ $(III.57)$ *for the Cauchy problem associated with the nonlinear fractional differential equation involving the Riemann-Liouville fractional derivative. Specifically, we examine the equation:*

$$({}^{RL}D^{\vartheta}_{\ell^+}\Psi)(\tau) = \Lambda(\tau, \Psi(\tau)), \quad (\vartheta > 0), \tag{III.50}$$

The initial conditions are as follows:

$$({}^{RL}D^{\vartheta}_{\ell^+}\Psi)(\ell^+) = r_k, \quad r_k \in \mathbb{N}, (k = 1, ..., m = -[\vartheta]), \tag{III.51}$$

The Volterra type integral equation corresponding to problem $(III.50) - (III.51)$ *is*

$$\Psi(\tau) = \Psi_0(\tau) + \frac{1}{\Gamma(\vartheta)}\int_{\ell}^{\tau}(\tau - \xi)^{\vartheta-1}\Lambda(\xi, \Psi(\xi))d\xi, \quad (\tau > \ell), \tag{III.52}$$

where

$$\Psi_0(\tau) = \sum_{k=1}^{m} = \frac{r_k}{\Gamma(\vartheta - k + 1)}(\tau - \ell)^{\vartheta - k}. \tag{III.53}$$

and ${}^{RL}D^{\vartheta}_{\ell+}$ *denotes the Riemann-Liouville left-sided fractional derivative. When* $0 < \vartheta < 1$, *problem* $(III.50) - (III.51)$ *can be expressed as:*

$$\begin{cases} ({}^{Rl}D^{\vartheta}_{\ell+}\Psi)(\tau) = \Lambda(\tau, \Psi(\tau)), \quad (0 < \vartheta < 1), \\ \\ ({}^{RL}I^{1-\vartheta}_{\ell+}\Psi)(\ell^+) = r \quad (r \in \mathbb{R}). \end{cases} \tag{III.54}$$

and this problem can be rewritten as the weighted Cauchy type problem

$$\begin{cases} ({}^{Rl}D^{\vartheta}_{\ell+}\Psi)(\tau) = \Lambda(\tau, \Psi(\tau)), \\ \\ \lim_{\tau \to \ell^+}(\tau - \ell)^{1-\vartheta}\Psi(\tau) = C \quad (C \in \mathbb{R}). \end{cases} \tag{III.55}$$

The corresponding integral equation to the problem $(III.54)$ *has the form:*

$$\Psi(\tau) = \frac{r(\tau - \ell)^{\delta - 1}}{\Gamma(\vartheta)} + \frac{1}{\Gamma(\vartheta)}\int_{\ell}^{\tau}(\tau - \xi)^{\vartheta - 1}\Lambda(\xi, \Psi(\xi))d\xi, \quad (\tau > \ell, 0 < \vartheta \leq 1). \tag{III.56}$$

Definition III.3.1 *[1, 9] For* $m - 1 < \vartheta \leq m$ $(m \in \mathbb{N})$, *we define the space* $C^{\vartheta}_{m-\vartheta}[\ell, r]$ *as follows:*

$$C^{\vartheta}_{m-\vartheta}[\ell, r] = \left\{\Lambda(t) \in C_{m-\vartheta}[\ell, r] : \quad ({}^{RL}D^{\vartheta}_{\ell+}\Lambda)(t) \in C_{m-\vartheta}[\ell, r]\right\} \tag{III.57}$$

Here $C_{m-\vartheta}[\ell, r]$ *denotes a weighted space of continuous functions described as follows:*

$$C_{m-\vartheta}[\ell, r] = \left\{\Lambda : (\ell, r] \to \mathbb{R}; (t - \ell)^{m-\vartheta}\Lambda(t) \in C[\ell, r]\right\}, \tag{III.58}$$

On this space, we establish the norm $\|.\|_{C_{m-\vartheta}}$ *defined as:*

$$\|\Lambda\|_{C_{m-\vartheta}} = \|(t - \ell)^{m-\vartheta}\Lambda(t)\|_C,$$

III.3.1 Equivalence of Cauchy-Type Problems Involving Riemann-Liouville Fractional Derivatives and Volterra Integral Equations

In this subsection, we establish the equivalence between the Cauchy-type problem $(III.50) - (III.51)$ *and the integral equation* $(III.52)$ *in the space* $(III.58)$. *For this purpose, we require the following lemmas.*

Lemma III.3.1 *[1, 9] If $\varrho \in \mathbb{R}$ $(0 \leqslant \varrho < 1)$, then the fractional integration operator ${}^{RL}I^{\vartheta}_{\ell+}$ with order $\vartheta \in \mathbb{R}$ $(\vartheta > 0)$ is a mapping from $C_\varrho[\ell, r]$ to $C_\varrho[\ell, r]$, and it satisfies the inequality:*

$$\|{}^{RL}I^{\vartheta}_{\ell+}\Lambda\|_{C_\varrho} \leq (r-\ell)^{\vartheta}\frac{\Gamma(1-\varrho)}{\Gamma(1+\vartheta-\varrho)}\|\Lambda\|_{C_\varrho}, \tag{III.59}$$

where ${}^{RL}I^{\vartheta}_{\ell+}$ represents the Riemann-Liouville fractional integral operator and $\Lambda \in C_\varrho[\ell, r]$.

Remark III.3.1 *The result established in Lemma III.3.1 is a specific case of the outcome presented in LemmaIII.1.1. By substituting $\theta(t) = t$, $\pi(t) = 1$, $\delta = \vartheta$, and $\kappa = \Lambda$ into Lemma III.1.1, we obtain the result described in Lemma III.3.1.*

Lemma III.3.2 *[9] When $\mu > 0$, $\eta > 0$, and $0 \leq \varrho < 1$, the following statements hold:*

(A1) If $\Lambda(u) \in C_\varrho[\ell, r]$, then the first and second relations in $(I.23)$ are valid for any point $u \in (\ell, r]$ and $u \in [\ell, r)$, respectively. If $\Lambda(u) \in C[\ell, r]$, these relations hold for any point $u \in [\ell, r]$.

(A2) If $\Lambda(u) \in C_\varrho[\ell, r]$, then the first and second relations in $(I.34)$ are satisfied for any point $u \in (\ell, r]$ and $u \in [\ell, r)$, respectively. When $\Lambda(u) \in C[\ell, r]$, these equalities hold for any point $u \in [\ell, r]$.

(A3) Let $\mu > \eta > 0$. If $\Lambda(u) \in C[\ell, r]$, then the first and second relations in $(I.35)$ are satisfied for any point $u \in (\ell, r]$ and $u \in [\ell, r)$, respectively. When $\Lambda(u) \in C[\ell, r]$, these equalities hold for any point $u \in [\ell, r]$. Specifically, when $\eta = i \in \mathbb{N}$ and $\mu > i$, the relations in $(I.36)$ hold in their respective cases.

(A4) Let $m = [\vartheta] + 1$. Additionally, denote $\Lambda_{m-\vartheta}(u) = ({}^{RL}\Im^{m-\vartheta}_{\ell+}\Lambda)(u)$ as the fractional integral $(I.19)$, and $\chi_{m-\vartheta}(u) = ({}^{RL}\Im^{m-\vartheta}_{r-}\Lambda)(u)$ as the fractional integral $(I.20)$ of order $m - \vartheta$.

- *If $\Lambda(u) \in C_\varrho[\ell, r]$ and $\Lambda_{m-\vartheta}(u) \in C^m_\varrho[\ell, r]$, then the relation $(I.38)$ is satisfied at any point $u \in (\ell, r]$. Specifically, when $0 < \vartheta < 1$ and $\Lambda_{1-\vartheta}(u) \in C^1_\varrho[\ell, r]$, the equality $(I.39)$ holds true.*
- *If $\chi(u) \in C_\varrho[\ell, r]$ and $\chi_{m-\vartheta}(u) \in C^m_\varrho[\ell, r]$, then the equality $(I.42)$ is satisfied at any point $u \in (\ell, r]$. Particularly, when $0 < \vartheta < 1$ and $\chi_{1-\vartheta}(u) \in C^1_\varrho[\ell, r]$, the equality $(I.43)$ holds true.*
- *If $\Lambda(u) \in C[\ell, r]$ and $\Lambda_{m-\vartheta}(u) \in C^m[\ell, r]$, then both $(I.38)$ and $(I.42)$ are satisfied at any point $u \in [\ell, r]$. Specifically, if $\Lambda(u) \in C^m[\ell, r]$, the relations $(I.40)$ and $(I.44)$ hold true for any point $u \in [\ell, r]$.*

Remark III.3.2 *It's noteworthy that when setting $\theta(t) = t$, $\pi(t) = 1$, $\delta = \vartheta$, and $\kappa = \Lambda$, Lemma III.3.2 becomes a special case of lemma III.1.2.*

Theorem III.3.1 *[1, 9] Let $\vartheta > 0$, and set $m = -[\vartheta]$. Consider a function $\Lambda : (\ell, r] \times \mathbb{R} \to \mathbb{R}$ such that $\Lambda(\tau, \Psi(\tau)) \in C_{m-\vartheta}[\ell, r]$ for any $\Psi(\tau) \in C_{m-\vartheta}[\ell, r]$. If $\Psi(\tau) \in C_{m-\vartheta}[\ell, r]$, then $\Psi(\tau)$ satisfies equations $(III.50)$ and $(III.51)$ if and only if it satisfies the Volterra integral equation $(III.52)$.*

Remark III.3.3 *We observe that the findings in Theorem III.1.1 represent an extension of the results obtained in Theorem III.3.1.*

Corollary III.3.1 *[1, 9] Let $0 < \vartheta < 1$, and consider a function $\Lambda : (\ell, r] \times \mathbb{R} \to \mathbb{R}$ such that $\Lambda(\tau, \Psi(\tau)) \in C_{1-\vartheta}[\ell, r]$ for any $\Psi(\tau) \in C_{1-\vartheta}[\ell, r]$. If $\Psi(\tau) \in C_{1-\vartheta}[\ell, r]$, then $\Psi(\tau)$ satisfies equations $(III.54)$ if, and only if, it satisfies the integral equation $(III.56)$.*

Remark III.3.4 *Lemma III.3.1, Lemma III.3.2, Corollary III.3.1, and Theorem III.3.1 were proven in [9].*

III.3.2 Existence and Uniqueness of Solutions to Cauchy-Type Problems Involving Riemann-Liouville Fractional Differential Equations

In this subsection, we demonstrate the existence of a unique solution to the Cauchy-type problem (III.50)-(III.51). $(III.50) - (III.51)$ within the space $C^{\vartheta}_{m-\vartheta}[\ell, r]$, as defined in $(III.57)$, employing Banach's fixed-point theorem. To accomplish this, we rely on the following preliminary assertion.

Lemma III.3.3 *[1] The fractional integration operator ${}^{RL}I^{\vartheta}_{\ell+}$ with order $\vartheta \in \mathbb{R}$ $(\vartheta > 0)$ maps functions from $C[\ell, r]$ to $C[\ell, r]$. It is bounded by:*

$$\|{}^{RL}\Im^{\vartheta}_{\ell+}\Lambda\|_{C[\ell,r]} \leq \frac{(r-\ell)^{\vartheta}}{\vartheta\Gamma(\vartheta)}\|\Lambda\|_{C[\ell,r]}. \tag{III.60}$$

where $\Lambda \in C[\ell, r]$.

Proof. *(In this part, we will repeat the same proof method found in [1] for the sake of clarification and dissemination of interest).First, we establish that if $\Lambda \in C[\ell, r]$, then $({}^{RL}\Im^{\vartheta}_{\ell+}\Lambda)(u)$ is also in $C[\ell, r]$. For any $u \in [\ell, r]$ and*

$\Pi u > 0$ *such that* $u + \Pi u \leq r$, *we obtain:*

$$\begin{aligned} |({}^{RL}\Im_{\ell+}^{\vartheta}\Lambda)(u+\Pi u) - ({}^{RL}\Im_{\ell+}^{\vartheta}\Lambda)(u)| &= |\frac{1}{\Gamma(\vartheta)}\int_{\ell}^{u+\Pi u}(u+\Pi u-\xi)^{\vartheta-1}\Lambda(\xi)d\xi \\ &- \frac{1}{\Gamma(\vartheta)}\int_{\ell}^{u}(u-\xi)^{\vartheta-1}\Lambda(\xi)d\xi| \\ &\leq \frac{1}{\Gamma(\vartheta)}\{|\int_{\ell}^{u}\Lambda(u)[(u+\Pi u-\xi)^{\vartheta-1}-(u-\xi)^{\vartheta-1}]d\xi| \\ &+ |\int_{u}^{u+\Pi u}(u+\Pi u-\xi)^{\vartheta-1}\Lambda(\xi)d\xi|\} \\ &\leq \frac{\|\Lambda\|_{C[\ell,r]}}{\vartheta\Gamma(\vartheta)}\{[(u+\Pi u-\ell)^{\vartheta}-(u-\ell)^{\vartheta}]+(\Pi u)^{\vartheta}+(\Pi u)^{\vartheta}\}. \end{aligned}$$

As $\Pi u \to 0^+$, *it's evident that:*

$$|({}^{RL}\Im_{\ell+}^{\vartheta}\Lambda)(u+\Pi u) - ({}^{RL}\Im_{\ell+}^{\vartheta}\Lambda)(u)| \to 0.$$

Likewise, we can demonstrate that as $\Pi u \to 0^-$, *we obtain:*

$$|({}^{RL}\Im_{\ell+}^{\vartheta}\Lambda)(u+\Pi u) - ({}^{RL}\Im_{\ell+}^{\vartheta}\Lambda)(u)| \to 0.$$

Therefore, ${}^{RL}\Im_{\ell+}^{\vartheta}\Lambda \in C[\ell, r]$.

Now we establish the estimate. Indeed,

$$\begin{aligned} \|{}^{RL}I_{\ell+}^{\vartheta}\Lambda\|_{C[\ell,r]} &= \max_{u\in[\ell,r]}|\frac{1}{\Gamma(\vartheta)}\int_{\ell}^{u}(u-\xi)^{\vartheta-1}\Lambda(\xi)d\xi| \\ &\leq \frac{(\|\Lambda\|_{C[\ell,r]}}{\vartheta\Gamma(\vartheta)}\int_{\ell}^{u}(u-\xi)^{\vartheta-1}d\xi \\ &\leq \frac{(r-\ell)^{\vartheta}}{\vartheta\Gamma(\vartheta)}\|\Lambda\|_{C[\ell,r]}. \end{aligned}$$

This concludes the proof of LemmaIII.3.3. ■

Lemma III.3.4 *[1] Let* $\xi \in [0, \infty)$, $\ell_1 < \ell^* < \ell_2$, $\Lambda \in C_{\xi}[\ell_1, \ell^*]$, *and* $\Lambda \in C[\ell^*, \ell_2]$. *Then* $\Lambda \in C_{\xi}[\ell_1, \ell_2]$ *and*

$$\|\Lambda\|_{C_{\xi}[\ell_1,\ell_2]} \leq \max\{\|\Lambda\|_{C_{\xi}[\ell_1,\ell^*]}, (\ell_2-\ell_1)^{\xi}\|\Lambda\|_{C[\ell^*,\ell_2]}\}. \tag{III.61}$$

Proof. *Given that*$\Lambda \in C_{\xi}[\ell_1, \ell^*]$ *and* $\Lambda \in C[\ell^*, \ell_2]$, *we can conclude that*

$$\Lambda \in C(\ell_1, \ell_2] \quad and \quad \Lambda \in C_{\xi}[\ell_1, \ell_2].$$

Now, we proceed with proving the estimate. Given that $\Lambda \in C_{\xi}[\ell_1, \ell_2]$, *there exists* $\tau^* \in [\ell_1, \ell_2]$ *such that*

$$\|\Lambda\|_{C_{\xi}[\ell_1,\ell_2]} = |(\tau^*-\ell_1)^{\xi}\Lambda(\tau^*)|. \tag{III.62}$$

Assuming $\tau^* \in [\ell_1, \ell^*]$, we obtain the following:

$$\|\Lambda\|_{C_\xi[\ell_1,\ell_2]} \leq \|\Lambda\|_{C_\xi[\ell_1,\ell^*]}.$$

In a similar manner, if we consider $\tau^* \in [\ell^*, \ell_2]$, then we have

$$\|\Lambda\|_{C_\xi[\ell_1,\ell_2]} \leq (\ell_2 - \ell_1)^\xi \|\Lambda\|_{C[\ell^*,\ell_2]}.$$

Now, we can express this finding

$$\|\Lambda\|_{C_\xi[\ell_1,\ell_2]} \leq \max\{\|\Lambda\|_{C_\xi[\ell_1,\ell^*]}, (\ell_2 - \ell_1)^\xi \|\Lambda\|_{C[\ell^*,\ell_2]}\}.$$

The proof of this lemma is now concluded. ■

Theorem III.3.2 *[1] Let $\vartheta > 0$ and $m = -[-\vartheta]$. Suppose $\Lambda : (\ell, r] \times \mathbb{R} \to \mathbb{R}$ is a function such that $\Lambda(\xi, \Psi(\xi)) \in C_{m-\vartheta}[\ell, r]$ for any $\Psi(\xi) \in C_{m-\vartheta}[\ell, r]$, and it satisfies the Lipschitz condition $(I.4)$. Then, there exists a unique solution $\Psi(\xi)$ to the Cauchy-type problem $(III.50) - (III.51)$ in the space $C^{\vartheta}_{m-\vartheta}[\ell, r]$.*

Proof. *To begin, we establish the existence of a unique solution $\Psi(\xi) \in C_{m-\vartheta}[\ell, r]$. Based on Theorem III.3.1, establishing the existence of a unique solution $\Psi(\xi) \in C_{m-\vartheta}[\ell, r]$ to the nonlinear Volterra integral equation $(III.52)$ is sufficient. Equation $(III.52)$ is valid within any interval $(\ell, \xi_1] \subset (\ell, r]$ $(\ell < \xi < r)$. Select ξ_1 such that*

$$\varpi(\xi_1 - \ell)^\vartheta \frac{\Gamma(\vartheta - m + 1)}{\Gamma(2\vartheta - m + 1)} < 1, \tag{III.63}$$

where $\varpi > 0$ represents the Lipschitzian coefficient. Subsequently, we proceed to demonstrate the existence of a unique solution $\Psi(\xi) \in C_{m-\vartheta}[\ell, \xi_1]$ to$(III.52)$ on the interval $(\ell, \xi_1]$. To achieve this, we apply the Banach fixed-point theorem to the space $C_{m-\vartheta}[\ell, \xi_1]$, which is a complete metric space with the distance defined as

$$\Delta(\Psi_1, \Psi_2) = \|\Psi_1 - \Psi_2\|_{C_{m-\ell}[\ell,\xi_1]} = \max_{\xi\in[\ell,\xi_1]} \left|(\xi - \ell)^{m-\ell} [\Psi_1(\xi) - \Psi_2(\xi)]\right|. \tag{III.64}$$

We express the integral $(III.52)$ as:

$$\Psi(\xi) = (F\Psi)(\xi), \tag{III.65}$$

where

$$(F\Psi)(\xi) = \Psi_0(\xi) + \frac{1}{\Gamma(\vartheta)} \int_\ell^\xi \frac{\Lambda[u, \Psi(u)]du}{(\xi - u)^{1-\vartheta}}. \tag{III.66}$$

To utilize Theorem I.2.1, we need to establish the following:

(1) If $\Psi(\xi) \in C_{m-\vartheta}[\ell, \xi_1]$, then $(F\Psi)(\xi) \in C_{m-\vartheta}[\ell, \xi_1]$.

(2) For any $\Psi_1, \Psi_2 \in C_{m-\vartheta}[\ell, \xi_1]$, the following inequality holds:

$$\|F\Psi_1 - F\Psi_2\|_{C_{m-\vartheta}[\ell,\xi_1]} \leq \Upsilon \|\Psi_1 - \Psi_2\|_{C_{m-\vartheta}[\ell,\xi_1]}, \quad \Upsilon = \varpi (\xi_1 - \ell)^{\vartheta} \frac{\Gamma(\vartheta - m + 1)}{\Gamma(2\vartheta - m + 1)}. \tag{III.67}$$

From $(III.53)$, we can conclude that $\Psi_0(\xi) \in C_{m-\vartheta}[\ell, \xi_1]$. Because $\Lambda[\xi, \Psi(\xi)] \in C_{m-\vartheta}[\ell, \xi_1]$ for any $\Psi(\xi) \in C_{m-\vartheta}[\ell, \xi_1]$, Therefore, according to Lemma III.3.1 (with $\varrho = m - \vartheta$, $r = \xi_1$, and $\Lambda(\xi) = \Lambda[\xi, \Psi(\xi)]$), the integral on the right-hand side of $(III.65)$ also falls within $C_{m-\vartheta}[\ell, \xi_1]$, implying that $(F\Psi)(\xi) \in C_{m-\vartheta}[\ell, \xi_1]$. Now we establish the estimate in $(III.67)$. Utilizing $(III.66)$, employing the Lipschitzian condition, and applying the relation $(III.59)$ (with $\varrho = m - \ell$, $r = \xi_1$, and $\Lambda(\xi) = \Lambda[\xi, \Psi_1(\xi)] - \Lambda[\xi, \Psi_2(\xi)]$), we obtain:

$$\begin{aligned} \|F\Psi_1 - F\Psi_2\|_{C_{m-\vartheta}[\ell,\xi_1]} &\leq \left\| {}^{RL}\Im^{\vartheta}_{\ell+} \left[|\Lambda[\xi, \Psi_1(\xi)] - \Lambda[\xi, \Psi_2(\xi)]| \right] \right\|_{C_{m-\vartheta}[\ell,\xi_1]} \\ &\leq \varpi \left\| {}^{RL}\Im^{\vartheta}_{\ell+} \left[|\Psi_1(\xi) - \Psi_2(\xi)| \right] \right\|_{C_{m-\vartheta}[\ell,\xi_1]} \\ &\leq \varpi (\xi_1 - \ell)^{\vartheta} \frac{\Gamma(\vartheta - m + 1)}{\Gamma(2\vartheta - m + 1)} \|\Psi_1 - \Psi_2\|_{C_{m-\vartheta}[\ell,\xi_1]}, \end{aligned} \tag{III.68}$$

This results in deriving the estimate $(III.67)$.According to $(III.63)$,where $0 < \varpi < 1$, by Theorem I.2.1, there exists a unique solution $\Psi^*(\xi) \in C_{m-\vartheta}[\ell, \xi_1]$ to $(III.52)$ on the interval $[\ell, \xi_1]$.

Using Theorem I.2.1, this solution $\Psi^*(\xi)$ is obtained as the limit of a convergent sequence $(F^n\Psi_0^*)(\xi)$:

$$\lim_{n\to\infty} \|F^n\Psi_0^* - \Psi^*\|_{C_{m-\vartheta}[\ell,\xi_1]} = 0, \tag{III.69}$$

where $\Psi_0^*(\xi) \in C_{m-\vartheta}[ell, \xi_1]$.

If there exists at least one $r_k \neq 0$ in the initial condition $(III.51)$, then we can choose $\Psi_0^*(\xi) = \Psi_0(\xi)$, where $\Psi_0(\xi)$ defined by $(III.53)$.

The last equation can be reformulated as follows:

$$\lim_{n\to\infty} \|\Psi_n - \Psi^*\|_{C_{m-\vartheta}[\ell,\xi_1]} = 0, \tag{III.70}$$

where

$$\Psi_n(\xi) = (F^n\Psi_0^*)(\xi) = \Psi_0(\xi) + \frac{1}{\Gamma(\vartheta)} \int_{\ell}^{\xi} \frac{\Lambda\left[u, \left(F^{n-1}\Psi_0^*\right)(u)\right] du}{(\xi - u)^{1-\vartheta}}, \quad (n \in \mathbb{N}). \tag{III.71}$$

Next, we examine the interval $[\xi_1, r]$.We express equation $(III.52)$ as:

$$\Psi(\xi) = \Psi_{01}(\xi) + \frac{1}{\Gamma(\vartheta)} \int_{\xi_1}^{\xi} \frac{\Lambda[u, \Psi(u)]du}{(\xi - u)^{1-\vartheta}}, \tag{III.72}$$

where $\Psi_{01}(\xi)$*is given by*

$$\Psi_{01}(\xi) = \sum_{i=1}^{m} \frac{r_i}{\Gamma(\vartheta - i + 1)}(\xi - \ell)^{\vartheta - i} + \frac{1}{\Gamma(\vartheta)} \int_{\ell}^{\xi_1} \frac{\Lambda[u, \Psi(u)]du}{(\xi - u)^{1-\vartheta}}. \tag{III.73}$$

We establish that $\Psi_{01}(\xi) \in C\,[\xi_1, r]$.

Next, we establish the existence of a unique solution $\Psi(\xi) \in C\,[\xi_1, r]$ *to* $(III.52)$ *on the interval* $[\xi_1, r]$. *To achieve this, we utilize the Banach fixed-point theorem for the space* $C\,[\xi_1, \xi_2]$, *where* ξ_2 *is chosen such that it satisfies the condition:*

$$\frac{\varpi\,(\xi_2 - \xi_1)^{\vartheta}}{\vartheta\Gamma(\vartheta)} < 1. \tag{III.74}$$

$C\,[\xi_1, \xi_2]$ *forms a complete metric space, where the distance between two functions* Ψ_1 *and* Ψ_2 *is defined as*

$$\Delta\,(\Psi_1 - \Psi_2) = \|\Psi_1 - \Psi_2\|_{C[\xi_1,\xi_2]} = \max_{\xi\in[\xi_1,\xi_2]} |\Psi_1(\xi) - \Psi_2(\xi)|\,. \tag{III.75}$$

We express the integral equation $(III.72)$ *as:*

$$\Psi(\xi) = (F\Psi)(\xi), \tag{III.76}$$

where

$$(F\Psi)(\xi) = \Psi_{01}(\xi) + \frac{1}{\Gamma(\vartheta)} \int_{\xi_1}^{\xi} \frac{\Lambda[u, \Psi(u)]du}{(\xi - u)^{1-\vartheta}}. \tag{III.77}$$

To utilize Theorem I.2.1, we must demonstrate the following:

(1) If $\Psi(\xi) \in C\,[\xi_1, \xi_2]$, *then* $(F\Psi)(\xi) \in C\,[\xi_1, \xi_2]$

(2) For any $\Psi_1, \Psi_2 \in C\,[\xi_1, \xi_2]$, *the following inequality holds:*

$$\|F\Psi_1 - F\Psi_2\|_{C[\xi_1,\xi_2]} \le \Upsilon\,\|\Psi_1 - \Psi_2\|_{C[\xi_1,\xi_2]} \quad \Upsilon = \frac{\varpi\,(\xi_2 - \xi_1)^{\vartheta}}{\vartheta\Gamma(\vartheta)}. \tag{III.78}$$

Given that $\Lambda[\xi, \Psi(\xi)] \in C_{m-\vartheta}[\ell, r]$ *for any* $\Psi(\xi) \in C_{m-\xi}[\ell, r]$, *the integral in the right-hand side of* $(III.77)$ *also belongs to* $C\,[\xi_1, \xi_2]$. *As a result,* $(F\Psi)(\xi) \in \mathrm{C}\,[\xi_1, \xi_2]$.

Now, we proceed to establish the estimate in $(III.78)$ *as follows:*

$$\begin{aligned}\|F\Psi_1 - F\Psi_2\|_{C[\xi_1,\xi_2]} &\le \left\|{}^{RL}\Im_{\ell+}^{\vartheta}\,[|\Lambda\,[\xi, \Psi_1(\xi)] - \Lambda\,[\xi, \Psi_2(\xi)]|]\right\|_{C[\xi_1,\xi_2]} \\ &\le \tfrac{\varpi(\xi_2-\xi_1)^{\vartheta}}{\vartheta\Gamma(\vartheta)}\,\|\Psi_1 - \Psi_2\|_{C[\xi_1,\xi_2]}\,,\end{aligned} \tag{III.79}$$

which implies the estimate $(III.78)$.*According to* $(III.74)$, *where* $0 < \varpi < 1$, *and hence by Theorem I.2.1, there exists a unique solution* $\Psi_1^*(\xi) \in C\,[\xi_1, \xi_2]$ *to* $(III.52)$ *on the interval* $[\xi_1, \xi_2]$. *As per Theorem I.2.1, this solution* *in $C[\xi_1,\xi_2]$ is a limit of a convergent sequence $(F^n\Psi_{01}^*)(\xi)$:*

$$\lim_{n\to\infty}\|F^n\Psi_{01}^*-\Psi_1^*\|_{C[\xi_1,\xi_2]}=0, \tag{III.80}$$

where $\Psi_{01}(\xi)$ represents any function in $C[\xi_1,\xi_2]$. If $\Psi_0(\xi)\neq 0$ on $[\xi_1,\xi_2]$, then we can choose $\Psi_{01}^*(\xi)=\Psi_0(\xi)$, where $\Psi_0(\xi)$ is defined by $(III.53)$. The last relation can be expressed as:

$$\lim_{n\to\infty}\|\Psi_m-\Psi_1^*\|_{C[\xi_1,\xi_2]}=0, \tag{III.81}$$

where

$$\Psi_n(\xi)=(F^n\Psi_{01}^*)(\xi)=\Psi_{01}(\xi)+\frac{1}{\Gamma(\vartheta)}\int_{\xi_1}^{\xi}\frac{\Lambda\left[\xi,\left(F^{n-1}\Psi_{01}^*\right)\right]du}{(\xi-u)^{1-\vartheta}},\quad (n\in\mathbb{N}). \tag{III.82}$$

Repeating the aforementioned process, we establish the existence of a unique solution $\Psi(\xi)\in C[\xi_1,r]$ to $(III.52)$ over the interval $[\xi_1,r]$. With Lemma III.3.4, we further conclude that there exists a singular solution $\Psi(\xi)\in C_{m-\vartheta}[\ell,r]$ to the Volterra integral equation $(III.52)$ spanning the entire interval $[\ell,r]$. Consequently, $\Psi(\xi)\in C_{m-\vartheta}[\ell,r]$ stands as the sole solution to the Cauchy-type problem $(III.50)-(III.51)$.

To finalize the proof of Theorem III.3.2, we need to demonstrate that the unique solution $\Psi(\xi)\in C_{m-\vartheta}[\ell,r]$ also belongs to the space $C^{\vartheta}_{m-\vartheta}[\ell,r]$. This requires showing that $\left({}^{RL}D^{\vartheta}_{\ell+}\Psi\right)(\xi)\in C_{m-\vartheta}[\ell,r]$. Based on the preceding proof, the solution $\Psi(\xi)\in C_{m-\vartheta}[\ell,r]$ can be represented as the limit of a sequence $\Psi_n(\xi)$, where $\Psi_n(\xi)=(F^n\Psi_0^*)\in C_{m-\vartheta}[\ell,r]$, such that:

$$\lim_{n\to\infty}\|\Psi_n-\Psi\|_{C_{m-\vartheta}[\ell,r]}=0, \tag{III.83}$$

with appropriate choices of $\Psi_0^(\xi)$ on each interval $[\ell,\xi_1],\ldots,[\xi_M,r]$. If $\Psi_0(\xi)\neq 0$, then we can take $\Psi_0^*(\xi)=\Psi_0(\xi)$. From $(III.50)$ and the Lipschitz condition, we obtain:

$$\left\|{}^{RL}D^{\vartheta}_{\ell+}\Psi_m-{}^{RL}D^{\vartheta}_{\ell+}\Psi\right\|_{C_{m-\vartheta}}=\|\Lambda[\xi,\Psi_n]-\Lambda[\xi,\Psi]\|_{C_{m-\vartheta}}\leq\varpi\|\Psi_n-\Psi\|_{C_{m-\vartheta}}. \tag{III.84}$$

Therefore,

$$\lim_{n\to\infty}\left\|D^a_{a+}y_m-D^a_{a+}y\right\|_{C_{n-a}}=0. \tag{III.85}$$

Using $\left({}^{RL}D^{\vartheta}_{\ell+}\Psi_n\right)(\xi)=\Lambda[\xi,\Psi_{n-1}(\xi)]$ and $\Lambda[\xi,\Psi(\xi)]\in C_{m-\vartheta}[\ell,r]$ for any $\Psi(\xi)\in C_{m-\vartheta}[\ell,r]$, we can deduce that $\Lambda[\xi,\Psi_{n-1}(\xi)]\in C_{m-\vartheta}[\ell,r]$, implying $\left({}^{RL}D^{\vartheta}_{\ell+}\Psi_n\right)(\xi)\in C_{m-\vartheta}[\ell,r]$. Therefore,

$$\left({}^{RL}D^{\vartheta}_{\ell+}\Psi\right)(\xi)\in C_{m-\vartheta}[\ell,r]$$

This concludes the proof of Theorem III.3.2. ■

Corollary III.3.2 *[9] Let $0 < \vartheta < 1$, let S be an open set in $\mathbb{R}$ and let $\Lambda : (\ell, r] \times S \to \mathbb{R}$ be a function such that $\Lambda(\xi, \Psi(\xi)) \in C_{1-\vartheta}[\ell, r]$ for any $\Psi(\xi) \in C_{1-\vartheta}[\ell, r]$ and $(I.4)$ holds.*

Then there exists a unique solution $\Psi(\xi)$ to the Cauchy type problem $(III.54)$in the space $C^{\vartheta}_{1-\vartheta}[\ell, r]$.

III.4 The weighted Cauchy Type Problem with R-L Fractional Derivatives

When $0 < \vartheta < 1$, the conclusion of Corollary III.3.2 holds for the weighted Cauchy-type problem $(III.55)$, where $C \in \mathbb{R}$:

$$({}^{Rl}D^{\vartheta}_{\ell+}\Psi)(\xi) = \Lambda(\xi, \Psi(\xi)); \quad \lim_{\xi \to \ell^+} (\xi - \ell)^{1-\vartheta}\Psi(\xi) = c \quad (c \in \mathbb{R}), (0 < \vartheta < 1). \tag{III.86}$$

Its proof relies on the following lemma:

Lemma III.4.1 *[9] Let $\vartheta \in \mathbb{C}(0 < \Re(\vartheta) < 1)$ and let $\Psi(\xi) \in C_{1-\vartheta}[\ell, r]$.*

(a) If a limit exists almost everywhere

$$\lim_{\xi \to \ell^+} \left[(\xi - \ell)^{1-\vartheta}\Psi(\xi)\right] = c \quad (c \in \mathbb{C}), \tag{III.87}$$

then there also exists a limit almost everywhere.

$$\left({}^{RL}\Im^{1-\alpha}_{\ell+}\Psi\right)(\ell^+) := \lim_{\xi \to \ell^+} \left({}^{RL}\Im^{1-\alpha}_{\ell+}\Psi\right)(\xi) = c\Gamma(\vartheta). \tag{III.88}$$

(b) If a limit exists almost everywhere

$$\lim_{\xi \to \ell^+} \left({}^{RL}\Im^{1-\alpha}_{\ell+}\Psi\right)(\xi) = r \quad (r \in \mathbb{C}), \tag{III.89}$$

and if the limit $\lim_{\xi \to \ell^+} \left[(\xi - \ell)^{1-\vartheta}\Psi(\xi)\right]$ exists, then

$$\lim_{\xi \to \ell^+} \left[(\xi - \ell)^{1-\vartheta}\Psi(\xi)\right] = \frac{r}{\Gamma(\vartheta)}. \tag{III.90}$$

Combining Corollary III.3.2 with Lemma III.4.1, we establish the existence and uniqueness of solutions for the weighted Cauchy-type problem $(III.86)$.

Theorem III.4.1 *[9] Let $0 < \vartheta < 1$, let S be an open set in $\mathbb{R}$ and let $\Lambda : (\ell, r] \times S \to \mathbb{R}$ be a function such that $\Lambda(\xi, \Psi(\xi)) \in C_{1-\vartheta}[\ell, r]$ for any $\Psi(\xi) \in C_{1-\vartheta}[\ell, r]$ and the Lipschitzian condition $(I.4)$ holds.*

Then there exists a unique solution $\Psi(x)$ to the weighted Cauchy type problem $(III.86)$ in the space $C^{\vartheta}_{1-\vartheta}[\ell, r]$.

Remark III.4.1 *A significant observation in this study is the broader scope delineated in Lemmas III.1.5, III.1.4, and III.2.1 compared to Lemmas III.3.3, III.3.4, and III.4.1. Additionally, the outcomes presented in Theorems III.3.2 and III.4.1 are considered specific instances of the broader result established in Theorems III.1.2 and III.2.1. This generalization is achieved by setting* $\theta(u) = u$*,* $\pi(u) = 1$*,* $\delta = \vartheta$*, and* $\kappa = \Lambda$*.*

CHAPTER

IV

COMPARISON RESULTS

In this chapter, we will introduce a new estimate for the weighted Riemann-Liouville fractional derivative of a function with respect to functions at their extreme points. Utilizing this estimate, we establish comparison theorems for fractional differential inequalities, both strict and non-strict, involving weighted Riemann-Liouville differential operators of a function with respect to functions of order ϑ, where $0 < \vartheta < 1$.

IV.1 Comparison Theorems for Fractional Differential Equation Involving Riemann-Liouville Differential Operators

In this section, we establish comparison theorems for fractional differential inequalities, encompassing both strict and non-strict cases, with Riemann-Liouville differential operators. The proof relies on an estimate of the R-L fractional derivative of a function at extreme points.

We consider the initial value problem (IVP) associated with the fractional differential equation represented as follows:

$$^{RL}D^{\vartheta}\Psi = \Lambda(u, \Psi(u)), \quad \Psi(u_0) = \Psi^0 = \Psi(u)(u-u_0)^{1-\vartheta}|_{u=u_0}, \quad u_0 \leq t \leq S, \quad S > 0, \tag{IV.1}$$

where $\Lambda \in C([u_0, S] \times \mathbb{R}, \mathbb{R})$, $^{RL}D^{\vartheta}\Psi$ represents the Riemann-Liouville fractional derivative of order ϑ , with $0 < \vartheta < 1$.

Definition IV.1.1 *[33, 34] Let $0 < \vartheta < 1$ and $1 - \vartheta = \rho$. We define the function space $C_\rho([\ell, r], \mathbb{R})$ as follows:*

$$C_\rho([u_0, S], \mathbb{R}) = \{\Psi \in C((u_0, S], \mathbb{R}), \quad (u-u_0)^\rho \Psi(u) \in C([u_0, S], \mathbb{R})\}. \tag{IV.2}$$

Lemma IV.1.1 *[33, 34] Let $k(u) \in C_\rho([u_0, S), R)$ be locally Hölder continuous with exponent $\mu > \vartheta$ .For any $u_1 \in (u_0, S]$, we have*

$$k(u_1) = 0 \quad \textit{and} \quad k(u) \leq 0 \quad \textit{for} \quad u_0 \leq u \leq u_1. \tag{IV.3}$$

Then we can deduce that:

$$^{RL}D^{\vartheta}k(u_1) \geq 0. \tag{IV.4}$$

Proof. *We can express the Riemann-Liouville fractional derivative as:*

$$^{RL}D^{\vartheta}k(u) = \frac{1}{\Gamma(\rho)}\frac{d}{du}\int_{u_0}^{u}(u-\omega)^{\rho-1}k(\omega)d\omega \tag{IV.5}$$

We define$L(u) = \int_{u_0}^{u}(u-\omega)^{\rho-1}k(\omega)d\omega$. For small $\zeta > 0$, we can consider:

$$\begin{aligned} L(u_1) - L(u_1-\zeta) &= \int_{u_0}^{u_1-\zeta}\left[(u_1-\zeta)^{\rho-1} - (u_1-\zeta-\omega)^{\rho-1}\right]k(\omega)d\omega \\ &+ \int_{u_1-\zeta}^{u_1}(u_1-\omega)^{\rho-1}k(\omega)d\omega = \jmath_1 + \jmath_2. \end{aligned}$$

Since $\left[(u_1-\zeta)^{\rho-1}-(u_1-\zeta-\omega)^{\rho-1}\right]<0$ *for* $u_0\le\omega\le u_1-\zeta$ *and* $k(u)\le 0$ *according to the hypothesis, we have* $\jmath_1\ge 0$*. This implies*

$$L(u_1)-L(u_1-\zeta)\ge\int_{u_1-\zeta}^{u_1}(u_1-\omega)^{\rho-1}k(\omega)d\omega=\jmath_2.$$

Given that $k(u)$ *is locally Hölder continuous, there exists a* $k(u_1)>0$ *such that for* $u_1-\zeta\le\omega\le u_1+\zeta$,

$$-k(u_1)(u_1-\omega)^{\mu}\le k(u_1)-k(\omega)\le k(u_1)(u_1-\omega)^{\mu},$$

where $0<\zeta<1$ *is such that* $\mu>\vartheta$*. Then, from* $(IV.3)$*, we have*

$$\jmath_2\ge -k(u_1)\int_{u_1-\zeta}^{u_1}(u_1-\omega)^{\rho-1+\mu}d\omega=k(u_1)\frac{\zeta^{\rho+\mu}}{\rho+\mu}.$$

Therefore,

$$L(u_1)-L(u_1-\zeta)+k(u_1)\frac{\zeta^{\rho+\mu}}{\rho+\mu}\ge 0,$$

holds for sufficiently small $\zeta>0$*.*

Taking the limit as ζ *approaches 0, we get* $\frac{d}{du}L(u_1)\ge 0$*, which implies* ${}^{RL}D^{\vartheta}k(u_1)\ge 0$*, and thus, the proof is complete.* ■

Example IV.1.1 *Consider the function* $k(u)=-u^2+4u-4$ *defined on the interval* $[u_0,S]=[0,4]$*. This function is locally Hölder continuous on the given interval.*

For any $u_1\in(u_0,S]$*, , let's choose* $u_1=2$ *for this example:*

- $k(u_1)=k(2)=-(2)^2+4(2)-4=0.$
- $k(u)=-u^2+4u-4\le 0$ *for* $0\le u\le 2.$

The Riemann-Liouville fractional derivative of order $\vartheta=0.5$ *at* $u_1=2:$

$${}^{RL}D^{0.5}k(u_1)\ge 0.$$

Theorem IV.1.1 *[33, 34] Let* $\Psi,\Phi\in C_{\rho}([u_0,S],\mathbb{R})$ *be locally Hölder continuous with an exponent* $0<\mu<1$ *and* $\mu>\vartheta$*. Additionally, suppose* $\Lambda\in C([u_0,S]\times\mathbb{R},\mathbb{R})$*, and consider the following conditions:*

(Q1) ${}^{RL}D^{\vartheta}\Phi(u)\le\Lambda(u,\Phi(u)),$

(Q2) ${}^{RL}D^{\vartheta}\Psi(u)\ge\Lambda(u,\Psi(u)),\quad u_0<u\le S,$

Assuming that one of the inequalities $(Q1)$ *or* $(Q2)$ *is strict, and if*

$$\Phi^0 < \Psi^0, \tag{IV.6}$$

where $\Phi^0 = \Phi(u)(u-u_0)^{1-\vartheta}|_{u=u_0}$ *and* $\Psi^0 = \Psi(u)(u-u_0)^{1-\vartheta}|_{u=u_0}$*, then we have*

$$\Phi(u) < \Psi(u), \quad u_0 \leq u \leq S. \tag{IV.7}$$

Proof. *Assume that inequality* $(IV.7)$ *does not hold. Then, according to the definitions of* Φ^0, Ψ^0, $(IV.6)$ *and the continuity of the functions* $\Phi(u)(u-u_0)^{1-\vartheta}$,$\Psi(u)(u-u_0)^{1-\vartheta}$*, there exists a point* u_1 *such that* $u_0 < u_1 \leq S$*, satisfying*

$$\Phi(u_1) = \Psi(u_1) \quad and \quad \Phi(u) < \Psi(u), \quad u_0 \leq u < u_1. \tag{IV.8}$$

If we define $k(u) = \Phi(u) - \Psi(u)$ *for* $u_0 \leq u \leq S$*, we observe that*

$$k(u) = 0 \quad and \quad k(u) < 0 \quad for \quad u_0 \leq u_1 < S.$$

Therefore, by Lemma IV.1.1, we have

$$^{RL}D^{\vartheta}k(u_1) \geq 0.$$

This implies, assuming the strict inequality $(Q2)$*, for example,*

$$\Lambda(u_1, \Phi(u_1)) \geq^{RL} D^{\vartheta}\Phi(u_1) \geq^{RL} D^{\vartheta}\Psi(u_1) > \Lambda(u_1, \Psi(u_1)),$$

which contradicts $(IV.8)$*. Therefore, the relation* $(IV.7)$ *holds, and the proof is complete.* ■

Theorem IV.1.2 *[33, 34] Assuming that the conditions of Theorem IV.1.1 are satisfied with non-strict inequalities* $(Q1)$ *and* $(Q2)$*, and additionally, supposing that* Λ *satisfies the Lipschitz condition:*

$$\Lambda(u, \phi) - \Lambda(u, \psi) \leq \beta(\phi - \psi), \quad \phi \geq \psi \quad and \quad \beta > 0. \tag{IV.9}$$

Then, if $\phi^0 \leq \psi^0$*, we have:*

$$\Phi(u) \leq \Psi(u), \quad u_0 \leq u \leq S. \tag{IV.10}$$

Proof. *First, we define the function* $\Psi_\zeta(u)$ *for small* ζ *as follows:*

$$\Psi_\zeta(u) = \Psi(\zeta) + \zeta\Delta(u), \tag{IV.11}$$

where $\Delta(u) = (u-u_0)^{\vartheta-1}E_{\vartheta,\vartheta}[2\beta(u-u_0)^{\vartheta}]$, *with* $\Psi_\zeta \in C_\rho([u_0,S],\mathbb{R})$.

This implies that

$$\Psi_\zeta(u)(u-u_0)^{1-\vartheta}|_{u=u_0} = \Psi(u)(u-u_0)^{1-\vartheta}|_{u=u_0} + \zeta\Delta(u)(u-u_0)^{1-\vartheta}|_{u=u_0}.$$

So, we obtain $\Psi^0\zeta = \Psi^0 + \zeta\Delta^0$. *This leads to*

$$\Psi^0_\zeta > \Psi^0 \geq \Phi^0 \quad and \quad \Psi_\zeta(u) > \Psi(u). \tag{IV.12}$$

Next, by applying the Lipschitz condition $(IV.9)$, *we can deduce*

$$\begin{aligned} {}^{RL}D^{\vartheta}\Psi_\zeta(u) &= {}^{RL}D^{\vartheta}\Psi(u) + \zeta^{RL}D^{\vartheta}\Delta(u) \\ &\geq \Lambda(u,\Psi(u)) + 2\zeta\beta\Delta(u) \\ &> \Lambda(u,\Psi_\zeta(u) - \beta\zeta\Delta(u) + 2\zeta\beta\Delta(u) \\ &> \Lambda(u,\Psi_\zeta(u)), \quad u_0 < u \leq S. \end{aligned}$$

Since $\Delta(u)$ *is the solution to (IVP), we have made use of this information in this instance:*

$${}^{RL}D^{\vartheta}\Delta(u) = 2\theta\Delta(u), \quad \Delta(u)(u-u_0)^{1-\vartheta}|_{u=u_0} = \Delta^0 > 0.$$

Clearly there is no assumption on the growth of $\theta > 0$. *Applying Theorem IV.1.1 to* $\Phi(u)$ *and* $\Psi(u)$, *we find* $\Phi(u) < \Psi_\zeta(u)$, $u_0 \leq u \leq T$, *for every* $\zeta > 0$. *Consequently, by letting* $\zeta > 0$, *we obtain* $\Phi(u) \leq \Psi(u)$ *for* $u_0 \leq u \leq S$, *as required.* ■

Example IV.112 *Choose* ϑ *and* μ:

- *Let* $\vartheta = 0.2$ *and* $\mu = 1$. *This ensures* $0 < \vartheta < \mu < 1$.

2) *Choose the interval* $[u_0, S]$:

- *Let* $u_0 = 2.1$ *and* $S = 2.2$.

3) *Define* $\Phi(u)$ *and* $\Psi(u)$:

- *Let* $\Phi(u) = 0.9u^2$.
- *Let* $\Psi(u) = 1.1u^3$. *Both function are locally Holder continuous with exponent* $\mu = 1$.

4) *Define* $\Lambda(x,y)$:

- *Let* $\Lambda(x,y) = y - 1 + 0.5y$. *Thus function satisfies the Lipschitz condition* $(IV.9)$ *with* $\beta = 1.5$.

5) Verify condition $(Q1)$ and $(Q2)$:

- *For $u \in [2.1, 2.2]$:*
- $(Q1)$ $\quad {}^{RL}D^{\vartheta}\Phi(u) ={}^{RL} D^{0.2}(0.9u^2) = 1.018u^{1.8} \leq \Lambda(u, \Phi(u)) = 1,35u^2 - 1.$
- $(Q2)$ $\quad {}^{RL}D^{\vartheta}\Psi(u) ={}^{RL} D^{0.2}(1.1u^3) = 1.745u^{2.8} \geq \Lambda(u, \Psi(u)) = 1.65u^3 - 1.$

6) Verify the initial condition $\Phi^0 \leq \Psi^0$:

- *At $u_0 = 2.1$:*

$$\Phi^0 = 3.969 \leq \Psi^0 = 10.1871$$

Since $3.969 \leq 10.1871$, the initial condition is satisfied .

Therefore, the functions $\Phi(u) = 0.9u^2$, $\Psi(u) = 1.1u^3$, and $\Lambda(x, y) = y - 1 + 0.5y$ fulfill all the conditions of Theorem IV.1.2, then for all $u \in [2.1, 2.2]$,

$$\Phi(u) = 0.9u^2 \leq \Psi(u) = 1.1u^3.$$

IV.2 Comparison Theorems for Caputo Fractional Differential Equation with Initial Condition

In this section, we revisit a comparison theorem for a Caputo fractional differential equation of order ϑ, where $0 < \vartheta < 1$, with initial condition and an estimate of Caputo fractional derivatives at extreme points.

First, let's consider the initial value problem presented in the following form:

$$\begin{cases} {}^{c}D^{\vartheta}\omega(u) = \Lambda(u, \omega(u)), \quad (0 < \vartheta < 1), \\ \omega(0) = \omega_0. \end{cases} \tag{IV.13}$$

Here, ${}^{c}D^{\vartheta}\Psi(u)$ denotes the Caputo derivative of order ϑ for $u \in [\ell, r]$.

Now, we proceed to demonstrate the estimate of Caputo fractional derivatives at extreme points, as presented in the following lemma.

Lemma IV.2.1 *[35] If $k(u) \in C^1([0,S],\mathbb{R})$, and there exists $u_1 \in [0,S]$ such that $k(u_1)=0$ and $k(u) \leq 0$ on $[0,S]$, then we can conclude that*

$$^{c}D^{\vartheta}k\left(u_1\right) \geq 0. \tag{IV.14}$$

Proof. *Let $u_1 \in [0,S]$. Utilizing the relationship between the Caputo fractional derivative and the Riemann-Liouville fractional derivative $(I.52)$, we obtain*

$$^{c}D^{\vartheta}k(u_1) = ^{RL}D^{\vartheta}k(u_1) - \frac{k(0)}{\Gamma(1-\vartheta)}u^{-\vartheta} \geq ^{RL}D^{\vartheta}k(u_1). \tag{IV.15}$$

According to Lemma IV.1.1, it is established that

$$^{RL}D^{\vartheta}k(u_1) \geq 0,$$

this implies that $^{c}D^{\vartheta}k(u_1) \geq 0$. ■

Example IV.2.1 *1) Function Definition:*

Define the function $\kappa(u) = -(u-1)^2$ for $u \in [0,2]$. This function belongs to $C^1([0,2],\mathbb{R})$ because it is differentiable, and its derivative is continuous on $[0,2]$.

2) *Zero Point:*

Choose $u_1 = 1$. At this point,

$$\kappa(1) = -(1-1)^2 = 0$$

3) *Non-Positive Condition:*

For $u \in [0,2]$,

$$\kappa(u) = -(u-1)^2 \leq 0$$

4) *Caputo Fractional Derivative:*

The Caputo fractional derivative of $\kappa(u)$ of order ϑ at $u_1 = 1$ is:

$$^{C}D^{\vartheta}\kappa(u_1) = \frac{1}{\Gamma(1-\vartheta)}\int_0^{u_1}(u_1-\tau)^{-\vartheta}\kappa'(\tau)\,d\tau \geq 0.$$

Thus, the function $\kappa(u) = -(u-1)^2$ for $u \in [0,2]$ satisfies all the conditions of Lemma $IV.2.1$.

Now, we proceed to establish the following comparison theorem.

Theorem IV.2.1 *[35] Let $D = [\ell, r]$, $\Lambda \in C[D\times\mathbb{R}, \mathbb{R}]$, $\Phi, \Psi \in C^1[D, \mathbb{R}]$, and for $u \in D$ the following inequalities hold:*

$$\begin{aligned} {}^cD^\vartheta\Phi(u) &\leq \Lambda(u, \Phi(u)), \quad \Phi(0) \leq \omega_0, \\ {}^cD^\vartheta\Psi(u) &\geq \Lambda(u, \Psi(u)), \quad \Psi(0) \geq \omega_0. \end{aligned} \tag{IV.16}$$

Additionally, suppose that $\Lambda(u, \omega)$ satisfies the Lipschitz condition:

$$\Lambda(u, \chi_1) - \Lambda(u, \chi_2) \leq \theta(\chi_1 - \chi_2), \quad \chi_1 \geq \chi_2 \quad and \quad \theta > 0, \tag{IV.17}$$

then if$\Phi(0) \leq \Psi(0)$, we have:

$$\Phi(u) \leq \Psi(u), \quad for \quad 0 \leq u \leq S. \tag{IV.18}$$

Proof. *Assume that one of the inequalities in $(IV.16)$ is strict, for example:*

$${}^cD^\vartheta\Phi(u) < \Lambda(u, \Phi(u)) \quad and \quad \Phi_0 < \Psi_0,$$

where $\Phi(0) = \Phi_0$ and $\Psi(0) = \Psi_0$.We will show that $\Phi(u) < \Psi(u)$ for $u \in D$. Suppose, to the contrary, that there exists u_1 such that $0 < u_1 \leq S$ for which

$$\Phi(u_1) = \Psi(u_1) \quad and \quad \Phi(u) < \Psi(u), \quad u_0 \leq u < u_1, \tag{IV.19}$$

Setting $k(u) = \Phi(u) - \Psi(u)$, we observe that

$$k(u_1) = 0 \quad and \quad k(u) < 0 \quad for \quad u < u_1.$$

Then, based on the hypothesis and LemmaIV.2.1, we conclude that ${}^cD^\vartheta k(u_1) \geq 0$. Therefore,

$$\Lambda(u_1, \Phi(u_1)) >^c D^\vartheta\Phi(u_1) \geq^c D^\vartheta\Psi(u_1) \geq \Lambda(u_1, \Psi(u_1)),$$

This contradicts the assumption $\Phi(u_1) = \Psi(u_1)$. Hence, $\Phi(u) < \Psi(u)$. Now assume that the inequalities $(IV.16)$ are non-strict. We will prove that $\Phi(u) \geq \Psi(u)$. We define the function $\Psi_\xi(u)$ as follows:

$$\Psi_\xi(u) = \Phi(u) + \xi\Theta(u),$$

where $\xi > 0$ and $\Theta(u) = E_\vartheta[2\theta u^\vartheta]$, with $E_\vartheta(.)$ representing the one-parameter Mittag-Leffler function. This indicates that $\Psi_{\xi,0} = \Phi_0 + \xi$ and $\Psi_\xi(u) > \Phi(u)$ for $u \in (0, S]$. Utilizing $(IV.16)$ and the Lipschitz condition

$(IV.17)$, *we can deduce that*

$$\begin{aligned} {}^cD^\vartheta \Psi_\xi(u) &= {}^cD^\vartheta \Psi(u) + \xi {}^cD^\vartheta \Theta(u) \\ &\geq \Lambda(u, \Psi(u)) + 2\xi\theta\Theta(u) \\ &> \Lambda(u, \Psi_\xi(u) - \xi\theta\Theta(u) + 2\xi\theta\Theta(u) \\ &> \Lambda(u, \Psi_\xi(u)), \quad 0 < u \leq S. \end{aligned}$$

Here we've used the property that $\Theta(u)$ *satisfies the initial value problem*

$${}^cD^\vartheta \Theta(u) = 2\theta\Theta(u), \quad \Delta(0) = 1 > 0.$$

Now, applying the result for strict inequalities to $\Phi(u)$ *and* $\Psi_\xi(u)$, *we infer that* $\Phi(u) < \Psi_\xi(u)$ *for* $u \in D$, *for every* $\xi > 0$. *Subsequently, letting* $\xi \to 0$, *we obtain* $\Phi(u) \leq \Psi(u)$ *for* $u \in D$. ■

Example IV.2.2 *1) Interval: Let* $I = [0, 1]$.

2) *Choose* $\Phi(u)$ *and* $\Psi(u)$:

   Define $\Phi(u) = -2u$ *and* $\Psi(u) = 3u$. *Both functions are in* $C^1[I, \mathbb{R}]$.

3) *Choose* $\Lambda(u, \omega)$:

   Define $\Lambda(u, \omega) = \frac{1}{2}\omega$. *This function is in* $C(I \times \mathbb{R}, \mathbb{R})$

4) *Lipschitz condition:*

$$\Lambda(u, \chi_1) - \Lambda(u, \chi_2) = \frac{1}{2}\chi_1 - \frac{1}{2}\chi_2 = \frac{1}{2}(\chi_1 - \chi_2).$$

   This satisfies the Lipschitz condition with $\theta = \frac{1}{2}$.

5) *Initial conditions:*

$$\Phi(0) = 0 \leq \omega_0 = 0, \qquad \Psi(0) = 0 \geq \omega_0 = 0.$$

6) *Verify* $\Phi(0) \leq \Psi(0)$:

$$\Phi(0) = 0 \leq 0 = \Psi(0).$$

7) *Caputo Fractional Derivative:*

   The Caputo fractional derivative ${}^cD^\vartheta \Phi$ *and* ${}^cD^\vartheta \Psi$ *for* $0 < \vartheta < 1$:

- *For* $\Phi(u) = -2u$*:*

$$\begin{aligned} {}^{c}D^{\vartheta}\Phi(u) &= \frac{1}{1-\vartheta}\int_{0}^{u}(u-\tau)^{-\vartheta}(-2)d\tau \\ &= -\frac{2u^{1-\vartheta}}{\Gamma(2-\vartheta)}. \end{aligned}$$

- *For* $\Psi(u) = 3u$*:*

$$\begin{aligned} {}^{c}D^{\vartheta}\Phi(u) &= \frac{1}{1-\vartheta}\int_{0}^{u}(u-\tau)^{-\vartheta}(3)d\tau \\ &= \frac{3u^{1-\vartheta}}{\Gamma(2-\vartheta)}. \end{aligned}$$

8) Differential Inequalities:

$${}^{c}D^{\vartheta}\Phi(u) = -\tfrac{2u^{1-\vartheta}}{\Gamma(2-\vartheta)} \leq \Lambda(u,\Phi(u)) = -u.$$

$${}^{c}D^{\vartheta}\Psi(u) = \tfrac{3u^{1-\vartheta}}{\Gamma(2-\vartheta)} \geq \Lambda(u,\Psi(u)) = \tfrac{3}{2}u.$$

This is true for $u \in [0,1]$.

9) Conclusion: Thus, the functions $\Phi(u) = -2u$, $\Psi(u) = 3u$, *and* $\Lambda(u,\omega) = \frac{1}{2}\omega$ *satisfy all the condition of Theorem IV.2.1, and the conclusion* $\Phi(u) = -2u \leq \Psi(u) = 3u$ *for* $0 \leq u \leq 1$ *holds.*

IV.3 Comparison Theorems for Fractional Differential Equation with weighted R-L fractional derivatives of a function with respect to another function

In this section, we develop the comparison theorems of fractional differential inequalities, strict as well as non-strict, involving weighted Riemann-Liouville differential operators of a function with respect to functions of order ϑ, $0 < \vartheta < 1$. *The proof relies on an estimate of the weighted R-L fractional derivative of a function at extreme points, which will be demonstrated subsequently.*

We begin by examining the initial value problem (IVP) *associated with the fractional differential equation expressed as:*

$${}^{RL}D^{\vartheta,\theta(u)}_{\pi(u)}\Phi = \Lambda(u,\Phi(u)), \quad \Phi(u_0) = \Phi^0 = \Phi(u)\pi(u)(\theta(u)-\theta(u_0))^{1-\vartheta}|_{u_0}, \quad u_0 \leq u \leq S, \quad S > 0, \tag{IV.20}$$

$\Lambda \in C([u_0, S] \times \mathbb{R}, \mathbb{R})$, ${}^{RL}D^{\vartheta,\theta(u)}_{\pi(u)}\Phi$ *is the weighted Riemann Liouville fractional derivative of order* ϑ *of* Φ*, such that* $0 < \vartheta < 1$.

Definition IV.3.1 *Let* $0 < \vartheta < 1$ *and* $\rho = 1 - \vartheta$*. We denote by* $C^{\pi}_{\rho,\theta}([\ell, r], \mathbb{R})$*, the function space*

$$C^{\pi}_{\rho,\theta}([u_0, S], \mathbb{R}) = \{\Phi \in C^{\pi}((u_0, S], \mathbb{R}), \quad (\theta(u) - \theta(u_0))^{\rho}\pi(u)\Phi(u) \in C([t_0, S], \mathbb{R})\}. \tag{IV.21}$$

Definition IV.3.2 *(**Locally Hölder continuous with respect to** θ) Let* Λ *be a real function. We say that* Λ *is locally Hölder continuous with respect to* θ *at a point* u_1*, with exponent* $\lambda \in (0, 1]$*, if there exist a real number* $M > 0$*, such that for all* $\xi > 0$*, small enough, we have*

$$|\Lambda(u_1) - \Lambda(u)| \leq M|\theta(u_1) - \theta(u)|^{\lambda} \quad \forall u \in]u_1 - \xi, u_1 + \xi[\cap dom(\Lambda), \quad \xi > 0, \tag{IV.22}$$

where θ *is a strictly increasing* C^1 *function. A function* Λ *is simply said to be locally Hölder continuous with respect to* θ*, if it is locally Hölder continuous with respect to* θ *at all points in* $dom(\Lambda)$*.*

Lemma IV.3.1 *Let* $0 < \vartheta, \rho < 1$ *, and suppose* θ *is a* C^1 *function that is strictly increasing, and* $\pi(u) \neq 0$ *for* $u \in [\ell, r]$*. We define the function*

$$\kappa(u) = \frac{(\theta(u) - \theta(\ell))^{\rho-1}}{\pi(u)} E_{\vartheta,\rho}\left[\mu(\theta(u) - \theta(\ell))^{\vartheta}\right],$$

where $E_{\vartheta,\mu}(.)$ *is the Mittag-Leffler function with two parameters. Then,*

$${}^{RL}D^{\vartheta,\theta(u)}_{\ell^+\pi(u)}\kappa(u) = \mu\kappa(u). \tag{IV.23}$$

Proof. *Using the definition of the Mittag Leffler function and Property II.2.1, we have*

$$\begin{aligned}
{}^{RL}D^{\vartheta,\theta(u)}_{\ell^+,\pi(u)}\kappa(u) &= {}^{RL}D^{\vartheta,\theta(u)}_{\ell^+,\pi(u)}\left[\frac{(\theta(u) - \theta(\ell))^{\rho-1}}{\pi(u)} E_{\vartheta,\rho}\left[\mu(\theta(u) - \theta(\ell))^{\vartheta}\right]\right] \\
&= {}^{RL}D^{\vartheta,\theta(u)}_{\ell^+,\pi(u)}\left[\frac{(\theta(u) - \theta(\ell))^{\rho-1}}{\pi(u)} \sum_{i=0}^{\infty} \frac{\mu^i(\theta(u) - \theta(\ell))^{\vartheta i}}{\Gamma(\vartheta i + \rho)}\right] \\
&= \sum_{i=0}^{\infty} \frac{\mu^i}{\Gamma(\vartheta i + \rho)} {}^{RL}D^{\vartheta,\theta(u)}_{\ell^+,\pi(u)}\left[\frac{(\theta(t) - \theta(\ell))^{\delta i+\beta-1}}{\pi(t)}\right] \\
&= \mu\frac{(\theta(u) - \theta(\ell))^{\rho-1}}{\pi(t)} \sum_{i=1}^{\infty} \frac{\mu^{i-1}(\theta(u) - \theta(\ell))^{\vartheta(i-1)}}{\Gamma(\vartheta(i-1) + \rho)} \\
&= \mu\kappa(t).
\end{aligned}$$

This completes the proof of the lemma. ■

IV.3.1 Estimates on Weighted R-L Fractional Derivatives of a Function with Respect to Functions at their Extreme Points

The following Lemma as a generalization and an extension of the Lemmas IV.1.1 and IV.2.1 introduced in [33] and [35] for the Riemann Liouville and Caputo derivatives, respectively. These Lemmas as a starting point for developing comparison theorems.

Lemma IV.3.2 *Let $m \in C^{\pi}_{\rho,\theta}([u_0, S], \mathbb{R})$, such that π is a positive function in $L^{\infty}((u_0,\, S))$. Assume that m is locally Hölder continuous with respect to θ at $u^* \in (u_0,\, S]$ and exponent $\lambda > 1-\rho$. If u^* satisfies the conditions:*

$$m(u^*) = 0 \quad and \quad m(u) \leq 0 \quad for \quad u_0 \leq u \leq u^*, \tag{IV.24}$$

then it follows that

$${}^{RL}D^{\vartheta,\theta(u)}_{\pi(u)} m(u^*) \geq 0, \tag{IV.25}$$

where $0 < \vartheta < 1$ and $\rho = 1-\vartheta$.

Proof. *From $(I.4)$, it is clear that*

$${}^{RL}D^{1,\theta(u)}_{\pi(u)} \left(\frac{G(u)}{\pi(u)}\right) = \frac{G^{'}(u)}{\theta'(u)\pi(u)}. \tag{IV.26}$$

Thus, according to $(II.10)$, we find that

$$\begin{aligned}
({}^{RL}D^{\vartheta,\theta(u)}_{\ell^+,\pi(u)} m)(u) &= ({}^{RL}D^{1,\theta(u)}_{\pi(u)}\, {}^{RL}\Im^{1-\vartheta,\theta(u)}_{\ell^+,\pi(u)} m)(u)\\
&= {}^{RL}D^{1,\theta(u)}_{\pi(u)} \left[\frac{1}{\Gamma(\rho)\pi(u)} \int_{u_0}^{u} (\theta(u)-\theta(s))^{\rho-1}\theta^{'}(s)\pi(s)m(s)ds\right]\\
&= \frac{1}{\Gamma(\rho)\pi(u)\theta'(u)} \frac{d}{du} \int_{u_0}^{u} (\theta(u)-\theta(s))^{\rho-1}\theta^{'}(s)\pi(s)m(s)ds,
\end{aligned}$$

we set, $G(u) = \int_{u_0}^{u} (\theta(u)-\theta(s))^{\rho-1}\theta^{'}(s)\pi(s)m(s)ds$. Consider the following for a small $\eta > 0$:

$$\begin{aligned}
G(u^*) - G(u^*-\eta) &= \int_{u_0}^{u^*-\eta} [(\theta(u^*)-\theta(s))^{\rho-1} - (\theta(u^*-\eta)-\theta(s))^{\rho-1}]\theta^{'}(s)\pi(s)m(s)ds\\
&+ \int_{u^*-\eta}^{u^*} (\theta(u^*)-\theta(s))^{\rho-1}\theta^{'}(s)\pi(s)m(s)ds.\\
&= \Delta_1 + \Delta_2.
\end{aligned}$$

Since $u_0 \leq s \leq u^* - \eta$ and $\rho - 1 < 0$, then from $(IV.24)$, we obtain

$$[(\theta(u^*) - \theta(s))^{\rho-1} - (\theta(u^* - \eta) - \theta(s))^{\rho-1}] < 0 \quad and \quad m(s) \leq 0.$$

Thus implying that $\Delta_1 \geq 0$. Therefore,

$$G(u^*) - G(u^* - \eta) \geq \int_{u^*-\eta}^{u^*} (\theta(u^*) - \theta(s))^{\rho-1}\theta^{'}(s)\pi(s)m(s)ds = \Delta_2.$$

Since $m(u)$ is locally Hölder continuous with respect to θ and exponent λ, there exists a real number $N(u^*) > 0$ such that for $u^* - \eta \leq s \leq u^* + \eta$

$$-N(u^*)(\theta(u^*) - \theta(s))^{\lambda} \leq m(u^*) - m(s) \leq N(u^*)(\theta(u^*) - \theta(s))^{\lambda},$$

where $0 < \lambda < 1$ is such that $\lambda > 1 - \rho$. Knowing that π is a positive function, then by $(IV.24)$ we have

$$\begin{aligned} \Delta_2 &\geq -N(u^*)\|\pi\|_{L^\infty} \int_{u^*-\eta}^{u^*} (\theta(u^*) - \theta(s))^{\rho-1+\lambda}\theta^{'}(s)ds \\ &= -\frac{N(u^*)\|\pi\|_{L^\infty}(\theta(u^*) - \theta(u^* - \eta))^{\rho+\lambda}}{\rho + \lambda}. \end{aligned}$$

Hence, for sufficiently small $\eta > 0$

$$\frac{G(u^*) - G(u^* - \eta)}{\eta} \geq -\frac{N(u^*)\|\pi\|_{L^\infty}}{\rho + \lambda} \left(\frac{\theta(u^*) - \theta(u^* - \eta)}{\eta}\right)^{\rho+\lambda} \eta^{\rho+\lambda-1}.$$

Letting $\eta \to 0$, we obtain $\frac{d}{du}G(u^*) \geq 0$, which implies ${}^{RL}D^{\vartheta,\theta(u)}_{\pi(u)}m(u^*) \geq 0$, and the proof is complete. ■

Theorem IV.3.1 *Let $\Phi, \Psi \in C^{\pi}_{\rho,\theta}([u_0, S], \mathbb{R})$, such that π is a positive function in $L^\infty((u_0, S))$ and $\theta \in C^1$ be a strictly increasing function on $[u_0, S]$, $\Lambda \in C([u_0, S] \times \mathbb{R}, \mathbb{R})$. Assume that Φ, Ψ are locally Hölder continuous with respect to θ for respectively an exponent λ_1 and λ_2 in $]0, 1]$ such that $\min\{\lambda_1, \lambda_2\} + \rho > 1$ and*

(R1) ${}^{RL}D^{\vartheta,\theta(u)}_{\pi(u)}\Phi(u) \leq \Lambda(u, \Phi(u))$

(R2) ${}^{RL}D^{\vartheta,\theta(u)}_{\pi(u)}\Psi(u) \geq \Lambda(u, \Psi(u)), \quad u_0 < u \leq S,$

one of the inequalities $(R1)$ or $(R2)$ being strict. Then

$$\Phi^0 < \Psi^0, \tag{IV.27}$$

where $\Phi^0 = \Phi(u)\pi(u)(\theta(u) - \theta(u_0))^{1-\vartheta}|_{u=u_0}$ and $\Psi^0 = \Psi(t)\pi(u)(\theta(u) - \theta(u_0))^{1-\vartheta}|_{u=u_0}$, implies

$$\Phi(u) < \Psi(u), \quad u_0 \leq u \leq S. \tag{IV.28}$$

Proof. *Assume that the conclusion* $(IV.28)$ *is not true. Then, since* $\Phi^0 < \Psi^0$ *and* $\Phi(u)\pi(u)(\theta(u)-\theta(u_0))^{1-\vartheta}$, $\Psi(u)\pi(u)(\theta(u)-\theta(u_0))^{1-\vartheta}$ *are continuous functions, there exists a* ξ *such that* $u_0 < \xi \leq S$

$$\Phi(\xi) = \Psi(\xi) \quad and \quad \Phi(u) < \Psi(u) \quad u_0 \leq u < \xi. \tag{IV.29}$$

Define $m(u) = \Phi(u) - \Psi(u)$, $u \in [u_0, S]$. *Then, we find that* $m(\xi) = 0$ *and* $m(u) < 0 \quad u_0 \leq u < \xi$, *with* $m \in C^{\pi}_{\rho,\theta}([u_0, S], \mathbb{R})$. *Hence by Lemma IV.3.2, we obtain*

$$^{RL}D^{\vartheta,\theta(u)}_{\pi(u)} m(\xi) \geq 0.$$

This gives

$$^{RL}D^{\vartheta,\theta(u)}_{\pi(u)} \Phi(\xi) \geq^{RL} D^{\Phi,\theta(u)}_{\pi(u)} \Psi(\xi).$$

Suppose that the inequality $(R2)$ *is strict, then we have*

$$\Lambda(\xi, \Phi(\xi)) \geq^{RL} D^{\rho,\theta(u)}_{\pi(u)} \Phi(\xi) \geq^{RL} D^{\rho,\theta(u)}_{\pi(u)} \Psi(\xi) > \Lambda(\xi, \Psi(\xi)),$$

which is a contradiction with $\Phi(\xi) = \Psi(\xi)$. *Hence, the conclusion* $(IV.28)$ *is valid and the proof is complete.* ■

The next result is for non-strict fractional differential inequalities, which demand a Lipschitz-type condition.

Theorem IV.3.2 *Assume that the condition of Theorem IV.3.1 holds with non-strict inequalities* $(R1)$ *and* $(R2)$. *Further, assume that* Λ *satisfies the Lipschitz condition*

$$\Lambda(u, \chi_1) - \Lambda(u, \chi_2) \leq \sigma(\chi_1 - \chi_2), \quad \chi_1 \geq \chi_2 \quad and \quad \sigma > 0. \tag{IV.30}$$

Then, $\Phi^0 \leq \Psi^0$, *implies*

$$\Phi(u) \leq \Psi(u), \quad u_0 \leq u \leq S. \tag{IV.31}$$

Proof. *For small* h, *we define*

$$\Psi_h(u) = \Psi(u) + h\Delta(u), \tag{IV.32}$$

where $\Delta(u) = \pi^{-1}(u)(\theta(u) - \theta(u_0))^{\vartheta-1} E_{\vartheta,\vartheta}[2\sigma(\theta(u) - \theta(u_0))^{\vartheta}]$, *with* $\Psi_h \in C^{\pi}_{\vartheta,\theta}([u_0, S], \mathbb{R})$. *It follows from this*

$$\Psi_h(u)\pi(u)(\theta(u)-\theta(u_0))^{1-\vartheta}|_{u=u_0} = \Psi(u)\theta(u)(\theta(u)-\theta(u_0))^{1-\vartheta}|_{u=u_0} + h\Delta(u)\pi(u)(\theta(u)-\theta(u_0))^{1-\vartheta}|_{u=u_0}.$$

So, we obtain, $\Psi^0_h = \Psi^0 + h\Delta^0$. *This leads to*

$$\Psi^0_h > \Psi^0 \geq \Phi^0 \quad and \quad \Psi_h(u) > \Psi(u). \tag{IV.33}$$

Next, by applying the Lipschitz condition $(IV.30)$ *and Lemma IV.3.1(with* $\mu = 2\sigma, \rho = \vartheta$ *and* $\ell = u_0$*), we deduce*

$$\begin{aligned} {}^{RL}D^{\vartheta,\theta(u)}_{\pi(u)}\Psi_h(u) &= {}^{RL}D^{\vartheta,\theta(u)}_{\pi(u)}\Psi(u) + h^{RL}D^{\vartheta,\theta(u)}_{\pi(u)}\Delta(u) \\ &\geq \Lambda(u, \Psi(u)) + 2h\sigma\Delta(u) \\ &> \Lambda(u, \Psi_h(u) - \sigma h\Delta(u) + 2h\sigma\Delta(u) \\ &> \Lambda(u, \Psi_h(u)), \quad u_0 < u \leq S. \end{aligned}$$

Therefore,

$${}^{RL}D^{\vartheta,\theta(u)}_{\pi(u)}\Psi_h(u) > \Lambda(u, \Psi_h(u)), \quad u_0 \leq u \leq S.$$

In this case, we have made use of the fact that $\Delta(u)$ *is the linear weighted weighted Riemann-Liouville fractional differential equation*

$${}^{RL}D^{\vartheta,\theta(u)}_{\pi(u)}\Delta(u) = 2\sigma\Delta(u), \quad u_0 < u_1 \leq S \qquad \Delta(u)\pi(u)(\theta(u) - \theta(u_0))^{1-\vartheta}|_{u=u_0} = \Delta^0 > 0.$$

Utilizing $(IV.33)$*, we can apply Theorem IV.3.1 to* $\Psi(t)$ *and* $\Psi_h(u)$*. As a result, we have*

$$\Phi(t) < \Psi_h(u), \quad u \in [u_0, S], \quad h > 0. \tag{IV.34}$$

By taking the limit as $h \to 0$*, in the above inequality and using* $(IV.32)$*, we deduce that*

$$\Phi(u) \leq \Psi(u), \quad u \in [u_0, S].$$

Hence, then the proof is complete. ■

Remark IV.3.1 *We observe that the comparison results found in Theorems IV.3.1 and IV.3.2 are a generalization of the results established in Theorems IV.1.1 and IV.1.2, respectively, as demonstrated in [33, 34].*

CONCLUSION

In this thesis, we investigate the uniqueness and existence of solutions to the Cauchy problem for nonlinear differential equations of fractional order that involve the weighted Riemann-Liouville fractional derivative of one function with respect to another function, using Banach's fixed point theorem and the method of successive approximations. We not only establish these results but also develop comparison theorems, known as strict and non-strict inequality theorems. These theorems rely on estimates of the weighted Riemann-Liouville fractional derivative of one function with respect to another at their extreme points. This estimation has been rigorously validated.

This work is considered important because it is a generalization of many of the results that have been reached in previous research. This work has opened new horizons for us, as well as many questions that need further research and that must be considered because they are interesting, this is what made us highlight the most important points that deserve Research and delve deeper. Consequently, we outline several directions for prospective research:

1) *Develop comparison theorems for weighted Caputo fractional operators with respect to another function*

2) *Generalization of this work in the case of nonlinear systems of fractional differential equations containing the weighted fractional Riemann-Liouville derivative of a function with respect to another function.*

It is essential to outline the significant challenges and questions that arose during our research. These include:

- *Comparison of solutions for nonlinear systems of multi-order fractional differential equations, i.e. the orders of derivation for these systems that were chosen, whether they contain a Riemann-Liouville or Caputo derivative, or a weighted fractional Riemann-Liouville derivative of a function with respect to another function, is different, unlike previous research in which the orders of derivation were unified, and this is what made us stop and ask ourselves a question: Does previous work, comparison results, and the principle of maximum help us in this work, and do the results remain the same as before?*

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