ON NEW EXTENSIONS OF F-CONTRACTION WITH AN APPLICATION TO INTEGRAL INCLUSIONS

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Abstract

The purpose of this study is to present some fixed point theorems by combining the contractions of Geraghty and Hardy-Rogers with F-contraction and α-admissible concepts in the setting of set-valued mappings under weaker conditions. We give also an example and an application to support new theories.

Key words: *F-contraction, Geraghty contraction,Hardy-Rogers contraction, α-admissible, semi lower continuous, integral inclusion.*

1. Introduction

The multivalued fixed point theory has many and different applications as in integral or differential inclusions, economy, optimization, etc. The contraction principle due to Banach has been generalized in different directions and one of such generalizations is due to Nadler[2], where he used the Pompeiu-Hausdorff metric to establish some fixed point results of multivalued mappings in metric spaces. Later many authors established some results in nonlinear analysis concerning the multivalued fixed point theory and its applications using the Pompeiu-Hausdorff distance.

Samet et al [4] introduced a new concept called as α-admissible, they obtained some fixed point results for $\alpha-\psi$-contractive mappings, later many authors invested such concepts to establish some results. Recently, Wardowski [6] introduced a new type of contractions called as F-contraction to show the existence of fixed points for such contraction by more simple method of proof than Banach's one. After that, several authors studied on different variations of F-contraction for single-valued and multivalued mappings.

In this study, we combine the notion of α admissible with Wardowski contraction and the contractions of Geraghty and Hardy-Rogers in order to introduce new types of multivalued contractions to establish some fixed point theorems in the setting of complete metric spaces. We derive new fixed point results on a metric space endowed with a graph by using the results obtained herein. Finally, we give an example and an application of the existence of solution for an integral inclusion to illustrate our results.

2. Preliminaries

Let (X,d) be a metric space, and let $CB(X)$ be a set of non empty, closed and bounded subsets of X, the Hausdorff-Pompeiu metric is defined as:

$$H(A,B)=\max\{\sup_{a\in A} d(a,B),\sup_{b\in B} d(b,A)\},$$

for all $A,B\in CB(X)$ and $d(a,B)=\inf\{d(a,b),b\in B\}$. Also, denote the family of nonempty and closed subsets of X by $CL(X)$ and the family of nonempty and compact subsets of X by $K(X)$.

Lemma. Let (X,d) be a metric space and $A,B\in CL(X)$ with $H(A,B)>0$. Then, for each $h>1$ and for each $a\in A$, there exists $b=b(a)\in B$ such that $d(a,b)<hH(A,B)$.

Definition [6]. Let $F:\mathbb{R}_+\to\mathbb{R}$ be a function satisfying:

(F_1) : F is strictly increasing.

(F_2) : For each sequence $\{\alpha_n\}$ in X, $\lim_{n\to\infty}\alpha_n=0$ if and only if $\lim_{n\to\infty}F(\alpha_n)=-\infty$.

(F_3) : There exists $k\in(0,1)$ satisfying $\lim_{\alpha\to0^+}\alpha^kF(\alpha)=0$.

Example Let $F_i:\mathbb{R}_+\to\mathbb{R}$, $i\in\{1,2,3\}$, defined by

1. $F_1(t)=\ln t$,
2. $F_2(t)=t+\ln t$,
3. $F_3(t)=-\frac{1}{\sqrt{t}}$.

Then F_i, satisfying $(F_1),(F_2),(F_3)$ for each $i\in\{1,2,3\}$.

Now, denote by:

$\mathcal{F}$**:** the set of all functions F satisfying the conditions $(F_1)-(F_3)$.

$\mathcal{F}_*$**:** the set of all functions F satisfying $(F_1)-(F_3)$ and (F_4) : F is right continuous.

Ω: the set of all functions $\beta:[0,\infty)\to[0,1)$ satisfying $\lim_{n\to\infty}\beta(t_n)=1$ implies $\lim_{n\to\infty}t_n=0$

Theorem of Geraghty:

Let (X,d) be a complete metric space and $T:X\longrightarrow X$ be a mapping. Assume that there exists a function $\beta\in\Omega$ and

$$d(Tx,Ty)\leqslant\beta((d(x,y))d(x,y)$$

for all $x,y\in X$. Then T has a unique fixed point.

Theorem

Let (X,d) be a metric space and let $T:X\to CB(X)$ be a multivalued F-contraction of Hardy-Rogers type, that is, there exist $F\in\mathcal{F}_*$, $\tau>0$ and non-negative real numbers $\alpha,\beta,\gamma,\delta,L$ with $\alpha+\beta+\gamma+2\delta=1$ and $\gamma\neq1$ such that

$$2\tau+F(H(Tx,Ty))\leq F(\alpha d(x,y)+\beta d(x,Tx)+\gamma d(y,Ty)+\delta d(x,Ty)+Ld(y,Tx)),$$

for all $x,y\in X$ with $H(Tx,Ty)>0$. Then T has a fixed point in X.

Definition :

Let (X,d) be a metric space and $\alpha\colon X\times X\to[0,+\infty)$ be a given mapping. A mapping $T\colon X\to CL(X)$ is an

1. α_*-admissible, if $\alpha(x,y)\geq1$ implies $\alpha_*(Tx,Ty)\geq1$, where
$$\alpha_*(Tx,Ty)=\inf\{\alpha(a,b):\ a\in Tx,b\in Ty\},$$
2. α-admissible, if for each $x\in X$ and $y\in Tx$ with $\alpha(x,y)\geq1$, we have $\alpha(y,z)\geq1$ for all $z\in Ty$,
3. α-lower semi-continuous, if for $x\in X$ and a sequence $\{x_n\}$ in X with $\lim_{n\to\infty}d(x_n,x)=0$ and $\alpha(x_n,x_{n+1})\geq1$, for all $n\in\mathbb{N}$, implies
$$\liminf_{n\to\infty}d(x_n,Tx_n)\geq d(x,Tx).$$

3. Results

Definition:

Let (X,d) be a metric space and $\alpha:X\times X\to\mathbb{R}$. A mapping $T:X\to CL(X)$ is called α-F-Geraghty contraction of Hardy-Rogers type if there exist $F\in\mathcal{F}$, $\beta\in\Omega$ and $\tau>0$ and non-negative real numbers a_1,a_2,a_3,a_4,a_5 with $a_1+a_2+a_3+2a_4=1$ and $a_3\neq1$ such that

$$\tau+F(H(Tx,Ty))\leq F(\beta(N(x,y))N(x,y)), \tag{3.1}$$

for all $x,y\in X$ with $\alpha(x,y)\geq1$ and $H(Tx,Ty)>0$ where

$$N(x,y)=a_1d(x,y)+a_2d(x,Tx)+a_3d(y,Ty)+a_4d(x,Ty)+a_5d(y,Tx).$$

Theorem 1:

Let (X,d) be a complete metric space and $T:X\to K(X)$ be an α-F-Geraghty contraction of Hardy-Rogers type. Assume that the following conditions are satisfied:

1. T is an α-admissible;
2. There exist $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\geq1$;
3. T is an α-lower semi-continuous mapping, or X is α-regular, that is, for every sequence $\{x_n\}$ in X such that $x_n\to x\in X$ and $\alpha(x_n,x_{n+1})\geq1$ for all $n\in\mathbb{N}$, then $\alpha(x_n,x)\geq1$ for all $n\in\mathbb{N}$.

Then T has a fixed point.

In the next theorem, we replace $K(X)$ with $CB(X)$ by considering in the setting of $\mathcal{F}_*$.

Theorem 2:

Let (X,d) be a complete metric space and $T:X\to CB(X)$ be an α-F-Geraghty contraction of Hardy-Rogers type with $F\in\mathcal{F}_*$. Assume that the following conditions are satisfied:

1. T is an α-admissible;
2. There exist $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\geq1$;
3. T is an α-lower semi-continuous mapping, or X is α-regular.

Then T has a fixed point.

Example:

Let $X=\{1,2,3,4\}$ and $d(x,y)=|x-y|$. Define $T\colon X\to CB(X)$ and $\alpha:X\times X\to[0,\infty)$ by

$$Tx=\begin{cases}\{2\}, & x\in\{1,2\}\\ \{1\}, & x=3\\ \{3\}, & x=4\end{cases}$$

and

$$\alpha(x,y)=\begin{cases}0, & (x,y)\in\{(2,3),(3,2),(3,4),(4,3)\}\\ 1, & \text{otherwise.}\end{cases}$$

By taking $F(x)=\ln x$, $\beta(t)=\frac{t}{1+t}$, $\tau=\frac{1}{4}$, $a_1=1$ and $a_2=a_3=a_4=a_5=0$. Then all conditions of Theorem 1 (resp. Theorem 2) are satisfied. Then T has a fixed point which is 2.

4. Some Consequences

Now, we present the existence of fixed point for multivalued mappings from a metric space X, endowed with a graph, into the space of nonempty closed and bounded subsets of the metric space. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops; that is, $E(G)\supseteq\Delta$, where $\Delta=\{(x,x):x\in X\}$. We assume G has no parallel edges, so we can identify G with the pair $(V(G),E(G))$.

If we define the function

$$\alpha\colon X\times X\to[0,+\infty),\quad \alpha(x,y)=\begin{cases}1, & \text{if } (x,y)\in E(G),\\ 0, & \text{otherwise,}\end{cases}$$

then the following result is a direct consequence of our results.

Theorem

Let (X,d) be a complete metric space endowed with a graph G and $T\colon X\to CB(X)$ (resp. $K(X)$) be a multivalued mapping. Assume that the following conditions are satisfied:

1. For each $x\in X$ and $y\in Tx$ with $(x,y)\in E(G)$, we have $(y,z)\in E(G)$ for all $z\in Ty$;
2. There exist $x_0\in X$ and $x_1\in Tx_0$ such that $(x_0,x_1)\in E(G)$;
3. T is G-lower semi-continuous, that is, for $x\in X$ and a sequence $\{x_n\}$ in X with $\lim_{n\to\infty}d(x_n,x)=0$ and $(x_n,x_{n+1})\in E(G)$ for all $n\in\mathbb{N}$, implies
$$\liminf_{n\to\infty}d(x_n,Tx_n)\geq d(x,Tx)$$
or, for every sequence $\{x_n\}$ in X such that $x_n\to x\in X$ and $(x_n,x_{n+1})\in E(G)$ for all $n\in\mathbb{N}$, we have $(x_n,x)\in E(G)$ for all $n\in\mathbb{N}$;
4. There exist $F\in\mathcal{F}_*$, $\beta\in\Omega$ and $\tau>0$ and non-negative real numbers a_1,a_2,a_3,a_4,a_5 with $a_1+a_2+a_3+2a_4=1$ and $a_3\neq1$ such that
$$\tau+F(H(Tx,Ty))\leq F(\beta(N(x,y))N(x,y)), \tag{4.1}$$
for all $x,y\in X$ with $(x,y)\in E(G)$ and $H(Tx,Ty)>0$ where
$$N(x,y)=a_1d(x,y)+a_2d(x,Tx)+a_3d(y,Ty)+a_4d(x,Ty)+a_5d(y,Tx).$$

Then T has a fixed point.

5. Application

Let $X:=C([a,b],\mathbb{R})$ be the space of all continuous realvalued functions on $[a,b]$. Clearly X with uniform metric $d(x,y)=\sup_{t\in[a,b]}|x(t)-y(t)|$ is a complete metric space.

Consider now the following problem

$$x(t)\in f(t)+\int_a^b Q(t,s,x(s))ds,\qquad t\in J=[a,b]. \tag{5.1}$$

where $f\in X$ and $Q:J\times J\times\mathbb{R}\to CB(\mathbb{R})$. Our hypotheses are on the following data :

(A) for each $x\in X$, the multivalued operator $Q_x(t,s):=Q(t,s,x(s))$, $t,s\in J\times J$, is lower semi-continuous;

(B) there exists a continuous mapping $\rho:J\times J\to[0,+\infty)$ such that

$$|Q(t,s,u(s))-Q(t,s,v(s))|\leq\rho(t,s)\cdot\ln(|u(s)-v(s)|+1),$$

for all $u,v\in X$ with $(u,v)\in E(G)$ and $u\neq v$ and for each $(t,s)\in J\times J$;

(C) there exists $\tau>0$ such that

$$\sup_{t\in J}\int_a^b\rho(t,s)ds\leq e^{-\tau};$$

(D) there exist $x_0\in X$ and $x_1\in Tx_0$ such that $(x_0,x_1)\in E(G)$;

(E) for each $x\in X$ and $y\in Tx$ with $(x,y)\in E(G)$, we have $(y,z)\in E(G)$ for all $z\in Ty$;

for every sequence $\{x_n\}$ in X such that $x_n\to x\in X$ and $(x_n,x_{n+1})\in E(G)$ for all $n\in\mathbb{N}$, we have $(x_n,x)\in E(G)$ for all $n\in\mathbb{N}$;

Theorem:

Under assumptions $(A)-(F)$ the integral inclusion (5.1) has a solution in X.

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