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Thème

On the stability and periodicity for some differential equations with delay

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الإهداء

إلى من علمه شديد القُوى، سيّدنا محمّد صلى اللّه عليه وسلّم

إلى والدينا وفاءً وعرفاناً

إلى الأحبة من أهلنا وصحبتنا حبّاً وحناناً

إلى من ساهم في بلوغ اللّحظة التي نخط فيها هذي الكلمات

وأمدّنا بحرف ننفع به أنفسنا وأمتنا .

إلى أهل العلم كافة أهدي هذا الجهد المتواضع

قطار شيماء طواهري خولة***

ملخص

يلعب الاستقرار دورا مهما في الانظمة الديناميكية والتحكم يميز خاصية المسارات غير المضطربة بحيث أن الإضطرابات الصغيرة تسبب فقط تغيرات صغيرة في سلوك النظام.في هذه المذكرة نستخدم تقنيات النقطة الثابتة للحصول على نتائج الاستقرار للمعادلة التفاضلية التكاملية مع التأخير الوظيفي مع تقديم بعض الفرضيات من اجل ايجاد حل دوري موجب للمعادلة التفاضلية ذات التأخير.مع اعطاء امثلة للتوضيح.

Abstract

Stability plays an important role in dynamic sestems and contol characterizes the characteristic of non turbulent paths so that small perturbations cause only small changes in the behavior of the systeme .In this memiory we use fixed point techniques to obtain the stability results of the integral differential equation with functional delays with the introduction of some hypotheses in order to find a periodic solution positive for the delay differential equation. With exemples to be given for clarification.

Résumé

La stabilité joue un rôle important dans les systèmes dynamiques et le contrôle caractérise la caractéristique des trajectoires non turbulentes de sorte que de petites perturbations ne provoquent que de petits changements dans le comportement du système. Dans cette mémoire nous utilisons des techniques de virgule fixe pour obtenir les résultats de stabilité de la différentielle intégral équation avec retard fonctionnel tout en présentant quelques hypothèses afin de trouve une solution périodique positive pour l'équation retard différentielle. Avec exemple de clarification.

Contents

Notations générales

$\mathbb{S}$	Un ensemble non vide.
ρ	Une distance sur $\mathbb{S}$.
$\mathbb{R}_+$	Ensemble des nombres réels positifs ou nuls.
$\mathbb{R}^n$	Espace vectoriel de dimension n construit sur le corps des réels.
$\Vert\cdot\Vert$	La norme.
$\vert\cdot\vert$	Valeur absolue d'un nombre réel.
$C\left([a,b],\mathbb{R}^n\right)$	Ensemble des fonctions continues de $[a,b]$ dans $\mathbb{R}^n$.
$(\mathbb{S},\rho)$	un espace métrique complet.

Introduction

Why study this subject ?. Why study differential equations with time delays when so much is known about equations without delays, and they are so much easier ?. The answer is because so many of the processes, both natural and manmade, in biology, medicine, chemistry, physics, engineering, economics, etc., involve time delays. Like it or not, time delays occur so often, in almost every situation, that to ignore them is to ignore reality. A simple example in nature is reforestation. A cut forest, after replanting, will take at least 20 years before reaching any kind of maturity. For certain species of trees (redwoods, for example) it would be much longer. Hence, any mathematical model of forest harvesting and regeneration clearly must have time delays built into it. A concrete exemple of this begins with volterra (1928) who sought to model a certain biological problem by means of the equation.

$$x'(t) = -\int_{t-L}^{t} a(t-s)g(x(s))ds \tag{.0.1}$$

in which $xg(x) > 0$ if $x \neq 0$, $L > 0$, and $a(t)$ is twice differentiable with $a''(t)$ not identically zero. Levin and Nohel (1964) extended that Liapunov functional to the case of (.0.1), itself. Here is thier result.Let $L > 0$, $xg(x) > 0$ when $x \neq 0$,$a : [0, L] \to \mathbb{R}$ and consider the IVP

$$x'(t) = -\int_{t-L}^{t} a(t-s)g(x(s))ds, \psi : [-L, 0] \to \mathbb{R}. \tag{.0.2}$$

We need an initial function ,ψ,because we must define $x'(0)$:

$$x'(0) = -\int_{-L}^{0} a(-s)g(\psi(s))ds,$$

This requires the intial datum of a function defined over the entire interval $[-r, 0]$ of $\mathbb{R}$. When $I = [-r, 0]$ these are finite delay differential equation in which case the phase space C is the space of continuous functions from $[-r, 0]$ in $\mathbb{R}$ endowed with unifom convergence. So the solution x (.0.1) must depend on a function such that $x(t) = \psi(t)$,$t \in [-r, 0]$. When $\psi : (-\infty, 0] \to \mathbb{R}$ and $x(t) = \psi(t)$,$t \in (-\infty, 0]$,then to solve equation (.0.1) for $t \geq 0$, we will associate an infinite functional delay which is not necessarily constant.

In first chapter of this memoire include some concepts, definition and theoremes related to the fixed point results about differential equation with a delay. For more details, we refer to the following references(see [22] ; [2] ; [7] ; [18] ; [23] ; [5] ; [10] ; [11] ; and [32]). The objective in the second chapter is to present an extension of the results and an improvement of the conditions used in [24], [3], [31], [27]. This task relies on the definition of a good continuous functio H and on the choice of assumptions on this functin , Without restrictions on the inverse of $t-\tau(t)$, so that ,for a continuous function initially given a fixed point map P associated with(II.0.1) is built on a complete metric space S,in itself allowinf P to have a fixed point this approach will make it possible to establish and proove an asymptotic stability theorem of thhe zero solutin of(II.0.1). With less restrictive necessary and sufficient conditions

$$x'(t) = -\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t} a_j(t,s)x(s)ds + \sum_{j=1}^{n} c_j(t)x'(t-\tau_j(t)) \qquad (.0.3)$$

with the initial condition

$$x(t) = \varphi(t) \text{ for } t \in [m(0), 0]$$

where $\varphi \in C([m(0), 0], \mathbb{R})$ and

$$m_j(0) = \inf\{t-\tau_j(t), t \geq t \geq\}, m(0) = \min\{m_j(0) 1 \leq j \leq N\}$$

to talk about new condition on b_j, τ_j and c_j. Without contraints on $t-\tau_j(t)$ so that , for initially given functin ψ a fixed point map can be defined on a carefullyb chosen complete metric space in which this application has a fixed point offering a bounded and stable solution to the problem posed .these conditions were proposed by Ardjouni and Djoudi [1], jin and luo [30]. Ardiouni and Djoudi have already,in [30] pushed the context of this talk to a more advanced stage which supports copletely nonlinear differential equations of neutral type and which clearly improves the previous results particularly those of Burton, jin, Raffoul and Zhang [24],[3],[31],[33]. with fewer restrictions and more rigors . The application of this method has also shown significant advantages of this method over that of Liapunov when the coefficients are unbounded and or if the delay is unbounded. The conditions of the first method are on average on the other hand those of the second are always punctual the fundamental idea exposed here making it possible to obtain asymptotic stability is based on the data of continuous function H subjected to an asymptotic hypothesis.

In the last chapter, we consider existence, multiplicity and non existence of positive ω_periodic solutions for the following second-order neutral functional differential equation :

$$(x(t) - cx(t-\delta))'' + \alpha(t)x(t) = \lambda b(t) f(x(t-\tau(t))) \qquad (.0.4)$$

where λ is a positive parameter, c and δ are constants and $|c| \neq 1$. The existence of periodic solutions for functional differential equations has been derived from many fields such as physics, biology and mechanics [10],[32]. Many results were obtained by Kuang [32], Freedman and Wu [9]. Wang [27] and many others by applying fixed point index theory, theory of Fourier series, fixed point theorems in cones, Leray–Schauder continuation theorem, coincidence degree theory and so on. We refer to [29],[16] for some recent results in this field. Among the previous results on this problem many of them concern neutral systems (Lu and Ge [20],Lu,Ge and Zheng [19] and Chen [8]). In this part, we aim to establish existence, multiplicity and non existence of positive ω_periodic solutions for second-order neutral functional differential equation(.0.4). Our approach is based on a fixed point theorem in cones as well as some analysis techniques used in [27],[28]. The main goal in the sequel is to study some qualitative and quantitative properties of solutions such as stability, asymptotic stability and the existence of periodic solutions. We are interested in the asymptotic stability of the zero solution. First, we will briefly present the theory of the stability of differential equations with functional delay of the neutral type. In a second step we present two examples of delayed differential equations, the first being linear and the other nonlinear. These examples are part of the problems that have resisted the study by Lyapunov's method, supports fully nonlinear-type integro-differential equations not covered here. In addition, our fixed point method clearly inproves the previous results,particularly those of Becker and Burton [27],[23],[25]with far fewer restrictions and with more rigor. The fixed point method used for stability purposes has been used in a number of recent works like [16],[6],[23],[20],[33]. This method is based on three fundamental elements. Remember that the method is based on three fundamental elements :

- ■ set M consisting of points which would be acceptable solutions ;
- ■ a mapping $P : M \to M$ with the property that a fixed point solves the problem ;
- ■ a fixed-point theorem stating that this mapping on this set will have a fixed point.

Chapter I

Preliminaries

Sommaire

In this section we discus compact sets. The following elements have been gathered from several analysis and some specialized books on the functional analysis and the most is in the following bibliography ([22],[2],[7])

I.1 Norm and mertic

Definition I.1.1 *(Metric) Let X be a nonempty set and $\rho : X \times X \to [0, \infty)$ a function. Then ρ is called a metric on X if the following properties hold*

i $\rho(x, y) = 0$ *if and only of* $x = y$, *for all* $x, y \in X$,
ii $\rho(x, y) = \rho(y, x)$ *for all* $x, y \in X$,
iii $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ *for all* $x, y, z \in X$.

The value of metric ρ at (x, y) is called distance between x and y, and the ordered pair (X, ρ) is called metric space.

Definition I.1.2 *(Norm) Let $(X, +, .)$ be a linear space over a field $\Bbbk$ ($\mathbb{R}$ or $\mathbb{C}$) and $N : X \to [0, +\infty)$ a function. Then, N is said to be a norm if the following properties hold*

i $N(x) = 0$ *if and only if* $x = 0$,
ii $N(\lambda x) = |\lambda| N(x)$ *for all* $x \in X$ *and* $\lambda \in \Bbbk$,
iii $N(x + y) \leq N(x) + N(y)$ *for all* $x, y \in X$.

The ordered pair (X, N) is called a normed space. We use the notation $\|\cdot\|$ for norm. Then every normed space $(X, \|\cdot\|)$ is a vector space and it is a metric space (X, ρ) with induced metric $\rho(x, y) = \|x - y\|$. But a vector space with a metric is not always a normed space.

Definition I.1.3 *Let X be a nonempty set and ρ is a metric on X, a sequence $\{x_n\} \subset X$ is a Cauchy sequence if for each $\epsilon > 0$ there exists $n_0 \in N$ such that $\rho(x_n, x_m) < \epsilon$ for all $n, m > n_0$, i.e $\lim_{n,m\to+\infty} \rho(x_n, x_m) = 0$.*

Definition I.1.4 *The metric space (X, ρ) is complete if every Cauchy sequence in (X, ρ) has a limit in that space.*

Definition I.1.5 *A Banach space is a complete normed space. We often say a complete normed vector space.*

Theorem I.1.1 *A closed subspace of a Banach space is a Banach space.*

Example I.1.1 *The linear space $C([a, b])$ of continuous functions on the closed and bounded interval $[a, b]$ is a Banach space with the uniform convergence norm*

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|.$$

Definition I.1.6 *Let (X, ρ) be a metric space. Recal that a subset Ω of X is called compact if every open cover of Ω has a finite subcover. Equivalently, a subset Ω of X is compact if every sequence in Ω contains a convergent subsequence with a limit in Ω.*

Example I.1.2 *Let $\Phi : [a, b] \to \mathbb{R}^n$ be continuous and let X the set of continuous functions $f : [a, c] \to \mathbb{R}^n$ with $c > b$ and $f(t) = \Phi(t)$for $a \le t \le b$. Define $\rho(f, g) = \sup_{a\le t\le c} |f(t) - g(t)|$ for $f, g \in X$. Then (X, ρ) is complete metric space but not a Banach space because $f + g$ is not in X.*

Definition I.1.7 *Let (X, ρ) be a metric space. A subset Ω of X is said to be totally bounded if for each $\epsilon > 0$, there exists a finite number of elements $x_1, x_2, ..., x_n$ in X such that $\Omega \subseteq \cup_{i=1}^n \mathbf{B}_\epsilon(x_i)$.*

Proposition I.1.1 *Let (X, ρ) be a metric space. Then the following are equivalent*
i) X is compact.
ii) Every sequence in X has a convergent subsequence.
iii) X is complete and totally bounded.

Definition I.1.8 *A set Ω in a metric space (X, ρ) is relatively compact if its closure is compact, i.e. $\overline{\Omega}$ is compact.*

Proposition I.1.2 *Let Ω be a closed subset of a complete metric space (X, ρ). Then Ω is compact if and only if it is relatively compact.*

Definition I.1.9 *Let X be a compact metric space and Ω be a subset of $C(X)$.*
* *Ω is uniformly bounded if there exists $M > 0$ such that $\|u\| \le M$ for all $u \in \Omega$.*
* *Ω is equicontinuous if for any $\epsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in X$ and $\rho(t_1, t_2) < \delta$ imply $|u(t_1) - u(t_2)| < \epsilon$ for all $u \in \Omega$.*

The following result gives the main method of proving compactness in the spaces in which we are interested.

Theorem I.1.2 *(Ascoli-Arzela [7]) Let X be a compact metric space. If Ω is an equicontinuous, uniformly bounded subset of $C(X)$, then Ω is relatively compact.*

Definition I.1.10 *Let S be a mapping from a metric space (X, ρ) into another metric space (Y, ρ'). Then S is said to satisfy Lipschitz condition on X if there existe a constant $L > 0$ such that*

$$\rho'(Sx, Sy) \leq L\rho(x, y) \text{ for all } x, y \in X$$

The following proposition guarantees the existence of Lipschitzian mappings.

Proposition I.1.3 *Let $S : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function on (a, b). Suppose S is differentiable on $[a, b]$. Then, S is a Lipschitz continuous function (and hence is uniformly continuous). Let X and Y be two Banach spaces and let the mapping $S : X \to X$. Then the mapping S is said to be*

1) *S is said to be bounded if Ω is bounded in X implies $S(\Omega)$ is bounded.*
2) *locally bounded if each point in X has a bounded neighborhood V such that $S(V)$ is bounded;*
3) *S is said to be closed if $x_n \to x$ in X and $Sx_n \to y$ in Y imply $Sx = y$.*
4) *S is said to be compact if Ω is bounded in X implies $S(\Omega)$ is relatively compact ($\overline{S(\Omega)}$ is compact), i.e. for every bounded sequence $\{x_n\}$ in X, $\{Sx_n\}$ has convergent subsequence in Y.*
5) *S is said to be completely continuous if it is continuous and compact. In the case of linear mappings, the concepts of continuity and boundedness are equivalent, but it is not true in general*

I.2 Fixed point theorems

In this section we state some fixed point theorems that we employ to help us in proving existence and stability of solutions(see [21],[13],[23],[22],[15],[5])

Definition I.2.1 *Let S be a mapping in the set Ω. we call fixed point of S any point x satisfying $S(x) = x$. If there exists such x, we say that S has a fixed point, which is equivalent to saying that the equation $S(x) - x = 0$ has a solution. Fixed point theorems guarantee the existence of a fixed point under appropriate conditions on the map S and the set Ω.*

I.2.1 Banach fixed point theorem

Let the classical Cauchy problem on existence and uniqueness of the solution to the differential equation satisfying a given initial condition,

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases} \tag{I.2.1}$$

can be expressed as an integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s; x(s))ds, \tag{I.2.2}$$

from which a sequence of functions $\{x_n\}$ may be inductively defined by

$$x_0(t) = x_0; x_1(t) = x_0 + \int_{t_0}^{t} f(s; x_0)ds$$

and, in general

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s; x_n(s))ds : \tag{I.2.3}$$

This is called Picard method of successive approximations and, under liberal conditions on f , one can show that $\{x_n\}$ converges uniformly on some interval $|t - t_0| \leq k$ to some continuous function, say x. Taking the limit in the equation de ning x_{n+1}, we pass the limit through the integral and have

$$x(t) = x_0 + \int_{t_0}^{t} f(s; x(s))ds;$$

so that $x(t_0) = x_0$ and, upon differentiation, we obtain $x_0(t) = f(t; x(t))$. Thus, x is asolution of the initial value problem. Banach realized that this was actually a fixed-point theorem with wide application. For if we de ne an operator S on a complete metric space $C([t_0; t_0 + k]; \mathbb{R})$ with the supremum norm $\|.\|$ by $x \in C$ implies

$$(Sx)(t) = x_0 + \int_{t_0}^{t} f(s; x(s))ds; \tag{I.2.4}$$

then a fixed point of S , say $S\phi = \phi$, is a solution of the initial value problem. The idea had two outstanding features. First, it had application to problems in every area of mathematics which used complete metric spaces. And it was clean. For example, the standard muddy and shaky proofs of implicit function theorems became clear and solid using the fixed-point theory. We will use it here to prove existence of solutions of various kinds of differential equations.

Theorem I.2.1 *(Contraction mapping principle [5]) Let $(X; \rho)$ be a complete metric space and $S : X \to X$ a contraction operator. Then there is a unique point $x \in X$ with $Sx = x$. Furthermore, if $y \in X$ and if $\{y_n\}$ is defined inductively by $y_1 = Sy$ and $y_{n+1} = Sy_n$, then $y_n \to x$, the unique fixed point. In particular, the equation $Sx = x$ has one and only*

one solution. In applying this result to I.2.1, a distressing event occurred which we now briey describe. Assume that f is continuous and satisfies a global Lipschitz condition in x, say

$$|f(t, x_1) - f(t, x_2)|\, |x_1 - x_2|\, ;$$

for $t \in \mathbb{R}$ and $x_1; x_2 \in \mathbb{R}^n$. Then by I.2.4 we obtain (for $t \geq t_0$)

$$\begin{aligned} |Sx_1(t) - Sx_2(t)| &= \left| \int_{t_0}^{t} f(s, x_1(s)) - f(s, x_2(s))ds \right| \\ &\leq \int_{t_0}^{t} \alpha\, |x_1(s) - x_2(s)|\, ds, \end{aligned}$$

so that if $\|.\|$ is the sup norm on continuous functions on $[t_0; t_0 + k]$, then

$$\|Sx_1 - Sx_2\| \leq \alpha k\, \|x_1 - x_2\|$$

This is a contraction if $\alpha k = \lambda < 1$. Now is fixed and we take k small enough that $\alpha k < 1$. This gives a fixed point which is a solution of I.2.1 on $[t_0; t_0 + k]$.

I.2.2 Schauder's theorem

Definition I.2.2 *Let X be a normed vector space and $F = \{x_1, x_2, ..., x_n\}$ a finite subset of X. Then conv(F), the convex hull of F, is defined by*

$$Conv(F) = \left\{ \sum_{j=1}^{n} t_j x_j = 1 : \sum_{j=1}^{n} t_j = 1,\ t_j \geq 0) \right\}$$

For future applications, we will need a more general definition to handle the case in which F is infinite

Theorem I.2.2 *(Schauder's fixed point theorem [5]) Let Ω be a closed, convex and nonempty subset of a Banach space X. Let $S : \Omega \to \Omega$ be a continuous mapping such that $S\Omega$ is a relatively compact subset of X. Then S has at least one fixed point in Ω. That is there exists an $x \in \Omega$ such that $Sx = x$*

I.2.3 Krasnoselskii's fixed point theorem

In 1955 Krasnoselskii's observed that in a good number of problems, the integration of a perturbed differential operator gives rise to a sum of two applications, ($S\varphi := A\varphi + B\varphi$) a contraction and a compact application. It declares then Principle : the integration of a perturbed differential operator can produce a sum of two applications, a contraction and a compact operator. Krasnoselskii's found the solution by combining the two theorems of Banach and that of Schauder in one hybrid theorem which bears its name. In light, it establishes the following result .

Theorem I.2.3 *(Krasnoselskii's fixed point theorem [5]) Let Ω be a closed bounded convex nonempty subset of a Banach space $(X, \|\ .\ \|)$. Suppose that A and B map Ω into X such that*

(i) A is a contraction mapping,
(ii) B is compact and continuous,
(iii) $x, y \in \Omega$, implies $Ax + By \in \Omega$, Then there exists $z \in \Omega$ with $z = Az + Bz$.

Remark I.2.1 *Note that if $B = 0$, the theorem becomes the theorem of Banach. If $A = 0$, then the theorem is not other than the theorem of Schauder. We first state the well-known fixed point theorem in cones [15],[6]. For the proof, we refer to the classical works [20],[27].*

Theorem I.2.4 *Krasnoselskii [15]) Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{u \in K : \|u\| < r\}$ Assume that $T : \bar{K_r} \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K : \|u\| = r\}$*

1) $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then $i\,(T, K_r, K) = 0$;
2) $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then $i\,(T, K_r, K) = 1$.

I.3 Delay differential equations

Delayed problems are innumerable in literature. Some of these problems have been of particular interest. We have chosen in this some models intervening in different real world domains (see [18],[10],[11],[32]).

I.3.1 Definition of delay differential equations

For $\tau \geq 0$ a given real number, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|\ .\ |, C([a,b], \mathbb{R}^n)$ is the Banach space of continuous functions mapping in the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. If $[a,b] = [-\tau, 0]$ we let $C = C([-\tau, 0], \mathbb{R}^n)$ and designate the norm of an element ψ in C by $\|\psi\| = \sup_{-\tau<s<0} ||\psi(s)||$. Even though single bars are used for norms in different spaces, no confusion should arise. If $t_0 \in \mathbb{R}$, $A \geq 0$ and $x \in C([t_0 - \tau, t_0 + A], \mathbb{R}^n)$, then for any $t \in [t_0, t_0 + A]$, we let $x_t \in C$ be defined by $x_t(s) = x(t+s), -\tau \leq s \leq 0$.

Definition I.3.1 *If Ω is a subset of $\mathbb{R} \times C$, $f : \Omega \to \mathbb{R}^n$ is a given function and represents the right-hand derivative, we say that the relation*

$$x'(t) = f(t, x_t), \tag{I.3.1}$$

where

$$x_t(s) = x(t+s), -\tau \leq s \leq 0,$$

is a delay differential equation on Ω and will denote this equation by DDE. A function x is said to be a solution of Equation (I.3.1) on $[t_0 - \tau, t_0 + A)$ if there are $t_0 \in \mathbb{R}$ and $A > 0$ such that $x \in C([t_0 - \tau, t + A], \mathbb{R}^n)$, $(t, x_t) \in \Omega$ and $x(t)$ satisfies Equation (I.3.1) for $t \in [t_0, t_0 + A)$. For given $t_0 \in \mathbb{R}, \psi \in C$, we say $x(t, t_0, \psi)$ is a solution of Equation (I.3.1) with initial value ψ at t_0 or simply a solution through (t_0, ψ) if there is an $A > 0$ such that $x(t, t_0, \psi)$ is a solution of Equation (I.3.1) on $[t_0 - \tau, t_0 + A)$ and $x_{t_0}(t, t_0, \psi) = \psi$.

If $\tau = 0$, Equation (I.3.1) is a very general type of equation and includes ordinary differential equations.

If $f(t, \psi) = L(t, \psi) + h(t)$ where $L(t, \psi)$ is linear in ψ, we say Equation (I.3.1) is linear and is homogeneous if $h \equiv 0$ and nonhomogeneous if $h \neq 0$.

If $f(t, \psi) = g(\psi)$ where g does not depend on t, we claim Equation (I.3.1)is autonomous.

Example I.3.1 *The following equations are delay differential equations*

$$x'(t) = 3x(t) + 2x(t-5), \tag{I.3.2}$$

$$x'(t) = a(t)x(t) + b(t)x'(t - \tau(t)) + h(t), \tag{I.3.3}$$

$$x'(t) = \int_{-\tau}^{0} x(t+s)ds, \tag{I.3.4}$$

where a, b, τ are continuous functions. Equation (I.3.2) is an linear autonomous delay differential equation with constant $\tau = 5$. Equation (I.3.3) is nonhomogeneous, linear nonautonomous delay functional differential equations and Equation (I.3.4) is a delay linear integro-differential equation. If $t_0 \in \mathbb{R}$, $\psi \in C$ are given and $f(t, \psi)$ is continuous, then finding a solution of Equation(I.3.1) through (t_0, ψ) is equivalent to solving the integral equation

$$x(t_0) = \psi, \tag{I.3.5}$$

$$x(t) = \psi(0) + \int_{t_0}^{t} f(s, x_s)ds, t \geq t_0.$$

We define Sx by

$$(Sx)(t) = \psi(0) + \int_{t_0}^{t} f(s, x_s)ds, t \geq t_0, \quad x = \psi \text{ on } [t_0 - \tau, t_0]$$

To prove the existence of the solution through a point $(t_0, \psi) \in \mathbb{R} \times C$, we consider an $\eta > 0$ and all functions x on $[t_0 - \tau, t_0 + A]$ which are continuous and coincide with ψ on $[t_0 - \tau, t_0]$, that is, $xt_0 = \psi$. The values of these functions on $[t_0, t_0 + \eta]$ are restricted to the class of x such that $|x(t) - \psi(0)| < \delta$ or $t \in [t_0, t_0 + \eta]$. The usual mapping S obtained from the corresponding integral equation is defined and it is then shown that η and δ can be so chosen that S maps this class into itself and is completely continuous. Thus, Schauder's fixed-point theorem implies existence.(for exemple details see the books [10],[11],[32]).

I.3.2 Existence and Uniqueness of the solutions

Theorem I.3.1 *(**Existence [11]**) In (I.3.1), suppose Ω is an open subset in $\mathbb{R} \times C$ and f is continuous on Ω. If $(t_0, \psi) \in \Omega$, then there is a solution of (I.3.1) passing through (t_0, ψ). We say that $f(t, \psi)$ is Lipschitz in ψ in a compact set X of $\mathbb{R} \times C$ if there is a constant $k > 0$ such that, for any $(t, \psi_i) \in K$, $i = 1, 2$,*

$$|f(t, \psi_1) - f(t, \psi_2)| \leq k\, |\psi_1 - \psi_2|$$

***Theorem I.3.2 (Uniqueness [11])** Suppose Ω is an open set in $\mathbb{R} \times C$, $f : \Omega \to \mathbb{R}^n$ is continuous, and $f(t, \psi)$ is Lipschitz in ψ in each compact set in Ω. If $(t_0, \psi) \in \Omega$ then there is a unique solution of (I.3.1) through (t_0, ψ).*

I.3.3 Neutral functional differential equations

In order to define a general class of neutral delay differential equations NDDEs (or neutral functional differential equations NFDEs), we need the definition of atomic.

Definition I.3.2 *Suppose $\Omega \subseteq \mathbb{R} \times C$ is open with elements (t, ψ). A function $\Psi : \Omega \to \mathbb{R}^n$ is said to be atomic at β on Ω if Ψ is continuous together with its first and second Fréchet derivatives with respect to ψ and $\Psi\psi$ the derivative with respect to ψ, is atomic at β on Ω.*

Definition I.3.3 *Suppose $\Omega \subseteq \mathbb{R} \times C$ is open, $f : \Omega \to \mathbb{R}^n$, $\Psi : \Omega \to \mathbb{R}^n$ are given continuous functions with Ψ atomic at zero. The equation*

$$\frac{d}{dt}\Psi(t, x_t) = f(t, x_t), \tag{I.3.6}$$

is called the neutral delay differential equation $NDDE(\Psi, f)$.

Definition I.3.4 *A function x is said to be a solution of the $NDDE(\Psi, f)$ or equation (I.3.6), if there are $t_0 \in \mathbb{R}$, $A > 0$, such that $x \in C([t_0 - \tau, t_0 + A), \mathbb{R}^n)$, $(t, x_t) \in \Omega$, $t \in [t_0, t_0 + A), \Psi(t, x_t)$ is continuously differentiable and satisfies (I.3.6) on $[t_0, t_0 + A)$. For a given $t_0 \in \mathbb{R}, \psi \in C$, and $(t_0, \psi) \in \Omega$, we say $x(t_0, \psi)$ is a solution of (I.3.6) with initial value ψ at t_0, or simply a solution through (t_0, ψ) ,if there is an $A > 0$ such that $x(t_0, \psi)$, is a solution of (I.3.6) on $[t_0 - \tau, t_0 + A)$ and $x_{t_0}(t_0, \psi) = \psi$.*

Theorem I.3.3 *(Existence [11]) If Ω is an open set in $\mathbb{R} \times C$ and $(t_0, \psi) \in \Omega$, then there exists a solution of the $NDDE(\Psi, f)$through (t_0, ψ).*

Theorem I.3.4 *(Uniqueness [11]) :If $\Omega \subseteq \mathbb{R} \times C$ is open and $f : \Omega \to \mathbb{R}^n$ as Lipschitz in ψ on compact sets of Ω, then, for any $(t_0, \psi) \in \Omega$, there exists a unique solution of the $NDDE(\Psi, f)$ through (t_0, ψ).*

Example I.3.2 *The equations*

$$x'(t) = 3x'(t-5),$$

$$x'(t) = -x(t) + [x'(t-4)+1]^2,$$
$$x'(t) = x(t) + x'(t-2) - x'(t-7),$$

are neutral differential equations.

I.4 Stability of delay differential equations

The theory of stability was created by the Russian mathematician Lyapunov (1857-1918). This theory has found wide application in various fields of physics and mathematical sciences. From a mathematical point of view, the theory of stability presents a particular case of the qualitative theory of differential equations. Lyapunov method was the ultimate object for studying stability for differential equations and partial differential equations. Nevertheless, this method has encountered serious obstacles and there are still a lot of problems that resist this technique. The simplest notion of stability is the one related to stability of equilibrium points.

Definition I.4.1 *A point $x(t) = x_e$ in the state space is said to be an equilibrium point of the autonomous system $x' = f(x)$ if and only if it has the property that when ever the state of the system starts at x_e, it remains at x_e for all future time.According to the definition, the equilibrium points are the real roots of the equation $f(x_e) = 0$. This is made clear by noting that if $x'_e = f(x_e) = 0$, then it follows that x_e is constant and, by definition, an equilibrium point. Without loss of generality, we assume that 0 is an equilibrium point of the system. If the equilibrium point under study, x_e, is not at zero we may define a new (shifted) coordinate system $x_s(t) = x(t) - x_e$ and note that*

$$x'_s(t) = x'(t) = f(x(t)) = f(x_s(t) + x_e) =: f_s(x_s(t)), x_s(0) = x(0) - x_e.$$

The claim follows by noting that $f_s(0) = f(x_e) = 0$. In summary, the study of the zero equilibrium point of $x\prime_s(t) = f_s(x_s(t))$ is equivalent to the study of the nonzero equilibrium point xe of $x'(t) = f(x(t))$. We consider the system

$$x' = f(t, xt) \text{ for all } t \geq t_0, \tag{I.4.1}$$

$$x(t) = \psi(t) \text{ for all } t_0 - \tau \leq t \leq t_0, \tag{I.4.2}$$

where $f : (-\infty, +\infty) \times C \to \mathbb{R}^n$, with $C = C([-\tau, 0], \mathbb{R}^n)$ the Banach space of continuous functions $\psi : [t_0 - \tau, t_0] \to \mathbb{R}^n, \tau > 0$ equipped with the supremum norm $\|\psi\| = \sup_{-\tau \leq t \leq 0} |\psi(t)|$. We suppose that f is continuous and is supposed to satisfy all the conditions which guarantee a solution $x(t, t_0, \psi)$ through of the problem (I.4.1)-(I.4.2) and to be continuous in (t, t_0, ψ) of the definition domain of f (see [9])

Definition I.4.2 *Suppose that $f(t,0)=0$ for all $t\in\mathbb{R}$.*

1) *The trivial solution $x(t)=0$ of (I.4.1) is Stable in $t_0(t_0\in\mathbb{R})$, if for every $\epsilon>0$ there exists a $\delta=\delta(\epsilon,t_0)>0$ such that if $\|\psi\|<\delta$, the solution of (I.4.1)-(I.4.2) exists on $[t_0-\tau,\infty)$ and $|x(t,t_0,\psi)|<\epsilon$ for all $t\geq t_0$. Otherwise we will say that the solution is unstable in t_0.*
2) *The solution $x=0$ of (I.4.1) is said to be uniformly stable if the number $\delta=\delta(\epsilon)$ is independent of t_0.*
3) *The trivial solution $x(t)=0$ of (I.4.1) is Asymptotically stable in t_0 if it is stable in t_0 and if there exists $\delta_1=\delta_1(t_0)>0$ such that whenever all $\|\psi\|<\delta_1$, the solution of the problem (I.4.1)-(I.4.2) satisfies*

$$|x(t,t_1,\psi)|\to 0 \text{ as } t\to\infty$$

Definition I.4.3 *4) The trivial solution $x(t)=0$ of (I.4.1) is Asymptotically uniformly stable in t_0 if it is uniformly stable and if there exist $\delta_1>0$ (independent of t_0) as for all t_0 and $\|\psi\|<\delta_1$ the solution x of the problem (I.4.1)-(I.4.2) satisfies the condition $x(t,t_0,\psi)\to 0$ when $t\to\infty$, in the following way : for all $\eta>0$ there is a $T=T(\eta)>0$ such that $|x(t,t_0,\psi)|<\eta$ for all $t\geq t_0+T$.*

Remark I.4.1 *If all the solutions tend to zero, then $x=0$ is globally asymptotically stable.*

Example I.4.1 *:Consider, for $t\geq 1$, the differential delay equation*

$$x'=\frac{1}{t}x(t)-\frac{27t}{(t+2)^3}x^3\left(\frac{t+2}{3}\right)=\frac{1}{t}x\left(t\right)-\frac{27t}{(t+2)^3}x^3(t-\tau(t))$$

with $\tau(t)=\frac{2}{3}(t-1)$, with the initial condition $x(1)=x_0$. We easily check that the unique solution of this problem is $x(t)=x_0t\exp\left(-x_0^2\left(t-1\right)\right)$, for all $t\geq 1$. Thus, $\lim_{t\to\infty}x(t)=0$, for all x_0. Suppose that $|x_0|=\delta$, then

$$\left|x(1+\frac{1}{\delta^2})\right|=\frac{1}{e}\left(\delta+\frac{1}{\delta}\right)\geq\frac{2}{e}.$$

Therefore, for all δ the solution is outside the bali $|x|<\frac{2}{e}$, in times $t=1+\frac{1}{\delta^2}$, and the solution is unstable.

I.5 Stability by method of the fixed point

When one wants to study the stability of the trivial solution of a differential equation with delay by the method of fixed point one will have to proceed as follows

(1) A delay differential equation requires primarily a an initial function defined on an appropriate initial interval I_{t_0} i.e. $\psi: I_{t_0}\to\mathbb{R}^n$. We must fall immediately after a suitable space C of functions $\varphi: I_{t_0}\cup[t_0,+\infty)\to\mathbb{R}^n$ which coincide on I_{t_0} with ψ. According to the case of needs we can always add other restrictions to the functions φ of C such as the boundary or the condition $\varphi(t)\to\infty$ when $t\to\infty$. This last condition is necessary if we wish to study asymptotic stability.

(2) Then we have to invert the differential equation to define what we call a fixed point application i.e., a mapping $S : C \to C$ whose fixed point is the solution of the given delay equation (the original equation).Nevertheless, this inversion can be a delicate task in many cases. For example if the equation does not have a linear term in its structure we will not be able to use the variation of the parameters. It is therefore essential to act differently and to try if a trans formation of this equation is possible.

(3) A fixed point theorem must be chosen allowing the equation $S(x) = x$ to have a solution. Specially if S is a contraction we can apply the Banach fixed point theorem, if S is compact then we will apply the theorem of Schauder or Schaeffer and if S is puts in the form of a sum of a contraction and a compact application then the Krasnoselski ihybrid theorem can give satisfaction. It thus becomes clear that the stability method by the fixed point method relies on three essential things, the variation of the parameters,a complete space and a fixed point theorem.n one stage we can conclude the existence (or even uniqueness) and stability.n addition, it will be seen that this method always requires conditions on average however the conditions of the Lyapunov method are always punctual.

Chapter II

Stability asymptotic of delay integro-differential equation

Sommaire

In this chapter we study a linear neutral integro-differential equation with variable delays (II.0.1) and give suitable conditions to obtain asymptotic stability of the zero solution, by means of fixed point technique. To do this we define a suitable continuous function H and find conditions for H, with no need of further assumptions on the inverse of delays $t-\tau_j(t)$, so that for a given continuous initial function ψ a mapping P for (II.0.1) is constructed in such a way to map a complete metric space S_ψ into itself and in which P possesses a fixed point. This procedure will enable us to establish and prove an asymptotic stability theorem for the zero solution of (II.0.1) with a necessary and sufficient condition and with less restrictive conditions.

We consider the following linear neutral integro-differential equation with variable delays

$$x'(t) = -\sum_{j=1}^{n} \int_{t-\tau_j(t)}^{t} a_j(t,s)x(s)ds + \sum c_j(t)x'(t-\tau_j(t)) \tag{II.0.1}$$

with the initial condition

$$x(t) = \varphi(t) \text{ for } t \in [m(0), 0] \tag{II.0.2}$$

where $\varphi \in \mathcal{C}([m(0), 0)], \mathbb{R})$ and

$$m_j(0) = \inf\{t - \tau_j(t), t \geq t \geq\}, m(0) = \min\{m_j(0), 1 \leq j \leq N\}$$

With $a_j \in \mathcal{C}(\mathbb{R}^+ \times [m(0); +\infty); \mathbb{R})$, $c_j \in \mathcal{C}^1(\mathbb{R}^+; \mathbb{R})$, and $\tau_j \in \mathcal{C}(\mathbb{R}^+\mathbb{R}^+)$ with $t-\tau_j(t) \to +\infty$ as $t \to \infty$ for $j = 1...N$. Special cases of equation (II.0.1) have been investigated by many authors. For example, Burton in [26], Becker and Burton in [14], Jin and Luo in [4] studied the equation

$$x'(t) = -\int_{t-\tau(t)}^{t} a(t,s)x(s)ds.$$

It is known that studying the stability of an equation using a fixed point technic involves the construction of a suitable fixed point mapping. This can be an arduous task. So, to construct our mapping, we begin by transforming (II.0.1) to a more tractable, but equivalent, equation, which we then invert to obtain an equivalent integral equation from which we derive a fixed point mapping. After that, we define a suitable complete space, depending on the initial condition, so that the mapping is a contraction. Using Banach's contraction mapping principle, we obtain a solution for this mapping, and hence a solution for (II.0.1), which is asymptotically stable.

II.1 Inversion and transformation of the equation

First, we have to trans form (II.0.1) into an equivalent equation that possesses the same basic structure and properties to which we apply the variation of parameters to define a fixed point mapping.

Lemma II.1.1 *Equation (II.0.1) is equivalent to*

$$\begin{aligned} x'(t) &= \sum_{j=1}^{N} B_j(t, t-\tau_j(t))\left(1-\tau_j'(t)\right) x(t-\tau_j(t)) \\ &\quad + \sum_{j=1}^{N} \frac{d}{dt}\int_{t-\tau_j(t)}^{t} B_j(t,s)\, x(s)\, ds + \sum_{j=1}^{N} c_j(t) x'(t-\tau_j(t)) \end{aligned} \tag{II.1.1}$$

where

$$B_j(t,s) = \int_t^s a_j(u,s)\, du \text{ and } B_j(t, t-\tau_j(t)) = \int_t^{t-\tau_j(t)} a_j(u, t-\tau_j(t))\, du$$

Proof: Differentiating the following integral,

$$\frac{d}{dt}\int_{t-\tau_j(t)}^{t} B_j(t,s)\, x(s)\, ds$$

becomes

$$\begin{aligned} &\frac{d}{dt}\int_{t-\tau_j(t)}^{t} B_j(t,s)\, x(s)\, ds \\ = &\; B_j(t,t)\, x(t) - B_j(t, t-\tau_j(t))\left(1-\tau_j'(t)\right) x(t-\tau_j(t)) + \int_{t-\tau_j(t)}^{t} \frac{\partial}{\partial t} B_j(t,s)\, x(s)\, ds \end{aligned}$$

Substituting this into (II.1.1),it follows that (II.1.1) is equivalent to (II.0.1) provided B_j satisfies the following conditions

$$B_j(t,t) = 0 \text{ and } \frac{\partial}{\partial t} B_j(t,s) = -a_j(t,s) \tag{II.1.2}$$

This equality implies

$$B_j(t,t) = -\int_0^t a_j(t,t)\, dt + \phi(t), \tag{II.1.3}$$

for some function ϕ, and $B_j(t,s)$ must satisf

$$B_j(t,s) = -\int_0^t a_j(u,t)\, du + \phi(t) = 0.$$

Consequently

$$\phi(t) = \int_0^t a_j\,(u,t)\,du.$$

Substituting this into(II.1.3), we obtain

$$B_j\,(t,s) = -\int_0^t a_j\,(u,s)\,du + \int_0^s a_j\,(u,t)\,du = \int_t^s a_j\,(u,s)\,du$$

This definition of B_j satisfies (II.1.2). Consequently,(II.0.1) is equivalent to(II.1.1) ■

Lemma II.1.2 *Suppose that τ_j is twice differentiable and $\tau_j'(t) \neq 1$ for all $t \in \mathbb{R}^+$, and there exist continuous functions $h_j : [m_j(0), \infty) \to \mathbb{R}$ for , $j = 1..N$. Then, x is a solution of equation (II.0.1)-(II.0.2) if and only if x is a solution of the following integral equation.*

$$\begin{aligned} x(t) \;&=\; \left(\psi(0) - \sum_{j=1}^{N} \frac{c_j(0)}{1-\tau_j'(0)}\psi(-\tau_j(0)) - \sum_{j=1}^{N}\int_{-\tau_j(0)}^{0} \left[h_j(s) + B_j(0,s)\right]\psi(s)ds\right) e^{-\int_0^t H(u)du} \\ &+ \sum_{j=1}^{N} \frac{c_j(t)}{1-\tau_j'(t)} x(t-\tau_j(t)) + \sum_{j=1}^{N}\int_{t-\tau_j(t)}^{t} \left[h_j(s) + B_j(t,s)\right]x(s)ds \\ &+ \sum_{j=1}^{N}\int_0^t e^{-\int_s^t H(u)du}\left\{\left[h_j(s-\tau_j(s)) + B_j(s,s-\tau_j(s))\right](1-\tau_j'(s)) - r_j(s)\right\} x(s-\tau_j(s))ds \\ &- \sum_{j=1}^{N}\int_0^t e^{-\int_s^t H(u)du} H(s)\left(\int_{s-\tau_j(s)}^{s} \left[h_j(u) + B_j(s,u)\right]x(u)du\right)ds \end{aligned}$$

Where

$$H(t) = \sum_{j=1}^{N} h_j(t)\ ,\ r_j(t) = \frac{[c_j(t)H(t) + c_j'(t))](1-\tau'(t)) + c_j(t)\tau''_j(t)}{(1-\tau_j'(t))^2}$$

and

$$B(t,s) = \int_t^s a_j(u,s)du \ \textit{with}\ B(t,t-\tau_j(t)) = \int_t^{t-\tau_j(t)} a_j(u,t-\tau_j(t))du.$$

Proof: Rewrite (II.0.1) in the following equivalent form

$$\begin{aligned} x'(t) \;&=\; B_j(t,t-\tau_j(t))(1-\tau_j'(s))x(t-\tau_j(t)) \\ &+ \sum_{j=1}^{N} \frac{d}{dt}\int_{t-\tau_j(t)}^{t} B_j(t,s)x(s)ds + \sum_{j=1}^{N} c_j(t)x'(t-\tau_j(t)) \end{aligned} \qquad \text{(II.1.4)}$$

Multiplying both sides of(II.1.4) by $e^{\int_0^t H(u)du}$ and integrating with respect from 0 to t, we obtain

$$\begin{aligned} x(t) &= \psi(0)e^{-\int_0^t H(u)du} + \int_0^t e^{-\int_s^t H'(u)du}\sum_{j=1}^N h_j(s)x(s)ds \\ &+ \int_0^t e^{-\int_s^t H(u)du}\sum_{j=1}^N \frac{d}{ds}\int_{s-\tau_j(s)}^t B_j(s,u)x(u)du \\ &+ \int_0^t e^{-\int_s^t H(u)du}\sum_{j=1}^N B_j(s,s-\tau_j(s))(1-\tau_j'(s))x(s-\tau_j(s))ds \\ &+ \int_0^t e^{-\int_s^t H(u)du}\sum_{j=1}^N c_j(s)x'(s-\tau_j(s))ds. \end{aligned}$$

Thus

$$\begin{aligned} x(t) &= \psi(0)e^{-\int_0^t H(u)du} + \sum_{j=1}^N\int_0^t e^{-\int_s^t H(u)du} h_j(s)x(s)ds \\ &+ \sum_{j=1}^N\int_0^t e^{-\int_s^t H(u)du}\frac{d}{ds}\int_{s-\tau_j(s)}^s B_j(s,u)x(u)du \\ &+ \sum_{j=1}^N\int_0^t e^{-\int_s^t H(u)du} B_j(s,s-\tau_j(s))(1-\tau_j'(s))x(s-\tau_j(s))ds \\ &+ \sum_{j=1}^N\int_0^t e^{-\int_s^t H(u)du} c_j(s)x'(s-\tau_j(s))ds. \end{aligned}$$

Performing an integration by parts, we obtain

$$\begin{aligned} x(t) &= \psi(0)e^{-\int_0^t H(u)du} + \sum_{j=1}^N\int_0^t e^{-\int_s^t H(u)du} d\left(\int_{s-\tau_j(s)}^s \left[h_j(u)+B_j(s,u)\right]x(u)du\right) \\ &+ \sum_{j=1}^N\int_0^t e^{-\int_s^t H(u)du}\left[h_j(s-\tau_j(s))+B_j(s,s-\tau_j(s))\right]\left(1-\tau_j'(s))x(s-\tau_j(s)\right)ds \\ &+ \sum_{j=1}^N\int_0^t \frac{c_j(s)}{1-\tau_j'(s)}e^{-\int_s^t H(u)du}dx(s-\tau_j(s)) \\ &= \left(\psi(0)-\sum_{j=1}^N\frac{c_j(0)}{1-\tau_j'(0)}\psi(-\tau_j(0))-\sum_{j=1}^N\int_{-\tau_j(0)}^0\left[h_j(s)+B_j(0,s)\right]\psi(s)ds\right)e^{-\int_0^t H(u)du} \\ &+ \sum_{j=1}^N\frac{c_j(t)}{1-\tau_j'(t)}x(t-\tau_j(t)) + \sum_{j=1}^N\int_{t-\tau_j(t)}^t\left[h_j(s)+B_j(t,s)\right]x(s)ds \\ &+ \sum_{j=1}^N\int_0^t e^{-\int_s^t H(u)du}\left\{\left[h_j(s-\tau_j(s))+B_j(s,s-\tau_j(s))\right](1-\tau_j'(s))-r_j(s)\right\}x(s-\tau_j(s))ds \\ &- \sum_{j=1}^N\int_0^t e^{-\int_s^t H(u)du}H(s)\left(\int_{s-\tau_j(s)}^s\left[h_j(u)+B_j(s,u)\right]x(u)du\right)ds \end{aligned} \tag{II.1.5}$$

■

II.2 Boundedness and stability of the solutions

Next, to obtain stability of the zero solution of (II.0.1), we will define a mapping directly from (II.1.5) to map bounded functions into bounded functions. For that, we let $(\mathcal{C}, \|\cdot\|)$ to be the Banach space of real-valued bounded continuous functions on $[m(0), \infty)$ with the supremum norm $\|\cdot\|$, that is for $\varphi \in \mathcal{C}$

$$\|\varphi\| := \sup\left\{|\varphi(t)| \ ; t \in [m(0), \infty)\right\}.$$

Our investigations will be carried out on the complete metric space $(\mathcal{C}, \rho)$, where ρ is the uniform metric. That is, for $\varphi, \phi \in \mathcal{C}$ we set $\rho(\varphi, \phi) = \|\varphi - \phi\|$. Let $\psi \in \mathcal{C}([m(0), 0], \mathbb{R})$ be fixed and define

$$S_\psi := \{\varphi : [m(0), \infty) \to \mathbb{R} \mid \varphi \in \mathcal{C},\ \varphi(t) = \psi(t) \text{ for } t \in [m(0), 0],\ \varphi(t) \to 0 \text{ as } t \to \infty\}.$$

Being closed in $\mathcal{C}$, (S_ψ, ρ) is itself complete. There is no confusion if we use the norm $\|\cdot\|$ on $[m(0), 0]$ or on $[m(0), \infty)$. In other words, we carry out our investigations in the complete metric space(S_ψ, ρ) where ρ is supremum metric

$$\rho(x, y) = \sup_{t \geq m(0)} |x(t) - y(t)| = \|x - y\| \text{ for } x, y \in S_\psi$$

Now use this equality to define the operator $P : S_\psi \to S_\psi$ by $(P\varphi)(t) = \psi(t)$ if $t \in [m(0), 0]$ and for $t \geq 0$ we let

$$\begin{aligned}
(P\varphi)(t) \quad = \quad & \left(\psi(0) - \sum_{j=1}^{N} \frac{c_j(0)}{1 - \tau_j'(0)}\psi(-\tau_j(0)) - \sum_{j=1}^{N} \int_{-\tau_j(0)}^{0} [h_j(s)B_j(0, s)]\psi(s)ds\right) e^{-\int_0^t H(u)du} \\
& + \sum_{j=1}^{N} \frac{c_j(t)}{1 - \tau_j'(t)}\varphi(t - \tau_j(t)) + \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} [h_j(s) + B_j(t, s)]\varphi(s)ds \\
& + \sum_{j=1}^{N} \int_0^t e^{-\int_s^t H(u)du} \left\{[h_j(s - \tau_j(s) + B_j(s, s - \tau_j(s))] \left(1 - \tau_j'(s)\right) - r_j(s)\right\} \varphi(s - \tau_j(s))ds \\
& - \sum_{j=1}^{N} \int_0^t e^{-\int_s^t H(u)du} H(s) \left(\int_{s-\tau_j(s)}^{s} [h_j(u) + B_j(s, u)]\varphi(u)du\right) ds \qquad \text{(II.2.1)}
\end{aligned}$$

For that reason we assume that the followings condition hold

Theorem II.2.1 *(H1) there exist a constant $\alpha \in (0.1)$ such that for $t \geq 0$*

$$\lim_{t\to\infty} \inf \int_0^t H(s)ds > -\infty \qquad \text{(II.2.2)}$$

$$\begin{aligned}
&\sum_{j=1}^{N}\left|\frac{c_j(t)}{1-\tau_j'(t)}\right| + \sum_{j=1}^{N}\int_{t-\tau_j(t)}^{t}|h_j(s)+B_j(t,s)|ds \\
&+\sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_s^t H(u)du}|[h_j(s-\tau_j(s)+B_j(s,s-\tau_j(s))](1-\tau_j'(s))-r_j(s)| + \\
&+\sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_s^t H(u)du}|H(s)|\left(\int_{s-\tau_j(t)}^{s}|h_j(u)+B_j(u,s)|du\right)ds \\
\leq\ & \alpha \qquad \text{(II.2.3)}
\end{aligned}$$

Theorem II.2.2 *Under the hypotheses of Lemma II.1.2 and (H1) we have the zero solution of (II.0.1) is asymptotically stable if and only if*

$$\int_0^t H(s)ds \to \infty \ as\ t \to \infty \qquad \text{(II.2.4)}$$

Proof: First, suppose that (II.2.4) holds. We set

$$K = \sup_{t\geq 0}\left\{e^{-\int_s^t H(s)ds}\right\} \qquad \text{(II.2.5)}$$

$$|P\varphi| \leq K(1+\sum_{j=1}^{N}\frac{c_j(0)}{1-\tau_j(0)}+\sum_{j=1}^{N}\int_{t_0-\tau_j(t_0)}^{t_0}h(s)ds\,\|\psi\| + \|\varphi\|\,\alpha < \infty$$

It is clear that $P\varphi \in [m(t_0), t_0]$. We will show that $(P\varphi)(t) \to 0$ as $t \to \infty$. To this end, denote the five terms on the right hand side of (II.2.1) by $I_1, I_2, ...I_5$, respectively. It is obvious that the first term I_1 tends to zero as $t \to \infty$, by condition (II.2.4). Also, due to the facts that $\varphi(t) \to 0$ and $t - \tau_j(t) \to \infty$ for $j = 1, 2, ..., N$ as $t \to \infty$, the second term I_2 in (II.2.1) tends to zero as $t \to \infty$. What is left to show is that each of the remaining terms in (II.2.1) go to zero at infinity. Let $\varphi \in S_\psi$ be fixed. For a given $\epsilon > 0$ we choose $T_0 > 0$ large enough such that $t - \tau_j(t) \geq T_0$,$j = 1, 2..., N$ implies $|\varphi(s)| < \epsilon$ if $s \geq t - \tau_j.(t)$. There fore, the third term I_3 in (II.2.1) satisfies

$$\begin{aligned}
|I_3| &= \left|\sum_{j=1}^{N}\int_{t-\tau_j(t)}^{t}[h_j(s)+B_j(t,s)]\,\varphi(s)ds\right| \\
&\leq \epsilon\sum_{j=1}^{N}\int_{t-\tau_j(t)}^{t}|h_j(s)+B_j(t,s)|\,ds \\
&\leq \alpha\epsilon < \epsilon
\end{aligned}$$

Thus, $I_3 \to 0$ as $t \to \infty$. Now consider I_4. For the given $\epsilon > 0$, there exists a $T_1 > 0$ such that $s \geq T_1$ implies $|\varphi(s - \tau_j(s))| < \epsilon$ for $j = 1, 2, ..., N$. Thus, for $t \geq T_1$, the term I_4

in (II.2.1) satisfies

$$\begin{aligned}
|I_4| &= \left|\sum_{j=1}^{N}\int_0^t e^{-\int_s^t H(u)du}\left\{[h_j(s-\tau_j(s))+B_j(s,s-\tau_j(s))](1-\tau_j'(s))-r_j(s)\right\}\varphi(s-\tau_j(s))ds\right| \\
&\leq \sum_{j=1}^{N}\int_0^{T_1} e^{-\int_s^t H(u)du}\left|h_j(s-\tau_j(s))+B_j(s,s-\tau_j(s))(1-\tau_j'(s))-r_j(s)||\varphi(s-\tau_j(s))\right| ds \\
&\quad +\sum_{j=1}^{N}\int_{T_1}^{t} e^{-\int_s^t H(u)du}\left|[h_j(s-\tau_j(s))+B_j(s,s-\tau_j(s))]\,(1-\tau_j'(s))-\tau_j(s)||\varphi(s-\tau_j(s))\right| ds \\
&\leq \sup_{\alpha\geq m(0)}|\varphi(\sigma)|\sum_{j=1}^{N}\int_0^{T_1} e^{-\int_s^t H(u)du}\left|[h_j(s-\tau_j(s))+B_j(s,s-\tau_j(s))]\,(1-\tau_j'(s))-r_j(s)\right| ds \\
&\quad +\epsilon\sum_{j=1}^{N}\int_{T_1}^{t} e^{-\int_s^t H(u)du}\left|\left[h_j(s-\tau_j(s))+B_j(s,s-\tau_j(s))(1-\tau_j'(s))-r_j(s)\right]\right| ds
\end{aligned}$$

By(II.2.4), we can find $T_2 > T_1$ such that $t \geq T_2$ implies

$$\begin{aligned}
&\sup_{\alpha\geq m(0)}|\varphi(\sigma)|\sum_{j=1}^{N}\int_0^{T_1} e^{-\int_s^t H(u)du}\left|\left[h_j(s-\tau_j(s))+B_j(s,s-\tau_j(s))(1-\tau_j'(s))-r_j(s)\right]\right| ds \\
= &\sup_{\alpha\geq m(t_0)}|\varphi(\sigma)|\,e^{-\int_{T_2}^t H(u)du}\sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_s^t H(u)du}\left|[h_j(s-\tau_j(s))+B_j(s,s-\tau_j(s))]\,(1-\tau_j'(s))-r_j(s)\right| ds \\
< &\ \epsilon.
\end{aligned}$$

Now, apply(II.2.3) to have $|I_4| < \epsilon + \alpha\epsilon$. Thus, $I_4 \to 0$ as $t \to \infty$. Similarly, by using (II.2.3), then, if $t \geq T_1$ then term I_5 in (II.2.1) (II.2.1) satisfies

$$\begin{aligned}
|I_5| &= \left|\sum_{j=1}^{N}\int_0^t e^{-\int_s^t H(u)du}H(s)\left(\int_{s-\tau_j(s)}^{s}[h_j(u)+B_j(s,u)]\,\varphi(u)du\right)ds\right| \\
&\leq \sup_{\sigma\geq m(0)}|\varphi(\sigma)|\,e^{-\int_{T_2}^t H(u)du}\sum_{j=1}^{N}\int_0^{T_1} e^{-\int_s^{T_2} H(u)du}|H(s)| \\
&\quad\times\left(\int_{s-\tau_j(s)}^{s}|h_j(u)|+B_j(s,u)du\right)ds \\
&\quad+\epsilon\sum_{j=1}^{N}\int_{T_1}^{t} e^{-\int_s^t H(u)du}\,|H(s)|\left(\int_{s-\tau_j(s)}^{s}|h_j(u)+B_j(s,u)du|\right)ds \\
&< \epsilon+\alpha\epsilon < 2\epsilon.
\end{aligned}$$

Thus, $I_5 \to 0$ as $t \to \infty$. In conclusion $(P\varphi)(t) \to 0$ as $t \to \infty$, as required. Hence P maps S_ψ into S_ψ. Also, by condition(II.2.3), P is a contraction mapping with contraction constant α. Indeed, for $\phi, \eta \in S_\psi$ and $t > 0$

$$|P(\varphi)(t) - (P\eta)(t)|$$
$$\leq \sum_{j=1}^{N} \left|\frac{c_j(t)}{1-\tau_j'(t)}\right| |\varphi(t-\tau_j(t)) - \eta(t-\tau_j(t))| + \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{s} |h_j(s) + B_j(t,s)||\varphi(s) - \eta(s)|ds$$
$$+ \sum_{j=1}^{N} \int_0^t e^{-\int_s^t H(u)du} |[h_j(s\text{-}\tau_j(s)){+}B_j(s, s\text{-}\tau_j(s))]| \left(1\text{-}\tau_j'(s)\right) \text{-}r_j(s)| \, |\varphi(s\text{-}\tau_j(s))\text{-}\eta\,(s\text{-}\tau_j(s))| \, ds$$
$$+ \sum_{j=1}^{N} \int_0^t e^{-\int_s^t H(u)du} H(s) \left(\int_{s-\tau_j(s)}^{s} |h_j(u) + B_j(s,u)||\varphi(u) - \eta(u)|du\right) ds.$$

this

$$\begin{aligned} &|P(\varphi)(t) - (P\eta)(t)| \\ &\leq \sum_{j=1}^{N} \Bigg[\left|\frac{c_j(t)}{1-\tau_j'(t)}\right| + \int_{t-\tau_j(t)}^{t} \sum_{j=1}^{N} |h_j(u) + B_j(t,s)| \, ds \\ &+ \int_0^t e^{-\int_s^t H(u)du} \left|[h_j(s-\tau_j(s)) + B_j(s, s-\tau_j(s))] \, (1-\tau_j'(s) - r_j(s)\right| ds \\ &+ \int_0^t e^{-\int_s^t H(u)du} H(s) \left(\int_{s-\tau_j(s)}^{s} |h_j(u) + B_j(s,u)|du\right) ds\Bigg] \|\varphi - \eta\| \\ &\leq \alpha \parallel \varphi - \eta \parallel . \end{aligned}$$

From condition (II.2.3) P is a contraction mapping with constant α. By the contraction mapping principle, P has a unique fixed point x in S_ψ which is a solution of (II.0.1) with $x(t) = \psi(t)$ on $[m(0), 0]$ and $x(t) = x(t, 0, \psi) \to 0$ as $t \to \infty$. To obtain the asymptotic stability, we need to show that the zero solution of (II.0.1) is stable. Let $\epsilon > 0$ be given and choose $\delta > 0$ $(\delta < \epsilon)$ satisfying $2\delta K + \alpha\epsilon < \epsilon$. If $x(t) = x(t, 0, \psi)$ is a solution of (II.0.1)with , $\|\psi(t)\| < \delta$ then $x(t) = (Px)(t)$ defined in (II.2.1). We claim that $|x(t)| < \epsilon$ for all $t \geq t_0$. Notice that $|x(s)| < \epsilon$ on $[m(0), 0]$. If there exists $t^* > 0$ such that $|x(t^*)| = \epsilon$ and $|x(s)| < \epsilon$ for $m(0) \leq s < t^*$, then it follows from (II.2.1) that

$$\begin{aligned} & |x(t^*)| \\ \leq\ & \|\psi\| \left(1 + \sum_{j=1}^{N} \left|\frac{c_j(0)}{1-\tau_j'(0)}\right| + \sum_{j=1}^{N} \int_{-\tau_j(0)}^{0} |h_j(s) + B_j(0,s)| \, ds\right) e^{-\int_0^{t^*} H(u)du} \\ & +\epsilon \sum_{j=1}^{N} \left|\frac{c_j(t^*)}{1-\tau_j'(t^*)}\right| + \epsilon \sum_{j=1}^{N} \int_{t^*-\tau_j(t^*)}^{t^*} |h_j(s) + B_j(t^*, s)|ds \\ & +\epsilon \int_{t_0}^{t^*} e^{-\int_s^{t^*} H(u)du} |[h_j(s - \tau_j(s) + B_j(s, s - \tau_j(s))](1 - \tau_j'(s)) - r_j(s)|ds \\ & +\epsilon \sum_{j=1}^{N} \int_{t_0}^{t^*} e^{-\int_s^{t^*} H(u)du} \, |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u) + B_j(s,u)| \, du\right) ds \\ \leq\ & 2\delta K + \alpha\epsilon < \epsilon \end{aligned}$$

wich contradicts the definition of t^*. Thus $|x(t)| < \epsilon$ for all $t \geq 0$, and the zero solution of (II.0.1) is stable. This shows that the trivial solution of eqution (II.0.1) is asymptoyically stable if (II.0.1) stable if (II.2.4) holds. Conversely, suppose (II.2.4) fails. Then, by (II.2.2) there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty}\left(\int_0^{t_n} H(u)du = l\right)$ for some $l \in \mathbb{R}$. We may also choose a positive constant J satisfying

$$-J \leq \int_0^{t_n} H(u)du \leq J$$

for all $n \geq 1$. To simplify our expressions, we define

$$\begin{aligned}
\omega(s) &= \sum_{j=1}^{N} \left|\left[h_j\left(s-\tau_j(s)\right)+B_j\left(s, s-\tau_j(s)\right)\right]\left(1-\tau_j'(s)\right)-r_j(s)\right| ds \\
&\quad + \sum_{j=1}^{N} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)+B_j(s,u)|\, du\right)
\end{aligned}$$

for all $s \geq 0$. By(II.2.3) , we have

$$\int_0^{t_n} e^{-\int_s^{t_n} H(u)du}\omega(s)ds \leq \alpha$$

This yields

$$\int_0^{t_n} e^{-\int_0^s H(u)du}\omega(s)ds \leq \alpha e^{\int_0^{t_n} H(u)du} \leq J.$$

The sequence $\left\{\int_0^{t_n} e^{-\int_0^s H(u)du}\omega(s)ds\right\}$ is bounded, so there exists a convergent subsequence. For brevity of notation, we may assume that

$$\lim_{n\to\infty}\int_0^{t_n} e^{-\int_0^s H(u)du}\omega(s)ds = \gamma,$$

for some $\gamma \in \mathbb{R}^+$ and choose a positive integer m so large that

$$\int_{t_m}^{t_n} e^{-\int_0^s H(u)du}\omega(s)ds < \frac{\delta_0}{4k},$$

for all $n \geq m$, where $\delta_0 > 0$ satisfies $2\delta_0 K e^J .+\alpha \leq 1$. By (II.2.2) K in (II.2.5) is well defined. We now consider the solution $x(t) = x(t, t_m, \psi)$ of (II.0.1) with $\psi(t_m) = \delta_0$ and $\psi(s) \leq \delta_0$ for $s \leq t_m$. We may choose ψ so that $|x(t)| \leq 1$ for $t \geq t_m$ and

$$\psi(t_m) - \sum_{j=1}^{N}\left[\frac{c_j(t_m)}{1-\tau_j'(t_m)}\psi(t_m-\tau_j(t_m)) + \int_{t_m-\tau_j(t_m)}^{t_m}[h_j(s)+B_j(t_m,s)\psi(s)ds\right] \geq \frac{1}{2}\delta_0$$

It follows from (II.2.1) with $x(t) = (P_X)(t)$ that for $n \geq m$

$$\begin{aligned}
&\left| x(t_n) - \sum_{j=1}^{N} \left[\frac{c_j(t_n)}{1-\tau_j'(t_n)} \psi(t_n - \tau_j(t_n)) + \int_{t_n-\tau_j(t_n)}^{t_n} h_j(s) + B_j(t_n, s) x(s) ds \right] \right| \\
\geq\ & \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} H(u)du} - \int_{t_m}^{t_n} e^{-\int_s^{t_n} H(u)du} \omega(s) ds \\
=\ & \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} H(u)du} - e^{-\int_0^{t_n} H(u)du} \int_{t_m}^{t_n} e^{\int_0^s H(u)du} \omega(s) ds \\
=\ & e^{-\int_{t_m}^{t_n} H(u)du} \left(\frac{1}{2}\delta_0 - e^{-\int_0^{t_m} H(u)du} \int_{t_m}^{t_n} e^{-\int_0^s H(u)du} \omega(s) ds \right) \\
\geq\ & e^{-\int_{t_m}^{t_n} H(u)du} \left(\frac{1}{2}\delta_0 - K \int_{t_m}^{t_n} e^{\int_0^s H(u)du} \omega(s) ds \right) \\
\geq\ & \frac{1}{4}\delta_0 e^{-\int_{t_m}^{t_n} H(u)du} \geq \frac{1}{4}\delta_0 e^{-2J} > 0. \qquad \text{(II.2.6)}
\end{aligned}$$

On the other hand, if the zero solution of(II.0.1) is asymptotically stable, then $x(t) = x(t, t_m, \psi) \to 0$ as $t \to \infty$. Since $t_n - \tau_j(t_n) \to \infty$ as $n \to \infty$ and(II.2.3) holds,we have

$$x(t_n) - \sum_{j=1}^{N} \left[\frac{c_j(t_n)}{1-\tau_j'(t_n)} x(t_n - \tau_j(t_n)) + \int_{t_n-\tau_j(t_n)}^{t_n} \left[h_j(s) + B\left(t_n, s\right)\right] x(s) ds \right] \to 0$$

as $n \to \infty$, which contradicts (II.2.6) . Hence condition (II.2.4) is necessary for the asymptotic stability of the zero solution of(II.0.1) The proof is complete. ■

Remark II.2.1 *It follows from the first part of the proof of (II.2.2)that the zero solution of (II.0.1) is stable under (II.2.2) and (II.2.3) . Moreover,(II.2.2) still holds if (II.2.3) is satisfied for $t \geq t_\sigma$ for some $t_\sigma \in .\mathbb{R}^+$. For the special case $N = 1$ and $c_1 = 0$, we have the following result*

Corollary II.2.1 *Suppose that τ_1 is differentiable and there exist continuous function $h_1 : [m_1(0), \infty) \to \mathbb{R}$ and a constant $\alpha \in (0, 1)$ such that for $t \geq 0$*

$$\liminf_{t\to\infty} \int_0^t h_1(s) ds > -\infty \qquad \text{(II.2.7)}$$

and

$$\begin{aligned}
&\int_{t-\tau_1(t)}^{t} \left|h_1(s) + B_1(t, s)\right| ds \\
&+ \int_0^t e^{-\int_s^t h_1(u)du} \left|h_1(s - \tau_1(s))\right| + \left|B_1(s, s - \tau_1(s))\right| \left|1 - \tau'(s)\right| ds \\
&+ \int_0^t e^{-\int_s^t h_1(u)du} |h_1(s)| \left(\int_{s-\tau_1(s)}^{s} |h_1(u) + B_1(s, u)| du \right) ds \\
&\leq \alpha \qquad \text{(II.2.8)}
\end{aligned}$$

Then the zero solution of II.0.1 is asymptotically stable if and only if

$$\int_0^t h_1(s) ds \to \infty, \quad \text{as } t \to \infty \qquad \text{(II.2.9)}$$

Remark II.2.2 *when $h_j(s) = b_j(g_j(s))$ for $j = 1, 2, ..N$, then corollary(II.2.1) reduced to the (II.2.2)*

II.3 Applications

Example II.3.1 *Consider the linear neutral integro-differential equation with variable delays.*

$$x'(t) = -\sum_{j=1}^{2} \int_{t-\tau_j(t)}^{t} a_j(t,s)x(s)ds + \sum_{j=1}^{2} c_j(t)x'(t-\tau j(t)) \tag{II.3.1}$$

where $\tau_1(t) = 0.489t, \tau_2(t) = 0.478t, a_1(t,s) = \frac{0.48}{s^2+1}, a_2(t,s) = \frac{0.52}{s^2+1}$, $c_1(t) = 0.015$, $c_2(t) = 0.017$ *Then the zero solution of (II.3.1) is asymptotically stable.*

Proof: we have

$$B_1(t,s) = \int_t^s \frac{0.48}{s^2+1}du = \frac{0.48(s-t)}{s^2+1}, \quad B_2(t,s) = \int_t^s \frac{0.52}{s^2+1}du = \frac{0.52(s-t)}{s^2+1}.$$

Choosing $h_1(t) = \frac{0.52t}{t^2+1}$ and $h_2(t) = \frac{0.48t}{t^2+1}$ in (II.2.2) we have $H(t) = \frac{t}{t^2+1}$ and

$$\begin{aligned}
\sum_{j=1}^{2} \left| \frac{c_j(t)}{1-\tau_j'(t)} \right| &= \left| \frac{0.015}{1-0.489} \right| + \left| \frac{0.017}{1-0.478} \right| < 0.062, \\
\sum_{j=1}^{2} \int_{t-\tau_j(t)}^{t} |h_j(s) + B_j(t,s)|\, ds &= \int_{0.511t}^{t} \left| \frac{s-0.48t}{s^2+1} \right| ds + \int_{0.522t}^{t} \left| \frac{s-0.52t}{s^2+1} \right| ds \\
&= \int_{0.511t}^{t} \frac{s-0.48t}{s^2+1} ds + \int_{0.522t}^{t} \frac{s-0.52t}{s^2+1} ds \\
&= t[0.48 \arctan 0.511t + 0.52 \arctan 0.522t - \arctan t] \\
&\quad + \ln(t^2+1) - \frac{1}{2}[\ln(0.511^2 t^2 + 1) + \ln(0.522^2 t^2 + 1) \\
&= \omega(t)
\end{aligned}$$

Since the function ω is increasing in $[0.\infty)$ and

$$\lim_{t\to\infty} \omega(t) = \frac{1-0.48}{0.511} - \frac{0.52}{0.522} - \ln(0.511 \times 0.522) \simeq 0.386,$$

then

$$\sum_{j=1}^{2} \int_{t-\tau_j(t)}^{t} |h_j(s)+B_j(t,s)| ds < 0.386, \sum_{j=1}^{2} \int_0^t e^{-\int H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{t} |h_j(u) + B_j(s,u)|\, du \right) ds < 0.386$$

and

$$\begin{aligned}
&\sum_{j=1}^{2}\int_0^t e^{-\int_s^t H(u)du}|[h_j(s-\tau_j(s))+B(s,s-\tau_j(s))](1-\tau_j'(s))-r_j(s)|ds\\
=\;& \int_0^t e^{-\int_s^t \frac{u}{u^2+1}du}\left|0.511\left(\frac{0.52\times 0.511s}{0.511^2s^2+1}+\frac{0.48(0.511s-s)}{0.511^2s^2+1}\right)-\frac{0.015s}{0.511(s^2+1)}\right|ds\\
&+\int_0^t e^{-\int_s^t \frac{u}{u^2+1}du}\left|0.522\left(\frac{0.48\times 0.522s}{0.522^2s^2+1}+\frac{0.52(0.522s-s)}{0.522^2s^2+1}\right)-\frac{0.017s}{0.522(s^2+1)}\right|ds\\
\le\;& \left(1-\frac{0.48}{0.511}\right)\int_0^t e^{-\int_s^t \frac{u}{u^2+1}du}\frac{s}{\frac{s^2+1}{0.511^2}}ds+\frac{0.015}{0.511}\int_0^t e^{-\int_s^t \frac{u}{u^2+1}du}\frac{s}{s^2+1}ds\\
&+\left(1-\frac{0.52}{0.522}\right)\int_0^t e^{-\int_s^t \frac{u}{u^2+1}du}\frac{s}{s^2+\frac{1}{0.522^2}}ds+\frac{0.017}{0.522}\int_0^t e^{-\int_s^t \frac{u}{u^2+1}du}\frac{s}{s^2+1}ds\\
<\;& 1-\frac{0.48}{0.511}+\frac{0.015}{0.511}+1-\frac{0.52}{0.522}+\frac{0.017}{0.522}<0.127
\end{aligned}$$

It is easy to see that all the conditions of (II.2.2) hold for $\alpha = 0.062+0.386+0.386+0.127 = 0.961 < 1$. Thus, (II.2.2) implies that the zero solution of (II.3.1) is asymptotically stable. ■

Chapter III

Existence of positive periodic solution of delay differential equation

Sommaire

In this chapter,we consider existence, multiplicity and non existence of positive ω_periodic solutions for the following second-order neutral functional differential equation :

$$\left(x\left(t\right)-cx\left(t-\delta\right)\right)^{''}+\alpha\left(t\right)x\left(t\right)=\lambda b\left(t\right)f\left(x\left(t-\tau\left(t\right)\right)\right) \tag{III.0.1}$$

where λ is a positive parameter, c and δ are constants and $|c|\neq 1$. The existence of periodic solutions for functional differential equations has been derived from many fields such as physics, biology and mechanics [8] ,[32]. Many results were obtained by Kuang [32], Freedman and Wu [28], Wang [27] and many others by applying fixed point index theory, theory of Fourier series, fixed point theorems in cones, Leray Schauder continuation theorem, coincidence degree theory and so on.We refer to [29],[16] for some recent results in this field. Among the previous results on this problem many of them concern neutral systems (Lu and Ge [20], Lu, Ge and Zheng [19] and Chen [8]). In this section, we aim to establish existence, multiplicity and nonexistence of positive ω_periodic solutions for second-order neutral functional differential equation(III.0.1).Our approach is based on a fixed point theorem in cones as well as some analysis techniques used in [27] and [28]. Let

1. $f_0=\lim_{u\to0^+}\frac{f(u)}{u}$ and $f_\infty=\lim_{u\to\infty}\frac{f(u)}{u}$
2. $i_0=$ number of zeros in the set $\{f_0,f_\infty\}$
3. $i_\infty=$ number of infinities in the set $\{f_0,f_\infty\}$

It is clear that i_0 or, $i_\infty=0,1$ 2. We will show that (III.0.1) has $i_{0,}$ or $i_{\infty,}$ positive $\omega-$periodic solution(s) for certain λ, respectively. Let

$$\overline{f}=\frac{1}{\omega}\int_0^\omega f\left(s\right)ds$$

where a is a continuous ω_ periodic function. In what follows we set

$$X=\{x\backslash\quad x\in\mathcal{C}(\mathbb{R},\mathbb{R}),x(t+\omega)=x(t)\}$$

with the norm defined by $\|x\|_X=\max\{|x(t)|:t\in[0;\omega]\}$.Then $(x,\|x\|_X)$ is a Banach space. We consider the following assumptions

(H1) $a, b \in \mathcal{C}(\mathbb{R}, (0, +\infty))$ are ω_periodic functions,

$$\max\{a(t) : t \in [0, \omega]\} < (\frac{\pi}{\omega})^2$$

and $\tau \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ is a positive ω_periodic function.
(H2) $f \in \mathcal{C}([0, \infty), [0, \infty))$ and $f(u) > 0$ for $u > 0$.
Next, we denote by

$$\begin{aligned} M &= \max\{a(t) : t \in [0, \omega]\}, m = \{\min a(t) : t \in [0, \omega]\} \\ \beta &= \sqrt{M}, \quad L = \frac{1}{2\beta \sin \frac{\beta\omega}{2}}, l = \frac{cos \frac{\beta\omega}{2}}{2\beta \sin \frac{\beta\omega}{2}} \\ k &= l(M + m) + LM, \alpha = \frac{l[m - |c|(M + m)]}{LM(1 - |c|).} \end{aligned}$$

If the assumption (H1) holds, then $M < \left(\frac{\pi}{\omega}\right)^2$. Thus we can see that $L \geq l > 0$. Additionally, define

$$\begin{aligned} M(r) &= \max\left\{f(t) : 0 \leq t \leq \frac{r}{1 - |c|}\right\} \\ m(r) &= \min\left\{f(t) : \alpha r \leq t \leq \frac{r}{1 - |c|}\right\}, k_1 = \frac{k - \sqrt{k_2 - 4LlMm}}{2LM} \end{aligned}$$

The rest here is organized as follows Section 2 is about statement of the method (a fixed point theorem in cones) and some prior estimations in order to prove our main results; in Section 3, we give the proofs of our main results by using our lemmas and present an example.

Next, we transfer existence of positive ω_periodic solutions of neutral equation(III.0.1) into existence of positive fixed points of some fixed point mapping.

III.1 Some Lemmas

the following lemmas are used to study existence of In order to establish existence, multiplicity and nonexistence of positive ω_periodic solutions for (III.0.1) we need the following lemmas. for this purpose, let $A : X \to X$ defined by

$$Ax(t) = x(t) - cx(t - \delta).$$

Then, we have

Lemma III.1.1 *If $|c| \neq 1$, then A has continuous bounded inverse A^{-1} on X and for all $x \in X$*

$$A^{-1}x(t) = \begin{cases} \sum_{j\geq 0} c^j x(t - j\delta) \; if \; |c| < 1 \\ -\sum_{j\geq 1} c^{-j} x(t + j\delta) \; if \; |c| > 1 \end{cases} \tag{III.1.1}$$

and

$$\left\|A^{-1}x\right\|_X \leq \frac{\|x\|_X}{|1 - |c||}$$

Proof: According to [15], we can get the equality(III.1.1) and then verify (III.1.1). We consider the following equation

$$y\text{''}\left(t\right)+a\left(t\right)\left(A^{-1}y\right)\left(t\right)=\lambda b\left(t\right)f\left(\left(A^{-1}y\right)\left(t-\tau\left(t\right)\right)\right) \tag{III.1.2}$$

where A^{-1} is defined by (III.1.1). By Lemma (III.1.1) we conclude that ■

Lemma III.1.2 *$y(t)$ is an $\omega\_$ periodic solution of (III.1.2) if and only if $(A^{-1}y)$ is an $\omega\_$ periodic solution of (III.0.1).*

Aiming to apply (**??**) to Eq. (III.1.2) , we rewrite((III.1.2) as

$$y''\left(t\right)+a\left(t\right)y(\left(t\right)-a\left(t\right)G\left(y\left(t\right)\right)=\lambda b\left(t\right)f\left(\left(A^{-1}y\right)\left(t-\tau\left(t\right)\right)\right)$$

where

$$G\left(y\left(t\right)\right)=y\left(t\right)-\left(A^{-1}y\right)y\left(t\right)=-c\left(A^{-1}y\right)\left(t-\delta\right)$$

Set $K=\{x\in X:x\left(t\right)\geq\alpha\left\|x\right\|_{X}\}$. Clearly, K is a cone in X. Note that $\Omega_r=\{x\in K:\left\|x\right\|_{X}<r\}$ and $\partial\Omega_r=\{x\in K:\left\|x\right\|_{X}=r\}$. Additionally, we let $C_\omega=\{x\in\mathcal{C}\left(\mathbb{R},\mathbb{R}^+\right):x\left(t+\omega\right)=x\left(t\right)\}$. By solving the inequality $|c|<\frac{l[m-|c|(M+m)]}{LM(1-|c|)}$,we can obtain the following result immediately.

Lemma III.1.3 *If* $|c|<\min\left\{k_1,\frac{m}{M+m}\right\}$ *,then* $|c|<\alpha$

Lemma III.1.4 *If* $y\in K$ *and* $c\in(-\min\left\{k_1,\frac{m}{M+m}\right\},0]$*, then*

(a) $\frac{\alpha-|c|}{1-c2}\left\|y\right\|_X\leq(A^{-1}y)(t)\leq\frac{1}{1-|c|}\left\|y\right\|_X$

(b) $\frac{|c|(\alpha-|c|)}{1-c_2}\left\|y\right\|_X\leq G(y(t))\leq\frac{|c|}{1-|c|}\left\|y\right\|_X$, $t\in[0,\omega]$

Proof: **(a) :** For $y\in K$ and $c\in(-\min\left\{k_1,\frac{m}{M+m}\right\},0]$, by Lemma (III.1.1), we have

$$\begin{aligned}\left(A^{-1}y\right)\left(t\right) &= \sum_{j\geq0}c^jy\left(t-j\delta\right)=\sum_{j\geq0}c^jy\left(t-j\delta\right)-\sum_{j\geq0}\left|c\right|^jy\left(t-j\delta\right)\geq\frac{\alpha-|c|}{1-c2}\left\|y\right\|_X\\ \left(A^{-1}y\right)\left(t\right) &\leq \frac{1}{1-|c|}\left\|y\right\|_X\end{aligned}$$

(b) : From the definition of $G\left(y\left(t\right)\right)$ and **(a)**, we have

$$\frac{|c|\left(\alpha-|c|\right)}{1-c^2}\left\|y\right\|_X\leq Gy\left(t\right)\leq\frac{|c|}{1-|c|}\left\|y\right\|_X$$

The proof of (III.1.4) is completed.Firstly, we consider the following equation :

$$y''\left(t\right)+My\left(t\right)=\lambda h\left(t\right),h\in\mathcal{C}_\omega. \tag{III.1.3}$$

Define $G(t,s)$ by

$$G\left(t,s\right)=\frac{\cos\beta\left(t+\frac{\omega}{2}-s\right)}{2\beta\sin\frac{\beta\omega}{2}},\;t\in\mathbb{R},\;t\leq s\leq t+\omega$$

Thus,

$$\begin{aligned}\int_t^{t+\omega} G(t,s)ds &= \int_t^{t+\omega} \frac{\cos\beta\left(t+\frac{\omega}{2}-s\right)}{2\beta\sin\frac{\beta\omega}{2}}ds = \left.\frac{\sin\beta\left(t+\frac{\omega}{2}-s\right)}{2\beta\sin\frac{\beta\omega}{2}}\right|_t^{t+\omega} \\ &= \frac{1}{2\beta^2}+\frac{1}{2\beta^2}=\frac{1}{\beta^2}=\frac{1}{M}. \\ 0 &< \frac{\cos\beta\frac{\omega}{2}}{2\beta\sin\beta\frac{\omega}{2}} \leq G(t,s) \leq \frac{1}{2\beta\sin\frac{\beta\omega}{2}} = L\end{aligned}$$

since $M < \left(\frac{\pi}{\omega}\right)^2$. ■

Directly from (III.1.3) we define a mapping as follows

$$T_\lambda h(t) = \lambda \int_t^{t+\omega} G(t,s)\, h(s)\, ds$$

It is easy to show that $T_\lambda h(t) > 0$ for $h(t) > 0$. And by the properties of $G(t,s)$ and $h(t), T_\lambda$ is completely continuous. Also, by simple computations and the maximum principle, we establish the following lemma

Lemma III.1.5 *.Suppose the assumptions (H1),(H2) hold and $c \in \left(-\min\left\{k_1, \frac{m}{M+m}\right\}, 0\right]$. For any $h \in \mathcal{C}_\omega, y(t) = T_\lambda h(t)$ is the unique positive ω_periodic solution of (III.1.3). Meanwhile,$\|T_\lambda\| = \frac{\lambda}{M}$*

Secondly, we study the following equation corresponding to (III.1.3).

$$y''(t) + a(t)\, y(t) - a(t)\, G(y(t)) = \lambda h(t), h \in \mathcal{C}_\omega \tag{III.1.4}$$

Let

$$By(t) = \frac{1}{\lambda}\left[(M - a(t))y(t) + a(t)\, G(y(t))\right]$$

Clearly,

$$\|B\| \leq \frac{1}{\lambda}\left(M - m + \frac{M}{(1-|c|)}\right)$$

Then, from (III.1.5), we have

$$y(t) = T_\lambda h(t) + T_\lambda By(t).$$

$|c| < \min\left\{k_1, \frac{m}{M+m}\right\}$ implies that $\frac{M-m+m|c|}{M(1-|c|)} < 1$.So $\|T_\lambda B\| \leq \|T_\lambda\|\,\|B\| \leq \frac{M-m+m|c|}{M(1-|c|)} < 1$. Thus we have

$$y(t) = (I - T_\lambda B)^{-1}\, T_\lambda h(t) \tag{III.1.5}$$

Let

$$P_\lambda h(t) = (I - T_\lambda B)^{-1}\, T_\lambda h(t)$$

Then we can make the following conclusion.

Lemma III.1.6 *Suppose the assumptions (H1), (H2) hold and $c \in \left(-\min\left\{k_1, \frac{m}{M+m}\right\}, 0\right]$. For any $h \in \mathcal{C}_\omega$, $y(t) = P_\lambda h(t)$ is the unique positive ω_periodic solution of(III.1.4) ; P_λ is completely continuous and satisfies*

$$T_\lambda h(t) \leq P_\lambda h(t) \leq \frac{M(1-|c|)}{m-(M+m)|c|}\,\|T_\lambda h\|_X\,,\ h \in \mathcal{C}_\omega$$

Proof: By expansions of P_λ

$$\begin{aligned} P_\lambda &= (I - T_\lambda B)^{-1} T_\lambda = \left(I + T_\lambda B + (T_\lambda B)^2 + \cdots + (T_\lambda B)^2 + \cdots\right) T_\lambda \\ &= T_\lambda + T_\lambda B + (T_\lambda B)^2 T_\lambda + \cdots + (T_\lambda B)^n + T_\lambda \cdots \end{aligned} \tag{III.1.6}$$

P_λ is completely continuous since T_λ is completely continuous. From (III.1.6), we get

$$T_\lambda h(t) \le P_\lambda h(t) \le \frac{M(1-|c|)}{m-(M+m)|c|} \|T_\lambda h\|_X \, h \in \mathcal{C}_\omega$$

The proof is completed(III.1.5).and (III.1.6) are obtained similarly with(III.1.1) and (**??**) in [25]. Let $Q_\lambda y(t) = P_\lambda(b(t) f((A^{-1}y)(t-\tau(t))))$. Since P_λ is completely continuous, Q_λ is completely continuous by the continuity of $b(\cdot)$ and $f(\cdot)$. Also, from the definition of T_λ and P_λ, it follows that is Q_λ continuous about λ. From the above arguments, we can obtain the following lemma immediately ■

Lemma III.1.7 *.Suppose the assumptions (H1), (H2) hold and* $c \in (-\min\left\{k_1, \frac{m}{M+m}\right\}, 0]$. *Then* $Q_\lambda(K) \subset K$.

Proof: From the above arguments, it is easy to verify that $Q_\lambda y(t+\omega) = Q_\lambda y(t)$.For $y \in K$, we have

$$\begin{aligned} Q_\lambda y(t) &= P_\lambda(b(t) f((A^{-1}y)(t-\tau(t))) \ge T_\lambda(b(t) f((A^{-1}y)(t-\tau(t)))ds \\ &= \lambda \int_t^{t+\omega} G(t,s) b(s) f((A^{-1}y)(s-\tau(s)))ds \\ &\ge \lambda l \int_0^\omega b(s) f((A^{-1}y)(s-\tau(s))ds \\ Q_\lambda y(t) &= P_\lambda(b(t) f((A^{-1}y)(t-\tau(t))) \\ &\le \frac{M(1-|c|)}{m-(M+m)|c|} \left\|T_\lambda b(s) f((A^{-1}y)(-\tau(s)))\right\|_X \\ &= \lambda \frac{M(1-|c|)}{m-(M+m)|c|} \max_{t\in[0,\omega]} L \int_t^{t+\omega} G(t,s) b(s) f((A^{-1}y)(s-\tau(s)))ds \\ &\le \lambda \frac{M(1-|c|)}{m-(M+m)|c|} L \int_0^\omega b(s) f((A^{-1}y)(s-\omega(s)))ds \end{aligned}$$

Therefore

$$Q_\lambda y(t) \ge \frac{I[m-(M+m)|c|]}{LM(1-|c|)} \|Q_\lambda y\|_X = \alpha \|Q_\lambda y\|_X$$

So Q_λ $(K) \subset k$ This completes the proof ■

Lemma III.1.8 *. Suppose the assumptions hold and (H1), (H2) hold and* $c \in (-\min\left\{k_1, \frac{m}{M+m}\right\}, 0]$. *Then* $y(t)$ *is a positive fixed point of* Q_λ *if and only if* $(A^{-1}y)$ *is a positive* $\omega\_$ *solution of(III.0.1).*

Proof: *If* $y(t)$ *is a positive fixed point of* Q_λ, *then* $y(t) = P_\lambda(b(t) f((A^{-1}y)(t-\tau(t))))$ *and* $y \in K$ *from (III.1.7). By (III.1.6),* $y(t)$ *is a positive* $\omega\_$ *periodic solution of the equation*

$$y''(t) + a(t) y(t) - a(t) G((y(t)) = \lambda b(t) f((A^{-1}y)(t-\tau(t)).$$

That is, $y(t)$ *is a positive* $\omega\_$ *periodic solution of (III.1.2). Since* $y \in K$, *it follows from (III.1.4) that* $(A^{-1}y)(t) > 0$. *Therefore,* $(A^{-1}y)(t)$ *is a positive* $\omega\_$ *periodic solution* *of(III.0.1) by (**??**).Suppose that there exists $y \in X$ such that $(A^{-1}y)(t)$ is a positive $\omega_$periodic solution of (III.0.1).(**??**) tells that $y(t)$ is an $\omega_$periodic solution of (III.1.2) , that is, $y(t)$ is an $\omega\_$ periodic solution of the equation*

$$y''(t) + a(t)\, y(t) - a(t)\, G((y(t)) = \lambda b(t)\, f((A^{-1}y)(t - \tau(t))$$

Additionally, $y(t) = (A^{-1}y)(t) - c(A^{-1}y)(t-\delta) > 0$ since $c \in (-\min\left\{k_1, \frac{m}{M+m}\right\}, 0]$. It follows from (III.1.6)that $y(t) = P_\lambda(b(t)\, f((A^{-1}y)(t-\tau(t)))) = Q_\lambda y(t)$. Thus $y(t)$ is a positive fixed point of Q_λ From (III.1.2)(III.1.8), in order to discuss existence of positive $\omega\_$ periodic solutions of(III.0.1), it is sufficient to consider existence of positive fixed points of Q_λ. The following is about our prior estimations which play important roles in the proofs of our main results ■

Lemma III.1.9 .. *Suppose the assumptions (H1), (H2) hold and $c \in (-\min\left\{k_1, \frac{m}{M+m}\right\}, 0]; \eta > 0$. If $f((A^{-1}y)(t-\tau(t))) \geq (A^{-1}y)(t-\tau(t))\eta$ for $t \in [0,\omega]$ and $y \in K$, then*

$$\|Q_\lambda y\|_X \geq \lambda \bar{b} l \omega \eta \frac{\alpha - |c|}{1 - c^2} \|y\|_X$$

Proof: *For $c \in (-\min\left\{k_1, \frac{m}{M+m}\right\}, 0]$ and $y \in K$, we have*

$$\begin{aligned}
Q_\lambda y(t) &= P_\lambda(b(t)\, f((A^{-1}y)(t-\tau(t))) \geq T_\lambda(b(t)\, f((A^{-1}y)(t-\tau(t)))ds \\
&= \lambda \int_t^{t+\omega} G(t,s)\, b(s)\, f((A^{-1}y)(x - \tau(s)\, ds \\
&\geq l\lambda\eta \int_0^\omega b(s)\, f((A^{-1}y)(s-\tau(s)))ds \\
&\geq I\lambda \bar{b} l \omega \eta \frac{\alpha - |c|}{1-c^2} \|y\|_X
\end{aligned}$$

Hence

$$\|Q_\lambda y\|_X \geq I\lambda \bar{b} l \omega \eta \frac{\alpha - |c|}{1-c^2} \|y\|_X$$

■

Lemma III.1.10 . *Suppose the assumptions hold and (H1), (H2) $c \in (-\min\left\{k_1, \frac{m}{M+m}\right\}, 0]$. If there exists $\epsilon > 0$ such that $f((A^{-1}y)(t-\tau(t)) \leq (A^{-1}y)(t-\tau(t))\epsilon$ for $t \in [0,\omega]$, then*

$$\|Q_\lambda y\|_X \leq \lambda \bar{b} \omega \epsilon \frac{LM}{m - (M+m)|c|} \|y\|_X$$

Proof: In view of (III.1.1), (III.1.4) and(III.1.6), we have

$$\begin{aligned}
\|Q_\lambda y\|_X &\leq \lambda \frac{M(1-|c|)}{m - |c|(M+m)} L \int_0^\omega b(s)\, f\left((A^{-1}y)(s - \tau(s)\right) ds \\
&\leq \lambda \frac{M(1-|c|)}{m - |c|(M+m)} L\epsilon \int_0^\omega b(s)\, f\left((A^{-1}y)(s - \tau(s)\right) ds \\
&\leq \lambda \bar{b} \omega \epsilon \frac{M(1-|c|)}{m - |c|(M+m)} \|y\|_X
\end{aligned}$$

■

Lemma III.1.11 *Suppose the assumptions(H1), (H2) hold and* $c \in (-\min\left\{k_1, \frac{m}{M+m}\right\}, 0]$. *If* $y \in \partial\Omega_r$, $r > 0$, *then*

$$\|Q_\lambda y\|_X \geq I\lambda\bar{b}\omega m(r)$$

Proof: *Since* $y \in \partial\Omega_r$ *and* $c \in (-\min\left\{k_1, \frac{m}{M+m}\right\}, 0]$,*by (III.1.4), we obtain* $\frac{\alpha-|c|}{1-c_2}r \leq (A^{-1}y)(s-\tau(s)) \leq \frac{r}{1-|c|}$. *Thus* $f(A^{-1}y)(s-\tau(s) \geq m(r)$. *Then it is easy to see that this lemma can be proved in a similar manner as in (III.1.9)* ∎

Lemma III.1.12 . *Suppose the assumptions(H1), (H2)hold and* $c \in (-\min\{k_1, \frac{m}{M+m}\}, 0]$. *If* $y \in \partial\Omega_r$, $r > 0$, *then*

$$\|Q_\lambda y\|_x \leq L\lambda\bar{b}\omega\frac{M(1-|c|)}{m-|c|(M+m)}M(r)$$

Proof: *Since* $y \in \partial\Omega_r$ *and* $c \in (-\min\left\{k_1, \frac{m}{M+m}\right\}, 0]$, *by (III.1.4), we obtain* $0 \leq (A^{-1}y)(t-\tau(t)) \leq \frac{r}{1-|c|}$. *Thus* $f((A^{-1}y)(t-\tau(t))) \leq M(r)$. *Then it is easy to see that this lemma can be shown in a similar manner as in (III.1.10).* ∎

III.2 Existence of positive periodic solutions

In this section, we give the proofs of main results based on lemmas in Section 1. We discuss existence of positive $\omega\_$ periodic solutions of Eq (III.0.1). when $c \in (-\min\left\{k_1, \frac{m}{M+m}\right\}; 0]$.

Theorem III.2.1 *Suppose the assumptions (H1), (H2) hold and*

$$-\min\left\{\frac{m}{M+m}, k_1\right\} < c \leq 0$$

(a) If $i_0 = 1$ *or* 2, *then (III.0.1) has* i_0 *positive* $\omega\_$ *periodic solutions for* $\lambda > \frac{1}{m(1)\bar{b}l\omega} > 0$.
(b) If $i_\infty = 1$*or* 2, *then (III.0.1) has* i_∞ *positive* $\omega\_$ *periodic solutions for* $0 < \lambda < \frac{m-(M+m)|c|}{\lambda L\bar{b}\omega(M-M|c|)M(1)}$.
(c) If $i_\infty = 0$ *or* $i_0 = 0$, *then (III.0.1) has no positive* $\omega\_$ *periodic solution for sufficiently small or large* $\lambda > 0$, *respectively.*

Proof: Part (a) : Take $r_1 = 1$ and $\lambda_0 = \frac{1}{m(r_1)\bar{b}l\omega} > 0$. By(III.1.11) for $y \in \partial\Omega_r$ and $\lambda > \lambda_0$,

$$\|Q_\lambda y\|_X > \|y\|_X$$

From(III.2.1), $i(Q_\lambda, \Omega_{r_1}, K) = 0$.

Case I : If $f_0 = 0$, we can choose $0 < \overline{r_2} < r_1$ so that $f(u) \leq \epsilon u$ for $0 \leq u \leq \overline{r_2}$, where the constant $\epsilon > 0$ satisfies

$$\lambda\bar{b}\omega\epsilon\frac{LM}{m-|c|(M+m)} < 1 \tag{III.2.1}$$

Let $r_2 = (1-|c|)\overline{r_2}$. Since $0 \leq (A^{-1}y)(t-\tau(t) \leq \frac{1}{1-|c|}\|y\|_X \leq \overline{r_2}$. for $y \in \partial\Omega_{r_2}$ by(III.1.4), we obtain $f((A^{-1}y)(t-\tau(t))) \leq \epsilon(A^{-1}y)(t-\tau(t))$. Thus we have by(III.1.10) and(III.2.1) that, for , $y \in \partial\Omega_r$

$$\|Q_\lambda y\|_X \leq \lambda\bar{b}\omega\epsilon\frac{LM}{m-|c|(M+m)}\|y\|_X \leq \|y\|_X$$

It follows from (III.2.1) that $i(Q_\lambda, \Omega_{r_3}, K) = 1$. Thus $i(Q_\lambda, \Omega_{r_1} \backslash \overline{\Omega_{r_2}}, K) = -1$ and $Q_\lambda$has a fixed point y in $\Omega_{r_1} \backslash \overline{\Omega_{r_2}}$. By (III.1.8), we see that $(A^{-1}y)(t)$ is a positive ω_periodic solution of (III.0.1) for $\lambda > \lambda_0$.

Case II : If $f_\infty = 0$, there exists a constant $\overline{H} > 0$ such that $f(u) \leq \epsilon u$ for $u \geq \overline{H}$, where the constant $\epsilon > 0$ satisfies inequality(III.2.1) Let $r_3 = \max\left\{2r_1, \frac{\overline{H}(1-c^2)}{\alpha-|c|}\right\}$. Since $(A^{-1}y)(t-\tau(t)) \geq \frac{\alpha-|c|}{1-c^2}\|y\|_X \geq \bar{H}$ for $y \in \partial\Omega_r$, we obtain $f((A^{-1}y)(t-\tau(t))) \leq \epsilon(A^{-1}y)(t-\tau(t))$. Thus we have by (III.1.10) and (III.2.1) that, for $y \in \partial\Omega_r$

$$\|Q_\lambda y\|_X \leq \lambda\bar{b}\omega\epsilon\frac{LM}{m-|c|(M+m)}\|y\|_X \leq \|y\|_X$$

It follows from (III.2.1) that $i(Q_\lambda, \Omega_{r_3}, K) = 1$. Thus $i(Q_\lambda, \Omega_{r_3} \backslash \overline{\Omega_{r_1}}, K) = -1$ and Q_λ has a fixed point y in $\Omega_{r_3} \backslash \overline{\Omega_{r_1}}$ .By(III.1.8) we see that $(A^{-1}y)$ is a positive ω_ periodic solution of (III.0.1) for $\lambda > \lambda_0$.

Case III : If $f_\infty = f_0 = 0$, from the above arguments, there exist $0 < r_2 < r_1 < r_3$ such that Q_λ has a fixed point $y_1(t)$ in $\Omega_{r_1} \backslash \overline{\Omega_{r_2}}$ and a fixed point $y_2(t)$ in $\Omega_{r_3} \backslash \overline{\Omega_{r_1}}$. Consequently, $(A^{-1}y_1)(t)$ and $(A^{-1}y_2)(t)$ are two positive ω_periodic solutions of(III.0.1) for $\lambda > \lambda_0$.

Part (b) : Let $r_1 = 1$. Take $\lambda_1 = \frac{m-(M+m)|c|}{L\bar{b}\omega(M-M|c|)M(r_1)} > 0$. By(III.1.12), for $y \in \partial\Omega_r$ and $0 < \lambda < \lambda_1$,

$$\|Q_\lambda y\|_X < \|y\|_X$$

By(III.2.1), $(Q_\lambda, \Omega_{r_1}, K) = 1$.

Case I : If $f_0 = \infty$, we can choose $0 < \overline{r_2} < r_1$ so that $f(u) \geq \eta u$ for $0 \leq u \leq \overline{r_2}$, where the constant $\eta > 0$ satisfies

$$\lambda\bar{b}\omega\eta\frac{\alpha-|c|}{1-c_2} > 1. \tag{III.2.2}$$

Let $r_2 = (1-|c|)\overline{r_2}$. Since $0 \leq (A^{-1}y)(t-\tau(t)) \leq \frac{1}{1-|c|}\|y\|_X \leq r_2$ for $y \in \partial\Omega_r$, we obtain $(A^{-1}y)(t-\tau(t)) \geq \eta(A^{-1}y)(t-\tau(t))$. Thus we have by (III.1.9)and(III.2.2) that, for $y \in \partial\Omega_{r_2}$,

$$\|Q_\lambda y\|_X \geq \lambda\bar{b}\omega\eta\frac{\alpha-|c|}{1-c^2}\|y\|_X > \|y\|_X$$

It follows from (III.2.1) that $i(Q_\lambda, \Omega_{r_3}, K) = 1$ Thus $i(Q_\lambda, \Omega_{r_1} \backslash \overline{\Omega_{r_2}}, K) = -1$ and Q_λ has a fixed point y in $\Omega_{r_1} \backslash \overline{\Omega_{r_2}}$. By (III.1.2), we see that $(A^{-1}y)(t)$ is a positive ω_ periodic solution of (III.0.1)for $\lambda \in (0, \lambda_1)$.

Case II : If $f_\infty = \infty$, there exists a constant $\overline{H} > 0$ such that $f(u) \geq \eta u$ for $u \geq \overline{H}$, where the constant $\eta > 0$ satisfies inequality(III.2.2). Let $r_3 = \max\left\{2r_1, \frac{\bar{H}(1-|c|^2)}{\alpha-|c|}\right\}$. Since $(A^{-1}y)(t-\tau(t)) \geq \frac{\alpha-|c|}{1-c^2}\|y\|_X \geq \overline{H}$ for $y \in \partial\Omega_{r_3}$ by(III.1.4), we obtain $f(A^{-1}y)(t-\tau(t))$ $\geq \eta(A^{-1}y)(t-\tau(t))$. Thus we have by(III.1.9) and inequality(III.2.2) that, for $y \in \partial\Omega_{r_3}$

$$\|Q_\lambda y\|_X \geq \lambda\bar{b}l\omega\eta\frac{\alpha-|c|}{1-c^2}\|y\|_X > \|y\|_X$$

It follows from (III.2.1) that $i(Q_\lambda, \Omega_{r_3}, K) = 1$. Thus $i(Q_\lambda, \Omega_{r_3} \backslash \overline{\Omega_{r_1}}, K) = -1$ and Q_λ has a fixed point y in $\Omega_{r_3} \backslash \overline{\Omega_{r_1}}$. By (III.1.8), we see that $(A^{-1}y)$ is a positive ω_periodic solution of(III.0.1)for $\lambda \in (0, \lambda_1)$.

Case III : If $f_\infty = f_0 = \infty$, it is clear from the above proofs that Q_λ has a fixed point y_1 in $\Omega_{r_1} \backslash \overline{\Omega_{r_2}}$ and a fixed point y_2 in $\Omega_{r_3} \backslash \overline{\Omega_{r_1}}$. Consequently, $(A^{-1}y_1)(t)$ and $(A^{-1}y_2)(t)$ are two positive $\omega\_$ periodic solutions of (III.0.1) for $\lambda \in (0, \lambda_1)$.

Part (c) : By (III.1.4), $(A^{-1}y)(t-\tau(t)) \geq \frac{\alpha-|c|}{1-c^2} \|y\|_X > \|y\|_X$ for $t \in [0,\omega]$ and $y \in K$.

Case I : If $i_0 = 0$, we have $f_0 > 0$ and $f_\infty > 0$. Let $c_1 = \min\left\{\frac{f(u)}{u} : u > 0\right\} > 0$, then we obtain $f(u) \geq c_1 u$, $u \in [0, +\infty)$. Assume $y(t)$ is a positive $\omega\_$ periodic solution of (III.0.1) for $\lambda > \lambda_2$, where $\lambda_2 = \frac{1-c_2}{\bar{b}l\omega c_1(\alpha-|c|)}$. Since $Q_\lambda y(t) = y(t)$ for $t \in [0,\omega]$, it follows from (III.1.9) that, for $\lambda > \lambda_2$,

$$\|y\|_X = \|Q_\lambda y\|_X \geq \lambda \bar{b} l \omega c_1 \frac{\alpha - |c|}{1-c^2} \|y\|_X > \|y\|_X$$

which is a contradiction.

Case II : If $i_\infty = 0$, we have $f_0 < \infty$ and $f_\infty < \infty$. Let $c_2 = \max\left\{\frac{f(u)}{u} : u > 0\right\} > 0$, then we obtain $f(u) \leq c_2 u, u \in [0, +\infty)$. Assume $y(t)$ is a positive $\omega\_$ periodic solution of (III.0.1) for $\lambda \in (0, \lambda_3)$, where $\lambda_3 = \frac{m-(M+m)|c|}{\bar{b}\omega c_2 LM}$. Since $Q_\lambda y(t) = y(t)$ for $t \in [0,\omega]$ it follows from (III.1.10) that, for $\lambda \in (0, \lambda_3)$

$$\|y\|_X = \|Q_\lambda y\|_X \geq \lambda \bar{b} l \omega c_2 \frac{LM}{m - |c|(M+m)} \|y\|_X > \|y\|_X$$

which is a contradiction. ∎

Theorem III.2.2 . *Suppose the assumptions (H1), (H2) hold and* $-\min\left\{\frac{m}{M+m}, k_1\right\} < c \leq 0$.

(a) *If there exists a constant* $c_1 > 0$ *such that* $f(u) \geq c_1 u$ *for* $u \in [0, +\infty)$, *then (III.0.1) has no positive* $\omega\_$ *periodic solution for* $\lambda > \frac{1-c_2}{\bar{b}l\omega c1(\alpha-|c|)}$

(b) *If there exists a constant* $c_2 > 0$ *such that* $f(u) \leq c_2 u$ *for* $u \in [0, +\infty)$, *then (III.0.1) has no positive* $\omega\_$ *periodic solution for* $0 < \lambda < \frac{m-(M+m)|c|}{\bar{b}\omega c_2 LM}$.

Proof: From the proof of part (c) in(III.2.1), we obtain this theorem immediately ∎

Theorem III.2.3 *Suppose the assumptions (H1), (H2) hold,* $c \in (-\min\left\{k_1, \frac{m}{M+m}\right\}, 0]$ *and* $i_0 = i_\infty = 0$. *If*

$$\frac{1-c_2}{\max\{f_0, f_\infty\}\bar{b}\omega l(\alpha - |c|)} < \lambda < \frac{m-(M+m)|c|}{\max\{f_0, f_\infty\}\bar{b}\omega LM}$$

then (III.0.1) has one positive $\omega_$periodic solution.

Proof: Case I : $f_0 \leq f_\infty$. In this case, we have

$$\frac{1-c_2}{l\bar{b}\omega f_\infty(\alpha - |c|)} \leq \lambda \leq \frac{m-(M+m)|c|}{f_0 \bar{b}\omega LM}$$

It is clear that there exists an $0 < \epsilon < f_\infty$ such that

$$!\frac{1-c_2}{l\bar{b}\omega(f_\infty - \epsilon)(\alpha - |c|)} \leq \lambda \leq \frac{m-(M+m)|c|}{(f_0 - \epsilon)\bar{b}\omega LM}$$

For the above ϵ, we choose $0 < \overline{r_1}$ so that $f(u) \leq (f_0 - \epsilon)u$ for $0 \leq u \leq \overline{r_1}$. Let $r_1 = (1-|c|)\overline{r_1}$. Since $0 \leq (A^{-1}y)(t-\tau(t)) \leq \frac{1}{1-|c|}\|y\|_X \leq \overline{r_1}$ for $y \in \partial\Omega_{r_1}$ by(III.1.4), we obtain $f(A^{-1}y)(t-\tau(t)) \leq (f_0-\epsilon)(A^{-1}y)(t-\tau(t))$. Thus we have by(III.1.10) that, for $y \in \partial\Omega_{r_1}$,

$$\|Q_\lambda y\|_X \leq \lambda\bar{b}\omega(f_0+\epsilon)\frac{LM}{m-|c|(M+m)}\|y\|_X < \|y\|_X$$

On the other hand, there exists a constant $\overline{H} > 0$ such that $f(u) \geq (f_0 - \epsilon)u$ for $u \geq \overline{H}$ Let $r_2 = \max\left\{2r_1, \frac{\overline{H}(1-c^2)}{\alpha-|c|}\right\}$. Since $(A^{-1}y)(t-\tau(t)) \geq \frac{\alpha-|c|}{1-c^2}\|y\|_X > \overline{H}$. for $y \in \partial\Omega_{r_2}$, by(III.1.4), we obtain $f((A^{-1}y)(t-\tau(t))(f_\infty - \epsilon)(A^{-1}y))(t-\tau(t))$. Thus we have by(III.1.9) that, for $y \in \partial\Omega_{r_2}$

$$\|Q_\lambda y\|_X \geq \lambda\bar{b}\omega(f_0-\epsilon)\frac{\alpha-|c|}{1-c^2}\|y\|_X > \|y\|_X$$

It follows from(III.2.1) that $i(Q_\lambda, \Omega_{r_1}, K) = 1, i(Q_\lambda, \Omega_{r_2}, K) = 1$ Thus $i(Q_\lambda, \Omega_{r_2}\backslash\overline{\Omega_{r_1}}, K) = -1$ and Q_λ has a fixed point y in $\Omega_{r_2}\backslash\overline{\Omega_{r_1}}$. By(III.1.8), we see that $(A^{-1}y)(t)$ is a positive $\omega\_$ periodic solution of (III.0.1)

Case II : $f_0 > f_\infty$. In this case, we have

$$\frac{1-c_2}{\bar{b}lf_\infty\omega(\alpha-|c|)} < \lambda < \frac{m-(M+m)|c|}{f_0\bar{b}\omega LM}$$

It is clear that there exists an $0 < \epsilon < f_0$ such that

$$\frac{1-c_2}{\bar{b}l\omega(f_\infty-\epsilon)(\alpha-|c|)} < \lambda < \frac{m-(M+m)|c|}{(f_0-\epsilon)\bar{b}\omega LM}$$

For the above ϵ, we choose $0 < r_1$ so that $f(u)(f_0-\epsilon)u$ for $0 \leq u \leq \overline{r_1}$. Let $r_1 = (1-|c|)\overline{r_1}$. Since $0 \leq (A^{-1}y)(t-\tau(t)) \leq \frac{1}{1-c}\|y\|_X \leq \overline{r_1}$ for $y \in \partial\Omega_{r_1}$. We obtain $f(A^{-1}y)(t-\tau(t))$ $\geq (f_0-\epsilon)(A^{-1}y)(t-\tau(t))$ Thus we have by (III.1.9) that, for $y \in \partial\Omega_{r_1}$,

$$\|Q_\lambda y\|_X \geq \lambda\bar{b}\omega(f_0-\epsilon)\frac{\alpha-|c|}{1-c^2}\|y\|_X < \|y\|_X$$

On the other hand, there exists a constant $\overline{H} > 0$ such that $f(u) \leq (f_0 + \epsilon)u$ for $u \geq \overline{H}$. Let $r_2 = \max\left\{2r_1, \frac{\overline{H}(1-c^2)}{\alpha-|c|}\right\}$. Since $(A^{-1}y)(t-\tau(t)) \geq \frac{\alpha-|c|}{1-c^2}\|y\|_X \geq \overline{H}$ for $y \in \partial\Omega_{r_2}$ we obtain $f((A^{-1}y)(t-\tau(t))(f_\infty+\epsilon)(A^{-1}y))(t-\tau(t))$. Thus we have by (III.1.10) that, for $y \in \partial\Omega_{r_2}$

$$\|Q_\lambda y\|_X \leq \lambda\bar{b}\omega(f_0+\epsilon)\frac{LM}{m-(M+m)|c|}\|y\|_X < \|y\|_X$$

It follows from (III.2.1) that $i(Q_\lambda, \Omega_{r_1}, K) = 1$. $i(Q_\lambda, \Omega_{r_2}, K) = 1$. Thus $i(Q_\lambda, \Omega_{r_2}\backslash\overline{\Omega_{r_1}}, K) = 1$ and Q_λ has a fixed point y in $\Omega_{r_2}\backslash\overline{\Omega_{r_1}}$. By (III.1.8), we see that $(A^{-1}y)(t)$ is a positive $\omega\_$ periodic solution of(III.0.1). Our results are applicable to consider existence problem of periodic solutions of many neutral differential systems. ■

III.3 Applications

Example III.3.1 *We consider the following neutral functional differential equation :*

$$\left(u\,(t)+\frac{1}{3}u(t-\frac{\pi}{2})\right)''+\frac{1}{4}u\,(t)=\lambda[1-\sin 2t]u^{a}(t-\tau\,(t))e^{-u(t-\tau(t))} \tag{III.3.1}$$

where λ and a are positive parameters, $\tau(t+\pi)\equiv\tau\,(t)$. We see that $\delta=\frac{\pi}{2}$,

$$c=-\frac{1}{3},\quad a\,(t)\equiv\frac{1}{4},\quad b\,(t)=1-\sin 2t,\quad f(u)=u^{a}e^{-u},\qquad M=m=\frac{1}{4}$$

Additionally,

$$\max_{u\in[0,\infty)} f(u)=f(a)$$

Clearly, $M=\frac{1}{4}<(\frac{\pi}{\pi})^{2}=1$. The assumptions (H1) and (H2) are satisfied and $f_\infty=0$. Then we conclude.

Conclusion 1 *(a) If $a\in(0,1)$, then (III.3.1) has one positive π_periodic solution for $\lambda>\frac{1}{\pi r_0}>0$ or $0<\lambda<\frac{\sqrt{2}}{4\pi f(a)}$, where $r_0=\min\left\{f(\frac{\sqrt{2}}{4}),f(\frac{3}{2})\right\}$;*

(b) If $a=1$, then (III.3.1) has one positive π_periodic solution for $\lambda>\frac{1}{\pi r_0}>0$

(c) If $a>1$, then (III.3.1) has two positive π_periodic solutions for $\lambda>\frac{1}{\pi r_0}>0$.

In fact, by simple computations, we have

$$\begin{aligned}
\beta &= \frac{1}{2},\qquad L=\frac{1}{2\beta\sin\frac{\beta\pi}{2}}=\sqrt{2},\qquad l=\frac{\cos\frac{\beta\pi}{2}}{2\beta\sin\frac{\beta\pi}{2}}\\
k &= \frac{2+\sqrt{2}}{4}\qquad \alpha=\frac{\sqrt{2}}{4},\qquad k_1=\frac{\sqrt{2}+1-\sqrt{3}}{2},\\
|c| &= \frac{1}{3}<\min\left\{k_1,\frac{m}{m+M}\right\}=\frac{\sqrt{2}+1-\sqrt{3}}{2},|c|=\frac{1}{3}<\frac{\sqrt{2}}{4}=\alpha
\end{aligned}$$

Let $t_0=\min\left\{a,\frac{3}{2}\right\}$ and $r_0=\min\left\{f(\frac{\sqrt{2}}{4}),f(\frac{3}{2})\right\}$, we have

$$M(1)=\max\left\{f\,(t):0\le t\le\frac{3}{2}\right\}=f(t_0),$$

$$m(1)=\min\left\{f\,(t):\frac{\sqrt{2}}{4}\le t\le\frac{3}{2}\right\}=\min\left\{f(\frac{\sqrt{2}}{4}),f(\frac{3}{2})\right\}=r_0$$

$$\frac{1}{m(1)l\omega\bar{b}}=\frac{1}{\pi r_0},.\ \frac{m-(M+m)\,|c|}{L\bar{b}\omega(M-M\,|c|)M(1)}=\frac{\sqrt{2}}{4\pi f(t_0)}$$

Additionally, if $a\in(0,1)$, $f_0=+\infty$; if $a=1$, $f_0=1$ and $f_\infty=0$; if $a>1$, $f_0=f_\infty=0$. From Theorem (III.2.1), we can obtain Conclusion (1).

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