People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

University of El Oued

Faculty of Exact Sciences

Laboratory of Operator Theory and PDEs

A Doctoral Thesis

Submitted in Fulfillment of the Requirements for Degree of Doctor (LMD)

Field: Mathematics

Specialization: Applied Mathematics

On the time behavior of some thermos-elastic problems with various laws of thermal effects

Presented by: BELKHIR Naziha

Defended on [21/02/2026], in the presence of the Examination Committee:

Name	Rank	Affiliation	Role
Dehda Bachir	Professor	Univ. El Oued	Chair
Fareh Abdelfeteh	Professor	Univ. El Oued	Supervisor
Messaoudi Salim	Professor	Univ. Sharjah UAE	Examiner
Zarai Abderrahmane	Professor	Univ. Tebassa	Examiner
Doudi Nadjet	MCA	Univ. El Oued	Examiner
Touati Brahim Med Said	MCA	Univ. El Oued	Examiner

Academic Year: 2025-2026 / 1447-1448 AH

Dedication

I dedicate this work to:

My beloved father, may his soul rest in peace.

My dearest mother.

My dear husband.

My dear daughter.

My dear brothers and sisters.

My kind parents-in-law.

My family, my teachers, and my friends.

Acknowledgements

All praise is due to Allah, who granted me the strength, patience, and perseverance to complete this research.

I would like to express my deepest gratitude to my family, especially my late **father**, who could not witness my graduation but was always optimistic and proud of my academic journey; to my dear **mother**, whose patient support and constant encouragement guided me throughout this path; and to my beloved brother **Saeed**, who eagerly awaited this day but passed away shortly before my defense. This achievement is dedicated to their memory, their love, and their unwavering belief in me.

Secondly, I would like to express my sincere appreciation to my supervisor, Professor **Fareh Abdelfeteh**, for his continuous guidance, invaluable advice, and unwavering support since the beginning of my research. I am grateful for his patience and care, as he was always available to discuss and answer my questions. May Allah reward him abundantly.

I would also like to express my sincere gratitude to Professor **Salim Messaoudi** for honoring me with his virtual presence at my defense, as well as for his time and valuable contributions, which I deeply appreciate. .

I am very grateful to the president of the jury, Professor **Dehda Bachir**, and to the examiners: Professors **Zarai Abderrahmane**, **Doudi Nadjet**, and **Touati Brahim Mohammed Said**, for agreeing to evaluate this thesis and for their insightful comments and suggestions.

I also wish to express my sincere appreciation to Dr. **Askri Souhila** for her guidance and kind support.

In addition, I am grateful to all the teachers of the Department of Mathematics for their valuable assistance and support.

Finally, I extend my heartfelt thanks to my husband, **Nacer Halouadji**, for his patience, understanding, and constant encouragement, to my little daughter, **Maouadda**, whose smile brightened every step of this journey, and to my father-in-law, Dr.**Ali Halouadji**, for his continuous support and wise counsel.

Abstract

In this thesis, we study the long-time behavior of certain truncated porous thermoelastic systems, which are free from the adverse effects associated with the second spectrum of the frequencies. We examine four problems involving different heat effect conduction laws, and prove exponential stability result for each system.

In the first problem, heat conduction is governed by the classical Fourier law, resulting in a hyperbolic-parabolic coupling. In the second problem, the thermal disturbance is described by the Cattaneo law, leading to a problem characterized bysecond-sound heat conduction. The third problem consists hereditary heat conduction governed by the Gurtin-Pipkin law. Finally, the fourth system addresses a porous thermoelastic system in which heat conduction follows the type III model introduced by Green and Naghedi.

For all the problems, well-posedness is established using a semigroup approach along with the Hille-Yosida and Lax-Milgram theorems. Additionally, stability is achieved by employing the multiplier method and Lyapunov functionals.

Keywords: Porosity, thermoelasticity, well-posedness, exponential stability, second spectrum, truncated equation.

Résumé

Dans cette thèse, nous étudions le comportement à long terme de certains systèmes thermoélastiques poreux tronqués, non affectés par les effets indésirables liés au second spectre de fréquences. Nous analysons quatre problèmes présentant des effets thermiques distincts et démontrons un résultat de stabilité exponentielle pour chacun de ces systèmes.

Dans le premier problème, la conduction thermique est régie par la loi classique de Fourier, ce qui conduit à un couplage hyperbolique-parabolique. Dans le deuxième problème, la perturbation thermique est décrite par la loi de Cattaneo, aboutissant à un modèle caractérisé par une conduction de chaleur du second son. Le troisième problème carractérisé par la conduction thermique héréditaire, régie par la loi de Gurtin-Pipkin. Enfin, le quatrième système traite d'un milieu thermoélastique poreux dans lequel la conduction thermique suit le modèle de type III, tel que décrit par Green et Naghedi.

Pour tous ces problèmes, l'existence et l'unicité ont été établies grâce à une approche par semi-groupes ainsi qu'aux théorèmes de Hille-Yosida et de Lax-Milgram. De plus, la stabilité a été démontrée par la méthode des multiplicateurs et les fonctionnelles de Lyapunov.

Mots clés: Porosité, thermo-élasticité, second spectre, existence et unicité, décroissance exponentielle, équation tronquée.

ملخص

تتناول هذه الأطروحة دراسة السلوك بعيد المدى لبعض أنظمة المرونة الحرارية المسامية المبتورة، والتي تخلو من التأثيرات السلبية المرتبطة بالطيف الثاني للترددات. تم تحليل أربع مسائل تختلف في قوانين التأثيرات الحرارية، مع اثبات خاصية الاستقرار الأسي لكل نظام على حدة.

في المسألة الأولى، تخضع عملية انتقال الحرارة لقانون فورييه الكلاسيكي (law Fourier) ، مما يؤدي اقتران زائدي-قطعي. أما في المسألة الثانية، فيوصف الاضطراب الحراري وفق قانون كاتانيو (law Cattaneo) ، مما ينتج نموذجا يمثل ظاهرة الصوت الثاني في المرونة الحرارية. وتتناول المسألة الثالثة انتقال الحرارة بتأثير الذاكرة المحكوم بقانون غورتن- بيبكين (Gurtin-Pipkin). في حين يدرس النظام الرابع نموذجا لمسألة حرارية مسامية لزجة يعتمد انتقال الحرارة فيها على نموذج من النوع الثالث الذي قدمه غرين وناجدي (Naghdi and Green).

تم إثبات وجود ووحدانية الحلول لجميع الأنظمة المدروسة باستخدام منهج أنصاف الزمر، بالاستعانة بنظريتي (Hille-Yocida) و (Lax-Milgram). كما تم تحقيق نتائج الاستقرار الأسي باستعمال طريقة المضاعف ودوال ليابونوف (functionals Lyapunov).

الكلمات المفتاحية: المسامية، المرونة الحرارية، وجود ووحدانية، الإستقرار الأسي، الطيف الثاني، المبتورة.

Notation

H^m	Sobolev Spaces,
$H_0^1(0,L)$	the closure of $C_0^\infty(0,L)$ in $H^1(0,L)$,
$H^{-1}(0,L)$	$= \mathscr{L}\left(H_0^1(0,L),\mathbb{R}\right)$ the dual space of $H_0^1(0,L)$,
$C_0^\infty(0,L)$	the test functions space,
$L^p(0,L),\ L^\infty(0,L)$	the Lebesgue space,
$L^q(0,L) = (L^p(0,L))'$	the dual space of $L^p(0,L)$, $\dfrac{1}{p}+\dfrac{1}{q}=1$,
$D(\mathcal{A})$	the domain of the operator $\mathcal{A}$,
$\sigma(\mathcal{A})$	The spectrum of operator $\mathcal{A}$,
$\rho(\mathcal{A})$	The resolvent set of the operator $\mathcal{A}$,
$N(\mathcal{A}) = ker(\mathcal{A})$	The kernel of the operator $\mathcal{A}$,
$R(\mathcal{A})$	The range of the operator $\mathcal{A}$,
$\lvert\cdot\rvert$	The euclidean norm on $\mathbb{R}^d$,
$\lVert\cdot\rVert_H$	The norm on a normed space H,
$\langle\cdot,\cdot\rangle$	the inner Product in a Hilbert space,
∂	The operator of partial differentiation,
$\mathscr{L}(H)$	The space of bounded linear operators from H into H,
H'	The dual space of H,
$Re\langle .,.\rangle$	The real part of the inner product,
$(T(t))_{t\geqslant 0}$	A semigroup of linear operators,
$C(X,Y)$	The space of all continuous functions from X into Y,
$a.e.$	almost everywhere (except on a negligible set).

Contents

Introduction

Porous media theory

In recent years, the study of the asymptotic behavior of solutions of mathematical models describing elastic materials has garnered increasing attention from researchers in mathematical analysis and theoretical mechanics. This growing interest stems from the crucial role these models play in explaining the behavior of materials with complex internal structures, such as porous materials or materials containing voids.

Due to their significant practical relevance, a substantial body of recent research has focused on the rigorous analysis of the existence and stability properties of solutions to these elasticity systems.

This theory envisions the material as a composite structure consisting of a solid matrix interlaced with a network of interconnected empty pores. Its development has been shaped by successive advancements and key milestones that have progressively refined and strengthened its theoretical framework.

To the best of our knowledge, the first contribution to describing these materials was made by Goodman and Cowin [25]. In 1972, they developed a model for granular materials based on the assumption that the bulk density is the product of the material matrix density γ and the volume fraction ν; that is, $\rho = \gamma\nu$. This representation endowed the theory with enhanced kinematic flexibility, enabling a more accurate description of the influence of the micro-void structure on the material overall deformation.

Later, in 1979, Nunziato and Cowin [40] extended the previous model by proposing a nonlinear theory to describe the behavior of porous materials, incorporating finite deformations and nonlinear effects.

By 1983, the theory of porous elasticity had matured, culminating in the linear formulation proposed by Cowin and Nunziato [14]. Subsequent advancements by Ieşan [29] incorporated thermoelastic and micro-temperature effects [30, 31], thereby completing the theoretical framework underpinning modern porous thermoelasticity.

To the best of our knowledge, the investigation of the long-term behavior of porous media problems began with the work of Quintanilla. In [44], he examined the behavior of the system:

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + \beta \varphi_x, \\ J\varphi_{tt} = \alpha \varphi_{xx} - \beta u_x - \xi \varphi - \tau \varphi_t, \end{cases} \tag{1}$$

and showed that the solution of system (1) decays slowly. We note that, later in 2017, Apalara [4] and Santos et al. [48] independently proved an exponential rate of decay for system (1) provided that

$$\frac{\mu}{\rho} = \frac{\alpha}{J}$$

Parallel to advancements in the study of porous materials, researchers have increasingly recognized the crucial role of thermal effects in the mechanical behavior of deformable media. This understanding has led to the incorporation of heat conduction effects to porous elastic problems. In the next section, we provide an overview of the fundamental laws governing heat transfer within a material body.

Thermoelasticity

Thermoelasticity offers a comprehensive theoretical framework that describes the interaction between thermal and mechanical fields in deformable media. It examines how temperature variations induce mechanical deformation and how this deformation, in turn, affects the material thermal state. Over time, various thermoelastic models have been developed from classical formulations based on Fourier's law to advanced theories incorporating relaxation, memory effects, and finite-speed heat propagation.

Fourier's Law

Heat transfer within a material body is typically described by Fourier's law of heat conduction, which establishes a linear relationship between the heat flux vector $\mathbf{q}$ and the temperature gradient $\nabla\theta$. This law is expressed as

$$\mathbf{q} = -\kappa \nabla \theta,$$

and, together with the conservation of energy law

$$c\theta_t + \nabla \cdot \mathbf{q} = 0,$$

leads to the well-known parabolic heat equation

$$c\theta_t = \kappa \Delta \theta,$$

where κ denotes the thermal conductivity coefficient and c is the specific heat capacity.

Within the framework of classical porous thermoelasticity, Quintanilla and his coauthors [11, 12, 34] investigated system (1) coupled with the heat equation governed by Fourier's law. They demonstrated that an exponential decay rate can be achieved when porous and thermal damping act together, or when elastic and microthermal damping operate simultaneously. Conversely, when these damping mechanisms do not act simultaneously, the system fails to achieve exponential stabilization and instead exhibits a slower decay behavior. Further insights can be found in [5, 20, 35, 36, 39, 49].

However, the parabolic heat equation derived from Fourier's law leads to a physical inconsistency: it predicts an infinite speed of heat propagation, implying that thermal signals are felt simultaneously throughout the entire domain, which contradicts physical reality. To address this inconsistency, scientists have proposed several alternative models that transform the coupling from a hyperbolic-parabolic type to a hyperbolic-hyperbolic type, thereby resolving the issue of infinite heat propagation speed.

Second sound thermoelasticity

The first model was proposed by Cattaneo [13] in 1958. He introduced the concept of thermal relaxation time, τ, as a key factor in heat transfer. This modification fundamentally altered the nature of heat propagation, changing it from instantaneous diffusion to wave-like behavior. The modified constitutive equation, known as Cattaneo's law, is expressed as follows:

$$q + \tau q_t = -\kappa \nabla \theta.$$

This model represents a significant advancement in understanding heat transfer as a wave phenomenon, as observed in the second sound effect.

Second sound is regarded as one of the most significant phenomena challenging the classical model. It was first experimentally observed in 1944 by Peshkov in superfluid helium II (He II) at cryogenic temperatures [43]. However, its theoretical modeling only became feasible following advancements in the law.

This extension resolves the paradox of infinite propagation speed by introducing a wave-like mechanism of heat transport, similar to mechanical or acoustic waves. Graphically, it is represented by hyperbolic partial differential equations rather than parabolic ones, contradicting the infinite-speed prediction of Fourier's law.

The corresponding heat transport system is described as follows:

$$\begin{cases} c_0\rho\theta_t + \nabla\cdot q = 0 \\ q + \tau q_t + \kappa\nabla\theta = 0, \end{cases}$$

which predicts a finite speed of heat propagation, given by $\nu = \sqrt{\dfrac{\kappa}{c_0\rho\tau}}$.

Within this context, Messaoudi and Fareh [37] considered the system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + \gamma u_{xxt} & \text{in } (0,1)\times(0,+\infty), \\ J\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi + \beta\theta_x & \text{in } (0,1)\times(0,+\infty), \\ c\theta_t = -q_x - \beta\varphi_{xt} - \delta\theta & \text{in } (0,1)\times(0,+\infty), \\ \tau q_t + q + \kappa\theta = 0 & \text{in } (0,1)\times(0,+\infty), \end{cases} \tag{2}$$

They proved an exponential decay result without imposing any restrictions on the system coefficients. In [17], the same authors investigated the system (2) for $\gamma = 0$ and $b^2 = \xi\mu$. They introduced a stability number χ, which depends on the system coefficients, and demonstrated an exponential rate of decay provided that $\chi = 0$.

Recently, Li and Feng [21] examined the system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x & \text{in } (0,1)\times(0,+\infty), \\ J\phi_{tt} = \alpha\phi_{xx} - bu_x - \xi\phi + \beta\theta_x & \text{in } (0,1)\times(0,+\infty), \\ \rho_1\theta_t + q_x + \beta\phi_{xt} = 0 & \text{in } (0,1)\times(0,+\infty), \\ \tau q_t + \gamma q + \kappa\theta = 0 & \text{in } (0,1)\times(0,+\infty), \end{cases} \tag{3}$$

along with the boundary conditions

$$u_x(0,t) = \phi(0,t) = q(0,t) = u(1,t) = \phi_x(1,t) = \theta(1,t) = 0, \;\; t \in (0,+\infty).$$

They presented the stability number

$$\pi = \left(\frac{\rho}{\mu} - \frac{J}{\alpha}\right)\left(1 - \frac{\tau\rho_1\mu}{\rho}\right) - \frac{\tau\beta^2}{\alpha}$$

and established an exponential rate of decay provided that $\pi = 0$ and a polynomial decay in the opposite case.

Green and Naghdi theories

Green and Naghdi [26] proposed a series of pioneering theories aimed at reinterpreting heat conduction within the framework of thermomechanics. Their formulation departs from classical approaches by introducing a new state variable, the thermal displacement, defined as

$$\alpha(x,t) = \alpha_0 + \int_0^t \theta(x,s)\,ds,$$

where α_0 denotes the initial value at the reference time.

$$\frac{d\alpha}{dt}(x,t) = \theta(x,t).$$

This approach offers a natural kinematic interpretation of thermal evolution, ensuring that heat propagation aligns with the mechanical concepts of motion and finite speed. Through this formulation, Green and Naghdi developed a generalized framework for the fundamental balance laws based on novel constitutive relations involving both the temperature θ and the thermal displacement α.

These relations form the foundation of three distinct thermomechanical theories—Type I, Type II, and Type III.

Thermoelasticity of Type I

The standard assumption in this version is that the heat flux vector q and the entropy flux vector h are linearly related through the reference temperature T_0; that is

$$q = T_0\,h,$$

Here, T_0 represents the absolute temperature that links the thermal and entropic fluxes. This assumption results in the same constitutive law as that derived from classical Fourier theory.

$$q = -\kappa \nabla \theta,$$

where κ denotes the thermal conductivity. As a result, the heat equations corresponding to the classical heat conduction theory and the linearized Green and Naghedi Type I formulation appear mathematically identical. However, their physical interpretations differ: the classical model addresses heat conduction phenomena, whereas the Type I formulation incorporates these phenomena within a more rigorous thermodynamic framework.

Thermoelasticity of Type II

This model is based on the constitutive assumption that the entropy flux vector h depends on the gradient of the thermal displacement α; namely,

$$h = -\kappa_2 \, \nabla \alpha,$$

where κ_2 is a material constant. The corresponding evolution equation takes the form:

$$\rho c \, \theta_t = \rho c \, \alpha_{tt} = \kappa_2 \, \Delta \alpha,$$

This is a hyperbolic-type heat equation describing the propagation of thermal disturbances as finite-speed waves. In this case, heat propagates without energy dissipation at a characteristic speed.

$$v = \sqrt{\frac{\kappa_2}{c\rho}}.$$

This formulation offers a theoretical explanation for phenomena such as the second sound observed in certain solids and super-fluids.

Thermoelasticity of Type III

The most comprehensive version of the Green–Naghdi models incorporates the effects of both the temperature gradient and the thermal displacement gradient. In this case, the heat flux vector q is expressed as

$$q = -\kappa_3 \, \nabla \alpha \; - \; \kappa_4 \, \nabla \theta$$

Where κ_3 and κ_4 are material constants characterizing the coupling between thermal and mechanical effects.

The corresponding evolution equation is expressed as follows:

$$\rho c\, \theta_t = \rho c\, \alpha_{tt} = \kappa_3\, \Delta\alpha \; - \; \kappa_4\, \Delta\theta.$$

This mixed formulation captures both wave-like propagation and dissipative effects within a unified framework, offering a more realistic description of heat transport in heterogeneous materials.

Gurtin–Pipkin Theory

In an ongoing effort to deepen the understanding of heat conduction phenomena, Gurtin and Pipkin [27] developed a comprehensive nonlinear theory of heat conduction that effectively incorporates the concept of material memory and allows for the occurrence of second sound in thermoelastic media. A key aspect of their formulation is the introduction of memory kernels, which enable the heat flux to depend on the entire "past" history of the temperature gradient. In the linear approximation, their constitutive law for heat conduction is expressed in integral form:

$$q(x,t) = -\int_{-\infty}^{t} \kappa(t-s)\, \nabla\theta(x,s)\, ds, \tag{4}$$

where $\kappa(t)$ is a relaxation function that characterizes the material memory response. This model extends beyond Fourier's law by accounting for finite propagation speeds and non-local temporal effects. It provides a rigorous framework for studying heat transport in materials with memory.

In this regard, Pata and Vuk [41] studied the linear thermoelastic system within the framework of the Gurtin-Pipkin thermal law. Precisely, they looked into the system

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) - \theta_x(x,t), \\ \theta_t\,(x,t) = -u_{tx}\,(x,t) - q_x\,(x,t)\,, \end{cases}$$

where the heat flux q is modeled by equation (4). They employed a semigroup method and established an exponential stability result under certain assumptions on the kernel $\mu(s). = -g'(s)$.

Next, Fatori and Muñoz Rivera [19] analyzed the system

$$\begin{cases} u_{tt} - a u_{xx} + \alpha \theta_x = 0 & \text{in } (0;L) \times \mathbb{R}_+ \\ \theta_t - k * \theta_{xx} + \alpha u_{xt} = 0 & \text{in } (0;L) \times \mathbb{R}_+, \end{cases}$$

where

$$(k * \theta_{xx})(t) = \int_0^t k(t-\tau)\,\theta_{xx}(\tau)\,d\tau.$$

As long as the kernel k is positive definite and decays exponentially, they obtained an exponential decay result.

In the framework of Gurtin-Pipkin's theory, Timoshenko-type systems have been studied. For instance, Dell'Oro and Pata [16] studied the following system

$$\begin{cases} \rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x = 0, \\ \rho_3 \theta_t - \dfrac{1}{\beta}\displaystyle\int_0^{+\infty} g(s)\,\theta_{xx}(t-s)\,ds + \delta\psi_{tx} = 0. \end{cases} \tag{5}$$

They introduced the stability number

$$\chi_g = \left[\frac{\rho_1}{\rho_3\kappa} - \frac{\beta}{g(0)}\right]\left[\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right] - \frac{\beta}{g(0)}\frac{\rho_1\delta^2}{\rho_3\kappa b}$$

and established exponential stability provided that $\chi_g = 0$.

Fareh [18] examined the porous thermoelastic system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x, \\ J\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi + m\theta - \tau\varphi_t, \\ c\theta_t = -\displaystyle\int_{-\infty}^0 k(t-s)\,\theta_{xx}(s)\,ds - \beta u_{tx} - m\varphi_t. \end{cases} \tag{6}$$

An exponential decay rate was determined regardless to any restrictions on the coefficients of the system (6).

Recently, Hao and Yang [28] investigated the porous thermoelastic system

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\varphi_x - \gamma u_{txx} = 0, & \text{in } (0,1) \times \mathbb{R}^+, \\ J\varphi_{tt} - \delta\varphi_{xx} + bu_x + \xi\varphi + \beta\theta_x = 0, & \text{in } (0,1) \times \mathbb{R}^+, \\ c\theta_t + q_x + \beta\varphi_{xt} = 0, & \text{in } (0,1) \times \mathbb{R}^+, \end{cases} \tag{7}$$

where the heat flux vector q is governed by the Gurtin–Pipkin thermal law (equation 4), and established an exponential decay rate independent of the system coefficients.

Porous thermoelasticity free from second spectrum frequency

It is generally acknowledged that Timoshenko-type systems are affected by the damaging effects associated with the second spectrum of frequencies. In this section, we will demonstrate that this phenomenon also influences porous systems.

If we study the undamped version of system 1 and eliminate the variable φ, we obtain

$$u_{xxxx} + \frac{\beta^2 - \xi\mu}{\mu\alpha} u_{xx} - \frac{J\mu + \rho\alpha}{\mu\alpha} u_{xxtt} + \frac{\xi\rho}{\mu\alpha} u_{tt} + \frac{J\rho}{\mu\alpha} u_{tttt} = 0. \tag{8}$$

By substituting $u = Ae^{i(\gamma x + \omega t)}$ into equation (2.8), where A denotes the amplitude of the harmonic solution u, ω represents the frequency, and γ is the wave number, we arrive at

$$\omega^4 - \left[\left(\frac{\alpha}{J} + \frac{\mu}{\rho} \right) \gamma^2 + \frac{\xi}{J} \right] \omega^2 + \frac{\mu\xi - \beta^2}{\rho J} \gamma^2 + \frac{\alpha\mu}{\rho J} \gamma^4 = 0.$$

The above equation admits frequency bands, denoted as ω_1 and ω_2, which are given by

$$\omega_{1,2}(\gamma) = \frac{1}{2} \left(\frac{\alpha}{J} + \frac{\mu}{\rho} \right) \gamma^2 + \frac{\xi}{2J} \pm \frac{1}{2} \sqrt{\left(\left(\frac{\alpha}{J} - \frac{\mu}{\rho} \right) \gamma^2 + \frac{\xi}{J} \right)^2 + \frac{4\beta^2}{J\rho} \gamma^2}. \tag{9}$$

Let us assume that ω and γ are related by the identity $\omega = c\gamma$, where c is the phase velocity. This yields

$$c_{1,2} = \frac{1}{2} \left(\frac{\alpha}{J} + \frac{\mu}{\rho} \right) + \frac{\xi}{2J\gamma^2} \pm \frac{1}{2} \sqrt{\left(\frac{\alpha}{J} - \frac{\mu}{\rho} + \frac{\xi}{J\gamma^2} \right)^2 + \frac{4\beta^2}{J\rho\gamma^2}}. \tag{10}$$

To simplify our results, we use the asymptotic representation of the discriminant, from which we derive

$$c_1^2 \approx \frac{\alpha\mu\gamma^2 + (\mu\xi - \beta^2)}{(\alpha\rho + J\mu)\gamma^2 + \rho\xi}$$

$$c_2^2 \approx \frac{\alpha\mu\gamma^2 + (\mu\xi - \beta^2)}{J\rho} \frac{1}{c_1^2\gamma^2} + c_1^2.$$

It is important to note that when $\mu\xi \approx \beta^2$, c_1^2 goes to zero as γ goes to zero, while c_1^2 remains bounded as γ tends to infinity. The curve c_2^2 represents a hyperbola inversely proportional to c_1^2, which leads us to conclude that c_2^2 increases significantly at lower frequencies. This behavior is characteristic of the impairment associated with the so-called second spectrum, which was observed early in the Timoshenko system (see [2, 50]). Now, let us consider the truncated system

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - \beta \varphi_x = 0, \\ -J u_{xtt} - \alpha \varphi_{xx} + \beta u_x + \xi \varphi = 0. \end{cases} \tag{11}$$

The one variable equation corresponds to (11) is

$$u_{xxxx} + \frac{\beta^2 - \xi\mu}{\mu\alpha} u_{xx} - \frac{J\mu + \rho\alpha}{\mu\alpha} u_{xxtt} + \frac{\xi\rho}{\mu\alpha} u_{tt} = 0,$$

and the frequency equation becomes

$$\alpha\mu\gamma^4 + (\xi\mu - \beta^2)\gamma^2 - \left((J\beta + \rho\alpha)\gamma^2 + \xi\rho\right)\omega^2 = 0. \tag{12}$$

which has a natural frequency given by

$$\omega^2 = \frac{\alpha\mu\gamma^4 + (\xi\mu - \beta^2)\gamma^2}{(J\beta + \rho\alpha)\gamma^2 + \xi\rho}.$$

Given that the identity $\omega = c\gamma$, we deduce

$$c^2 = \frac{\alpha\mu\gamma^2 + (\xi\mu - \beta^2)}{(J\beta + \rho\alpha)\gamma^2 + \xi\rho},$$

Hence, the phase velocity remains bounded as γ tends to infinity and

$$c \to \sqrt{\frac{\mu}{\rho + (J\beta)/\alpha}}$$

Therefore, the simplified porous elastic system exhibits a single phase velocity and is free from the undesirable effects associated with the second frequency spectrum.

Within the framework of porous systems unaffected by the damage related to the second spectrum, Ramos et al. [45] analyzed a simplified system that incorporates frictional damping in the porous equation

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - \beta \varphi_x = 0, \\ -J u_{ttx} - \alpha \varphi_{xx} + \beta u_x + \xi \varphi + \tau \varphi_t = 0, \end{cases} \tag{13}$$

and established an exponential decay result regardless of the coefficients of the system.

As for stabilization via heat dissipation, Apalara et al. in [6] examined the simplified Timoshenko system coupled with the heat equation described by Fourier's law

$$\begin{cases} \rho_1 \varphi_{tt} - \kappa(\varphi_x - \psi)_x = 0, \\ -\rho_2 \varphi_{ttx} - b\psi_{xx} + \kappa(\varphi_x - \psi) + \gamma\theta_x = 0, \\ \rho_3 \theta_t - \beta\theta_{xx} + \gamma\psi_{tx} = 0, \end{cases} \tag{14}$$

The study yielded an exponential decay regardless to any relationship between the system coefficients. Also, Keddi et al.[32] extended this result to the case of the truncated Timoshenko system with second sound thermoelasticity.

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, & \text{in } (0,1)\times(0,\infty) \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa(\varphi_x+\psi) + \delta\theta_x = 0, & \text{in } (0,1)\times(0,\infty) \\ c\theta_t + q_x + \delta\psi_{xt} = 0, & \text{in } (0,1)\times(0,\infty). \\ \tau q_t + \beta q + \theta_x = 0, & \text{in } (0,1)\times(0,\infty). \end{cases} \tag{15}$$

In the context of the Gurtin-Pipkin thermal law, the same authors [38] studied the following Timoshenko-type system

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, & \text{in } (0,1)\times(0,\infty) \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa(\varphi_x+\psi) + \delta\theta_x = 0, & \text{in } (0,1)\times(0,\infty) \\ c\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^\infty g(s)\theta_{xx}(t-s)ds + \delta\psi_{tx} = 0 & \text{in } (0,1)\times(0,\infty). \end{cases} \tag{16}$$

They provided a rigorous proof of well-posedness and established exponential decay rates for the systems (15) and (16), independent of the coefficients of the systems.

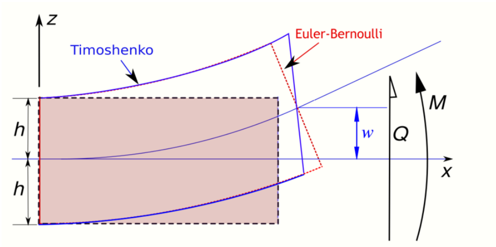

Figure 1: Comparison between Timoshenko and Euler-Bernoulli beams

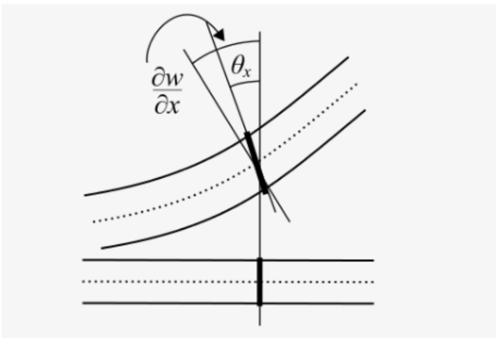

Figure 2: The slope w_x for Timoshenko beam

The remainder of the thesis is organized as follows: Chapter 1 presents the fundamental definitions and essential theoretical background used throughout the thesis. In Chapter 2, we examine a porous thermoelastic system modeled by Fourier's law of heat conduction. Chapter 3 investigates a similar porous thermoelastic system governed by Cattaneo's law, which characterizes the second-sound effect. Chapter 4 focuses on a system described by the Gurtin–Pipkin heat conduction law. Finally, Chapter 5 addresses a porous thermoelastic system based on the Green–Naghedi Type III theory. This model, along with the previous one, explores the same porous structures but with different laws of heat propagation.

chapter

1 Preliminear

In this chapter, we review a few mathematical ideas that will be used in this thesis.

1.1 L^p Spaces

Definition 1.1. *[10] Let $p \in [1, +\infty[$, we define the space*

$$L^p(0,L) = \left\{ g : [0,L] \to \mathbb{R};\, g \ \text{ is measurable and } \int_0^L |g(x)|^p \, dx < +\infty \right\},$$

which is a Banach space with respect to the norm.

$$\|g\|_{L^p} = \|g\|_p = \left[\int_0^L |g(x)|^p \, dx \right]^{1/p}, \quad 1 \le p < +\infty.$$

For $p = \infty$,

$$L^\infty(0,L) = \left\{ \begin{array}{c} g : [0,L] \to \mathbb{R};\ g \text{ is measurable and } \exists \ M > 0 \text{ suth that} \\ |g(x)| \le M \ \text{ a.e. on } [0,L] \end{array} \right\},$$

which is a Banach space with respect to the norm

$$\|g\|_{L^\infty} = \|g\|_\infty = \inf \{ M;\ |g(x)| \le M \ \ \text{a.e. on } [0,L] \}.$$

Theorem 1.1. ***(Young's inequality)****[10] Let p, q be real conjugates, then*

$$\forall a, b \in \mathbb{R}, \quad |ab| \le \frac{|a|^p}{p} + \frac{|b|^q}{q}.$$

In particular if $g, h \in L^2(0,L)$ we have, for any $\varepsilon > 0$,

$$\int_0^L |gh| dx \le \varepsilon \int_0^L |g|^2 dx + \frac{1}{4\varepsilon} \int_0^L |h|^2 dx.$$

Theorem 1.2. ***(Hölder's inequality)****[10] Suppose that $g \in L^p$ and $h \in L^q$ where $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, $gh \in L^1$ and*

$$\int |gh| \le \|g\|_p \|h\|_q .$$

Theorem 1.3. ***(Cauchy-Schwarz inequality)****[10] Let X be a Hilbert space and $\langle \cdot, \cdot \rangle$ be its inner product, then*

$$|\langle g, h\rangle| \leq \langle g, g\rangle^{\frac{1}{2}} \langle h, h\rangle^{\frac{1}{2}}, \quad \forall f, g \in X.$$

Remark 1.1. *For $p = 2$, $L^2(0, L)$ is a Hilbert space with respect to the inner product*

$$\langle g, h\rangle_{L^2} = \int_0^L g\,(x)\, h(x) dx.$$

Lemma 1.1. *The subspace $L^2_*(0, L)$ of $L^2(0, L)$, defined by*

$$L^2_*(0, L) = \left\{ g \in L^2(0, L);\ \int_0^L g(x) dx = 0 \right\},$$

is a Hilbert space.

Proof. Let $(\phi_n) \subset L^2_*(0, L)$ be a convergent sequence to ϕ in $L^2(0, L)$, then

$$\begin{aligned} \left| \int_0^L \phi\,(x)\, dx \right| &= \left| \int_0^L \phi\,(x)\, dx - \int_0^L \phi_n\,(x)\, dx \right| \\ &= \left| \int_0^L [\phi\,(x) - \phi_n\,(x)]\, dx \right|. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we arrive

$$\left| \int_0^L \phi\,(x)\, dx \right| \leq L \left[\int_0^L |\phi\,(x) - \phi_n\,(x)|^2\, dx \right]^{1/2}.$$

We have $\lim\limits_{n\to\infty} \int_0^L |\phi\,(x) - \phi_n\,(x)|^2\, dx = 0$, so

$$\int_0^L \phi\,(x)\, dx = 0,$$

which implies that $\phi \in L^2_*(0, L)$. The proof is thus finished since $L^2_*(0, L)$ is closed in $L^2(0, L)$. □

Remark 1.2. *The above result, as well as its proof, remains valid in any space dimension.*

1.2 Sobolev spaces

Definition 1.2. *[51] For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, we define the Sobolev space $W^{k,p}(0, L)$ by*

$$W^{k,p}(0, L) = \left\{ \phi \in L^p(0, L);\ \frac{\partial^j \phi}{\partial x^j} \in L^p(0, L),\ \forall j \in \mathbb{N} \text{ with } j \leq k \right\},$$

where $\dfrac{\partial^j \phi}{\partial x^j} = \varphi$ *is the* $j-$*weak derivative of* ϕ *defined as*

$$\int_0^L \phi(x) \frac{\partial^j \psi}{\partial x^j}(x)dx = (-1)^j \int_0^L \varphi(x)\psi(x)\,dx, \qquad \forall \psi \in C_0^\infty(0,L).$$

The space $W^{k,p}(0,L)$*, equipped with the norm*

$$\|\phi\|_{k,p} = \left(\sum_{j\leq k} \left\|\frac{\partial^j \phi}{\partial x^j}\right\|_p^p\right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty$$

and

$$\|\phi\|_{k,\infty} = \max_{j\leq k} \left\|\frac{\partial^j \phi}{\partial x^j}\right\|_\infty$$

are Banach spaces.

The space $W^{k,2}(0,L)$ *is denoted by* $H^k(0,L)$ *and it is a Hilbert space with respect to the inner product*

$$\langle \phi, \psi\rangle_{H^k} = \sum_{j\leq k} \int_0^L \frac{\partial^j \phi}{\partial x^j}(x) \frac{\partial^j \psi}{\partial x^j}(x)dx, \qquad \forall \phi, \psi \in H^k(0,L).$$

Definition 1.3. *[10] Given* $1 \leq p < \infty$*, we denote by* $W_0^{k,p}(0,L)$ *the closure of* $C_0^\infty(0,L)$ *in* $W^{k,p}(0,L)$*.For* $p = 2$*, we denote* $W_0^{k,2}(0,L)$ *by* $H_0^k(0,L)$*.*

In this thesis, we are focus on the case where $p = 2$ and $k = 1$, so we frequently use the space $H_0^1(0,L)$.

Definition 1.4. *The space* $H^{-1}(0,L)$ *is the dual space of* $H_0^1(0,L)$*, namely,*

$$H^{-1}(0,L) = \big(H_0^1(0,L)\big)' = \mathscr{L}\big(H_0^1(0,L), \mathbb{R}\big).$$

In addition, we denote by $H_*^1(0,L), H_*^2(0,L)$ *and* $H_*^3(0,L)$*, the Hilbert spaces*

$$H_*^1(0,L) = \left\{\phi \in H^1(0,L); \int_0^L \phi(x)dx = 0\right\},$$

$$H_*^2(0,L) = \Big\{\phi \in H^2(0,L); \quad \phi_x(0) = \phi_x(L) = 0\Big\},$$

$$H_*^3(0,L) = \Big\{\phi \in H^3(0,L) \cap H_0^1(0,L); \quad \phi_{xx}(0) = \phi_{xx}(L) = 0\}.$$

Theorem 1.4. ***(Poincaré's inequality)*** *Suppose that* $u \in H_0^1(0,L)$. *Then there exists a constant* $M > 0$, *depending only on* L, *such that*

$$\|\phi\|_{L^2(0,L)} \leq M\|\phi_x\|_{L^2(0,L)}, \qquad \forall \phi \in H_0^1(0,L).$$

Remark 1.3. *Poincaré's inequality also holds for all* $\phi \in H^1(0,L)$ *that satisfy*

$$\int_0^L \phi(x)\,dx = 0.$$

Definition 1.5. ***(Bilinear form)*** *Let* H *be a Hilbert space over* $\mathbb{R}$.

$$A : H \times H \longrightarrow \mathbb{R}.$$

is called a bilinear form *if it is linear in each component separately, i.e.,*

$$A(\phi + \psi, \varphi) = A(\phi, \varphi) + A(\psi, \varphi), \quad A(\mu\ \phi, \psi) = \mu\ A(\phi, \psi),$$

and similarly in the second argument.

Definition 1.6. ***(Continuity)*** *A bilinear form* $A : H \times H \to \mathbb{R}$ *is said to be* continuous *(or bounded) if there exists a constant* $M > 0$ *such that*

$$|A(\phi, \psi)| \leq M\,\|\phi\|\,\|\psi\|, \quad \forall \phi, \psi \in H.$$

Definition 1.7. ***(Coercivity)*** *A bilinear form* $A : H \times H \to \mathbb{R}$ *is said to be* coercive *if there exists a constant* $\alpha > 0$ *such that*

$$A(\phi, \phi) \geq \alpha\,\|\phi\|^2, \quad \forall \phi \in H.$$

Theorem 1.5. ***(Lax Milgram)*** *[10] Assume that* A *is a continuous and coercive bilinear form on* H. *Then, for any* $\phi \in H'$, *there exists a unique element* $x \in H$ *such that*

$$A\,(x, y) = \langle \phi, y \rangle, \quad \forall y \in H.$$

Moreover, if A *is symmetric, then* x *is characterized by the property*

$$\frac{1}{2}A(x,x) - \langle \phi, x \rangle = \min_{y \in X} \left\{ \frac{1}{2}A(y,y) - \langle \phi, y \rangle \right\}.$$

Theorem 1.6. ***(Rellich-Kondrachov)****[1] Suppose that* $I \subset \mathbb{R}$ *is a bounded interval. Then, we have the following compact embedding:*

$$H^m(I) \subset H^n(I), \quad \forall n \leq m.$$

1.3 Spectral Theory of Operators

Definition 1.8. *Let H be a Hilbert space and $\mathcal{A} : H \to H$ an operator,*

1. *The operator $\mathcal{A}$ is said to be positive, if*

$$\langle \mathcal{A}\phi, \phi \rangle \geq 0, \quad \forall \phi \in H.$$

2. *The operator $\mathcal{A}$ is said to be self-adjoint, if*

$$\langle \mathcal{A}\phi, \psi \rangle = \langle \phi, \mathcal{A}\psi \rangle, \quad \forall \phi, \psi \in H.$$

Theorem 1.7. ***(Invertibility of positive self-adjoint operators)*** *[46][Theorem VI.3 p. 194] Let $\mathcal{A} : H \to H$ be a bounded, self-adjoint operator. Suppose that there exists $c > 0$ such that*

$$\langle \mathcal{A}\phi, \phi \rangle \geq c \|\phi\|^2, \quad \forall \phi \in H,$$

then $\mathcal{A}$ is invertible and

$$\|A^{-1}\| \leq \frac{1}{c}.$$

1.4 Some Semigroup arguments

Definition 1.9. *We consider the bounded operator $\mathcal{A} \in \mathscr{L}(H)$. The resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ is the set of all α in $\mathbb{C}$ for which $(\mathcal{A} - \alpha I)^{-1}$ exists and bounded*

$$\rho(\mathcal{A}) = \left\{\alpha \in \mathbb{C}; \quad (\mathcal{A} - \alpha I)^{-1} \in \mathscr{L}(H)\right\}.$$

The spectrum of $\mathcal{A}$, denoted by $\sigma(\mathcal{A})$, is the complement of the resolvent set, i.e.,

$$\sigma(\mathcal{A}) = \mathbb{C} \backslash \rho(\mathcal{A}).$$

A complex number α is an eigenvalue of $\mathcal{A}$ if

$$N(\mathcal{A} - \alpha I) \neq \{0\}.$$

Theorem 1.8. *Let H be a Banach space and $\mathcal{A} \in \mathscr{L}(H)$. If the operator $\mathcal{A}$ satisfies $\|\mathcal{A}\| < 1$, then $I - \mathcal{A}$ is invertible and its inverse is given by*

$$(I-\mathcal{A})^{-1} = \sum_{m=0}^{\infty} \mathcal{A}^m.$$

Definition 1.10. *[42] Let H be a Banach space. A semigroup of bounded linear operators is a family of linear operators $T(t) \in \mathscr{L}(H)$, which depend on a parameter $0 \leq t < \infty$ and that fulfills the following characteristics.*

(i) $T(0) = I$, (I is the identity operator on H).

(ii) $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$ (the semigroup property).

A semigroup of bounded linear operators $T(t)$ is uniformly continuous if

$$\lim_{t\to 0^+} \|T(t) - I\| = 0.$$

The infinitesimal generator of a semigroup $T(t)$ is the operator $\mathcal{A}$ defined on

$$D(\mathcal{A}) = \left\{ x \in H : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \quad \text{exists} \right\}$$

by

$$\mathcal{A}x = \lim_{t\to 0^+} \frac{T(t)x - x}{t}, \quad \text{for} \quad x \in D(\mathcal{A}).$$

Definition 1.11. ***(Maximal Monotone Operators)****[10]: Let H be a Hilbert space, an unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ is said to be monotone (accretive) if it satisfies*

$$Re \langle \mathcal{A}u, u \rangle \geq 0, \quad \forall u \in D(\mathcal{A}).$$

In addition, the operator is said to be maximal monotone, if $R(I + \mathcal{A}) = H$ i.e.,

$$\forall g \in H, \ \exists v \in D(\mathcal{A}), \text{such that } v + \mathcal{A}v = g.$$

Remark 1.4. *If $-\mathcal{A}$ is monotone, we say that $\mathcal{A}$ is dissipative.*

Proposition 1.1. *[10] Let $\mathcal{A}$ be a maximal monotone operator on a Hilbert space. Then*

(i) $D(\mathcal{A})$ is dense in H.

(ii) $\mathcal{A}$ *is a closed operator.*

(iii) *For all* $\alpha > 0$, $(I + \alpha\mathcal{A})$ *is bijective from* $D(\mathcal{A})$ *into* H, $(I + \alpha\mathcal{A})^{-1}$ *is a bounded operator, and* $\|(I + \alpha\mathcal{A})^{-1}\|_{\mathscr{L}(H)} \leq 1$.

Remark 1.5. *If the operator* $\mathcal{A}$ *is coercive (strictly monotone) and maximal then*

$$\|(I + \alpha\mathcal{A})^{-1}\|_{\mathscr{L}(H)} < 1.$$

Theorem 1.9. ***(Hille-Yosida)****[10] Let* $\mathcal{A}$ *be a maximal monotone operator. Then, given any* $v_0 \in D(\mathcal{A})$ *there exists a unique function*

$$v \in C^1([0, +\infty); H) \cap C([0, +\infty); D(\mathcal{A}))$$

satisfying

$$\begin{cases} \dfrac{dv}{dt} + \mathcal{A}v = 0 \quad \text{on } [0, +\infty), \\ v(0) = v_0. \end{cases} \tag{1.1}$$

Theorem 1.10. ***(Lumer-Phillips)*** *[42, 51] Let* $\mathcal{A} : D(\mathcal{A}) \subset H \longrightarrow H$ *be a densely defined operator on a Hilbert space* H*. Then* $\mathcal{A}$ *generates a* C_0*-semigroup of contractions on* H *if and only if*

(i) $\mathcal{A}$ *is dissipative;*

(ii) *there exists* $\alpha > 0$ *such that* $\alpha I - \mathcal{A}$ *is surjective.*

Theorem 1.11. *[33] Let* $\mathcal{A} : D(\mathcal{A}) \subset H \longrightarrow H$*, be a linear operator and* H *Hilbert space.*

Suppose that $D(\mathcal{A})$ *is dense in* H*,* $\mathcal{A}$ *is dissipative and* $0 \in \rho(\mathcal{A})$*. Then,* $\mathcal{A}$ *is the infinitesimal generator of a* C_0-*semi-group of contractions on* H*.*

Theorem 1.12. *[51] Let* H *be a Banach space and* $\mathcal{A} : D(\mathcal{A}) \subset H \longrightarrow H$ *be the infinitesimal generator of a* C_0*-semigroup* $\{S(t); t \geq 0\}$ *on* H*. Then, for each* $\xi \in D(\mathcal{A})$ *and each* $t \geq 0$*, we have* $S(t)\xi \in D(\mathcal{A})$*, and the mapping*

$$t \longrightarrow S(t)\xi$$

is of class C^1 *on* $[0, +\infty)$ *and satisfies*

$$\frac{d}{dt}(S(t)\xi) = \mathcal{A}S(t)\xi = S(t)\mathcal{A}\xi. \tag{1.2}$$

1.5 Exponential stability notions

Definition 1.12. *The solution $U(t) = e^{\mathcal{A}}U_0$ of (1.1) is said to be exponentially stable if there exist two positive constants α and $C > 1$ such that*

$$\|U(t)\| \leq Ce^{-\lambda t}, \quad \forall t \geq 0.$$

Theorem 1.13 (Gerhard-Bruss). *[23, 33] A C_0-semigroup of contractions $S\left(t\right) = e^{\mathcal{A}t}$ generated by an operator $\mathcal{A}$ in a Hilbert space H is exponentially stable if and only if*

i) $i\mathbb{R} = \{i\lambda, \lambda \in \mathbb{R}\} \subset \rho(\mathcal{A})$,

ii) $\overline{\lim\limits_{|\lambda|\to\infty}} \|(i\lambda I - \mathcal{A})^{-1}\| < \infty.$

Theorem 1.14. *[3] Let $\mathcal{A}$ (unbounded operator) be the infinitesimal generator of a semigroup of contractions $S(t) = e^{\mathcal{A}t}$. Then, $S\left(t\right)$ is exponentially stable if and only if there exists a positive constant c such that*

$$\inf_{\lambda\in\mathbb{R}}\|(i\lambda I - \mathcal{A})U\| \geq c\|U\|, \ \forall U \in D(\mathcal{A}). \tag{1.3}$$

chapter 2

Exponential Stability of a Truncated Porous Elastic System with Classical Thermoelasticity

2.1 Introduction

In this chapter, we investigate a truncated porous thermoelastic system in which heat conduction is governed by the classical Fourier law. The system is free from the adverse effects associated with the secondary spectrum of frequencies. We employ a non-classical approach based on operator theory, utilizing a novel technique developed by Messaoudi and Keddi [32, 38], who formulated the problem using the semigroup method.

We prove the well-posedness result using the Hille–Yosida theorem, and establish exponential stability by applying the multiplier method.

We are concerned with the following problem

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - \beta \varphi_x = 0, \\ -J u_{ttx} - \alpha \varphi_{xx} + \beta u_x + \xi \varphi + \delta \theta_x = 0, \\ c\theta_t - \kappa \theta_{xx} + \delta \varphi_{tx} = 0. \end{cases} \tag{2.1}$$

where u, φ, θ are the displacement of a elastic material, the volume fraction and the temperature difference, respectively. The coefficients $\kappa, \rho, J, \mu, \alpha, \xi, c, \delta, \beta$ are positive constants and defined as follows: ρ expresses the mass density of the material, $J = \rho\kappa$, where κ is the porous inertia coefficient, μ is the elastic modulus, α denotes the porous diffusivity, ξ denotes the weight of the restoring force, $c = \rho c_p$, where c_p is the specific heat capacity. Finally, β and δ are the coupling coefficients, assume to satisfy

$$\mu\xi - \beta^2 > 0.$$

The system (2.1) is supplemented by the following initial and boundary conditions

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \varphi(x,0) = \varphi_0(x), \theta(x,0) = \theta_0(x),\ x \in [0,L] \tag{2.2}$$

$$u(0,t) = u(L,t) = \varphi_x(0,t) = \varphi_x(L,t) = \theta(0,t) = \theta(L,t) = 0,\ t \in (0,+\infty) \tag{2.3}$$

Observe that Neumann boundary conditions are imposed on φ, which, in turn, prevent the applicability of Poincaré's inequality . However, from the second equation of (2.1) and the boundary conditions (2.3), we obtain

$$\int_0^L \varphi(x,t)\,dx = 0,$$

which allows the use of Poincaré's inequality.

2.2 well-posdeness

In this section, we start by applying a series of transformations to the system (2.1)–(2.3) to recast it within the framework of the Hille–Yosida theorem. This approach enables us to establish the existence and uniqueness of the solution.

First, multiply all equations in (2.1) except the first one by β, and differentiate the second equation of (2.1) with respect to x. We then find

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - \beta\varphi_x = 0, \\ -J\beta u_{xxtt} - \alpha\beta\varphi_{xxx} + \beta^2 u_{xx} + \xi\beta\varphi_x + \delta\beta\theta_{xx} = 0, \\ c\beta\theta_t - \kappa\beta\theta_{xx} + \delta\beta\varphi_{tx} = 0. \end{cases} \tag{2.4}$$

From the first equation of (2.1) we have

$$\beta\varphi_x = \rho u_{tt} - \mu u_{xx}. \tag{2.5}$$

Next, substitute equation (2.5) into the second and third equations of (2.4); we obtain

$$\begin{cases} -J\beta u_{xxtt} - \alpha(\rho u_{tt} - \mu u_{xx})_{xx} + \beta^2 u_{xx} + \xi(\rho u_{tt} - \mu u_{xx}) + \delta\beta\theta_{xx} = 0, \\ c\beta\theta_t - \kappa\beta\theta_{xx} + \delta(\rho u_{tt} - \mu u_{xx})_t = 0. \end{cases} \tag{2.6}$$

This corresponds to

$$\begin{cases} \left[\xi\rho I - (J\beta + \alpha\rho)\partial_{xx}\right] u_{tt} + \alpha\mu u_{xxxx} + (\beta^2 - \xi\mu)u_{xx} + \delta\beta\theta_{xx} = 0, \\ c\beta\theta_t - \kappa\beta\theta_{xx} + \delta\rho u_{ttt} - \delta\mu u_{xxt} = 0. \end{cases} \tag{2.7}$$

Putting $B = \xi\rho I - (J\beta + \alpha\rho)\partial_{xx}$, with domain $D(B) = H^2(0, L) \cap H_0^1(0, L)$. we get

$$\begin{cases} Bu_{tt} + \alpha\mu u_{xxxx} + (\beta^2 - \xi\mu)u_{xx} + \delta\beta\theta_{xx} = 0, \\ c\beta\theta_t - \kappa\beta\theta_{xx} + \delta\rho u_{ttt} - \delta\mu u_{xxt} = 0, \end{cases} \tag{2.8}$$

$$\partial_{xx} = \frac{1}{\alpha\rho + J\beta}\left(\xi\rho I - B\right), \tag{2.9}$$

and

$$\xi\rho I = B + (\alpha\rho + J\beta)\partial_{xx}. \tag{2.10}$$

Remark 2.1. *Note that B is invertible because it is coercive. By applying the Lax-Milgram Theorem, for any $f \in L^2([0, L])$, the problem*

$$\begin{cases} -(\alpha\rho + J\beta)u_{xx} + \xi\rho u = f, \quad in\ (0, L), \\ u(0) = u(L) = 0, \end{cases} \tag{2.11}$$

admits a unique solution $u \in H_0^1([0, L])$ that satisfies the inequality

$$\|u\|_{H_0^1(0,L)} \leq C\|f\|_{L^2(0,L)}.$$

Next, by using standard elliptic regularity theory, we infer that the solution u belongs to $\big(H^2(0, L) \cap H_0^1(0, L)\big)$.

From the equation in (2.11), we have

$$u_{xx} = \frac{1}{\alpha\rho + J\beta}(\xi\rho u - f).$$

Therefore,

$$\|u_{xx}\|_{L^2(0,L)} \leq \widehat{C}\big(\|u\|_{L^2(0,L)} + \|f\|_{L^2(0,L)}\big) \leq C'\|f\|_{L^2(0,L)}.$$

Consequently,

$$\|u\|_{H^2(0,L)} \leq C'\|f\|_{L^2(0,L)}.$$

which confirms that the operator B is invertible with inverse $B^{-1} : L^2(0, L) \to H^2(0, L) \cap H_0^1(0, L)$. In addition B^{-1} commutes with ∂_{xx} since B does.

Remark 2.2. *Note that $A = -\partial_{xx}$ with Dirichlet boundary conditions is strictly monotone, because*

$$\langle -\partial_{xx}u, u\rangle = \|u_x\|^2 \geq \frac{1}{C_p}\|u\|_{L^2}^2,$$

then,

$$\left\langle I - \frac{\alpha\rho + J\beta}{\rho\xi}\partial_{xx}u, u\right\rangle \geq \left(1 + \frac{\alpha\rho + J\beta}{\rho\xi C_p}\right)\|u\|_{L^2}^2.$$

Therefore, if we set $v = Bu = \rho\xi u - (\alpha\rho + J\beta)u_{xx}$*, we arrive at*

$$\|v\|_{L^2}\|u\|_{L^2} \geq \langle v, u\rangle \geq \left(\rho\xi + \frac{\alpha\rho + J\beta}{C_p}\right)\|u\|_{L^2}^2.$$

Consequently, since $u = B^{-1}v$*, we have*

$$\|v\|_{L^2} \geq \left(\rho\xi + \frac{\alpha\rho + J\beta}{C_p}\right)\|B^{-1}v\|_{L^2},$$

which yields

$$\|B^{-1}\| \leq \frac{1}{\rho\xi + \frac{\alpha\rho+J\beta}{C_p}} < \frac{1}{\rho\xi}.$$

Thus, there exists $\delta > 0$*, (for instance* $\delta = \frac{1}{2}\left(\frac{1}{\rho\xi} - \|B^{-1}\|\right)$*), such that*

$$\|B^{-1}\| < \frac{1}{\rho\xi} - \delta.$$

Now, returning to the first equation of (2.8), we have

$$u_{ttt} = -\alpha\mu B^{-1}u_{xxxxt} + (\xi\mu - \beta^2)B^{-1}u_{xxt} - \delta\beta B^{-1}\theta_{xxt}. \tag{2.12}$$

Substitute $P = \partial_{xx} \circ B^{-1} = B^{-1} \circ \partial_{xx}$, we get

$$u_{ttt} = -\alpha\mu P u_{xxt} + (\xi\mu - \beta^2)Pu_t - \delta\beta P\theta_t. \tag{2.13}$$

In addition, from (2.9) and since $B^{-1} : L^2(0, L) \to H^2(0, L) \cap H_0^1(0, L)$, we have

$$P = \frac{1}{\alpha\rho + J\beta}\left(\xi\rho B^{-1} - I\right) : L^2(0, L) \to L^2(0, L).$$

Substituting (2.13) in the second equation of (2.8), we get

$$\left(c\beta I - \delta^2\rho\beta P\right)\theta_t - \kappa\beta\theta_{xx} + \delta\mu\left(\xi\rho I - \alpha\rho\partial_{xx}\right)Pu_t - \delta\rho\beta^2 Pu_t - \delta\mu u_{txx} = 0.$$

As shown in (2.10), we derive

$$\left(c\beta I - \delta^2\rho\beta P\right)\theta_t - \kappa\beta\theta_{xx} + \delta\mu\left[(B + J\beta\partial_{xx}) + \alpha\rho\partial_{xx} - \alpha\rho\partial_{xx}\right]Pu_t - \delta\rho\beta^2 Pu_t - \delta\mu u_{xxt} = 0,$$

$$\left(c\beta I - \delta^2\rho\beta P\right)\theta_t - \kappa\beta\theta_{xx} + \delta\mu\left(B + J\beta\partial_{xx}\right)Pu_t - \delta\rho\beta^2 Pu_t - \delta\mu u_{xxt} = 0.$$

We conclude from the definition of P that $BPu_t = u_{xxt}$, so

$$(c\beta I - \delta^2\rho\beta P)\theta_t - \kappa\beta\theta_{xx} - \beta\delta\left(\rho\beta I - J\mu\partial_{xx}\right)Pu_t = 0. \tag{2.14}$$

Finally, by multiplying system (2.8) by B^{-1}, we obtain the following auxiliary problem:

$$\begin{cases} u_{tt} + \alpha\mu P u_{xx} - (\xi\mu - \beta^2)Pu + \delta\beta P\theta = 0, \\ S\theta_t - \kappa\beta\theta_{xx} + \delta\beta T u_t = 0, \end{cases} \tag{2.15}$$

where $S, T : L^2(0,L) \to L^2(0,L)$ are self-adjoint operators defined by

$$\begin{cases} S = c\beta I - \delta^2\rho\beta P, \\ T = -RP = -\big(\rho\beta I - J\mu\partial_{xx}\big)B^{-1}\circ\partial_{xx}, \\ R = \rho\beta I - J\mu\partial_{xx}. \end{cases}$$

Note that since $B^{-1} : L^2(0,L) \to H^2(0,L)\cap H_0^1(0,L)$ then $P : H^2(0,L)\cap H_0^1(0,L) \subset L^2(0,L) \to H^2(0,L)\cap H_0^1(0,L)$, therefore we can apply R to Pu for any $u \in D(P) = H^2(0,L)\cap H_0^1(0,L)$ Consequently $D(T) = H^2(0,L)\cap H_0^1(0,L)$.
On the other hand, by inserting $P = \dfrac{1}{\alpha\rho + J\beta}\left(\xi\rho B^{-1} - I\right)$ into the expression of T, we get

$$T = -RP = -\frac{1}{\alpha\rho + J\beta}\big(\rho\beta I - J\mu\partial_{xx})(\xi\rho B^{-1} - I)\big),$$

that is

$$T = -\frac{1}{\alpha\rho + J\beta}\left[(\rho\beta I - J\mu\partial_{xx})\xi\rho B^{-1} - \rho\beta I + J\mu\partial_{xx}\right],$$

equivalently, we get

$$T = -\frac{1}{\alpha\rho + J\beta}\left[\left((\rho\beta I - J\mu\partial_{xx})\xi\rho I - \rho\beta B\right)B^{-1} + J\mu\partial_{xx}\right].$$

Substituting B, we arrive at

$$T = -\frac{1}{\alpha\rho + J\beta}\left[\big(\alpha\beta\rho^2 - J\rho(\xi\mu - \beta^2)\big)B^{-1} + J\mu I\right]\partial_{xx}. \tag{2.16}$$

Remark 2.3. *Note that we have*

$$\langle Sv, v\rangle = c\beta\|v\|_{L^2}^2 + \delta^2\rho\beta\langle B^{-1}v_x, v_x\rangle.$$

Since B^{-1} is positive self-adjoint, the second term is nonnegative, so

$$\langle Sv, v\rangle \geq c\beta\|v\|_{L^2}^2,\ \forall v \in L^2(0,L).$$

Therefore, S is coercive and consequently, is invertible.

In fact, $R = \rho\beta I - J\mu\partial_{xx}$ is clearly positive definite. Moreover, since both B and $-\partial_{xx}$ are positive definite operators, we deduce that $-P$ is also positive definite, and consequently $S = c\beta I - \delta^2\rho\beta P$ is also positive definite. In addition, we have $T = -(\rho\beta I - J\mu\partial_{xx})B^{-1}\partial_{xx}$ on $H^2(0,L) \cap H_0^1(0,L)$. Then, for any $\phi \in D(T)$, recalling that B^{-1} is positive, we have

$$\begin{aligned}\langle T\phi, \phi\rangle &= \left\langle -(\rho\beta I - J\mu\partial_{xx})B^{-1}\phi_{xx}, \phi\right\rangle \\ &= \rho\beta\langle B^{-1}\phi_x, \phi_x\rangle + J\mu\langle B^{-1}\phi_{xx}, \phi_{xx}\rangle \geq 0.\end{aligned}$$

Moreover, if $\langle T\phi, \phi\rangle = 0$, then $\langle B^{-1}\phi_x, \phi_x\rangle = 0$. Since B^{-1} is positive definite, this implies that $\phi_x = 0$. Because $\phi \in H_0^1(0,L)$, it follows that $\phi = 0$. Therefore, T is definite.

Consequently, R, S, and T admit well-defined square roots, denoted by $R^{1/2}$, $S^{1/2}$, and $T^{1/2}$, with domains $D(R^{1/2}) = D(T^{1/2}) = H_0^1(0,L)$ and $D(S^{1/2}) = L^2(0,L)$.

Remark 2.4. *Because B^{-1} commute with ∂_{xx} and $D(R) = D(P) = H^2(0,L) \cap H_0^1(0,L)$, we have*

$$\begin{aligned}T = -RP &= -(\rho\beta I - J\mu\partial_{xx})B^{-1}\partial_{xx} = -(\rho\beta B^{-1}\partial_{xx} - J\mu\partial_{xx}B^{-1}\partial_{xx}) \\ &= -(\rho\beta\partial_{xx}B^{-1} - J\mu\partial_{xx}B^{-1}\partial_{xx}) = -\partial_{xx}B^{-1}(\rho\beta I - J\mu\partial_{xx})) \\ &= -B^{-1}\partial_{xx}(\rho\beta I - J\mu\partial_{xx}) = -PR.\end{aligned}$$

Thus, T is symmetric because R and P commute, and the product of two self-adjoint operators is self-adjoint if and only if the operators commute.

Let $\mathcal{H}$ be the Hilbert space

$$\mathcal{H} = (H^2(0,L) \cap H_0^1(0,L)) \times H_0^1(0,L) \times L^2(0,L),$$

equipped with the inner product

$$\langle \Phi, \Phi^*\rangle = \alpha\mu\left\langle Tu_x, u_x^*\right\rangle + \langle Rv, v^*\rangle + \langle S\theta, \theta^*\rangle + (\xi\mu - \beta^2)\left\langle Tu, u^*\right\rangle,$$

where $\Phi = (u, v, \theta)^T, \Phi^* = (u^*, v^*, \theta^*)^T \in \mathcal{H}$.

Next, system (2.15) can be represented as a Cauchy problem by introducing the new variable $v = u_t$ and defining $\Phi = (u, v, \theta)^T$ as follows:

$$\begin{cases} \Phi'(t) + \mathcal{A}\Phi(t) = 0, \quad \forall t \geq 0, \\ \Phi(0) = (u_0, u_1, \theta_0), \end{cases} \tag{2.17}$$

where the operator $\mathcal{A} : \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$\mathcal{A}\Phi = \begin{pmatrix} -v \\ \alpha\mu P u_{xx} - (\xi\mu - \beta^2)Pu + \delta\beta P\theta \\ \delta\beta S^{-1}Tv - \kappa\beta S^{-1}\theta_{xx} \end{pmatrix},$$

with domain

$$D(\mathcal{A}) = H^3_*(0, L) \times (H^2(0, L) \cap H^1_0(0, L)) \times (H^2(0, L) \cap H^1_0(0, L)).$$

where

$$H^3_*(0.L) = \left\{ \phi \in H^3(0.L) \cap H^1_0(0.L) : \phi_{xx} \in H^1_0(0.L) \right\}.$$

We present the well-posedness result for the auxiliary problem (2.15) as follows:

Theorem 2.1. *For any $\Phi_0 \in \mathcal{H}$, there exists a unique weak solution $\Phi \in C(\mathbb{R}^+, \mathcal{H})$ of problem (2.17). Moreover, if $\Phi_0 \in D(\mathcal{A})$, then $\Phi \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Proof. The proof of Theorem 2.1 will be given by applying the Hille-Yosida theorem 1.9, so it suffices to show that $\mathcal{A}$ is monotone and maximal. □

First, we prove that $\mathcal{A}$ is monotone.
For any $\Phi \in D(\mathcal{A})$, using the inner product and the definitions of the operators R, S and T , we have

$$\begin{aligned} \langle \mathcal{A}\Phi, \Phi \rangle_{\mathcal{H}} =& - \alpha\mu \langle Tv_x, u_x \rangle + \left\langle \alpha\mu RPu_{xx} - (\xi\mu - \beta^2)RPu + \delta\beta RP\theta, v \right\rangle \\ &+ \langle -\kappa\beta\theta_{xx} + \delta\beta Tv, \theta \rangle - (\xi\mu - \beta^2) \langle Tv, u \rangle \\ =& - \alpha\mu \langle Tv_x, u_x \rangle - \alpha\mu \langle Tu_{xx}, v \rangle + (\xi\mu - \beta^2) \langle Tu, v \rangle - \delta\beta \langle T\theta, v \rangle \\ &- \kappa\beta \langle \theta_{xx}, \theta \rangle + \delta\beta \langle Tv, \theta \rangle - (\xi\mu - \beta^2) \langle Tv, u \rangle . \end{aligned}$$

Note that T is symmetric (self-adjoint) on $H^2(0, L) \cap H^1_0(0, L)$, since B^{-1} is positive self-adjoint and ∂_{xx} is symmetric with Dirichlet boundary conditions. Therefore, we can freely interchange T between the two entries of the inner product. An integration by parts leads to

$$\langle \mathcal{A}\Phi, \Phi \rangle_{\mathcal{H}} = \kappa\beta \langle \theta_x, \theta_x \rangle \geq 0. \tag{2.18}$$

Hence $\mathcal{A}$ is monotone.

Next, to establish the maximality of $\mathcal{A}$, let $F = (f_1, f_2, f_3)^T \in \mathcal{H}$, we prove that there exists $\Phi \in D(\mathcal{A})$ satisfying $(I + \mathcal{A})\Phi = F$, that is

$$\begin{cases} u - v = f_1, \\ v + \alpha\mu P u_{xx} - (\xi\mu - \beta^2) P u + \delta\beta P\theta = f_2, \\ \delta\beta T v - \kappa\beta\theta_{xx} + S\theta = S f_3. \end{cases} \tag{2.19}$$

Form the first equation of (2.19), we obtain

$$v = u - f_1. \tag{2.20}$$

Next, by substituting (2.20) into the second and third equations of (2.19), we get

$$\begin{cases} u + \alpha\mu P u_{xx} - (\xi\mu - \beta^2) P u + \delta\beta P\theta = f_1 + f_2, \\ \delta\beta T u - \kappa\beta\theta_{xx} + S\theta = S f_3 + \delta\beta T f_1. \end{cases} \tag{2.21}$$

Introducing the space

$$\mathcal{V} = (H^2(0, L) \cap H_0^1(0, L)) \times H_0^1(0, L).$$

We multiply formally, the first equation from system (2.21) by Ru^* with $u^* \in H^2(0, L) \cap H_0^1(0, L)$ and the second equation by $\theta^* \in H_0^1(0, L)$, and using the symmetry of R, we obtain the variational formulation corresponding to (2.21), which can be expressed as follows:

$$A((u, \theta), (u^*, \theta^*)) = L(u^*, \theta^*), \tag{2.22}$$

with $A : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is the bilinear form given by:

$$\begin{aligned} A((u, \theta), (u^*, \theta^*)) =& \alpha\mu \langle T u_x, u_x^* \rangle + (\xi\mu - \beta^2) \langle T u, u^* \rangle + \langle R u, u^* \rangle \\ & - \delta\beta \langle T\theta, u^* \rangle + \delta\beta \langle T u, \theta^* \rangle + \langle S\theta, \theta^* \rangle + \kappa\beta \langle \theta_x, \theta_x^* \rangle, \end{aligned}$$

and $L : \mathcal{V} \to \mathbb{R}$ is the linear functional defined by

$$L(u^*, \theta^*) = \langle R(f_1 + f_2), u^* \rangle + \langle S f_3, \theta^* \rangle + \delta\beta \langle T f_1, \theta^* \rangle.$$

It is clear that A and L are bounded. In addition, by substituting T from (2.16), we get

$$\begin{aligned}A((u,\theta),(u,\theta)) =& \frac{\alpha\mu}{\alpha\rho+J\beta}\Big((\alpha\beta\rho^2 - J\rho(\xi\mu-\beta^2))\langle B^{-1}u_{xx},u_{xx}\rangle + J\mu\|u_{xx}\|^2\Big)\\ &+(\xi\mu-\beta^2)\langle Tu,u\rangle + \rho\beta\|u\|^2 + J\mu\|u_x\|^2 + \langle S\theta,\theta\rangle + \kappa\beta\|\theta_x\|^2,\end{aligned}$$

then,

$$\begin{aligned}A((u,\theta),(u,\theta)) =& \frac{\alpha\mu}{\alpha\rho+J\beta}\Big((\alpha\beta\rho^2 + J\rho\beta^2)\langle B^{-1}u_{xx},u_{xx}\rangle - J\xi\mu\rho\langle B^{-1}u_{xx},u_{xx}\rangle + J\mu\|u_{xx}\|^2\Big)\\ &+(\xi\mu-\beta^2)\langle Tu,u\rangle + \rho\beta\|u\|^2 + J\mu|u_x\|^2 + \langle S\theta,\theta\rangle + \kappa\beta\|\theta_x\|^2.\end{aligned}$$

By virtue of Remark 2.2 and using the positiveness of B^{-1}, we infer that

$$\begin{aligned}A((u,\theta),(u,\theta)) \geq& \frac{\delta J\alpha\mu^2}{\alpha\rho+J\beta}\|u_{xx}\|^2 + \rho\beta\|u\|^2\\ &+(\xi\mu-\beta^2)\langle Tu,u\rangle + J\mu|u_x\|^2 + \langle S\theta,\theta\rangle + \kappa\beta\|\theta_x\|^2.\end{aligned}$$

Consequently, there exists a positive constant m such that

$$\langle A(u,\theta),(u,\theta)\rangle \geq m\left(\|u\|^2 + \|u_x\|^2 + \|u_{xx}\|^2 + \|\theta_x\|^2\right) = m\|(u,\theta)\|^2_{\mathcal{V}},$$

which shows that A is coercive. Consequently, by Lax-Milgram Lemma, Eq. (2.22) has a unique solution

$$(u,\theta) \in (H^2(0,L)\cap H^1_0(0,L)) \times H^1_0(0,L).$$

By substituting u into the equation (2.20), we infer that

$$v \in H^2(0,L)\cap H^1_0(0,L). \tag{2.23}$$

Furthermore, by taking $u^* \equiv 0$ in (2.22), we arrive at

$$\kappa\beta\langle\theta_x,\theta^*_x\rangle + \langle S\theta,\theta^*\rangle + \delta\beta\langle Tu,\theta^*\rangle = \langle Sf_3,\theta^*\rangle + \delta\beta\langle Tf_1,\theta^*\rangle, \ \forall\theta^* \in H^1_0(0,L),$$

which still valid for $\theta^* = \phi \in C^1_0(0,L)$, then we get

$$\kappa\beta\int_0^L \theta_x\phi_x dx = -\int_0^L (S\theta + \delta\beta Tu - Sf_3 - \delta\beta Tf_1)\,\phi dx, \ \forall\phi \in C^1_0(0,L). \tag{2.24}$$

Since, $S\theta + \delta\beta Tu - Sf_3 - \delta\beta Tf_1 \in L^2(0,L)$, (2.24) shows that $\theta_x \in H^1(0,L)$, with

$$\theta_{xx} = \frac{1}{\kappa\beta}(S\theta - \delta\beta Tu + Sf_3 + \delta\beta Tf_1) \in L^2(0,L),$$

consequently, $\theta \in H^2(0,L)$. In addition, the integration by parts of the left-hand side of (2.24) and the density of $C_0^1(0,L)$ in $H_0^1(0,L)$ lead to

$$\kappa\beta\theta_{xx} = S\theta - \delta\beta Tu + Sf_3 + \delta\beta T f_1.$$

By using (2.20), we arrive at

$$\kappa\beta\theta_{xx} = S\theta + \delta\beta Tv - Sf_3,$$

which shows that θ satisfies the third equation of (2.19).

Similarly, by taking $\theta^* \equiv 0$ in (2.22) and using the definition of R, we obtain

$$\langle \alpha\mu Tu_x - J\mu(f_2+f_1)_x, u_x^*\rangle = \left\langle \rho\beta(f_1+f_2) - (\xi\mu-\beta^2)Tu - Ru + \delta\beta T\theta, u^*\right\rangle, \tag{2.25}$$

for all $u^* \in H^2(0,L)\cap H_0^1(0,L)$, which also holds for $u^* \in C_0^1(0,L) \subset H^2(0,L)\cap H_0^1(0,L)$. Therefore,

$$\alpha\mu Tu_x - J\mu(f_2+f_1)_x \in H^1(0,L),$$

with

$$(\alpha\mu Tu_x - J\mu(f_2+f_1)_x)_x = -(\rho\beta(f_1+f_2) - (\xi\mu-\beta^2)Tu - Ru + \delta\beta T\theta) \in L^2(0,L).$$

An integration by part in (2.25) and the fact that $Rf_2 \in H^{-1}(0,L)$ yield

$$-\alpha\mu Tu_{xx} = R(f_2+f_1) - (\xi\mu-\beta^2)Tu - Ru + \delta\beta T\theta \in H^{-1}(0,L)$$

Moreover, bearing in mind that the operator $-\partial_{xx}$ defines an isomorphism between $H_0^1(0,L)$ and $H^{-1}(0,L)$, we infer that $u_{xx} \in H_0^1(0,L)$ and consequently,

$$u \in H_*^3(0.L).$$

Substituting $f_1 = u - v$ from (2.20) in the last equality, we get

$$-\alpha\mu Tu_{xx} + Rv + (\xi\mu-\beta^2)Tu - \delta\beta T\theta = Rf_2. \tag{2.26}$$

Finally, by applying R^{-1} to the terms in equation (2.26), we deduce that u satisfies the third equation of (2.19).

We therefore conclude that (u, v, θ) demonstrates that $\mathcal{A}$ is maximal. Consequently, by the Hille–Yosida theorem, problem (2.17) has a unique solution. This completes the proof of Theorem 2.1. Next, we put

$$\mathscr{H} = H^2(0,L) \cap H_0^1(0,L) \times H_0^1(0,L) \times H_*^1(0,L) \times L^2(0,L),$$

and define the set

$$\mathscr{D} = \left\{ (u, v, \varphi, \theta) \in \mathscr{H}; \left| \begin{array}{ll} u \in H_*^3(0,L), & v \in H^2(0,L) \cap H_0^1(0,L) \\ \varphi \in H_*^2(0,L), & \theta \in H^2(0,L), \end{array} \right. \right\},$$

where

$$H_*^1(0,L) := \left\{ \phi \in H^1(0,L); \int_0^L \phi(x)dx = 0 \right\},$$

and

$$H_*^2(0.L) = \left\{ \phi \in H^2(0.L) : \psi_x(0) = \phi_x(L) = 0 \right\}.$$

The well-posedness result of problem (2.1)-(2.3) is given by the following theorem:

Theorem 2.2. *Let $(u_0, v_0, \varphi_0, \theta_0) \in \mathscr{D}$ and satisfies the following compatibility condition*

$$B\big(\varphi_0 + \frac{\beta}{\xi} u_{0x}\big) = -\left[\frac{\beta}{\xi} \left(J\beta + \alpha\rho - \frac{J\mu\xi}{\beta} \right) u_{0xxx} + \delta\rho\theta_{0x} \right].$$

Then there exists a unique solution (u, v, φ, θ) of problem (2.1) that satisfies

$$(u, v, \varphi, \theta) \in C(\mathbb{R}^+, \mathscr{D})) \cap C^1(\mathbb{R}^+, \mathscr{H}).$$

2.2.1 Proof of theorem 2.2

From Theorem 2.1, we have $(u, u_t, \theta) \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$. Particularly,

$$u \in C\left(\mathbb{R}^+; H^3(0,L) \cap H_0^1(0,L)\right) \cap C^1\left(\mathbb{R}^+; H^2(0,L) \cap H_0^1(0,L)\right) \cap C^2\left(\mathbb{R}^+; H_0^1(0,L)\right).$$

Now, returning to (2.13) we have

$$u_{ttt} \in C\left(\mathbb{R}^+; L^2(0,L)\right),$$

that is

$$u_{tt} \in C^1\left(\mathbb{R}^+; L^2(0,L)\right).$$

Next, we define

$$\varphi(x,t) = -\frac{\mu}{\beta}u_x(x,t) + \frac{\rho}{\beta}\int_0^x u_{tt}(y,t)dy. \tag{2.27}$$

Since $u \in H_*^3(0,L)$ and is a solution of the auxiliary problem (2.15), then $u_x \in C(\mathbb{R}^+; H^2(0,L))$. We easily check that $\varphi_x(0) = \varphi_x(L) = 0$, and consequently,

$$\varphi \in C(\mathbb{R}^+; H_*^2(0,L)),$$

and

$$\rho u_{tt} - \mu u_{xx} - \beta\varphi_x = 0. \tag{2.28}$$

In addition $\int_0^L \varphi(x,t) = 0$, because by using $(2.6)_1$, we infer

$$\int_0^L \varphi(x,t) = -\frac{\mu}{\beta}\int_0^L u_x(x,t)dx + \frac{J\beta+\alpha\rho}{\xi\beta}\int_0^L\int_0^x u_{xxtt}(y,t)dydx - \frac{\alpha\mu}{\xi\beta}\int_0^L\int_0^x u_{xxxx}(y,t)dydx$$
$$+ \frac{(\xi\mu-\beta^2)}{\xi\beta}\int_0^L\int_0^x u_{xx}(y,t)dydx - \frac{\delta}{\xi}\int_0^L\int_0^x \theta_{xx}(y,t)dydx$$

Since $(u, u_t, \theta) \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$ and the boundary condition, we have

$$\int_0^L \varphi(x,t) = \left[-\frac{\mu}{\beta}u(x,t) + \frac{J\beta+\alpha\rho}{\xi\beta}u_{tt} - \frac{\alpha\mu}{\xi\beta}u_{xx} + \frac{(\xi\mu-\beta^2)}{\xi\beta}u - \frac{\delta}{\xi}\theta\right]_0^L = 0.$$

Using (2.28), system (2.6) becomes

$$\begin{cases} -Ju_{xxtt} - \alpha\varphi_{xxx} + \beta u_{xx} + \xi\varphi_x + \delta\theta_{xx} = 0, \\ c\theta_t - \kappa\theta_{xx} + \delta\varphi_{xt} = 0. \end{cases} \tag{2.29}$$

The second equation clearly shows that $\varphi_{xt} \in C(\mathbb{R}^+, L^2(0,L))$ and, hence, $\varphi \in C^1(\mathbb{R}^+, H^1(0,L))$. Moreover, a straightforward integration of the first equation of (2.29) over $(0,x)$ yields

$$- J\rho u_{xtt} + J\rho u_{0xtt} - \alpha\rho\varphi_{xx} + \alpha\rho\varphi_{0xx} + \beta\rho u_x - \beta\rho u_{0x} + \xi\rho\varphi - \xi\rho\varphi_0$$
$$+ \delta\rho\theta_x - \delta\rho\theta_{0x} + J\beta\varphi_{0xx} - J\beta\varphi_{0xx} + \frac{\beta}{\xi}(J\beta+\alpha\rho)u_{0xxx} - \frac{\beta}{\xi}(J\beta+\alpha\rho)u_{0xxx}$$
$$+ J\mu u_{0xxx} - J\mu u_{0xxx} = C,$$

For some constant C that depends only on t.

After simplification, we obtain

$$\begin{aligned}
&-J\rho u_{xtt} + J\rho u_{0xtt} - \alpha\rho\varphi_{xx} + \beta\rho u_x + \xi\rho\varphi + \delta\rho\theta_x \\
&- B\left(\frac{\beta}{\xi}u_{0x} + \varphi_0\right) - \delta\rho\theta_{0x} - J\beta\varphi_{0xx} \\
&- \frac{\beta}{\xi}(J\beta + \alpha\rho)u_{0xxx} + J\mu u_{0xxx} - J\mu u_{0xxx} = C,
\end{aligned}$$

using the compatibility condition and $J\rho u_{0xtt} = J\beta\varphi_{0xx} + J\mu u_{0xxx}$, we get

$$-Ju_{xtt} - \alpha\varphi_{xx} + \beta u_x + \xi\varphi + \delta\theta_x = C.$$

Integrating over $(0, L)$, we obtain

$$\int_0^L \left(-Ju_{xtt} - \alpha\varphi_{xx} + \beta u_x + \xi\varphi + \delta\theta_x\right)\, dx = L\,C(t).$$

So

$$-J\big[u_{tt}\big]_0^L - \alpha\big[\varphi_x\big]_0^L + \beta\big[u\big]_0^L + \xi\int_0^L \varphi\, dx + \delta\big[\theta\big]_0^L = L\,C(t).$$

By the boundary conditions (2.3) and $\int_0^L \varphi(x,t) = 0$ all terms vanish, hence

$$L\,C(t) = 0,$$

which implies $C(t) = 0$. Therefore, (u, φ, θ) solves the problem (2.1) with the initial and boundary conditions (2.2), (2.3), which completes the proof of Theorem 2.2.

2.3 Exponential Stability

This section is devoted to proving the exponential decay of the solution to the problem (2.1)-(2.3).

First, we define the energy associated to the solution of (2.1) by

$$\mathcal{E}(t) := \frac{1}{2}\int_0^L \left(\frac{J\rho}{\beta}u_{tt}^2 + \frac{J\mu}{\beta}u_{xt}^2 + \rho u_t^2 + \mu u_x^2 + \alpha\varphi_x^2 + 2\beta u_x\varphi + \xi\varphi^2 + c\theta^2\right) dx.$$

Note that since $\mu\xi > \beta^2$, we have

$$\mathcal{E}(t) = \frac{J\rho}{2\beta}\int_0^L u_{tt}^2 dx + \frac{J\mu}{2\beta}\int_0^L u_{xt}^2 dx + \frac{\rho}{2}\int_0^L u_t^2 dx + \frac{1}{2}\left(\mu - \frac{\beta^2}{\xi}\right)\int_0^L u_x^2 dx + \frac{\alpha}{2}\int_0^L \varphi_x^2 dx$$

$$+\frac{1}{2}\int_0^L \left(\frac{\beta}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)^2 dx+\frac{c}{2}\int_0^L \theta^2 dx,$$

which shows that $\mathcal{E}(t)$ is a positive form.

Remark 2.5. *Differentiating the first equation of (2.1) with respect to t, we obtain*

$$\varphi_{xt}=\frac{\rho}{\beta}u_{ttt}-\frac{\mu}{\beta}u_{xxt}. \tag{2.30}$$

Using integration by parts and substituting (2.30), we get

$$\begin{aligned}-J\int_0^L u_{xtt}\varphi_t dx =& J\int_0^L u_{tt}\varphi_{xt}dx\\ =& J\int_0^L \left[\frac{\rho}{\beta}u_{ttt}-\frac{\mu}{\beta}u_{xxt}\right]u_{tt}dx\\ =& \frac{J\rho}{\beta}\int_0^L u_{ttt}u_{tt}dx-\frac{J\mu}{\beta}\int_0^L u_{xxt}u_{tt}dx,\end{aligned}$$

using integration by parts for the integral $\int_0^L u_{xxt}u_{tt}dx$*, so*

$$-J\int_0^L u_{xtt}\varphi_t dx=\frac{J\rho}{2\beta}\frac{d}{dt}\int_0^L u_{tt}^2 dx+\frac{J\mu}{2\beta}\frac{d}{dt}\int_0^L u_{xt}^2 dx.$$

Remark 2.6. *Clearly, we have*

$$\begin{aligned}&\beta\frac{d}{dt}\int_0^L u_x\varphi dx+\frac{\mu}{2}\frac{d}{dt}\int_0^L u_x^2 dx+\frac{\xi}{2}\frac{d}{dt}\int_0^L \varphi^2 dx\\ =&\frac{1}{2}\frac{d}{dt}\int_0^L\left(\sqrt{\xi}\varphi+\frac{\beta}{\sqrt{\xi}}u_x\right)^2 dx+\frac{1}{2}\left(\mu-\frac{\beta^2}{\xi}\right)\frac{d}{dt}\int_0^L u_x^2 dx.\end{aligned}$$

Theorem 2.3. *The energy functional $\mathcal{E}(t)$ satisfies, along the solution of (2.1),(2.3), the estimate*

$$\mathcal{E}(t)\le A\mathcal{E}(0)e^{-\lambda t},\ \forall t\ge 0,$$

where A and λ are two positive constants.

The proof of Theorem 2.3 will be established with the help of several lemmas.

Lemma 2.1. *The energy $\mathcal{E}(t)$ satisfies, along the solution (u,φ,θ) of (2.1), the estimate*

$$\mathcal{E}'(t)\le -\kappa\int_0^L \theta_x^2 dx\le 0. \tag{2.31}$$

Proof. Taking the L^2-inner product of the equation of (2.1) by u_t, φ_t and θ, respectively, the first equation yields

$$\frac{\rho}{2}\frac{d}{dt}\int_0^L u_t^2 dx + \beta\int_0^L u_{xt}\varphi dx + \frac{\mu}{2}\frac{d}{dt}\int_0^L u_x^2 dx = 0 \tag{2.32}$$

and the second equation leads to

$$-J\int_0^L u_{ttx}\varphi_t + \frac{\alpha}{2}\frac{d}{dt}\int_0^L \varphi_x^2 dx + \beta\int_0^L u_x\varphi_t + \frac{\xi}{2}\frac{d}{dt}\int_0^L \varphi^2 dx + \delta\int_0^L \theta_x\varphi_t = 0.$$

By invoking Remark 2.5, we get

$$\begin{aligned}&\frac{J\rho}{2\beta}\frac{d}{dt}\int_0^L u_{tt}^2 dx + \frac{J\mu}{2\beta}\frac{d}{dt}\int_0^L u_{xt}^2 dx + \frac{\alpha}{2}\frac{d}{dt}\int_0^L \varphi_x^2 dx\\ &+\beta\int_0^L u_x\varphi_t + \frac{\xi}{2}\frac{d}{dt}\int_0^L \varphi^2 dx + \delta\int_0^L \theta_x\varphi_t = 0.\end{aligned} \tag{2.33}$$

Finally, the third equation produces

$$\frac{c}{2}\frac{d}{dt}\int_0^L \theta^2 dx - \delta\int_0^L \varphi_t\theta_x dx = -\kappa\int_0^L \theta_x^2 dx, \tag{2.34}$$

From equations (2.32)–(2.34), we obtain

$$\begin{aligned}&\frac{\rho}{2}\frac{d}{dt}\int_0^L u_t^2 dx + \beta\int_0^L u_{xt}\varphi dx + \frac{\mu}{2}\frac{d}{dt}\int_0^L u_x^2 + \frac{J\rho}{2\beta}\frac{d}{dt}\int_0^L u_{tt}^2 dx + \frac{J\mu}{2\beta}\frac{d}{dt}\int_0^L u_{xt}^2 dx\\ &+\frac{\alpha}{2}\frac{d}{dt}\int_0^L \varphi_x^2 dx + \beta\int_0^L u_x\varphi_t + \frac{\xi}{2}\frac{d}{dt}\int_0^L \varphi^2 dx + \delta\int_0^L \theta_x\varphi_t + \frac{c}{2}\frac{d}{dt}\int_0^L \theta^2 dx - \delta\int_0^L \varphi_t\theta_x dx\\ &= -\kappa\int_0^L \theta_x^2 dx.\end{aligned}$$

Remark 2.6 implies that

$$\mathcal{E}'(t) = -\kappa\int_0^L \theta_x^2 dx \le 0.$$

□

Lemma 2.2. *Let* (u, φ, θ) *be the solution of* (2.1)–(2.3)*. The functional*

$$\begin{aligned}\mathcal{F}_1(t) = &-\frac{J}{\sqrt{\xi}}\int_0^L u_{xt}\left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)dx\\ &+\left(\frac{Jc}{\delta} + \frac{c\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L u_t\theta dx + \frac{\alpha\beta\rho}{\xi\mu}\int_0^L \varphi_x u_t dx,\end{aligned}$$

satisfies, for any positive constant ε, the estimate

$$\mathcal{F}_1'(t) \le -\frac{\alpha}{\mu}\left(\mu-\frac{\beta^2}{\xi}\right)\int_0^L \varphi_x^2 dx - \frac{1}{2}\int_0^L \left(\frac{\beta}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)^2 dx$$
$$-\frac{J\beta}{2\xi}\int_0^L u_{xt}^2 dx + C_\varepsilon \int_0^L \theta_x^2 dx + \varepsilon\int_0^L u_{tt}^2 dx. \tag{2.35}$$

Proof. From the first and second equations of (2.1), we have

$$u_{xx} = \frac{\rho}{\mu}u_{tt} - \frac{\beta}{\mu}\varphi_x \tag{2.36}$$

and

$$-Ju_{xxt} = \alpha\varphi_{xx} - \beta u_x - \xi\varphi - \delta\theta_x. \tag{2.37}$$

By differentiating $\mathcal{F}_1(t)$ and using (2.37), we arrive at

$$\begin{aligned}
\mathcal{F}_1'(t) =& -\frac{J}{\sqrt{\xi}}\int_0^L u_{xtt}\left(\frac{\beta}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)dx - \frac{J}{\sqrt{\xi}}\int_0^L u_{xt}\left(\frac{\beta}{\sqrt{\xi}}u_{xt}+\sqrt{\xi}\varphi_t\right)dx \\
&+\left(\frac{Jc}{\delta}+\frac{c\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L u_{tt}\theta dx + \left(\frac{Jc}{\delta}+\frac{c\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L u_t\theta_t dx \\
&+\frac{\alpha\beta\rho}{\xi\mu}\int_0^L \varphi_x u_{tt}dx + \frac{\alpha\beta\rho}{\xi\mu}\int_0^L \varphi_{xt}u_t dx \\
=& \frac{\alpha}{\sqrt{\xi}}\int_0^L \varphi_{xx}\left(\frac{\beta}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)dx - \int_0^L\left(\frac{\beta}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)^2 dx \\
&-\frac{\delta}{\sqrt{\xi}}\int_0^L \theta_x\left(\frac{\beta}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)dx - \frac{J\beta}{\xi}\int_0^L u_{xt}^2 dx - J\int_0^L u_{xt}\varphi_t dx \\
&+\left(\frac{Jc}{\delta}+\frac{c\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L u_{tt}\theta dx + \left(\frac{Jc}{\delta}+\frac{c\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L u_t\theta_t dx \\
&+\frac{\alpha\beta\rho}{\xi\mu}\int_0^L \varphi_x u_{tt}dx + \frac{\alpha\beta\rho}{\xi\mu}\int_0^L \varphi_{xt}u_t dx.
\end{aligned}$$

Using the third equation of (2.1) and applying integration by parts, we get

$$\begin{aligned}
\mathcal{F}_1'(t) =& -\frac{\alpha\beta}{\xi}\int_0^L \varphi_x u_{xx}dx - \alpha\int_0^L \varphi_x^2 dx - \int_0^L\left(\frac{\beta}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)^2 dx \\
&-\frac{\delta}{\sqrt{\xi}}\int_0^L \theta_x\left(\frac{\beta}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)dx - \frac{J\beta}{\xi}\int_0^L u_{xt}^2 dx + J\int_0^L u_t\varphi_{xt}dx \\
&+\left(\frac{Jc}{\delta}+\frac{c\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L u_{tt}\theta dx - \left(\frac{J\kappa}{\delta}+\frac{\kappa\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L u_{xt}\theta_x dx - \left(J+\frac{\alpha\beta\rho}{\xi\mu}\right)\int_0^L u_t\varphi_{xt}dx \\
&+\frac{\alpha\beta\rho}{\xi\mu}\int_0^L \varphi_x u_{tt}dx + \frac{\alpha\beta\rho}{\xi\mu}\int_0^L \varphi_{xt}u_t dx.
\end{aligned}$$

Recalling (2.36); we obtain

$$\begin{aligned}
\mathcal{F}_1'(t) =& -\frac{\alpha\beta\rho}{\xi\mu}\int_0^L \varphi_x u_{tt}dx + \frac{\alpha\beta^2}{\xi\mu}\int_0^L \varphi_x^2 dx - \alpha\int_0^L \varphi_x^2 dx - \int_0^L \left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx \\
& -\frac{\delta}{\sqrt{\xi}}\int_0^L \theta_x\left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)dx - \frac{J\beta}{\xi}\int_0^L u_{xt}^2 dx + \left(\frac{Jc}{\delta} + \frac{c\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L u_{tt}\theta dx \\
& -\left(\frac{J\kappa}{\delta} + \frac{\kappa\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L u_{xt}\theta_x dx + \frac{\alpha\beta\rho}{\xi\mu}\int_0^L \varphi_x u_{tt}dx. \\
=& -\frac{\alpha}{\mu}\left(\mu - \frac{\beta^2}{\xi}\right)\int_0^L \varphi_x^2 dx - \int_0^L \left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx - \frac{\delta}{\sqrt{\xi}}\int_0^L \theta_x\left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)dx \\
& -\frac{J\beta}{\xi}\int_0^L u_{xt}^2 dx - \left(\frac{J\kappa}{\delta} + \frac{\alpha\beta\rho}{\xi\mu}\right)\int_0^L u_{xt}\theta_x dx + \left(\frac{Jc}{\delta} + \frac{c\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L u_{tt}\theta dx. \quad (2.38)
\end{aligned}$$

By applying Young's and Poincaré's inequalities, we arrive at

$$-\frac{\delta}{\sqrt{\xi}}\int_0^L \theta_x\left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)dx \leq \frac{1}{2}\int_0^L \left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx + \frac{\delta^2}{2\xi}\int_0^L \theta_x^2 dx, \quad (2.39)$$

$$\left(\frac{J\kappa}{\delta} + \frac{\alpha\beta\rho}{\xi\mu}\right)\int_0^L u_{xt}\theta_x dx \leq \frac{J\beta}{2\xi}\int_0^L u_{xt}^2 dx + \frac{\xi}{2J\beta}\left(\frac{J\kappa}{\delta} + \frac{\alpha\beta\rho}{\xi\mu}\right)^2\int_0^L \theta_x^2 dx, \quad (2.40)$$

and

$$\left(\frac{Jc}{\delta} + \frac{c\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L u_{tt}\theta dx \leq \varepsilon\int_0^L u_{tt}dx + \frac{1}{4\varepsilon}\left(\frac{J\kappa}{\delta} + \frac{\alpha\beta\rho c_p}{\xi\mu}\right)^2\int_0^L \theta_x dx. \quad (2.41)$$

By substituting inequalities (2.39), (2.40), and (2.41) into equation (2.38), the estimate (2.35) is confirmed, with

$$C_\varepsilon = \frac{\delta^2}{2\xi} + \frac{\xi}{2J\beta}\left(\frac{J\kappa}{\delta} + \frac{\alpha\beta\rho}{\xi\mu}\right)^2 + \frac{1}{4\varepsilon}\left(\frac{J\kappa}{\delta} + \frac{\alpha\beta\rho c_p}{\xi\mu}\right)^2.$$

□

Remark 2.7. *Notice that*

$$\begin{aligned}
\mu u_x^2 - 2\beta u_x\varphi - \xi\varphi^2 &= \mu u_x^2 - 2\beta u_x\varphi - \xi\varphi^2 + \frac{\beta^2}{\xi}u_x^2 - \frac{\beta^2}{\xi}u_x^2 \\
&= -\left(\mu - \frac{\beta^2}{\xi}\right)u_x^2 - \left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2
\end{aligned}$$

Lemma 2.3. *Let (u,φ,θ) be a solution of system (2.1-(2.3),the functional*

$$\mathcal{F}_2(t) = \rho\int_0^L u_t u dx - J\int_0^L u_{xt}\varphi dx + \frac{Jc}{\delta}\int_0^L u_t\theta dx,$$

satisfies, for any positive constant ε, the estimate

$$\mathcal{F}_2'(t) \leq -\left(\mu - \frac{\beta^2}{\xi}\right)\int_0^L u_x^2 dx - \frac{\alpha}{2}\int_0^L \varphi_x^2 dx \\ - \int_0^L \left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx + \rho\int_0^L u_t^2 dx + C_\varepsilon'\int_0^L \theta_x^2 dx \\ + \varepsilon\int_0^L u_{tt}^2 dx + \varepsilon\int_0^L u_{xt}^2 dx. \tag{2.42}$$

Proof. By directly differentiating $\mathcal{F}_2$, we obtain

$$\mathcal{F}_2'(t) = \rho\int_0^L u_{tt}u dx + \rho\int_0^L u_t^2 dx - J\int_0^L u_{xtt}\varphi dx - J\int_0^L u_{xt}\varphi_t dx \\ + \frac{Jc}{\delta}\int_0^L u_{tt}\theta dx + \frac{Jc}{\delta}\int_0^L u_t\theta_t dx.$$

Using (2.1), we arrive at

$$\mathcal{F}_2'(t) = \beta\int_0^L \varphi_x u dx + \mu\int_0^L u_{xx}u dx + \rho\int_0^L u_t^2 dx + \alpha\int_0^L \varphi_{xx}\varphi dx - \beta\int_0^L u_x\varphi dx \\ - \xi\int_0^L \varphi^2 dx - \delta\int_0^L \theta_x\varphi dx - J\int_0^L u_{xt}\varphi_t dx + \frac{Jc}{\delta}\int_0^L u_{tt}\theta dx \\ + \frac{J\kappa}{\delta}\int_0^L u_t\theta_{xx} dx - J\int_0^L u_t\varphi_{xt} dx.$$

Applying integration by parts yields

$$\mathcal{F}_2'(t) = -\mu\int_0^L u_x^2 dx + \rho\int_0^L u_t^2 dx - \alpha\int_0^L \varphi_x^2 dx - 2\beta\int_0^L u_x\varphi dx \\ - \xi\int_0^L \varphi^2 dx + \delta\int_0^L \theta\varphi_x dx + J\int_0^L u_t\varphi_{xt} dx + \frac{Jc}{\delta}\int_0^L u_{tt}\theta dx \\ - \frac{J\kappa}{\delta}\int_0^L u_{xt}\theta_x dx - J\int_0^L u_t\varphi_{xt} dx.$$

By applying Remark2.7, we obtain the following equality

$$\mathcal{F}_2'(t) = -\left(\mu - \frac{\beta^2}{\xi}\right)\int_0^L u_x^2 dx + \rho\int_0^L u_t^2 dx - \int_0^L \left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx \\ - \alpha\int_0^L \varphi_x^2 dx + \delta\int_0^L \theta\varphi_x + \frac{Jc}{\delta}\int_0^L u_{tt}\theta dx - \frac{J\kappa}{\delta}\int_0^L u_{xt}\theta_x dx. \tag{2.43}$$

Thus, by applying Young's and Poincaré's inequalities, we deduce

$$\begin{aligned}\delta\int_0^L \theta\varphi_x dx &\leq \frac{\alpha}{2}\int_0^L \varphi_x^2 dx + \frac{\delta^2}{2\alpha}\int_0^L \theta^2 dx\\ &\leq \frac{\alpha}{2}\int_0^L \varphi_x^2 dx + \frac{\delta^2 c_p}{2\alpha}\int_0^L \theta_x^2 dx, \qquad (2.44)\\ \frac{Jc}{\delta}\int_0^L u_{tt}\theta dx &\leq \varepsilon\int_0^L u_{tt}^2 dx + \frac{J^2c^2}{4\delta^2\varepsilon}\int_0^L \theta^2 dx\\ &\leq \varepsilon\int_0^L u_{tt}^2 dx + \frac{J^2c^2c_p}{4\delta^2\varepsilon}\int_0^L \theta_x^2 dx, \qquad (2.45)\end{aligned}$$

and

$$-\frac{J\kappa}{\delta}\int_0^L u_{xt}\theta_x dx \leq \varepsilon\int_0^L u_{xt}^2 dx + \frac{J^2\kappa^2}{4\delta^2\varepsilon}\int_0^L \theta_x^2 dx. \qquad (2.46)$$

By substituting inequalities (2.44), (2.45), and (2.46) into equation (2.43), the estimate (2.42) established, with

$$C'_\varepsilon = \frac{\delta^2 c_p}{2\alpha} + \frac{1}{\varepsilon}\left(\frac{J^2c^2c_p}{4\delta^2} + \frac{J^2\kappa^2}{4\delta^2}\right).$$

□

Lemma 2.4. *Let (u,φ,θ) be the solution of (2.1)-(2.3), then the functional*

$$\mathcal{F}_3(t) = \frac{\alpha\rho}{\beta}\int_0^L u_t\left(\varphi_x + \frac{c}{\delta}\theta\right)dx$$

satisfies, for any positive constant ε the following estimate

$$\begin{aligned}\mathcal{F}'_3(t) \leq &-\frac{J\rho}{2\beta}\int_0^L u_{tt}^2 dx - \frac{1}{2}\int_0^L\left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx\\ &+\varepsilon\int_0^L u_{xt}^2 dx + C''_\varepsilon\int_0^L \theta_x^2 dx + C\int_0^L u_x^2 dx. \qquad (2.47)\end{aligned}$$

Proof. By integrating the second equation of (2.1) over $(0,x)$ and using (2.3), we easily see that

$$\alpha\varphi_x = -Ju_{tt} + \beta u + \xi\int_0^x \varphi(y)dy + \delta\theta. \qquad (2.48)$$

Taking the derivative of $\mathcal{F}_3(t)$, we get

$$\begin{aligned}\mathcal{F}'_3(t) &= \frac{\alpha\rho}{\beta}\int_0^L u_{tt}\left(\varphi_x + \frac{c}{\delta}\theta\right) + \frac{\alpha\rho}{\beta}\int_0^L u_t\left(\varphi_{xt} + \frac{c}{\delta}\theta_t\right)\\ &= \frac{\alpha\rho}{\beta}\int_0^L u_{tt}\varphi_x dx + \frac{c\alpha\rho}{\beta\delta}\int_0^L u_{tt}\theta dx + \frac{\alpha\rho}{\beta}\int_0^L u_t\varphi_{xt}dx + \frac{c\alpha\rho}{\beta\delta}\int_0^L u_t\theta_t dx.\end{aligned}$$

Using the third equation from (2.1) and equation (2.48), we obtain

$$\mathcal{F}_3'(t) = -\frac{J\rho}{\beta}\int_0^L u_{tt}^2 dx + \frac{\rho}{\beta}\int_0^L u_{tt}\left(\beta u + \xi\int_0^x \varphi(y)dy\right)dx + \frac{\rho}{\beta}\left(\delta + \frac{c\alpha}{\delta}\right)\int_0^L u_{tt}\theta dx + \frac{\alpha\rho\kappa}{\delta\beta}\int_0^L u_t\theta_{xx}dx. \tag{2.49}$$

Next, we put

$$f_1 = \frac{\rho}{\beta}\int_0^L u_{tt}\left(\beta u + \xi\int_0^x \varphi(y)dy\right)dx,\ \ f_2 = \frac{\rho}{\beta}\left(\delta + \frac{c\alpha}{\delta}\right)\int_0^L u_{tt}\theta dx,\ \ f_3 = \frac{\alpha\rho\kappa}{\delta\beta}\int_0^L u_t\theta_{xx}dx.$$

Using $\rho u_{tt} = (\mu u_x + \beta\varphi)_x$ and integration by parts, we get

$$\begin{aligned} f_1 = \frac{\rho}{\beta}\int_0^L u_{tt}\left(\beta u + \xi\int_0^x \varphi(y)dy\right)dx &= \frac{1}{\beta}\int_0^L (\mu u_x + \beta\varphi)_x\left(\beta u + \xi\int_0^x \varphi(y)dy\right)dx \\ &= -\int_0^t \left(\frac{\mu}{\beta}u_x + \varphi\right)(\beta u_x + \xi\varphi)\,dx \\ &= -\int_0^L \left(\frac{\mu}{\beta}u_x + \varphi\right)\sqrt{\xi}\left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)dx \\ &= -\int_0^L \left(\frac{\mu\sqrt{\xi}}{\beta}u_x + \sqrt{\xi}\varphi\right)\left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)dx, \end{aligned}$$

so

$$f_1 = -\left(\frac{\mu\sqrt{\xi}}{\beta} - \frac{\beta}{\sqrt{\xi}}\right)\int_0^L u_x\left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)dx - \int_0^L\left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx,$$

We apply Young's inequality to the first part of the last inequality, we infer

$$f_1 \leq -\frac{1}{2}\int_0^L\left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 + \frac{(\xi\mu - \beta^2)^2}{2\xi\beta^2}\int_0^L u_x^2 dx. \tag{2.50}$$

Next, we apply integration by parts on f_2 and f_3, then use Young's and Poincaré's inequalities again, to arrive at

$$f_2 = \frac{\rho}{\beta}\left(\delta + \frac{c\alpha}{\delta}\right)\int_0^L u_{tt}\theta dx \leq \frac{J\rho}{2\beta}\int_0^L u_{tt}^2 dx + \frac{\rho}{2J\beta}\left(\delta + \frac{c\alpha}{\delta}\right)^2 c_p\int_0^L \theta_x^2 dx. \tag{2.51}$$

$$f_3 = \frac{\alpha\rho\kappa}{\delta\beta}\int_0^L u_t\theta_{xx}dx = -\frac{\alpha\rho\kappa}{\delta\beta}\int_0^L u_{xt}\theta_x$$

$$f_3 \leq \varepsilon\int_0^L u_{xt}^2 dx + \frac{\alpha^2\rho^2\kappa^2}{4\delta^2\beta^2\varepsilon}\int_0^L \theta_x^2 dx. \tag{2.52}$$

Finally, by taking

$$C = \frac{(\xi\mu - \beta^2)^2}{2\xi\beta^2} \quad \textit{and} \quad C''_\varepsilon = \frac{\rho}{2J\beta}\left(\delta + \frac{c\alpha}{\delta}\right)^2 + \frac{\alpha^2\rho^2\kappa^2}{4\delta^2\beta^2\varepsilon}.$$

and substituting equations (2.50), (2.51), and (2.52) into (2.49), we establish the estimate (2.47). Now, we define the Lyapunov functional $\mathcal{L}$ by

$$\mathcal{L}(t) := NE(t) + N_1\mathcal{F}_1(t) + N_2\mathcal{F}_2(t) + \mathcal{F}_3(t). \tag{2.53}$$

where $N_{(i=1,2)}$ are positive constants to be fixed later and $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ are given in Lemmas 2.2, 2.3 and 2.4, respectively.

Lemma 2.5. *There exists a positive constant χ such that*

$$(N-\chi)\mathcal{E}(t) \leq \mathcal{L}(t) \leq (N+\chi)\mathcal{E}(t) \quad \forall t > 0.$$

Proof. We have

$$\begin{aligned}\Big|\mathcal{L}(t) - N\mathcal{E}(t)\Big| \leq & \frac{JN_1}{\sqrt{\xi}}\int_0^L \Big|u_{xt}\left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)\Big|dx + \left(\frac{Jc}{\delta} + \frac{c\alpha\beta\rho}{\xi\mu\delta}\right)N_1\int_0^L \Big|u_t\theta\Big|dx \\ & + \frac{\alpha\beta\rho N_1}{\xi\mu}\int_0^L \Big|\varphi_x u_t\Big|dx + \rho N_2\int_0^L \Big|u_t u\Big|dx + JN_2\int_0^L \Big|u_{xt}\varphi\Big|dx \\ & + \frac{JcN_2}{\delta}\int_0^L \Big|u_t\theta\Big|dx + \frac{\alpha\rho}{\beta}\int_0^L \Big|u_t\left(\varphi_x + \frac{c}{\delta}\theta\right)\Big|dx.\end{aligned}$$

Applying Young's inequality, we deduce that there exists a a positive constant χ so that

$$\begin{aligned}\Big|\mathcal{L}(t) - N\mathcal{E}(t)\Big| \leq & \chi\left(\int_0^L u_t^2dx + \int_0^L u_x^2dx + \int_0^L \varphi^2dx + \int_0^L \varphi_x^2dx + \int_0^L \theta^2dx\right) \\ & + \chi\left(\int_0^L u_{xt}^2dx + \int_0^L \left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2dx\right) \leq \chi\mathcal{E}(t).\end{aligned}$$

So

$$\begin{aligned}-\chi\mathcal{E}(t) \leq \mathcal{L}(t) - N\mathcal{E}(t) \leq \chi\mathcal{E}(t), \\ N\mathcal{E}(t) - \chi\mathcal{E}(t) \leq \mathcal{L}(t) \leq N\mathcal{E}(t) + \chi\mathcal{E}(t).\end{aligned}$$

Thus,

$$(N-\chi)\mathcal{E}(t) \leq \mathcal{L}(t) \leq (N+\chi)\mathcal{E}(t), \ \forall t > 0.$$

□

Proof theorem 2.5

By differentiating equation (2.53) and substituting equations (2.35), (2.42) and (2.47), then applying Poincaré's inequality, we obtain

$$\begin{aligned}\mathcal{L}'(t) \leq & -\left[\frac{J\rho}{2\beta} - \varepsilon\left(N_2 + N_1\right)\right]\int_0^L u_{tt}^2 dx - \left[\frac{J\beta}{4\xi}N_1 - \varepsilon\left(N_2 + 1\right)\right]\int_0^L u_{xt}^2 dx \\ & - \left[\frac{J\beta}{4\xi c_p}N_1 - \rho N_2\right]\int_0^L u_t^2 dx - \left[\left(\mu - \frac{\beta^2}{\xi}\right)N_2 - C\right]\int_0^L u_x^2 dx \\ & - \left[\frac{\alpha}{\mu}(\mu - \frac{\beta^2}{\xi})N_1 + \frac{\alpha}{2}N_2\right]\int_0^L \varphi_x^2 dx - \left[\frac{1}{2}N_1 + N_2 + \frac{1}{2}\right]\int_0^L \left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx \\ & - \frac{\kappa}{2c_p}N\int_0^L \theta dx - \left[\frac{\kappa}{2}N - C_\varepsilon N_1 - C'_\varepsilon N_2 - C''_\varepsilon\right]\int_0^L \theta_x^2 dx.\end{aligned}$$

First, we select $N_2 > \dfrac{C}{\left(\mu - \dfrac{\beta^2}{\xi}\right)}$ and $N_1 > \dfrac{4\xi\rho N_2 c_p}{J\beta}$,

then choose $\varepsilon = \min\left\{\dfrac{J\rho}{2\beta(N_1+N_2)}, \dfrac{J\beta}{4\xi(N_2+1)}\right\}$.

Finally, pick N large such that

$$\left[\frac{\kappa}{2}N - C_\varepsilon N_1 - C'_\varepsilon N_2 - C''_\varepsilon\right] > 0 \quad \text{and} \quad N > \chi.$$

Then, we conclude that there exists a positive constant ζ such that

$$\mathcal{L}'(t) \leq -\zeta\mathcal{E}(t), \quad \forall t \geq 0 \tag{2.54}$$

Moreover, from Lemma 2.5, we have $\mathcal{L}(t)$ and $\mathcal{E}(t)$ are equivalents. So

$$\mathcal{L}'(t) \leq -\lambda\mathcal{L}(t) \Longleftrightarrow \mathcal{L}(t) \leq \mathcal{L}(0)\, e^{-\lambda t}, \quad \forall t \geq 0.$$

Therefore, the desired result follows by the equivalence of $\mathcal{L}(t)$ and $\mathcal{E}(t)$. This completes the proof of Theorem 2.3. □

chapter 3

On the Exponential Stability of a Truncated Porous Elastic System with Second Sound Thermoelasticity

The results presented in this chapter have been published in our paper [7].

3.1 Introduction

In this chapter, we examine the well-posedness and long-time behavior of the solution to a porous thermoelastic system free of the second spectrum. We employ a non-classical approach based on operator theory to demonstrate the existence of a unique solution. Additionally, we utilize the multiplier method to establish the exponential decay of the solution, independent of any relationships between the coefficients of the system.

The problem studied in this chapter is the following:

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - \beta \varphi_x = 0 & \text{in } (0, L) \times (0, +\infty), \\ -J u_{xtt} - \alpha \varphi_{xx} + \beta u_x + \xi \varphi + \delta \theta_x = 0 & \text{in } (0, L) \times (0, +\infty), \\ c\theta_t + q_x + \delta \varphi_{tx} = 0 & \text{in } (0, L) \times (0, +\infty), \\ \tau q_t + q + \theta_x = 0 & \text{in } (0, L) \times (0, +\infty), \end{cases} \tag{3.1}$$

where the heat conduction in system (3.1) is governed by Cattaneo's law.

The system (3.1) is subjected to the following initial and boundary conditions:

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \varphi(x,0) = \varphi_0(x), \theta(x,0) = \theta_0(x), q(x,0) = q_0(x), \tag{3.2}$$

for $x \in [0, L]$, and

$$u(0,t) = u(L,t) = \varphi_x(0,t) = \varphi_x(L,t) = \theta(0,t) = \theta(L,t) = 0, \ t \in (0, +\infty), \tag{3.3}$$

where u is the displacement, φ is the volume fraction, θ denotes the difference of temperature and q is the heat flux of a one-dimensional porous material of length L. Regarding the positive coefficients $\rho, \mu, J, \alpha, \xi, c, \beta$ and δ, they are the same as those introduced and defined in the previous chapter. The parameter τ represents the relaxation time.

Furthermore, to ensure that the energy $\mathcal{E}(t)$ associated with the system (3.1) is positive definite, we assume that

$$\xi\mu > \beta^2. \tag{3.4}$$

It is important to note that the applicability of Poincaré's inequality to q cannot be established by the absence of boundary conditions for q. To address this issue, we proceed as follows:
From the fourth equation of (3.1) and the boundary conditions in (3.3), we deduce that

$$\frac{d}{dt}\int_0^L q(x,t)dx + \frac{1}{\tau}\int_0^L q(x,t)dx = 0. \tag{3.5}$$

By solving the differential equation (3.5), we get

$$\int_0^L q(x,t)dx = e^{\frac{-t}{\tau}}\int_0^L q_0(x)dx.$$

Next, we introduce the new variable

$$\widetilde{q}(x,t) = q(x,t) - \frac{e^{\frac{-t}{\tau}}}{L}\int_0^L q_0(x)dx,$$

then, we have

$$\int_0^L \widetilde{q}(x,t)dx = 0. \tag{3.6}$$

In addition, by integrating the second equation of (3.1) and applying the boundary conditions in (3.3), we obtain

$$-J\frac{d^2}{dt^2}\int_0^L u_x(x,t)dx + \xi\int_0^L \varphi(x,t)dx = 0.$$

From the boundary conditions (3.3), the first integral on the left-hand side vanishes, leading to the following result:

$$\int_0^L \varphi(x,t)dx = 0. \tag{3.7}$$

Thus, Poincaré's inequality can be applied for $\widetilde{q}$ and φ. Moreover, $(u, \varphi, \theta, \widetilde{q})$ satisfies (3.1)–(3.3). For simplicity, we will henceforth write (u, φ, θ, q) instead of $(u, \varphi, \theta, \widetilde{q})$.

3.2 Well-posedness

In this section, we prove that the problem presented by (3.1)–(3.3) has a unique solution. First, we apply several transformations to convert the original problem it into an auxiliary boundary value problem, and then we apply the Hille–Yosida theorem.

First, we differentiate the second equation of (3.1) with respect to x, and substitute φ_x from the first equation of (3.1) into the remaining equations. As result, we find

$$\begin{cases} Bu_{tt} + \alpha\mu u_{xxxx} + (\beta^2 - \xi\mu)u_{xx} + \delta\beta\theta_{xx} = 0, \\ c\beta\theta_t + \beta q_x + \delta\rho u_{ttt} - \delta\mu u_{xxt} = 0, \\ \tau q_t + q + \theta_x = 0, \end{cases} \tag{3.8}$$

where B is a positive, self-adjoint, and invertible operator defined on $L^2(0, L)$ with domain $D(B) = H^2(0, L) \cap H_0^1(0, L)$, given by

$$B = \xi\rho I - (J\beta + \alpha\rho)\partial_{xx}.$$

Clearly, we have

$$\partial_{xx} = \frac{1}{\alpha\rho + J\beta}\left(\xi\rho I - B\right). \tag{3.9}$$

Then, from the first equation of (3.8) we have

$$u_{ttt} = -\alpha\mu B^{-1}u_{xxxxt} + (\xi\mu - \beta^2)B^{-1}u_{xxt} - \delta\beta B^{-1}\theta_{xxt}. \tag{3.10}$$

Inserting (3.10) into the second equation of (3.8), we get

$$\left(c\beta I - \delta^2\rho\beta B^{-1} \circ \partial_{xx}\right)\theta_t + \beta q_x + \delta\beta\left[-\frac{\alpha\mu\rho}{\beta}B^{-1} \circ \partial_{xx} + \frac{\rho}{\beta}(\xi\mu - \beta^2)B^{-1} - \frac{\mu}{\beta}I\right] \circ \partial_{xx}u_t = 0,$$

substituting (3.9) into the last equation, we arrive at

$$S\theta_t+\beta q_x+\frac{\delta\beta}{J\beta+\alpha\rho}\left[-\frac{\alpha\rho^2\xi\mu}{\beta}B^{-1}+\frac{\alpha\mu\rho}{\beta}I+J\rho(\xi\mu-\beta^2)B^{-1}\right]\circ\partial_{xx}u_t$$
$$+\frac{\delta\beta}{J\beta+\alpha\rho}\left[\frac{\alpha\rho^2\xi\mu}{\beta}B^{-1}-\alpha\rho^2\beta B^{-1}-J\mu I-\frac{\rho\alpha\mu}{\beta}I\right]\circ\partial_{xx}u_t=0,$$

which gives, after simplification,

$$S\theta_t+\beta q_x+\frac{\delta\beta}{J\beta+\alpha\rho}\left[J\rho(\xi\mu-\beta^2)B^{-1}-\alpha\rho^2\beta B^{-1}-J\mu I\right]\circ\partial_{xx}u_t=0. \tag{3.11}$$

we put

$$\begin{aligned}T&=-\frac{1}{J\beta+\alpha\rho}\left[\alpha\rho^2\beta B^{-1}-J\rho(\xi\mu-\beta^2)B^{-1}+J\mu I\right]\circ\partial_{xx},\\
&=-\frac{1}{J\beta+\alpha\rho}\left[-J\rho I(\xi\mu-\beta^2)+\alpha\rho^2\beta I+J\mu B\right]B^{-1}\circ\partial_{xx},\\
&=-\frac{1}{J\beta+\alpha\rho}\left[-J\rho I(\xi\mu-\beta^2)+\alpha\rho^2\beta I+J\mu\left(\xi\rho I-(J\beta+\alpha\rho)\partial_{xx}\right)\right]B^{-1}\circ\partial_{xx},\\
&=-\frac{1}{J\beta+\alpha\rho}\left[J\rho\beta^2 I+\alpha\rho^2\beta I-J^2\mu\beta\partial_{xx}-J\mu\alpha\rho\partial_{xx}\right]B^{-1}\circ\partial_{xx},\\
&=-\frac{1}{J\beta+\alpha\rho}\left[(J\beta+\alpha\rho)\left(\rho\beta I-J\mu\partial_{xx}\right)\right]B^{-1}\circ\partial_{xx}.\\
&=-\left(\rho\beta I-J\mu\partial_{xx}\right)B^{-1}\circ\partial_{xx}.\end{aligned}$$

Accordingly, (3.11) can be rewritten in the following form:

$$S\theta_t+\beta q_x+\delta\beta Tu_t=0, \tag{3.12}$$

where $S:L^2(0,L)\to L^2(0,L)$, and $P,T,R:H^2(0,L)\cap H_0^1(0,L)\subset L^2(0.L)\to L^2(0.L)$ are the operators defined as follows

$$\begin{cases}P=B^{-1}\circ\partial_{xx},\\ S=c\beta I-\delta^2\rho\beta P,\\ R=\rho\beta I-J\mu\partial_{xx},\\ T=-RP=-\dfrac{1}{\alpha\rho+J\beta}\left[\left(\alpha\beta\rho^2-J\rho(\xi\mu-\beta^2)\right)B^{-1}+J\mu I\right]\partial_{xx},\end{cases}$$

with domains $D(R)=D(T)=H^2(0,L)\cap H_0^1(0,L)$ and $D(S)=L^2(0,L)$.

Returning to the first equation from (3.8) and multiplying it by B^{-1}, we obtain

$$\begin{cases}u_{tt}+\alpha\mu Pu_{xx}-(\xi\mu-\beta^2)Pu+\delta\beta P\theta=0,\\ S\theta_t+\beta q_x+\delta\beta Tu_t=0,\\ \tau q_t+q+\theta_x=0.\end{cases} \tag{3.13}$$

Let $\mathcal{H}$ be the Hilbert space

$$\mathcal{H} = H^2(0,L) \cap H_0^1(0,L) \times H_0^1(0,L) \times L^2(0,L) \times L_*^2(0,L),$$

where

$$L_*^2(0,L) = \left\{ q \in L^2(0,L) : \int_0^L q dx = 0 \right\}.$$

We equip $\mathcal{H}$ by the inner product

$$\langle \Psi, \Psi^* \rangle = \alpha\mu \langle T u_x, u_x^* \rangle + \langle R v, v^* \rangle + \langle S\theta, \theta^* \rangle + (\xi\mu - \beta^2) \langle T u, u^* \rangle + \beta\tau \langle q, q^* \rangle ,$$

where $\Psi = (u, v, \theta, q)^T$ and $\Psi^* = (u^*, v^*, \theta^*, q^*)^T$. The associated norm is

$$\|\Psi\|_{\mathcal{H}}^2 = \alpha\mu \|T^{1/2} u_x\|^2 + \|R^{1/2} v\|^2 + \|S^{1/2}\theta\|^2 + \left(\xi\mu - \beta^2\right) \|T^{1/2} u\|^2 + \beta\tau \|q\|^2.$$

Next, in order to apply the Hille-Yosida Theorem 1.9, we introduce the new variable $v = u_t$, Therefore we get

$$\begin{cases} u_t - v = 0, \\ v_t + \alpha\mu P u_{xx} - (\xi\mu - \beta^2) P u + \delta\beta P\theta = 0, \\ S\theta_t + \beta q_x + \delta\beta T v = 0, \\ \tau q_t + q + \theta_x = 0. \end{cases} \tag{3.14}$$

The system (3.14) can be formulated as a Cauchy problem in the following manner:

$$\begin{cases} \Psi'(t) + \mathcal{A}\Psi(t) = 0, \qquad \forall t \geq 0, \\ \Psi(0) = (u_0, u_1, \theta_0, q_0), \end{cases} \tag{3.15}$$

where $\mathcal{A} : \mathcal{H} \longrightarrow \mathcal{H}$ is the operator defined by

$$\mathcal{A}\Psi = \begin{pmatrix} -v \\ \alpha\mu P u_{xx} + (\beta^2 - \xi\mu) P u + \delta\beta P\theta \\ \delta\beta S^{-1} T v + \beta S^{-1} q_x \\ \frac{1}{\tau} q + \frac{1}{\tau}\theta_x \end{pmatrix}, \tag{3.16}$$

with domain

$$D(\mathcal{A}) = \left\{ (u, v, \theta, q) \in \mathcal{H}, \left| \begin{array}{ll} u \in H_*^3(0,L), & v \in H^2(0,L) \cap H_0^1(0,L), \\ \theta \in H_0^1(0,L), & q \in H_0^1(0,L) \cap L_*^2(0,L) \end{array} \right. \right\},$$

where

$$H_*^2(0,L) := \left\{\psi \in H^2(0,L);\ \psi_x(0) = \psi_x(L) = 0\right\},$$

and

$$H_*^3(0.L) := \left\{\psi \in H^3(0.L) \cap H_0^1(0.L) : \psi_{xx}(0) = \psi_{xx}(L) = 0\right\}.$$

The following theorem shows the well-posedness of problem (3.15).

Theorem 3.1. *For every $\Psi_0 \in \mathcal{H}$, there exists a unique weak solution $\Psi \in C(\mathbb{R}^+, \mathcal{H})$ to the problem (3.15). Furthermore, if $\Psi_0 \in D(\mathcal{A})$, then Ψ belongs to $C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

The proof of Theorem 3.1 is based on the Hille-Yosida Theorem 1.9 and the Lax-Milgram Theorem 1.5, and it will be established through several lemmas.

Lemma 3.1. *The operator $\mathcal{A}$ defined by (3.16) is monotone.*

Proof. A direct calculation gives, for $\Psi \in D(\mathcal{A})$,

$$\begin{aligned}\langle \mathcal{A}\Psi, \Psi\rangle_{\mathcal{H}} =& -\alpha\mu \langle Tv_x, u_x\rangle + \left\langle \alpha\mu RPu_{xx} + (\beta^2 - \xi\mu)RPu + \delta\beta RP\theta, v\right\rangle \\ &+ \langle \delta\beta Tv + \beta q_x, \theta\rangle - (\xi\mu - \beta^2)\langle Tv, u\rangle + \beta\langle q + \theta_x, q\rangle.\end{aligned}$$

Utilizing the properties of the operators $T = -RP$, we arrive at

$$\begin{aligned}\langle \mathcal{A}\Psi, \Psi\rangle_{\mathcal{H}} =& -\alpha\mu \langle Tv_x, u_x\rangle + \alpha\mu \langle Tu_x, v_x\rangle + (\xi\mu - \beta^2)\langle Tu, v\rangle - \delta\beta \langle T\theta, v\rangle \\ &+ \beta \langle q_x, \theta\rangle + \delta\beta \langle Tv, \theta\rangle - (\xi\mu - \beta^2)\langle Tv, u\rangle + \beta\langle q, q\rangle + \beta\langle \theta_x, q\rangle\end{aligned}$$

The operator T is self-adjoint on $H^2(0,L) \cap H_0^1(0,L)$; therefore, it can be shifted between the arguments of the inner product. Integration by parts yields

$$\langle \mathcal{A}\Psi, \Psi\rangle_{\mathcal{H}} = \beta\langle q, q\rangle \geq 0.$$

Thus, $\mathcal{A}$ is monotone. □

Lemma 3.2. *The operator $\mathcal{A}$ defined by (3.16) is maximal.*

Proof. Let $G = (g_1, g_2, g_3, g_4)^T \in \mathcal{H}$, and we will seek the existence of $\Psi \in D(\mathcal{A})$ that satisfies the equation

$$(I + \mathcal{A})\Psi = G, \tag{3.17}$$

that is

$$\begin{cases} u - v = g_1, \\ v + \alpha\mu P u_{xx} - (\xi\mu - \beta^2)Pu + \delta\beta P\theta = g_2, \\ \delta\beta T v + S\theta + \beta q_x = S g_3, \\ \theta_x + (\tau+1)q = \tau g_4. \end{cases} \tag{3.18}$$

From the first and fourth equations of (3.18), we deduce that

$$\begin{aligned} v &= u - g_1, \\ \theta_x &= -(\tau+1)q + \tau g_4. \end{aligned} \tag{3.19}$$

The integration of the second equation in (3.19) yields

$$\theta = -(\tau+1)\int_0^x q(y)dy + \tau \int_0^x g_4(y)dy \in L^2(0.L). \tag{3.20}$$

Substituting (3.19) into (3.18), we obtain

$$\begin{cases} u + \alpha\mu P u_{xx} - (\xi\mu - \beta^2)Pu - \delta\beta(\tau+1)P\displaystyle\int_0^x q(y)dy \\ \qquad\qquad\qquad\qquad = g_1 + g_2 - \tau\delta\beta \displaystyle\int_0^x g_4(y)dy, \\ \delta\beta T u - (\tau+1)S\displaystyle\int_0^x q(y)dy + \beta q_x = S g_3 + \delta\beta T g_4 - \tau S\displaystyle\int_0^x g_4(y)dy. \end{cases} \tag{3.21}$$

At this stage, we define the space

$$\mathcal{W} = (H^2(0,L) \cap H_0^1(0,L)) \times L_*^2(0,L).$$

By formally multiplying the equations in (3.21) by $Ru^* \in H^2(0,L) \cap H_0^1(0,L)$ and by $-(\tau+1)\int_0^x q^*(y)\,dy \in L_*^2(0,L)$, we derive the associated variational problem corresponding to the system (3.21).

$$A((u,q),(u^*,q^*)) = L(u^*,q^*), \tag{3.22}$$

where A and L are, respectively, the bilinear and linear forms defined on $\mathcal{W}$ by:

$$\begin{aligned} A((u,q),(u^*,q^*)) =& \alpha\mu \left\langle T^{1/2}u_x, T^{1/2}u_x^* \right\rangle + (\xi\mu - \beta^2)\left\langle T^{1/2}u, T^{1/2}u^* \right\rangle \\ &+ \left\langle R^{1/2}u, R^{1/2}u^* \right\rangle + \delta\beta(\tau+1)\left\langle T^{1/2}\int_0^x q(y)dy, T^{1/2}u^* \right\rangle \\ &- \delta\beta(\tau+1)\left\langle T^{1/2}u, T^{1/2}\int_0^x q^*(y)dy \right\rangle \\ &+ (\tau+1)^2\left\langle S^{1/2}\int_0^x q(y)dy, S^{1/2}\int_0^x q^*(y)dy \right\rangle + \beta(\tau+1)\left\langle q, q^* \right\rangle, \end{aligned}$$

and

$$L(u^*, q^*) = \langle R(g_1+g_2), u^* \rangle + \delta\beta \left\langle T\int_0^x g_4(y)dy, u^* \right\rangle - \delta\beta(\tau+1)\left\langle Tg_1, \int_0^x q^*(y)dy \right\rangle$$
$$- (\tau+1)\left\langle S\left(g_3 - \tau\int_0^x g_4(y)dy\right), \int_0^x q^*(y)dy \right\rangle.$$

A and L are bounded. Additionally, we have

$$\begin{aligned} A((u,q),(u,q)) =& \alpha\mu \langle Tu_x, u_x\rangle + (\xi\mu - \beta^2)\langle Tu, u\rangle + \langle Ru, u\rangle \\ &+ \delta\beta(\tau+1)\left\langle T\int_0^x q(y)dy, u \right\rangle - \delta\beta(\tau+1)\left\langle Tu, \int_0^x q(y)dy \right\rangle \\ &+ (\tau+1)^2\left\langle S\int_0^x q(y)dy, \int_0^x q(y)dy \right\rangle + \beta(\tau+1)\langle q, q\rangle \\ =& \alpha\mu \langle Tu_x, u_x\rangle + (\xi\mu - \beta^2)\langle Tu, u\rangle + \langle Ru, u\rangle \\ &+ (\tau+1)^2\left\langle S\int_0^x q(y)dy, \int_0^x q(y)dy \right\rangle + \beta(\tau+1)\langle q, q\rangle. \end{aligned}$$

The definition of T and R, and integration by parts yield

$$\begin{aligned} A((u,q),(u,q)) =& \frac{\alpha\mu}{\alpha\rho + J\beta}\Big((\alpha\beta\rho^2 - J\rho(\xi\mu - \beta^2))\langle B^{-1}u_{xx}, u_{xx}\rangle + J\mu\|u_{xx}\|^2\Big) \\ &+ (\xi\mu - \beta^2)\langle Tu, u\rangle + \rho\beta\|u\|^2 + J\mu\|u_x\|^2 + (\tau+1)^2\left\langle S\int_0^x q(y)dy, \int_0^x q(y)dy \right\rangle \\ &+ \beta(\tau+1)\|q\|^2. \end{aligned}$$

then,

$$\begin{aligned} A((u,\theta),(u,\theta)) =& \frac{\alpha\mu}{\alpha\rho + J\beta}\Big((\alpha\beta\rho^2 + J\rho\beta^2)\langle B^{-1}u_{xx}, u_{xx}\rangle - J\xi\mu\rho\langle B^{-1}u_{xx}, u_{xx}\rangle + J\mu\|u_{xx}\|^2\Big) \\ &+ (\xi\mu - \beta^2)\langle Tu, u\rangle + \rho\beta\|u\|^2 + J\mu\|u_x\|^2 + (\tau+1)^2\left\langle S\int_0^x q(y)dy, \int_0^x q(y)dy \right\rangle \\ &+ \beta(\tau+1)\|q\|^2. \end{aligned}$$

By virtue of Remark 2.2 and using the positiveness of B^{-1}, we infer that

$$\begin{aligned} A((u,\theta),(u,\theta)) \geq& \frac{J\delta\alpha\mu^2}{\alpha\rho + J\beta}\|u_{xx}\|^2 + (\xi\mu - \beta^2)\langle Tu, u\rangle + \rho\beta\|u\|^2 \\ &+ J\mu\|u_x\|^2 + (\tau+1)^2\left\langle S\int_0^x q(y)dy, \int_0^x q(y)dy \right\rangle \\ &+ \beta(\tau+1)\|q\|^2. \end{aligned}$$

Consequently, there exists a positive constant m such that

$$A((u,\theta),(u,\theta)) \geq m(\|u_x\|^2 + \|u_{xx}\|^2 + \|q\|^2) = m\|(u,q)\|^2_{\mathcal{W}},$$

with a positive constant m, which ensures that A is coercive, it follows from the Lax-Milgram Theorem that equation (3.22) admits a unique solution

$$(u,q) \in (H^2(0,L) \cap H^1_0(0,L)) \times L^2_*(0,L).$$

Furthermore, by taking $u^* \equiv 0$ in (3.22), we get

$$\begin{aligned} &-\delta\beta \left\langle Tu, \int_0^x q^*(y)dy \right\rangle + (\tau+1) \left\langle S\int_0^x q(y)dy, \int_0^x q^*(y)dy \right\rangle + \beta \langle q, q^* \rangle \\ &= -\left\langle S\left(g_3 - \tau\int_0^x g_4(y)dy\right), \int_0^x q^*(y)dy \right\rangle - \delta\beta \left\langle Tg_1, \int_0^x q^*(y)dy \right\rangle, \end{aligned}$$

for all $q^* \in L^2_*(0,L)$. In particular, for $q^* = \psi_x$, with $\psi \in C^1_0(0,L)$, we get

$$\langle q, \psi_x \rangle = \left\langle \delta Tu - \frac{(\tau+1)}{\beta} S\int_0^x q(y)dy - \frac{1}{\beta}S\left(g_3 - \tau\int_0^x g_4(y)dy\right) - \delta Tg_1, \psi \right\rangle, \quad (3.23)$$

for all $\psi \in C^1_0(0,L)$. From the definitions of S and T we have that

$$\delta Tu - \frac{(\tau+1)}{\beta} S\int_0^x q(y)dy - \frac{1}{\beta}S\left(g_3 - \tau\int_0^x g_4(y)dy\right) - \delta Tg_1 \in L^2(0,L),$$

this shows that $q \in H^1_*(0,L)$. In addition, an integration by parts in (3.23) shows that

$$\beta\delta Tu - (\tau+1)S\int_0^x q(y)dy - S\left(g_3 - \tau\int_0^x g_4(y)dy\right) + \beta q_x = \delta\beta Tg_1.$$

By using (3.19) and (3.20), we obtain

$$\delta\beta Tv + S\theta + \beta q_x = Sg_3,$$

which shows that q solves the third equation of (3.18).

Similarly, by taking $q^* \equiv 0$ in (3.22), we obtain

$$\begin{aligned} &\alpha\mu \langle Tu_x, u^*_x \rangle + (\xi\mu - \beta^2) \langle Tu, u^* \rangle + \langle Ru, u^* \rangle + \delta\beta(\tau+1) \left\langle T\int_0^x q(y)dy, u^* \right\rangle \\ &= \langle R(g_1+g_2), u^* \rangle + \delta\beta \left\langle T\int_0^x g_4(y)dy, u^* \right\rangle, \end{aligned}$$

Consequently

$$\alpha\mu \langle Tu_x, u^*_x \rangle =$$

$$-\left\langle(\xi\mu-\beta^2)Tu+R(u-g_1-g_2)+\delta\beta(\tau+1)T\int_0^x q(y)dy-\delta\beta\tau T\int_0^x g_4(y)dy, u^*\right\rangle,$$

for all $u^* \in C_0^1$, which implies that

$$\alpha\mu Tu_{xx}=(\xi\mu-\beta^2)Tu+R(u-g_1-g_2)+\delta\beta(\tau+1)T\int_0^x q(y)dy-\delta\beta\tau T\int_0^x g_4(y)dy. \quad (3.24)$$

Replacing (3.19) and (3.20) into (3.24) we find

$$\alpha\mu Tu_{xx}=(\xi\mu-\beta^2)Tu+R(v-g_2)-\delta\beta T\theta.$$

Recall that If $g \in H_0^1(0,L)$, the functional $-\partial_{xx}g : H_0^1(0,L) \to \mathbb{R}$ operates on $H_0^1(0,L)$ through the bilinear pairing

$$\langle -\partial_{xx}g, u\rangle = \int_0^L \frac{\partial g}{\partial x}\frac{\partial u}{\partial x}dx.$$

This operator is bounded on $H_0^1(0,L)$, which implies that $-\partial_{xx}g \in H^{-1}(0,L)$. Furthermore, by applying the Lax–Milgram theorem, it can be rigorously established that the operator $-\partial_{xx}$ defines an isomorphism between $H_0^1(0,L)$ onto $H^{-1}(0,L)$. We have the identity $T \equiv -\left(I+B^{-1}\right)\partial_{xx}$, with domain $D\left(I+B^{-1}\right)=L^2(0,L)$. Moreover, since $g_2 \in H_0^1(0,L)$, it follows that

$$Rg_2=\left(\rho\beta I-J\mu\partial_{xx}\right)g_2 \in H^{-1}(0,L).$$

It is also clear that Tu, Rv, and $T\theta$ belong to $H^{-1}(0,L)$. Consequently, $\alpha\mu Tu_{xx} \in H^{-1}(0,L)$. Based on the preceding analysis and the definition of the operator T, we can deduce that

$$u_{xx} \in H_0^1(0,L).$$

which implies that $u_{xx}(0)=u_{xx}(L)=0$. Consequently,

$$u \in H_*^3(0.L).$$

Next, by applying R^{-1}, we obtain

$$v+\alpha\mu Pu_{xx}-(\xi\mu-\beta^2)Pu+\delta\beta P\theta=g_2.$$

As a result, $(u,v,\theta,q) \in D(\mathcal{A})$ and it satisfies system (3.18), demonstrating that $\mathcal{A}$ is maximal. □

Thanks to the Hile–Yosida Theorem, problem (3.15) has a unique solution. This completes the proof of Theorem 3.1.

At this stage, we denote by $\mathscr{H}$ the Hilbert space

$$\mathscr{H} = H^2(0,L) \cap H_0^1(0,L) \times H_0^1(0,L) \times H_*^1(0,L) \times L^2(0,L) \times L_*^2(0,L),$$

and define the set

$$\mathscr{D} = \left\{ (u,v,\varphi,\theta,q) \in \mathscr{H};\left| \begin{array}{ll} u \in H_*^3(0,L), & v \in H^2(0,L) \cap H_0^1(0,L) \\ \varphi \in H_*^2(0,L), & \theta \in H_0^1(0,L), \\ q \in H_*^1(0.L), & \end{array} \right. \right\}.$$

The well-posedness result of problem (3.1)-(3.3) is given by the following theorem:

Theorem 3.2. *Let* $(u_0,u_1,\varphi_0,\theta_0,q_0) \in \mathscr{D}$*, then the problem (3.1) has a unique solution* $(u,u_t,\varphi,\theta,q) \in C\left(\mathbb{R}^+;\mathscr{D}\right) \cap C^1\left(\mathbb{R}^+;\mathscr{H}\right)$.

Proof of Theorem 3.2

Based on Theorem 3.1, there exists a unique solution (u,u_t,θ,q) in the space $C\left(\mathbb{R}^+;D(\mathcal{A})\right)\cap C^1\left(\mathbb{R}^+;\mathcal{H}\right)$ to the problem (3.15). Thus, the problem (3.1)-(3.3) can be reduced to finding a solution to the following problem:

$$\begin{cases} -\alpha\varphi_{xx} + \xi\varphi = f \quad \text{in } (0.L), \\ \varphi_x(0) = \varphi_x(L) = 0, \end{cases} \tag{3.25}$$

where,

$$f = Ju_{xtt} - \beta u_x - \delta\theta_x.$$

From Theorem 3.1, we have

$$u \in \left(\begin{array}{c} C\left(\mathbb{R}^+;H_*^3(0,L)\cap H_0^1(0,L)\right) \cap C^1\left(\mathbb{R}^+,H^2(0,L)\cap H_0^1(0,L)\right) \\ \cap C^2\left(\mathbb{R}^+,H_0^1(0,L)\right) \end{array} \right),$$

and

$$\theta \in C\left(\mathbb{R}^+;H_0^1(0,L)\right) \cap C^1\left(\mathbb{R}^+;L^2(0,L)\right).$$

On the other hand, from (3.10), we have

$$u_{ttt} \in C\left(\mathbb{R}^+; L^2(0,L)\right).$$

Therefore, $u \in C^3\left(\mathbb{R}^+, L^2(0,L)\right)$.

The function f is formed by summing several terms that are continuous over time and take values in the space $L^2(0,L)$. Since the term u_{xtt}, u_x and θ_x each possess the required smoothness and are in $L^2(0,L)$, and because multiplication by constants does not affect their smoothness, it follows that:

$$f = Ju_{xtt} - \beta u_x - \delta\theta_x \in C^1(\mathbb{R}^+, L^2(0,L)).$$

To prove that the problem (3.25) admits a unique solution,we apply the Lax-Milgram theorem. Let be consider the following variational formulation

$$\alpha \int_0^L \varphi_x\phi_x dx + \xi \int_0^L \varphi\phi dx = \int_0^L f\phi dx, \ \forall\phi \in H^1(0,L) \tag{3.26}$$

and denote by

$$\widetilde{A}(\varphi,\phi) = \alpha \int_0^L \varphi_x\phi_x dx + \xi \int_0^L \varphi\phi dx$$

and

$$\widetilde{L}(\phi) = \int_0^L f\phi dx.$$

It is clear that $\widetilde{A}$ is a bilinear form and $\widetilde{L}$ is linear form. Both $\widetilde{A}$ and $\widetilde{L}$ are bounded. Moreover

$$\begin{aligned}\widetilde{A}(\varphi,\varphi) &= \alpha \int_0^L \varphi_x^2\, dx + \xi \int_0^L \varphi^2\, dx,\\ &\geq C\left(\int_0^L \varphi_x^2\, dx + \int_0^L \varphi^2\, dx\right).\end{aligned}$$

By choosing $C = \min(\alpha,\xi)$, we get $\widetilde{A}(\varphi,\varphi) \geq C\|\varphi\|^2_{H_0^1}$. We conclude that, the bilinear form $\widetilde{A}$ is coercive.

Thanks to the Lax-Milgram theorem, the problem (3.26) has a unique solution $\varphi \in H_0^1(0,L)$. Furthermore, by substituting $\phi \in C_0^1(0,L) \subset H^1(0,L)$ into equation (3.26), we obtain:

$$\int_0^L \varphi_x\phi_x\, dx = -\frac{1}{\alpha}\int_0^L (\xi\varphi - f)\phi\, dx, \quad \forall\phi \in C_0^1(0,L).$$

This shows that $\varphi \in H^2(0,L)$. Next, we integrate equation (3.26) by parts, we get

$$\alpha\varphi_x(L)\phi(L) - \alpha\varphi_x(0)\phi(0) = \int_0^L (\alpha\varphi_{xx} - \xi\varphi + f)\phi\, dx, \quad \forall \phi \in H^1(0,L). \tag{3.27}$$

Taking $\phi \in H_0^1(0,L)$, we get

$$-\alpha\varphi_{xx} + \xi\varphi = f.$$

Substituting in (3.27), we arrive at

$$\alpha\varphi_x(L)\phi(L) - \alpha\varphi_x(0)\phi(0) = 0, \quad \forall \phi \in H^1(0,L).$$

Since ϕ is arbitrary, we deduce that:

$$\varphi_x(0) = \varphi_x(L) = 0. \tag{3.28}$$

Consequently, $\varphi \in H_*^2(0,L)$, and it satisfies the equation given in (3.25).

Therefore, $(u, u_t, \varphi, \theta, q)$ belongs to the domain $\mathscr{D}$. Moreover, since (u, u_t, θ, q) is a solution of equation (3.13), we conclude that $(u, u_t, \varphi, \theta, q)$ is the unique solution to problem (3.1) with the initial and boundary conditions given by (3.2) and (3.3). This completes the proof of Theorem 3.2.

3.3 Exponential stability

In this section, we analyze the time-behavior of the solution to problem (3.1)–(3.3). We demonstrate that the solution exhibits exponential decay without imposing any restrictions on the coefficients of the system (3.1).

First, we introduce the energy associated with the solution (u, φ, θ, q) of problem (3.1)–(3.3), defined as:

$$\mathcal{E}(t) = \frac{J\rho}{2\beta}\int_0^L u_{tt}^2 dx + \frac{J\mu}{2\beta}\int_0^L u_{xt}^2 dx + \frac{\rho}{2}\int_0^L u_t^2 dx + \left(\mu - \frac{\beta^2}{\xi}\right)\int_0^L u_x^2 dx + \frac{\alpha}{2}\int_0^L \varphi_x^2 dx$$
$$+ \int_0^L \left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx + \frac{c}{2}\int_0^L \theta^2 dx + \frac{\tau}{2}\int_0^L q^2 dx.$$

We have the following result.

Lemma 3.3. *Let (u, φ, θ, q) be a solution to (3.1)-(3.3). Then the energy functional $\mathcal{E}(t)$ satisfies*

$$\mathcal{E}'(t) = -\int_0^L q^2 dx \leq 0, \ \forall t > 0. \tag{3.29}$$

Proof. Performing the L^2-inner product of the four equations in (3.1) with u_t, φ_t, θ, and q, respectively, and applying integration by parts, we obtain

$$\begin{aligned}
&\frac{\rho}{2}\frac{d}{dt}\int_0^L u_t^2 dx + \frac{\mu}{2}\frac{d}{dt}\int_0^L u_x^2 dx + \beta\int_0^L u_{xt}\varphi dx = 0,\\
&\frac{\alpha}{2}\frac{d}{dt}\int_0^L \varphi_x^2 dx - J\int_0^L u_{xtt}\varphi_t dx + \beta\int_0^L u_x\varphi_t dx + \frac{\xi}{2}\frac{d}{dt}\int_0^L \varphi^2 dx - \delta\int_0^L \varphi_{xt}\theta dx = 0,\\
&\frac{c}{2}\frac{d}{dt}\int_0^L \theta^2 dx - \int_0^L q\theta_x dx + \delta\int_0^L \varphi_{xt}\theta dx = 0,\\
&\frac{\tau}{2}\frac{d}{dt}\int_0^L q^2 dx + \int_0^L q^2 dx + \int_0^L q\theta_x dx = 0.
\end{aligned} \tag{3.30}$$

We now that

$$\beta\int_0^L u_{xt}\varphi dx + \beta\int_0^L u_x\varphi_t dx = \beta\frac{d}{dt}\int_0^L u_x\varphi dx. \tag{3.31}$$

Next, applying integration by parts and using the first equation from (3.1) leads to

$$\begin{aligned}
-J\int_0^L u_{xtt}\varphi_t dx =& J\int_0^L u_{tt}\varphi_{xt} dx = J\int_0^L \left[\frac{\rho}{\beta}u_{ttt} - \frac{\mu}{\beta}u_{xxt}\right]u_{tt} dx\\
=&\frac{J\rho}{\beta}\int_0^L u_{ttt}u_{tt} dx - \frac{J\mu}{\beta}\int_0^L u_{xxt}u_{tt} dx\\
=&\frac{J\rho}{2\beta}\frac{d}{dt}\int_0^L u_{tt}^2 dx + \frac{J\mu}{2\beta}\frac{d}{dt}\int_0^L u_{xt}^2 dx.
\end{aligned} \tag{3.32}$$

Substituting equations (3.31)–(3.32), we find the following:

$$\begin{aligned}
&\frac{\rho}{2}\frac{d}{dt}\int_0^L u_t^2 dx + \frac{\mu}{2}\frac{d}{dt}\int_0^L u_x^2 dx + \beta\frac{d}{dt}\int_0^L u_x\varphi dx + \frac{J\rho}{2\beta}\frac{d}{dt}\int_0^L u_{tt}^2 dx + \frac{J\mu}{2\beta}\frac{d}{dt}\int_0^L u_{xt}^2 dx\\
&+\frac{\alpha}{2}\frac{d}{dt}\int_0^L \varphi_x^2 dx + \frac{\xi}{2}\frac{d}{dt}\int_0^L \varphi^2 dx + \frac{c}{2}\frac{d}{dt}\int_0^L \theta^2 dx + \frac{\tau}{2}\frac{d}{dt}\int_0^L q^2 dx + \int_0^L q^2 dx.
\end{aligned} \tag{3.33}$$

In addition

$$\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\int_0^L u_x^2 dx + \beta\frac{d}{dt}\int_0^L u_x\varphi dx + \frac{\xi}{2}\frac{d}{dt}\int_0^L \varphi^2 dx\\
=&\frac{\mu}{2}\frac{d}{dt}\int_0^L u_x^2 dx + \beta\frac{d}{dt}\int_0^L u_x\varphi dx + \frac{\xi}{2}\frac{d}{dt}\int_0^L \varphi^2 dx + \frac{\beta^2}{2\xi}\frac{d}{dt}\int_0^L u_x^2 dx - \frac{\beta^2}{2\xi}\frac{d}{dt}\int_0^L u_x^2 dx\\
=&\frac{1}{2}\frac{d}{dt}\int_0^L \left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx + \frac{1}{2}\frac{d}{dt}\left(\mu - \frac{\beta^2}{\xi}\right)\int_0^L u_x^2 dx.
\end{aligned} \tag{3.34}$$

Finally, by substituting equation (3.34) into equation (3.33), we obtain equation (3.29). □

Lemma 3.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be the functionals defined by*

$$\mathcal{M}(t) = \tau c \int_0^L \theta \int_0^x q(s)dsdx - \tau\delta \int_0^L q\varphi dx - \frac{\delta\mu\tau}{\beta} \int_0^L qu_x dx + \frac{\delta\rho}{\beta} \int_0^L \theta u_t dx,$$

and

$$\mathcal{N}(t) = \frac{\rho}{\beta} \int_0^L (\alpha\varphi_x - \delta\theta)\, u_t dx.$$

Then, we define the functional

$$\mathcal{I}_1(t) = \kappa_1 \mathcal{M}(t) + \kappa_2 \mathcal{N}(t),$$

where $\kappa_1 = \left(\frac{\alpha\rho}{\beta} + \frac{\delta^2\rho}{c\beta}\right)$ *and* $\kappa_2 = \frac{\delta^2\rho}{c\beta}$.

$\mathcal{I}_1(t)$ *satisfies, for any positive constant* ε_1*, the estimate along the solution* (u, φ, θ, q) *of (3.1)*

$$\begin{aligned} \mathcal{I}_1'(t) \le & -\frac{\rho}{2\beta}\left(\delta^2 + c\alpha\right) \int_0^L \theta^2 dx - \frac{J\delta^2\rho^2}{c\beta^2} \int_0^L u_{tt}^2 dx + \varepsilon_1 \int_0^L u_{xt}^2 dx + \varepsilon_1 \int_0^L \varphi_x^2 dx \\ & - \frac{\delta^2\rho}{2c\beta}\left(\mu - \frac{\beta^2}{\xi}\right) \int_0^L u_x^2 dx - \frac{\delta^2\rho}{c\beta} \int_0^L \left(\frac{\beta}{\sqrt{\xi}} u_x + \sqrt{\xi}\varphi\right)^2 dx + C_{\varepsilon_1} \int_0^L q^2 dx, \quad (3.35) \end{aligned}$$

where C_{ε_1} *is a positive constant.*

Proof. By differentiating $\mathcal{M}(t)$, we get

$$\begin{aligned} \mathcal{M}(t) =& \tau c \int_0^L \theta_t \int_0^x q(s)dsdx + \tau c \int_0^L \theta \int_0^x q_t(s)dsdx - \tau\delta \int_0^L q_t\varphi dx - \tau\delta \int_0^L q\varphi_t dx \\ & - \frac{\delta\mu\tau}{\beta} \int_0^L q_t u_x dx - \frac{\delta\mu\tau}{\beta} \int_0^L q u_{xt} dx + \frac{\delta\rho}{\beta} \int_0^L \theta_t u_t dx + \frac{\delta\rho}{\beta} \int_0^L \theta u_{tt} dx, \end{aligned}$$

From the third and fourth equations of (3.1), we arrive

$$\begin{aligned} \mathcal{M}'(t) =& -c \int_0^L \theta^2 dx - c \int_0^L \theta \int_0^x q(s)dsdx - \tau \int_0^L q_x \int_0^x q(s)dsdx \\ & - \delta\tau \int_0^L \varphi_{xt} \int_0^x q(s)dsdx + \delta \int_0^L q\varphi dx + \delta \int_0^L \theta_x \varphi dx - \delta\tau \int_0^L q\varphi_t dx \\ & + \frac{\delta\mu}{\beta} \int_0^L q u_x dx + \frac{\delta\mu}{\beta} \int_0^L \theta_x u_x dx - \frac{\delta\mu\tau}{\beta} \int_0^L q u_{xt} dx \\ & - \frac{\delta\rho}{c\beta} \int_0^L q_x u_t dx - \frac{\delta^2\rho}{c\beta} \int_0^L \varphi_{xt} u_t dx + \frac{\delta\rho}{\beta} \int_0^L \theta u_{tt} dx. \end{aligned}$$

Integration by parts along with the boundary conditions yield

$$\begin{aligned}\mathcal{M}'(t) = &- c\int_0^L \theta^2 dx - c\int_0^L \theta \int_0^x q(s)dsdx + \tau\int_0^L q^2 dx + \delta\tau\int_0^L \varphi_t q dx + \delta\int_0^L q\varphi dx \\ &- \delta\int_0^L \theta\varphi_x dx - \delta\tau\int_0^L \varphi_t q dx + \frac{\delta\mu}{\beta}\int_0^L qu_x dx - \frac{\delta\mu}{\beta}\int_0^L \theta u_{xx} dx - \frac{\delta\mu\tau}{\beta}\int_0^L qu_{xt} dx \\ &+ \frac{\delta\rho}{c\beta}\int_0^L qu_{xt} dx - \frac{\delta^2\rho}{c\beta}\int_0^L \varphi_{xt} u_t dx + \frac{\delta\rho}{\beta}\int_0^L \theta u_{tt} dx. \end{aligned} \tag{3.36}$$

By substituting $\varphi_x = \dfrac{1}{\beta}(\rho u_{tt} - \mu u_{xx})$ into (3.36), we obtain

$$\begin{aligned}\mathcal{M}'(t) = &- c\int_0^L \theta^2 dx - c\int_0^L \theta \int_0^x q(s)dsdx + \tau\int_0^L q^2 dx + \delta\int_0^L q\varphi dx \\ &- \frac{\delta}{\beta}\int_0^L \theta(\rho u_{tt} - \mu u_{xx})\, dx + \frac{\delta\mu}{\beta}\int_0^L qu_x dx - \frac{\delta\mu}{\beta}\int_0^L \theta u_{xx} dx \\ &- \frac{\delta\mu\tau}{\beta}\int_0^L qu_{xt} dx + \frac{\delta\rho}{c\beta}\int_0^L qu_{xt} dx - \frac{\delta^2\rho}{c\beta}\int_0^L \varphi_{xt} u_t dx + \frac{\delta\rho}{\beta}\int_0^L \theta u_{tt} dx, \\ = &- c\int_0^L \theta^2 dx - c\int_0^L \theta \int_0^x q(s)dsdx + \tau\int_0^L q^2 dx + \delta\int_0^L q\varphi dx \\ &- \frac{\delta\rho}{\beta}\int_0^L \theta u_{tt} dx + \frac{\delta\mu}{\beta}\int_0^L \theta u_{xx} dx + \frac{\delta\mu}{\beta}\int_0^L qu_x dx - \frac{\delta\mu}{\beta}\int_0^L \theta u_{xx} dx \\ &- \frac{\delta\mu\tau}{\beta}\int_0^L qu_{xt} dx + \frac{\delta\rho}{c\beta}\int_0^L qu_{xt} dx - \frac{\delta^2\rho}{c\beta}\int_0^L \varphi_{xt} u_t dx + \frac{\delta\rho}{\beta}\int_0^L \theta u_{tt} dx,\end{aligned}$$

which can be simplified to

$$\begin{aligned}\mathcal{M}'(t) = &- c\int_0^L \theta^2 dx - c\int_0^L \theta \int_0^x q(s)dsdx + \tau\int_0^L q^2 dx + \delta\int_0^L q\varphi dx \\ &+ \frac{\delta\mu}{\beta}\int_0^L qu_x dx + \frac{\delta}{\beta}\left(\frac{\rho}{c} - \mu\tau\right)\int_0^L qu_{xt} dx - \kappa_2\int_0^L \varphi_{xt} u_t dx.\end{aligned} \tag{3.37}$$

On the other hand, by differentiating $\mathcal{N}(t)$, we get

$$\begin{aligned}\mathcal{N}(t) &= \frac{\rho}{\beta}\int_0^L (\alpha\varphi_{xt} - \delta\theta_t)\, u_t dx + \frac{\rho}{\beta}\int_0^L (\alpha\varphi_x - \delta\theta)\, u_{tt} dx \\ &= \frac{\alpha\rho}{\beta}\int_0^L \varphi_{xt} u_t dx - \frac{\delta\rho}{\beta}\theta_t u_t dx + \frac{\alpha\rho}{\beta}\int_0^L \varphi_x u_{tt} dx - \frac{\delta\rho}{\beta}\theta u_{tt} dx.\end{aligned}$$

The integration of the second equation in (3.1) over the interval $(0, x)$, together with the boundary conditions in (3.3), yields

$$\alpha\varphi_x = -Ju_{tt} + \beta u + \delta\theta + \xi\int_0^x \varphi(y)dy, \tag{3.38}$$

then using (3.38) and (3.1), we obtain

$$\begin{aligned}\mathcal{N}'(t) =&\frac{\alpha\rho}{\beta}\int_0^L \varphi_{xt}u_t dx + \frac{\rho}{\beta}\int_0^L u_{tt}\left(-Ju_{tt}+\beta u+\delta\theta+\xi\int_0^x \varphi(y)dy\right)dx\\ &-\frac{\delta\rho}{\beta}\int_0^L \theta_t u_t dx - \frac{\delta\rho}{\beta}\int_0^L u_{tt}\theta dx,\\ =&-\frac{J\rho}{\beta}\int_0^L u_{tt}^2 dx - \mu\int_0^L u_x^2 dx + \frac{\alpha\rho}{\beta}\int_0^L u_t\varphi_{xt}dx - \beta\int_0^L u_x\varphi dx\\ &+\frac{\xi}{\beta}\int_0^L (\mu u_{xx}+\beta\varphi_x)\left(\int_0^x \varphi(y)dy\right)dx + \frac{\delta\rho}{c\beta}\int_0^L u_t\left(q_x+\delta\varphi_{xt}\right)dx.\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}\mathcal{N}'(t) =&-\frac{J\rho}{\beta}\int_0^L u_{tt}^2 dx - \mu\int_0^L u_x^2 dx - \xi\int_0^L \varphi^2 dx + \kappa_1\int_0^L u_t\varphi_{xt}dx\\ &-\left(\frac{\mu\xi}{\beta}+\beta\right)\int_0^L u_x\varphi dx - \frac{\delta\rho}{c\beta}\int_0^L qu_{xt}dx.\end{aligned} \tag{3.39}$$

By substituting the results of the derivatives from equations (3.37) and (3.39) into $\mathcal{I}_1'(t)$, we obtain

$$\begin{aligned}\mathcal{I}_1'(t) =&-c\kappa_1\int_0^L \theta^2 dx - c\kappa_1\int_0^L \theta\int_0^x q(s)dsdx + \tau\kappa_1\int_0^L q^2 dx + \delta\kappa_1\int_0^L q\varphi dx\\ &-\frac{J\rho}{\beta}\kappa_2\int_0^L u_{tt}^2 dx + \frac{\delta\mu}{\beta}\kappa_1\int_0^L qu_x dx + \frac{\delta\rho}{c\beta}\left(\kappa_1 - \frac{c\mu\tau}{\rho}\kappa_1 - \kappa_2\right)\int_0^L qu_{xt}dx\\ &-\mu\kappa_2\int_0^L u_x^2 dx - \kappa_2\left(\frac{\mu\xi}{\beta}+\beta\right)\int_0^L u_x\varphi dx - \xi\kappa_2\int_0^L \varphi^2 dx.\end{aligned}$$

To approximate the final three terms, we apply the inequality $\dfrac{\xi\mu}{\beta} > \beta$ to obtain

$$\begin{aligned}-\mu\int_0^L u_x^2 dx - &\left(\beta+\frac{\mu\xi}{\beta}\right)\int_0^L u_x\varphi dx - \xi\int_0^L \varphi^2 dx\\ &\leq -\mu\int_0^L u_x^2 dx - \xi\int_0^L \varphi^2 dx - 2\beta\int_0^L u_x\varphi dx\\ &\leq -\left(\mu - \frac{\beta^2}{\xi}\right)\int_0^L u_x^2 dx - \int_0^L \left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx.\end{aligned} \tag{3.40}$$

So

$$\begin{aligned}\mathcal{I}_1'(t) \leq&-c\kappa_1\int_0^L \theta^2 dx - c\kappa_1\int_0^L \theta\int_0^x q(s)dsdx + \tau\kappa_1\int_0^L q^2 dx + \delta\kappa_1\int_0^L q\varphi dx\\ &-\frac{J\rho}{\beta}\kappa_2\int_0^L u_{tt}^2 dx + \frac{\delta\mu}{\beta}\kappa_1\int_0^L qu_x dx + \frac{\delta\rho}{c\beta}\left(\kappa_1 - \frac{c\mu\tau}{\rho}\kappa_1 - \kappa_2\right)\int_0^L qu_{xt}dx\\ &-\kappa_2\left(\mu - \frac{\beta^2}{\xi}\right)\int_0^L u_x^2 dx - \kappa_2\int_0^L \left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx.\end{aligned} \tag{3.41}$$

At this stage, we utilize Young's, Cauchy–Schwarz, and Poincaré's inequalities.

$$-c\kappa_1 \int_0^L \theta \int_0^x q(s)dsdx \leq \frac{c\kappa_1}{2} \int_0^L \theta^2 dx + \frac{c\kappa_1}{2} \int_0^L q^2 dx. \tag{3.42}$$

$$\frac{\delta\mu}{\beta}\kappa_1 \int_0^L q u_x dx \leq \frac{\kappa_2}{2}\left(\mu - \frac{\beta^2}{\xi}\right) \int_0^L u_x^2 dx + \frac{\kappa_1^2\delta^2\mu^2}{2\kappa_2\beta^2\left(\mu - \frac{\beta^2}{\xi}\right)} \int_0^L q^2 dx. \tag{3.43}$$

Similarly, for any $\varepsilon_1 > 0$, we have

$$\delta\kappa_1 \int_0^L q\varphi dx \leq \varepsilon_1 \int_0^L \varphi_x dx + \frac{\delta^2\kappa_1^2}{4\varepsilon_1} \int_0^L q^2 dx. \tag{3.44}$$

$$\frac{\delta\rho}{c\beta}\left(\kappa_1 - \frac{c\mu\tau}{\rho}\kappa_1 - \kappa_2\right) \int_0^L q u_{xt} dx \leq \varepsilon_1 \int_0^L u_{xt}^2 dx + \frac{\delta^2\rho^2}{4\varepsilon_1 c^2\beta^2}\left(\kappa_1 - \frac{c\mu\tau}{\rho}\kappa_1 - \kappa_2\right)^2 \int_0^L q^2 dx. \tag{3.45}$$

We put

$$C_{\varepsilon_1} = \frac{c\kappa_1}{2} + \frac{\kappa_1^2\delta^2\mu^2}{2\kappa_2\beta^2\left(\mu - \frac{\beta^2}{\xi}\right)} + \frac{\delta^2\kappa_1^2}{4\varepsilon_1} + \frac{\delta^2\rho^2}{4\varepsilon_1 c^2\beta^2}\left(\kappa_1 - \frac{c\mu\tau}{\rho}\kappa_1 - \kappa_2\right)^2 + \tau\kappa_1.$$

Using the inequalities (3.42), (3.43, 3.44) and (3.45), and substituting the values of κ_1 and κ_2 into (3.41), we obtain (3.35). □

Lemma 3.5. *Let (u, φ, θ, q) be the solution of (3.1). Then, there exists a positive constant C such that the functional*

$$\mathcal{I}_2(t) = -\frac{J}{\sqrt{\xi}} \int_0^L u_{xt}\left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right) dx + \frac{Jc}{\delta} \int_0^L u_t\theta dx,$$

satisfies the estimate

$$\begin{aligned} \mathcal{I}_2'(t) \leq &-\frac{\alpha}{2\mu}\left(\mu - \frac{\beta^2}{\xi}\right) \int_0^L \varphi_x^2 dx - \int_0^L \left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx \\ &-\frac{J\beta}{2\xi} \int_0^L u_{xt}^2 dx + C\left(\int_0^L \theta^2 dx + \int_0^L q^2 dx + \int_0^L u_{tt}^2 dx\right). \end{aligned} \tag{3.46}$$

Proof.

$$\begin{aligned} \mathcal{I}_2'(t) = &-\frac{J}{\sqrt{\xi}} \int_0^L u_{xtt}\left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right) - \frac{J}{\sqrt{\xi}} \int_0^L u_{xt}\left(\frac{\beta}{\sqrt{\xi}}u_{xt} + \sqrt{\xi}\varphi_t\right) dx \\ &+\frac{Jc}{\delta} \int_0^L u_{tt}\theta dx + \frac{Jc}{\delta} \int_0^L u_t\theta_t dx, \end{aligned}$$

using the second equation of (3.1), we arrive at

$$\begin{aligned}\mathcal{I}_2'(t) =&\frac{\alpha}{\sqrt{\xi}}\int_0^L \varphi_{xx}\left(\frac{\beta}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)dx-\int_0^L\left(\frac{\beta}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)^2dx\\&-\frac{\delta}{\sqrt{\xi}}\int_0^L\theta_x\left(\frac{\beta}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)dx-\frac{J\beta}{\xi}\int_0^L u_{xt}^2dx-J\int_0^L u_{xt}\varphi_t dx\\&+\frac{Jc}{\delta}\int_0^L u_{tt}\theta dx+\frac{Jc}{\delta}\int_0^L u_t\theta_t dx.\end{aligned}$$

Then, integration by parts yields

$$\begin{aligned}\mathcal{I}_2'(t) =&-\frac{\alpha\beta}{\xi}\int_0^L\varphi_x u_{xx}dx-\alpha\int_0^L\varphi_x^2dx-\int_0^L\left(\frac{\beta}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)^2dx\\&+\frac{\delta}{\sqrt{\xi}}\int_0^L\theta\left(\frac{\beta}{\sqrt{\xi}}u_{xx}+\sqrt{\xi}\varphi_x\right)dx-\frac{J\beta}{\xi}\int_0^L u_{xt}^2dx\\&+\frac{J}{\delta}\int_0^L u_t\left(c\theta_t+\delta\varphi_{xt}\right)dx+\frac{Jc}{\delta}\int_0^L u_{tt}\theta dx.\end{aligned}$$

Afterward, we apply the first and third equations of (3.1) to obtain the result

$$\begin{aligned}\mathcal{I}_2'(t) =&-\frac{\alpha\beta\rho}{\mu\xi}\int_0^L\varphi_x u_{tt}dx-\frac{\alpha}{\mu}\left(\mu-\frac{\beta^2}{\xi}\right)\int_0^L\varphi_x^2dx-\int_0^L\left(\frac{\beta}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)^2dx\\&+\frac{\delta\beta}{\xi\mu}\int_0^L\theta\left(\rho u_{tt}-\beta\varphi_x\right)dx+\delta\int_0^L\theta\varphi_x dx-\frac{J\beta}{\xi}\int_0^L u_{xt}^2dx\\&-\frac{J}{\delta}\int_0^L u_t q_x dx+\frac{Jc}{\delta}\int_0^L u_{tt}\theta dx,\end{aligned}$$

which simplifies to

$$\begin{aligned}\mathcal{I}_2'(t) =&-\frac{\alpha\beta\rho}{\xi\mu}\int_0^L\varphi_x u_{tt}dx-\frac{\alpha}{\mu}\left(\mu-\frac{\beta^2}{\xi}\right)\int_0^L\varphi_x^2dx\\&-\int_0^L\left(\frac{\beta}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)^2dx+\frac{\delta}{\xi\mu}\left(\xi\mu-\beta^2\right)\int_0^L\theta\varphi_x\\&-\frac{J\beta}{\xi}\int_0^L u_{xt}^2dx+\frac{J}{\delta}\int_0^L u_{xt}q dx+\left(\frac{Jc}{\delta}+\frac{\delta\beta\rho}{\xi\mu}\right)\int_0^L u_{tt}\theta dx.\end{aligned} \tag{3.47}$$

As the final step, we apply Young's inequality

$$-\frac{\alpha\beta\rho}{\xi\mu}\int_0^L\varphi_x u_{tt}dx\le\frac{\alpha}{4\mu}\left(\mu-\frac{\beta^2}{\xi}\right)\int_0^L\varphi_x^2dx+\frac{\alpha\beta^2\rho^2}{\mu\xi(\xi\mu-\beta^2)}\int_0^L u_{tt}^2dx. \tag{3.48}$$

$$\frac{\delta}{\xi\mu}\left(\xi\mu-\beta^2\right)\int_0^L\theta\varphi_x\le\frac{\alpha}{4\mu}\left(\mu-\frac{\beta^2}{\xi}\right)\int_0^L\varphi_x^2dx+\frac{\delta^2\left(\xi\mu-\beta^2\right)}{\alpha\mu\xi}\int_0^L\theta^2dx. \tag{3.49}$$

$$\frac{J}{\delta}\int_0^L u_{xt}q dx\le\frac{J\beta}{2\xi}\int_0^L u_{xt}^2dx+\frac{J\xi}{2\beta\delta^2}\int_0^L q^2dx. \tag{3.50}$$

$$\left(\frac{Jc}{\delta}+\frac{\delta\beta\rho}{\xi\mu}\right)\int_0^L u_{tt}\theta dx\le\left(\frac{Jc}{\delta}+\frac{\delta\beta\rho}{\xi\mu}\right)^2\int_0^L u_{tt}^2dx+\frac{1}{4}\int_0^L\theta^2dx. \tag{3.51}$$

We take

$$C = \max\left(\frac{\alpha\beta^2\rho^2}{\mu\xi(\xi\mu-\beta^2)} + \left(\frac{Jc}{\delta}+\frac{\delta\beta\rho}{\xi\mu}\right)^2, \frac{\delta^2(\xi\mu-\beta^2)}{\alpha\mu\xi} + \frac{1}{4}, \frac{J\xi}{2\beta\delta^2},\right)$$

By substituting results (3.48),(3.49) ,(3.50), and (3.51) into (3.47), we obtain the estimate (3.46). □

At this point, we introduce the Lyapunov functional defined as

$$\mathcal{L}(t) := N\mathcal{E}(t) + N_1\mathcal{I}_1(t) + \mathcal{I}_2(t), \tag{3.52}$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ the two expressions defined in Lemmas 3.4, 3.5, and N, N_1 are positive constants to be determined later.

First, we have the following result:

Lemma 3.6. *There exists a positive constant χ such that*

$$(N-\chi)\mathcal{E}(t) \le \mathcal{L}(t) \le (N+\chi)\mathcal{E}(t),\ \forall t > 0.$$

Proof. We determine that there is a positive constant χ by using Young's inequality, so that

$$\begin{aligned}\Big|\mathcal{L}(t) - N\mathcal{E}(t)\Big| \le& \chi\left(\int_0^L u_t^2 dx + \int_0^L u_x^2 dx + \int_0^L \varphi^2 dx + \int_0^L \varphi_x^2 dx + \int_0^L \theta^2 dx\right) \\ &+ \chi\left(\int_0^L q^2 dx + \int_0^L u_{xt}^2 dx + \int_0^L \left(\frac{\beta}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx\right) \le \chi\mathcal{E}(t).\end{aligned}$$

So

$$-\chi\mathcal{E}(t) \le \mathcal{L}(t) - N\mathcal{E}(t) \le \chi E(t),$$
$$N\mathcal{E}(t) - \chi\mathcal{E}(t) \le \mathcal{L}(t) \le N\mathcal{E}(t) + \chi\mathcal{E}(t)$$

Consequently,

$$(N-\chi)\mathcal{E}(t) \le \mathcal{L}(t) \le (N+\chi)\mathcal{E}(t),\ \forall t > 0.$$

□

Now, we are ready to cite our mean theorem, which reads as follows:

Theorem 3.3. *There exist positive constants A and ω such that the energy $\mathcal{E}(t)$ associated to the solution (u, φ, θ, q) of the problem (3.1)-(3.3), satisfies*

$$\mathcal{E}(t) \le A\mathcal{E}(0)e^{-\omega t},\ \textit{for all } t \ge 0. \tag{3.53}$$

Proof of Theorem 3.3

By differentiating $\mathcal{L}(t)$ and replacing (3.36), (3.46) in (3.52) and applying Poincaré's inequality, we get

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\frac{J\delta^2\rho^2}{c\beta^2}N_1 - C\right]\int_0^L u_{tt}^2 dx - \left[\frac{J\beta}{4\xi} - \varepsilon_1 N_1\right]\int_0^L u_{xt}^2 dx \\
& - \frac{J\beta c_p}{4\xi}N_1 \int_0^L u_t^2 dx - \frac{\delta^2\rho}{2c\beta}\left(\mu - \frac{\beta^2}{\xi}\right) N_1 \int_0^L u_x^2 dx \\
& - \left[\frac{\alpha}{2\mu}\left(\mu - \frac{\beta^2}{\xi}\right) - \varepsilon_1 N_1\right]\int_0^L \varphi_x^2 dx - \left[\frac{\delta^2\rho}{c\beta}N_1 + 1\right]\int_0^L \left(\frac{1}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx \\
& - \left[\frac{c}{2}\Big(\frac{\alpha\rho}{\beta} + \frac{\delta^2\rho}{c\beta}\Big)N_1 - C\right]\int_0^L \theta^2 dx - [N - CN_1 - C]\int_0^L q^2 dx.
\end{aligned}$$

At this stage, we must choose the constants N, N_1 and ε_1 carefully, to obtain the estimate (3.53).

First, we select N_1 large enough such that

$$\frac{J\delta^2\rho^2}{c\beta^2}N_1 > C \quad \text{and} \quad \frac{c}{2}\Big(\frac{\alpha\rho}{\beta} + \frac{\delta^2\rho}{c\beta}\Big)N_1 > C,$$

We proceed by setting

$$c_0 = \frac{J\delta^2\rho^2}{c\beta^2}N_1 - C \;\text{ and }\; c_1 = \frac{c}{2}\Big(\frac{\alpha\rho}{\beta} + \frac{\delta^2\rho}{c\beta}\Big)N_1 - C.$$

Next, we choose ε_1 so small such that

$$\varepsilon_1 < \min\left\{\frac{J\beta}{4N_1\xi}, \frac{\alpha}{2N_1\mu}\left(\mu - \frac{\beta^2}{\xi}\right)\right\},$$

then, we set

$$c_2 = \frac{J\beta}{4\xi} - \varepsilon_1 N_1 \;\text{ and }\; c_3 = \frac{\alpha}{2\mu}\left(\mu - \frac{\beta^2}{\xi}\right) - \varepsilon_1 N_1.$$

Finally, we pick N so large such that

$$N > C(N_1 + 1) \;\text{ and } N > \chi,$$

then, we put

$$c_4 = N - C(N_1 + 1), \quad c_5 = \frac{J\beta c_p}{4\xi}N_1, \; c_6 = \frac{\delta^2\rho}{2c\beta}\left(\mu - \frac{\beta^2}{\xi}\right)N_1,$$

and

$$c_7 = \frac{\delta^2 \rho}{c\beta} N_1 + 1.$$

Accordingly, $\mathcal{L}(t) \sim \mathcal{E}(t)$, and

$$\begin{aligned}\mathcal{L}'(t) \leq & - c_0 \int_0^L u_{tt}^2 dx - c_2 \int_0^L u_{xt}^2 dx - c_5 \int_0^L u_t^2 dx - c_6 \int_0^L u_x^2 dx - c_3 \int_0^L \varphi_x^2 dx \\ & - c_7 \int_0^L \left(\frac{\beta}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx - c_1 \int_0^L \theta^2 dx - c_4 \int_0^L q^2 dx. \end{aligned} \tag{3.54}$$

By taking $\lambda = \min\{c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$, leads to

$$\mathcal{L}'(t) \leq -\lambda \mathcal{E}(t), \ \ \forall t \geq 0. \tag{3.55}$$

Moreover, from Lemma 3.6, we have $\mathcal{L}(t) \sim \mathcal{E}(t)$. It follows that , there exists a positive constant ω, for which (3.55) becomes

$$\mathcal{L}'(t) \leq -\omega \mathcal{L}(t), \ \forall t \geq 0.$$

The integration over $(0, t)$ gives

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-\omega t} \quad \forall t \geq 0.$$

Finally, the estimate (3.53) follows from the equivalence between $\mathcal{L}(t)$ and $\mathcal{E}(t)$. The proof of Theorem 3.3 is then completed.

chapter

4 Well posedness and decay estimate for a truncated porous thermoelastic system with Gurtin-Pipkin thermal effect and free of second spectrum

The results presented in this chapter have been published in our paper [8].

4.1 Introduction

In this chapter, we examine a truncated porous thermoelastic system, which is modeled by the memory-type Gurtin-Pipkin thermal law. The system is free from the adverse effects associated with the second spectrum of frequencies. Our results represent a significant improvement over those presented in [18, 28] for classical porous thermoelastic problems.

We are concerned with the following system

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\varphi_x = 0 & \text{in } (0,L)\times(0,+\infty)\,, \\ -Ju_{ttx} - \alpha\varphi_{xx} + bu_x + \xi\varphi + \delta\theta_x = 0 & \text{in } (0,L)\times(0,+\infty)\,, \\ c\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^s g(s)\theta_{xx}(s-t)ds + \delta\varphi_{tx} = 0 & \text{in } (0,L)\times(0,+\infty)\,, \end{cases} \tag{4.1}$$

where u denotes the transverse displacement, φ the volume fraction, and θ represents of the temperature difference from a reference confguration of a porous material of length L. The coefficients $J, \rho, \mu, \alpha, \beta, b, \xi, c$ and δ are positive constants, and μ, ξ, b satisfy

$$\mu\xi > b^2. \tag{4.2}$$

4.2 Preliminaries

To formulate the problem (4.1) in the setting of semigroup theory, we follow Giorgi et *et al.* [24] and introduce the new variable

$$\theta^t(x,s) := \theta(x,t-s),\ s \geq 0$$

and

$$\eta(x,s) = \eta^t(x,s) := \int_0^s \theta^t(x,\tau)d\tau,\ s \geq 0,$$

which represent the past history and the integrated past history of θ up to t, respectively. Clearly, we have

$$\eta_t(x,s) = \theta(x,t) - \eta_s(x,s). \tag{4.3}$$

The system (4.1) is completed with the following initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), \varphi(x,0) = \varphi_0(x), \theta(x,0) = \theta_0(x),\ x \in (0,L)\,, \\ \theta(x,-s) = h(x,s), s > 0,\ \eta^0(x,s) = \displaystyle\int_0^s h(x,\tau)d\tau = \eta_0(x,s), \\ \eta(x,0) = \displaystyle\lim_{s \longrightarrow 0} \eta^t(x,s) = 0. \end{cases} \tag{4.4}$$

$$\begin{cases} u(0,t) = u(L,t) = \varphi_x(0,t) = \varphi_x(L,t) = \theta(0,t) = \theta(L,t) = 0 \\ \eta^t(0,s) = \eta^t(L,s) = 0,\ t \in (0,+\infty)\,,\ s > 0, \end{cases} \tag{4.5}$$

where, $h \in C\left((0,+\infty); H^1(0,L)\right)$ expresses the history of θ.

Note that since Neumann boundary conditions are considered for φ, which may prevent the application of Poincaré's inequality. However, from the second equation of (4.1) and the boundary conditions (4.5), we have

$$\int_0^L \varphi(x,t)\,dx = 0,$$

which allows the use of Poincaré's inequality.

Regarding the memory kernel g, we assume that $\lim_{s\to\infty} g\,(s) = 0$, and that there exists a function κ such that $g'(s) = -\kappa(s)$, with the following hypotheses:

(h1) $\kappa \in C\,(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, (h2) $\kappa(s) > 0$, $\kappa'(s) \leq 0$, $\forall s \geq 0$,

(h3) $\int_0^\infty \kappa(s)ds = g(0)$,

(h4) there exists $\lambda > 0$ such that $\kappa'(s) \leq -\lambda\kappa(s)$, $\forall s \geq 0$.

A formal integration by part yields

$$q = -\int_{-\infty}^{t} g(t-s)\,\theta_x(x,s)\,ds = -\int_0^{+\infty} \kappa(s)\,\eta_x^t(x,s)\,ds.$$

Thus, the system (4.1) becomes

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\varphi_x = 0 & \text{in } (0,L)\times(0,\infty), \\ -Ju_{ttx} - \alpha\varphi_{xx} + bu_x + \xi\varphi + \delta\theta_x = 0 & \text{in } (0,L)\times(0,\infty), \\ c\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^{+\infty} \kappa(s)\eta_{xx}(s)ds + \delta\varphi_{tx} = 0 & \text{in } (0,L)\times(0,\infty), \\ \eta_t + \eta_s - \theta = 0 & \text{in } (0,L)\times(0,\infty). \end{cases} \tag{4.6}$$

Let define the weighted Hilbert space

$$\mathcal{M} = L^2_\kappa(\mathbb{R}^+, H_0^1(0,L)) = \{\eta : \mathbb{R}^+ \to H_0^1(0,L); \int_0^{+\infty} \kappa(s)\|\eta_x(s)\|_2^2 ds < +\infty\},$$

and the inner product

$$\langle \eta, \zeta\rangle_{\mathcal{M}} = \int_0^{+\infty} \kappa(s)\langle \eta_x, \zeta_x\rangle ds,$$

with the associated norm

$$\|\eta\|^2_{\mathcal{M}} = \int_0^{+\infty} \kappa(s)\|\eta_x(s)\|_2^2 ds,$$

where $\langle .,. \rangle$ denotes the L^2- inner product

$$\langle \psi, \tilde{\psi}\rangle = \int_0^L \psi(x)\tilde{\psi}(x)dx,$$

and $\|\cdot\|_2$ is the associated L^2-norm defined by

$$\|\psi\|_2^2 = \int_0^L |\psi|^2 dx.$$

Next, let $\mathscr{H}$ be the Hilbert space

$$\mathscr{H} = H^2(0,L)\cap H_0^1(0,L) \times H_0^1(0,L) \times H^1(0,L) \times L^2(0,L) \times \mathcal{M},$$

and

$$\mathscr{D} = \left\{ (u, v, \varphi, \theta, \eta) \in \mathscr{H}; \left| \begin{array}{ll} u \in H_*^3(0, L), & v \in H^2(0, L) \cap H_0^1(0, L), \\ \varphi \in H_*^2(0, L), & \theta \in H_0^1(0, L), \\ \eta \in \mathcal{N}, & \int_0^s \kappa(s)\eta_{xx}(s)ds \in L^2(0, L) \end{array} \right. \right\},$$

where

$$H_*^2(0, L) := \left\{ \psi \in H^2(0, L);\ \psi_x(0) = \psi_x(L) = 0 \right\},$$

$$H_*^3(0.L) := \left\{ \psi \in H^3(0.L) \cap H_0^1(0.L) : \phi_{xx}(0) = \phi_{xx}(L) = 0 \right\},$$

and

$$\mathcal{N} := \{ \eta \in \mathcal{M} : \eta_s \in \mathcal{M}, \eta(0) = 0 \}.$$

4.3 Well-posedness

In this section we demonstrate that problem (4.6) has a unique solution. The proof relies on semigroup approach and the Hille-Yosida Theorem.

Our well-posedness result reads as follow:

Theorem 4.1. *Let* $(u_0, u_1, \varphi_0, \theta_0, \eta_0) \in D$. *Then the problem* (4.6) *has a unique solution* $(u, u_t, \varphi, \theta, \eta) \in C(\mathbb{R}^+; D) \cap C^1(\mathbb{R}^+; \mathscr{H})$.

Before proving Theorem 4.1, we will address an auxiliary problem. To formulate this problem associated with the system (4.6) we proceed as follows:

First, we multiply the second and third equations in (4.6) by b. Next, we differentiate the second equation with respect to x. Then, we substitute φ_x from the first equation of (4.6) into the second and third equations to obtain

$$\begin{cases} Bu_{tt} + \alpha\mu u_{xxxx} - (\xi\mu - b^2)u_{xx} + \delta b\theta_{xx} = 0, \\ bc\theta_t - \dfrac{b}{\beta}\displaystyle\int_0^{+\infty} \kappa(s)\eta_{xx}(s)ds + \delta(\rho u_{ttt} - \mu u_{xxt}) = 0, \\ \eta_t + \eta_s - \theta = 0, \end{cases} \tag{4.7}$$

where $B = \xi\rho I - (J\beta + \alpha\rho)\partial_{xx}$ is a self-adjoint, positive operator defined on $L^2(0, L)$ with domain $H^2(0, L) \cap H_0^1(0, L)$. Note that B is invertible because it is coercive. For any

$f \in L^2(0,L)$, the equation $Bu = f$ admits a unique solution $u \in H^2(0,L) \cap H_0^1(0,L)$. Moreover, by differentiating the first equation of (4.7) with respect to t and then applying B^{-1}, we obtain

$$u_{ttt} = -\alpha\mu(B^{-1} \circ \partial_{xx})u_{xxt} + (\xi\mu - b^2)(B^{-1} \circ \partial_{xx})u_t - \delta b(B^{-1} \circ \partial_{xx})\theta_t. \tag{4.8}$$

Inserting equation (4.8) into the second equation of (4.7), we get

$$\begin{aligned}
&cb\theta_t - \frac{b}{\beta}\int_0^{+\infty} \kappa(s)\eta_{xx}(s)ds - \delta\alpha\mu\rho(B^{-1} \circ \partial_{xx})u_{xxt} + \delta\rho(\xi\mu - b^2)(B^{-1} \circ \partial_{xx})u_t \\
&- \delta^2 b\rho(B^{-1} \circ \partial_{xx})\theta_t - \delta\mu u_{xxt} = 0, \\
&S\theta_t - \frac{b}{\beta}\int_0^{+\infty} \kappa(s)\eta_{xx}(s)ds - \delta\alpha\mu\rho(B^{-1} \circ \partial_{xx})u_{xxt} + \delta\rho(\xi\mu - b^2)(B^{-1} \circ \partial_{xx})u_t - \delta\mu\rho u_{xxt} = 0.
\end{aligned}$$

From the definition of B we have

$$B^{-1} : L^2(0,L) \longrightarrow H^2(0,L) \cap H_0^1(0.L),$$

and

$$\partial_{xx} = \frac{1}{Jb + \alpha\rho}(\xi\rho I - B).$$

So, we put

$$F = B^{-1} \circ \partial_{xx} = \frac{\xi\rho B^{-1} - I}{Jb + \alpha\rho} : L^2(0,L) \longrightarrow L^2(0,L).$$

By substituting this expression into the last equality, we obtain the following:

$$\begin{aligned}
0 =& S\theta_t - \frac{1}{\beta}\int_0^{+\infty} \kappa(s)\eta_{xx}(s)ds - \delta\alpha\mu\rho\left[\frac{\xi\rho B^{-1} - I}{Jb + \alpha\rho}\right]u_{xxt} + \delta\rho(\xi\mu - b^2)(B^{-1} \circ \partial_{xx})u_t - \delta\mu u_{xxt} \\
=& S\theta_t - \frac{b}{\beta}\int_0^{+\infty} \kappa(s)\eta_{xx}(s)ds + \frac{\delta b}{Jb + \alpha\rho}\left[-\frac{\alpha\rho^2\xi\mu B^{-1}}{b}u_{xxt} + \frac{\alpha\mu\rho}{b}u_{xxt} + J\rho(\xi\mu - b^2)B^{-1}u_{xxt}\right] \\
&+ \frac{\delta b}{Jb + \alpha\rho}\left[\frac{\alpha\rho^2\xi\mu}{b}B^{-1}u_{xxt} - \alpha\rho^2 bB^{-1}u_{xxt} - J\mu u_{xxt} - \frac{\rho\alpha\mu}{b}u_{xxt}\right] \\
=& S\theta_t - \frac{b}{\beta}\int_0^{+\infty} \kappa(s)\eta_{xx}(s)ds + \frac{\delta b}{Jb + \alpha\rho}\left[\left(J\rho(\xi\mu - b^2) - \alpha\rho^2 b\right)B^{-1} - J\mu I\right]\partial_{xx}u_t \\
=& S\theta_t - \frac{b}{\beta}\int_0^{+\infty} \kappa(s)\eta_{xx}(s)ds + \frac{\delta b}{Jb + \alpha\rho}\left[J\rho(\xi\mu - b^2) - \alpha\rho^2 b - J\mu B\right]B^{-1} \circ \partial_{xx}u_t
\end{aligned}$$

then, we get

$$S\theta_t - \frac{b}{\beta}\int_0^{+\infty} \kappa(s)\eta_{xx}(s)ds + \frac{\delta b}{Jb + \alpha\rho}\left[(Jb + \alpha\rho)(J\mu\partial_{xx} - \rho b)\right]B^{-1} \circ \partial_{xx}u_t = 0.$$

Accordingly, the second equation in (4.7) can be rewritten as follows:

$$S\theta_t - \frac{b}{\beta}\int_0^{+\infty} \kappa(s)\eta_{xx}(s)ds + \delta b T u_t = 0, \tag{4.9}$$

where $S : L^2(0,L) \longrightarrow L^2(0.L)$ and $T : H^2(0,L) \cap H_0^1(0,L) \longrightarrow L^2(0.L)$ are the operators defined by

$$S = cbI - \delta^2 \rho b\ F, \quad T = -RP = -\frac{1}{\alpha\rho + Jb}\left[\left(\alpha b\rho^2 - J\rho(\xi\mu - b^2)\right)B^{-1} + J\mu I\right]\partial_{xx},$$

with $R = \rho b I - J\mu\partial_{xx}$ and

$$D(S) = L^2(0,L), \quad D(R) = D(T) = H^2(0,L) \cap H_0^1(0,L).$$

Finally, multiplying the first equation of (4.7) by B^{-1}, we arrive at

$$\begin{cases} u_{tt} + \alpha\mu F u_{xx} - (\xi\mu - b^2)Fu + \delta b F\theta = 0, \\ S\theta_t - \dfrac{b}{\beta}\displaystyle\int_0^{+\infty} \kappa(s)\eta_{xx}(s)ds + \delta b T u_t = 0, \\ \eta_t + \eta_s - \theta = 0. \end{cases} \tag{4.10}$$

In order to formulate the auxiliary problem within the semigroup framework, we introduce the Hilbert space

$$\mathcal{H} = H^2 \cap H_0^1(0,L) \times H_0^1(0,L) \times L^2(0,L) \times \mathcal{M},$$

equipped with the inner product

$$\langle \Psi, \Psi^* \rangle_{\mathcal{H}} = \alpha\mu \langle T u_x, u_x^* \rangle + \langle Rv, v^* \rangle + \langle S\theta, \theta^* \rangle + (\xi\mu - b^2)\langle Tu, u^* \rangle + \frac{b}{\beta}\langle \eta, \eta^* \rangle_{\mathcal{M}}.$$

Moreover, we define a new independent variable $u_t = v$ and let $\Psi = (u, v, \theta, \eta)^T$, consequently, the system (4.10) can then be expressed as follows:

$$\begin{cases} \Psi'(t) + \mathcal{A}\Psi(t) = 0, \forall t \geq 0, \\ \Psi(0) = (u_0, u_1, \theta_0, \eta_0)^T, \end{cases} \tag{4.11}$$

where $\mathcal{A}$ is the operator defined on $\mathcal{H}$ by

$$\mathcal{A} = \begin{pmatrix} 0 & -I & 0 & 0 \\ \alpha\mu F\partial_{xx} - (\xi\mu - b^2)FI & 0 & +\delta b F I & 0 \\ 0 & +\delta b S^{-1} T I & 0 & -\dfrac{b}{\beta}S^{-1}\displaystyle\int_0^{+\infty} \kappa(s)\partial_{xx} ds \\ 0 & 0 & -I & \partial_s \end{pmatrix}, \tag{4.12}$$

with domain

$$D(\mathcal{A}) = \left\{ \Psi = (u,v,\theta,\eta)^T \in \mathcal{H}, \middle| \begin{array}{l} u \in H^3_*(0,L),\ v \in (H^2(0,L) \cap H^1_0(0,L)), \\ \theta \in H^1_0(0,L),\ \eta \in \mathcal{N}, \displaystyle\int_0^{+\infty} \mu(s)\eta_{xx}(s)ds \in L^2(0,L) \end{array} \right\}.$$

Lemma 4.1. *[24] For any $\eta \in \mathcal{N}$ and $\theta \in L^2(0,L)$, we have*

$$-\int_0^{+\infty} \kappa(s)\frac{d}{ds}\|\eta_x(s)\|_2^2 ds = \int_0^{+\infty} \kappa'(s)\|\eta_x(s)\|_2^2 ds \tag{4.13}$$

and

$$-\int_0^{+\infty} \kappa(s)\langle\theta, \eta_s(s)\rangle ds = \int_0^{+\infty} \kappa'(s)\langle\theta, \eta(s)\rangle ds. \tag{4.14}$$

Proof. First, we differentiate the square of the norm:

$$\frac{d}{ds}\|\eta_x(s)\|_2^2 = 2\langle\eta_x(s), \eta_{sx}(s)\rangle.$$

Substituting this equality into the left-hand side of equation (4.13), we obtain:

$$-\int_0^{+\infty} \kappa(s)\frac{d}{ds}\|\eta_x(s)\|_2^2 \, ds = -\int_0^{+\infty} \kappa(s)2\langle\eta_x(s), \eta_{sx}(s)\rangle \, ds.$$

Now, by applying integration by parts, we consider:

$$-\int_0^{+\infty} \kappa(s)\frac{d}{ds}\|\eta_x(s)\|_2^2 \, ds = -\kappa(s)\|\eta_x(s)\|_2^2\Big|_0^{+\infty} + \int_0^{+\infty} \kappa'(s)\|\eta_x(s)\|_2^2 \, ds.$$

Since $\eta(0) = 0$ and $\eta(s) \to 0$ as $s \to +\infty$, leading to the desired conclusion:

$$-\int_0^{+\infty} \kappa(s)\frac{d}{ds}\|\eta_x(s)\|_2^2 \, ds = \int_0^{+\infty} \kappa'(s)\|\eta_x(s)\|_2^2 \, ds.$$

In a similar manner, by applying integration by parts to the left-hand side of equation (4.14), we obtain

$$-\int_0^{+\infty} \kappa(s)\langle\theta, \eta_s(s)\rangle \, ds = -\kappa(s)\langle\theta, \eta(s)\rangle\Big|_0^{+\infty} + \int_0^{+\infty} \kappa'(s)\langle\theta, \eta(s)\rangle \, ds.$$

And assuming boundary terms vanish, so we find

$$-\int_0^{+\infty} \kappa(s)\langle\theta, \eta_s(s)\rangle \, ds = \int_0^{+\infty} \kappa'(s)\langle\theta, \eta(s)\rangle \, ds.$$

□

The following theorem establishes the well-posedness of the auxiliary problem (4.10).

Theorem 4.2. *For any $\Psi_0 \in \mathcal{H}$, The problem (4.11) has a weak unique solution $\Psi \in C(\mathbb{R}^+, \mathcal{H})$. Moreover, if $\Psi_0 \in D(\mathcal{A})$, then $\Psi \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Proof. According to the Hille-Yosida theorem 1.9, it suffices to prove that $\mathcal{A}$ is maximal and dissipative First, we prove that $\mathcal{A}$ is monotone. Let $\Psi \in D(\mathcal{A})$, we have

$$\begin{aligned}\langle \mathcal{A}\Psi, \Psi \rangle_{\mathcal{H}} =& -\alpha\mu \langle Tv_x, u_x \rangle + \langle \alpha\mu RFu_{xx} - (\xi\mu - b^2)FRu + \delta\beta RF\theta, v \rangle \\ &+ \left\langle \delta\beta Tv - \frac{b}{\beta}\int_0^{+\infty} \kappa(s)\eta_{xx}(s)ds, \theta \right\rangle - (\xi\mu - \beta^2)\langle Tv, u \rangle + \frac{b}{\beta}\langle -\theta + \eta_s, \eta \rangle_{\mathcal{M}}.\end{aligned}$$

From definition $FR = -T$, so

$$\begin{aligned}\langle \mathcal{A}\Psi, \Psi \rangle_{\mathcal{H}} =& -\alpha\mu \langle Tv_x, u_x \rangle - \alpha\mu \langle Tu_{xx}, v \rangle + (\xi\mu - b^2)\langle Tu, v \rangle - \delta\beta \langle T\theta, v \rangle \\ &- \delta\beta \langle Tv, \theta \rangle - \frac{b}{\beta}\int_0^{+\infty} \kappa(s)\langle \eta_{xx}(s), \theta \rangle ds - (\xi\mu - b^2)\langle Tv, u \rangle - \langle \theta, \eta \rangle_{\mathcal{M}} + \langle \eta_s, \eta \rangle_{\mathcal{M}}.\end{aligned}$$

Integration by parts, we get

$$\begin{aligned}\langle \mathcal{A}\Psi, \Psi \rangle_{\mathcal{H}} =& -\alpha\mu \langle Tv_x, u_x \rangle + \alpha\mu \langle Tu_x, v_x \rangle + (\xi\mu - b^2)\langle Tu, v \rangle - \delta\beta \langle T\theta, v \rangle \\ &- \delta\beta \langle Tv, \theta \rangle + \frac{b}{\beta}\int_0^{+\infty} \kappa(s)\langle \eta_x(s), \theta_x \rangle ds - (\xi\mu - b^2)\langle Tv, u \rangle \\ &- \frac{b}{\beta}\langle \theta, \eta \rangle_{\mathcal{M}} + \frac{b}{\beta}\langle \eta_s, \eta \rangle_{\mathcal{M}}. \\ =& \frac{b}{\beta}\int_0^{+\infty} \kappa(s)\langle \eta_x(s), \theta_x \rangle ds - \frac{b}{\beta}\langle \theta, \eta \rangle_{\mathcal{M}} + \frac{b}{\beta}\langle \eta_s, \eta \rangle_{\mathcal{M}}.\end{aligned}$$

We have

$$\langle \theta, \eta \rangle_{\mathcal{M}} = \int_0^{+\infty} \kappa(s)\langle \theta_x, \eta_x(s) \rangle ds. \tag{4.15}$$

So

$$\langle \mathcal{A}\Psi, \Psi \rangle_{\mathcal{H}} = +\frac{b}{\beta}\langle \eta_s, \eta \rangle_{\mathcal{M}} = \frac{b}{2\beta}\int_0^{+\infty} \kappa(s)\left\langle \frac{d}{ds}\eta_x, \eta_x \right\rangle ds,$$

using equation (4.13), find

$$\langle \mathcal{A}\Psi, \Psi \rangle_{\mathcal{H}} = -\frac{b}{2\beta}\int_0^{+\infty} \kappa'(s)\|\eta_x(s)\|_2^2 ds \geq 0,$$

which shows that $\mathcal{A}$ is monotone.

Next, to demonstrate that $\mathcal{A}$ is maximal, we consider $H = (h_1, h_2, h_3, h_4) \in \mathcal{H}$ and seek $\Psi \in D(\mathcal{C})$ that satisfies

$$(I + \mathcal{A})\Psi = H. \tag{4.16}$$

which is expressed in terms of components

$$\begin{cases} u - v &= h_1, \\ -\alpha\mu T u_{xx} + (\xi\mu - b^2)Tu + Rv - \delta\beta T\theta &= Rh_2, \\ \delta bTv + S\theta - \dfrac{b}{\beta}\displaystyle\int_0^{+\infty} \kappa(s)\eta_{xx} ds &= Sh_3, \\ -\theta + \eta + \eta_s &= h_4. \end{cases} \tag{4.17}$$

From the first equation of (4.17), we have

$$v = u - h_1, \tag{4.18}$$

with the following lemma

Lemma 4.2. *Solving the fourth equation of (4.17) yields*

$$\eta(s) = (1 - e^{-s})\theta + \int_0^s e^{r-s} h_4(r) dr. \tag{4.19}$$

Proof. Let us consider the first-order linear differential equation:

$$\frac{d\eta}{ds} + \eta = \theta + h_4(s).$$

Next, by multiplying both sides of the equation by e^s, we obtain:

$$e^s \frac{d\eta}{ds} + e^s \eta = e^s \theta + e^s h_4(s),$$

the left-hand side is the derivative of a product:

$$\frac{d}{ds}(e^s \eta) = e^s \theta + e^s h_4(s),$$

integrating both sides from 0 to s:

$$e^s \eta(s) - \eta(0) = \int_0^s e^r \theta \, dr + \int_0^s e^r h_4(r) \, dr.$$

Since θ is constant with respect to r, and $\eta(0) = 0$ we have:

$$e^s \eta(s) = (e^s - 1)\theta + \int_0^s e^r h_4(r) \, dr,$$

dividing by e^s:

$$\eta(s) = e^{-s}\eta(0) + (1 - e^{-s})\theta + e^{-s}\int_0^s e^r h_4(r)\, dr.$$

We rewrite the last integral using the change of variables $r \to s - \tau$:

$$e^{-s}\int_0^s e^r h_4(r)\, dr = \int_0^s e^{-(s-r)} h_4(r)\, dr,$$

hence, the solution simplifies to:

$$\eta(s) = (1 - e^{-s})\theta + \int_0^s e^{r-s} h_4(r)\, dr.$$

□

Replacing v and η from (4.18)and (4.19) into the second and third equations of (4.17), we obtain

$$\begin{cases} -\alpha\mu T u_{xx} + (\xi\mu - b^2)Tu + Ru - \delta b T\theta = R(h_1 + h_2), \\ \delta b T u + S\theta - \dfrac{b}{\beta}\displaystyle\int_0^{+\infty} \kappa(s)\left[(1 - e^{-s})\theta_{xx} + \int_0^s e^{-(s-r)} h_{4xx}(r)\, dr\right] ds = Sh_3 + \delta b T h_1, \end{cases} \tag{4.20}$$

we put

$$c_\kappa = \frac{b}{\beta}\int_0^{+\infty} \kappa(s)(1 - e^{-s}) ds < \frac{b}{\beta}\, g(0),$$

therefore, (4.20) can be expressed as follows:

$$\begin{cases} -\alpha\mu T u_{xx} + (\xi\mu - b^2)Tu + Ru - \delta b T\theta = R(h_1 + h_2) \in H^{-1}(0, L), \\ \delta b T u + S\theta - c_\kappa \theta_{xx} = Sh_3 + \delta b T h_1 + \dfrac{b}{\beta}\displaystyle\int_0^{+\infty} e^{-s}\kappa(s)\left(\int_0^s e^r h_{4xx}(r) dr\right) ds \in H^{-1}(0, L). \end{cases} \tag{4.21}$$

Let $\mathcal{W} = (H^2(0, L) \cap H_0^1(0, L)) \times H_0^1(0, L)$. By 'formally' multiplying the equations of (4.21) by $\overset{*}{u} \in H^2(0, L) \cap H_0^1(0, L)$ and $\overset{*}{\theta} \in H_0^1(0, L)$, respectively, we arrive at the following variational formulation

$$A((u, \theta), (\overset{*}{u}, \overset{*}{\theta})) = L(\overset{*}{u}, \overset{*}{\theta}), \tag{4.22}$$

where $A(\cdot, \cdot)$ and l are the bilinear and linear forms defined over $\mathcal{W}$ by:

$$\begin{aligned} A((u, \theta), (\overset{*}{u}, \overset{*}{\theta})) =& \alpha\mu \left\langle T u_x, \overset{*}{u}_x \right\rangle + (\xi\mu - b^2)\left\langle Tu, \overset{*}{u} \right\rangle + \left\langle Ru, \overset{*}{u} \right\rangle \\ & - \delta b \left\langle T\theta, \overset{*}{u} \right\rangle + \delta b \left\langle Tu, \overset{*}{\theta} \right\rangle + \left\langle S\theta, \overset{*}{\theta} \right\rangle + c_\kappa \left\langle \theta_x, \overset{*}{\theta}_x \right\rangle, \end{aligned}$$

and

$$L(\overset{*}{u},\overset{*}{\theta}) = \left\langle R(f_1+f_2),\overset{*}{u}\right\rangle + \left\langle Sf_3,\overset{*}{\theta}\right\rangle + \delta b \left\langle Tf_1,\overset{*}{\theta}\right\rangle$$
$$+ \frac{b}{\beta}\int_0^{+\infty}\kappa(s)\left\langle \int_0^s e^{r-s} f_{4xx}(r)dr,\overset{*}{\theta}\right\rangle ds.$$

A straightforward calculation shows that A and L are bounded. In addition, we have

$$\begin{aligned}A((u,\theta),(u,\theta)) =& \alpha\mu\left\langle Tu_x,u_x\right\rangle + (\xi\mu-b^2)\left\langle Tu,u\right\rangle + \left\langle Ru,u\right\rangle \\ &+ \left\langle S\theta,\theta\right\rangle + c_\kappa\left\langle \theta_x,\theta_x\right\rangle \\ =& \frac{\alpha\mu}{\alpha\rho+J\beta}\Big((\alpha\beta\rho^2 - J\rho(\xi\mu-\beta^2))\langle B^{-1}u_{xx},u_{xx}\rangle + J\mu\|u_{xx}\|^2\Big) \\ &+ (\xi\mu-\beta^2)\langle Tu,u\rangle + \rho b\|u\|^2 + J\mu\|u_x\|^2 + \langle S\theta,\theta\rangle + c_\kappa\|\theta_x\|^2,\end{aligned}$$

then,

$$\begin{aligned}A((u,\theta),(u,\theta)) =& \frac{\alpha\mu}{\alpha\rho+J\beta}\Big((\alpha\beta\rho^2 + J\rho\beta^2)\langle B^{-1}u_{xx},u_{xx}\rangle - J\xi\mu\rho\langle B^{-1}u_{xx},u_{xx}\rangle + J\mu\|u_{xx}\|^2\Big) \\ &+ (\xi\mu-\beta^2)\langle Tu,u\rangle + \rho b\|u\|^2 + J\mu\|u_x\|^2 + \langle S\theta,\theta\rangle + c_\kappa\|\theta_x\|^2.\end{aligned}$$

By virtue of Remark 2.2 and using the positiveness of B^{-1}, we infer that

$$\begin{aligned}A((u,\theta),(u,\theta)) \geq& \frac{J\delta\alpha\mu^2}{\alpha\rho+J\beta}\|u_{xx}\|^2 + (\xi\mu-\beta^2)\langle Tu,u\rangle + \rho b\|u\|^2 \\ &+ J\mu\|u_x\|^2 + \langle S\theta,\theta\rangle + c_\kappa\|\theta_x\|^2.\end{aligned}$$

Consequently, there exists a positive constant m such that

$$A((u,\theta),(u,\theta)) \geq m(\| u_x \|^2 + \|u_{xx}\|^2 + \| \theta_x \|^2) = m \| (u,\theta) \|^2_{\mathcal{W}} .$$

This demonstrates that $A(\cdot,\cdot)$ is coercive. Consequently, the Lax-Milgram Theorem guarantees that the equation (4.22) has a unique solution

$$(u,\theta) \in (H^2(0,L)\cap H_0^1(0,L)) \times H_0^1(0,L).$$

By substituting u into the first equation (4.18), we infer that

$$v \in H^2(0,L)\cap H_0^1(0,L). \tag{4.23}$$

From (4.19) we have $\eta(0)=0$. On the other hand, using (4.19) along with Cauchy-Schwarz inequality, we get

$$\begin{aligned}\int_0^{+\infty}\kappa(s)\|\eta_x\|_2^2 ds &\leq 2\int_0^{\infty}\kappa(s)(1-e^{-s})^2\|\theta_x\|_2^2 ds\\ &+2\int_0^{\infty}\kappa(s)\left(\int_0^s e^{2(r-s)}dr\right)\left(\int_0^s\|h_{4x}(r)\|_2^2 dr\right)ds\\ &\leq 2g(0)\|\theta_x\|_2^2+2\int_0^{\infty}\kappa(s)\left(\frac{1}{2}-\frac{e^{-2s}}{2}\right)\left(\int_0^s\|h_{4x}\|_2^2 dr\right)ds,\\ &\leq 2g(0)\|\theta_x\|_2^2+\int_0^{\infty}\kappa(s)\left(\int_0^s\|h_{4x}\|_2^2 dr\right)ds\\ &\leq 2g(0)\|\theta_x\|_2^2+\int_0^{\infty}\|h_{4x}\|_2^2\int_r^{+\infty}\kappa(s)dsdr.\end{aligned}$$

Using the hypothesis (h4), we arrive at

$$\begin{aligned}\int_0^{+\infty}\kappa(s)\|\eta_x\|_2^2 ds &\leq 2g(0)\|\theta_x\|_2^2-\frac{1}{\lambda}\int_0^{\infty}\|h_{4x}\|_2^2\int_r^{+\infty}\kappa'(s)dsdr\\ &\leq 2g(0)\|\theta_x\|_2^2+\frac{1}{\lambda}\int_0^{\infty}\kappa(r)\|h_{4x}\|_2^2 dr<+\infty,\end{aligned}$$

which shows that $\eta\in\mathcal{M}$. Finally,

$$\eta_s(s)=e^{-s}\theta+h_4(s)-\int_0^s e^{r-s}h_4(r)dr=\theta+h_4-\eta(s)\in\mathcal{M}. \tag{4.24}$$

Therefore, $\eta\in\mathcal{N}$.

Next, by taking $\overset{*}{\theta}\equiv 0$ in (4.22), we arrive at

$$\alpha\mu\left\langle Tu_x,\overset{*}{u}_x\right\rangle=-\left\langle(\xi\mu-b^2)Tu+Ru-R(h_1+h_2),\overset{*}{u}\right\rangle,\ \forall\overset{*}{u}\in C_0^1(0,L),$$

which gives

$$\alpha\mu Tu_{xx}=(\xi\mu-b^2)Tu+Ru-R(h_1+h_2)\in H^{-1}(0,L).$$

The regularity theory of elliptic equations and the definition of the operator T yield

$$u_{xx}\in H_0^1(0,L).$$

Consequently

$$u\in H_*^3(0.L).$$

Similarly, by taking $\overset{*}{u}\equiv 0$ in (4.22), we obtain for any $\overset{*}{\theta}\in C_0^1(0,L)$:

$$\left\langle c_\kappa\theta_x+\frac{b}{\beta}\int_0^{+\infty}\kappa(s)\int_0^s e^{r-s}h_{4x}(r)\,dr\,ds,\overset{*}{\theta}_x\right\rangle=\left\langle-\delta bTu-S\theta+Sh_3+\delta bTh_1,\overset{*}{\theta}\right\rangle.$$

Given the definitions of the operators S and T, and recalling that $u, h_1 \in H^2(0,L) \cap H^1(0,L)$, $\theta \in H_0^1(0,L)$, and $h_3 \in L^2(0,L)$, it follows that:

$$-\delta b T u - S\theta + S h_3 + \delta b T h_1 \in L^2(0,L),$$

which implies that

$$c_\kappa \theta_x + \frac{b}{\beta} \int_0^{+\infty} \kappa(s) \int_0^s e^{r-s} h_{4x}(r)\, dr\, ds \in H^{-1}(0,L),$$

with

$$c_\kappa \theta_{xx} + \frac{b}{\beta} \int_0^{+\infty} \kappa(s) \int_0^s e^{r-s} h_{4xx}(r)\, dr\, ds = \delta b T u + S\theta - S h_3 - \delta b T h_1. \tag{4.25}$$

By virtue of (4.19), we arrive at

$$\int_0^{+\infty} \kappa(s) \eta_{xx}(s)\, ds = \delta b T u + S\theta - S h_3 - \delta b T h_1 \in L^2(0,L). \tag{4.26}$$

Therefore, the solution (u, v, θ, η) of system (4.11) belongs to $D(\mathcal{A})$, which shows that $\mathcal{A}$ is maximal. This completes the proof of Theorem 4.2. □

At this stage, we turn to the main problem (4.6).

Proof of Theorem 4.1

Let $(u_0, u_1, \theta_0, \eta_0) \in D(\mathcal{A})$, then from theorem 4.2, there exists a unique solution $(u, u_t, \theta, \eta) \in C(\mathbb{R}^+; D(\mathcal{A})) \cap C^1(\mathbb{R}^+; \mathcal{H})$ to the problem (4.6). As a result, we have

$$u \in C\left(\mathbb{R}^+; H_*^3(0,L) \cap H_0^1(0,L)\right) \cap C^1\left(\mathbb{R}^+; H^2(0,L) \cap H_0^1(0,L)\right) \cap C^2\left(\mathbb{R}^+; H_0^1(0,L)\right).$$

Thus, (4.8) obtains

$$u_{ttt} = -\alpha\mu (B^{-1} \circ \partial_{xx}) u_{xxt} + (\xi\mu - b^2)(B^{-1} \circ \partial_{xx}) u_t - b\delta (B^{-1} \circ \partial_{xx}) \theta_t \in C\left(\mathbb{R}^+; L^2(0,L)\right).$$

Next, let define φ by

$$\varphi(x,t) = -\frac{\mu}{b} u_x(x,t) + \frac{\rho}{b} \int_0^x u_{tt}(y,t) dy, \tag{4.27}$$

then we get

$$\rho u_{tt} - \mu u_{xx} - b\varphi_x = 0. \tag{4.28}$$

Replacing (4.28) into the second equation of (4.7), we arrive at

$$c\theta_t - \frac{1}{\beta}\int_0^s \kappa(s)\eta_{xx}(s)ds + \delta\varphi_{tx} = 0. \tag{4.29}$$

Consequently,

$$\varphi_{xt} \in C\left(\mathbb{R}^+, L^2(0,L)\right),$$

which implies that

$$\varphi \in C^1(\mathbb{R}^+, H_0^1(0,L)).$$

Moreover, from (4.27), we have

$$\varphi_x = -\frac{\mu}{b}u_{xx} + \frac{\rho}{b}u_{tt}.$$

Bearing in mind that $u \in H_*^3(0,L)$, we infer that $\varphi \in H_*^2(0,L)$. Therefore, (u,φ,θ,η) solves the problem (4.6) with the initial and boundary conditions (4.4) and (4.5). which completes the proof of Theorem 4.1.

4.4 Exponential Stability

In this section, we establish an the exponential decay of the solution of the problem (4.6).First, we define the energy associated with the solution (4.6) by

$$\begin{aligned}\mathcal{E}(t) := &\frac{1}{2}\int_0^L \left(\frac{J\rho}{b}u_{tt}^2 + \frac{J\mu}{b}u_{xt}^2 + \rho u_t^2 + \mu u_x^2 + \alpha\varphi_x^2 + 2bu_x\varphi + \xi\varphi^2 + c\theta^2\right)dx \\ &+ \frac{1}{2b}\int_0^{+\infty}\kappa(s)\|\eta_x(s)\|_2^2 ds.\end{aligned}$$

Note that as $\mu\xi > b^2$, we have

$$\begin{aligned}\mathcal{E}(t) = &\frac{J\rho}{2b}\int_0^L u_{tt}^2dx + \frac{J\mu}{2b}\int_0^L u_{xt}^2dx + \frac{\rho}{2}\int_0^L u_t^2dx + \left(\mu - \frac{b^2}{\xi}\right)\int_0^L u_x^2dx + \frac{\alpha}{2}\int_0^L \varphi_x^2dx \\ &+ \int_0^L \left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx + \frac{c}{2}\int_0^L \theta^2dx + \frac{1}{2\beta}\int_0^{+\infty}\kappa(s)\|\eta_x(s)\|_2^2ds,\end{aligned}$$

which shows that $\mathcal{E}(t)$ is a positive definite form.

Our stability result reads as follows.

Theorem 4.3. *The energy functional $\mathcal{E}(t)$ satisfies, along the solution of (4.4),(4.5)-(4.6), the estimate*

$$\mathcal{E}((t) \leq A\mathcal{E}(0)e^{-\zeta t} \ \forall t \geq 0, \tag{4.30}$$

where A and ζ are two positive constants.

The proof of Theorem 4.3 will be established through the following two lemmas.

Lemma 4.3. *The energy $\mathcal{E}(t)$ satisfies, along the solution (u, φ, θ) of (4.6), the estimate*

$$\mathcal{E}'(t) = \frac{1}{2\beta}\int_0^{+\infty} \kappa'(s)\|\eta_x(s)\|_2^2 ds \leq 0. \tag{4.31}$$

Proof. Performing the L^2-inner product of the first three equations of (4.6) with u_t, φ_t, and θ, respectively, and applying integration by parts, we arrive at

$$\frac{\rho}{2}\frac{d}{dt}\int_0^L u_t^2 dx + \frac{\mu}{2}\frac{d}{dt}\int_0^L u_x^2 dx + b\int_0^L u_{xt}\varphi dx = 0, \tag{4.32}$$

$$J\int_0^L u_{tt}\varphi_{xt} dx + \frac{\alpha}{2}\frac{d}{dt}\int_0^L \varphi_x^2 dx + b\int_0^L u_x\varphi_t dx + \frac{\xi}{2}\frac{d}{dt}\int_0^L \varphi^2 dx + \delta\int_0^L \theta_x\varphi_t dx = 0, \tag{4.33}$$

and

$$\frac{c}{2}\frac{d}{dt}\int_0^L \theta^2 dx - \frac{1}{\beta}\int_0^L \theta\left(\int_0^{+\infty} \kappa(s)\eta_{xx}(s) ds\right) dx - \delta\int_0^L \varphi_t\theta_x dx = 0. \tag{4.34}$$

Referring back to Remark 2.5 and equation (4.33), we find that

$$\begin{aligned} \frac{J\rho}{2b}\frac{d}{dt}\int_0^L u_{tt}^2 dx + \frac{J\mu}{2b}\frac{d}{dt}\int_0^L u_{xt}^2 dx + \frac{\alpha}{2}\frac{d}{dt}\int_0^L \varphi_x^2 dx + b\int_0^L u_x\varphi_t dx \\ + \frac{\xi}{2}\frac{d}{dt}\int_0^L \varphi^2 dx + \delta\int_0^L \theta_x\varphi_t dx = 0. \end{aligned} \tag{4.35}$$

Moreover, integration by parts and the boundary conditions (4.5), we obtain

$$\int_0^L \theta\left(\int_0^{+\infty} \kappa(s)\eta_{xx}(s) ds\right) dx = -\int_0^{+\infty}\int_0^L \kappa(s)\theta_x\eta_x(s) dx ds.$$

From the last equation of (4.6), we get

$$\begin{aligned} -\int_0^{+\infty}\int_0^L \kappa(s)\theta_x\eta_x(s) dx ds &= -\int_0^{+\infty}\int_0^L \kappa(s)\left(\eta_t(s) + \eta_s(s)\right)\eta_x(s) dx ds \\ &= -\frac{1}{2}\frac{d}{dt}\int_0^{+\infty} \kappa(s)\|\eta_x(s)\|_2^2 ds - \frac{1}{2}\int_0^{+\infty} \kappa(s)\frac{d}{ds}\|\eta_x(s)\|_2^2 ds. \end{aligned}$$

Returning to Lemma 4.1 and using (4.13), we obtain

$$\int_0^L \theta\left(\int_0^{+\infty}\kappa(s)\eta_{xx}(s)ds\right)dx = -\frac{1}{2}\frac{d}{dt}\int_0^{+\infty}\kappa(s)\|\eta_x(s)\|_2^2 ds + \frac{1}{2}\int_0^{+\infty}\kappa'(s)\|\eta_x(s)\|_2^2 ds. \tag{4.36}$$

Consequently, (4.34) becomes

$$\begin{aligned}&\frac{c}{2}\frac{d}{dt}\int_0^L \theta^2 dx + \frac{1}{2\beta}\frac{d}{dt}\int_0^{+\infty}\kappa(s)\|\eta_x(s)\|_2^2 ds + \frac{1}{2\beta}\int_0^{+\infty}\kappa'(s)\|\eta_x(s)\|_2^2 ds \\ &- \delta\int_0^L \varphi_t\theta_x dx = 0. \end{aligned}\tag{4.37}$$

Finally, equations (4.32), (4.35), and (4.37) yield the desired estimate (4.31). □

Lemma 4.4. *For any $\phi \in L^2(0,L)$ and any ε positive, we have*

$$-\int_0^L \phi\int_0^\infty \kappa'(s)\eta(s)dsdx \le \varepsilon\int_0^L \phi^2 dx - \frac{\lambda c_p g(0)}{4\varepsilon}\int_0^\infty \kappa'(s)\|\eta_x(s)\|^2 ds. \tag{4.38}$$

and

$$\int_0^L\left(\int_0^{+\infty}\kappa(s)\eta_x(s)ds\right)^2 dx \le -\frac{g(0)}{\lambda}\int_0^\infty \kappa'(s)\|\eta_x(s)\|_2^2 ds. \tag{4.39}$$

.

Proof. By applying Young's and Cauchy–Schwarz inequalities, we infer

$$\begin{aligned}-\int_0^L \phi\int_0^\infty \kappa'(s)\eta(s)dsdx &\le \varepsilon\int_0^L \phi^2 dx + \frac{1}{4\varepsilon}\int_0^L\left(\int_0^\infty \kappa'(s)\eta(s)ds\right)^2 dx,\\ &\le \varepsilon\int_0^L \phi^2 dx + \frac{1}{4\varepsilon}\int_0^L\int_0^\infty \kappa'(s)ds\int_0^\infty \kappa'(s)\eta^2(s)dsdx.\end{aligned}$$

From (h4) and Poincaré's inequality, we get

$$\begin{aligned}-\int_0^L \phi\int_0^\infty \kappa'(s)\eta(s)dsdx &\le \varepsilon\int_0^L \phi^2 dx - \frac{\lambda c_p}{4\varepsilon}\int_0^L\int_0^\infty \kappa(s)ds\int_0^\infty \kappa'(s)\eta_x^2 dsdx\\ &\le \varepsilon\int_0^L \phi^2 dx - \frac{\lambda c_p g(0)}{4\varepsilon}\int_0^\infty \kappa'(s)\|\eta_x(s)\|^2 ds.\end{aligned}$$

Next, based on assumptions (h3) and (h4), we have

$$\begin{aligned}\int_0^L\left(\int_0^{+\infty}\kappa(s)\eta_x(s)ds\right)^2 dx &\le \int_0^L\left(\int_0^\infty \kappa(s)ds\right)\int_0^\infty \kappa(s)\eta_x^2(s)dsdx\\ &\le -\frac{g(0)}{\lambda}\int_0^\infty \kappa'(s)\|\eta_x(s)\|_2^2 ds.\end{aligned}$$

□

Remark 4.1. *In this remark we use* $\frac{\mu\xi}{b} > b$*, so*

$$\begin{aligned}
&-\mu \int_0^L u_x^2 dx - \xi \int_0^L \varphi^2 dx - \left(b + \frac{\mu\xi}{b}\right) \int_0^L u_x\varphi dx \\
\leq &-\mu \int_0^L u_x^2 dx - \xi \int_0^L \varphi^2 dx - 2b \int_0^L u_x\varphi dx \\
= &-\mu \int_0^L u_x^2 dx - \xi \int_0^L \varphi^2 dx - 2b \int_0^L u_x\varphi dx + \frac{b^2}{\xi} \int_0^L u_x^2 dx - \frac{b^2}{\xi} \int_0^L u_x^2 dx \\
\leq &-\left(\mu - \frac{b^2}{\xi}\right) \int_0^L u_x^2 dx - \int_0^L \left(\frac{b}{\sqrt{\xi}} u_x + \sqrt{\xi}\varphi\right)^2 dx. \qquad (4.40)
\end{aligned}$$

Lemma 4.5. *Let* $K_1(t)$ *and* $K_2(t)$ *be the functionals*

$$\begin{aligned}
K_1(t) =& \frac{-cb}{g(0)} \int_0^L \theta \int_0^{+\infty} \kappa(s)\eta(s) ds dx + \delta\rho \int_0^L u_t\theta dx \\
&- \frac{\delta\rho}{g(0)} \int_0^L u_{tt} \int_0^{+\infty} \kappa(s)\eta(s) ds dx, \\
K_2(t) =& \frac{\alpha\rho}{b} \int_0^L \varphi_x u_t dx - \frac{\delta\rho}{b} \int_0^L \theta u_t dx,
\end{aligned}$$

and let the functional

$$\mathcal{G}_1 := \gamma_1 K_1(t) + \gamma_2 K_2(t),$$

where

$$\gamma_1 = \left(\frac{\alpha\rho}{b} + \frac{\delta^2\rho}{cb}\right) \quad \text{and} \quad \gamma_2 = \frac{\delta^2\rho}{c}.$$

The estimate below holds for all $\varepsilon > 0$ *along the solution* $(u, \varphi, \theta, \eta)$ *of equation (4.6):*

$$\begin{aligned}
\mathcal{G}_1'(t) \leq &- \frac{\gamma_1 cb}{2} \int_0^L \theta^2 dx - \frac{\gamma_2 J\rho}{2cb} \int_0^L u_{tt}^2 dx + \varepsilon \int_0^L u_{xt}^2 dx \\
&- \gamma_2 \left(\mu - \frac{b^2}{\xi}\right) \int_0^L u_x^2 dx - \gamma_2 \int_0^L \left(\frac{b}{\sqrt{\xi}} u_x + \sqrt{\xi}\varphi\right)^2 dx \\
&- m\left(1 + \frac{1}{\varepsilon}\right) \int_0^\infty \kappa'(s) \|\eta_x(s)\|_2^2 ds, \qquad (4.41)
\end{aligned}$$

where m *is a positive constant.*

Proof. Direct differentiation of $K_1(t)$, using equation (4.6), yields

$$\begin{aligned}
K_1'(t) =& \frac{-cb}{g(0)} \int_0^L \theta_t \int_0^{+\infty} \kappa(s)\eta(s) ds dx - \frac{cb}{g(0)} \int_0^L \theta \int_0^{+\infty} \kappa(s)\eta_t(s) ds dx + \delta\rho \int_0^L u_{tt}\theta dx \\
&+ \delta\rho \int_0^L u_t\theta_t dx - \frac{\delta\rho}{g(0)} \int_0^L u_{ttt} \int_0^{+\infty} \kappa(s)\eta(s) ds dx - \frac{\delta\rho}{g(0)} \int_0^L u_{tt} \int_0^{+\infty} \kappa(s)\eta_t(s) ds dx, \\
=& \frac{b}{g(0)} \int_0^L \left[-\frac{1}{\beta} \int_0^{\infty} \kappa(s)\eta_{xx}(s) ds + \delta\varphi_{xt} \right] \int_0^{+\infty} \kappa(s)\eta(s) ds dx \\
&- \frac{cb}{g(0)} \int_0^L \theta \int_0^{+\infty} \kappa(s) \left(\theta - \eta_s(s)\right) ds dx \\
&+ \delta\rho \int_0^L \theta u_{tt} dx + \frac{\rho\delta}{c} \int_0^L u_t \left(\frac{1}{\beta} \int_0^{\infty} \kappa(s)\eta_{xx}(s) ds - \delta\varphi_{tx} \right) dx \\
&- \frac{\delta}{g(0)} \int_0^L \left(\mu u_{xxt} + b\varphi_{tx}\right) \int_0^{+\infty} \kappa(s)\eta(s) ds dx \\
&- \frac{\rho\delta}{g(0)} \int_0^L u_{tt} \int_0^{\infty} \kappa(s) \left(\theta - \eta_s(s)\right) ds dx.
\end{aligned}$$

Integration by parts and the boundary conditions (4.5) yield

$$\begin{aligned}
K_1'(t) =& \frac{b}{\beta g(0)} \int_0^L \left(\int_0^{\infty} \kappa(s)\eta_x(s) ds \right)^2 dx + \frac{\delta b}{g(0)} \int_0^L \varphi_{xt} \int_0^{+\infty} \kappa(s)\eta(s) ds dx \\
&- \frac{cb}{g(0)} \int_0^L \left(\int_0^{+\infty} \kappa(s) ds \right) \theta^2 dx + \frac{cb}{g(0)} \int_0^L \theta \int_0^{+\infty} \kappa(s)\eta_s(s) ds dx \\
&+ \delta\rho \int_0^L \theta u_{tt} dx - \frac{\rho\delta}{c\beta} \int_0^L u_{xt} \int_0^{\infty} \kappa(s)\eta_x(s) ds dx - \frac{\delta^2\rho}{c} \int_0^L u_t\varphi_{tx} dx \\
&+ \frac{\delta\mu}{g(0)} \int_0^L u_{xt} \int_0^{+\infty} \kappa(s)\eta_x(s) ds dx - \frac{\delta b}{g(0)} \int_0^L \varphi_{tx} \int_0^{+\infty} \kappa(s)\eta(s) ds dx \\
&- \frac{\rho\delta}{g(0)} \int_0^L u_{tt} \left(\int_0^{\infty} \kappa(s) ds \right) \theta dx + \frac{\rho\delta}{g(0)} \int_0^L u_{tt} \int_0^{\infty} \kappa(s)\eta_s(s) ds dx.
\end{aligned}$$

Using hypothesis (h3) and equation (4.13), we derive the following

$$\begin{aligned}
K_1'(t) =& \frac{b}{\beta g(0)} \int_0^L \left(\int_0^{+\infty} \kappa(s)\eta_x(s) ds \right)^2 dx - cb \int_0^L \theta^2 dx \\
&- \frac{cb}{g(0)} \int_0^L \theta \int_0^{+\infty} \kappa'(s)\eta(s) ds dx - \frac{\delta^2\rho}{c} \int_0^L u_t\varphi_{xt} dx \\
&+ \left(\frac{\delta\mu}{g(0)} - \frac{\delta\rho}{c\beta} \right) \int_0^L u_{xt} \int_0^{+\infty} \kappa(s)\eta_x(s) ds dx \\
&- \frac{\delta\rho}{g(0)} \int_0^L u_{tt} \int_0^{+\infty} \kappa'(s)\eta(s) ds dx.
\end{aligned}$$

Next, we have the expression from equation (4.39)

$$\frac{b}{\beta g(0)}\int_0^L\left(\int_0^{+\infty}\kappa(s)\eta_x(s)ds\right)^2dx\leq-\frac{b}{\beta\lambda}\int_0^{+\infty}\kappa'(s)\|\eta_x(s)\|_2^2dsdx, \tag{4.42}$$

and from equation (4.38)

$$-\frac{cb}{g(0)}\int_0^L\theta\int_0^{+\infty}\kappa'(s)\eta(s)dsdx\leq\varepsilon\int_0^L\theta^2dx-\frac{c^2b^2\lambda c_p}{4\varepsilon g(0)}\int_0^L\int_0^{+\infty}\kappa'(s)\|\eta_x(s)\|_2^2dsdx.$$

We choose $\varepsilon=\dfrac{cb}{2}$, and we get

$$-\frac{cb}{g(0)}\int_0^L\theta\int_0^{+\infty}\kappa'(s)\eta(s)dsdx\leq\varepsilon\int_0^L\theta^2dx-\frac{cb\lambda c_p}{2g(0)}\int_0^L\int_0^{+\infty}\kappa'(s)\|\eta_x(s)\|_2^2dsdx. \tag{4.43}$$

Hence, by substituting the two expressions (4.42) and (4.43), we obtain the following:

$$\begin{aligned}K_1'(t)\leq&\frac{-b}{\beta\lambda}\int_0^{\infty}\kappa'(s)\|\eta_x(s)\|_2^2ds-\frac{cb}{2}\int_0^L\theta^2dx\\&-\frac{\lambda cbc_p}{2g(0)}\int_0^{\infty}\kappa'(s)\|\eta_x(s)\|^2ds-\frac{\delta^2\rho}{c}\int_0^L u_t\varphi_{xt}dx\\&+\left(\frac{\delta\mu}{g(0)}-\frac{\delta\rho}{c\beta}\right)\int_0^L u_{xt}\int_0^{+\infty}\kappa(s)\eta_x(s)dsdx\\&-\frac{\delta\rho}{g(0)}\int_0^L u_{tt}\int_0^{+\infty}\kappa'(s)\eta(s)dsdx.\end{aligned} \tag{4.44}$$

On the other hand, the integration of the second equation of (4.6) over $(0,x)$, and boundary conditions (4.5), yield

$$\alpha\varphi_x=-Ju_{tt}+bu+\delta\theta+\xi\int_0^x\varphi(y)dy. \tag{4.45}$$

Next, differentiating $K_2(t)$ with the use of (4.6) and (4.45) yields

$$\begin{aligned}K_2'(t)=&\frac{\alpha\rho}{b}\int_0^L\varphi_{xt}u_tdx+\frac{\alpha\rho}{b}\int_0^L\varphi_xu_{tt}dx-\frac{\delta\rho}{b}\int_0^L\theta_tu_tdx-\frac{\delta\rho}{b}\int_0^L\theta u_{tt}dx,\\=&\frac{\alpha\rho}{b}\int_0^L u_t\varphi_{xt}dx+\frac{\rho}{b}\int_0^L u_{tt}\left(-Ju_{tt}+bu+\delta\theta+\xi\int_0^x\varphi(y)dy\right)dx\\&-\frac{\delta\rho}{b}\int_0^L u_t\theta_tdx-\frac{\delta\rho}{b}\int_0^L u_{tt}\theta dx\\=&-\frac{J\rho}{b}\int_0^L u_{tt}^2dx+\frac{\alpha\rho}{b}\int_0^L u_t\varphi_{xt}dx+\int_0^L(\mu u_{xx}+b\varphi_x)\,udx\\&+\frac{\xi}{b}\int_0^L(\mu u_{xx}+b\varphi_x)\left(\int_0^x\varphi(y)dy\right)dx\\&-\frac{\delta\rho}{cb}\int_0^L u_t\left(\frac{1}{\beta}\int_0^{\infty}\kappa(s)\eta_{xx}(s)ds-\delta\varphi_{xt}\right)dx.\end{aligned}$$

Integration by parts, we obtain

$$\begin{aligned}K_2'(t) =& -\frac{J\rho}{b}\int_0^L u_{tt}^2dx + \left(\frac{\alpha\rho}{b}+\frac{\delta^2\rho}{cb}\right)\int_0^L u_t\varphi_{xt}dx - \mu\int_0^L u_x^2dx\\&-\left(\frac{\mu\xi}{b}+b\right)\int_0^L u_x\varphi dx - \xi\int_0^L \varphi^2dx + \frac{\delta\rho}{cb\beta}\int_0^L u_{xt}\int_0^\infty \kappa(s)\eta_x(s)dsdx.\end{aligned}$$

By returning to equation (4.40), we obtain

$$\begin{aligned}K_2'(t) \leq& -\frac{J\rho}{b}\int_0^L u_{tt}^2dx + \left(\frac{\alpha\rho}{b}+\frac{\delta^2\rho}{cb}\right)\int_0^L u_t\varphi_{xt}dx - \left(\mu-\frac{b^2}{\xi}\right)\int_0^L u_x^2dx\\&-\int_0^L\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)^2dx + \frac{\delta\rho}{cb\beta}\int_0^L u_{xt}\int_0^\infty \kappa(s)\eta_x(s)dsdx.\end{aligned} \tag{4.46}$$

Substituting (4.44) and (4.46) into $\mathcal{G}_1'(t)=\gamma_1K_1'(t)+\gamma_2K_2'(t)$, we arrive at

$$\begin{aligned}\mathcal{G}_1'(t) \leq& -\frac{b\gamma_1}{\beta\lambda}\int_0^\infty \kappa'(s)\|\eta_x(s)\|_2^2ds - \frac{cb\gamma_1}{2}\int_0^L\theta^2dx\\&-\frac{\lambda cbc_p\gamma_1}{2g(0)}\int_0^\infty \kappa'(s)\|\eta_x(s)\|^2ds - \gamma_1\gamma_2\int_0^L u_t\varphi_{xt}dx\\&+\gamma_1\left(\frac{\delta\mu}{g(0)}-\frac{\delta\rho}{c\beta}\right)\int_0^L u_{xt}\int_0^{+\infty}\kappa(s)\eta_x(s)dsdx\\&-\frac{\delta\rho\gamma_1}{g(0)}\int_0^L u_{tt}\int_0^{+\infty}\kappa'(s)\eta(s)dsdx - \frac{J\rho\gamma_2}{b}\int_0^L u_{tt}^2dx\\&+\gamma_2\gamma_1\int_0^L u_t\varphi_{xt}dx - \gamma_2\left(\mu-\frac{b^2}{\xi}\right)\int_0^L u_x^2dx\\&-\gamma_2\int_0^L\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)^2dx + \frac{\delta\rho\gamma_2}{cb\beta}\int_0^L u_{xt}\int_0^\infty\kappa(s)\eta_x(s)dsdx.\end{aligned}$$

After simplification, we obtain

$$\begin{aligned}\mathcal{G}_1'(t) \leq& -\gamma_1\frac{\lambda^2\beta cbc_p+2bg(0)}{2g(0)\beta\lambda}\int_0^\infty \kappa'(s)\|\eta_x(s)\|_2^2ds - \frac{cb\gamma_1}{2}\int_0^L\theta^2dx\\&+\left[\gamma_1\left(\frac{\delta\mu}{g(0)}-\frac{\delta\rho}{c\beta}\right)+\gamma_2\frac{\delta\rho}{cb\beta}\right]\int_0^L u_{xt}\int_0^{+\infty}\kappa(s)\eta_x(s)dsdx\\&-\frac{\delta\rho\gamma_1}{g(0)}\int_0^L u_{tt}\int_0^{+\infty}\kappa'(s)\eta(s)dsdx - \frac{J\rho\gamma_2}{b}\int_0^L u_{tt}^2dx\\&-\gamma_2\left(\mu-\frac{b^2}{\xi}\right)\int_0^L u_x^2dx - \gamma_2\int_0^L\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)^2dx.\end{aligned} \tag{4.47}$$

By applying Young's inequality and using equation (4.39), we obtain:

$$\begin{aligned}
&\left[\gamma_1\left(\frac{\delta\mu}{g(0)}-\frac{\delta\rho}{c\beta}\right)+\gamma_2\frac{\delta\rho}{cb\beta}\right]\int_0^L u_{xt}\int_0^{+\infty}\kappa(s)\eta_x(s)dsdx\\
\leq&\varepsilon\int_0^L u_{xt}^2dx+\frac{1}{4\varepsilon}\left[\gamma_1\left(\frac{\delta\mu}{g(0)}-\frac{\delta\rho}{c\beta}\right)+\gamma_2\frac{\delta\rho}{cb\beta}\right]^2\int_0^L\left(\int_0^{+\infty}\kappa(s)\eta_x(s)ds\right)^2dx.\\
\leq&\varepsilon\int_0^L u_{xt}^2dx-\frac{g(0)}{4\lambda\varepsilon}\left[\gamma_1\left(\frac{\delta\mu}{g(0)}-\frac{\delta\rho}{c\beta}\right)+\gamma_2\frac{\delta\rho}{cb\beta}\right]^2\int_0^L\int_0^{\infty}\kappa'(s)\|\eta_x(s)\|_2^2dsdx. \qquad (4.48)
\end{aligned}$$

On the other hand, by selecting $\varepsilon_1=\dfrac{J\rho\gamma_2}{2b}$, equation (4.38) yields

$$\begin{aligned}
-\frac{\delta\rho\gamma_1}{g(0)}\int_0^L u_{tt}\int_0^{+\infty}\kappa'(s)\eta(s)dsdx&\leq\varepsilon_1\int_0^L u_{tt}^2dx-\frac{1}{4\varepsilon_1}\frac{\delta^2\rho^2\gamma_1^2\lambda c_p}{g(0)}\int_0^{+\infty}\kappa'(s)\|\eta_x(s)\|_2^2ds\\
&\leq\frac{J\rho\gamma_2}{2b}\int_0^L u_{tt}^2dx-\frac{b\rho\lambda c_p\delta^2\gamma_1^2}{2Jg(0)\gamma_2}\int_0^{+\infty}\kappa'(s)\|\eta_x(s)\|_2^2ds.
\end{aligned} \qquad (4.49)$$

We put $m=\max\left(\frac{g(0)}{4\lambda}\left[\gamma_1\left(\frac{\delta\mu}{g(0)}-\frac{\delta\rho}{c\beta}\right)+\gamma_2\frac{\delta\rho}{cb\beta}\right]^2,\frac{b\rho\lambda c_p\delta^2\gamma_1^2}{Jg(0)\gamma_2},\frac{\lambda^2\beta cbc_p+2bg(0)}{g(0)\beta\lambda}\gamma_1\right)$.

In the final step, we substitute m and equations (4.48) and (4.49) into equation (4.47); the estimate (4.41) follows immediately. □

Lemma 4.6. *Let (u,φ,θ,η) be the solution of (4.6), then, the functional*

$$\begin{aligned}
\mathcal{G}_2(t)=&-\frac{J}{\sqrt{\xi}}\int_0^L u_{xt}\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)dx\\
&+\left(\frac{Jc}{\delta}+\frac{c\alpha b\rho}{\xi\mu\delta}\right)\int_0^L u_t\theta dx+\frac{\alpha b\rho}{\xi\mu}\int_0^L\varphi_x u_t dx,
\end{aligned}$$

satisfies the following estimate

$$\begin{aligned}
\mathcal{G}_2'(t)\leq&-\frac{\alpha}{2\mu}\left(\mu-\frac{b^2}{\xi}\right)\int_0^L\varphi_x^2dx-\int_0^L\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)^2dx\\
&-\frac{Jb}{2\xi}\int_0^L u_{xt}^2dx+C_1\int_0^L\theta^2dx+\int_0^L u_{tt}^2dx-C_2\int_0^{+\infty}\kappa'(s)\|\eta_x(s)\|_2^2dxds. \qquad (4.50)
\end{aligned}$$

Proof. By differentiating $\mathcal{G}_2(t)$ and applying the relation $-Ju_{ttx}=\alpha\varphi_{xx}-bu_x-\xi\varphi-\delta\theta_x$, we obtain

$$
\begin{aligned}
\mathcal{G}_2'(t) =& -\frac{J}{\sqrt{\xi}}\int_0^L u_{xtt}\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)dx-\frac{J}{\sqrt{\xi}}\int_0^L u_{xt}\left(\frac{b}{\sqrt{\xi}}u_{xt}+\sqrt{\xi}\varphi_t\right)dx\\
&+\left(\frac{Jc}{\delta}+\frac{c\alpha b\rho}{\xi\mu\delta}\right)\int_0^L u_{tt}\theta dx+\left(\frac{Jc}{\delta}+\frac{c\alpha b\rho}{\xi\mu\delta}\right)\int_0^L u_t\theta_t dx\\
&+\frac{\alpha b\rho}{\xi\mu}\int_0^L \varphi_{xt}u_t dx+\frac{\alpha b\rho}{\xi\mu}\int_0^L \varphi_x u_{tt}dx\\
=&\frac{\alpha}{\sqrt{\xi}}\int_0^L \varphi_{xx}\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)dx-\int_0^L\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)^2dx\\
&-\frac{\delta}{\sqrt{\xi}}\int_0^L \theta_x\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)dx-\frac{Jb}{\xi}\int_0^L u_{xt}^2dx-J\int_0^L u_{xt}\varphi_t dx\\
&+\left(\frac{Jc}{\delta}+\frac{c\alpha b\rho}{\xi\mu\delta}\right)\int_0^L u_{tt}\theta dx+\left(\frac{Jc}{\delta}+\frac{c\alpha b\rho}{\xi\mu\delta}\right)\int_0^L u_t\theta_t dx\\
&+\frac{\alpha b\rho}{\xi\mu}\int_0^L \varphi_x u_{tt}dx+\frac{\alpha b\rho}{\xi\mu}\int_0^L \varphi_{xt}u_t dx.
\end{aligned}
$$

Integration by parts, then use the third equation of (4.6) yield

$$
\begin{aligned}
\mathcal{G}_2'(t) =&\frac{\alpha b}{\xi}\int_0^L \varphi_{xx}u_x dx-\alpha\int_0^L\varphi_x^2dx-\int_0^L\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)^2dx\\
&+\frac{b\delta}{\xi}\int_0^L u_{xx}\theta dx+\delta\int_0^L\varphi_x\theta dx-\frac{Jb}{\xi}\int_0^L u_{xt}^2dx+J\int_0^L u_t\varphi_{xt}dx\\
&+\left(\frac{Jc}{\delta}+\frac{c\alpha b\rho}{\xi\mu\delta}\right)\int_0^L u_{tt}\theta dx+\left(\frac{J}{\delta\beta}+\frac{\alpha b\rho}{\xi\mu\delta\beta}\right)\int_0^{+\infty}\kappa(s)\int_0^L u_t\eta_{xx}(s)dxds\\
&-\left(J+\frac{\alpha b\rho}{\xi\mu}\right)\int_0^L u_t\varphi_{xt}dx+\frac{\alpha b\rho}{\xi\mu}\int_0^L\varphi_x u_{tt}dx+\frac{\alpha b\rho}{\xi\mu}\int_0^L\varphi_{xt}u_t dx.
\end{aligned}
$$

. Using integration by parts and the relation $u_{xx}=\frac{\rho}{\mu}u_{tt}-\frac{b}{\mu}\varphi_x$, we obtain

$$
\begin{aligned}
\mathcal{G}_2'(t) =& -\frac{\alpha}{\mu}\left(\mu-\frac{b^2}{\xi}\right)\int_0^L\varphi_x^2dx-\int_0^L\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)^2dx-\frac{Jb}{\xi}\int_0^L u_{xt}^2dx\\
&-\left(\frac{\delta b^2}{\xi\mu}-\delta\right)\int_0^L\varphi_x\theta dx+\left(\frac{Jc}{\delta}+\frac{c\alpha b\rho}{\xi\mu\delta}+\frac{\delta b\rho}{\xi\mu}\right)\int_0^L u_{tt}\theta dx\\
&-\left(\frac{J}{\delta\beta}+\frac{\alpha b\rho}{\xi\mu\delta\beta}\right)\int_0^L u_{xt}\int_0^{+\infty}\kappa(s)\eta_x(s)dsdx.
\end{aligned}
\tag{4.51}
$$

By applying Young's inequality, we derive

$$
-\left(\frac{\delta b^2}{\xi\mu}-\delta\right)\int_0^L\varphi_x\theta dx\le\frac{\alpha}{2\mu}\left(\mu-\frac{b^2}{\xi}\right)\int_0^L\varphi_x^2dx+\frac{\mu\left(\frac{\delta b^2}{\xi\mu}-\delta\right)^2}{2\alpha\left(\mu-\frac{b^2}{\xi}\right)}\int_0^L\theta^2dx, \tag{4.52}
$$

$$\left(\frac{Jc}{\delta}+\frac{c\alpha b\rho}{\xi\mu\delta}+\frac{\delta b\rho}{\xi\mu}\right)\int_0^L u_{tt}\theta dx \leq \int_0^L u_{tt}^2 dx+\frac{1}{4}\left(\frac{Jc}{\delta}+\frac{c\alpha b\rho}{\xi\mu\delta}+\frac{\delta b\rho}{\xi\mu}\right)^2\int_0^L \theta^2 dx, \quad (4.53)$$

and

$$\begin{aligned}&-\left(\frac{J}{\delta\beta}+\frac{\alpha b\rho}{\xi\mu\delta\beta}\right)\int_0^L u_{xt}\int_0^{+\infty}\kappa(s)\eta_x(s)dsdx\\ &\leq \frac{Jb}{2\xi}\int_0^L u_{xt}^2dx+\frac{\xi}{2Jb}\left(\frac{J}{\delta\beta}+\frac{\alpha b\rho}{\xi\mu\delta\beta}\right)^2\int_0^L\left(\int_0^{+\infty}\kappa(s)\eta_x(s)ds\right)^2dx,\end{aligned}$$

by applying (4.39), we obtain

$$\leq \frac{Jb}{2\xi}\int_0^L u_{xt}^2dx-\frac{\xi g(0)}{2Jb\lambda}\left(\frac{J}{\delta\beta}+\frac{\alpha b\rho}{\xi\mu\delta\beta}\right)^2\int_0^{+\infty}\kappa'(s)\|\eta_x(s)\|_2^2ds. \quad (4.54)$$

Then, we take

$$C_1=\max\left(\frac{\mu\left(\frac{\delta b^2}{\xi\mu}-\delta\right)^2}{2\alpha\left(\mu-\frac{b^2}{\xi}\right)},\frac{1}{4}\left(\frac{Jc}{\delta}+\frac{c\alpha b\rho}{\xi\mu\delta}+\frac{\delta b\rho}{\xi\mu}\right)^2\right) \quad \text{and} \quad C_2=\frac{\xi g(0)}{2Jb\lambda}\left(\frac{J}{\delta\beta}+\frac{\alpha b\rho}{\xi\mu\delta\beta}\right)^2.$$

Substituting expressions (4.52), (4.53), and (4.54) into (4.51), estimate (4.50) follows immediately. □

At this stage, we introduce the Lyapunov functional $\mathcal{L}$ as follows:

$$\mathcal{L}(t):=N_1\mathcal{E}(t)+N_2\mathcal{G}_1(t)+\mathcal{G}_2(t), \quad (4.55)$$

where N_1 and N_2 are two positive constants to be determined later, $\mathcal{G}_1$ and $\mathcal{G}_2$ are defined in Lemmas 4.5 and 4.6.

Lemma 4.7. *Let $\mathcal{K}(t)=\mathcal{L}(t)-N_1\mathcal{E}(t)$, then there exists a positive constant $\widetilde{C}$, such that*

$$\left|\mathcal{K}(t)\right|\leq \widetilde{C}\mathcal{E}(t) \quad \forall t>0.$$

Proof. Simple calculation shows that

$$\begin{aligned}\left|\mathcal{K}(t)\right| \leq & W\left\{\int_0^L\left|u_t\theta\right|dx+\int_0^L\left|\varphi_x u_t\right|dx+\int_0^L\left|u_{xt}\left(\frac{\beta}{\sqrt{\xi}}u_x+\sqrt{\xi}\varphi\right)\right|dx\right\}\\ &+W\left\{\int_0^L\left|\theta\int_0^{+\infty}\kappa(s)\eta(s)ds\right|dx+\int_0^L\left|u_{tt}\int_0^{+\infty}\kappa(s)\eta(s)ds\right|dx\right\},\end{aligned}$$

where W is a positive constant that depends on N_2, γ_1 and γ_2.

Thanks to Young's and Cauchy–Schwarz inequalities, we conclude the existence of a positive constant $\widetilde{C}$ such that

$$\left|\mathcal{K}(t)\right| \leq \widetilde{C}\mathcal{E}(t),\ \forall t > 0.$$

□

proof of Theorem 4.3

A direct substitution leads to the result

$$\begin{aligned}\mathcal{L}'(t) \leq & -\left[\frac{J\rho\gamma_2}{2b}N_2 - 1\right]\int_0^L u_{tt}^2 dx - \left[\frac{Jb}{4\xi} - \varepsilon N_2\right]\int_0^L u_{xt}^2 dx \\ & - \frac{Jbc_p}{2\xi}\int_0^L u_t^2 dx - \gamma_2 N_2\left(\mu - \frac{b^2}{\xi}\right)\int_0^L u_x^2 dx - \frac{\alpha}{2\mu}\left(\mu - \frac{b^2}{\xi}\right)\int_0^L \varphi_x^2 dx \\ & - (1+\gamma_2 N_2)\int_0^L \left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx - \left[\frac{cb\gamma_1}{2}N_2 - C_1\right]\int_0^L \theta^2 dx \\ & + \left[N_1 - mN_2\left(1+\frac{1}{\varepsilon}\right) - C_2\right]\int_0^L\int_0^{+\infty} \kappa'(s)\|\eta_x(s)\|_2^2 ds.\end{aligned}$$

Now, we select N_1, N_2, and ε carefully.
First, we choose $N_2 > \max\left\{\frac{2C_1}{cb\gamma_1}, \frac{2b}{J\rho\gamma_2}\right\}$, then we choose $\varepsilon = \frac{J\beta}{8\xi N_2}$.
Finally, we pick for N_1 large such that

$$N_1 > mN_2\left(1+\frac{1}{\varepsilon}\right) + C_2. \tag{4.56}$$

Therefore, there exists a positive constant $\chi \geq 0$ such that As a result, , there existssuch that

$$\mathcal{L}'(t) \leq -\chi\mathcal{E}(t) \quad \forall t \geq 0. \tag{4.57}$$

Also, we choose N to be large enough so that (4.56) is satisfied and $N > \widetilde{C}$, where $\widetilde{C}$ is the constant that satisfy Lemma 4.7. Consequently, $\mathcal{L}(t) \sim \mathcal{E}(t)$. Thus, there exists a constant $\zeta > 0$ such that

$$\mathcal{L}'(t) \leq -\zeta\mathcal{L}(t) \quad \forall t \geq 0,$$

an integration with respect to t leads to

$$\mathcal{L}\left(t\right) \leq \mathcal{L}\left(0\right) e^{-\zeta t} \quad \forall t \geq 0.$$

Finally, the desired estimate (4.30) follows from the equivalence between $\mathcal{L}(t)$ and $\mathcal{E}(t)$. This completes the proof of Theorem 4.3.

chapter

5 Exponential Stability of a Truncated Porous System of Thermoelasticity with Type III

Remark. After completing our thesis, we discovered that this problem had already been solved by H. Zougheib and T. Arwadi [52]. However, for establishing well-posedness, they used the Faedo-Galerkin method; however, we employed the semigroup approach.

5.1 Introduction

In this chapter, we focus on the following truncated porous system with thermoelasticity of Type III,

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - \beta \varphi_x = 0, \\ -J u_{ttx} - \alpha \varphi_{xx} + \beta u_x + \xi \varphi + \delta \theta_x = 0, \\ \gamma \theta_{tt} - \kappa \theta_{xx} + \delta \varphi_{ttx} - \tau \theta_{txx} = 0, \end{cases} \tag{5.1}$$

subjected to the following initial and boundary conditions

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \varphi(x,0) = \varphi_0(x), \theta(x,0) = \theta_0(x), \theta_t(x,0) = \theta_1(x), \; x \in [0,L] \tag{5.2}$$

$$u(0,t) = u(L,t) = \varphi_x(0,t) = \varphi_x(L,t) = \theta(0,t) = \theta(L,t) = 0, \; t \in (0,+\infty), \tag{5.3}$$

where the coefficients are positive constant and satisfy the following conditions:

$$\xi\mu > \beta^2.$$

Note that although φ satisfies the Neumann-boundary conditions, simple integration of $(5.1)_2$ and use (5.3) give

$$\int_0^L \varphi(x,t)\,dx = 0,$$

which allows the use of Poincaré's inequality.

In this chapter we investigate the well-posedness of the problem (5.1)–(5.3), using semi-group theory. Additionally, we prove the exponential stability by employing the multiplier method.

Now, by differentiating the first two equations of (5.1) with respect to t and taking $u_t = v$ and $\varphi_t = \psi$, the system (5.1)-(5.3) gives

$$\begin{cases} \rho v_{tt} - \mu v_{xx} - \beta \psi_x = 0, \\ -J v_{ttx} - \alpha \psi_{xx} + \beta v_x + \xi \psi + \delta \theta_{xt} = 0, \\ c\theta_{tt} - \kappa \theta_{xx} + \delta \psi_{xt} - \tau \theta_{xxt} = 0 \end{cases} \tag{5.4}$$

$$v(x,0) = v_0(x),\ v_t(x,0) = \mu u_{xx}(x,0) + \beta \varphi_x(x,0) = v_1(x),\ \psi(x,0) = \psi_0(x),$$

$$\theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \theta_1(x)\ x \in [0,L] \tag{5.5}$$

$$v(0,t) = v(L,t) = \psi_x(0,t) = \psi_x(L,t) = \theta(0,t) = \theta(L,t) = 0,\ t \in (0,+\infty) \tag{5.6}$$

5.2 Well-posedness

In this section, we prove the existence and uniqueness of a solution to problem (5.4)–(5.6), using the semi-group theory, utilizing certain operators introduced in previous chapters.

First, we differentiate the second equation of (5.4) with respect to x, then, we substitute ψ from the first equation of (5.4) into the second and third equations of the same system to obtain

$$\begin{cases} B v_{tt} + \alpha\mu v_{xxxx} - (\xi\mu - \beta^2) v_{xx} + \delta\beta\theta_{xxt} = 0, \\ c\beta\theta_{tt} - \kappa\beta\theta_{xx} + \delta\left(\rho v_{ttt} - \mu v_{xxt}\right) - \tau\beta\theta_{xxt} = 0, \end{cases} \tag{5.7}$$

where B is a positive, self-adjoint, and invertible operator defined on $L^2(0,L)$ with domain $H^2(0,L) \cap H_0^1(0,L)$, given by

$$B = \xi\rho I - (J\beta + \alpha\rho)\partial_{xx}.$$

Through a series of calculations, we examine the following auxiliary problem

$$\begin{cases} v_{tt} + \alpha\mu P v_{xx} - (\xi\mu - \beta^2) P v + \delta\beta P \theta_t = 0, \\ S\theta_{tt} - \kappa\beta\theta_{xx} + \delta\beta T v_t - \tau\beta\theta_{xxt} = 0, \end{cases} \tag{5.8}$$

with $S, P, T : L^2(0,L) \to L^2(0,L)$ are the operators defined as follows

$$\begin{cases} P = B^{-1} \circ \partial_{xx}, \\ S = c\beta I - \delta^2 \rho\beta P, \\ R = \rho\beta I - J\mu\partial_{xx}, \\ T = -\dfrac{1}{\alpha\rho + J\beta} \left[\left(\alpha\beta\rho^2 - J\rho(\xi\mu - \beta^2)\right) B^{-1} + J\mu I \right] \partial_{xx} = -\left(\rho\beta I - J\mu\partial_{xx}\right) B^{-1} \circ \partial_{xx} = -RP, \end{cases}$$

with domains $D(R) = D(T) = H^2 \cap H_0^1(0,L)$ and $D(S) = L^2(0,L)$.

Note that S is invertible, and the operators R, B, and $-\partial_{xx}$ are positive definite. It follows that $-P$, and consequently S and T, are also positive definite and well-defined. Therefore, we can define their square roots $R^{1/2}$, $S^{1/2}$, and $T^{1/2}$.

Now, to rewrite problem (5.8) in the semigroup setting, we introduce the new variables: $w = v_t$, $q = \theta_t$, and $\Psi = (v, w, \theta, q)^T$, and set the Hilbert space

$$\mathcal{H} = \left[H^2(0,L) \cap H_0^1(0,L)\right] \times H_0^1(0,L) \times H_0^1(0,L) \times L^2(0,L),$$

equipped with the inner product

$$\langle \Psi, \Psi^* \rangle = \alpha\mu \langle T v_x, v_x^* \rangle + \langle R w, w^* \rangle + (\xi\mu - \beta^2) \langle T v, v^* \rangle + \beta\kappa \langle \theta_x, \theta_x \rangle + \langle S q, q^* \rangle .$$

Then, the system (5.8) can be written as follows

$$\begin{cases} \Psi'(t) + \mathcal{A}\Psi(t) = 0, \qquad \forall t \geq 0, \\ \Psi(0) = (v_0, v_1, \theta_0, \theta_1), \end{cases} \tag{5.9}$$

where the operator $\mathcal{A} : \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$\mathcal{A}\Psi = \begin{pmatrix} -w \\ \alpha\mu P v_{xx} - (\xi\mu - \beta^2) P v + \delta\beta P q \\ -q \\ \delta\beta S^{-1} T w - \beta S^{-1} \left(\kappa\theta + \tau q\right)_{xx} \end{pmatrix},$$

with domain

$$D(\mathcal{A}) = \left\{ \Psi = (v, w, \theta, q)^T \in \mathcal{H}, \middle| \begin{array}{l} v \in H^3_*(0, L), \; w \in H^2(0, L) \cap H^1_0(0, L)), \\ \kappa\theta + \tau q \in H^2(0, L) \cap H^1_0(0, L), \end{array} \right\}.$$

where

$$H^3_*(0.L) = \left\{ \phi \in H^3(0.L) \cap H^1_0(0.L) : \phi_{xx} \in H^1_0(0.L) \right\}.$$

The well-posedness result is given by the following theorem.

Theorem 5.1. *For any $\Phi_0 \in \mathcal{H}$, there exists a unique weak solution $\Phi \in C(\mathbb{R}^+, \mathcal{H})$ of problem (5.9). Moreover, if $\Phi_0 \in D(\mathcal{A})$, then $\Phi \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Proof. According to the Hille–Yosida theorem 1.9, it suffices to show that $\mathcal{A}$ is monotone and maximal. □

First, we prove that $\mathcal{A}$ is monotone. For any $\Psi \in D(\mathcal{A})$, we have

$$\begin{aligned} \langle \mathcal{A}\Psi, \Psi \rangle_{\mathcal{H}} = & -\alpha\mu \langle Tw_x, v_x \rangle + \left\langle \alpha\mu RPv_{xx} - (\xi\mu - \beta^2)RPv + \delta\beta RPq, w \right\rangle \\ & - (\xi\mu - \beta^2) \langle Tw, v \rangle - \beta\kappa \langle q_x, \theta_x \rangle \\ & - \beta \left\langle (\kappa\theta + \tau q)_{xx}, q \right\rangle + \delta\beta \langle Tw, q \rangle . \end{aligned}$$

Using the definitions of the operators T and integration by parts, we obtain

$$\langle \mathcal{A}\Psi, \Psi \rangle_{\mathcal{H}} = \kappa\beta \langle q_x, q_x \rangle \geq 0. \tag{5.10}$$

Accordingly, it follows that $\mathcal{A}$ is monotone.

Next, to establish the maximality of $\mathcal{A}$, let $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, we seek $\Psi \in D(\mathcal{A})$ that satisfies $(I + \mathcal{A})\Phi = F$. That is,

$$\begin{cases} v - w = f_1, \\ w + \alpha\mu Pv_{xx} - (\xi\mu - \beta^2)Pv + \delta\beta Pq = f_2, \\ \theta - q = f_3, \\ \delta\beta Tw - \beta(\kappa\theta + \tau q)_{xx} + Sq = Sf_4. \end{cases} \tag{5.11}$$

Form the first and fourth equations of (5.11), we get

$$w = v - f_1, \tag{5.12}$$

$$q = \theta - f_3. \tag{5.13}$$

Next, by substituting equations (5.12)–(5.13) into the second and third equations of (5.11), we obtain

$$\begin{cases} v + \alpha\mu P v_{xx} - (\xi\mu - \beta^2) P v + \delta\beta P\theta = g_1, \\ \delta\beta T v - \kappa\beta\theta_{xx} + S\theta - \tau\beta\theta_{xx} = g_2, \end{cases} \tag{5.14}$$

where $g_1 = f_1 + f_2 + \delta\beta P f_3 \in L^2(0, L)$ and $g_2 = S(f_4 + f_3) + \delta\beta T f_1 - \tau\beta f_{3xx} \in H^{-1}(0, L)$. Now, we introduce the space

$$\mathcal{V} = H^2(0, L) \cap H_0^1(0, L) \times H_0^1(0.L).$$

The variational formulation corresponding to equation (5.14) can be expressed as follows:

$$A((v, \theta), (v^*, \theta^*)) = L(v^*, \theta^*). \tag{5.15}$$

with $A : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is the bilinear form given by:

$$\begin{aligned} A((v, \theta), (v^*, \theta^*)) =& \alpha\mu \left\langle T^{\frac{1}{2}} v_x, T^{\frac{1}{2}} v_x^* \right\rangle + (\xi\mu - \beta^2) \left\langle T^{\frac{1}{2}} v, T^{\frac{1}{2}} v^* \right\rangle + \left\langle R^{\frac{1}{2}} v, R^{\frac{1}{2}} v^* \right\rangle \\ & - \delta\beta \left\langle T^{\frac{1}{2}} \theta, T^{\frac{1}{2}} v^* \right\rangle + \delta\beta \left\langle T^{\frac{1}{2}} v, T^{\frac{1}{2}} \theta^* \right\rangle + \left\langle S^{\frac{1}{2}} \theta, S^{\frac{1}{2}} \theta^* \right\rangle + \kappa\beta \left\langle \theta_x, \theta_x^* \right\rangle \\ & + \tau\beta \left\langle \theta_x, \theta_x^* \right\rangle, \end{aligned}$$

and $L : \mathcal{V} \to \mathbb{R}$ is the linear functional defined by

$$L(u^*, \theta^*) = \langle R(f_1 + f_2), v^* \rangle - \delta\beta \langle T f_3, v^* \rangle + \langle S(f_3 + f_4), \theta^* \rangle + \delta\beta \langle T f_1, \theta^* \rangle + \tau\beta \langle f_{3x}, \theta_x^* \rangle .$$

It is clear that A and L are bounded. In addition, A is also coercive because

$$\begin{aligned} A((v, \theta), (v, \theta)) =& \alpha\mu \langle T v_x, v_x \rangle + (\xi\mu - \beta^2) \langle T v, v \rangle + \langle R v, v \rangle \\ & + \langle S\theta, \theta \rangle + \kappa\beta \langle \theta_x, \theta_x \rangle + \tau\beta \langle \theta_x, \theta_x \rangle, \end{aligned}$$

From the definitions of the operators R and T, and by applying integration by parts, we obtain

$$\begin{aligned} A((v, \theta), (v, \theta)) =& \frac{\alpha\mu}{\alpha\rho + J\beta} \Big((\alpha\beta\rho^2 - J\rho(\xi\mu - \beta^2)) \langle B^{-1} v_{xx}, v_{xx} \rangle + J\mu \|v_{xx}\|^2 \Big) \\ & + (\xi\mu - \beta^2) \langle T v, v \rangle + \rho\beta \|v\|^2 + J\mu \|v_x\|^2 + \langle S\theta, \theta \rangle + \kappa\beta \|\theta_x\|^2 + \tau\beta \|\theta_x\|^2 \end{aligned}$$

then,

$$A((u,\theta),(u,\theta)) = \frac{\alpha\mu}{\alpha\rho + J\beta}\Big((\alpha\beta\rho^2 + J\rho\beta^2)\langle B^{-1}v_{xx}, v_{xx}\rangle - J\xi\mu\rho\langle B^{-1}v_{xx}, v_{xx}\rangle + J\mu\|v_{xx}\|^2\Big)$$
$$+ (\xi\mu - \beta^2)\langle Tv, v\rangle + \rho\beta\|v\|^2 + J\mu\|v_x\|^2 + \langle S\theta, \theta\rangle + \kappa\beta\|\theta_x\|^2 + \tau\beta\|\theta_x\|^2$$

By virtue of Remark 2.2 and using the positiveness of B^{-1}, we infer that

$$A((u,\theta),(u,\theta)) \geq \frac{J\delta\alpha\mu^2}{\alpha\rho + J\beta}\|v_{xx}\|^2 + (\xi\mu - \beta^2)\langle Tv, v\rangle + \rho\beta\|v\|^2 + J\mu\|v_x\|^2$$
$$+ \langle S\theta, \theta\rangle + \kappa\beta\|\theta_x\|^2 + \tau\beta\|\theta_x\|^2$$

Consequently, there exists a positive constant m such that

$$A((v,\theta),(v,\theta)) \geq m(\|v_x\|^2 + \|v_{xx}\|^2 + \|\theta_x\|^2) = m\|(v,\theta)\|^2_{\mathcal{V}}.$$

Thus, A is coercive, and consequently, the Lax-Milgram Theorem shows that equation (5.15) has a unique solution

$$(v,\theta) \in H^2(0,L) \cap H_0^1(0,L) \times H_0^1(0,L).$$

By substituting v, θ into the equation (5.12)-(5.13), we deduce

$$w \in H^2(0,L) \cap H_0^1(0,L), \tag{5.16}$$
$$q \in H_0^1(0,L). \tag{5.17}$$

Moreover, by taking $v^* \equiv 0$, and replace $f_3 = \theta - q$, (5.15) becomes

$$\beta\left\langle (\kappa\theta + \tau q)_x, \theta_x^*\right\rangle + \langle S\theta, \theta^*\rangle + \delta\beta\left\langle Tv, \theta^*\right\rangle = \left\langle S(f_3 + f_4), \theta^*\right\rangle + \delta\beta\left\langle Tf_1, \theta^*\right\rangle$$

for all $\theta^* \in H_0^1(0,L)$, if $\theta^* = v, v \in C_0^1$ this implies

$$\beta\left\langle (\kappa\theta + \tau q)_x, \theta_x^*\right\rangle = \left\langle S(f_3 + f_4) + \delta\beta Tf_1 - S\theta - \delta\beta Tv, \theta^*\right\rangle,$$

which shows that $\kappa\theta + \tau q \in H^2(0,L)$, with

$$\beta(\kappa\theta + \tau q)_{xx} = -S(f_3 + f_4) - \delta\beta Tf_1 + S\theta + \delta\beta Tv.$$

Therefore

$$(\kappa\theta + \tau q) \in H^2(0,L) \cap H_0^1(0,L).$$

In addition, bearing in mind that $f_1 = v - w$ and $f_3 = \theta - q$, we arrive at

$$\delta\beta Tw - \beta(\kappa\theta + \tau q)_{xx} + Sq = Sf_4.$$

Similarly, by letting $\theta^* \equiv 0$, we obtain, from(5.15),

$$\alpha\mu \langle Tv_x, v_x^*\rangle + (\xi\mu - \beta^2)\langle Tv, v^*\rangle + \langle Rv, v^*\rangle - \delta\beta \langle T\theta, v^*\rangle = \langle R(f_1 + f_2), v^*\rangle - \delta\beta \langle Tf_3, v^*\rangle$$

for all $v^* \in H^2 \cap H_0^1$, If $v^* \in C_0^1$, then

$$-\alpha\mu Tv_{xx} = R(f_1 + f_2) - \delta\beta Tf_3 - (\xi\mu - \beta^2)Tv - Rv + \delta\beta T\theta,$$

Then, using (5.12) and (5.13), we obtain

$$-\alpha\mu Tv_{xx} = Rf_2 - (\xi\mu - \beta^2)Tv - Rw + \delta\beta Tq \in H^{-1}(0, L).$$

Since T is an isomorphism from $H_0^1(0, L)$ into $H^{-1}(0, L)$, we infer that $v_{xx} \in H_0^1(0, L)$, consequently,

$$v \in H_*^3(0, L),$$

and

$$-\alpha\mu Tv_{xx} + (\xi\mu - \beta^2)Tv + Rw - \delta\beta Tq = Rf_2.$$

We therefore conclude that $(v, w, \theta, q) \in D(\mathcal{A})$ and it solves system (5.11), which demonstrates that $\mathcal{A}$ is maximal. Consequently, by the Hille–Yosida theorem, problem (5.9) has a unique solution. This completes the proof of Theorem 5.1.

Next, we introduce

$$\mathscr{H} = \left[H^2(0, L) \cap H_0^1(0, L)\right] \times H_0^1(0, L) \times H_*^1(0, L) \times H^2(0, L) \cap H_0^1(0, L) \times L^2(0, L),$$

where

$$H_*^1(0, L) := \left\{\phi \in H^1(0, L); \int_0^L \phi(x)dx = 0\right\}.$$

and define the domain $\mathscr{D} \subset \mathscr{H}$, by

$$\mathscr{D} = \left\{ (v, w, \psi, \theta, q) \in \mathscr{H}; \middle| \begin{array}{ll} v \in H_*^3(0, L), & w \in H^2(0, L) \cap H_0^1(0, L) \\ \psi \in H_*^2(0, L), & \kappa\theta + \tau q \in H^2(0, L) \cap H_0^1(0, L) \end{array} \right\},$$

where

$$H_*^2(0.L) = \left\{\psi \in H^2(0.L) : \psi_x(0) = \psi_x(L) = 0\right\}.$$

The well-posedness of problem (5.4) is given by the following theorem:

Theorem 5.2. *Let* $(v_0, w_0, \psi_0, \theta_0, q_0) \in \mathscr{D}$. *Then there exists a unique solution* $(v, w, \psi, \theta, q) \in C(\mathbb{R}^+, \mathscr{D})) \cap C^1(\mathbb{R}^+, \mathscr{H})$, *of problem(5.4).*

5.2.1 Proof of theorem 5.2

From Theorem 5.1, we have $(v, v_t, \theta, \theta_t) \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$. As a result, we get

$$v \in C\left(\mathbb{R}^+; H_*^3(0,L)\right) \cap C^1\left(\mathbb{R}^+; H^2(0,L) \cap H_0^1(0,L)\right) \cap C^2\left(\mathbb{R}^+; H_0^1(0,L)\right).$$

Now, from (5.7) we have

$$Bv_{tt} = -\alpha\mu v_{xxxx} + \left(\xi\mu - \beta^2\right) v_{xx} - \delta\beta\theta_{xxt},$$

so

$$v_{ttt} = -\alpha\mu\left(B^{-1} \circ \partial_{xx}\right) v_{xxt} + \left(\xi\mu - \beta^2\right)\left(B^{-1} \circ \partial_{xx}\right) v_t - \delta\beta\left(B^{-1} \circ \partial_{xx}\right)\theta_{tt} \in C\left(\mathbb{R}^+; L^2(0,L)\right)$$
$$\Rightarrow v_{tt} \in C^1\left(\mathbb{R}^+; L^2(0,L)\right).$$

Let

$$\psi(x,t) = -\frac{\mu}{\beta} v_x(x,t) + \frac{\rho}{\beta}\int_0^x v_{tt}(y,t)dy, \tag{5.18}$$

by differentiation, we get

$$\rho v_{tt} - \mu v_{xx} - \beta\psi_x = 0. \tag{5.19}$$

By inserting (5.19) into the second equation of (5.7), we obtain

$$c\theta_{tt} - \kappa\theta_{xx} + \delta\psi_{tx} - \tau\theta_{xxt} = 0. \tag{5.20}$$

Consequently,

$$\psi_{xt} \in C\left(\mathbb{R}^+, L^2(0,L)\right),$$

which implies that

$$\psi \in C^1(\mathbb{R}^+, H_0^1(0,L)).$$

Finally, since $\psi_x = -\frac{\mu}{\beta} v_{xx} + \frac{\rho}{\beta} v_{tt}$ and $v \in H_*^3(0,L)$, we conclude that $\psi \in H_*^2(0,L)$. Therefore, (v, ψ, θ) solves the problem (5.4) with the initial and boundary conditions (5.5), (5.6), which completes the proof of Theorem 5.2.

5.3 Exponential Stability

In this section, we prove the exponential stability of the solution (v, ψ, θ) to the system (5.4)-(5.6).

Note that, since $\mu\xi > \beta^2$, the energy associated with the solution of equation (5.4) defined by

$$\mathcal{E}(t) = \frac{J\rho}{2\beta}\int_0^L v_{tt}^2 dx + \frac{J\mu}{2\beta}\int_0^L v_{xt}^2 dx + \frac{\rho}{2}\int_0^L v_t^2 dx + \left(\mu - \frac{\beta^2}{\xi}\right)\int_0^L v_x^2 dx + \frac{\alpha}{2}\int_0^L \psi_x^2 dx$$
$$+ \int_0^L \left(\frac{\beta}{\sqrt{\xi}} v_x + \sqrt{\xi}\psi\right)^2 dx + \frac{\kappa}{2}\int_0^L \theta_x^2 dx + \frac{c}{2}\int_0^L \theta_t^2 dx,$$

is a positive definite form.

Lemma 5.1. *Let (v, ψ, θ) be a solution to (5.4)-(5.6). Then the energy functional $\mathcal{E}(t)$ satisfies*

$$\mathcal{E}'(t) = -\tau \int_0^L \theta_{xt}^2 dx \leq 0, \ \forall t > 0. \tag{5.21}$$

Proof. Taking the L^2-inner product of (5.4) by v_t, ψ_t and θ_t respectively, then appling the remark 2.5 to the second equation, we have

$$\frac{\rho}{2}\frac{d}{dt}\int v_t^2 dx + \beta \int v_{xt}\psi dx + \frac{\rho}{2}\frac{d}{dt}\int v_x^2 dx = 0,$$
$$\frac{J\rho}{2\beta}\frac{d}{dt}\int v_{tt}^2 dx + \frac{J\mu}{2\beta}\frac{d}{dt}\int v_{xt}^2 dx + \frac{\alpha}{2}\frac{d}{dt}\int \psi_x^2 dx + \beta\int v_x\psi_t + \frac{\xi}{2}\frac{d}{dt}\int \psi^2 dx + \delta\int \theta_{xt}\psi_t = 0,$$
$$\frac{c}{2}\frac{d}{dt}\int \theta_t^2 dx + \frac{\kappa}{2}\frac{d}{dt}\int \theta_x^2 dx + \delta\int \psi_{xt}\theta_t dx + \tau\int \theta_{xt}^2 dx = 0.$$

Consequently

$$\frac{d}{dt}\mathcal{E}(t) = \frac{1}{2}\frac{d}{dt}\int_0^L \left(\frac{J\rho}{\beta}v_{tt}^2 + \frac{J\mu}{\beta}v_{xt}^2 + \rho v_t^2 + \mu v_x^2 + \alpha\psi_x^2 + 2\beta v_x\psi + \xi\psi^2 + \kappa\theta_x^2 + c\theta_t^2\right) dx$$
$$= -\tau\int \theta_{xt}^2 dx.$$

□

Lemma 5.2. *The functional*

$$\mathcal{K}_1(t) = c\int_0^L \theta_t\theta dx + \frac{\tau}{2}\int_0^L \theta_x^2 dx + \delta\int_0^L \psi_x\theta dx$$

satisfies, along the solution of problem (5.4) and for any positive constant ε, the estimate

$$\mathcal{K}_1'(t) \leq -\kappa \int_0^L \theta_x^2 dx + c \int_0^L \theta_t^2 dx + \varepsilon \int_0^L \psi_x^2 dx + C_\varepsilon \int_0^L \theta_{xt}^2 dx, \tag{5.22}$$

where C_ε is positive constant that depends on ε.

Proof.

$$\mathcal{K}_1'(t) = c \int_0^L \theta_{tt}\theta dx + c \int_0^L \theta_t^2 + \tau \int_0^L \theta_{xt}\theta_x dx + \delta \int_0^L \psi_x \theta_t dx + \delta \int_0^L \psi_{xt}\theta dx.$$

By virtue of the third equation in (5.4), we find

$$\mathcal{K}_1'(t) = \kappa \int_0^L \theta_{xx}\theta dx + \tau \int_0^L \theta_{xxt}\theta dx + c \int_0^L \theta_t^2 + \tau \int_0^L \theta_{xt}\theta_x dx + \delta \int_0^L \psi_x \theta_t dx.$$

The integration by part yields

$$\mathcal{K}_1'(t) = -\kappa \int_0^L \theta_x^2 dx + c \int_0^L \theta_t^2 + \delta \int_0^L \psi_x \theta_t dx. \tag{5.23}$$

Using Young's and Poincaré's inequalities for the last term, we have, for any $\varepsilon > 0$:

$$\begin{aligned} \delta \int_0^L \psi_x \theta_t dx &\leq \varepsilon \int_0^L \psi_x^2 dx + \frac{\delta^2}{4\varepsilon} \int_0^L \theta_t^2 dx, \\ &\leq \varepsilon \int_0^L \psi_x^2 dx + \frac{\delta^2}{4\varepsilon} \int_0^L \theta_{xt}^2 dx. \end{aligned} \tag{5.24}$$

We set $C_\varepsilon = \dfrac{\delta^2}{4\varepsilon}$ and substitute inequality (5.24) into equation (5.23) to obtain the estimate (5.22). □

Lemma 5.3. *Let us introduce the functional*

$$\mathcal{K}_2(t) = \rho \int_0^L v_t v dx - J \int_0^L v_{xt}\psi dx + \frac{Jc}{\delta} \int_0^L v_t \theta_t dx,$$

along the solution of problem (5.4). Then, there exist two positive constants C_2) and C_3, such that $\mathcal{K}_2$ satisfies the following estimate:

$$\begin{aligned} \mathcal{K}_2'(t) \leq &- \left(\mu - \frac{\beta^2}{\xi}\right) \int_0^L v_x^2 dx - \frac{\alpha}{2} \int_0^L \psi_x^2 dx \\ &- \int_0^L \left(\frac{\beta}{\sqrt{\xi}} v_x + \sqrt{\xi}\psi\right)^2 dx + \rho \int_0^L v_t^2 dx + \frac{C_2}{\varepsilon_1} \int_0^L \theta_x^2 dx \\ &+ C_3 \int_0^L \theta_{xt}^2 dx + \varepsilon \int_0^L v_{tt}^2 dx + \varepsilon_1 \int_0^L v_{xt}^2 dx. \end{aligned} \tag{5.25}$$

Proof. We differentiate $\mathcal{K}_2(t)$ then use (5.4), we get

$$\begin{aligned}\mathcal{K}_2'(t) =&\rho\int_0^L v_{tt}v dx+\rho\int_0^L v_t^2dx-J\int_0^L v_{xtt}\psi dx-J\int_0^L v_{xt}\psi_t dx\\&+\frac{Jc}{\delta}\int_0^L v_{tt}\theta_t dx+\frac{Jc}{\delta}\int_0^L v_t\theta_{tt}dx,\\=&\mu\int_0^L v_{xx}vdx+\beta\int_0^L\psi_x v dx+\rho\int_0^L v_t^2dx+\alpha\int_0^L\psi_{xx}\psi dx-\beta\int_0^L v_x\psi\\&-\xi\int_0^L\psi^2dx-\delta\int_0^L\theta_{xt}\psi dx-J\int_0^L v_{xt}\psi_t dx+\frac{Jc}{\delta}\int_0^L v_{tt}\theta_t dx+\frac{J\kappa}{\delta}\int_0^L v_t\theta_{xx}dx\\&-J\int_0^L v_t\psi_{xt}+\frac{J\tau}{\delta}\int_0^L v_t\theta_{xxt}dx.\end{aligned}$$

For convenience, we add and subtract the term $\dfrac{\beta^2}{\xi}v_x$. Using integration by parts, the expression can be rewritten as

$$\begin{aligned}\mathcal{K}_2'(t)=&-\left(\mu-\frac{\beta^2}{\xi}\right)\int_0^L v_x^2dx+\rho\int_0^L v_t^2dx-\int_0^L\left(\frac{\beta}{\sqrt{\xi}}v_x+\sqrt{\xi}\psi\right)^2dx\\&-\alpha\int_0^L\psi_x^2dx-\delta\int_0^L\theta_{xt}\psi dx+\frac{Jc}{\delta}\int_0^L v_{tt}\theta_t dx-\frac{J\kappa}{\delta}\int_0^L v_{xt}\theta_x dx-\frac{J\tau}{\delta}\int_0^L v_{xt}\theta_{xt}dx.\end{aligned}\tag{5.26}$$

Thus, using Young's and Poincaré's inequalities, we deduce

$$\begin{aligned}-\delta\int_0^L\theta_{xt}\psi dx&\le\frac{\alpha}{2}\int_0^L\psi dx+\frac{\delta^2}{2\alpha}\int_0^L\theta_{xt}^2dx\\&\le\frac{\alpha}{2}\int_0^L\psi_x dx+\frac{\delta^2}{2\alpha}\int_0^L\theta_{xt}^2dx,\end{aligned}\tag{5.27}$$

$$\begin{aligned}\frac{Jc}{\delta}\int_0^L v_{tt}\theta_t dx&\le\varepsilon\int_0^L v_{tt}^2dx+\frac{J^2c^2}{4\delta^2\varepsilon}\int_0^L\theta_t^2dx\\&\le\varepsilon\int_0^L v_{tt}^2dx+\frac{J^2c^2}{4\delta^2\varepsilon}\int_0^L\theta_{xt}^2dx,\end{aligned}\tag{5.28}$$

$$-\frac{J\tau}{\delta}\int_0^L v_{xt}\theta_{xt}dx\le\frac{\varepsilon_1}{2}\int_0^L v_{xt}^2dx+\frac{J^2\tau^2}{2\delta^2\varepsilon_1}\int_0^L\theta_{xt}^2dx,\tag{5.29}$$

$$-\frac{J\kappa}{\delta}\int_0^L v_{xt}\theta_x dx\le\frac{\varepsilon_1}{2}\int_0^L v_{xt}^2dx+\frac{J^2\kappa^2}{2\delta^2\varepsilon_1}\int_0^L\theta_x^2dx.\tag{5.30}$$

By taking $C_2=\dfrac{J^2\kappa^2}{2\delta^2}$ and $C_3=\dfrac{\delta^2}{2\alpha}+\dfrac{J^2c^2}{4\delta^2\varepsilon}+\dfrac{J^2\tau^2}{2\delta^2\varepsilon_1}$, and substituting equations (5.27)–(5.30) into (5.26), we obtain the desired result (5.25). □

Lemma 5.4. *The functional*

$$\begin{aligned}\mathcal{K}_3(t) = & -\frac{J}{\sqrt{\xi}}\int_0^L v_{xt}\left(\frac{\beta}{\sqrt{\xi}}v_x+\sqrt{\xi}\psi\right)dx \\ & +\left(\frac{Jc}{\delta}+\frac{c\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L v_t\theta_t dx+\frac{\alpha\beta\rho}{\xi\mu}\int_0^L \psi_x v_t dx,\end{aligned}$$

satisfies, along the solution and for any positive constant ε, *the estimate*

$$\begin{aligned}\mathcal{K}_3'(t) \leq & -\frac{\alpha}{\xi\mu}(\xi\mu-\beta^2)\int_0^L \psi_x^2 dx-\frac{1}{2}\int_0^L\left(\frac{\beta}{\sqrt{\xi}}v_x+\sqrt{\xi}\psi\right)^2 dx \\ & -\frac{J\beta}{2\xi}\int_0^L v_{xt}^2 dx+\varepsilon\int_0^L v_{tt}^2+C_4\int_0^L \theta_{xt}^2 dx+C_5\int_0^L \theta_x^2 dx. \qquad (5.31)\end{aligned}$$

Proof. By differentiating $\mathcal{K}_3(t)$ and employing the second equation from (5.4), we obtain

$$\begin{aligned}\mathcal{K}_3'(t) = & -\frac{J}{\sqrt{\xi}}\int_0^L v_{xtt}\left(\frac{\beta}{\sqrt{\xi}}v_x+\sqrt{\xi}\psi\right)dx-\frac{J}{\sqrt{\xi}}\int_0^L v_{xt}\left(\frac{\beta}{\sqrt{\xi}}v_{xt}+\sqrt{\xi}\psi_t\right)dx \\ & +\left(\frac{Jc}{\delta}+\frac{c\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L v_{tt}\theta_t dx+\left(\frac{Jc}{\delta}+\frac{c\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L v_t\theta_{tt}dx \\ & +\frac{\alpha\beta\rho}{\xi\mu}\int_0^L \psi_x v_{tt}dx+\frac{\alpha\beta\rho}{\xi\mu}\int_0^L \psi_{xt}v_t dx \\ = & \frac{\alpha}{\sqrt{\xi}}\int_0^L \psi_{xx}\left(\frac{\beta}{\sqrt{\xi}}v_x+\sqrt{\xi}\psi\right)dx-\int_0^L\left(\frac{\beta}{\sqrt{\xi}}v_x+\sqrt{\xi}\psi\right)^2 dx \\ & -\frac{\delta}{\sqrt{\xi}}\int_0^L \theta_{xt}\left(\frac{\beta}{\sqrt{\xi}}v_x+\sqrt{\xi}\psi\right)dx-\frac{J\beta}{\xi}\int_0^L v_{xt}^2 dx-J\int_0^L v_{xt}\psi_t dx \\ & +\left(\frac{Jc}{\delta}+\frac{c\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L v_{tt}\theta_t dx-\left(J+\frac{\alpha\beta\rho}{\xi\mu}\right)\int_0^L v_t\psi_{xt}dx+\left(\frac{J\tau}{\delta}+\frac{\tau\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L v_t\theta_{xxt}dx \\ & +\left(\frac{J\kappa}{\delta}+\frac{\kappa\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L v_t\theta_{xx}dx+\frac{\alpha\beta\rho}{\xi\mu}\int_0^L \psi_x v_{tt}dx+\frac{\alpha\beta\rho}{\xi\mu}\int_0^L \psi_{xt}v_t dx.\end{aligned}$$

Integration by parts yields

$$\begin{aligned}\mathcal{K}_3'(t) = & -\frac{\alpha\beta}{\xi}\int_0^L \psi_x v_{xx}dx-\alpha\int_0^L \psi_x^2 dx-\int_0^L\left(\frac{\beta}{\sqrt{\xi}}v_x+\sqrt{\xi}\psi\right)^2 dx \\ & -\frac{\delta}{\sqrt{\xi}}\int_0^L \theta_{xt}\left(\frac{\beta}{\sqrt{\xi}}v_x+\sqrt{\xi}\psi\right)dx-\frac{J\beta}{\xi}\int_0^L v_{xt}^2 dx+\left(\frac{Jc}{\delta}+\frac{c\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L v_{tt}\theta_t dx \\ & -\left(\frac{J\tau}{\delta}+\frac{\tau\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L v_{xt}\theta_{xt}dx-\left(\frac{J\kappa}{\delta}+\frac{\kappa\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L v_{xt}\theta_x dx+\frac{\alpha\beta\rho}{\xi\mu}\int_0^L \psi_x v_{tt}dx.\end{aligned}$$

From the first equation in (5.4), we have $v_{xx} = \frac{\rho}{\mu} v_{tt} - \frac{\beta}{\mu}\psi_x$. So, we infer

$$\begin{aligned}
\mathcal{K}_3'(t) =& -\frac{\alpha}{\xi\mu}\left(\xi\mu - \beta^2\right)\int_0^L \psi_x^2 dx - \int_0^L \left(\frac{\beta}{\sqrt{\xi}}v_x + \sqrt{\xi}\psi\right)^2 dx - \frac{\delta}{\sqrt{\xi}}\int_0^L \theta_{xt}\left(\frac{\beta}{\sqrt{\xi}}v_x + \sqrt{\xi}\psi\right) dx \\
& - \frac{J\beta}{\xi}\int_0^L v_{xt}^2 dx - \left(\frac{J\kappa}{\delta} + \frac{\kappa\alpha\beta\rho}{\delta\xi\mu}\right)\int_0^L v_{xt}\theta_x dx - \left(\frac{J\tau}{\delta} + \frac{\tau\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L v_{xt}\theta_{xt} dx \\
& + \left(\frac{Jc}{\delta} + \frac{c\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L v_{tt}\theta_t dx. \qquad (5.32)
\end{aligned}$$

Now, by using Young's inequality, we arrive at

$$-\frac{\delta}{\sqrt{\xi}}\int_0^L \theta_{xt}\left(\frac{\beta}{\sqrt{\xi}}v_x + \sqrt{\xi}\psi\right) dx \le \frac{1}{2}\int_0^L \left(\frac{\beta}{\sqrt{\xi}}v_x + \sqrt{\xi}\psi\right) + \frac{\delta^2}{2\xi}\int_0^L \theta_{xt}^2 dx, \qquad (5.33)$$

$$-\left(\frac{J\kappa}{\delta} + \frac{\kappa\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L v_{xt}\theta_x dx \le \frac{J\beta}{4\xi}\int_0^L v_{xt}^2 dx + \frac{\xi}{J\beta}\left(\frac{J\kappa}{\delta} + \frac{\kappa\alpha\beta\rho}{\delta\xi\mu}\right)^2\int_0^L \theta_x^2 dx, \quad (5.34)$$

$$-\left(\frac{J\tau}{\delta} + \frac{\tau\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L v_{xt}\theta_{xt} dx \le \frac{J\beta}{4\xi}\int_0^L v_{xt}^2 dx + \frac{\xi}{J\beta}\left(\frac{J\tau}{\delta} + \frac{\tau\alpha\beta\rho}{\xi\mu\delta}\right)^2\int_0^L \theta_{xt}^2 dx, \qquad (5.35)$$

$$\left(\frac{Jc}{\delta} + \frac{c\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L v_{tt}\theta_t dx \le \varepsilon\int_0^L v_{tt}^2 dx + \frac{1}{4\varepsilon}\left(\frac{Jc}{\delta} + \frac{c\alpha\beta\rho}{\xi\mu\delta}\right)^2\int_0^L \theta_{xt}^2 dx. \qquad (5.36)$$

Then, by substituting equations (5.33)-(5.36) into (5.32), we obtain equation (5.31), for $C_4 = \frac{\delta^2}{2\xi} + \frac{\xi}{J\beta}\left(\frac{J\tau}{\delta} + \frac{\tau\alpha\beta\rho}{\xi\mu\delta}\right)^2 + \frac{1}{4\varepsilon}\left(\frac{Jc}{\delta} + \frac{c\alpha\beta\rho}{\xi\mu\delta}\right)^2$, and $C_5 = \frac{\xi}{J\beta}\left(\frac{J\kappa}{\delta} + \frac{\kappa\alpha\beta\rho}{\delta\xi\mu}\right)^2$. □

Remark 5.1. *From the first equation in (5.4), we have*

$$\rho v_{tt} = \left(\mu v_x + \beta\psi\right)_x. \qquad (5.37)$$

Using equation (5.37) and integration by parts, we get

$$\begin{aligned}
\frac{\rho}{\beta}\int_0^L v_{tt}\left(\beta v + \xi\int_0^x \psi(y)dy\right) dx =& -\int_0^L \left(\frac{\mu\sqrt{\xi}}{\beta}v_x + \sqrt{\xi}\psi\right)\left(\frac{\beta}{\sqrt{\xi}}v_x + \sqrt{\xi}\psi\right) dx \\
=& -\int_0^L \left(\frac{\mu\sqrt{\xi}}{\beta}v_x - \frac{\beta}{\sqrt{\xi}}v_x + \left(\frac{\beta}{\sqrt{\xi}}v_x + \sqrt{\xi}\psi\right)\right)\left(\frac{\beta}{\sqrt{\xi}}v_x + \sqrt{\xi}\psi\right) dx \\
=& -\int_0^L \left(\frac{\xi\mu - \beta^2}{\beta\sqrt{\xi}}\right)\left(\frac{\beta}{\sqrt{\xi}}v_x + \sqrt{\xi}\psi\right) v_x dx - \int_0^L \left(\frac{\beta}{\sqrt{\xi}}v_x + \sqrt{\xi}\psi\right)^2 dx.
\end{aligned}$$

Lemma 5.5. *The functional given by*

$$\mathcal{K}_4(t) = \frac{\alpha\rho}{\beta}\int_0^L v_t\left(\psi_x + \frac{c}{\delta}\theta_{tt}\right) dx,$$

satisfies, along the solution of problem (5.4), and for any positive constant ε, the estimate

$$\begin{aligned}\mathcal{K}_4'(t) \leq &-\frac{J\rho}{2\beta}\int_0^L v_{tt}^2dx - \frac{1}{2}\int_0^L\left(\frac{\beta}{\sqrt{\xi}}v_x + \sqrt{\xi}\psi\right)^2dx\\ &+\varepsilon_1\int_0^L v_{xt}^2dx + \frac{C_6}{\varepsilon_1}\int_0^L\theta_x^2dx + C_7\int_0^L\theta_{xt}^2 + C_8\int_0^L v_x^2dx. \end{aligned} \tag{5.38}$$

Proof. From the second equation in (5.4) and the boundary condition, we have

$$\alpha\psi_x = -Jv_{tt} + \beta v + \xi\int_0^x\psi(y)dy + \delta\theta_t. \tag{5.39}$$

Differentiating $\mathcal{K}_4(t)$ and using equation (5.39) along with the third equation of (5.4), we obtain

$$\begin{aligned}\mathcal{K}_4'(t) =&\frac{\alpha\rho}{\beta}\int_0^L v_{tt}\left(\psi_x + \frac{c}{\delta}\theta_t\right) + \frac{\alpha\rho}{\beta}\int_0^L v_t\left(\psi_{xt} + \frac{c}{\delta}\theta_{tt}\right)\\ =&-\frac{J\rho}{\beta}\int_0^L v_{tt}^2dx + \frac{\rho}{\beta}\int_0^L v_{tt}\left(\beta v + \xi\int_0^x\psi(y)dy\right)dx + \frac{\rho}{\beta}\left(\delta + \frac{c\alpha}{\delta}\right)\int_0^L v_{tt}\theta_tdx\\ &+\frac{\alpha\rho\kappa}{\delta\beta}\int_0^L v_t\theta_{xx}dx + \frac{\alpha\rho\tau}{\delta\beta}\int_0^L v_t\theta_{xxt}dx.\end{aligned}$$

Applying integration by parts, as noted in Remark 5.1, we arrive at

$$\begin{aligned}\mathcal{K}_4'(t) =& -\frac{J\rho}{\beta}\int_0^L v_{tt}^2dx - \left(\frac{\xi\mu-\beta^2}{\beta\sqrt{\xi}}\right)\int_0^L\left(\frac{\beta}{\sqrt{\xi}}v_x + \sqrt{\xi}\psi\right)v_xdx - \int_0^L\left(\frac{\beta}{\sqrt{\xi}}v_x + \sqrt{\xi}\psi\right)^2dx\\ &+\frac{\rho}{\beta}\left(\delta + \frac{c\alpha}{\delta}\right)\int_0^L v_{tt}\theta_tdx - \frac{\alpha\rho\kappa}{\delta\beta}\int_0^L v_{xt}\theta_xdx - \frac{\alpha\rho\tau}{\delta\beta}\int_0^L v_{xt}\theta_{xt}dx.\end{aligned} \tag{5.40}$$

Now, using Young's and Poincaré's inequalities, we have, for any $\varepsilon > 0$:

$$-\left(\frac{\xi\mu-\beta^2}{\beta\sqrt{\xi}}\right)\int_0^L\left(\frac{\beta}{\sqrt{\xi}}v_x + \sqrt{\xi}\psi\right)v_xdx \leq \frac{1}{2}\int_0^L\left(\frac{\beta}{\sqrt{\xi}}v_x + \sqrt{\xi}\psi\right)^2 + \frac{1}{2}\left(\frac{\xi\mu-\beta^2}{\beta\sqrt{\xi}}\right)^2\int_0^L v_x^2. \tag{5.41}$$

$$\begin{aligned}\frac{\rho}{\beta}\left(\delta + \frac{c\alpha}{\delta}\right)\int_0^t v_{tt}\theta_tdx &\leq \frac{J\rho}{2\beta}\int_0^L v_{tt}^2 + \frac{\beta}{2J\rho}\left(\frac{\rho\delta}{\beta} + \frac{c\alpha\rho}{\delta\beta}\right)^2\int_0^L\theta_t^2\\ &\leq \frac{J\rho}{2\beta}\int_0^L v_{tt}^2 + \frac{\beta}{2J\rho}\left(\frac{\rho\delta}{\beta} + \frac{c\alpha\rho}{\delta\beta}\right)^2\int_0^L\theta_{xt}^2.\end{aligned} \tag{5.42}$$

$$-\frac{\alpha\rho\kappa}{\delta\beta}\int_0^t v_{xt}\theta_xdx \leq \frac{\varepsilon_1}{2}\int_0^L v_{xt}^2dx + \frac{\alpha^2\rho^2\kappa^2}{2\delta^2\beta^2\varepsilon_1}\int_0^L\theta_x^2dx. \tag{5.43}$$

$$-\frac{\alpha\rho\tau}{\delta\beta}\int_0^t v_{xt}\theta_{xt}dx \leq \frac{\varepsilon_1}{2}\int_0^L v_{xt}^2dx + \frac{\alpha^2\rho^2\tau^2}{2\delta^2\beta^2\varepsilon_1}\int_0^L\theta_{xt}^2dx. \tag{5.44}$$

Setting $C_6 = \frac{\alpha^2\rho^2\kappa^2}{2\delta^2\beta^2}$, $C_7 = \frac{\beta}{2J\rho}\left(\frac{\rho\delta}{\beta} + \frac{c\alpha\rho}{\delta\beta}\right)^2 + \frac{\alpha^2\rho^2\tau^2}{2\delta^2\beta^2\varepsilon_1}$, $C_8 = \frac{1}{2}\left(\frac{\xi\mu-\beta^2}{\beta\sqrt{\xi}}\right)^2$ and substituting equations (5.41)–(5.44) into equation (5.40), we obtain the estimate (5.38). □

At this stage, we define the Lyapunov functional $\mathcal{L}$ as follows:

$$\mathcal{L}(t) := N_1\mathcal{E}(t) + N_2\mathcal{K}_1(t) + N_3\mathcal{K}_2(t) + N_3\mathcal{K}_4(t) + \mathcal{K}_4(t), \tag{5.45}$$

where $N_i (i = 1, 2, 3, 4)$ are positive constants to be determined later, and $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ and $\mathcal{K}_4$ are given in Lemmas 5.2-5.5, respectively.

Lemma 5.6. *For sufficiently large N_1, there exist positive constants m_1 and m_2 such that*

$$m_1\mathcal{E}(t) \leq \mathcal{L}(t) \leq m_2\mathcal{E}(t). \tag{5.46}$$

Proof. A simple calculation leads to

$$\begin{aligned}
\Big|\mathcal{L}(t) - N_1\mathcal{E}(t)\Big| \leq & c\int_0^L \Big|\theta_t\theta\Big| dx + \frac{\tau}{2}\int_0^L \Big|\theta_x^2\Big| dx + \delta\int_0^L \Big|\psi_x\theta\Big| dx + \rho N_2\int_0^L \Big|v_t v\Big| dx \\
& + JN_2\int_0^L \Big|v_{xt}\psi\Big| dx + \frac{JcN_2}{\delta}\int_0^L \Big|v_t\theta_t\Big| dx + \frac{JN_3}{\sqrt{\xi}}\int_0^L \Big|v_{xt}\left(\frac{\beta}{\sqrt{\xi}}v_x + \sqrt{\xi}\psi\right)\Big| dx \\
& + N_3\left(\frac{Jc}{\delta} + \frac{c\alpha\beta\rho}{\xi\mu\delta}\right)\int_0^L \Big|v_t\theta_t\Big| dx + \frac{N_3\alpha\beta\rho}{\xi\mu}\int_0^L \Big|\psi_x v_t\Big| dx \\
& + \frac{\alpha\rho N_4}{\beta}\int_0^L \Big|v_t\left(\psi_x + \frac{c}{\delta}\theta_{tt}\right)\Big| dx.
\end{aligned}$$

Applying Young's inequality, we deduce that there exists a positive constant χ such that

$$\begin{aligned}
\Big|\mathcal{L}(t) - N_1\mathcal{E}(t)\Big| \leq & \chi\left(\int_0^L v_t^2 dx + \int_0^L v_x^2 dx + \int_0^L \psi^2 dx + \int_0^L \psi_x^2 dx + \int_0^L \theta_t^2 dx\right) \\
& + \chi\left(\int_0^L \theta_x^2 dx + \int_0^L v_{tt}^2 dx + \int_0^L v_{xt}^2 dx + \int_0^L \left(\frac{\beta}{\sqrt{\xi}}v_x + \sqrt{\xi}\psi\right)^2 dx\right) \leq \chi E(t).
\end{aligned}$$

So

$$\begin{aligned}
-\chi\mathcal{E}(t) \leq \mathcal{L}(t) - N_1\mathcal{E}(t) \leq \chi\mathcal{E}(t), \\
N_1\mathcal{E}(t) - \chi\mathcal{E}(t) \leq \mathcal{L}(t) \leq N_1\mathcal{E}(t) + \chi\mathcal{E}(t).
\end{aligned}$$

Consequently,

$$(N_1 - \chi)\mathcal{E}(t) \leq \mathcal{L}(t) \leq (N_1 + \chi)\mathcal{E}(t), \ \forall t > 0.$$

It suffices to choose N_1 large enough so that $N_1 - \chi > 0$ and then set $m_1 = N_1 - \chi$ and $m_2 = N_1 + \chi$. □

Theorem 5.3. *Let (v, ψ, θ) be the solution of the system (5.4)-(5.6), then there exist positive constants A and ω such that the energy $\mathcal{E}(t)$ associated to the solution satisfies*

$$\mathcal{E}(t) \leq A\mathcal{E}(0)e^{-\omega t}, \text{ for all } t \geq 0. \tag{5.47}$$

Proof of Theorem 5.3

By differentiating $\mathcal{L}(t)$, substituting equations (5.22), (5.25),(5.31) and (5.38) into (5.45), and applying Poincaré's inequality, we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\frac{J\rho}{2\beta} - \varepsilon\left(N_3 + N_4\right)\right]\int_0^L v_{tt}^2 dx - \left[\frac{J\beta}{4\xi}N_4 - \varepsilon_1\left(N_3+1\right)\right]\int_0^L v_{xt}^2 dx \\
& - \left[\frac{J\beta}{4\xi c_p}N_4 - \rho N_3\right]\int_0^L v_t^2 dx - \left[\left(\mu - \frac{\beta^2}{\xi}\right)N_3 - C_8\right]\int_0^L v_x^2 dx \\
& - \left[\frac{\alpha}{\xi\mu}(\xi\mu - \beta^2)N_4 + \frac{\alpha}{2}N_3 - \varepsilon N_2\right]\int_0^L \psi_x^2 dx - \left[N_3 + \frac{1}{2}N_4 + \frac{1}{2}\right]\int_0^L \left(\frac{\beta}{\sqrt{\xi}}v_x + \sqrt{\xi}\psi\right)^2 dx \\
& - \left[\frac{\tau}{2c_p}N_1 - cN_2\right]\int_0^L \theta_t^2 dx - \left[\kappa N_2 - \frac{C_2}{\varepsilon_1}N_3 - C_5N_4 - \frac{C_6}{\varepsilon_1}\right]\int_0^L \theta_x^2 dx \\
& - \left[\frac{\tau}{2}N_1 - C_1N_2 - C_3N_3 - C_4N_4 - C_7\right]\int_0^L \theta_{xt} dx.
\end{aligned}$$

First, we put $\varepsilon_1 = 1$, then we select $N_3 > \dfrac{C_8}{\left(\mu - \dfrac{\beta^2}{\xi}\right)}$.

Subsequently, we choose $N_4 \geq \max\left\{\dfrac{4\xi c_p \rho}{J\beta}N_3, \dfrac{4\xi}{J\beta}(N_3+1)\right\}$.

Furthermore, we pick N_2 large such that

$$[\kappa N_2 - C_2N_3 - C_5N_4 - C_6] > 0$$

Having completed the above steps, we can now proceed to make a choice ε positive, such that

$$\varepsilon \leq \min\left\{\frac{J\rho}{4\beta(N_3+N_4)}, \frac{\alpha(\xi\mu - \beta^2)N_4}{\xi\mu N_2}\right\}.$$

Finally, we pick N_1 large such that

$$\left[\frac{\tau}{2}N_1 - C_1N_2 - C_3N_3 - C_4N_4 - C_7\right] > 0, \quad \left[\frac{\tau}{2c_p}N_1 - cN_2\right] > 0 \text{ and } N > \chi.$$

Therefore, $\mathcal{L} \sim \mathcal{E}$, and if we define the constants $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7, \zeta_8, \zeta_9 > 0$, as

$$\begin{aligned}
\zeta_1 &= \frac{J\rho}{2\beta} N_4 - \varepsilon(N_2 + N_3),\\
\zeta_2 &= \frac{J\beta}{4\xi} N_3 - \varepsilon_1(N_2 + N_4),\\
\zeta_3 &= \frac{J\beta}{4\xi c_p} N_3 - \rho N_2,\\
\zeta_4 &= \left(\mu - \frac{\beta^2}{\xi}\right) N_2 - C_8 N_4,\\
\zeta_5 &= \frac{\alpha}{\xi\mu}(\xi\mu - \beta^2) N_3 + \frac{\alpha}{2} N_2 - \varepsilon,\\
\zeta_6 &= N_2 + \frac{1}{2} N_3 + \frac{1}{2} N_4,\\
\zeta_7 &= \frac{\tau}{2c_p} N_1 - c,\\
\zeta_8 &= \kappa - \frac{C_2}{\varepsilon_1} N_2 - C_5 N_3 - \frac{C_6}{\varepsilon_1} N_4,\\
\zeta_9 &= \frac{\tau}{2} N_1 - C_1 - C_3 N_2 - C_4 N_3 - C_7 N_4,
\end{aligned}$$

we get

$$\begin{aligned}
\mathcal{L}'(t) \leq &- \zeta_1 \int_0^L v_{tt}^2 dx - \zeta_2 \int_0^L v_{xt}^2 dx - \zeta_3 \int_0^L v_t^2 dx - \zeta_4 \int_0^L u_x^2 dx - \zeta_5 \int_0^L \psi_x^2 dx\\
&- \zeta_6 \int_0^L \left(\frac{\beta}{\sqrt{\xi}} v_x + \sqrt{\xi}\psi\right)^2 dx - \zeta_7 \int_0^L \theta_t^2 dx - \zeta_8 \int_0^L \theta_x^2 dx - \zeta_9 \int_0^L \theta_{xt}^2 dx. \qquad (5.48)
\end{aligned}$$

By taking $\eta = \min\{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7, \zeta_8, \zeta_9\}$, we get

$$\mathcal{L}'(t) \leq -\eta \mathcal{E}(t), \ \ \forall t \geq 0. \qquad (5.49)$$

Moreover, from Lemma 5.6, we have $\mathcal{L}(t) \sim \mathcal{E}(t)$. It follows that equation (5.49) becomes

$$\mathcal{L}'(t) \leq -\omega \mathcal{L}(t) \quad \Longleftrightarrow \quad \mathcal{L}(t) \leq \mathcal{L}(0) e^{-\omega t} \quad \forall t \geq 0.$$

Therefore, the desired result follows from the equivalence between $\mathcal{L}(t)$ and $\mathcal{E}(t)$. This completes the proof of Theorem 5.3.

Conclusion

In this thesis, we conducted a systematic study of the long-time behavior of several truncated porous thermoelastic systems. A key feature of these systems is that they are free from the adverse effects associated with the second spectrum of frequencies, which often complicates the analysis of classical porous thermoelastic models. By focusing on truncated formulations, we rigorously analyzed the stability properties of the resulting dynamics. Specifically, we examined four distinct problems, each characterized by a different heat conduction law. For each system, we proved the exponential stability of the corresponding solutions.

The first problem addressed in this thesis concerns heat conduction governed by the classical Fourier law. In this context, the coupling between the porous structure and the thermal field results in a hyperbolic-parabolic system. The parabolic nature of the heat equation provides a strong dissipative mechanism, which facilitates the stability analysis. Despite the hyperbolic character of the elastic component, the overall system has been shown to be exponentially stable.

In the second problem, the classical Fourier law was replaced by the Cattaneo law of heat conduction. This law introduces a relaxation time for thermal disturbances, resulting in a phenomenon known as second-sound heat conduction. Unlike the Fourier case, both the mechanical and thermal equations in this model are hyperbolic. This characteristic adds complexity, as the dissipative effect is weaker. Nevertheless, by carefully constructing appropriate Lyapunov functionals, we were able to establish exponential stability for this system as well.

The third problem addresses a more general scenario, specifically hereditary heat conduction governed by the Gurtin–Pipkin law. This constitutive relation accounts for memory effects in the thermal response, rendering the system nonlocal in time. The presence of memory introduces additional mathematical challenges, particularly in the analysis of well-posedness. However, by employing semigroup theory and imposing suitable assumptions on the memory kernel, we have proven that the system is well-posed and that its solutions decay exponentially to equilibrium.

The fourth and final problem investigated in this thesis concerns a porous thermoelastic system in which heat conduction follows the type III model introduced by Green and Naghdi. This model interpolates between the Fourier and Cattaneo laws, incorporating both instantaneous and second-sound effects. The resulting system exhibits a combination of parabolic and hyperbolic characteristics. Using a combination of energy estimates and multiplier techniques, we demonstrated that exponential stability holds under appropriate conditions on the physical parameters.

For all four problems, the well-posedness of the systems was rigorously established. To achieve this, we employed a unified semigroup approach, which allowed us to reformulate each system as an abstract Cauchy problem on an appropriate Hilbert space. The Hille–Yosida theorem was then applied to verify that the associated semigroup is strongly continuous, thereby guaranteeing the existence and uniqueness of solutions. In certain cases, particularly for the hereditary and type III models, the Lax–Milgram theorem was also invoked to address the variational formulation of the elliptic components.

Regarding stability, we systematically applied the multiplier method in combination with Lyapunov functionals. The multiplier method proved particularly effective for deriving the necessary energy estimates, while the construction of appropriate Lyapunov functionals enabled us to establish the exponential decay of the total energy. For each problem, a suitable Lyapunov functional was carefully designed to account for the specific dissipative mechanisms inherent in the corresponding heat conduction law. The exponential stability results obtained are uniform with respect to the initial data and provide precise decay rates.

In summary, this thesis advances the mathematical theory of porous thermoelasticity by demonstrating that, under the considered truncation, exponential stability is a robust property that holds for a wide range of heat conduction models, including Fourier, Cattaneo, Gurtin–Pipkin, and Green–Naghdi type III. These results not only extend existing results in the field but also open the horizon for further investigations, such as the study of optimal decay rates, numerical simulations, and extensions to nonlinear or multidimensional settings.

Bibliography

[1] R.A. Adams, J.J. Fournier, Sobolev Spaces, Elsevier, 2^{nd} edition, 2003.

[2] D.S. Almeida Júniora, A.J.A. Ramos, , On the nature of dissipative Timoshenko systems at light of the second spectrum. Zeitschrift fúr angewandte Mathematik und Physik. 68 (2017) 145, 31.

[3] M. Aouadi, On uniform decay of a nonsimple thermoelastic bar with memory, J. Math. Anal. App. 402, 745-757 (2013).

[4] T.A. Apalara, Exponential decay in one dimensional porous dissipation elasticity, Quart. J. Mech. Appl. Math., 70 (4), 363-372 (2017).

[5] T.A. Apalara, On the stability of porous-elastic system with microtemperatures, J.Thermal Stresses, 42 (2019), 265-278.

[6] T.A. Apalara, C.A. Raposo, A. Ige, Thermoelastic Timoshenko system free of second spectrum, Applied Mathematics Letters 126,107793 (2022).

[7] N. Belkhir, A. Fareh, On the exponential stability of a truncated porous elastic system with second sound thermoelasticity, J. Thermal Stresses, (2025) https://doi.org/10.1080/01495739.2025.2558182.

[8] N. Belkhir, A. Fareh, Well posedness and energy estimate of a truncated porous thermoelastic system with a memory thermal law, Applicable Analysis, (2025). https://doi.org/10.1080/00036811.2025.2545614.

[9] B. A. Boley, J. H. Weiner, Theory of Thermal Stresses. John Wiley & Sons (1960).

[10] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universitext, Springer, New York, 2011.

[11] P. S. Casas, R. Quintanilla, Exponential decay in one-dimensional porous-thermoelasticity, Mech. Res. Commun., 32 (2005), 652-658

[12] P. S. Casas, R. Quintanilla, Exponential stability in thermoelasticity with microtemperatures, Inter. J. Engin. Scien. 43 (2005), 33-47.

[13] C. Cattaneo, A form of heat conduction equation which eliminates the paradox of instantaneous propagation. C. R. Acad. Sci. Paris, 247, 431–433 (1958).

[14] S.C. Cowin, J.W. Nunziato, Linear elastic materials with voids, J. Elasticity, 13. 125-147 (1983).

[15] J. L. Davidson, et al, Biothermoelasticity: The thermal response of biological tissues. Journal of Biomechanics, 33(10), 1237–1248 (2000).

[16] F. Dell'Oro, V. Pata, On the stability of Timoshenko systems with Gurtin-Pipkin thermal law, J. Diff. Equa. **257** (2014), 523–548 .

[17] A. Fareh, S.A. Messaoudi, Energy decay for a porous thermoelastic system with thermoelasticity of second sound and with a non-necessary positive definite energy, Appl. Math. Comput., 293 (2017), pp. $\sim$ 493-507.

[18] A. Fareh, Exponential stability of a porous thermoelastic system with Gurtin-Pipkin thermal law, RACSAM. **116** (2022), 6. https://doi.org/10.1007/s13398-021-01132-1.

[19] L.H. Fatori, J.E. Muñoz Rivera, Energy decay for hyperbolic thermoelastic systems of memory type, Quart. Appl. Math. **59** (2001), 441–458.

[20] B. Feng and T. A. Apalara, Optimal decay for a porous elasticity system with memory, J. Math. Anal. Appl., 470 (2019), 1108-1128.

[21] H. Li, B. Feng, Exponential and polynomial decay rates of a porous elastic system with thermal damping, J. Funct. Spaces, 2023.1 (2023): 3116936.

[22] J. B. J. Fourier, The Analytical Theory of Heat, Cambridge University Press, 1878 (English translation of original work, 1822).

[23] L. Gearhart, Spectral theory for contraction semigroups on Hilbert spaces, Trans. Amer. Math. Soc. 236, 385-394 (1978).

[24] C. Giorgi, V. Pata, and A. Marzocchi, Asymptotic behavior of a semilinear problem in heat conduction with memory. NoDEA Nonlinear Differential Equations Appl. 5 (1998), no. 3, 333-354 Zbl 0912.45009 MR 1638908.

[25] M.A. Goodman, S.C. Cowin, A continuum theory for granular materials. Archives for Rational Mechanics and Analysis. 44, 249-266 (1972).

[26] A.E. Green, P.M. Naghdi, A re-examination of the basic postulates of thermomechanics, *Proc. R. Soc. Lond. Ser. A Math. Phys. Sci.*, 432 (1991) 171–194. https://doi.org/10.1098/rspa.1991.0012.

[27] M.E. Gurtin, A.C. Pipkin, A general theory of heat conduction with finite wave speeds, Archive for Rational Mechanics and Analysis, vol. 31, no. 2, pp. 113–126, 1968.

[28] J. Hao, J. Yang, Exponential stability for porous thermoelastic system with Gurtin–Pipkin flux, Electronic Journal of Differential Equations, **2023** (2023), 44-17.

[29] D. Ieşan, A theory of thermoelastic materials with voids, Acta Meehanica, 60, 67–89 (1986). doi.org/10.1007/BF01302942.

[30] D. Ieşan, On a theory of micromorphic elastic solids with microtemperatures. J. Therm. Stress. 24, 737-752 (2001).

[31] D. Ieşan, Thermoelastic Models of Continua. Springer, Berlin, 2004.

[32] A. Keddi, S. A. Messaoudi, and M. Alahyane, On a thermoelastic Timoshenko system without the second spectrum: Existence and stability. Journal of Thermal Stresses 46 (2023), no. 8, 823?838.

[33] Z. Liu, S. Zheng, Semigroups Associated with Dissipative Systems. Chapman and Hall/CRC, Boca Raton, 1999.

[34] A. Magaña, R. Quintanilla, On the time decay of solutions in one-dimensional theories of porous materials, Int. J. Solids Struct., 43 (2006), 3414-3427.

[35] S.A. Messaoudi, A. Fareh, General decay for a porous-thermoelastic system with memory: The case of equal speeds, Nonlinear Analysis. TMA, 74 (2011), 6895-6906.

[36] S. A. Messaoudi and A. Fareh, General decay for a porous-thermoelastic system with memory: the case of nonequal speeds, Acta Math. Sci. Ser. B (Engl. Ed.), 33 (2013), 23-40.

[37] S.A. Messaoudi , A. Fareh, Exponential decay for linear damped porous thermoelastic systems with second sound, Disc. Cont. Dyn. Sys. Ser. 20 (2015) (2), pp.$\sim$ 599–612.

[38] S.A. Messaoudi, A. Keddi, On the well posedness and the stability of a thermoelastic Gurtin–Pipkin–Timoshenko system without the second spectrum, Z. Anal. Anwend. (2024), DOI 10.4171/ZAA/1759.

[39] J. Muñoz Rivera, R. Quintanilla, On the time polynomial decay in elastic solids with voids, J. Math. Anal. Appl., 338 (2008), 1296-1309.

[40] J.W. Nunziato, S.C. Cowin, A nonlinear theory of elastic materials with voids, Arch. Ration. Mech. Anal. 72, 175-201 (1979).

[41] V. Pata, E. Vuk, On the exponential stability of linear thermoelasticity, Contin. Mech. Thermodyn. **12** (2000), 121–130.

[42] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, New York, 1983.

[43] V. P. Peshkov, Second Sound in Helium II, Journal of Experimental and Theoretical Physics, vol. 10, 1944.

[44] R. Quintanilla, Slow decay for one-dimensional porous dissipation elasticity, Appl. Math. letters, 16, 487-491 (2003).

[45] A. J. A. Ramos, D.S. A. Júnior, M. M.Freitasa, M.J. Santos, A new exponential decay result for one-dimensional porous dissipation elasticity from second spectrum viewpoint, Applied Mathematics Letters(2019), doi: https://doi.org/10.1016/j.aml.2019.106061.

[46] M. Reed, B. Simon, Methods of Modern Mathematical Physics: Functional analysis: I, Gulf Professional Publishing, 1980.

[47] W. Rudin, Functional analysis 2nd ed. International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.

[48] M.L. Santos, A.D.S. Campelo, D.S.A. Júnior, Rates of decay for porous elastic system weakly dissipative. Acta Applicandae Mathematicae, 151, 1-26 (2017).

[49] M. L. Santos, A. D. S. Campelo, D. S. A. Júnior, On the Decay Rates of Porous Elastic Systems, J. Elasticity, 127 (2017): 79-101.

[50] R.W.M. Smith, Graphical representation of Timoshenko beam modes for clamped-clamped boundary conditions at high frequency: Beyond transverse deflection. Wave Motion. 45, 785-794 (2008).

[51] I I. Vrabie, C_0-semigroups and applications, Elsevier Science B.V., Amesterdam, 2003.

[52] H. Zougheib, T. El Arwadi, Energy decay analysis for porous elastic system with thermoelasticity of type III: a second spectrum approach. Results in Applied Mathematics, 21, (2024) 100435.