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**Specialty:** Mathematics

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# Global Existence and Stability in Porous-elasticity with Heat Conduction

**Presented by:** Abdelbaki CHOUCHA

In front of the jury:

| | | | |
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.


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Thank you all

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# Dedication

To my father, may Allah have mercy on him.

To my mather, may Allah prolong her life.

To my wife and children.

To all family and friends.

Abdelbaki Choucha

2022

# ملخص

الهدف من هذه الرسَالة هو إعطَاء بعض نتَائِج الوجود و الإنحلَال العَام، بَالإِضَافة إِلَى تزويد القَارِئ بَالطرق المستخدمة لدرَاسة بعض الأَنظمة المرنة، المطَاطية الحرَارية وَاللَزجة، وكذَلك بعض مشَاكل معَادلة الموجة مع وجود آليَات التبديد وَالتخميد المختلفة في ظل إفترَاضَات منَاسبة. لقد استخدمنَا عدة طرق منهَا: طريقة الطَاقة ودَالة ليَابونوف.

قسمنَا الرسَالة إِلَى ثلَاثة أجزَاء:

في الجزء الأَول، قمنَا بدرَاسة وجود الحلول لبعض الأَنظمة المسَامية وَالإِضمحلَال العَام لهَا.

أَمَّا بَالنسبة لِلجزء الثَاني، فقد قمنَا بتوسيع الدرَاسة لتشمل أَنظمة أُخرَى، وهي نظَام بريس و عَارضة مغلفة.

أَخيرًا الجزء الثَالث يختص بمشَاكل معَادلة الموجة بمَا في ذلك معَادلَات كيرشوف، نقوم بدرَاسة الإِنحلَال العَام وَالتفجير ونمو الحلول بآليَات التبديد وَالتخميد المختلفة في ظل ظروف منَاسبة.

**أَلكلمَات المفتَاحية:**

نظَام مسَامي؛ تورم نظَام مسَامي؛ إِستقرَار؛ تسوس عَام؛ أَلوضع الجيد؛ مصطلح التأَخير؛ مصطلح الذَاكرة؛ أَلتخميد الدقيق لدرجَات الحرَارة؛ أَلمرونة الحرَارية؛ لزوجة مطَاطية؛ نظَام تيموشينكو؛ نظَام بريس؛ عَارضة مغلفة؛ معَادلة الموجة؛ تفجير؛ أَلنمو الأَسي.



## Abstract

The aim of this thesis is to give some results of existence and general decay, as well as providing the reader with the methods used to study some elastic, thermoelastic and viscous systems, as well as some wave equation problems with the presence of various dissipation and damping mechanisms under suitable assumptions. We have used many methods, including: energy method and Lyapunov function.

We divided the thesis into three parts:

In the first part, we study the existence and general decay of solutions for some porous type systems.

As for the second part, we extend the study to other systems, namely Bresse system and Laminate beam.

Finally the third part is concerned with wave equation problems including Kirchhoff's equations, we study general decay, blow-up and growth of solutions with various dissipation and damping mechanisms under suitable conditions.



## Key-words

Porous system; Swelling porous system; Stability; General decay; Well-posedness; Delay term; Memory term; Microtemperature damping; Thermoelasticity; Viscoelasticity; Timoshenko system; Bresse system; Laminated beam; Wave equation; Blow-up; Exponential growth.

# Résumé

Le but de cette thése est de donner quelques résultats d'existence et de décroissance générale, ainsi que de fournir au lecteur les méthodes utilisées pour étudier certains systémes élastiques, thermoélastiques et visqueux, ainsi que quelques problémes d'équation d'onde avec la présence de divers mécanismes de dissipation et d'amortissement sous des hypothéses appropriées. Nous avons utilisé de nombreuses méthodes, notamment : la méthode de l'énergie et la fonction de Lyapunov.

Nous avons divisé la thése en trois parties:

Dans la premiére partie, nous étudions l'existence et la décroissance générale de solutions pour certains systémes de type poreux.

Quant à la deuxiéme partie, nous étendons l'étude á déautres systémes, á savoir le systéme Bresse et la poutre Laminate.

Enfin, la troisiéme partie concerne les problémes d'équation d'onde, y compris les équations de Kirchhoff, nous étudions la décroissance générale, l'explosion et la croissance de solutions avec divers mécanismes de dissipation et d'amortissement dans des conditions appropriées.



# Mots-clés

Système poreux; Système poreux gonflant; La stabilité; Délabrement générale; Bien posé; Terme de retard; Terme de mémoire; Amortissement à micro-température; Thermo-élasticité; Visco-élasticité; systm̀e Timochenko; Systm̀e Bresse; Poutre laminée; Équation d'onde; Exploser; Croissance exponentielle.

# Publications

1. **A. Choucha**, D. Ouchenane, and Kh. Zennir, General decay of solutions in one-dimensional porous-elastic with memory and distributed delay term, Tamkang Journal of Mathematics. Volume 52, Number 4, 479-495, (2021). DOI:10.5556/j.tkjm.52.2021.3519.

2. **A. Choucha**, D. Ouchenane, Kh. Zennir and B. Feng, Global well-posedness and exponential stability results of a class of Bresse-Timoshenko-type systems with distributed delay term. *Math Meth Appl Sci.* (2020) 1–26. DOI: 10.1002/mma.6437.

3. **A. Choucha**, D. Ouchenane and S. Boulaaras, Well Posedness and Stability result for a Thermoelastic Laminated Timoshenko Beam with distributed delay term, *Math Meth Appl Sci.* (2020) 1–22. DOI: 10.1002/mma.6673.

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# Contents

# Introduction

In this thesis, we consider a some hyperbolic systems with a different damping. The well-posedness is established, the general decay result , blow-up and growth of solutions is studied by the multiplier method.

We also tried to shed light on the various types of damping with the largest number of types of systemes, in order to provied the reader with a some what comprehensive study, and we hope that it will be a reference for researchers in the future.

At the outset, we give a brief overview of some of the systems that we are about to study.

### 0.0.1 Porous system

In 1972, Goodman and Cowin [66] introduced the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. This idea gave the relationship between the elasticity theory and the porous media theory and has attracted the attention of many researchers to produce and develop a lot of work in this area. For more details about porous-elastic theory, we refer the readers to the work of Cowin and Nunziato [48] in 1983, and [49] in 1985.

Firstly, we consider the following two basic evolution equations of the one-dimensional porous materials theory

$$\begin{aligned} \rho u_{tt} &= T_x, \\ J\phi_{tt} &= H_x + D. \end{aligned} \tag{0.0.1}$$

The variables $u$ is the displacement of the solid elastic material and $\phi$ is the volume fraction. $T, H$ and $D$ represent the stress tensor, the equilibrated stress vector, and the equilibrated body force, respectively. The parameter $\rho$ is the mass density and $J$ is the product of the equilibrated inertia by the mass density.

Quintanilla in [147] considered the system (0.0.1) with

$$T = \mu u_x + b\phi, \;\; H = \delta\phi_x, \;\; D = -bu_x - \xi\phi - \tau\phi_t, \tag{0.0.2}$$

where the coefficients $\mu, \delta, b, \xi, \tau$ are positive constants, with initial and mixed boundary conditions, and showed that the damping in the porous equation $(-\tau\phi_t)$ is not strong enough to obtain an exponential decay but only a slow (nonexponential) decay can be obtained. In [8] Apalara showed that the same system considered in [147] is exponentially stable for the case of equal speeds of wave propagation.
The viscoelastic damping (see [52] for details) is (according to the Boltzmann Principle) represented by a memory term in the form of a convolution which arises in the constitutive equation between the stress $\sigma(x,t)$ and the strain $\epsilon(x,t)$

$$\sigma(x,t) = \epsilon(x,t) + \int_0^t g(t-s)\,\epsilon(x,s)\,ds.$$

This type of viscoelastic dissipation (see [7] for details) could be said to coincide to viscosity with null initial history because it is assumed that the strains have been zero for $-\infty < t < 0$ or, equivalently, if any past strains have occurred sufciently long ago that the efect is trivial In other words, there will be a time prior to which all the strains which have previously occurred will have a trivial contribution. Thus, an experiment generally starts at some time $(t = 0)$ when the material is free of stresses. Ammar-Khodja et *al* in [4], considered a linear Timoshenko system with a viscoelastic damping term of the form

$$\int_0^t g(t-s)\,\phi_{xx}(x,s)\,ds. \tag{0.0.3}$$

In [7], T. Apalara considered the system (0.0.1) with

$$T = \mu u_x + b\phi, \;\; H = \delta\phi_x - \int_0^t g(t-s)\,\phi_x(x,s)\,ds, \;\; D = -bu_x - \xi\phi, \tag{0.0.4}$$

he proved that the unique dissipation given by the memory term is strong enough to exponentially stabilize the system, depending on the kernel of the memory term and the wave speeds of the system. Afer that, in [56], the authors considered the same system in [7], they extend the result to the case of non-equal wave speeds, which is more realistic from the physics point of view.
Mũnoz Rivera and Fernandez Sare [129], considered the same system in [4], but with a past history term (Infinite memory term) of the form

$$\int_0^\infty g(s)\,\phi_{xx}(x,t-s)\,ds, \tag{0.0.5}$$

instead of (0.0.3) and showed that the dissipation given by the history term is strong enough to stabilize the system exponentially if and only if the wave speeds are equal. They also proved that the solution decays polynomially for the case of different wave speeds. In [96], the authors suggested a one-dimensional porous-elastic system with infinite memory and a nonlinear damping term, given by the system (0.0.1) with

$$\begin{aligned} &T = \mu u_x + b\phi, \quad H = \delta\phi_x - \int_0^\infty g(s)\,\phi_x(x,t-s)\,ds, \\ &D = -bu_x - \xi\phi - \alpha(t) f(\phi_t), \end{aligned} \tag{0.0.6}$$

they established the well-posedness of the system using semigroup theory and showed the general decay for the case of nonequal speeds of wave propagation.

On the other hand, time delays arise in many applications because most phenomena naturally depend not only on the present state but also on some past occurrences. In recent years, the control of PDEs with time delay effects has become an active area of research. In many cases it was shown that delay is a source of instability unless additional condition or control terms are used, the stability problem of systems with delay is of theoretical and practical great importance. This term has several types: delay $(u_t(x,t-\tau))$, distributed delay $(\int_{\tau_1}^{\tau_2}\beta(s)u_t(t-s)ds)$ and time-varying delay $(u_t(x,t-\tau(t)))$ where $\beta$ is a $L^\infty$ function. See the papers ([132],[133],[134]).
In [95], the authors considered the system (0.0.1) with

$$T=\mu u_x+b\phi,\ \ H=\delta\phi_x,\ \ D=-bu_x-\xi\phi-\mu_1\phi_t-\int_{\tau_1}^{\tau_2}|\mu_2(s)|\phi_t(x,t-s)\,ds, \tag{0.0.7}$$

under suitable assumptions on the weight of distributed delay, they established the well-posedness of the system by using semigroup theory and they showed that the dissipation given by this complementary control stabilizes exponentially the system for the case of equal wave speeds of propagation.
In [28] A. Choucha et *al* considered the system (0.0.1) with

$$\begin{aligned} T&=\mu u_x+b\phi,\ \ H=\delta\phi_x-\int_0^t g(t-s)\,\phi_x(x,s)\,ds,\\ D&=-bu_x-\xi\phi-\mu_1\phi_t-\int_{\tau_1}^{\tau_2}|\mu_2(s)|\phi_t(x,t-s)\,ds, \end{aligned} \tag{0.0.8}$$

where they mixed the memory term and distributed delay term and under suitable suppositions on the weight of distributed delay and the kernel of memory, we established the exponential stability of the solution for the case of equal wave speeds of propagation.
Thermal effects were included in the work of Iesan [81]. In the one-dimensional case the evolution equations are:

$$\begin{aligned} \rho u_{tt} &= T_x,\\ J\phi_{tt} &= H_x+D,\\ d\eta_t &= q_x. \end{aligned} \tag{0.0.9}$$

Here $q,\eta$ are the heat flux vector and the entropy. The constitutive equations are

$$\begin{aligned} T=\mu u_x+b\phi-\gamma\theta \qquad & D=-bu_x-\xi\phi+m\theta,\\ H=\delta\phi_x \qquad & \rho\eta=\gamma u_x+c\theta+m\phi,\\ q=\kappa\theta_x, \qquad & \end{aligned} \tag{0.0.10}$$

where $\theta$ denote the temperature. In [21], the authors considered the couplying (0.0.9) and (0.0.10) with added the damping term $(-\tau\phi_t)$ on the functional $D$:

$$\begin{aligned} \rho u_{tt} &= \mu u_{xx} + b\phi_x - \gamma\theta_x, \\ J\phi_{tt} &= \delta\phi_{xx} - bu_x - \xi\phi + m\theta - \tau\phi_t, \\ c\theta_t &= \kappa\theta_{xx} - \gamma u_{tx} - m\phi_t, \end{aligned} \tag{0.0.11}$$

where

$$(x,t) \in (0,L) \times \mathbb{R}_+,$$

they proved that when both kinds of dissipation terms are taken into account in the evolution equations the solutions are exponentially stable. For more information and in depth in this type of problem, we refer the reader to the following papers ([108],[147]).

On the other hand, the basic evolution equations for one-dimensional theories of porous materials with temperature and microtemperature given by

$$\begin{aligned} \rho u_{tt} &= T_x, \\ J\phi_{tt} &= H_x + D, \\ \rho T_0 \eta_t &= q_x, \\ \rho E_t &= P_x + q - Q. \end{aligned} \tag{0.0.12}$$

Here $P$ is the first heat flux moment, $Q$ is the mean heat flux and $E$ is the first moment of energy and $T_0 > 0$ is the absolute temperature. The constitutive equations are

$$\begin{aligned} T = \mu u_x + b\phi - \gamma\theta \qquad & D = -bu_x - \xi\phi + m\theta, \\ H = \delta\phi_x - dw \qquad & \rho\eta = \gamma u_x + c\theta + m\phi, \\ q = \kappa\theta_x + k_1 w \qquad & P = -k_2 w_x, \\ \rho E = -aw - d\phi_x \qquad & Q = -k_3 w - k_1\theta_x, \end{aligned} \tag{0.0.13}$$

where $\theta, w$ denotes the temperature and the microtemperature vectors and $k_1, k_2, k_3$, $\mu, \delta, \xi, a, \kappa$ and $c$ are constitutive constants which are positive. As coupling is considered, $b$ must be different from zero and satisfies $\mu\xi > b^2$. The coefficients $\gamma, m$ and $d$ are constants that are not necessarily positive.

In [54], the authors considered the couplying (0.0.12) and (0.0.13) with added the damping term $(-\beta\phi_t)$ on the functional $D$, they studied the asymptotic behavior of solutions for the porous thermoelastic system with temperatures and microtemperatures effects, and they proved the exponential stability in case of zero thermal conductivity $(\kappa = 0)$ and without any condition on the coefficients of the system. For more inform ([9],[57],[84]-[86]).

Also, the model describing elastic solids with voids when viscoelasticity is present, and subjected to thermal effects. This theory established the first time by Cowin and Nunziato ([48], [49], [135]).
The linear theory of elastic materials with **voids** deals with small changes from a reference configuration of a porous body and it is distinguished from linear elasticity theory by the consideration of the void volume as an additional kinematic variable. In the one-dimensional case the evolution equations for the theory of elastic solids with voids are given by

$$\begin{aligned} \rho u_{tt} &= T_x \\ \rho\kappa\varphi_{tt} &= H_x + G \\ \rho T_0 \Xi_t &= q_x. \end{aligned} \tag{0.0.14}$$

Here, $T$ is the stress, $H$ is the equilibrated stress, $G$ is the equilibrated body force, $q$ is the heat flux and $T_0$ is the absolute temperature in the reference configuration which is assumed positive. The variables $u, \varphi$ and $\Xi$ are the displacement of the solid elastic material, the volume fraction and the entropy, respectively. We assumed that $\rho$ and $\kappa$ are positive constants whose physical meaning is well known. In general, we can considered several dissipation mechanisms in this theory (see [83]).
In [139], the authors restricted our attention to the case that the viscoelasticity is present and the viscosity at the microstructure is also present apart the temperature effect. They assumed the following constitutive equations (see [83])

$$\begin{aligned} T = \mu u_x + b\varphi - \beta\theta - \gamma u_{xt} \qquad & G = -b u_x - \xi\varphi + m\theta \\ H = \delta\varphi_x + \eta\varphi_{xt} + k_1\theta_x \qquad & \rho\Xi = \beta u_x + c\theta + m\varphi \\ q = k\theta_x + k_2\varphi_{xt} \qquad & . \end{aligned} \tag{0.0.15}$$

It is assumed that the internal mechanical energy density is a positive definite form. Thus, the constitutive coefficients satisfy the conditions

$$\mu > 0, \quad \delta > 0, \quad \mu\xi > b^2. \tag{0.0.16}$$

Thus, in particular when they assumed that $\eta$ or $k$ vanish then they also have $k_1 = k_2 = 0$. If introduce the constitutive equations in the evolution equations, they obtained the field equations

$$\begin{aligned} \rho u_{tt} &= \mu u_{xx} + b\varphi_x - \beta\theta_x - \gamma u_{xxt} \\ J\varphi_{tt} &= \delta\varphi_{xx} - b u_x - \xi\varphi + m\theta + \eta\varphi_{xxt} + k_1\theta_{xx} \\ c\theta_t &= k^*\theta_{xx} - \beta u_{xt} - m\varphi_t + k_2^*\varphi_{xxt}. \end{aligned} \tag{0.0.17}$$

Here $J = \rho\kappa$, $k^* = kT_0^{-1}$ and $k_2^* = k_2 T_0^{-1}$.
As coupling is considered, $b \neq 0$, but its sign does not matter in the analysis. As thermal effects is

considered, they assumed that the thermal capacity $c$ and the thermal conductivity $k$ are strictly positive. The sign of the coupling term $\beta$ does not matter in the analysis neither. And as viscoelastic dissipation is assumed in the system, $\gamma > 0$.

By assume that the porous dissipation is absent ($\eta = k_1 = k_2 = 0$). They studied the asymptotic behavior and the analyticity of the solutions of the problem (0.0.17) with thermal effect.

Recently, in [28] the authors considered the system (0.0.17) with added term of distributed delay

$$G = -bu_x - \xi\varphi + m\theta - \mu_1\varphi_t + \int_{\tau_1}^{\tau_2} |\mu_2(s)|\varphi_t(x,t-s)ds,$$

under suitable assumption they proved the exponential stability result. There are also many works from this type of model with many results with regard to the existence or stability , see ([53],[48],[152]). In the absence of temperature and in the presence of microtemperature the coupling (0.0.12) and (0.0.13) to get porous-elastic system with microtemperature, for example in [154] tha authors considered the following system

$$\begin{aligned} \rho u_{tt} &= \mu u_{xx} + b\phi_x, \\ J\phi_{tt} &= \delta\phi_{xx} - bu_x - \xi\phi - d\omega_x - \tau\phi_t, \\ \alpha\omega_t &= k_1\omega_{xx} - d\phi_{tx} - k_2\omega, \end{aligned} \tag{0.0.18}$$

they proved that the system (0.0.18) is exponentially stable if and only if $\frac{\mu}{\rho} = \frac{\delta}{J}$.

### 0.0.2 Swelling porous soils

Eringen was the first to present a theory in which a mixture of viscous liquid and solids mixed with gas [55]. Then, after studying this heat-resistant mixture, you get to the field equations [15].

Expansive (swelling) soils have also been classified under porous media theory which studies this type of problem. This is why this field is considered fertile for study, as there are many studies to reduce the damage caused by swelling soil, especially in civil engineering and architecture, for more depth see (see [18],[80],[82],[87],[91],[143]).

Where the basic field equations of the linear theory of swelling porous elastic soils were presented by

$$\begin{aligned} \rho_u u_{tt} &= P_{1x} + G_1 + H_1, \\ \rho_\phi \phi_{tt} &= P_{2x} + G_2 + H_2, \end{aligned} \tag{0.0.19}$$

where $u, \phi$ are the displacement of the fluid and the elastic solid material, $\rho_u, \rho_\phi > 0$ are the densities of each constituent. And $(P_1, G_1, H_1)$ are the partial tension, internal body forces, and eternal forces acting on the displacement, respectively. Similarly $(P_2, G_2, H_2)$ but acting on the elastic solid.

In addition, the constitutive equations of partial tensions are given by

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \underbrace{\begin{pmatrix} a_1, a_2 \\ a_2, a_3 \end{pmatrix}}_{A} \cdot \begin{pmatrix} u_x \\ \phi_x \end{pmatrix}, \tag{0.0.20}$$

where $a_1, a_3 > 0$ and $a_2 \neq 0$ is a real number. $A$ is matrix positive definite with $a_1 a_3 > a_2^2$.
Quintanilla [143] investigated (0.0.19) by taking

$$G_1 = G_2 = \xi(u_t - \phi_t), \quad H_1 = a_3 u_{xxt}, \quad H_2 = 0,$$

where $\xi > 0$, they obtained the stability is exponentially. Similarly, in [160] the authors are considered (0.0.19) with a different conditions

$$G_1 = G_2 = 0, \quad H_1 = -\rho_u \gamma(x) u_t, \quad H_2 = 0,$$

where $\gamma(x)$ is an internal viscous damping function with positive mean. By the spectral method they obtained the exponential stability result.
In [33] A. Choucha et *al* are interested in problem (0.0.19) with null internal body forces, but the eternal force acting only on the elastic solid is in the form of viscoelastic damping and distributed delay terms, that is

$$\begin{aligned} &G_1 = G_2 = H_1 = 0, \\ &H_2 = \int_0^t g(t-s)\phi_{xx}(x,s)\,ds - \beta_1 \phi_t - \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \phi_t(x, t-\sigma)\,d\sigma, \end{aligned}$$

they showed the exponential stability result of the system, for more detail of this type see ([10],[130], [143]-[146],[160]).
The basic evolution equations for one-dimensional theories of swelling porous materials with temperature and microtemperature ([54],[84]-[86]) given by

$$\begin{aligned} \rho_u u_{tt} &= T_x, \\ \rho_\varphi \phi_{tt} &= H_x + G, \\ \rho T_0 \eta_t &= q_x, \\ \rho E_t &= P_x^* + q - Q. \end{aligned} \tag{0.0.21}$$

Here $T, H, G, q, \eta, P^*, Q, E, T_0$ represents the stress, the equilibrated stress, the equilibrated body force, the heat flux vector, the entropy, the first heat flux moment, the mean heat flux, the first moment of energy and the reference temperature at the equilibrium.

The constitutive equations are

$$\begin{aligned} T &= P_1 + G_1 + H_1 & P^* &= -k_2 w_x, \\ H &= P_2 + P_3 & \rho\eta &= \gamma u_x + c_0\theta + m\phi, \\ G &= G_2 + H_2 & Q &= -k_3 w - k_1\theta_x, \\ q &= \kappa\theta_x + k_1 w & \rho E &= -\alpha w - d\phi_x. \end{aligned} \tag{0.0.22}$$

where $w$ is the microtemperature vector and $k_1, k_2, k_3, \alpha, \kappa, c_0, \mu_1, \gamma, m, d > 0$.

As coupling is considered, $a_2 \neq 0$ and satisfies

$$a = a_3 - \frac{a_2^2}{a_1} > 0. \tag{0.0.23}$$

In [35], the authors supposed that the heat capacity $c_0 > 0$, and for more excitement in posing the problem, they supposed that $T_0 = 1$ for simplicity, the thermal conductivity is non-existent $\kappa = 0$ and they considered

$$\begin{aligned} &G_1 = G_2 = 0, \quad P_3 = -dw \\ &H_1 = -\gamma\theta, \\ &H_2 = m\theta - \mu_1\phi_t - \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|\phi_t(x, t-\sigma)\, d\sigma \end{aligned} \tag{0.0.24}$$

By substituting (0.0.22)-(0.0.24) into (0.0.21), they arrived at the following problem:

$$\begin{cases} \rho_u u_{tt} - a_1 u_{xx} - a_2\phi_{xx} - \gamma\theta_x = 0, \\ \rho_\phi\phi_{tt} - a_3\phi_{xx} - a_2 u_{xx} - dw_x + m\theta + \mu_1\phi_t + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|\phi_t(x, t-\sigma)\, d\sigma = 0, \\ c_0\theta_t = -\gamma u_{tx} - m\phi_t - k_1 w_x, \\ \alpha w_t = k_2 w_{xx} - k_3 w - k_1\theta_x - d\phi_{tx}, \end{cases}$$

where

$$(x, \sigma, t) \in \mathcal{H} = (0,1) \times (\tau_1, \tau_2) \times (0, \infty),$$

they established the well-posedness of the system and they proved the exponential stability result under appropriate suppositions.

### 0.0.3 Thermoelasticity of type III

The constitutive law of the classical heat conduction is given by the Fourier law:

$$q(x,t) = -\kappa\nabla\theta(x,t), \tag{0.0.25}$$

where $x$ stands for the Lagrangian coordinates material point, $t$ is the time, $q$ is the heat flux, $\theta$ is the difference of temperature, measured with respect to a reference temperature and $\kappa$ is the thermal conductivity of the material which is a thermodynamic state property. Equation (0.0.25) states that the temperature gradient at a material point $x$ and at time $t$ gives rise to the heat flux at the same point $x$ and at the same time $t$. Equation (0.0.25) together with the conservation law (assuming for simplicity that no heat sources are present)

$$\rho\theta_t + \varrho div q = 0, \tag{0.0.26}$$

yields the classical heat equation (of parabolic type)

$$\rho\theta_t - \kappa\varrho\Delta\theta = 0, \tag{0.0.27}$$

Unfortunately, equation (0.0.27) predicts that thermal signals propagate at infinite speed. Clearly, such behavior is physically unrealistic and indicates that Fourier's law does not provide, in certain cases, an adequate description of the heat conduction. This has forced researchers to look for new constitutive relations. Consequently, a number of modifications of the basic assumption on the relation between the heat flux and the temperature have been made, such as Cattaneo's law, Gurtin and Pipkin's theory, Jefrey's law, Green and Naghdi's theory and others. The common feature of these theories is that all of them lead to hyperbolic differential equations and model heat flow as thermal waves traveling at finite speed. See ([24],[89]) for more details.

The Cattaneo law

$$\tau_q q_t + q + \kappa\nabla\theta = 0, \ \ (\tau_q > 0, \quad \text{relatively small}) \tag{0.0.28}$$

was proposed by Cattaneo in his famous paper [22]. It is perhaps the most obvious, most widely accepted and simplest generalization of Fourier's law that gives rise to a finite speed of heat propagation. When the Fourier law (0.0.25) is replaced by the Cattaneo law (0.0.28) for the heat conduction, the equations of thermoelasticity become purely hyperbolic and predict a finite signal speed leading to the system with **second sound**. In [94] studied the following system

$$\begin{cases} \rho_1 u_{tt} - \mu u_{xx} - b\phi_x = 0 \\ \rho_2\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^\infty g(s)\phi_{xx}(t-s)ds + \gamma\theta_x \\ +\mu_1\phi_t + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\phi_t(x,t-\varrho)d\varrho + \alpha(t)f(\phi_t) = 0 \\ \rho_3\theta_t - \kappa q_x + \gamma\phi_{tx} = 0 \\ \rho_4 q_t + dq + \kappa\theta_x = 0, \end{cases} \tag{0.0.29}$$

where

$$(x,\varrho,t) \in (0,1)\times(\tau_1,\tau_2)\times(0,\infty),$$

they established the well posedness of the system, and they proved the stability results of the system for the cases of equal and nonequal speeds of wave propagation. For more details on this theory, we refer to ([117],[118],[126],[149],[150],[158]).

Green and Naghdi ([70],[71]) developed a model of thermoelasticity which includes temperature gradient and thermal displacement gradient among the constitutive variables and proposed a heat conduction law as

$$q(x,t) = -[\kappa \nabla \theta(x,t) + \kappa^* \nabla v], \tag{0.0.30}$$

where $v_t = \theta$ and $v$ is the thermal displacement gradient, $\kappa$ and $\kappa^*$ are two positive constants. Equation (0.0.30) together with the energy balance law (0.0.25) lead to the equation

$$\rho \theta_{tt} - \varrho \kappa \Delta \theta_t - \varrho \kappa^* \Delta \theta = 0, \tag{0.0.31}$$

which permits propagation of thermal waves at finite speed.
The coupling of equation (0.0.31) with the equations of elasticity has been an active area of research in the last two decades.
In [137], the authors cosidered the following system

$$\begin{array}{rcl} \rho_1 u_{tt} &=& \mu u_{xx} + b\phi_x, \\ \rho_2 \phi_{tt} &=& \delta \phi_{xx} - b u_x - \xi \phi - \beta \theta_x - \mu_1 \phi_t - \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(s)| \phi_t(x, t-s)\, ds, \\ \rho_3 \theta_{tt} &=& l\theta_{xx} - \gamma \phi_{ttx} + k\theta_{txx}, \end{array} \tag{0.0.32}$$

where

$$(x,s,t) \in (0,L) \times (\tau_1, \tau_2) \times \mathbb{R}_+,$$

they established the well posedness of the system, and by the energy method combined with Lyapunov functional, they discussed the stability of system for both cases of equal and nonequal speeds of wave propagation. See in this connection ([112],[125],[147],[151],[166]).

### 0.0.4 Bresse system

The Bresse system, which is also known as the circular arch problem is a model for planar, linear shearable beam with initial curvature involving couplings of longitudinal, vertical and shear motions [19]. The evolution equations describing a classical Bresse system are

$$\begin{array}{l} \rho_1 \varphi_{tt} - S_x - lQ = 0, \\ \rho_2 \psi_{tt} - M_x + S = 0, \\ \rho_1 \omega_{tt} - Q_x + lS = 0, \end{array} \tag{0.0.33}$$

where

$$(x,t) \in (0,L) \times \mathbb{R}_+.$$

Here $\psi = \psi(x,t)$ is the shear angle displacement, $\varphi = \varphi(x,t)$ is the vertical angle displacement, $w = w(x,t)$ is the longitudinal angle displacement and $L$ is the length of the beam. The coefficients

$l = R^{-1}, \rho_1 = \rho A, \rho_2 = \rho I$ are physical parameters, where $I, A, R, \rho$ represent the second moment of the cross-section, the cross sectional curvature, the radius of curvature and the material density respectively. The constitutive laws are given by

$$\begin{aligned} S &= k_1(\varphi_x + \psi + l\omega), \\ Q &= k_3(\omega_x - l\varphi), \\ M &= k_2\psi_x, \end{aligned} \tag{0.0.34}$$

where $S, Q$ and $M$ stand for the shear force, the axial force and the bending moment respectively. For $k_1 = \kappa GA, k_2 = EI, k_3 = EA$, where $G, E, \kappa$ is the shear modulus, the modulus of elasticity and the shear factor respectively and substituting (0.0.33) and (0.0.34), we obtain the classical Bresse system

$$\begin{aligned} &\rho_1\varphi_{tt} - k_1(\varphi_x + \psi + l\omega)x - k_3 l(\omega_x - l\varphi) = 0, \\ &\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + l\omega) = 0, \\ &\rho_1\omega_{tt} - k_3(\omega_x - l\varphi)_x + k_1 l(\varphi_x + \psi + l\omega) = 0. \end{aligned} \tag{0.0.35}$$

In [120], the authors coupling (0.0.33) and (0.0.34) with

$$M = k_2\psi_x - \int_0^t g(t-s)\psi_x(x,s)ds, \tag{0.0.36}$$

they studied the system with Dirichlet–Neumann–Neumann boundary conditions and proved that the solutionnenergy has a general decay rate if

$$\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \text{ and } k_1 = k_3.$$

Furthermore, they established a weaker stability result for strong solution provided

$$\frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2} \text{ and } k_1 = k_3.$$

After that, in [128] the authors considered the thermoelastic dissipation effect on the bending moment of a Bresse system, where the heat is given by Fourier's law, we have

$$\rho_3\theta_t + q_x + \gamma\psi_{xt} = 0, \tag{0.0.37}$$

where $\theta = \theta(x,t)$ is the temperature difference, $q$ represents the heat flux, $\rho_3$ and $\gamma$ are capacity and adhesive stiffness, respectively. Coupling (0.0.33) and ( 0.0.37), we arrive at the thermoelastic Bresse system

$$\begin{aligned} &\rho_1\varphi_{tt} - S_x - lQ = 0, \\ &\rho_2\psi_{tt} - M_x + S = 0, \\ &\rho_1\omega_{tt} - Q_x + lS = 0, \\ &\rho_3\theta_t + q_x + \gamma\psi_{xt} = 0, \end{aligned} \tag{0.0.38}$$

with

$$\begin{aligned} S &= k_1(\varphi_x+\psi+l\omega)-k_1\int_0^t g(t-s)(\varphi_x+\psi+l\omega)(x,s)ds,\\ Q &= k_3(\omega_x-l\varphi),\\ M &= k_2\psi_x-\gamma\theta,\ \ q=-\beta\theta_x, \end{aligned} \tag{0.0.39}$$

they showed that with weaker conditions on the relaxation function and physical parameters, the solution energy has general and optimal decay rates.

In [127], the authors considered a Bresse system with viscoelastic law acting on the vertical angle displacement and thermoelastic dissipation govern by Maxwell–Cattaneo's law acting on the shear angle displacement. The constitutive laws in this case are given by

$$\begin{aligned} S &= k_1(\varphi_x+\psi+l\omega)-k_1\int_0^t g(t-s)(\varphi_x+\psi+l\omega)(x,s)ds,\\ Q &= k_3(\omega_x-l\varphi),\\ M &= k_2\psi_x-\gamma\theta, \end{aligned} \tag{0.0.40}$$

and the evolution equations describing such model is

$$\begin{aligned} &\rho_1\varphi_{tt}-S_x-lQ=0,\\ &\rho_2\psi_{tt}-M_x+S=0,\\ &\rho_1\omega_{tt}-Q_x+lS=0,\\ &\rho_3\theta_t+q_x+\gamma\psi_{xt}=0,\\ &\tau q_t+\beta q+\theta_x=0. \end{aligned} \tag{0.0.41}$$

By assuming very general conditions on the memory term and physical parameters, they proved the stability of the system, and they showed that the solution energy has a general and optimal decay estimate from which the exponential and polynomial decay estimates are only particular cases.

Silva et al. In [156] considered the Bresse system in

$$\begin{aligned} &\rho_1\varphi_{tt}-S_x-lQ=-\gamma_1\varphi_t,\\ &\rho_2\psi_{tt}-M_x+S=-\gamma_2\psi_t,\\ &\rho_1\omega_{tt}-Q_x+lS=0, \end{aligned} \tag{0.0.42}$$

with two frictional dissipations an exponential stability of system has been proved under conditions of equal speeds of wave propagation.

In [161], the authors by adding the delay term and the past history term considered the following system

$$\begin{aligned}
&\rho_1\varphi_{tt} - S_x - lQ = -\mu_1\varphi_t - \mu_2\varphi_t(x,t-\tau),\\
&\rho_2\psi_{tt} - M_x + S = 0,\\
&\rho_1\omega_{tt} - Q_x + lS = 0,\\
&\rho_3\theta_t + q_x + \gamma\psi_{xt} = 0,\\
&\tau q_t + \beta q + \theta_x = 0,
\end{aligned} \tag{0.0.43}$$

with

$$\begin{aligned}
S &= k_1(\varphi_x + \psi + l\omega),\\
Q &= k_3(\omega_x - l\varphi),\\
M &= k_2\psi_x - \gamma\theta - \int_0^\infty g(s)\psi_x(x,t-s)ds,
\end{aligned} \tag{0.0.44}$$

they established the well-posedness of problem using the semigroup method. By using the energy method, and they discussed the stability of the system for two cases. An exponential stability result of system is obtained in the case where the propagation velocities are equal in equation of vertical displacement. Furthermore, a result of algebraic stability is obtained in the case of the different propagation velocities.(See [93],[106],[167] for details).

### 0.0.5 Laminated beam

Laminate beam was introduced for the first time by Hansen and Spies, see ([76],[77]). And this is due to the relevance of the research topic to the applicability of these materials in the industry. Whereas, they presented a mathematical model of two-layer beams with structural damping due to the inter-slip obtained by

$$\begin{cases}
\rho\omega_{tt} + G(\psi - \omega_x)_x = 0\\
I_\rho(3s_{tt} - \psi_{tt}) - G(\psi - \omega_x) - D(3s_{xx} - \psi_{xx}) = 0\\
I_\rho s_{tt} - G(\psi - \omega_x) + \frac{4}{3}\gamma s + \frac{4}{3}\beta s_t - Ds_{xx} = 0.
\end{cases} \tag{0.0.45}$$

There are some results related to laminated beam equations, that studies global existence and stability from the relevant system. And by adding appropriate damping effects such as frictional damping (boundaries) viscoelastic or internal damping. Whereas, if we add linear damping terms to two of the three equations system (0.0.45) is exponentially stable under the assumption (equal wave speeds) $(\rho/I_\rho) = (G/D)$. But if we add these conditions in all equations, the system decays exponentially without assuming equal wave speedss, see ( [3], [20], [25],[58], [61], [131]). As for thermoelastic laminated Timoshenko beam, there are a few results including the work of Liu et al [109] and Apalara [6]. In [109], the authors considered that mexing the sytem (0.0.45) and the temperature with past history in the second equation of the form

$$\int_0^\infty g(s)(3\omega-\psi)_{xx}(t-s)ds.$$

They established the global well posedness of solutions to the system and the stability of the system. If $\beta \neq 0$, they proved the exponential and polynomial stabilities depending on the behavior of the kernel function $g$ and. If $\beta = 0$, they established exponential stability in case of equal wave speeds assumption and exponential stability not exist in case of nonequal wave speeds assumption. Apalara [6] considered a laminated beam with second sound (mexing by the second equation) and established the global well posedness and proved exponential and polynomial stabilities depending on the parameter $\chi$. In [59] Feng considered the same system in [6] but the mexing by the third equation, he studied the global well posedness and stability of systems.

A. Choucha et *al*, in [30] introduced a distributed delay term into the system of the paper [59] make problem different from those considered, they established the well posedness and proved the stability results of the system for the cases of equal and nonequal speeds of wave propagation.

After that, in [31] Choucha et *al* considered the same system in [30] by adding the infinite memory (past history) in the third equation

$$\begin{cases} \rho\omega_{tt} + G(\psi-\omega_x)_x + \delta\theta_x = 0 \\ I_\rho(3s-\psi)_{tt} - D(3s-\psi)_{xx} - G(\psi-\omega_x) = 0 \\ I_\rho s_{tt} - Ds_{xx} + G(\psi-\omega_x) + \frac{4}{3}\gamma s + \int_0^\infty g(\sigma)s_{xx}(x,t-\sigma)d\sigma \\ \qquad + \frac{4}{3}\beta s_t + \frac{4}{3}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| s_t(x,t-\varrho)d\varrho = 0 \\ \rho_3\theta_t + q_x + \delta\omega_{tx} = 0 \\ \tau q_t + \alpha q + \theta_x = 0, \end{cases} \tag{0.0.46}$$

where

$$(x,\varrho,t) \in (0,1)\times(\tau_1,\tau_2)\times(0,\infty),$$

with the Neumann-Dirichlet boundary conditions, they established the well posedness and proved the stability results of the system for the cases of equal and nonequal speeds of wave propagation.

### 0.0.6 Wave equation

The nonlinear wave equations may come from the modelizaton continuous media such as vibrating strings, elasticity and fluid flows. For example the one dimensional wave equation

$$u_{tt} = u_{xx}, \tag{0.0.47}$$

was introduced and analyzed by d'Alembert as a model of the vibrating string. Also, two and three dimensional wave equations

$$u_{tt} = \Delta u, \tag{0.0.48}$$

was studied by Bernoulli as a model of acoustic waves.

The study of single wave equation with the presence of different mechanisms of dissipation, damping and for more general forms of nonlinearities has been extensively studied and results concerning existence, nonexistence and namelly asymptotic behavior of solutions have been established by several authors and many results appeared in the literature over the past decades. See ([26],[44],[47],[64],[99],[104],[105]).

### 0.0.7 Blow-up

Hyperbolic equations, like ordinary differential equations, may experience solutions that do not exist globally.

We say then that they "blow up" in finite time. The diagnosis of such an occurrence proceeds always from the same principle: one exhibits a scalar quantity that cannot exist globally- due for instance to the fact that this quantity satisfies an ordinary differential equation whose solution blows up in finite time. This issue can be addressed from a different angle: for a given wave equation, under what circumstances will the solution blows up in a finite time? If blow-up does occur in a finite time, one may further ask: What is the set of the blow-up points? What is the blow-up rate of the solution when time approaches the blow-up time? These problems, which are also important from the point of view of applications, have become one of interesting topics of research since the middle of the 1980s.

The interaction between the term damping and the source makes the problem more interesting. This position was first considered by Levine ([102]), ([103]).

Recalling some results regarding wave equations similar to the following form:

$$u_{tt} - \Delta u + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \;\; x \in \Omega, t > 0, \tag{0.0.49}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^N$ and $a$ and $b$ are positive constants, together with initial and boundary conditions of Dirichlet type has been studied by Levine and Todorova [101], they showed that the solution of (0.0.49), blows-up in finite time, and then has been extensively studied and results concerning existence, blow up and asymptotic behaviour of smooth as well as weak solutions have been established by several authors over the past three decades see ([63],[64],[78],[102],[113],[119],[153]) and references therein).

Viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation. It is well known that viscoelastic materials show the damping. These damping effects are modeled by two integro-differentials operators such, the first is viscoelastic problems with finite memory

$$\int_0^t g(t-s)\Delta u(s)ds, \tag{0.0.50}$$

from the mathematical point of view. The study on viscoelastic wave equations has attracted many researchers. For example, Dafermos ([50],[51]) discussed a one dimensional viscoelastic equation and proved the well-posedness of the problem and showed that the solutions decay asymptotically to zero. However, no rate of decay has been specified. Guo et *al.* studied Hadamard well posedness, blow-up of the solutions for the history value problem of a viscoelastic wave equation which features a fading memory term as well as a supercritical source term and a frictional damping term in ([67],[68]). There exist many related works concerning the blow-up phenomenon of solutions for viscoelastic wave equations, the interested readers can refer to ([114],[157]) and the reference therein.

The second type is viscoelastic problems with infinite memory

$$\int_0^{\infty} g(s)\Delta u(t-s)ds, \tag{0.0.51}$$

There are a lot of results by many authors regarding the existence of solutions and their stability in addition to the blow-up and growth of solutions.

Recently, in [2], the authors considered the following problem

$$u_{tt} - \Delta u - \Delta u_{tt} + \int_0^{\infty} g(s)\Delta u(t-s)ds + |u_t|^{m-2}u_t = 0, \tag{0.0.52}$$

they showed that the stability of the system. More in depth see ([73],[74],[115],[116]).

In the case of logarithmic nonlinearity $f(u) = u\ln|u|^k$ , Many authors proved the existence of global solutions and infinite time blow-up. The results were obtained by use of the potential well method and the logarithmic Sobolev inequality. See ([111],[23],[27]).

### 0.0.8 Kirchhoff equation

The Kirchhoff equation belongs to the famous wave equation's models describing the transverse vibration of a string fixed in its ends. It has been introduced in 1876 by Kirchhoff [98] and it is more general than the D'Alembert equation. In one dimensional space it takes the following form:

$$u_{tt} - \left(\frac{P_0}{\rho h} + \frac{E}{2L\rho}\int_0^L \left|\frac{\partial u}{\partial x}(x,t)\right|^2 dx\right)u_{xx} = 0, \tag{0.0.53}$$

where the function $u(x,t)$ is the vertical displacement at the space coordinate $x$, varying in the segment $[0,L]$ and over time $t > 0$, $\rho$ is the mass density, $h$ is the area of the cross section of the string, $P_0$ is the initial tension on the string, $L$ is the length of the string and $E$ is the Young modulus of the material. The nonlinear coefficient

$$\int_0^L \left|\frac{\partial u}{\partial x}(x,t)\right|^2 dx,$$

is obtained by the variation of the tension during the deformation of the string. When we do not have an initial tension (i.e. $P_0 = 0$), we call that a degenerate case as opposed to the non-degenerate case. This problems type with different types of damping and sources has been extensively studied and results concerning existence, asymptotic behavior and blow-up have been established, we refer the

interested readers to ([123],[124]) and the references therein.
In [13], Balakrishnan and Taylor they proposed a new model of damping called it the Balakrishnan-Taylor damping , as it relates to the span problem and the plate equation. For more depth, here are some papers that focused on the study of this damping ([14],[17],[46],[110],[165] ).
Among the works on this type of problem, we mention what was mentioned in [37] , Choucha et *al* they mixed several dampings, including Balakrishnan-Taylor damping and distributed delay term, as they considered the following problem

$$\begin{cases} u_{tt} - \left(\zeta_0 + \zeta_1 \|\nabla u\|_2^2 + \sigma(\nabla u, \nabla u_t)_{L^2(\Omega)}\right)\Delta u(t) + \alpha(t)\int_0^t h(t-\varrho)\Delta u(\varrho)d\varrho \\ \quad +\beta_1 |u_t(t)|^{m-2}u_t(t) + \int_{\tau_1}^{\tau_2} |\beta_2(s)||u_t(t-s)|^{m-2}u_t(t-s)ds = 0. \\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), \;\; in \;\; \Omega \\ u_t(x,-t) = f_0(x,t), \;\; in \;\; \Omega \times (0,\tau_2) \\ u(x,t) = 0, \;\; in \;\; \partial\Omega \times (0,\infty). \end{cases} \tag{0.0.54}$$

By the energy method they established the general decay rate under suitable hypothesis.
Very recently, also Choucha et *al* in [36], considered a system of coupled nondegenerate Kirchhoff equations with distributed delay, by using the energy method and Faedo-Galerkin method, they proved the global existence of solutions. Furthermore, they proved the exponential stability result.
The problem [36] is a description of axially moving viscoelastic strings composed of two different materials (like the wires of electricity) that are nonhomogeneous and which will be of influence on its moving, specially on the acceleration. From the mathematical point of view, this influence is represented by $|u_t|^\rho u_{tt}$, where $|u_t|^\rho$ is the material density, varying the velocity. A lot of work has been published with this term, for example see [121] and [122], where we find different results about the global existence and nonexistence of solutions and the decay of energy.

### 0.0.9 Growth of solutions

The well known "Growth" phenomenon is one of the most important phenomena of asymptotic behavior, where many researches omit from its study especially when it comes from the evolution problems. It gives us very important information to know the behavior of equation when time arrives at infinity, it differs from global existence and blow up in both mathematically and in applications point of view. Although the interest of the scientific community for the study of delayed problems is fairly recent, multiple techniques have already been explored in depth. In this direction, in [38] considered the following delayed damped system

$$\begin{cases} u_{tt}-\Delta u-\omega\Delta u_t+\int_0^t g(t-s)\Delta u(s)ds \\ +\mu_1 u_t+\int_{\tau_1}^{\tau_2}|\mu_2(s)|u_t(x,t-s)ds=b|u|^{p-2}.u, \quad x\in\Omega, t>0, \\ u(x,t)=0,\ x\in\partial\Omega, \\ u_t(x,-t)=f_0(x,t),\ \ (x,t)\in\Omega\times(0,\tau_2), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ x\in\Omega. \end{cases} \tag{0.0.55}$$

They proved that if the initial energy $E(0)$ of the solutions is negative (this means that the initial data are large enough), then the local solutions in bounded.
More precisely, they proved that the $L_p$-norm of the solution grows as an exponential function.

$$\|u\|_p^p\to\infty, \text{ as } t\to+\infty. \tag{0.0.56}$$

For more information and to delve deeper into the subject, we refer the reader to the following papers, for example ([39],[40],[140],[141],[163],[164]).

After this comprehensive introduction, we present some concepts, definitions and propositions needed and used in the arguments in this thesis.

### 0.0.10 Semi-group

Let $(X,<.,.>,\|.\|)$ be a Hilbert space and let $\mathcal{A}:D(\mathcal{A})\subset X\to X$ be a linear operator. We introduce in this section some basic concepts that will be needed in the study of the initial value problem

$$\begin{cases} u_t(t)=\mathcal{A}u(t),\ \ t\geq 0 \\ u(0)=u_0(x), \end{cases} \tag{0.0.57}$$

the solution of (0.0.57) is given by

$$u(t)=e^{\mathcal{A}t}u_0, \tag{0.0.58}$$

where

$$e^{\mathcal{A}t}=\sum_{n\geq 0}\frac{\mathcal{A}^n t^n}{n!}. \tag{0.0.59}$$

The family of matrices $e^{\mathcal{A}t}$ is called a semigroup.

**Definition 0.1.** *Let $X$ be a Banach space. A family $(S(t))_{t\geq 0}$ of bounded linear operators from $X$ into $X$ is called a semigroup of bounded linear operators on $X$ if*

1. $S(0)=Id$.

2. $S(t+s)=S(t)+S(s),\ \forall t,s\geq 0$.

*If moreover the semigroup $S(t)$ also fulills*

$$\lim_{t\to 0^+} S(t)x = x,$$

*then $S(t)$ is called a strongly continuous semigroup ($C_0$-semigroup) of bounded linear operators on $X$.*

The definition of semigroup is an extension of $e^{\mathcal{A}t}$ defined in (0.0.59). For the matrix $\mathcal{A}$, one has

$$\mathcal{A}u = \lim_{t\to 0^+} \frac{e^{\mathcal{A}t}u - u}{t} = \lim_{t\to 0^+} \frac{e^{\mathcal{A}t}u - e^{\mathcal{A}0}u}{t-0} = \frac{de^{\mathcal{A}t}u}{dt}\bigg|_{t=0}. \tag{0.0.60}$$

We can say that the semigroup is generated by $\mathcal{A}$. The relation (0.0.60) can be generalized to the case of $C_0$-semigroup. Indeed, the linear operator $\mathcal{A}$ defined by

$$D(\mathcal{A}) = \left\{u \in X,\ \lim_{t\to 0^+} \frac{e^{\mathcal{A}t}u - u}{t} \text{ exists}\right\}, \tag{0.0.61}$$

and

$$\mathcal{A}u = \lim_{t\to 0^+} \frac{S(t)u - u}{t} = \frac{dS(t)u}{dt}\bigg|_{t=0},\quad for\ \ u \in X, \tag{0.0.62}$$

is called the infinitesimal generator of the semigroup $S(t)$. We usually write $S(t) = e^{\mathcal{A}t}$.

Now, we start by recalling the notion of $m$-dissipative (maximal-dissipative) operators.

**Definition 0.2.** *Let $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ be an unbounded linear operator. $\mathcal{A}$ is said to be dissipative (monotone) if $Re < \mathcal{A}u, u >_X \leq 0$ $\Big(Re < \mathcal{A}u, u >_X \geq 0\Big)$. The dissipative operator $\mathcal{A}$ is $m$-dissipative if $\lambda I - \mathcal{A}$ is surjective for some $\lambda > 0$.*

**Definition 0.3.** *[82] Let $V$ be a (real) Hilbert space with scalar product noted $(.,.)_v$ and associated norm $\|.\|_V$; we propose to solve the following problem*

*find $u \in V$ such as for any $v \in V$ we have: $\mathcal{A}(u,v) = \mathcal{L}(v)$,*

*the following conditions are imposed*

*I) $\mathcal{L}$ is an application defined on $V$; to values in $\mathbb{R}$ verifying moreover*

*1. $\mathcal{L}$ is linear.*

*2. $\mathcal{L}$ is continuous; i.e. there exists a constant $C > 0$ such that*

$$for\ all\ \ v \in V,\ \ |\mathcal{L}(v)| \leq C\|v\|_V.$$

*II) $\mathcal{A}$ is an application defined on $V \times V$; to values in $\mathbb{R}$ verifying moreover*

*1. $\mathcal{A}$ is bilinear.*

*2. $\mathcal{A}$ is continuous; i.e. there exists a constant $M > 0$ such that*

$$for\ all\ \ (u,v) \in V^2,\ \ |\mathcal{A}(u,v)| \leq M\|u\|_V\|v\|_V.$$

*3. $\mathcal{A}$ is coercive; i.e. $\exists \alpha > 0$ such that*

$$for\ all\ \ v \in V,\ \ |\mathcal{A}(v,v)| \geq \alpha\|v\|_V.$$

**Lemma 0.1.** *(Lax-Milgram) Let $V$ be a real Hilbert space, $\mathcal{A}$ be a bilinear form, continuous and coercive on $V$ and $\mathcal{L}$ a continuous linear form on $F$. So, there exists a unique element $u$ of $V$ solution of the variational problem. If the bilinear form is symmetric; (i.e. if $\mathcal{A}(u,v) = \mathcal{A}(v,u)$ for all $u,v \in V$ ), pose*

$$\text{for all } v \in V, \; E(v) = \frac{1}{2}\mathcal{A}(v,v) - L(v)$$

*such that is equivalent to a minimization problem for the quadratic functional $E$.*

**Theorem 0.1.** *(Lumer-Phillips) A linear operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ is the infinitesimal generator of a $C_0$-semigroup $(S(t))_{t\geq 0}$ if and only if $\mathcal{A}$ is $m$-dissipative.*

**Theorem 0.2.** *(Hille-Yosida) Let $(\mathcal{A}, D(\mathcal{A}))$ be an unbounded linear operator on $H$. Assume that $\mathcal{A}$ is the infinitesimal generator of a $C_0$-semigroup of contractions $S(t)t > 0$.*

*1. For $u_0 \in D(\mathcal{A})$, the problem (0.0.57) admits a unique strong solution*

$$u(t) = S(t)u_0 \in C^1([0,\infty[, H) \cap C([0,\infty[, D(\mathcal{A}))$$

*2. For $u_0 \in D(\mathcal{A})$, the problem (0.0.57) admits a unique weak solution*

$$u(t) \in C^0([0,\infty[, H).$$

*Moreover,*

$$|u(t)| \leq |u_0|, \;\; \text{and} \; \left|\frac{du}{dt}(t)\right| = |\mathcal{A}u(t)| \leq |\mathcal{A}u_0|, \;\; \forall t \geq 0.$$

### 0.0.11 Stability Consepts

Various types of stability have been defined for the solutions of differential equations describing dynamical systems. The most important one is that concerning the stability of solutions near to a point of equilibrium. This may be discussed by the theory of A. Lyapunov. In the simplest of terms, if the solutions that start out near an equilibrium point $u_e$ stay near $u_e$ forever, then $u_e$ is Lyapunov stable. More strongly, if $u_e$ is Lyapunov stable and all solutions that start out near $u_e$ converge to $u_e$, then $u_e$ is asymptotically stable. Consider in a Hilbert space $X$, the differential equation

$$u_t(t) = \mathcal{A}u(t), \tag{0.0.63}$$

where $\mathcal{A}$ is an unbounded operator with domain $D(\mathcal{A}) \subset X$. Suppose that the above differential equation subject to the condition $u(0) = u_0(x)$ is uniquely solvable and that $u \equiv 0$ is an equilibrium point for (0.0.63). We are interested in the asymptotic stability of the null solution in the sense.

**Definition 0.4.** *The equilibrium of (0.0.63) is:*

*1. exponentially stable if it exist constants $a, b, \varepsilon > 0$ such that if $\|u_0(x)\| < \varepsilon$, then*

$$\|u(t,x)\| \leq ae^{-bt}\|u_0(x)\|, \;\; \forall t \geq 0,$$

2. *polynomially stable if it exist constants* $\alpha, \beta, \varepsilon > 0$ *such that if* $\|u_0(x)\| < \varepsilon$*, then*

$$\|u(t,x)\| \leq \beta t^{-\alpha}\|u_0(x)\|, \ \forall t > 0,$$

The thesis beginning by general introduction and consists of three parts. It has 8 chapters.

**Part I**

- In the **chapter 1**, we consider a one-dimensional porous-elastic system with the presence of both memory and distributed delay terms in the second equation. Using the well known energy method combined with Lyapunov functional approach, we prove a general decay result. This result has been published in [28].

- In the **chapter 2**, the work deals with one-dimensional porous-elastic system with thermoelasticity of type III and distributed delay term. This model is dealing with dynamics of engineering structures and non-classical problems of math- ematical physics. We establish the well posedness of the system, and by the energy method combined with Lyapunov functionals, we discuss the stability of system for both cases of equal and nonequal speeds of wave propagation. This result has been published in [137].

- In the **chapter 3**, the swelling porous thermoelastic system with the presence of temperatures, microtemperature effect, and distributed delay terms is considered. We will establish the well posedness of the system, and we prove the exponential stability result. This result has been published in [35].

**Part II**

- In the **chapter 4**, a one-dimensional linear thermo-elastic system of Bresse type with past history and delay term is considered. We prove the well-posedness of the problem using the semigroup method. By using the energy method we discuss the stability of the system for two cases. An exponential stability result of the system is obtained in the case where the propagation velocities are equal in the equation of vertical displacement and the equation of the system rotation angle. On the other hand, a result of algebraic stability is obtained in the case of the different propagation velocities. This result has been published in [161].

- In the **chapter 5**, we consider a linear thermoelastic laminated timoshenko beam with distributed delay, where the heat conduction is given by cattaneoâs law. we establish the well posedness of the system. For stability results, we prove exponential and polynomial stabilities of the system for the cases of equal and nonequal speeds of wave propagation. This result has been published in [30].

**Part III**

- In the **chapter 6**, we are concerned with the problem of a nonlinear viscoelastic wave equation with distributed delay, strong damping and source terms. We obtain a blow-up result of solutions under suitable conditions. This result has been published in [41].

- In the **chapter 7**, a nonlinear viscoelastic wave equation with Balakrishnan-Taylor damping and distributed delay is studied. By the energy method we establish the general decay rate under suitable hypothesis. This result has been published in [37].

- In the **chapter 8**, we are concerned with a problem for a viscoelastic wave equation with strong damping, nonlinear source and distributed delay terms. We show the exponential growth of solution with $L_p$-norm, i.e., $\lim_{t\to\infty} \|u\|_p^p = \infty$. This result has been published in [38].

# Part I

# Porous-elastic Systems

Chapter 1

# General decay of solutions in one-dimensional porous-elastic with memory and distributed delay terms

## 1.1 Introduction

Our purpose in this chapter is to give a general decay result of solutions in one dimensional porous-elastic system with memory and distributed delay term.

Let $\mathcal{H} = (0,1) \times (\tau_1, \tau_2) \times (0, \infty)$, we consider the following problem

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^t g(t-s)\, \phi_{xx}(x,s)\, ds + \mu_1 \phi_t \\ \qquad + \int_{\tau_1}^{\tau_2} |\mu_2(s)| \phi_t(x, t-s)\, ds = 0, \end{cases} \tag{1.1.1}$$

where

$$(x,s,t) \in \mathcal{H}.$$

As in [133], taking the following new variable

$$z(x,\rho,s,t) = \phi_t(x, t - s\rho),$$

then we obtain

$$\begin{cases} sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \\ z(x,0,s,t) = \phi_t(x,t). \end{cases}$$

Consequently, the problem is equivalent to

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^t g(t-s)\phi_{xx}(x,s)\,ds + \mu_1\phi_t \\ \qquad\qquad + \int_{\tau_1}^{\tau_2} |\mu_2(s)|\phi_t(x,t-s\;)\,ds = 0, \\ sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \end{cases} \tag{1.1.2}$$

where

$$(x,\rho,s,t) \in (0,1) \times \mathcal{H}.$$

The system with memory and delay term acting only on the porous equation together with the initial data

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), \\ \phi(x,0) = \phi_0(x), \phi_t(x,0) = \phi_1(x), x \in (0,1), \end{cases} \tag{1.1.3}$$

and Neumann-Dirichlet boundary conditions

$$u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = 0, t \geq 0. \tag{1.1.4}$$

Here, $u$ is the longitudinal displacement, $\phi$ is the volume fraction of the solid elastic material and $\rho,\mu,b,J,\delta,\xi$ are positive constants with $\mu,\xi,b$ satisfying $\mu\xi > b^2$. The integral represents the memory and delay term with $\tau_1,\tau_2 > 0$ are a time delay, $\mu_1$ is positive constant, $\mu_2$ is an $L^\infty$ function and $g$ is the relaxation function satisfying

(H1) $g \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ is a non-increasing function satisfying

$$g(0) > 0,\; b - \int_0^\infty g(s)\,ds = l > 0. \tag{1.1.5}$$

(H2) There exists a positive non-increasing differentiable function $\vartheta \in (\mathbb{R}_+,\mathbb{R}_+)$ satisfying

$$g'(t) \leq -\vartheta(t)\,g(t), t \geq 0. \tag{1.1.6}$$

(H3) $\mu_2 : [\tau_1,\tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds < \mu_1. \tag{1.1.7}$$

In what follows, we consider $(u,\phi)$ to be a solution of system (1.1.2)-(1.1.4) with the regularity needed to justify the calculations in this chapter. We specify Section 2 to the statements and prove of our stability result. We use c throughout this chapter to denote a generic positive constant. Meanwhile, from (1.1.2) and (1.1.4), it follows that

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)\,dx = 0. \tag{1.1.8}$$

So, by solving (1.1.8) and using the initial data of $u$, we get

$$\int_0^1 u(x,t)\,dx = t\int_0^1 u_1(x)\,dx + \int_0^1 u_0(x)\,dx.$$

Consequently, if we let

$$\bar{u}(x,t) = u(x,t) - t\int_0^1 u_1(x)\,dx - \int_0^1 u_0(x)\,dx, \tag{1.1.9}$$

we get

$$\int_0^1 \bar{u}(x,t)\,dx = 0, \forall t \geq 0.$$

Therefore, the use of Poincare's inequality for $\bar{u}$ is justified. In addition, simple substitution shows that $\left(\bar{u},\phi\right)$ satisfies system (1.1.2) with initial data for $\bar{u}$ given as

$$\bar{u_0}(x) = u_0(x) - \int_0^1 u_0(x)\,dx \text{ and } \bar{u_1}(x) = u_1(x) - \int_0^1 u_1(x)\,dx.$$

Henceforth, we work with $\bar{u}$ instead of $u$ but write $u$ for simplicity of notation.

For the existence result we use the Faedo Galarkin method, for more information see ([79],[29]).

## 1.2 Main Result

In this section, we state and prove our decay result for the energy of the system (1.1.2)-(1.1.4) using the multiplier technique.

We need the following lemmas.

**Lemma 1.1.** *The energy functional $E$, defined by*

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \left[\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + \left(\delta - \int_0^t g(s)\,ds\right)\phi_x^2 + \xi\phi^2 + 2bu_x\phi\right]dx \\ &\quad + \frac{1}{2}g\circ\phi_x + \frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)\,dsd\rho dx, \end{aligned} \tag{1.2.1}$$

*satisfies*

$$E'(t) = \frac{1}{2}g'\circ\phi_x - \frac{1}{2}g(t)\int_0^1 \phi_x^2 dx - \left(\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\right)\int_0^1 \phi_t^2 dx, \tag{1.2.2}$$

*and*

$$E'(t) \leq \frac{1}{2}g'\circ\phi_x - \eta_0\int_0^1 \phi_t^2 dx \leq 0, \tag{1.2.3}$$

*where* $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(s)|ds > 0$ *and* $g\circ v = \int_0^1\int_0^t g(t-s)\left(v_x(t) - v_x(s)\right)^2 dsdx.$

*Proof.* Multiplying the first equation of (1.1.2) by $u_t$ and the second equation by $\phi_t$, then integration by parts over $(0,1)$, and using (1.1.4), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi\right]dx - \int_0^1 \phi_{xt}\int_0^t g(t-s)\phi_x(s)\,dsdx \\ &+\mu_1\int_0^1 \phi_t^2 dx + \int_0^1 \phi_t\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,s,t)\,dsdx = 0. \end{aligned} \tag{1.2.4}$$

The last term in the left hand side of (1.2.4) is estimated as follows.

$$\begin{aligned} -\int_0^1 \phi_{xt}\int_0^t g(t-s)\phi_x(s)\,dsdx &= \int_0^1 \phi_{xt}\int_0^t g(t-s)(\phi_x(t)-\phi_x(s))\,dsdx \\ &\quad -\int_0^t g(s)\,ds\int_0^1 \phi_{xt}\phi_x dx \\ &= \frac{1}{2}\frac{d}{dt}g\circ\phi_x-\frac{1}{2}\frac{d}{dt}\int_0^t g(s)\,ds\int_0^1 \phi_x^2 dx-\frac{1}{2}g'\circ\phi_x \\ &\quad +\frac{1}{2}g(t)\int_0^1 \phi_x^2 dx. \end{aligned} \tag{1.2.5}$$

Now, multiplying the last equation in (1.1.2) by $z|\mu_2(s)|$, and integrating the result over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx \\ &= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|zz_\rho(x,\rho,s,t)\,dsd\rho dx \\ &= -\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\frac{d}{d\rho}z^2(x,\rho,s,t)\,dsd\rho dx \\ &= \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|(z^2(x,0,s,t)-z^2(x,1,s,t))dsdx \\ &= \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds\int_0^1 \phi_t^2 dx-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|z^2(x,1,s,t)\,dsdx. \end{aligned}$$

Now, using Young's inequality, we have

$$\begin{aligned} E'(t) &\leq \frac{1}{2}g'\circ\phi_x-\frac{1}{2}g(t)\int_0^1 \phi_x^2 dx-\left(\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\right)\int_0^1 \phi_t^2 dx \\ &\leq \frac{1}{2}g'\circ\phi_x-\left(\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\right)\int_0^1 \phi_t^2 dx, \end{aligned} \tag{1.2.6}$$

then, by $(H3)$, there exists a positive constant $\eta_0$ such that

$$E'(t)\leq \frac{1}{2}g'\circ\phi_x-\eta_0\int_0^1 \phi_t^2 dx, \tag{1.2.7}$$

then we obtain $E$ is a non-increasing function. □

**Lemma 1.2.** *The functional*

$$D_1(t) := J\int_0^1 \phi_t\phi dx+\frac{b\rho}{\mu}\int_0^1 \phi\int_0^x u_t(y)\,dydx, \tag{1.2.8}$$

*satisfies*

$$\begin{aligned} D_1'(t) &\leq -\frac{l}{2}\int_0^1 \phi_x^2 dx-\mu_3\int_0^1 \phi^2 dx+\varepsilon_1\int_0^1 u_t^2 dx+c(1+\frac{1}{\varepsilon_1})\int_0^1 \phi_t^2 dx \\ &\quad +cg\circ\phi_x+c\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)\,dsdx, \end{aligned} \tag{1.2.9}$$

where $\mu_3=\xi-\frac{b^2}{\mu}>0$.

*Proof.* Direct computation using integration by parts and Young's inequality, for $\varepsilon_1 > 0$, yields

$$\begin{aligned} D_1'(t) &= -\delta\int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx + J\int_0^1 \phi_t^2 dx + \frac{b\rho}{\mu}\int_0^1 \phi_t \int_0^x u_t(y)\,dydx \\ &\quad + \int_0^1 \phi_x \int_0^t g(t-s)\,\phi_x(s)\,dsdx - \mu_1\int_0^1 \phi_t\phi dx - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t) dsdx \\ &\leq -\delta\int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 \phi_t^2 dx + \varepsilon_1\int_0^1 \left(\int_0^x u_t(y)\,dy\right)^2 dx \\ &\quad + \int_0^1 \phi_x \int_0^t g(t-s)\,\phi_x(s)\,dsdx - \mu_1\int_0^1 \phi_t\phi dx - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t) dsdx. \end{aligned} \tag{1.2.10}$$

By Cauchy-Schwartz inequality, it is clear that

$$\int_0^1 \left(\int_0^x u_t(y)\,dy\right)^2 dx \leq \int_0^1 \left(\int_0^1 u_t dx\right)^2 dx \leq \int_0^1 {u_t}^2 dx.$$

So, estimate (1.2.10) becomes

$$\begin{aligned} D_1'(t) \leq\ & -\delta\int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 \phi_t^2 dx + \varepsilon_1\int_0^1 {u_t}^2 dx \\ & -\mu_1\int_0^1 \phi_t\phi dx - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t) dsdx \\ & + \int_0^1 \phi_x \int_0^t g(t-s)\,\phi_x(s)\,dsdx. \end{aligned} \tag{1.2.11}$$

The last term in the RHS of (1.2.11) is estimated as follows:

$$\begin{aligned} & \int_0^1 \phi_x \int_0^t g(t-s)\,\phi_x(s)\,dsdx \\ & = \int_0^t g(s)\,ds \int_0^1 \phi_x^2 dx - \int_0^1 \phi_x \int_0^t g(t-s)\,(\phi_x(t)-\phi_x(s))\,dsdx \\ & \leq \left(\delta_1 + \int_0^t g(s)\,ds\right)\int_0^1 \phi_x^2 dx + \frac{1}{4\delta_1}\left(\int_0^t g(s)\,ds\right) g\circ\phi_x, \end{aligned} \tag{1.2.12}$$

where we have used Cauchy-Schwartz, Young's and Poincare's inequalities, for $\delta_1, \varepsilon_2, \varepsilon_3 > 0$.

By substituting (1.2.12) into(1.2.10), we obtain

$$\begin{aligned} D_1'(t) \leq\ & -\left(\delta - \int_0^t g(s)\,ds - \delta_1 - \mu_1 c\delta_2 - \mu_1 c\delta_3\right)\int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx \\ & + \varepsilon_1\int_0^1 u_t^2 dx + \left(c(1+\frac{1}{\varepsilon_1}) + \frac{\mu_1}{4\delta_2}\right)\int_0^1 \phi_t^2 dx + \frac{1}{4\delta_1}\left(\int_0^t g(s)\,ds\right) g\circ\phi_x \\ & + \frac{1}{4\delta_3}\int_0^t \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) dsdx. \end{aligned} \tag{1.2.13}$$

Bearing in mind that $\mu\xi > b^2$ and using the fact that $\delta - \int_0^t g(s)\,ds \geq l$,
and letting $\delta_1 = \dfrac{l}{6}$, $\delta_2 = \delta_3 = \dfrac{l}{6c\mu_1}$, we obtain estimate (1.2.9). □

In the following Lemma, we use the essential hypothesis that the wave speeds of the system are equal

$$\frac{\mu}{\rho} = \frac{\delta}{J}. \tag{1.2.14}$$

**Lemma 1.3.** *Assume that (H1) and (1.2.14) hold. Then the functional*

$$D_2(t) := \int_0^1 \phi_x u_t dx + \int_0^1 \phi_t u_x dx - \frac{\rho}{\mu J}\int_0^1 u_t \int_0^t g(t-s)\phi_x(s)\,dsdx,$$

*satisfies, for any* $\varepsilon_2 > 0$

$$\begin{aligned} D_2'(t) \leq & -\frac{b}{2J}\int_0^1 u_x^2 dx + c\left(1+\frac{1}{\varepsilon_2}\right)\int_0^1 \phi_x^2 dx + c\varepsilon_2 \int_0^1 u_t^2 dx \\ & + c\int_0^1 \phi_t^2 + cg\circ\phi_x - \frac{c}{\varepsilon_2} g'\circ\phi_x + c\mu_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)dx. \end{aligned} \tag{1.2.15}$$

*Proof.* By differentiating $D_2$, then using (1.1.2), integration by parts, and (1.1.4) we obtain

$$\begin{aligned} D_2'(t) = & -\frac{b}{J}\int_0^1 u_x^2 dx + \left(\frac{\delta}{J} - \frac{\mu}{\rho}\right)\int_0^1 u_x\phi_{xx} dx + \frac{b}{\rho}\int_0^1 \phi_x^2 dx - \frac{\rho g(0)}{\mu J}\int_0^1 u_t \phi_x dx \\ & -\frac{\xi}{J}\int_0^1 u_x \phi dx - \frac{b}{\mu J}\int_0^1 \phi_x \int_0^t g(t-s)\phi_x(s)\,dsdx \\ & -\frac{\rho}{\mu J}\int_0^1 u_t \int_0^t g'(t-s)\phi_x(s)\,dsdx \\ & -\frac{\mu_1}{J}\int_0^1 \phi_t u_x dx - \frac{1}{J}\int_0^1 u_x \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)dsdx. \end{aligned} \tag{1.2.16}$$

In what follows, we estimate the last five terms in the right hand side of (1.2.16), using Young's, Cauchy-Schwartz, and Poincare's inequalities. For $\delta_4, \delta_5, \varepsilon_2 > 0$, we have

$$-\frac{\xi}{J}\int_0^1 u_x \phi dx \leq \frac{\xi}{J}\delta_4 \int_0^1 u_x^2 dx + \frac{\xi}{4J\delta_4}\int_0^1 \phi^2 dx.$$

By letting $\delta_4 = \frac{b}{6\xi}$, using Poincar's inequality, we get

$$-\frac{\xi}{J}\int_0^1 u_x \phi dx \leq \frac{b}{6J}\int_0^1 u_x^2 dx + c\int_0^1 \phi^2 dx, \tag{1.2.17}$$

$$\begin{aligned} -\frac{b}{\mu J}\int_0^1 \phi_x \int_0^t g(t-s)\phi_x(s)\,dsdx = & \frac{b}{\mu J}\int_0^1 \phi_x \int_0^t g(t-s)(\phi_x(t) - \phi_x(s))\,dsdx \\ & -\frac{b}{\mu J}\int_0^t g(s)\,ds \int_0^1 \phi_x^2 dx \\ \leq & \left(\delta_5 - \frac{b}{\mu J}\right)\int_0^t g(s)\,ds \int_0^1 \phi_x^2 dx + \frac{c}{\delta_5} g\circ\phi_x. \end{aligned}$$

By letting $\delta_5 = \frac{b}{\mu J}$ we get

$$-\frac{b}{\mu J}\int_0^1 \phi_x \int_0^t g(t-s)\phi_x(s)\,dsdx \leq cg\circ\phi_x, \tag{1.2.18}$$

$$\begin{aligned}
-\frac{\rho}{\mu J}\int_0^1 u_t \int_0^t g'(t-s)\phi_x(s)\,dsdx &= \frac{b}{\mu J}\int_0^1 u_t\int_0^t g'(t-s)(\phi_x(t)-\phi_x(s))\,dsdx \\
&\quad -\frac{b}{\mu J}\int_0^t g'(s)\,ds\int_0^1 u_t\phi_x dx \\
&\le \frac{\rho\varepsilon_2}{2\mu J}\int_0^1 u_t^2 dx + \frac{\rho g(0)}{\mu J}\int_0^1 u_t\phi_x dx - \frac{\rho g(t)}{\mu J}\int_0^1 u_t\phi_x dx \\
&\quad +\frac{\rho}{2\mu J\varepsilon_2}\int_0^1 g'(s)\,ds\int_0^1\int_0^t g'(t-s)(\phi_x(t)-\phi_x(s))^2\,dsdx \\
&\le \frac{\rho\varepsilon_2}{\mu J}\int_0^1 u_t^2 dx + \frac{\rho}{2\mu J\varepsilon_2}\left(\int_0^1 g'(s)\,ds\right) g'\circ\phi_x. \\
&\quad +\frac{\rho g(0)}{\mu J}\int_0^1 u_t\phi_x dx + \frac{\rho g(t)}{2\mu J\varepsilon_2}\int_0^1 u_t\phi_x dx \\
&\le \frac{\rho\varepsilon_2}{\mu J}\int_0^1 u_t^2 dx + \frac{\rho}{2\mu J\varepsilon_2}\left(\int_0^1 g'(s)\,ds\right) g'\circ\phi_x. \\
&\quad +\frac{\rho g(0)}{\mu J}\int_0^1 u_t\phi_x dx + \frac{\rho (g(t))^2}{2\mu J\varepsilon_2}\int_0^1 \phi_x^2 dx \\
&\le c\varepsilon_2\int_0^1 u_t^2 dx + \frac{c}{\varepsilon_2}\int_0^1 \phi_x^2 dx + \frac{\rho g(0)}{\mu J}\int_0^1 u_t\phi_x dx \\
&\quad -\frac{c}{\varepsilon_2} g'\circ\phi_x, \qquad (1.2.19)
\end{aligned}$$

$$-\frac{\mu_1}{J}\int_0^1 \phi_t u_x dx \le \frac{\mu_1\delta_6}{2J}\int_0^1 \phi_t^2 dx + \frac{\mu_1}{2J\delta_6}\int_0^1 u_x^2 dx, \qquad (1.2.20)$$

$$\begin{aligned}
\frac{1}{J}\int_0^1 u_x\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,s,t)dsdx &\le \frac{\delta_7\mu_1}{2J}\int_0^1 u_x^2 dx \qquad (1.2.21)\\
&\quad +\frac{1}{2J\delta_7}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)ds.
\end{aligned}$$

The replacement of (1.2.17)-(1.2.21) into (1.2.16), and by letting $\delta_6=\delta_7=\dfrac{b}{4\mu_1}$, bearing in the mind (1.2.14), yields (1.2.15). □

**Lemma 1.4.** *The functional*

$$D_3(t) := -\rho\int_0^1 u_t u dx,$$

*satisfies*

$$D_3'(t) \le -\rho\int_0^1 u_t^2 dx + \frac{3\mu}{2}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx. \qquad (1.2.22)$$

*Proof.* Direct computations give

$$D_3'(t) = -\rho\int_0^1 u_t^2 dx + \mu\int_0^1 u_x^2 dx + b\int_0^1 u_x\phi dx.$$

Estimat (1.2.22) easily follows by using Young's and Poincaré inequalities.

$$\begin{aligned}
D_3'(t) &\le -\rho\int_0^1 u_t^2 dx + \mu\int_0^1 u_x^2 dx + b\varepsilon\int_0^1 u_x^2 dx + \frac{b}{4\varepsilon}\int_0^1 \phi^2 dx \\
&\le -\rho\int_0^1 u_t^2 dx + \mu\int_0^1 u_x^2 dx + b\varepsilon\int_0^1 u_x^2 dx + \frac{bc}{4\varepsilon}\int_0^1 \phi_x^2 dx,
\end{aligned}$$

by letting $\varepsilon = \dfrac{\mu}{2b}$, we obtain (1.2.22). □

Now, let us introduce the following functional used by

**Lemma 1.5.** *The functional*

$$D_4(t) := \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_2(s)|z^2(x,\rho,s,t)\,dsd\rho dx,$$

*satisfies,*

$$\begin{aligned} D_4'(t) &\leq -\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)\,dsd\rho dx + \mu_1\int_0^1\phi_t^2dx \\ &\quad -\eta_1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)\,dsdx, \end{aligned} \tag{1.2.23}$$

*where $\eta_1$ is a positive constant.*

*Proof.* By differentiating $D_4$, with respect to $t$ and using the last equation in $(H3)$, we have

$$\begin{aligned} D_4'(t) &= -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}|\mu_2(s)|zz_\rho(x,\rho,s,t)\,dsd\rho dx \\ &= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_2(s)|z^2(x,\rho,s,t)\,dsd\rho dx \\ &\quad -\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|[e^{-s}z^2(x,1,s,t)-z^2(x,0,s,t)]dsdx. \end{aligned}$$

Using the fact that $z(x,0,s,t)=\phi_t(x,t)$, and $e^{-s}\leq e^{-s\rho}\leq 1$, for all $0<\rho<1$, we obtain

$$\begin{aligned} D_4'(t) &= -\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)\,dsd\rho dx \\ &\quad -\int_0^1\int_{\tau_1}^{\tau_2} e^{-s}|\mu_2(s)|z^2(x,1,s,t)\,dsdx + \int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\int_0^1\phi_t^2dx. \end{aligned}$$

□

Because $-e^{-s}$ is a increasing function, we have $-e^{-s}\leq -e^{-\tau_2}$, for all $s\in[\tau_1,\tau_2]$. Finally, setting $\eta_1=e^{-\tau_2}$ and recalling (H3), we obtain (1.2.23). We are now ready to prove the main result.

**Theorem 1.1.** *Assume (H1), (H2), (H3) and (1.2.14) hold. Then, for any $t_0>0$, there exist positive constants $\alpha$ and $\beta$ such that the energy functional given by (1.2.1) satisfies*

$$E(t)\leq \alpha e^{-\beta\int_{t_0}^t\vartheta(s)ds}, \forall t\geq t_0. \tag{1.2.24}$$

*Proof.* We define a Lyapunov functional

$$\mathcal{L}(t) := NE(t)+N_1D_1(t)+N_2D_2(t)+D_3(t)+N_4D_4(t), \tag{1.2.25}$$

where $N$, $N_1$, $N_2$, and $N_4$ are positive constants to be selected later.

By differentiating (1.2.25) and using (1.2.1), (1.2.9), (1.2.15), (1.2.22), (1.2.23), we have

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\frac{lN_1}{2} - cN_2(1+\frac{1}{\varepsilon_2}) - c\right]\int_0^1 \phi_x^2 dx - [\rho - N_1\varepsilon_1 - N_2 c\varepsilon_2]\int_0^1 u_t^2 dx \\
& -\left[\frac{bN_2}{2J} - \frac{3\mu}{2}\right]\int_0^1 u_x^2 dx - \left[\eta_0 N - cN_1(1+\frac{1}{\varepsilon_1}) - N_2 c - \mu_1 N_4\right]\int_0^1 \phi_t^2 dx \\
& -N_1\mu_3\int_0^1 \phi^2 dx - [N_4\eta_1 - cN_1 - cN_2]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, ds dx \\
& +c[N_1+N_2]\, g\circ\phi_x + \left[\frac{N}{2} - \frac{cN_2}{\varepsilon_2}\right] g'\circ\phi_x \\
& -N_4\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t)\, ds d\rho dx.
\end{aligned}$$

By setting

$$\varepsilon_1 = \frac{\rho}{4N_1}, \varepsilon_2 = \frac{\rho}{4cN_2},$$

we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\frac{lN_1}{2} - cN_2(1+N_2) - c\right]\int_0^1 \phi_x^2 dx - \frac{\rho}{2}\int_0^1 u_t^2 dx \\
& -\left[\frac{bN_2}{2J} - \frac{3\mu}{2}\right]\int_0^1 u_x^2 dx - [\eta_0 N - cN_1(1+N_1) - cN_2 - \mu_1 N_4]\int_0^1 \phi_t^2 dx \\
& -N_1\mu_3\int_0^1 \phi^2 dx - [N_4\eta_1 - cN_1 - cN_2]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, ds dx \\
& +c[N_1+N_2]\, g\circ\phi_x + \left[\frac{N}{2} - cN_2^2\right] g'\circ\phi_x \\
& -N_4\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t)\, ds d\rho dx.
\end{aligned}$$

Next, we carefully choose our constants so that the terms inside the brackets are positive.

We choose $N_2$ large enough such that

$$\alpha_1 = \frac{bN_2}{2J} - \frac{3\mu}{2} > 0,$$

then we choose $N_1$ large enough such that

$$\alpha_2 = \frac{lN_1}{4} - cN_2(1+N_2) - c > 0,$$

then we choose $N_4$ large enough such that

$$\alpha_3 = N_4\eta_1 - cN_1 - cN_2 > 0,$$

thus, we arrive at

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\alpha_2\int_0^1 \phi_x^2 dx - \alpha_0\int_0^1 \phi^2 dx - \frac{\rho}{2}\int_0^1 u_t^2 dx - \alpha_1\int_0^1 u_x^2 dx - [\eta_0 N - c]\int_0^1 \phi_t^2 dx \\
& +\left[\frac{N}{2} - c\right] g'\circ\phi_x + cg\circ\phi_x - \alpha_3\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, ds dx \\
& -\alpha_4\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t)\, ds d\rho dx, \qquad (1.2.26)
\end{aligned}$$

where $\alpha_0 = \mu_3 N_1 = \left(\xi - \frac{b^2}{\mu}\right) N_1$. On the other hand, if we let

$$\mathfrak{L}(t) = N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t),$$

then

$$\begin{aligned} |\mathfrak{L}(t)| &\leq JN_1 \int_0^1 |\phi\phi_t|\,dx + N_2 \int_0^1 \left|\phi_x u_t + u_x \phi_t - \frac{\rho}{\mu J} u_t \int_0^t g(t-s)\,\phi_x(s)\,ds\right| dx \\ &\quad \frac{b\rho N_1}{\mu} \int_0^1 \left|\phi \int_0^x u_t(y)\,dy\right| dx + \rho \int_0^1 |u_t u|\,dx \\ &\quad + N_4 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x,\rho,s,t)\,ds\,d\rho\,dx. \end{aligned}$$

Exploiting Young's, Cauchy-Schwartz, and Poincaré inequalities, we obtain

$$\begin{aligned} |\mathfrak{L}(t)| &\leq c \int_0^1 \left(u_t^2 + \phi_t^2 + \phi_x^2 + u_x^2 + \phi^2\right) dx + c g \circ \phi_x \\ &+ c \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t)\,ds\,d\rho \\ &\leq cE(t). \end{aligned}$$

Consequently, we obtain

$$|\mathfrak{L}(t)| = |\mathcal{L}(t) - NE(t)| \leq cE(t),$$

that is

$$(N-c)E(t) \leq \mathcal{L}(t) \leq (N+c)E(t). \tag{1.2.27}$$

Now, by choosing $N$ large enough such that

$$\frac{N}{2} - c > 0, N - c > 0, N\eta_0 - c > 0,$$

and exploiting (1.2.1), estimates (1.2.26) and (1.2.27), respectively, give

$$\mathcal{L}'(t) \leq -k_1 E(t) + k_2 g \circ \phi_x, \forall t \geq t_0, \tag{1.2.28}$$

and

$$c_2 E(t) \leq \mathcal{L}(t) \leq c_3 E(t), \forall t \geq 0, \tag{1.2.29}$$

for some $k_1, k_2, c_2, c_3 > 0$.

By multiplying (1.2.28) by $\vartheta(t)$, we obtain

$$\vartheta(t)\mathcal{L}'(t) \leq -k_1 \vartheta(t) E(t) + k_2 \vartheta(t) g \circ \phi_x, \forall t \geq t_0. \tag{1.2.30}$$

The last term in (1.2.30) is estimated as following, using (1.1.6), we have

$$\begin{aligned} \vartheta(t) g \circ \phi_x &= \vartheta(t) \int_0^1 \int_0^t g(t-s)(\phi_x(t) - \phi_x(s))^2\,ds\,dx \\ &\leq \int_0^1 \int_0^t \vartheta(t-s) g(t-s)(\phi_x(t) - \phi_x(s))^2\,ds\,dx \\ &\leq -\int_0^1 \int_0^t g'(t-s)(\phi_x(t) - \phi_x(s))^2\,ds\,dx = -g' \circ \phi_x \\ &\leq -2E'(t). \end{aligned}$$

Thus, (1.2.30) becomes

$$\vartheta(t)\mathcal{L}'(t) \leq -k_1\vartheta(t)E(t) - 2k_2E'(t), \forall t \geq t_0,$$

which can be rewritten as

$$(\vartheta(t)\mathcal{L}(t) + 2k_2E(t))' - \vartheta'(t)\mathcal{L}(t) \leq -k_1\vartheta(t)E(t), \forall t \geq t_0,$$

using the fact that $\vartheta'(t) \leq 0, \forall t \geq 0$, we have

$$(\vartheta(t)\mathcal{L}(t) + 2k_2E(t))' \leq -k_1\vartheta(t)E(t), \forall t \geq t_0.$$

By exploiting (1.2.29), we notice that

$$\mathcal{R}(t) = \vartheta(t)\mathcal{L}(t) + 2k_2E(t) \sim E(t). \tag{1.2.31}$$

Consequently, for some positive constant $\lambda$, we obtain

$$\mathcal{R}'(t) \leq -\lambda\mathcal{R}(t)\vartheta(t), \forall t \geq t_0. \tag{1.2.32}$$

A simple integration of (1.2.32) over $(t_0, t)$ leads to

$$\mathcal{R}(t) \leq \mathcal{R}(t_0)e^{-\lambda\int_{t_0}^{t}\vartheta(s)ds}, \forall t \geq t_0. \tag{1.2.33}$$

Consequently, (1.2.24) is established by virtue of (1.2.29) and (1.2.33). □

**Remark 1.1.** *We give some examples to illustrate the energy decay rates obtained by Theorem 3.2. We consider the three different examples*

$$\textit{If } g(t) = \beta_1 e^{-\beta_2 t}, \ \textit{then} \ \ g'(t) = -\vartheta(t)g(t), \ \textit{where} \ \ \ \vartheta(t) = \beta_2,$$

*then*

$$E(t) \leq c_0 e^{-\beta_2 c_1 t}, \forall t \geq 0,$$

$$\textit{If } g(t) = \frac{\beta_1}{(1+t)^{\beta_2+1}}, \ \textit{then} \ \ g'(t) = -\vartheta(t)g(t), \ \textit{where} \ \ \vartheta(t) = \frac{\beta_2+1}{1+t},$$

*then*

$$E(t) \leq \frac{c_0}{(1+t)^{(\beta_2+1)c_1}}, \forall t \geq 0,$$

$$\textit{If } g(t) = \frac{\beta_1}{(e^{t(\frac{\pi}{2}-arctgt)}\sqrt{1+t^2})^{\beta_2}}, \ \textit{then} \ \ \ g'(t) = -\vartheta(t)g(t), \textit{where } \ \vartheta(t) = \beta_2(\frac{\pi}{2} - arctgt),$$

*then*

$$E(t) \leq \frac{c_0}{c_1(e^{t(\frac{\pi}{2}-arctgt)}\sqrt{1+t^2})^{\beta_2}}, \forall t \geq 0.$$

Chapter 2

# On the porous-elastic system with thermoelasticity of type III and distributed delay: Well-posedness and stability

## 2.1 Introduction

This chapter studied the asymptotic behavior of a one-dimensional porous-elastic with thermoelasticity of type III system combined by distributed time delay. We established the well-posedness of the system, and we proved stability estimates by means of appropriate Lyapunov functionals. Typically, under the assumption (2.1.4), the system keeps the same properties that the one without delay but only with a standard frictional damping $c\phi_t$, for some coefficient $c$.

Let $\mathcal{H} = (0,1)\times(\tau_1,\tau_2)\times(0,\infty), \tau_1,\tau_2 > 0$. For $(x,s,t)\in\mathcal{H}$, we consider the following porous-elastic system:

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b\phi_x \\ \rho_2 \phi_{tt} = \delta\phi_{xx} - bu_x - \xi\phi - \beta\theta_x - \mu_1\phi_t - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\phi_t(x,t-s)ds \\ \rho_3\theta_{tt} = l\theta_{xx} - \gamma\phi_{ttx} + k\theta_{txx}, \end{cases} \tag{2.1.1}$$

with the initial data

$$\begin{aligned} u(x,0) &= u_0(x), u_t(x,0) = u_1(x), \\ \phi(x,0) &= \phi_0(x), \phi_t(x,0) = \phi_1(x), \phi_t(x,-t) = f_0(x,t) \\ \theta(x,0) &= \theta_0(x), \theta_t(x,0) = \theta_1(x), x\in(0,1), t>0, \end{aligned} \tag{2.1.2}$$

and boundary conditions

$$u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, t \geq 0. \tag{2.1.3}$$

Here, $\phi$ is the volume fraction of the solid elastic material, $u$ is the longitudinal displacement and $\theta$ is the difference in temperatures. The parameters $\rho_1, \rho_2, \rho_3, \mu, b, \delta, \xi, l, \gamma, \beta, k$ are positive constants with $\mu\xi > b^2$. The integral represents the distributed delay term with $\tau_1, \tau_2$ are a time delays, $\mu_1$ is positive constant, $\mu_2$ is an $L^\infty$ function such that

(Hyp1) $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds < \mu_1. \tag{2.1.4}$$

This type of problem mainly based on the following equations for one-dimensional theories of porous materials with temperature

$$\begin{cases} \rho_1 u_{tt} - T_x = 0 \\ \rho_2 \phi_{tt} - H_x - G = 0 \\ \rho_3 \theta_t + q_x + \gamma \phi_{tx} = 0, \end{cases} \tag{2.1.5}$$

where $(x,t) \in (0,L) \times (0,\infty)$.

According to Green and Naghdi's theory, the constitutive equations of system (2.1.5) given by:

$$\begin{aligned} T &= \mu u_x + b\phi, \\ G &= -bu_x - \xi\phi - \mu_1\phi_t - \int_{\tau_1}^{\tau_2} |\mu_2(s)| \phi_t(x, t-s) ds, \\ H &= \delta\phi_x - \beta\theta, \\ q &= -l\Phi_x - k\Phi_{tx}, \end{aligned} \tag{2.1.6}$$

where $l, k > 0$ are the thermal conductivity, and $\Phi$ is the thermal displacement whose time derivative is the empirical temperature $\theta$, that is $\Phi_t = \theta$.

We substitute (2.1.6) in (2.1.5) with the condition $b \neq 0$, which results in us

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b\phi_x \\ \rho_2 \phi_{tt} = \delta\phi_{xx} - bu_x - \xi\phi - \mu_1\phi_t - \int_{\tau_1}^{\tau_2} |\mu_2(s)| \phi_t(x, t-s) ds - \beta\theta_x \\ \rho_3 \theta_t = l\Phi_{xx} - \gamma\phi_{tx} + k\Phi_{txx}. \end{cases} \tag{2.1.7}$$

By using $\Phi_t = \theta$ in the system (2.1.7) we find directly our system (2.1.1).

Our work differs from all of them, since we took the delay in the second equation where to make the distributed delay in the rotation angle of the filament, which makes the contributions clear and important. In addition, we established the well-posedness of the system, and we obtain the exponential decay rate when $\frac{\delta}{\rho_2} = \frac{\mu}{\rho_1}$ and the energy takes the algebraic rate for the case $\frac{\delta}{\rho_2} \neq \frac{\mu}{\rho_1}$, these results are mainely stated in Theorem 2.2.

In order to show the dissipativity of system (2.1.1)-(2.1.3), we introduce the new variables $\varphi = u_t$ and $\psi = \phi_t$. So, problem (2.1.1)-(2.1.3) takes the form

$$\begin{cases} \rho_1\varphi_{tt} = \mu\varphi_{xx} + b\psi_x \\ \rho_2\psi_{tt} = \delta\psi_{xx} - b\varphi_x - \xi\psi - \mu_1\psi_t - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\psi_t(x,t-s)ds - \beta\theta_{tx} \\ \rho_3\theta_{tt} = l\theta_{xx} - \gamma\psi_{tx} + k\theta_{txx}, \end{cases} \tag{2.1.8}$$

with the initial data

$$\begin{aligned} &\varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \psi(x,0) = \psi_0(x), \\ &\psi_t(x,0) = \psi_1(x), \theta(x,0) = \theta_0(x), \theta_t(x,0) = \theta_1(x), \\ &\psi_t(x,-t) = -f_0(x,t), \quad x \in (0,1), \end{aligned} \tag{2.1.9}$$

and boundary conditions

$$\varphi_x(0,t) = \varphi_x(1,t) = \psi(0,t) = \psi(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, t \geq 0. \tag{2.1.10}$$

First, as in [133], taking the following new variable

$$z(x,\rho,s,t) = \psi_t(x,t-s\rho),$$

then we obtain

$$\begin{cases} sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0 \\ z(x,0,s,t) = \psi_t(x,t). \end{cases}$$

Consequently, the problem rewritten as

$$\begin{cases} \rho_1\varphi_{tt} = \mu\varphi_{xx} + b\psi_x \\ \rho_2\psi_{tt} = \delta\psi_{xx} - b\varphi_x - \xi\psi - \mu_1\psi_t - \int_{\tau_1}^{\tau_2} |\mu_2(s)|z(x,1,s,t)ds - \beta\theta_{tx} \\ \rho_3\theta_{tt} = l\theta_{xx} - \gamma\psi_{tx} + k\theta_{txx} \\ sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \end{cases} \tag{2.1.11}$$

where

$$(x,\rho,s,t) \in (0,1) \times \mathcal{H},$$

with the boundary and the initial conditions

$$\begin{aligned} &\varphi_x(0,t) = \varphi_x(1,t) = \psi(0,t) = \psi(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, t \geq 0. \\ &\varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \psi(x,0) = \psi_0(x), \\ &\psi_t(x,0) = \psi_1(x), \theta(x,0) = \theta_0(x), \theta_t(x,0) = \theta_1(x), x \in (0,1), \\ &z(x,\rho,s,0) = -f_0(x,\rho s) = h_0(x,\rho s), \quad x \in (0.1), \rho \in (0.1), s \in (0,\tau_2). \end{aligned} \tag{2.1.12}$$

Meanwhile, from $(1.1.2)_1$ and (2.1.12), it follows that

$$\frac{d^2}{dt^2}\int_0^1 \varphi(x,t)\,dx = 0. \tag{2.1.13}$$

So, by solving (2.1.13) and using (2.1.12), we get

$$\int_0^1 \varphi(x,t)\,dx = t\int_0^1 \varphi_1(x)\,dx + \int_0^1 \varphi_0(x)\,dx.$$

Consequently, if we let

$$\bar{\varphi}(x,t) = \varphi(x,t) - t\int_0^1 \varphi_1(x)\,dx - \int_0^1 \varphi_0(x)\,dx, \tag{2.1.14}$$

we get

$$\int_0^1 \bar{\varphi}(x,t)\,dx = 0, \forall t \geq 0,$$

and from $(2.1.11)_3$, we have

$$\frac{d^2}{dt^2}\int_0^1 \theta(x,t)\,dx = 0. \tag{2.1.15}$$

So, by solving (2.1.15) and using (2.1.12), we get

$$\int_0^1 \theta(x,t)\,dx = t\int_0^1 \theta_1(x)\,dx + \int_0^1 \theta_0(x)\,dx.$$

Consequently, if we let

$$\bar{\theta}(x,t) = \theta(x,t) - t\int_0^1 \theta_1(x)\,dx - \int_0^1 \theta_0(x)\,dx, \tag{2.1.16}$$

we get

$$\int_0^1 \bar{\theta}(x,t)\,dx = 0, \forall t \geq 0.$$

Then, by using the Poincare's inequality for $\bar{\varphi}$ and $\bar{\theta}$ are justified. A simple substitution shows that $\left(\bar{\varphi}, \psi, \bar{\theta}\right)$ satisfies system (1.1.2) with initial data for $\bar{\varphi}$, and $\bar{\theta}$ given as

$$\bar{\varphi_0}(x) = \varphi_0(x) - \int_0^1 \varphi_0(x)\,dx \text{ and } \bar{\varphi_1}(x) = \varphi_1(x) - \int_0^1 \varphi_1(x)\,dx,$$

and

$$\bar{\theta_0}(x) = \theta_0(x) - \int_0^1 \theta_0(x)\,dx \text{ and } \bar{\theta_1}(x) = \theta_1(x) - \int_0^1 \theta_1(x)\,dx.$$

Now, we use $\bar{\varphi}, \bar{\theta}$ instead of $\varphi, \theta$ and writing $\varphi, \theta$ for simplicity.

## 2.2 Well-posedness

In this section, we give the existence and uniqueness result of the system (2.1.11)-(2.1.12) using the semigroup theory.

First, we introduce the vector function

$$U = (\varphi, \varphi_t, \psi, \psi_t, \theta, \theta_t, z)^T,$$

and the new dependent variables $u = \varphi_t, v = \psi_t, w = \theta_t$, then the system (2.1.11) can be written as follows:

$$\begin{cases} U_t = \mathcal{A}U \\ U(0) = U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \theta_1, h_0)^T, \end{cases} \tag{2.2.1}$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} :\rightarrow \mathcal{H}$ is the linear operator defined by

$$\mathcal{A}U = \begin{pmatrix} u \\ \frac{1}{\rho_1}[\mu\varphi_{xx} + b\psi_x] \\ v \\ \frac{1}{\rho_2}[\delta\psi_{xx} - b\varphi_x - \xi\psi - \beta w_x - \mu_1\psi_t - -\int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t)ds] \\ w \\ \frac{1}{\rho_3}[l\theta_{xx} - \gamma v_x + k w_{xx}] \\ -\frac{1}{s} z_\rho \end{pmatrix}, \tag{2.2.2}$$

and $\mathcal{H}$ is the energy space given by

$$\begin{aligned} \mathcal{H} &= H_*^1 \times L_*^2(0,1) \times H_0^1 \times L^2(0,1) \times H_*^1 \times L^2(0,1) \\ &\quad \times L^2((0,1) \times (0,1) \times (\tau_1, \tau_2)), \end{aligned}$$

where

$$\begin{aligned} L_*^2(0,1) &= \{\phi \in L^2(0,1) \ / \ \int_0^1 \phi(x)dx = 0\} \\ H_*^1(0,1) &= H^1(0,1) \cap L_*^2(0,1) \\ H_*^2(0,1) &= \{\phi \in H^2(0,1) \ / \ \phi_x(1) = \phi_x(0) = 0\}. \end{aligned}$$

For every

$$\begin{aligned} U &= (\varphi, u, \psi, v, \theta, w, z)^T \in \mathcal{H}, \\ \widehat{U} &= (\widehat{\varphi}, \widehat{u}, \widehat{\psi}, \widehat{v}, \widehat{\theta}, \widehat{w}, \widehat{z})^T \in \mathcal{H}, \end{aligned}$$

we equip $\mathcal{H}$ with the inner product defined by

$$\begin{aligned} < U, \widehat{U} >_{\mathcal{H}} &= \gamma\rho_1 \int_0^1 u\widehat{u}dx + \gamma\rho_2 \int_0^1 v\widehat{v}dx + \gamma\xi \int_0^1 \psi\widehat{\psi}dx \\ &\quad + \beta\rho_3 \int_0^1 w\widehat{w}dx + \gamma\mu \int_0^1 \varphi_x\widehat{\varphi}_x dx + \gamma\delta \int_0^1 \psi_x\widehat{\psi}_x dx \\ &\quad + \gamma b \int_0^1 (\varphi_x\widehat{\psi} + \psi\widehat{\varphi})dx + l\beta \int_0^1 \theta_x\widehat{\theta}_x dx \\ &\quad + \gamma \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z\widehat{z} ds d\rho dx. \end{aligned} \tag{2.2.3}$$

The domain of $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A})=\left\{\begin{array}{l} U\in\mathcal{H}\ \ /\varphi,\theta\in H^2_*(0,1)\cap H^1_*(0,1),\psi\in H^2(0,1)\cap H^1_0(0,1),\\ u,w\in H^1_*(0,1),v\in H^1_0(0,1),z(x,0,s,t)=v\\ z,z_\rho\in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)). \end{array}\right\}.$$

Clearly, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Now, we can give the following existence result.

**Theorem 2.1.** *Let $U_0\in\mathcal{H}$ and assume that (2.1.4) holds. Then, there exists a unique soltion $U\in\mathcal{C}(\mathbb{R}_+,\mathcal{H})$ of problem (2.1.11).*
*Moreover, if $U_0\in\mathcal{D}(\mathcal{A})$, then*

$$U\in\mathcal{C}(\mathbb{R}_+,\mathcal{D}(\mathcal{A}))\cap\mathcal{C}^1(\mathbb{R}_+,\mathcal{H}).$$

*Proof.* First, we prove that the operator $\mathcal{A}$ is dissipative. For any $U_0\in\mathcal{D}(\mathcal{A})$ and by using (2.2.3), we have

$$\begin{aligned} <\mathcal{A}U,U>_{\mathcal{H}} &= -\gamma\mu_1\int_0^1 v^2dx-\gamma\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|vz(x,1,s,t)dsdx\\ &\quad -\gamma\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z_\rho z\,dsd\rho dx-\beta k\int_0^1 w_x^2dx. \end{aligned}\tag{2.2.4}$$

For the third term of the right-hand side of (2.2.4), we have

$$\begin{aligned} -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z_\rho z\,dsd\rho dx &= -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\int_0^1|\mu_2(s)|\frac{d}{d\rho}z^2d\rho dsdx\\ &= -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx\\ &\quad +\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,0,s,t)dsdx. \end{aligned}\tag{2.2.5}$$

By using Young's inequality, we get

$$\begin{aligned} -\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|vz(x,1,s,t)dsdx &\le \frac{1}{2}(\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds)\int_0^1 v^2dx\\ &\quad +\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx. \end{aligned}\tag{2.2.6}$$

Substituting (2.2.5),(2.2.6) into (2.2.4), using the fact that $z(x,0,s,t)=v(x,t)$ and (2.1.4), we obtained

$$\begin{aligned} <\mathcal{A}U,U>_{\mathcal{H}} &\le -\gamma(\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds)\int_0^1 v^2dx-\beta k\int_0^1 w_x^2dx\\ &\le 0. \end{aligned}\tag{2.2.7}$$

Hence, the operator $\mathcal{A}$ is dissipative.

Next, we prove the operator $\mathcal{A}$ is maximal. It is sufficient to show that the operator $(Id-\mathcal{A})$ is surjective. Indeed, for any $F=(f_1,f_2,f_3,f_4,f_5,f_6,f_7)^T\in\mathcal{H}$, we prove that there exists a unique $V=(\varphi,u,\psi,v,\theta,w,z)\in\mathcal{D}(\mathcal{A})$ such that

$$(Id-\mathcal{A})V=F.\tag{2.2.8}$$

That is

$$\begin{cases} \varphi - u = f_1 \\ \rho_1 u - \mu\varphi_{xx} - b\psi_x = \rho_1 f_2 \\ \psi - v = f_3 \\ \rho_2 v - \delta\psi_{xx} + b\varphi_x + \xi\psi + \beta w_x + \mu_1 v \\ \qquad + \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t)ds = \rho_2 f_4 \\ \theta - w = f_5 \\ \rho_3 w - l\theta_{xx} + \gamma v_x - k w_{xx} = \rho_3 f_6 \\ s z_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = s f_7. \end{cases} \tag{2.2.9}$$

We note that the last equation in (2.2.9) with $z(x,0,s,t) = v(x,t)$ has a unique solution given by

$$z(x,\rho,s,t) = e^{-\rho s} v + s e^{s\rho} \int_0^\rho e^{s\sigma} f_7(x,\sigma,s,t) d\sigma, \tag{2.2.10}$$

then

$$z(x,1,s,t) = e^{-s} v + s e^{s} \int_0^1 e^{s\sigma} f_7(x,\sigma,s,t) d\sigma, \tag{2.2.11}$$

we have

$$u = \varphi - f_1, \;\; v = \psi - f_3, \;\; w = \theta - f_5. \tag{2.2.12}$$

Inserting (2.2.11) and (2.2.12) in $(2.2.9)_2$, $(2.2.9)_4$ and $(2.2.9)_6$, we get

$$\begin{cases} \rho_1 \varphi - \mu\varphi_{xx} - b\psi_x = h_1 \\ \mu_4 \psi - \delta\psi_{xx} + b\varphi_x + \beta\theta_x = h_2 \\ rho_3 \theta - (l+k)\theta_{xx} + \gamma\psi_x = h_3, \end{cases} \tag{2.2.13}$$

where

$$\begin{cases} \mu_4 = \rho_2 + \xi + \mu_1 + \frac{4}{3}\gamma + \int_{\tau_1}^{\tau_2} |\mu_2(s)| e^{-s} ds \\ h_1 = \rho_1(f_1 + f_2) \\ h_2 = \rho_2(f_3 + f_4) + \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| e^{-s} ds) f_3 ds \\ \qquad - \int_{\tau_1}^{\tau_2} s|\mu_2(s)| e^{s} \int_0^1 e^{s\sigma} f_7(x,\sigma,s,t) d\sigma ds + \beta f_{5x}. \\ h_3 = \rho_3(f_5 + f_6) + \gamma f_{3x} - k f_{5xx}. \end{cases} \tag{2.2.14}$$

We multiply (2.2.13) by $\widehat{\varphi}, \widehat{\psi}, \widehat{\theta}$ respectively, and integrate their sum over $(0,1)$ to get the following variational formulation:

$$B((\varphi,\psi,\theta),(\widehat{\varphi},\widehat{\psi},\widehat{\theta}) = \Gamma(\widehat{\varphi},\widehat{\psi},\widehat{\theta}), \tag{2.2.15}$$

where

$$B : (H_*^1(0,1) \times H_0^1(0,1) \times H_*^1(0,1))^2 \to \mathbb{R},$$

is the bilinear form defined by

$$\begin{aligned} B((\varphi,\psi,\theta),(\widehat{\varphi},\widehat{\psi},\widehat{\theta}) &= \gamma\rho_1\int_0^1 \varphi\widehat{\varphi}dx+\gamma\mu\int_0^1 \varphi_x\widehat{\varphi}_xdx \\ &+\gamma b\int_0^1(\psi\widehat{\varphi}_x+\varphi\widehat{\psi}_x)dx+\gamma\mu_4\int_0^1\psi\widehat{\psi}dx \\ &+\gamma\delta\int_0^1\psi_x\widehat{\psi}_xdx+\gamma\beta\int_0^1\theta_x\widehat{\psi}dx+\beta\gamma\int_0^1\psi_x\widehat{\theta}dx \\ &+\beta\rho_3\int_0^1\theta\widehat{\theta}dx+\beta(l+k)^2\int_0^1\theta_x\widehat{\theta}_xdx, \end{aligned} \tag{2.2.16}$$

and

$$\Gamma:(H^1_*(0,1)\times H^1_0(0,1)\times H^1_*(0,1))\to\mathbb{R},$$

is the linear functional given by

$$\Gamma(\widehat{\varphi},\widehat{\psi},\widehat{\theta}) = \int_0^1 h_1\widehat{\varphi}dx+\int_0^1 h_2\widehat{\psi}dx+\int_0^1 h_3\widehat{\theta}dx. \tag{2.2.17}$$

Now, for $V=H^1_*(0,L)\times H^1_0(0,L)\times H^1_*(0,L)$, equipped with the norm

$$\|(\varphi,\psi,\theta)\|_V^2 = \|\varphi\|_2^2+\|\varphi_x\|_2^2+\|\psi\|_2^2+\|\psi_x\|_2^2+\|\theta\|_2^2+\|\theta_x\|_2^2,$$

then, we have

$$\begin{aligned} B((\varphi,\psi,\theta),(\varphi,\psi,\theta)) &= \gamma\rho_1\int_0^1\varphi^2dx+\gamma\mu\int_0^1\varphi_x^2dx \\ &+\gamma\mu_4\int_0^1\psi^2dx+\gamma\delta\int_0^1\psi_x^2dx \\ &+\rho_3\beta\int_0^1\theta^2dx+\beta(l+k)\int_0^1\theta_x^2dx \\ &+2\gamma b\int_0^1\varphi_x\psi dx, \end{aligned} \tag{2.2.18}$$

we have

$$\begin{aligned} \mu\varphi_x^2+\mu_4\psi^2+2b\varphi_x\psi &= \frac{1}{2}\Big[\mu(\varphi_x+\frac{b}{\mu}\psi)^2+\mu_4(\psi+\frac{b}{\mu_4}\varphi_x)^2 \\ &+(\mu-\frac{b^2}{\mu_4})\varphi_x^2+(\mu_4-\frac{b^2}{\mu})\psi^2\Big] \\ &> \frac{1}{2}\Big[(\mu-\frac{b^2}{\mu_4})\varphi_x^2+(\mu_4-\frac{b^2}{\mu})\psi^2\Big], \end{aligned} \tag{2.2.19}$$

by the assume $\mu\xi-b^2>0$, we get

$$\mu-\frac{b^2}{\mu_4}>0,\ \ \mu_4-\frac{b^2}{\mu}>0,$$

then, for some $M_0>0$

$$B((\varphi,\psi,\theta),(\varphi,\psi,\theta))\geq M_0\|(\varphi,\psi,\theta)\|_V^2. \tag{2.2.20}$$

Thus $B$ is coercive. Consequently, using Lax-Milgram theorem, we conclude that the existence of a unique solution $((\varphi,\psi,\theta))$ in $V$ satisfying:

$$\begin{aligned} u &= \varphi - f_1 \in H^1_*(0,1) \\ v &= \psi - f_3 \in H^1_0(0,1) \\ w &= \theta - f_5 \in H^1_*(0,1). \end{aligned} \tag{2.2.21}$$

Substituting $\varphi,\psi,\theta$ into (2.2.11) and (2.2.12), respectively, we have

$$\begin{aligned} & u,\theta \in H^1_*(0,1) \\ & \psi \in H^1_0(0,1) \\ & z, z_\rho \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)), \end{aligned} \tag{2.2.22}$$

let $\widehat{\varphi} \in H^1_0(0,1)$ and denote

$$\widehat{\widehat{\varphi}} = \widehat{\varphi}(x) - \int_0^1 \widehat{\varphi}(\xi)d\xi, \tag{2.2.23}$$

which gives us $\widehat{\widehat{\varphi}} \in H^1_*(0,1)$. Now we replace $(\widehat{\varphi},\widehat{\psi},\widehat{\theta})$ by $(\widehat{\widehat{\varphi}},0,0)$ in (2.2.15) to obtain

$$\gamma\rho_1 \int_0^1 \varphi\widehat{\widehat{\varphi}}dx + \gamma\mu \int_0^1 \varphi_x \widehat{\widehat{\varphi}}_x dx + \gamma b \int_0^1 \psi_x \widehat{\widehat{\varphi}}dx = \int_0^1 h_1 \widehat{\widehat{\varphi}}dx, \tag{2.2.24}$$

we get

$$\gamma\mu \int_0^1 \varphi_x \widehat{\widehat{\varphi}}_x dx \quad = \quad \int_0^1 (h_1 - \gamma\rho_1\varphi - \gamma b\psi_x)\widehat{\widehat{\varphi}}dx, \tag{2.2.25}$$

which yields

$$\gamma\mu\varphi_{xx} = \gamma\rho_1\varphi - \gamma b\psi_x - h_1 \in L^2(0,1). \tag{2.2.26}$$

Thus

$$\varphi \in H^2(0,1). \tag{2.2.27}$$

Moreover, $(2.2.13)_1$ also holds for anyevery $\widehat{\varphi} \in C^1([0,1])$. Then, by using integration by parts, we obtain

$$\gamma\mu \int_0^1 \varphi_x \widehat{\varphi}_x dx \quad = \quad \int_0^1 (h_1 - \gamma\rho_1\varphi - \gamma b\psi_x)\widehat{\varphi}dx. \tag{2.2.28}$$

Then, we get for any $\widehat{\varphi} \in C^1([0,1])$

$$\varphi_x(1)\widehat{\varphi}(1) - \varphi_x(0)\widehat{\varphi}(0) = 0. \tag{2.2.29}$$

Since $\widehat{\varphi}$ is arbitrary, we get that $\varphi_x(0) = \varphi_x(1) = 0$. Hence, $\varphi \in H^2_*(0,1)$. Using similar arguments as above, we can obtain

$$\begin{aligned} & \psi \in H^2(0,1) \cap H^1_0(0,1) \\ & \theta \in H^2_*(0,1). \end{aligned} \tag{2.2.30}$$

Finally, the application of regularity theory for the linear elliptic equations garantees the existence of unique $U \in \mathcal{D}(\mathcal{A})$ such that (2.2.8) is satisfied.

Consequently, we conclude that $\mathcal{A}$ is a maximal dissipative operator. Hence by Lumer-Philips theorem (see [138]), we have the well-posedness result. This completes the proof. □

## 2.3 Stability results

We prepare the next Lemmas (Lemma 2.1-Lemma 2.6) which will be useful to introduce the Lyapunov functional in (2.3.18).

**Lemma 2.1.** *The energy functional $E$ associated with our problem, defined by*

$$\begin{aligned} E(t) &= \frac{\gamma}{2}\Big\{\int_0^1 \Big[\rho_1\varphi_t^2+\mu\varphi_x^2+\rho_2\psi_t^2+\delta\psi_x^2+\xi\psi^2+2b\varphi_x\psi\Big]\,dx\Big\} \\ &+\frac{\beta}{2}\Big\{\int_0^1\Big[l\theta_x^2+\rho_3\theta_t^2\Big]\,dx\Big\}+\frac{\gamma}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)\,dsd\rho dx, \end{aligned} \tag{2.3.1}$$

*satisfies*

$$E'(t) \;\le\; -k\beta\int_0^1\theta_{tx}^2dx-\gamma\eta_0\int_0^1\psi_t^2dx\le 0, \tag{2.3.2}$$

*where $\eta_0=\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\ge 0$.*

*Proof.* Multiplying $(2.1.11)_1$ by $\gamma\varphi_t$, $(2.1.11)_2$ by $\gamma\psi_t$, and $(2.1.11)_3$ by $\beta\theta_t$ then integration by parts over $(0,1)$, we get

$$\begin{aligned} &\frac{\gamma}{2}\frac{d}{dt}\int_0^1\Big[\rho_1\varphi_t^2+\mu\varphi_x^2+\rho_2\psi_t^2+\delta\psi_x^2+\xi\psi^2+2b\varphi_x\psi\Big]dx+\gamma\mu_1\int_0^1\psi_t^2dx \\ &+\frac{\beta}{2}\frac{d}{dt}\int_0^1\Big[l\theta_x^2+\rho_3\theta_t^2\Big]dx+\gamma\int_0^1\psi_t\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,s,t)\,dsdx=0. \end{aligned} \tag{2.3.3}$$

Now, multiplying $(2.1.11)_4$ by $z|\mu_2(s)|$, and integrating the result over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned} &\frac{d}{dt}\frac{\gamma}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx \\ =&\; -\gamma\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|zz_\rho(x,\rho,s,t)\,dsd\rho dx \\ =&\; -\frac{\gamma}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\frac{d}{d\rho}z^2(x,\rho,s,t)\,dsd\rho dx \\ =&\; \frac{\gamma}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|(z^2(x,0,s,t)-z^2(x,1,s,t))dsdx \\ =&\; \frac{\gamma}{2}\Big(\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\Big)\int_0^1\psi_t^2dx \\ &-\frac{\gamma}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)\,dsdx. \end{aligned} \tag{2.3.4}$$

From(2.3.3) and (2.3.4), we get (2.3.1) and (2.3.2) .

Now, using Young's inequality, (2.3.2) can be written as

$$E'(t) \leq -k\beta \int_0^1 \theta_{tx}^2 dx - \gamma\Big(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds\Big) \int_0^1 \psi_t^2 dx.$$

Then, by (2.1.4), there exists a positive constant $\eta_0$ such that

$$E'(t) \leq -k\beta \int_0^1 \theta_{tx}^2 dx - \gamma\eta_0 \int_0^1 \psi_t^2 dx.$$

Thus, the functional $E$ is a non-increasing. □

**Lemma 2.2.** *The functional*

$$F_1(t) := \rho_2 \int_0^1 \psi_t \psi dx + \frac{b\rho_1}{\mu} \int_0^1 \psi \int_0^x \varphi_t(y)\, dy dx + \frac{\mu_1}{2} \int_0^1 \psi^2 dx, \tag{2.3.5}$$

*satisfies*

$$\begin{aligned} F_1'(t) \leq & -\frac{\delta}{2} \int_0^1 \psi_x^2 dx - \mu_3 \int_0^1 \psi^2 dx + \varepsilon_1 \int_0^1 \varphi_t^2 dx + c\Big(1 + \frac{1}{\varepsilon_1}\Big) \int_0^1 \psi_t^2 dx \\ & + c \int_0^1 \theta_{tx}^2 dx + c \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, ds dx, \end{aligned} \tag{2.3.6}$$

*where* $\mu_3 = \xi - \frac{b^2}{\mu} > 0$.

*Proof.* Direct computation, using integration by parts and Young's inequality, for $\varepsilon_1 > 0$, yields

$$\begin{aligned} F_1'(t) = & -\delta \int_0^1 \psi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right) \int_0^1 \psi^2 dx + \rho_2 \int_0^1 \psi_t^2 dx \\ & + \frac{b\rho_1}{\mu} \int_0^1 \psi_t \int_0^x \varphi_t(y)\, dy dx - \beta \int_0^1 \psi \theta_{tx} dx \\ & - \int_0^1 \psi \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t) ds dx \\ \leq & -\delta \int_0^1 \psi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right) \int_0^1 \psi^2 dx + c\left(1 + \frac{1}{\varepsilon_1}\right) \int_0^1 \psi_t^2 dx \\ & + \varepsilon_1 \int_0^1 \left(\int_0^x \varphi_t(y)\, dy\right)^2 dx - \beta \int_0^1 \psi \theta_{tx} dx \\ & - \int_0^1 \psi \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t) ds dx. \end{aligned} \tag{2.3.7}$$

By Cauchy-Schwartz's inequality, it is clear that

$$\int_0^1 \left(\int_0^x \varphi_t(y)\, dy\right)^2 dx \leq \int_0^1 \left(\int_0^1 \varphi_t dx\right)^2 dx \leq \int_0^1 {\varphi_t}^2 dx.$$

So, estimate (2.3.7) becomes

$$\begin{aligned} F_1'(t) \leq & -\delta \int_0^1 \psi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right) \int_0^1 \psi^2 dx + c\left(1 + \frac{1}{\varepsilon_1}\right) \int_0^1 \psi_t^2 dx \\ & + \varepsilon_1 \int_0^1 {\varphi_t}^2 dx - \beta \int_0^1 \psi \theta_{tx} dx - \int_0^1 \psi \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t) ds dx, \end{aligned}$$

where the Cauchy-Schwartz, Young and Poincare's inequalities have been used, for $\varepsilon_1 > 0$.

By the fact that $\mu\xi > b^2$, we get the desired result (2.3.6). □

**Lemma 2.3.** *Assume that (2.1.4) hold. Then the functional*

$$F_2(t) := \int_0^1 \psi_x \varphi_t dx + \int_0^1 \psi_t \varphi_x dx,$$

*satisfies,*

$$\begin{aligned} F_2'(t) \leq & -\frac{b}{2\rho_2}\int_0^1 \varphi_x^2 dx + c\int_0^1 \psi_x^2 dx + c\int_0^1 \psi_t^2 \\ & + c\int_0^1 \theta_{tx}^2 + c\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) dx \\ & + \left(\frac{\delta}{\rho_2} - \frac{\mu}{\rho_1}\right)\int_0^1 \varphi_x \psi_{xx} dx. \end{aligned} \tag{2.3.8}$$

*Proof.* By differentiating $F_2$, then using (2.1.11), integration by parts gives

$$\begin{aligned} F_2'(t) = & -\frac{b}{\rho_2}\int_0^1 \varphi_x^2 dx + \left(\frac{\delta}{\rho_2} - \frac{\mu}{\rho_1}\right)\int_0^1 \varphi_x \psi_{xx} dx + \frac{b}{\rho_1}\int_0^1 \psi_x^2 dx \\ & -\frac{\xi}{\rho_2}\int_0^1 \varphi_x \psi dx - \frac{\mu_1}{\rho_2}\int_0^1 \psi_t \varphi_x dx - \frac{\beta}{\rho_2}\int_0^1 \theta_{tx}\varphi_x dx \\ & -\frac{1}{\rho_2}\int_0^1 \varphi_x \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds dx. \end{aligned} \tag{2.3.9}$$

Thanks to Young, Cauchy-Schwartz, and Poincar's inequalities to estimate terms in RHS of (2.3.9). For $\delta_1, \delta_2, \delta_3, \delta_4 > 0$, we have

$$-\frac{\xi}{\rho_2}\int_0^1 \varphi_x \psi dx \leq \delta_1 \int_0^1 \varphi_x^2 dx + \frac{c}{4\delta_1}\int_0^1 \psi^2 dx, \tag{2.3.10}$$

and

$$-\frac{\mu_1}{\rho_2}\int_0^1 \psi_t \varphi_x dx \leq \delta_2 \int_0^1 \varphi_x^2 dx + \frac{c}{4\delta_2}\int_0^1 \psi_t^2 dx, \tag{2.3.11}$$

and

$$-\frac{\beta}{\rho_2}\int_0^1 \theta_{tx}\varphi_x dx \leq \delta_3 \int_0^1 \varphi_x^2 dx + \frac{c}{4\delta_3}\int_0^1 \theta_{tx}^2 dx, \tag{2.3.12}$$

and

$$\begin{aligned} & -\frac{1}{\rho_2}\int_0^1 \varphi_x \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t) ds dx \\ \leq & \ \delta_4 \int_0^1 \varphi_x^2 dx \\ & + \frac{c}{4\delta_4}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds. \end{aligned} \tag{2.3.13}$$

The replacement of (2.3.10)-(2.3.13) into (2.3.9), and setting $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \dfrac{b}{8\rho_2}$, to obtain (2.3.8). □

**Lemma 2.4.** *The functional*

$$F_3(t) := -\rho_1 \int_0^1 \varphi_t \varphi dx,$$

*satisfies*

$$F_3'(t) \leq -\rho_1 \int_0^1 \varphi_t^2 dx + \frac{3\mu}{2}\int_0^1 \varphi_x^2 dx + c\int_0^1 \psi_x^2 dx. \tag{2.3.14}$$

*Proof.* Direct computations give

$$F_3'(t) = -\rho_1 \int_0^1 \varphi_t^2 dx + \mu \int_0^1 \varphi_x^2 dx + b \int_0^1 \varphi_x \psi dx.$$

Estimat (2.3.14) easily follows by using Young's and Poincaré inequalities

$$F_3'(t) \quad \leq \quad -\rho_1 \int_0^1 \varphi_t^2 dx + \mu \int_0^1 \varphi_x^2 dx + \delta_5 \int_0^1 \varphi_x^2 dx + \frac{c}{4\delta_5} \int_0^1 \psi_x^2 dx,$$

setting $\delta_5 = \frac{\mu}{2}$ to obtain (2.3.14). □

**Lemma 2.5.** *The functional*

$$F_4(t) := -\rho_3 \int_0^1 \theta_t \theta dx,$$

*satisfies*

$$F_4'(t) \leq -\frac{l}{2} \int_0^1 \theta_x^2 dx + c \int_0^1 \psi_t^2 dx + c \int_0^1 \theta_{tx}^2 dx. \tag{2.3.15}$$

*Proof.* Direct computations give

$$F_4'(t) \quad = \quad -l \int_0^1 \theta_x^2 dx + \gamma \int_0^1 \theta_x \psi_t dx - k \int_0^1 \theta_x \theta_{tx} dx + \rho_3 \int_0^1 \theta_t^2 dx.$$

By using Young and Poincaré's inequalities, we get (2.3.15). □

**Lemma 2.6.** *The functional*

$$F_5(t) := \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t) \, ds d\rho dx,$$

*satisfies*

$$\begin{aligned} F_5'(t) \quad \leq \quad & -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t) \, ds d\rho dx + \mu_1 \int_0^1 \psi_t^2 dx \\ & -\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) \, ds dx, \end{aligned} \tag{2.3.16}$$

*where $\eta_1$ is a given positive constant.*

*Proof.* By differentiating $F_5$, with respect to $t$ and using the last equation in $(Hyp1)$, we have

$$\begin{aligned} F_5'(t) \quad &= \quad -2 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_2(s)| z z_\rho(x, \rho, s, t) \, ds d\rho dx \\ &= \quad -\frac{d}{d\rho} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t) \, ds d\rho dx \\ & \quad\quad -\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| [e^{-s} z^2(x, 1, s, t) - z^2(x, 0, s, t)] ds dx. \end{aligned}$$

Using the fact that $z(x,0,s,t)=\psi_t(x,t-s)$, and $e^{-s}\leq e^{-s\rho}\leq 1$, for all $0<\rho<1$, we obtain

$$\begin{aligned} F_5'(t) &= -\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)\,dsd\rho dx \\ &\quad -\int_0^1\int_{\tau_1}^{\tau_2} e^{-s}|\mu_2(s)|z^2(x,1,s,t)\,dsdx+\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\int_0^1\psi_t^2dx. \end{aligned}$$

□

We have $-e^{-s}\leq -e^{-\tau_2}\forall s\in[\tau_1,\tau_2]$. Set $\eta_1=e^{-\tau_2}$ and by (2.1.4), we get (2.3.16).

We state and prove the decay result in the next Theorem

**Theorem 2.2.** *Let (2.1.4) hold. Then, there exist positive constants $\lambda_1$ and $\lambda_2$ such that the functional (2.3.1) satisfies, for any $t>0$*

$$\begin{aligned} E(t) &\leq \lambda_2 e^{-\lambda_1 t}, \qquad if\ \frac{\delta}{\rho_2}=\frac{\mu}{\rho_1}, \\ E(t) &\leq C(E_1(0)+E_2(0))t^{-1}, \quad if\ \frac{\delta}{\rho_2}\neq\frac{\mu}{\rho_1}. \end{aligned} \tag{2.3.17}$$

*Proof.* We define a class of an appropriate Lyapunov functional as

$$\mathcal{L}(t):=NE(t)+N_1F_1(t)+N_2F_2(t)+F_3(t)+F_4(t)+N_5F_5(t), \tag{2.3.18}$$

where $N$, $N_1$, $N_2$, and $N_5$ are positive constants to be selected later.

Differentiating (2.3.18) and by (2.3.2), (2.3.6); (2.3.8), (2.3.14),(2.3.15), (2.3.16), we have

$$\begin{aligned} \mathcal{L}'(t) &\leq -\left[\frac{\delta N_1}{2}-cN_2-c\right]\int_0^1\psi_x^2dx-[\rho_1-N_1\varepsilon_1]\int_0^1\varphi_t^2dx \\ &\quad -\left[\gamma\eta_0N-cN_1(1+\frac{1}{\varepsilon_1})-N_2c-\mu_1N_5-c\right]\int_0^1\psi_t^2dx \\ &\quad -\left[\frac{bN_2}{2\rho_2}-\frac{3\mu}{2}\right]\int_0^1\varphi_x^2dx-N_1\mu_3\int_0^1\psi^2dx \\ &\quad -[N_5\eta_1-cN_1-cN_2]\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)\,dsdx \\ &\quad -\frac{l}{2}\int_0^1\theta_x^2dx-N_5\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|z^2(x,\rho,s,t)\,dsd\rho dx \\ &\quad -[Nk\beta-cN_1-cN_2-c]\int_0^1\theta_{tx}^2dx+N_2(\frac{\delta}{\rho_2}-\frac{\mu}{\rho_1})\int_0^1\varphi_x\psi_{xx}dx. \end{aligned}$$

By setting $\varepsilon_1 = \dfrac{\rho_1}{2N_1}$, we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq\; & -\left[\frac{\delta N_1}{2} - cN_2 - c\right]\int_0^1 \psi_x^2 dx - \frac{\rho_1}{2}\int_0^1 \varphi_t^2 dx - \left[\frac{bN_2}{2\rho_2} - \frac{3\mu}{2}\right]\int_0^1 \varphi_x^2 dx \\
& - \left[\gamma\eta_0 N - cN_1(1+N_1) - cN_2 - \mu_1 N_5 - c\right]\int_0^1 \psi_t^2 dx \\
& - \left[N_5\eta_1 - cN_1 - cN_2\right]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, ds dx \\
& - N_1\mu_3 \int_0^1 \psi^2 dx - \left[Nk\beta - cN_1 - cN_2 - c\right]\int_0^1 \theta_{tx}^2 dx - \frac{l}{2}\int_0^1 \theta_x^2 dx \\
& - N_5\eta_1 \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t)\, ds d\rho dx \\
& + N_2\left(\frac{\delta}{\rho_2} - \frac{\mu}{\rho_1}\right)\int_0^1 \varphi_x \psi_{xx} dx.
\end{aligned}$$

Next, we carefully choose the constants. Starting by $N_2$ to be large enough such that

$$\alpha_1 = \frac{bN_2}{2J} - \frac{3\mu}{2} > 0,$$

and $N_1$ so that

$$\alpha_2 = \frac{\delta N_1}{2} - cN_2 - c > 0,$$

and $N_5$ large enough such that

$$\alpha_3 = N_5\eta_1 - cN_1 - cN_2 > 0.$$

We arrive at

$$\begin{aligned}
\mathcal{L}'(t) \leq\; & -\alpha_2\int_0^1 \psi_x^2 dx - \alpha_0\int_0^1 \psi^2 dx - \frac{\rho}{2}\int_0^1 \varphi_t^2 dx - \alpha_1\int_0^1 \varphi_x^2 dx \\
& - [\gamma\eta_0 N - c]\int_0^1 \psi_t^2 dx - [k\beta N - c]\int_0^1 \theta_{tx}^2 dx - \frac{l}{2}\int_0^1 \theta_x^2 dx \qquad (2.3.19) \\
& - \alpha_3\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, ds dx + \alpha_5\int_0^1 \varphi_x\psi_{xx} dx \\
& - \alpha_4\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t)\, ds d\rho dx, \qquad (2.3.20)
\end{aligned}$$

where $\alpha_0 = \mu_3 N_1 = \left(\xi - \dfrac{b^2}{\mu}\right)N_1, \alpha_4 = N_5\eta_1, \alpha_5 = N_2 k_0 = N_2\left(\dfrac{\delta}{\rho_2} - \dfrac{\mu}{\rho_1}\right)$.
Now, let us define the related functional

$$\mathfrak{L}(t) = N_1 F_1(t) + N_2 F_2(t) + F_3(t) + F_4(t) + N_5 F_5(t),$$

then

$$\begin{aligned}
|\mathfrak{L}(t)| \leq\; & JN_1\int_0^1 |\psi\psi_t|\, dx + \frac{b\rho_1 N_1}{\mu}\int_0^1 \left|\psi\int_0^x \varphi_t(y)\, dy\right| dx + \frac{\mu_1 N_1}{2}\int_0^1 \psi^2 dx \\
& + N_2\int_0^1 |\psi_x\varphi_t + \varphi_x\psi_t|\, dx + \rho_1\int_0^1 |\varphi_t\varphi|\, dx + \rho_3\int_0^1 |\theta_t\theta|\, dx \\
& + N_5\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s e^{-s\rho}|\mu_2(s)| z^2(x,\rho,s,t)\, ds d\rho dx.
\end{aligned}$$

Thanks to Young, Cauchy-Schwartz, and Poincaré's inequalities, to get

$$\begin{aligned} |\mathfrak{L}(t)| &\leq c\int_0^1 \left(\varphi_t^2+\psi_t^2+\psi_x^2+\varphi_x^2+\psi^2+\theta_t^2+\theta_x^2\right)dx \\ &\quad +c\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)\,dsd\rho \leq cE(t). \end{aligned}$$

Then

$$|\mathfrak{L}(t)| = |\mathcal{L}(t)-NE(t)| \leq cE(t).$$

Thus

$$(N-c)E(t) \leq \mathcal{L}(t) \leq (N+c)E(t). \tag{2.3.21}$$

One can now, choose $N$ large enough such that

$$N-c>0, k\beta N-c>0, N\gamma\eta_0-c>0,$$

we get

$$c_2E(t) \leq \mathcal{L}(t) \leq c_3E(t), \forall t\geq 0, \tag{2.3.22}$$

and using (2.3.1), (2.3.19), (2.3.21) and the fact that

$$\int_0^1 \theta_t^2 dx \leq \int_0^1 \theta_{tx}^2 dx,$$

which gives

$$\mathcal{L}'(t) \leq -k_1E(t)+\alpha_5\int_0^1 \varphi_x\psi_{xx}dx, \forall t\geq 0. \tag{2.3.23}$$

for some $k_1,c_2,c_3>0$.

**Case 1:** If, $k_0=\frac{\delta}{\rho_2}-\frac{\mu}{\rho_1}=0$, in this case, (2.3.23) takes the from

$$\mathcal{L}'(t) \leq -k_1E(t), \forall t\geq 0. \tag{2.3.24}$$

A combination with (2.3.22) and (2.3.24) gives

$$\mathcal{L}'(t) \leq -\lambda_1\mathcal{L}(t), \forall t\geq 0,\ \lambda_1=\frac{k_1}{c_2}. \tag{2.3.25}$$

Finally, by integrating (2.3.25), and recalling (2.3.22), we obtain the first result of (2.3.17).

**Case 2:** If, $k_0=\frac{\delta}{\rho_2}-\frac{\mu}{\rho_1}\neq 0$, and

$$\begin{cases} k_0<\frac{k_1\mu^2\gamma\delta}{2N_2(\rho_1+b)}, & if \quad k_0>0 \\ |k_0|<\frac{k_1\mu^2\gamma}{2N_2\rho_1}, & if \quad k_0<0. \end{cases}$$

Let

$$E(t)=E(\varphi,\psi,\theta,z)=E_1(t),$$

denotes by

$$E_2(t) = E(\varphi_t, \psi_t, \theta_t, z_t),$$

Then, we have

$$E_2'(t) \leq -k\beta \int_0^1 \theta_{ttx}^2 dx - \gamma\eta_0 \int_0^1 \psi_{tt}^2 dx. \tag{2.3.26}$$

The last term in (2.3.23), by using $(2.1.11)_1$, and Young's inequality, and by setting $K = \frac{-\rho_1\alpha_5}{\mu}$, we have

$$\begin{aligned}
\alpha_5 \int_0^1 \varphi_x \psi_{xx} dx &= -\frac{\alpha_5\rho_1}{\mu} \int_0^1 \psi_x \varphi_{tt} dx + \frac{b\alpha_5}{\mu} \int_0^1 \psi_x^2 dx \\
&= -K\Big(\frac{d}{dt}\Big[\int_0^1 \psi\varphi_{xt} dx - \int_0^1 \psi_t \varphi_x dx\Big]\Big) - K\int_0^1 \varphi_x \psi_{tt}^2 dx \\
&\quad + \frac{b\alpha_5}{\mu} \int_0^1 \psi_x^2 dx \\
&\leq -K\Big(\frac{d}{dt}\Big[\int_0^1 \psi\varphi_{xt} dx - \int_0^1 \psi_t \varphi_x dx\Big]\Big) + \frac{b\alpha_5}{\mu} \int_0^1 \psi_x^2 dx \\
&\quad + \frac{|K|}{4} \int_0^1 \psi_{tt}^2 dx + |K| \int_0^1 \varphi_x^2 dx.
\end{aligned} \tag{2.3.27}$$

Let

$$\mathcal{N}(t) = \int_0^1 \psi\varphi_{xt} dx - \int_0^1 \psi_t \varphi_x dx, \tag{2.3.28}$$

then (2.3.23)

$$\begin{aligned}
\mathcal{L}'(t) + K\mathcal{N}'(t) &\leq -k_1 E_1'(t) + \frac{b\alpha_5}{\mu} \int_0^1 \psi_x^2 dx \\
&\quad + \frac{|K|}{4} \int_0^1 \psi_{tt}^2 dx + |K| \int_0^1 \varphi_x^2 dx \\
&\leq -k_2 E_1'(t) + \frac{|K|}{4} \int_0^1 \psi_{tt}^2 dx,
\end{aligned} \tag{2.3.29}$$

where

$$k_2 = k_1 - \frac{2}{\mu\gamma}\Big(|K| + \frac{b\alpha_5}{\delta}\Big).$$

Let

$$G(t) = \mathcal{L}(t) + K\mathcal{N}(t) + N_3(E_1(t) + E_2(t)). \tag{2.3.30}$$

If $N_3 > max\{C_0|K| - c_1, |K|, \frac{|K|}{4C}\}$. Indeed,

$$\begin{aligned}
|\mathcal{N}(t)| &= \Big|\int_0^1 \psi\varphi_{xt} dx\Big| + \Big|\int_0^1 \psi_t \varphi_x dx\Big| \\
&\leq \frac{1}{2}\int_0^1 \varphi_{tx}^2 dx + \frac{1}{2}\int_0^1 \psi_t^2 dx + \frac{1}{2}\int_0^1 \psi^2 dx + \frac{1}{2}\int_0^1 \varphi_x^2 dx \\
&\leq E_2(t) + C_0 E_1(t),
\end{aligned}$$

where $C_0 = max\{\frac{2}{\gamma\xi}, \frac{2}{\gamma\mu}, \frac{2}{\gamma\rho_2}\}$. By (2.3.22), we obtain

$$\begin{aligned}
G(t) &\leq c_1 E_1(t) - |K|(E_2(t) + C_0 E_1(t)) + N_3(E_1(t) + E_2(t)) \\
&\leq (N_3 + c_1 - C_0|K|)E_1(t) + (N_3 - |K|)E_2(t).
\end{aligned}$$

It is not hard to prove

$$m_1(E_1(t)+E_2(t)) \leq G(t) \leq m_2(E_1(t)+E_2(t)), \tag{2.3.31}$$

where $m_1, m_2 > 0$. By using (2.3.29) and (2.3.28), we obtain

$$\begin{aligned} G'(t) &= \mathcal{L}'(t)+K\mathcal{N}'(t)+N_3(E_1'(t)+E_2'(t)) \\ &\leq -k_2E_1(t)+\Big(-CN_3+\frac{|K|}{4}\Big)\int_0^1 \psi_{tt}^2 dx. \end{aligned} \tag{2.3.32}$$

Choosing $N_3$ such that

$$CN_3-\frac{|K|}{4} > 0,$$

we have

$$G'(t) \leq -k_2E_1(t). \tag{2.3.33}$$

Integrating (2.3.33), we get

$$\int_0^t E_1(y)dy \leq \frac{1}{k_2}(G(0)-G(1)) \leq \frac{1}{k_2}G(0) \leq \frac{m_2}{k_2}(E_1(0)+E_2(0)), \tag{2.3.34}$$

using the fact that

$$(tE_1(t))' = tE_1'(t)+E_1(t) \leq E_1(t), \tag{2.3.35}$$

we get that

$$tE_1(t) \leq \frac{m_2}{C_2}(E_1(0)+E_2(0)), \tag{2.3.36}$$

which is desired the second result of (2.3.17). This completes the proof. □

Chapter 3

# Exponential stabilisation of a Swelling porous-elastic with microtemperature effect and distributed delay

## 3.1 Introduction

In this chapter, the swelling porous thermoelastic system with the presence of a temperatures, microtemperature effect and distributed delay terms is considered. We will established the well-posedness of the system and we prove the exponential stability result.

First, expansive (swelling) soils have also been classified under porous media theory which studies this type of problem. This is why this field is considered fertile for study, as there are many studies to reduce the damage caused by swelling soil, especially in civil engineering and architecture.

Where the basic field equations of the linear theory of swelling porous elastic soils were presented by

$$\begin{aligned} \rho_u u_{tt} &= P_{1x} + G_1 + H_1, \\ \rho_\phi \phi_{tt} &= P_{2x} + G_2 + H_2, \end{aligned} \tag{3.1.1}$$

where $u, \phi$ are the displacement of the fluid and the elastic solid material, $\rho_u, \rho_\phi > 0$ are the densities of each constituent. And $(P_1, G_1, H_1)$ are the partial tension, internal body forces, and eternal forces acting on the displacement, respectively. Similarly $(P_2, G_2, H_2)$ but acting on the elastic solid. In addition, the constitutive equations of partial tensions are given by

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \underbrace{\begin{pmatrix} a_1, a_2 \\ a_2, a_3 \end{pmatrix}}_{A} \cdot \begin{pmatrix} u_x \\ \phi_x \end{pmatrix}, \tag{3.1.2}$$

where $a_1, a_3 > 0$ and $a_2 \neq 0$ is a real number. $A$ is matrix positive definite with $a_1a_3 > a_2^2$.
The basic evolution equations for one-dimensional theories of swelling porous materials with temperature and microtemperature are given by

$$\begin{aligned} \rho_u u_{tt} &= T_x, \\ \rho_\varphi \phi_{tt} &= H_x + G, \\ \rho T_0 \eta_t &= q_x, \\ \rho E_t &= P_x^* + q - Q. \end{aligned} \tag{3.1.3}$$

Here $T, H, G, q, \eta, P^*, Q, E, T_0$ represents the stress, the equilibrated stress, the equilibrated body force, the heat flux vector, the entropy, the first heat flux moment, the mean heat flux, the first moment of energy and the reference temperature at the equilibrium (we assume $T_0 = 1$ for simplicity).
The constitutive equations are

$$\begin{aligned} T = P_1 + G_1 + H_1 &\quad P^* = -k_2 w_x, \\ H = P_2 + P_3 &\quad \rho\eta = \gamma u_x + c_0\theta + m\phi, \\ G = G_2 + H_2 &\quad Q = -k_3 w - k_1\theta_x, \\ q = \kappa\theta_x + k_1 w &\quad \rho E = -\alpha w - d\phi_x. \end{aligned} \tag{3.1.4}$$

where $w$ is the microtemperature vector and $k_1, k_2, k_3, \alpha, \kappa, c_0, \mu_1, \gamma, m, d > 0$.
As coupling is considered, $a_2 \neq 0$ and satisfies

$$a = a_3 - \frac{a_2^2}{a_1} > 0. \tag{3.1.5}$$

The goal of this work is the thermal effects, so we suppose that the heat capacity $c_0 > 0$, and for more excitement in posing the problem, we suppose that the thermal conductivity is non-existent $\kappa = 0$.
And by introduting the distributed delay term, form a new problem different from previous studies. Under appropriate suppositions the well posedness of the system is established and we prove the exponential stability result by the energy method.
We consider in this work:

$$\begin{aligned} &G_1 = G_2 = 0, \;\; P_3 = -dw \\ &H_1 = -\gamma\theta, \\ &H_2 = m\theta - \mu_1\phi_t - \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \phi_t(x, t-\sigma)\, d\sigma \end{aligned} \tag{3.1.6}$$

Now, by substituting (3.1.4)-(3.1.6) into (3.1.3), we arrive at the following problem:

$$\begin{cases} \rho_u u_{tt} - a_1 u_{xx} - a_2\phi_{xx} - \gamma\theta_x = 0, \\ \rho_\phi\phi_{tt} - a_3\phi_{xx} - a_2 u_{xx} - dw_x + m\theta + \mu_1\phi_t + \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|\phi_t(x,t-\sigma)\, d\sigma = 0, \\ c_0\theta_t = -\gamma u_{tx} - m\phi_t - k_1 w_x, \\ \alpha w_t = k_2 w_{xx} - k_3 w - k_1\theta_x - d\phi_{tx}, \end{cases}$$

where

$$(x,\sigma,t) \in \mathcal{H} = (0,1)\times(\tau_1,\tau_2)\times(0,\infty),$$

under the initial and boundary conditions

$$\begin{aligned} & u(x,0) = u_0(x), u_t(x,0) = u_1(x), \theta(x,0) = \theta_0(x) \\ & \phi(x,0) = \phi_0(x), \phi_t(x,0) = \phi_1(x), w(x,0) = w_0(x), x \in (0,1), \\ & \phi_t(x,-t) = f_0(x,t), \quad (x,t) \in (0,1)\times(0,\tau_2), \\ & u(0,t) = u(1,t) = \phi(0,t) = \phi(1,t) = 0, \\ & \theta(0,t) = \theta(1,t) = w_x(0,t) = w_x(1,t) = 0, t \geq 0. \end{aligned} \tag{3.1.7}$$

First, as in [133], we introduce the new variable

$$\mathcal{Y}(x,\rho,\sigma,t) = \phi_t(x,t-\sigma\rho),$$

then we get

$$\begin{cases} \sigma\mathcal{Y}_t(x,\rho,\sigma,t) + \mathcal{Y}_\rho(x,\rho,\sigma,t) = 0, \\ \mathcal{Y}(x,0,\sigma,t) = \phi_t(x,t). \end{cases}$$

Consequently, our problem is written in the form

$$\begin{cases} \rho_u u_{tt} - a_1 u_{xx} - a_2\phi_{xx} - \gamma\theta_x = 0, \\ \rho_\phi\phi_{tt} - a_3\phi_{xx} - a_2 u_{xx} - dw_x + m\theta + \mu_1\phi_t + \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|\mathcal{Y}(x,1,\sigma,t)\, d\sigma = 0, \\ c_0\theta_t = -\gamma u_{tx} - m\phi_t - k_1 w_x, \\ \alpha w_t = k_2 w_{xx} - k_3 w - k_1\theta_x - d\phi_{tx}, \\ \sigma\mathcal{Y}_t(x,\rho,\sigma,t) + \mathcal{Y}_\rho(x,\rho,\sigma,t) = 0, \end{cases} \tag{3.1.8}$$

where

$$(x,\rho,\sigma,t) \in (0,1)\times\mathcal{H},$$

with the initial data

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), \theta(x,0) = \theta_0(x), \\ \phi(x,0) = \phi_0(x), \phi_t(x,0) = \phi_1(x), w(x,0) = w_0(x) x \in (0,1), \\ \mathcal{Y}(x,\rho,\sigma,0) = f_0(x,\rho\sigma), \quad (x,\rho,\sigma) \in (0,1)\times(0,1)\times(0,\tau_2), \end{cases} \tag{3.1.9}$$

and the boundary conditions

$$\begin{cases} u(0,t) = u(1,t) = \phi(0,t) = \phi(1,t) = 0, \\ \theta(0,t) = \theta(1,t) = w_x(0,t) = w_x(1,t) = 0, \quad t \geq 0. \end{cases} \tag{3.1.10}$$

Here, the integral represent the distributed delay terms with $\tau_1, \tau_2 > 0$ are a time delay, $\mu_2$ is an $L^\infty$ function satisfying:

**(H1)** $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| d\sigma < \mu_1. \tag{3.1.11}$$

Meanwhile, from $(3.1.8)_4$ and (3.1.10), it follows that

$$\frac{d}{dt}\int_0^1 \omega(x,t)\,dx + \frac{k_3}{\alpha}\int_0^1 \omega(x,t)\,dx = 0. \tag{3.1.12}$$

So, by solving (3.1.12) and using the initial data of $u$, we get

$$\int_0^1 \omega(x,t)\,dx = \left(\int_0^1 \omega_0(x)\,dx\right) e^{-\frac{t}{\alpha}k_3}.$$

Consequently, if we let

$$\bar{\omega}(x,t) = \omega(x,t) - \left(\int_0^1 \omega_0(x)\,dx\right) e^{-\frac{t}{\alpha}k_3}, \tag{3.1.13}$$

we get

$$\int_0^1 \bar{\omega}(x,t)\,dx = 0, \forall t \geq 0.$$

Therefore, the use of Poincare's inequality for $\bar{\omega}$ is justified. In addition, simple substitution shows that $(u, \phi, \theta, \bar{\omega}, \mathcal{Y})$ satisfies system (3.1.8 ). Henceforth, we work with $\bar{\omega}$ instead of $\omega$ but write $\omega$ for simplicity of notation.

In this work, we consider $(u, \phi, \theta, w, \mathcal{Y})$ to be a solution of system (3.1.8)-(3.1.10) with the regularity needed to justify the calculations. In Section 2, the well-posedness is established, and in section 3 the exponential stability is proved. In all of the following we mention that $c > 0$.

**Remark 3.1.** *The coupling that we have proposed in this work with the presence of microtemperatures and distributed delay in problems of swelling in porous elasticity, we believe constitutes a new contribution and differs from the previous studies.*

## 3.2 Well-posedness

In this section, we established the well-posedness of the system ( 3.1.8)-(3.1.10).

First, introducing the vector function

$$U = (u, u_t, \phi, \phi_t, \theta, w, \mathcal{Y})^T,$$

and the variables $v = u_t, \varphi = \phi_t$, then the system (3.1.8) writes as follows:

$$\begin{cases} U_t = \mathcal{A}U \\ U(0) = U_0 = (u_0, u_1, \phi_0, \phi_1, \theta_0, w_0, f_0)^T, \end{cases} \tag{3.2.1}$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} :\to \mathcal{H}$ is the linear operator given by

$$\mathcal{A}U = \begin{pmatrix} v \\ -\frac{1}{\rho_u}[a_1 u_{xx} + a_2 \phi_{xx} - \gamma\theta_x] \\ \varphi \\ \frac{1}{\rho_\phi}[a_3\phi_{xx} + a_2 u_{xx} - dw_x + m\theta - \mu_1\varphi - \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|\mathcal{Y}(x,1,\sigma,t)d\sigma] \\ -\frac{1}{c_0}[\gamma v_x + m\varphi + k_1 w_x] \\ \frac{1}{\alpha}[k_2 w_{xx} - k_3 w - k_1\theta_x - d\varphi_x] \\ -\frac{1}{\sigma}\mathcal{Y}_\rho \end{pmatrix}, \tag{3.2.2}$$

and $\mathcal{H}$ is the energy space given by

$$\begin{aligned} \mathcal{H} &= H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times L^2(0,1) \times L_*^2(0,1) \\ &\quad \times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)), \end{aligned}$$

where

$$\begin{aligned} L_*^2(0,1) &= \{\psi \in L^2(0,1) \ / \ \int_0^1 \psi(x)dx = 0\} \\ H_*^1(0,1) &= H^1(0,1) \cap L_*^2(0,1) \\ H_*^2(0,1) &= \{\psi \in H^2(0,1) \ / \ \psi_x(1) = \psi_x(0) = 0\}. \end{aligned}$$

For any

$$\begin{aligned} U &= (u, v, \phi, \varphi, \theta, w, \mathcal{Y})^T \in \mathcal{H}, \\ \widehat{U} &= (\widehat{u}, \widehat{v}, \widehat{\phi}, \widehat{\varphi}, \widehat{\theta}, \widehat{w}, \widehat{\mathcal{Y}})^T \in \mathcal{H}, \end{aligned}$$

we equip $\mathcal{H}$ with the inner product defined by

$$\begin{aligned} < U, \widehat{U} >_{\mathcal{H}} &= \rho_u \int_0^1 v\widehat{v}dx + a_1 \int_0^1 u_x \widehat{u}_x dx + \rho_\phi \int_0^1 \varphi\widehat{\varphi}dx \\ &\quad + a_3 \int_0^1 \phi_x \widehat{\phi}_x dx + c_0 \int_0^1 \theta\widehat{\theta}dx + \alpha \int_0^1 w\widehat{w}dx \\ &\quad + a_2 \int_0^1 (u_x \widehat{\phi}_x + \widehat{u}_x \phi_x)dx \\ &\quad + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \sigma|\mu_2(\sigma)|\mathcal{Y}\widehat{\mathcal{Y}}d\sigma d\rho dx. \end{aligned} \tag{3.2.3}$$

The domain of $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} U \in \mathcal{H}/u, \phi \in H^2(0,1) \cap H_0^1(0,1), v, \varphi, \theta \in H_0^1(0,1), \\ w \in H_*^2(0,1) \cap H_*^1(0,1), \\ \mathcal{Y}, \mathcal{Y}_\rho \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)), \mathcal{Y}(x,0,\sigma,t) = \varphi \end{array} \right\}.$$

Clearly, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$.

**Theorem 3.1.** *Let $U_0 \in \mathcal{H}$ and assume that (3.1.11) holds. Then, there exists a unique solution $U \in \mathcal{C}(\mathbb{R}_+, \mathcal{H})$ of problem (3.2.1).*
*Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then*

$$U \in \mathcal{C}(\mathbb{R}_+, \mathcal{D}(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+, \mathcal{H}).$$

*Proof.* First, we prove that the operator $\mathcal{A}$ is dissipative. For any $U_0 \in \mathcal{D}(\mathcal{A})$ and by using (3.2.3), we have

$$\begin{aligned} < \mathcal{A}U, U >_{\mathcal{H}} &= -\mu_1 \int_0^1 \varphi^2 dx - \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \varphi \mathcal{Y}(x,1,\sigma,t) d\sigma dx \\ &\quad - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \mathcal{Y}_\rho \mathcal{Y} d\sigma d\rho dx \\ &\quad - k_2 \int_0^1 w_x^2 dx - k_3 \int_0^1 w^2 dx. \end{aligned} \tag{3.2.4}$$

For the third term of the RHS of (3.2.4), we have

$$\begin{aligned} -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \mathcal{Y}_\rho \mathcal{Y} d\sigma d\rho dx &= -\frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} \int_0^1 |\mu_2(\sigma)| \frac{d}{d\rho} \mathcal{Y}^2 d\rho d\sigma dx \\ &= -\frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \mathcal{Y}^2(x,1,\sigma,t) d\sigma dx \\ &\quad + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \mathcal{Y}^2(x,0,\sigma,t) d\sigma dx. \end{aligned} \tag{3.2.5}$$

By using Young's inequality, we get

$$\begin{aligned} -\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \varphi \mathcal{Y}(x,1,\sigma,t) d\sigma dx &\leq \frac{1}{2} \left( \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| d\sigma \right) \int_0^1 \varphi^2 dx \\ &\quad + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \mathcal{Y}^2(x,1,\sigma,t) d\sigma dx. \end{aligned} \tag{3.2.6}$$

Substituting (3.2.5),(3.2.6) into (3.2.4), using $\mathcal{Y}(x,0,\sigma,t) = \varphi(x,t)$ and (3.1.11), we find

$$\begin{aligned} < \mathcal{A}U, U >_{\mathcal{H}} &\leq -\eta_0 \int_0^1 \varphi^2 dx - k_2 \int_0^1 w_x^2 dx - k_3 \int_0^1 w^2 dx. \\ &\leq 0, \end{aligned} \tag{3.2.7}$$

where $\eta_0 = (\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| d\sigma) > 0$. Hence, $\mathcal{A}$ is dissipative operator.
Next, we prove $\mathcal{A}$ is maximal operator. It is sufficient to show that $(\lambda I - \mathcal{A})$ is surjective operator. Indeed, for any $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T \in \mathcal{H}$, we prove that there exists a unique $U = (u, v, \phi, \varphi, \theta, w, z) \in \mathcal{D}(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})U = F. \tag{3.2.8}$$

That is

$$\begin{cases} \lambda u - v = f_1 \in H_0^1(0,1) \\ \rho_u \lambda v - a_1 u_{xx} - a_2 \phi_{xx} + \gamma \theta_x = \rho_u f_2 \in L^2(0,1) \\ \lambda \phi - \varphi = f_3 \in H_0^1(0,1) \\ \rho_\phi \lambda \varphi - a_3 \phi_{xx} - a_2 u_{xx} + d w_x - m\theta + \mu_1 \varphi + \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \mathcal{Y}(x,1,\sigma,t) d\sigma = \rho_\phi f_4 \in L^2 \\ c_0 \lambda \theta + \gamma v_x + m\varphi + k_1 w_x = c_0 f_5 \in L^2(0,1) \\ \alpha \lambda w - k_2 w_{xx} + k_3 w + k_1 \theta_x + d\varphi_x = \alpha f_6 \in H_*^1(0,1) \\ \sigma \lambda \mathcal{Y}_t(x,\rho,\sigma,t) + \mathcal{Y}_\rho(x,\rho,\sigma,t) = \sigma f_7 \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)). \end{cases} \tag{3.2.9}$$

We note that the equation $(3.2.9)_7$ with $\mathcal{Y}(x,0,\sigma,t) = \varphi(x,t)$ has a unique solution defined by

$$\mathcal{Y}(x,\rho,\sigma,t) = e^{-\lambda\rho\sigma}\varphi + \sigma e^{\sigma\rho\lambda} \int_0^\rho e^{\lambda\sigma\varrho} f_7(x,\varrho,\sigma,t) d\varrho, \tag{3.2.10}$$

then

$$\mathcal{Y}(x,1,\sigma,t) = e^{-\lambda\sigma}\varphi + \sigma e^{\lambda\sigma} \int_0^1 e^{\lambda\sigma\varrho} f_7(x,\varrho,\sigma,t) d\varrho, \tag{3.2.11}$$

and we have

$$v = \lambda u - f_1, \quad \varphi = \lambda\phi - f_3. \tag{3.2.12}$$

Inserting (3.2.11) and (3.2.12) in $(3.2.9)_2$,$(3.2.9)_4$,

$(3.2.9)_5$ and $(3.2.9)_6$, we get

$$\begin{cases} \rho_u \lambda^2 u - a_1 u_{xx} - a_2 \phi_{xx} + \gamma \theta_x = h_1 \\ \mu_3 \phi - a_3 \phi_{xx} - a_2 u_{xx} + d w_x - m\theta = h_2 \\ c_0 \theta + \gamma u_x + m\phi + \frac{k_1}{\lambda} w_x = h_3 \\ \frac{\alpha\lambda + k_3}{\lambda} w - \frac{k_2}{\lambda} w_{xx} + \frac{k_1}{\lambda} \theta_x + d\phi_x = h_4, \end{cases} \tag{3.2.13}$$

where

$$\begin{cases} h_1 = \rho_u(\lambda f_1 + f_2) \\ h_2 = \rho_\phi f_4 + (\rho_\phi \lambda + \mu_1 + \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| e^{-\sigma\lambda} d\sigma) f_3 \\ \quad - \int_{\tau_1}^{\tau_2} \sigma |\mu_2(\sigma)| e^{\sigma\lambda} \int_0^1 e^{\lambda\sigma\varrho} f_7(x,\varrho,\sigma,t) d\varrho d\sigma. \\ h_3 = \frac{1}{\lambda}(\gamma f_{1x} + m f_3 + c_0 f_5) \\ h_4 = \frac{1}{\lambda}(\alpha f_6 + d\lambda f_3) \\ \mu_3 = \rho_\phi \lambda^2 + \mu_1 \lambda + \lambda \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| e^{-\lambda\sigma} d\sigma. \end{cases} \tag{3.2.14}$$

We multiply (3.2.13) by $\widehat{u}, \widehat{\phi}, \widehat{\theta}, \widehat{w}$, respectively, and integrate their sum over $(0,1)$ to find the following variational formulation:

$$B((u,\phi,\theta,w),(\widehat{u},\widehat{\phi},\widehat{\theta},\widehat{w})) = \Gamma(\widehat{u},\widehat{\phi},\widehat{\theta},\widehat{w}), \tag{3.2.15}$$

where

$$B : (H_0^1(0,1) \times H_0^1(0,1) \times L^2(0,1) \times H_*^1(0,1))^2 \to \mathbb{R},$$

is the bilinear form given by

$$\begin{aligned} B((u,\phi,\theta,w),(\widehat{u},\widehat{\phi},\widehat{\theta},\widehat{w})) &= \rho_u\lambda^2\int_0^1 u\widehat{u}dx + a_1\int_0^1 u_x\widehat{u}_xdx \\ &+a_2\int_0^1 \phi_x\widehat{u}_xdx + \gamma\int_0^1 \theta_x\widehat{u}dx \\ &+\mu_3\int_0^1 \phi\widehat{\phi}dx + a_3\int_0^1 \phi_x\widehat{\phi}_xdx \\ &+a_2\int_0^1 u_x\widehat{\phi}_xdx \\ &+d\int_0^1 w_x\widehat{\phi}dx - m\int_0^1 \theta\widehat{\phi}dx \\ &+c_0\int_0^1 \theta\widehat{\theta}dx + \gamma\int_0^1 u_x\widehat{\theta}dx \\ &+m\int_0^1 \phi\widehat{\theta}dx + \frac{k_1}{\lambda}\int_0^1 w_x\widehat{\theta}dx \\ &+\frac{\alpha\lambda+k_3}{\lambda}\int_0^1 w\widehat{w}dx + \frac{k_2}{\lambda}\int_0^1 w_x\widehat{w}_xdx \\ &+\frac{k_1}{\lambda}\int_0^1 \theta_x\widehat{w}dx + d\int_0^1 \phi_x\widehat{w}dx, \end{aligned} \tag{3.2.16}$$

and

$$\Gamma : (H_0^1(0,1)\times H_0^1(0,1)\times L^2(0,1)\times H_*^1(0,1)) \to \mathbb{R},$$

is the linear functional defined by

$$\Gamma(\widehat{u},\widehat{\phi},\widehat{\theta},\widehat{w}) = \int_0^1 h_1\widehat{u}dx + \int_0^1 h_2\widehat{\phi}dx + \int_0^1 h_3\widehat{\theta}dx + \int_0^1 h_4\widehat{w}dx. \tag{3.2.17}$$

Now, for $V = H_0^1(0,1)\times H_0^1(0,1)\times L^2(0,1)\times H_*^1(0,1)$, equipped with the norm

$$\|(u,\phi,\theta,w)\|_V^2 = \|u\|_2^2 + \|u_x\|_2^2 + \|\phi\|_2^2 + \|\phi_x\|_2^2 + \|\theta\|_2^2 + \|w_x\|_2^2 + \|w\|_2^2,$$

then, we have

$$\begin{aligned} B((u,\phi,\theta,w),(u,\phi,\theta,w)) &= \rho_u\lambda^2\int_0^1 u^2dx + a_1\int_0^1 u_x^2dx \\ &+\mu_3\int_0^1 \phi^2dx + a_3\int_0^1 \phi_x^2dx \\ &+2a_2\int_0^1 u_x\phi_xdx + c_0\int_0^1 \theta^2dx \\ &+\frac{\alpha\lambda+k_3}{\lambda}\int_0^1 w^2dx + \frac{k_2}{\lambda}\int_0^1 w_x^2dx. \end{aligned} \tag{3.2.18}$$

On the other hand, we can write

$$\begin{aligned} a_1u_x^2 + 2a_2u_x\phi_x + a_3\phi_x^2 &= \frac{1}{2}\Big[a_1(u_x + \frac{a_2}{a_3}\phi_x)^2 + a_3(\phi_x + \frac{a_2}{a_1}u_x)^2 \\ &+u_x^2(a_1 - \frac{a_2^2}{a_3}) + \phi_x^2(a_3 - \frac{a_2^2}{a_1})\Big]. \end{aligned} \tag{3.2.19}$$

Since (0.0.23), we deduce

$$a_1u_x^2+2a_2u_x\phi_x+a_3\phi_x^2 \quad > \quad \frac{1}{2}\Big[u_x^2(a_1-\frac{a_2^2}{a_3})+\phi_x^2(a_3-\frac{a_2^2}{a_1})\Big], \tag{3.2.20}$$

then, for some $M_0>0$

$$B((u,\phi,\theta,w),(u,\phi,\theta,w))\geq M_0\|(u,\phi,\theta,w)\|_V^2. \tag{3.2.21}$$

Thus $B$ is coercive. Hence, we use the Lax-Milgram theorem to conclude that (3.2.15) has a unique solution:

$$\begin{aligned} &u,\phi\in H_0^1(0,1)\\ &w\in H_*^1(0,1)\\ &\theta\in L^2(0,1). \end{aligned} \tag{3.2.22}$$

Substituting $u,\phi,\theta$, and $w$ into $(3.2.9)_{1,3}$, we have

$$v,\varphi\in H_0^1(0,1).$$

Similarly, the compensation of $v$ in (3.2.10) with $(3.2.9)_7$, gives

$$\mathcal{Y},\mathcal{Y}_\rho\in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)).$$

Moreover, if we take $\widehat{u}=\widehat{\theta}=\widehat{w}=0$ in (3.2.16), we get

$$\begin{aligned} &a_3\int_0^1\phi_x\widehat{\phi}_xdx+\mu_3\int_0^1\phi\widehat{\phi}dx+a_2\int_0^1u_x\widehat{\phi}_xdx\\ &+d\int_0^1w_x\widehat{\phi}dx-m\int_0^1\theta\widehat{\phi}dx=\int_0^1h_2\widehat{\phi}dx,\ \ \forall\widehat{\phi}\in H_0^1(0,1), \end{aligned}$$

which implies

$$a_3\int_0^1\phi_x\widehat{\phi}_xdx=\int_0^1\Big(h_2-\mu_3\phi+a_2u_{xx}-dw_x+m\theta\Big)\widehat{\phi}dx,\ \ \forall\widehat{\phi}\in H_0^1(0,1),$$

that is

$$a_3\phi_{xx}=\mu_3\phi-a_2u_{xx}+dw_x-m\theta-h_2\in L^2(0,1).$$

Consequently

$$\phi\in H^2(0,1)\cap H_0^1(0,1).$$

Similarly, we get

$$\begin{aligned} &u\in H^2(0,1)\cap H_0^1(0,1)\\ &\theta\in H_0^1(0,1), \end{aligned} \tag{3.2.23}$$

and, if we let $\widehat{u} = \widehat{\theta} = \widehat{\phi} = 0$ in (3.2.16), we get

$$\begin{aligned}&\frac{\alpha\lambda + k_3}{\lambda}\int_0^1 w\widehat{w}dx + \frac{k_2}{\lambda}\int_0^1 w_x\widehat{w}_x dx + \frac{k_1}{\lambda}\int_0^1 \theta_x\widehat{w}dx \\ &+d\int_0^1 \phi_x\widehat{w}dx - m\int_0^1 \theta\widehat{\phi}dx = \int_0^1 h_4\widehat{w}dx, \ \ \forall\widehat{w}\in H^1_*(0,1),\end{aligned}$$

which implies

$$\frac{k_2}{\lambda}\int_0^1 w_x\psi_x dx = \int_0^1 \Big(-h_4 + \frac{\alpha\lambda+k_3}{\lambda}w + \frac{k_1}{\lambda}\theta_x + d\phi_x\Big)\psi dx,$$
$$\forall\psi \in C^1(0,1) \subset H^1_*(0,1),$$

thus, using integration by parts, we get

$$w_x(1)\psi(1) - w_x(0)\psi(0) = 0, \ \ \forall\psi\in C^1(0,1),$$

therefore

$$w_x(1) = w_x(0) = 0.$$

Consequently

$$w \in H^2_*(0,1)\cap H^1_*(0,1).$$

Finally, the application of regularity theory for the linear elliptic equations guarantees the existence of unique $U \in \mathcal{D}(\mathcal{A})$ such that (3.2.8) is satisfied.

Consequently, we conclude that $\mathcal{A}$ is a maximal dissipative operator. Hence by Lumer-Philips theorem (see [138]), we have the well-posedness result. This completes the proof. □

## 3.3 Exponential decay

In this section, we prove our stability result of the system (3.1.8)-(3.1.10).

For this we have the following lemmas.

**Lemma 3.1.** *The energy functional $E$, defined by*

$$\begin{aligned}E(t) \ &= \ \frac{1}{2}\int_0^1 \Big[\rho_u u_t^2 + a_1 u_x^2 + \rho_\phi\phi_t^2 + a_3\phi_x^2 + 2a_2 u_x\phi_x + c_0\theta^2 + \alpha w^2\Big]dx \\ &+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\sigma|\mu_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)\,d\sigma d\rho dx,\end{aligned} \tag{3.3.1}$$

*satisfies*

$$E'(t) \ \le \ -k_2\int_0^1 w_x^2dx - k_3\int_0^1 w^2dx - \eta_0\int_0^1 \phi_t^2dx \le 0, \tag{3.3.2}$$

*where $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|d\sigma > 0$.*

*Proof.* Multiplying the equations $(3.1.8)_{1,2,3,4}$ by $u_t, \phi_t, \theta$ and $w$, integrating by parts over $(0,1)$, and using (3.1.10), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_u u_t^2 + a_1 u_x^2 + \rho_\phi \phi_t^2 + a_3 \phi_x^2 + 2a_2 u_x \phi_x + c_0\theta^2 + \alpha w^2\right] dx \\
&+\mu_1 \int_0^1 \phi_t^2 dx + \int_0^1 \phi_t \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \mathcal{Y}(x,1,\sigma,t)\, d\sigma dx \\
&+k_3 \int_0^1 w^2 dx + k_2 \int_0^1 w_x^2 dx = 0. \qquad (3.3.3)
\end{aligned}$$

Now, multiplying the equation $((3.1.8))_5$ by $\mathcal{Y}|\mu_2(\sigma)|$, and integrating the result over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\mu_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t) d\sigma d\rho dx \\
= & -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|\mathcal{Y}\mathcal{Y}_\rho(x,\rho,\sigma,t)\, d\sigma d\rho dx \\
= & -\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \frac{d}{d\rho}\mathcal{Y}^2(x,\rho,\sigma,t)\, d\sigma d\rho dx \\
= & \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| (\mathcal{Y}^2(x,0,\sigma,t) - \mathcal{Y}^2(x,1,\sigma,t)) d\sigma dx \\
= & \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| d\sigma \int_0^1 \phi_t^2 dx - \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|\mathcal{Y}^2(x,1,\sigma,t)\, d\sigma dx.
\end{aligned} \qquad (3.3.4)$$

Now, by substituting (3.3.4) into (3.3.3), and using Young's inequality, we have

$$E'(t) \le -k_3\int_0^1 w^2 dx - k_2 \int_0^1 w_x^2 dx - \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| d\sigma\right)\int_0^1 \phi_t^2 dx,$$

then, by (3.1.11), $\exists \eta_0 > 0$ so that

$$E'(t) \le -k_3\int_0^1 w^2 dx - k_2\int_0^1 w_x^2 dx - \eta_0 \int_0^1 \phi_t^2 dx, \qquad (3.3.5)$$

then we obtain (3.3.2) ($E$ is a non-increasing function ). □

**Remark 3.2.** *Using (3.1.5), we conclude that $E(t)$ satisfies*

$$\begin{aligned}
E(t) > & \frac{1}{2}\int_0^1 \left[\rho_u u_t^2 + a_4 u_x^2 + \rho_\phi \phi_t^2 + a_5\phi_x^2 + c_0\theta^2 + \alpha w^2\right] dx \\
&+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\mu_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)\, d\sigma d\rho dx,
\end{aligned}$$

*where*

$$\begin{aligned}
a_4 &= \frac{1}{2}\left(a_1 - \frac{a_2^2}{a_3}\right) > 0 \\
a_5 &= \frac{1}{2}\left(a_3 - \frac{a_2^2}{a_1}\right) > 0.
\end{aligned} \qquad (3.3.6)$$

*Then, the function $E(t)$ is non-negative.*

**Lemma 3.2.** *The functional*

$$D_1(t) := \rho_\phi \int_0^1 \phi_t \phi dx - \frac{a_2}{a_1}\rho_u \int_0^1 \phi u_t dx + \frac{\mu_1}{2}\int_0^1 \phi^2 dx, \tag{3.3.7}$$

*satisfies, for any* $\varepsilon_1 > 0$

$$\begin{aligned} D_1'(t) &\leq -\frac{a}{2}\int_0^1 \phi_x^2 dx + \varepsilon_1 \int_0^1 u_t^2 dx + c(1+\frac{1}{\varepsilon_1})\int_0^1 \phi_t^2 dx + c\int_0^1 w^2 dx \\ &\quad + c\int_0^1 \theta^2 dx + c\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \mathcal{Y}^2(x,1,\sigma,t) d\sigma dx. \end{aligned} \tag{3.3.8}$$

*Proof.* Direct computation using integration by parts and Young's inequality, yields

$$\begin{aligned} D_1'(t) &= -a_3 \int_0^1 \phi_x^2 dx + \rho_\phi \int_0^1 \phi_t^2 dx + \frac{a_2^2}{a_1}\int_0^1 \phi_x^2 dx + \frac{a_2}{a_1}\rho_u \int_0^1 \phi_t u_t dx \\ &\quad - d\int_0^1 w_x \phi dx + m\int_0^1 \theta\phi dx + \frac{a_2\gamma}{a_1}\int_0^1 \theta_x \phi dx \\ &\quad - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \mathcal{Y}(x,1,\sigma,t) d\sigma dx \\ &\leq -\left(a_3 - \frac{a_2^2}{a_1}\right)\int_0^1 \phi_x^2 dx + \rho_\phi \int_0^1 \phi_t^2 dx + \frac{a_2}{a_1}\rho_u \int_0^1 \phi_t u_t dx \\ &\quad - d\int_0^1 w_x \phi dx + m\int_0^1 \theta\phi dx + \frac{a_2\gamma}{a_1}\int_0^1 \theta_x \phi dx \\ &\quad - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \mathcal{Y}(x,1,\sigma,t) d\sigma dx, \end{aligned} \tag{3.3.9}$$

we use Cauchy-Schwartz, Young's and poincare's inequalities, for $\delta_1, \varepsilon_1 > 0$, we obtain

$$\begin{aligned} D_1'(t) &\leq -\left(a_3 - \frac{a_2^2}{a_1} - \mu_1 c\delta_1\right)\int_0^1 \phi_x^2 dx + \varepsilon_1 \int_0^1 u_t^2 dx + c(1+\frac{1}{\varepsilon_1})\int_0^1 \phi_t^2 dx \\ &\quad + d\int_0^1 w\phi_x dx + m\int_0^1 \theta\phi dx - \frac{a_2\gamma}{a_1}\int_0^1 \theta\phi_x dx \\ &\quad + \frac{1}{4\delta_1}\int_0^t \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \mathcal{Y}^2(x,1,\sigma,t) d\sigma dx. \end{aligned} \tag{3.3.10}$$

Bearing in mind that (3.1.5), and letting $\delta_1 = \dfrac{a}{2c}$, we obtain the estimate (3.3.8). □

**Lemma 3.3.** *The functional*

$$D_2(t) := a_2\left(\int_0^1 \phi_t u dx - \int_0^1 \phi u_t dx\right),$$

*satisfies,*

$$\begin{aligned} D_2'(t) &\leq -\frac{a_2^2}{2\rho_\phi}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx + c\int_0^1 \phi_t^2 dx + c\int_0^1 w^2 dx \\ &\quad + c\int_0^1 \theta^2 dx + c\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \mathcal{Y}^2(x,1,\sigma,t) d\sigma dx. \end{aligned} \tag{3.3.11}$$

*Proof.* By differentiating $D_2$, then using (3.1.8), integration by parts, and (3.1.10) we obtain

$$\begin{aligned} D_2'(t) &= -\frac{a_2^2}{\rho_\phi}\int_0^1 u_x^2dx + \frac{a_2^2}{\rho_u}\int_0^1 \phi_x^2dx - \left(\frac{a_2a_3}{\rho_\phi} - \frac{a_1a_2}{\rho_u}\right)\int_0^1 \phi_x u_x dx \\ &\quad -\frac{a_2\mu_1}{\rho_\phi}\int_0^1 u\phi_t dx + \frac{a_2 d}{\rho_\phi}\int_0^1 w u_x dx + \frac{a_2 m}{\rho_\phi}\int_0^1 \theta u dx \\ &\quad -\frac{a_2\gamma}{\rho_u}\int_0^1 \theta\phi_x dx - \frac{a_2}{\rho_\phi}\int_0^1 u\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathcal{Y}(x,1,\sigma,t)d\sigma dx. \end{aligned} \tag{3.3.12}$$

Now, we estimate the last six terms in the RHS of ( 3.3.12), using Young's, Cauchy-Schwartz, and Poincare's inequalities. For $\delta_2,\delta_3,\delta_4,\delta_5,\delta_6 > 0$, we have

$$-\left(\frac{a_2a_3}{\rho_\phi} - \frac{a_1a_2}{\rho_u}\right)\int_0^1 \phi_x u_x dx \le \delta_2\int_0^1 u_x^2dx + \left(\frac{a_2a_3}{\rho_\phi} - \frac{a_1a_2}{\rho_u}\right)^2\frac{1}{4\delta_2}\int_0^1 \phi^2 dx,$$

$$\frac{a_2 d}{\rho_\phi}\int_0^1 w u_x dx \le \delta_3\int_0^1 u_x^2dx + \frac{c}{4\delta_3}\int_0^1 w^2dx,$$

$$-\frac{a_2\mu_1}{\rho_\phi}\int_0^1 u\phi_t dx \le c\delta_4\int_0^1 u_x^2dx + \frac{c}{4\delta_4}\int_0^1 \phi_t^2dx,$$

and

$$\frac{a_2 m}{\rho_\phi}\int_0^1 \theta u dx \le \delta_5 c\int_0^1 u_x^2dx + \frac{c}{4\delta_5}\int_0^1 \theta^2dx,$$

$$\begin{aligned} -\frac{a_2}{\rho_\phi}\int_0^1 u\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathcal{Y}(x,1,\sigma,t)d\sigma dx &\le c\delta_6\int_0^1 u_x^2dx \\ &\quad -\frac{c}{4\delta_6}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathcal{Y}^2(x,1,\sigma,t)d\sigma dx. \end{aligned}$$

By letting $\delta_2 = \delta_3 = \frac{a_2}{10\rho_\phi}, \delta_4 = \delta_5 = \delta_6 = \frac{a_2}{10c\rho_\phi}$, and substituting into (3.3.12), we get (3.3.11). □

**Lemma 3.4.** *The functional*

$$D_3(t) := -\rho_u\int_0^1 u_t u dx,$$

*satisfies*

$$D_3'(t) \le -\rho_u\int_0^1 u_t^2dx + 3a_1\int_0^1 u_x^2dx + \frac{a_3}{4}\int_0^1 \phi_x^2dx + \frac{\gamma^2}{4a_1}\int_0^1 \theta^2dx. \tag{3.3.13}$$

*Proof.* Direct computations give

$$D_3'(t) = -\rho_u\int_0^1 u_t^2dx + a_1\int_0^1 u_x^2dx + a_2\int_0^1 u_x\phi_x dx - \gamma\int_0^1 u_x\theta dx.$$

Estimat (3.3.13) easily follows by using Young's inequality and (3.1.5). □

**Lemma 3.5.** *The functional*

$$D_4(t) := -c_0\alpha\int_0^1 \theta\left(\int_0^x w(y)dy\right)dx,$$

*satisfies*

$$D_4'(t) \leq -\frac{c_0k_1}{2}\int_0^1 \theta^2 dx + \varepsilon_2\int_0^1 u_t^2 dx + c(1+\frac{1}{\varepsilon_2})\int_0^1 w^2 dx$$
$$+c\int_0^1 \phi_t^2 dx + c\int_0^1 w_x^2 dx. \tag{3.3.14}$$

*Proof.* Direct computations give

$$D_4'(t) = -c_0k_1\int_0^1 \theta^2 dx + \alpha k_1\int_0^1 w^2 dx + \alpha\gamma\int_0^1 u_t w dx$$
$$+c_0 d\int_0^1 \theta\phi_t dx - \alpha m\int_0^1 \phi_t\Big(\int_0^x w(y)dy\Big)dx$$
$$+c_0k_2\int_0^1 w_x\theta dx - c_0k_3\int_0^1 \theta\Big(\int_0^x w(y)dy\Big)dx.$$

Estimat (3.3.14) easily follows by using Young's and Cauchy-Schwartz inequalities. □

Now, let us introduce the following functional used by

**Lemma 3.6.** *The functional*

$$D_5(t) := \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma e^{-\sigma\rho}|\mu_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)\, d\sigma d\rho dx,$$

*satisfies,*

$$D_5'(t) \leq -\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\mu_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)\, d\sigma d\rho dx + \mu_1\int_0^1 \phi_t^2 dx$$
$$-\eta_1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|\mathcal{Y}^2(x,1,\sigma,t)\, d\sigma dx, \tag{3.3.15}$$

*where* $\eta_1 > 0$.

*Proof.* By differentiating $D_5$, with respect to $t$ and using the last equation in (3.1.8), we have

$$D_5'(t) = -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-\sigma\rho}|\mu_2(\sigma)|\mathcal{Y}\mathcal{Y}_\rho(x,\rho,\sigma,t)\, d\sigma d\rho dx$$
$$= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma e^{-\sigma\rho}|\mu_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)\, d\sigma d\rho dx$$
$$-\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|[e^{-\sigma}\mathcal{Y}^2(x,1,\sigma,t) - \mathcal{Y}^2(x,0,\sigma,t)]d\sigma dx.$$

Using the fact that $\mathcal{Y}(x,0,\sigma,t) = \phi_t(x,t)$, and $e^{-\sigma} \leq e^{-\sigma\rho} \leq 1$, for all $0 < \rho < 1$, we obtain

$$D_5'(t) = -\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\mu_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)\, d\sigma d\rho dx$$
$$-\int_0^1\int_{\tau_1}^{\tau_2} e^{-\sigma}|\mu_2(\sigma)|\mathcal{Y}^2(x,1,\sigma,t)\, d\sigma dx + (\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|d\sigma)\int_0^1 \phi_t^2 dx.$$

□

Because $-e^{-\sigma}$ is a increasing function, we have $-e^{-\sigma} \leq -e^{-\tau_2}$, for all $\sigma \in [\tau_1, \tau_2]$.
Finally, setting $\eta_1 = e^{-\tau_2}$ and recalling (3.1.11), we find (3.3.15). We are now ready to prove the main result.

**Theorem 3.2.** *Assume (3.1.11) hold. Then,* $\forall t_0 > 0$, *there exist* $\beta_1, \beta_2 > 0$ *such that the energy functional given by (3.3.1) satisfies*

$$E(t) \leq \beta_1 e^{-\beta_2 t}, \quad \forall t \geq 0. \tag{3.3.16}$$

*Proof.* We define the functional of Lyapunov

$$\mathcal{L}(t) := NE(t) + N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t) + N_5 D_5(t), \tag{3.3.17}$$

where $N$, $N_1$, $N_2, N_4, N_5 > 0$ we will assign them later.
By differentiating (3.3.17) and using (3.3.1), (3.3.8), (3.3.11), (3.3.13),(3.3.14), (3.3.15), we have

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\frac{aN_1}{2} - cN_2 - \frac{a_3}{4}\right] \int_0^1 \phi_x^2 dx - [\rho_u - N_1\varepsilon_1 - N_4\varepsilon_2] \int_0^1 u_t^2 dx \\
& - \left[\frac{a_2^2 N_2}{2\rho_\phi} - 3a_1\right] \int_0^1 u_x^2 dx - [k_2 N - cN_4] \int_0^1 w_x^2 dx \\
& - \left[\eta_0 N - cN_1(1 + \frac{1}{\varepsilon_1}) - N_2 c - N_4 c - \mu_1 N_5\right] \int_0^1 \phi_t^2 dx \\
& - \left[k_3 N - cN_1 - cN_2 - cN_4(1 + \frac{1}{\varepsilon_2})\right] \int_0^1 w^2 dx \\
& - \left[\frac{c_0 k_1 N_4}{2} - cN_1 - cN_2 - \frac{\gamma^2}{4a_1}\right] \int_0^1 \theta^2 dx \\
& - [N_5\eta_1 - cN_1 - cN_2] \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \mathcal{Y}^2(x, 1, \sigma, t) \, d\sigma dx \\
& - N_5\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \sigma |\mu_2(\sigma)| \mathcal{Y}^2(x, \rho, \sigma, t) \, d\sigma d\rho dx.
\end{aligned}$$

By setting

$$\varepsilon_1 = \frac{\rho_u}{4N_1}, \quad \varepsilon_2 = \frac{\rho_u}{4N_4},$$

we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\frac{aN_1}{2} - cN_2 - \frac{a_3}{4}\right]\int_0^1 \phi_x^2 dx - \left[\frac{\rho_u}{2}\right]\int_0^1 u_t^2 dx \\
& -\left[\frac{a_2^2 N_2}{2\rho_\phi} - 3a_1\right]\int_0^1 u_x^2 dx - [k_2 N - cN_4]\int_0^1 w_x^2 dx \\
& -[\eta_0 N - cN_1(1+N_1) - N_2 c - N_4 c - \mu_1 N_5]\int_0^1 \phi_t^2 dx \\
& -[k_3 N - cN_1 - cN_2 - cN_4(1+N_4)]\int_0^1 w^2 dx \\
& -\left[\frac{c_0 k_1 N_4}{2} - cN_1 - cN_2 - \frac{\gamma^2}{4a_1}\right]\int_0^1 \theta^2 dx \\
& -[N_5\eta_1 - cN_1 - cN_2]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|\mathcal{Y}^2(x,1,\sigma,t)\, d\sigma dx \\
& -N_5\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\sigma|\mu_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)\, d\sigma d\rho dx.
\end{aligned}$$

At this point, choosing our constants.

We choose $N_2$ large enough so that

$$\alpha_1 = \frac{a_2^2 N_2}{2\rho_\phi} - 3a_1 > 0,$$

then we pick $N_1$ large enough such that

$$\alpha_2 = \frac{aN_1}{2} - cN_2 - \frac{a_3}{4} > 0,$$

then we select $N_4$ and $N_5$ large enough such that

$$\begin{aligned}
\alpha_3 &= \frac{c_0 k_1 N_4}{2} - cN_1 - cN_2 - \frac{\gamma^2}{4a_1} > 0 \\
\alpha_4 &= N_5\eta_1 - cN_1 - cN_2 > 0.
\end{aligned}$$

Thus, we arrive at

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\alpha_2\int_0^1 \phi_x^2 dx - \frac{\rho_u}{2}\int_0^1 u_t^2 dx - \alpha_1\int_0^1 u_x^2 dx - [\eta_0 N - c]\int_0^1 \phi_t^2 dx \\
& -[k_3 N - c]\int_0^1 w^2 dx - [k_2 N - c]\int_0^1 w_x^2 dx - \alpha_3\int_0^1 \theta^2 dx \\
& -\alpha_4\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|\mathcal{Y}^2(x,1,\sigma,t)\, d\sigma dx \\
& -\alpha_5\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\sigma|\mu_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)\, d\sigma d\rho dx. \qquad (3.3.18)
\end{aligned}$$

where $\alpha_5 = \eta_1 N_5$.

On the other hand, if we let

$$\mathfrak{L}(t) = N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t) + N_5 D_5(t),$$

then

$$\begin{aligned} |\mathfrak{L}(t)| \leq\ & N_1\rho_\phi\int_0^1|\phi\phi_t|\,dx+N_1\frac{a_2}{a_1}\rho_u\int_0^1|\phi u_t|\,dx+N_1\frac{\mu_1}{2}\int_0^1\phi^2dx\\ &+N_2a_2\int_0^1|\phi u_t-u\phi_t|\,dx+\rho_u\int_0^1|u_tu|\,dx\\ &+N_4c_0\alpha\int_0^1|\theta\Big(\int_0^x w(y)dy\Big)|dx\\ &+N_5\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\sigma e^{-\sigma\rho}|\mu_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)\,d\sigma d\rho dx. \end{aligned}$$

According Young's, Cauchy-Schwartz, and Poincaré inequalities, we find

$$\begin{aligned} |\mathfrak{L}(t)| \leq\ & c\int_0^1\Big(u_t^2+\phi_t^2+\phi_x^2+u_x^2+\theta^2+w^2\Big)\,dx\\ +\ & c\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\sigma|\mu_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)\,d\sigma d\rho. \end{aligned}$$

Hence, by (3.2.19) and (3.2.20), we get

$$|\mathfrak{L}(t)|=|\mathcal{L}(t)-NE(t)|\leq cE(t),$$

that is

$$(N-c)E(t)\leq\mathcal{L}(t)\leq(N+c)E(t). \tag{3.3.19}$$

At this point, we choose $N$ large enough such that

$$N-c>0, N\eta_0-c>0, Nk_3-c>0, Nk_2-c>0,$$

and exploiting (3.3.1), the estimates (3.3.18) and (3.3.19), respectively, gives

$$c_2E(t)\leq\mathcal{L}(t)\leq c_3E(t),\forall t\geq 0, \tag{3.3.20}$$

and

$$\mathcal{L}'(t)\leq -d_1E(t),\forall t\geq 0, \tag{3.3.21}$$

for some $d_1,c_2,c_3>0$.

Consequently, for some $\beta_2>0$, we find

$$\mathcal{L}'(t)\leq-\beta_2\mathcal{L}(t),\forall t\geq 0. \tag{3.3.22}$$

Integration of (3.3.22) over $(0,t)$ gives

$$\mathcal{L}(t)\leq\mathcal{L}(0)e^{-\beta_2 t},\forall t\geq 0. \tag{3.3.23}$$

Consequently, (3.3.16) is established by virtue of (3.3.20) and ( 3.3.23). □

# Part II

# Some PDEs of Hyperbolic type: Bresse system and Laminated beam

Chapter 4

# Stability result for thermo-elastic Bresse system of second sound with past history and delay terms

## 4.1 introduction

In the present chapter, a one-dimensional linear thermo-elastic system of Bresse type with past history and delay term is considered. We prove the well-posedness of the problem using the semigroup method. By using the energy method we discuss the stability of the system for two cases. An exponential stability result of the system (4.1.1) is obtained in the case where the propagation velocities are equal in the equation of vertical displacement and the equation of the system rotation angle in (4.3.31). On the other hand, a result of algebraic stability is obtained in the case of the different propagation velocities in (4.3.32).

Here, we are interested in the question of stability for the system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + lw + \psi)_x - k_0 l(w_x - l\varphi) + \mu_1\varphi_t(x,t) + \mu_2\varphi_t(x,t-\tau) = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + lw + \psi) + \int_0^\infty g(s)\psi_{xx}(x,t-s)\,ds + \gamma\theta_x = 0 \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + lw + \psi) = 0 \\ \rho_3\theta_t + \kappa q_x + \gamma\psi_{tx} = 0 \\ \alpha q_t + \beta q + \kappa\theta_x = 0, \end{cases} \tag{4.1.1}$$

where

$$(x,t) \in (0,1) \times (0,\infty),$$

with initial-boundary conditions

$$\begin{aligned} &\varphi(x,t) = \varphi_x(x,t) = \psi_x(x,t) = \psi(x,t) = 0 \\ &w_x(x,t) = w(x,t) = \theta(x,t) = q(x,t) = 0, \quad x = 0,1. \end{aligned} \tag{4.1.2}$$

and

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), & x \in (0,1) \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), & x \in (0,1) \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), & x \in (0,1) \\ \theta(x,0) = \theta_0(x), q(x,0) = q_0(x) \\ \varphi_t(x,t-\tau) = f_0(x,t-\tau), \end{cases} \tag{4.1.3}$$

with $\tau > 0$ is a time delay, $\mu_1 > 0$ and $\mu_2$ is a real constant. The function $\theta$ is the temperature difference, $q$ is the heat flux, $\rho_1, \rho_2, \rho_3, k, l, k_0, b, \gamma, \kappa, \alpha, \beta$ are positive constants. The relaxation function $g$ satisfies the folowing

(G1) The function $g \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfying

$$g(0) > 0, \; b - g_0 = L > 0, \; g_0 = \int_0^\infty g(s)\, ds > 0. \tag{4.1.4}$$

(G2) For some positive constant $\zeta$, we have

$$g'(t) \leq -\zeta g(t), \forall t \geq 0. \tag{4.1.5}$$

(G3) The following hypotheses

$$|\mu_2| < \mu_1. \tag{4.1.6}$$

Hold. We prove the well-posedness and establish a stabilities results related with the following

$$\tilde{\eta} = \left(\kappa^2 - \frac{\alpha k \rho_3}{\rho_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{L}\right) - \frac{\gamma^2 \alpha}{L} \text{ and } k = k_0. \tag{4.1.7}$$

**Remark 4.1.** *The case where $\tilde{\eta} \neq 0$ is very important from the application point of view, where waves are not necessarily of equal speeds. The stability result in this case will be shown in (4.3.32) which is new and very original. Never before have researchers reported this case, especially in thermo-elastic Bresse system.*

## 4.2 Basic concepts and Well-posedness

By using the semigroup theory, we will prove that systems (4.1.1)-(4.1.3) are well-posedness. Let us introduce a new variable as in [132]

$$z(x,\rho,t) = \varphi_t(x, t - \tau\rho), x \in (0,1), \rho \in (0,1), t > 0. \tag{4.2.1}$$

Then, we have

$$\tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0 \text{ in } (0,1)\times(0,1)\times(0,\infty). \tag{4.2.2}$$

Set an auxiliary variable as in [51]

$$\eta^t(x,s)=\psi(x,t)-\psi(x,t-s), s\geq 0.$$

Then,

$$\eta_t^t(x,s)+\eta_s^t(x,s)=\psi_t(x,t).$$

Therefore, problem (4.1.1) takes the form

$$\begin{cases} \rho_1\varphi_{tt}-k(\varphi_x+lw+\psi)_x-lk_0(w_x-l\varphi)+\mu_1\varphi_t(x,t)+\mu_2 z(x,1,t)=0 \\ \tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0 \\ \rho_2\psi_{tt}-L\psi_{xx}+k(\varphi_x+lw+\psi)-\int_0^\infty g(s)\eta_{xx}^t(x,s)\,ds+\gamma\theta_x=0 \\ \rho_1 w_{tt}-k_0(w_x-l\varphi)_x+lk(\varphi_x+lw+\psi)=0 \\ \rho_3\theta_t+\kappa q_x+\gamma\psi_{tx}=0 \\ \alpha q_t+\beta q+\kappa\theta_x=0 \\ \eta_t^t(x,s)+\eta_s^t(x,s)=\psi_t(x,t). \end{cases} \tag{4.2.3}$$

With the boundary and initial conditions

$$\begin{cases} \varphi(x,t)=\varphi_x(x,t)=\psi_x(x,t)=\psi(x,t)=w_x(x,t)=w(x,t)=0 \\ \theta(x,t)=q(x,t)=0, \ \ x=0,1, \ \ t\geq 0, \eta^t(0,s)=\eta^t(1,s)=0, \forall s\geq 0 \\ \varphi(x,0)=\varphi_0(x), \varphi_t(x,0)=\varphi_1(x), \psi(x,0)=\psi_0(x), \ \ x\in(0,1) \\ \psi_t(x,0)=\psi_1(x), w(x,0)=w_0(x), w_t(x,0)=w_1(x), x\in(0,1) \\ \theta(x,0)=\theta_0(x), q(x,0)=q_0(x), x\in(0,1) \\ \varphi_t(x,-t)=f_0(x,t) \text{ in } (0,1)\times(0,\tau) \\ z(x,1,t)=f_0(x,t-\tau) \text{ in } (0,1)\times(0,\tau) \\ \eta^t(x,0)=0, \qquad \forall t\geq 0 \\ \eta^0(x,s)=\eta_0(s)=0 \qquad \forall s\geq 0. \end{cases} \tag{4.2.4}$$

Let $\xi>0$ such that

$$\tau|\mu_2|<\xi<\tau(2\mu_1-|\mu_2|), \tag{4.2.5}$$

where, $\tau$ is a real number with $0<\tau$ and $\mu_1>0$ and $\mu_2$ is a real constant and $(\varphi_0,\varphi_1,f_0,\psi_0,\psi_1,w_0,w_1,\theta_0,q_0,\eta_0)$ belong to a suitable space. Let us set

$$U=\left(\varphi,\varphi_t,z,\psi,\psi_t,w,w_t,\theta,q,\eta^t\right)^T,$$

then

$$U'=\left(\varphi_t,\varphi_{tt},z_t,\psi_t,\psi_{tt},w_t,w_{tt},\theta_t,q_t,\eta_t^t\right)^T.$$

Then (4.2.3)-(4.2.4) can be take the form

$$\begin{cases} U'(t) - AU(t) = 0 \\ U(0) = (\varphi_0, \varphi_1, f_0(., -\tau), \psi_0, \psi_1, w_0, w_1, \theta_0, q_0, \eta_0), \end{cases} \tag{4.2.6}$$

and the new dependent variables $\varphi_t = u,\ \ \psi_t = v,\ \ \eta^t = \phi,\ \ \omega_t = \varpi,$

the operator $A$ is given by

$$A\begin{pmatrix} \varphi \\ u \\ z \\ \psi \\ v \\ w \\ \varpi \\ \theta \\ q \\ \phi \end{pmatrix} = \begin{pmatrix} u \\ \frac{k}{\rho_1}(\varphi_x + lw + \psi)_x + \frac{k_0 l}{\rho_1}(w_x - l\varphi) - \frac{\mu_1}{\rho_1}u - \frac{\mu_2}{\rho_1}z(.,1) \\ -\left(\frac{1}{\tau}\right) z_\rho \\ v \\ \frac{L}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + lw + \psi) + \frac{1}{\rho_2}\int_0^\infty g(s)\phi_{xx}(s)\,ds - \frac{\gamma}{\rho_2}\theta_x \\ \varpi \\ \frac{k_0}{\rho_1}(w_x - l\varphi)_x - \frac{kl}{\rho_1}(\varphi_x + lw + \psi) \\ -\frac{\kappa}{\rho_3}q_x - \frac{\gamma}{\rho_3}v_x \\ -\frac{\beta}{\alpha}q - \frac{\kappa}{\alpha}\theta_x \\ -\phi_s + v \end{pmatrix}. \tag{4.2.7}$$

We consider the following space

$$\begin{aligned} \mathcal{H} &= H_*^1(0,1) \times L^2(0,1) \times L^2\Big((0,1), H_*^1(0,1)\Big) \times H_0^1(0,1) \times L^2(0,1) \times H_*^1(0,1) \\ &\quad \times L^2(0,1) \times L^2(0,1) \times L^2(0,1) \times L_g^2\left(\mathbb{R}^+, H_*^1(0,1)\right), \end{aligned}$$

where

$$H_*^1(0,1) = \Big\{\phi \in H_0^1(0,1),\ \ \phi_x(0) = \phi_x(0) = 0\Big\},$$

and $L_g^2(\mathbb{R}^+, H_*^1(0,1))$ denotes the Hilbert space of $H_*^1-$valued functions on $\mathbb{R}^+$, endowed with the inner product

$$(V_1, V_2)_{L_g^2(\mathbb{R}^+, H_*^1(\Omega))} = \int_0^1 \int_0^1 g(s) V_{1x}(s) V_{2x}(s)\,ds dx.$$

We are now going to show that $A$ generates a $C_0$ semigroup on $\mathcal{H}$ under (4.2.5). For this end, we define on $\mathcal{H}$ for

$$U = (\varphi, u, z, \psi, v, w, \varpi, \theta, q, \phi)^T, \overline{U} = \left(\overline{\varphi}, \overline{u}, \overline{z}, \overline{\psi}, \overline{v}, \overline{w}, \overline{\varpi}, \overline{\theta}, \overline{q}, \bar{\phi}\right)^T,$$

the inner product

$$\begin{aligned} \left\langle U, \overline{U} \right\rangle_{\mathcal{H}} &= k\int_0^1 (\varphi_x + \psi + lw)\left(\overline{\varphi}_x + \overline{\psi} + l\overline{w}\right)dx + k_0\int_0^1 (w_x - l\varphi)(\overline{w}_x - l\overline{\varphi})\,dx \\ &\quad + \rho_1\int_0^1 u\overline{u}dx + \rho_2\int_0^1 v\overline{v}dx + \rho_1\int_0^1 \varpi\overline{\varpi}dx + L\int_0^1 \psi_x\overline{\psi_x}dx \\ &\quad + \rho_3\int_0^1 \theta\overline{\theta}dx + \alpha\int_0^1 q\overline{q}dx + \xi\int_0^1\int_0^1 z\overline{z}d\rho dx \\ &\quad + \int_0^1\int_0^\infty g(s)\phi_x(s)\bar{\phi}_x(s)\,dxds, \end{aligned} \tag{4.2.8}$$

for $l$ small enough since, it is easy to see that $\mathcal{H}$ is a Hilbert space. We define the domain of $A$ as

$$D(A)=\left\{\begin{array}{c} U\in\mathcal{H}/\varphi\in H^2\cap H^1_*;\psi,w\in H^2\cap H^1_*,u,v,\varpi\in H^1_*(0,1),\\ \theta,q\in H^1_0(0,1),u=z(.,0),z_\rho\in L^2((0,1),L^2(0,1)),\\ \phi_s\in L^2_g(\mathbb{R}^+,H^1_*(0,1)), \end{array}\right\}. \tag{4.2.9}$$

The following two Lemmas will be useful to prove that $A$ is a maximal monotone operator.

**Lemma 4.1.** *The operator $A$ is dissipative and satisfies, for any $U\in D(A)$,*

$$\begin{aligned} \langle AU,U\rangle_{\mathcal{H}} &\leq -\beta\int_0^1 q^2dx+\Big(-\mu_1+\frac{|\mu_2|}{2}+\frac{\xi}{2\tau}\Big)\int_0^1 u^2dx\\ &\quad+\Big(\frac{|\mu_2|}{2}-\frac{\xi}{2\tau}\Big)\int_0^1 z^2(x,1)\,dx\\ &\quad+\int_0^1\int_0^\infty g'(s)\,|\phi_x(x,s)|^2\,dsdx\\ &\leq 0. \end{aligned} \tag{4.2.10}$$

*Proof.* Using the inner product for any $U\in D(A)$

$$\langle AU,U\rangle_{\mathcal{H}}=$$
$$\left\langle\begin{pmatrix} u\\ \frac{k}{\rho_1}(\varphi_x+lw+\psi)_x+\frac{k_0l}{\rho_1}(w_x-l\varphi)-\frac{\mu_1}{\rho_1}u-\frac{\mu_2}{\rho_1}z(.,1)\\ -\left(\frac{1}{\tau}\right)z_\rho\\ v\\ \frac{L}{\rho_2}\psi_{xx}-\frac{k}{\rho_2}(\varphi_x+lw+\psi)+\frac{1}{\rho_2}\int_0^\infty g(s)\,\phi_{xx}(s)\,ds-\frac{\gamma}{\rho_2}\theta_x\\ -\varpi\\ \frac{k_0}{\rho_1}(w_x-l\varphi)_x-\frac{kl}{\rho_1}(\varphi_x+lw+\psi)\\ -\frac{\kappa}{\rho_3}q_x-\frac{\gamma}{\rho_3}v_x\\ -\frac{\beta}{\alpha}q-\frac{\kappa}{\alpha}\theta_x\\ -\phi_s+v \end{pmatrix},\begin{pmatrix}\varphi\\ u\\ z\\ \psi\\ v\\ w\\ \varpi\\ \theta\\ q\\ \phi\end{pmatrix}\right\rangle_{\mathcal{H}}$$

Then

$$\begin{aligned}
\langle AU,U\rangle_H &= k\int_0^1 \varphi_{tx}\left(\varphi_x+\psi+lw\right)+k\int_0^1 \psi_t\left(\varphi_x+\psi+lw\right)dx \\
&\quad +kl\int_0^1 w_t\left(\varphi_x+lw+\psi\right)+k_0\int_0^1 w_{tx}\left(w_x-l\varphi\right) \\
&\quad +\int_0^1 \varphi_t\left[k\left(\varphi_x+lw+\psi\right)_x+k_0l\left(w_x-l\varphi\right)-\mu_1\varphi_t-\mu_2 z\left(x,1\right)\right] \\
&\quad +\int_0^1 w_t\left[k_0\left(w_x-l\varphi\right)_x-lk\left(\varphi_x+lw+\psi\right)\right]-k_0l\int_0^1 \varphi_t\left(w_x-l\varphi\right) \\
&\quad +\int_0^1 \psi_t\left[L\psi_{xx}-k\left(\varphi_x+lw+\psi\right)+\int_0^\infty g\left(s\right)\phi_{xx}\left(s\right)ds-\gamma\theta_x\right] \\
&\quad +L\int_0^1 \psi_x\psi_{tx}-\frac{\xi}{\tau}\int_0^1\int_0^1 z\left(x,\rho\right)z_\rho\left(x,\rho\right)dxd\rho \\
&\quad +\int_0^1\int_0^\infty g\left(s\right)\phi_x\left(s\right)\left(-\phi_s+v\right)_x dxds \\
&\quad -\int_0^1 \theta q_x dx-\gamma\int_0^1 \theta\psi_{tx}-\beta\int_0^1 q^2-\int_0^1 \theta_x q \\
&= -\beta\int_0^1 q^2dx-\mu_1\int_0^1 u^2dx-\mu_2\int_0^1 z\left(x,1\right)udx \\
&\quad -\frac{\xi}{\tau}\int_0^1\int_0^1 z\left(x,\rho\right)z_\rho\left(x,\rho\right)d\rho dx+\int_0^1\int_0^\infty g'\left(s\right)\left|\phi_x\left(x,s\right)\right|^2 dsdx \\
&= -\beta\int_0^1 q^2dx-\mu_1\int_0^1 u^2dx-\mu_2\int_0^1 z\left(x,1\right)udx \\
&\quad -\frac{\xi}{2\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}z^2\left(x,\rho\right)d\rho dx+\int_0^1\int_0^\infty g'\left(s\right)\left|\phi_x\left(x,s\right)\right|^2 dsdx \\
&= -\beta\int_0^1 q^2dx-\mu_1\int_0^1 u^2dx-\mu_2\int_0^1 z\left(x,1\right)udx \\
&\quad -\frac{\xi}{2\tau}\int_0^1 \left\{z^2\left(x,1\right)-z^2\left(x,0\right)\right\}dx+\int_0^1\int_0^\infty g'\left(s\right)\left|\phi_x\left(x,s\right)\right|^2 dsdx \\
&= -\beta\int_0^1 q^2dx-\mu_1\int_0^1 u^2dx-\mu_2\int_0^1 z\left(x,1\right)udx-\frac{\xi}{2\tau}\int_0^1 z^2\left(x,1\right)dx \\
&\quad +\frac{\xi}{2\tau}\int_0^1 u^2dx+\int_0^1\int_0^\infty g'\left(s\right)\left|\phi_x\left(x,s\right)\right|^2 dsdx, \qquad (4.2.11)
\end{aligned}$$

we obtain by Young's inequality that

$$\begin{aligned}
\langle AU,U\rangle_H &\leq -\beta\int_0^1 q^2dx-\mu_3\int_0^1 u^2dx-\mu_4\int_0^1 z^2\left(x,1\right)dx \\
&\quad +\int_0^1\int_0^\infty g'\left(s\right)\left|\phi_x\left(x,s\right)\right|^2 dsdx.
\end{aligned}$$

where

$$\mu_3=\left(\mu_1-\frac{|\mu_2|}{2}-\frac{\xi}{2\tau}\right),\quad \mu_4=\left(\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right).$$

By condition (4.2.5), the desired result yields. □

**Lemma 4.2.** *The operator $I-A$ is surjective.*

*Proof.* For all

$$\mathcal{F}=\left(f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8,f_9,f_{10}\right)^T\in\mathcal{H}.$$

We show that there exists $U \in D(A)$ such that

$$U - AU = \mathcal{F} \tag{4.2.12}$$

that is

$$\begin{cases} -u+\varphi = f_1 \in H_*^1(0,1) \\ -k(\varphi_x+lw+\psi)_x - k_0 l(w_x - l\varphi) + \rho_1 u + \mu_1 u + \mu_2 z(.,1) = \rho_1 f_2 \in L^2(0,1) \\ z+\tau^{-1} z_\rho = \tau f_3 \in L^2((0,1), H^1(0,1)) \\ -v+\psi = f_4 \in H_*^1(0,1) \\ -L\psi_{xx} + k(\varphi_x+lw+\psi) + \rho_2 v - \int_0^\infty g(s)\phi_{xx}(s)\,ds + \gamma\theta_x = \rho_2 f_5 \in L^2(0,1) \\ -\varpi + w = f_6 \in H_*^1(0,1) \\ -k_0(w_x - l\varphi)_x + kl(\varphi_x+lw+\psi) + \rho_1\varpi = \rho_1 f_7 \in L^2(0,1) \\ q_x + \gamma v_x + \rho_3\theta = \rho_3 f_8 \in L^2(0,1) \\ (\beta+\alpha)q + \theta_x = \alpha f_9 \in L^2(0,1) \\ \phi + \phi_s - v = f_{10} \in L^2(0,1). \end{cases} \tag{4.2.13}$$

From (4.2.13), we define

$$\theta = \alpha \int_0^x f_9(y)\,dy - (\beta+\alpha)\int_0^x q(y)\,dy, \tag{4.2.14}$$

then $\theta(0,t) = 0$. Inserting

$$u = \varphi - f_1, v = \psi - f_4, \varpi = w - f_6,$$

and (4.2.13) into (4.2.14), we get

$$\begin{cases} -k(\varphi_x+lw+\psi)_x - k_0 l(w_x-l\varphi) + (\rho_1+\mu_1+\mu_2 e^{-\tau})\varphi = h_1 \in L^2(0.1) \\ -L\psi_{xx} + k(\varphi_x+lw+\psi) + \rho_2\psi - \int_0^\infty g(s)\phi_{xx}(s)\,ds - \gamma(\beta+\alpha)q = h_2 \in L^2 \\ -k_0(w_x-l\varphi)_x + kl(\varphi_x+lw+\psi) + \rho_1 w = h_3 \in L^2(0.1) \\ q_x + (\beta+\alpha)\int_0^x q(y)\,dy - \gamma\psi_x = h_4 \in L^2(0.1) \\ z + \tau^{-1} z_\rho = h_5 \in L^2(0.1) \\ \phi+\phi_s - v = h_6 \in L^2(0.1), \end{cases} \tag{4.2.15}$$

where

$$\begin{cases} h_1 = \rho_1(f_1+f_2) + (\mu_1 f_1 + \mu_2 z_0) \\ h_2 = \rho_2(f_4+f_5) - \alpha\gamma f_9 \\ h_3 = \rho_1(f_6+f_7) \\ h_4 = -\gamma f_{4x} - \rho_3(f_8 - \alpha\int_0^x f_9(y)\,dy) \\ h_5 = \tau f_3. \\ h_6 = f_{10}. \end{cases} \tag{4.2.16}$$

Furthermore, by (4.2.13) we can find as $z(x,0) = u(x)$ for $x \in (0,1)$.

As in [133], we obtain, by using equation for $z$ in (4.2.13)

$$z(x,\rho) = u(x)e^{-\tau\rho} + \tau e^{-\tau\rho}\int_0^\rho f_3(x,s)e^{\tau\rho s}ds.$$

By (4.2.13), we get

$$z(x,\rho)=\varphi(x)e^{-\tau\rho}-f_1e^{\tau\rho}+\tau e^{-\tau\rho}\int_0^\rho f_3(x,s)e^{\tau\rho s}ds,$$

then

$$z(x,1)=\varphi(x)e^{-\tau}+z_0(x),$$

such that

$$z_0(x)=-f_1e^{-\tau}+\tau e^{-\tau}\int_0^\rho f_3(x,s)e^{\tau s}ds.$$

Equation $(4.2.15)_6$ with $\phi(x,0)=0$ has a unique solution given as

$$\begin{aligned}\phi(x,s) &= \left(\int_0^x e^y\left(f_{10}(x,y)+v(x)\,dy\right)e^{-s}\right)\\ &= \left(\int_0^x e^y\left(f_{10}(x,y)+\psi(x)-f_4(x)\,dy\right)e^{-s}\right).\end{aligned} \tag{4.2.17}$$

To solve (4.2.16) let us consider

$$a\left((\varphi,\psi,w,q),\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}\right)\right)=L\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}\right), \tag{4.2.18}$$

where

$$a:\left[H_*^1(0,1)\times H_*^1(0,1)\times H_*^1(0,1)\times L^2(0,1)\right]^2\longrightarrow\mathbb{R},$$

is the bilinear form defined by

$$\begin{aligned}a\left((\varphi,\psi,w,q),\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}\right)\right) &= k\int_0^1(\varphi_x+lw+\psi)\left(\widetilde{\varphi}_x+l\widetilde{w}+\widetilde{\psi}\right)dx\\ &+(\beta+\alpha)\int_0^1 q\widetilde{q}dx+b\int_0^1\psi_x\widetilde{\psi}_xdx\\ &+\rho_2\int_0^1\psi\widetilde{\psi}dx-\gamma(\beta+\alpha)\int_0^1 q\widetilde{\psi}dx\\ &+\rho_1\int_0^1\psi\widetilde{\psi}dx+\gamma(\beta+\alpha)\int_0^1\psi\widetilde{q}dx\\ &+\rho_1\int_0^1 w\widetilde{w}dx+\int_0^1\varphi\widetilde{\varphi}\left(\rho_1+\mu_1+\mu_2e^{-\tau}\right)dx\\ &+k_0\int_0^1(w_x-l\varphi)(\widetilde{w}_x-l\widetilde{\varphi})\,dx\\ &+\rho_3(\beta+\alpha)\int_0^1\left(\int_0^x q(y)\,dy\int_0^x\widetilde{q}(y)\,dy\right)dx,\end{aligned} \tag{4.2.19}$$

and

$$L:\left[H_*^1(0,1)\times H_*^1(0,1)\times H_*^1(0,1)\times L^2(0,1)\right]\longrightarrow\mathbb{R},$$

is the linear form given by

$$\begin{aligned}L\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}\right) &= \int_0^1 h_1\widetilde{\varphi}dx+\int_0^1 h_2\widetilde{\psi}dx+\int_0^1 h_3\widetilde{w}dx\\ &+(\alpha+\beta)\int_0^1 h_4\int_0^x\widetilde{q}(y)\,dydx.\end{aligned} \tag{4.2.20}$$

It is not hard to see that $a$ is continuous and coercive and $L$ is continuous. So by using the Lax-Milgram theorem, we find that

$$\forall\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}\right)\in H_*^1(0,1)\times H_*^1(0,1)\times H_*^1(0,1)\times L^2(0,1)$$

the problem (**??**) admits a unique nsolution

$$(\varphi,\psi,w,q)\in H_*^1(0,1)\times H_*^1(0,1)\times H_*^1(0,1)\times L^2(0,1).$$

The existence of unique $U\in D(A)$ such that (4.2.6) is satisfied comes from the regularity theory for the linear elliptic equations. Then, by Lemmas 2.1 and Lemma 2.2, we conclude that $A$ is a maximal monotone operator. Hence, by Hille-Yosida theorem we can state well-posedness result (see [138]). □

**Theorem 4.1.** *Let $U_0\in\mathcal{H}$, then there exists a unique weak solution $U\in C(\mathbb{R}^+,\mathcal{H})$ of problem (4.1.1)-(4.1.3). Moreover, if $U_0\in D(A)$, then $U\in C(\mathbb{R}^+,D(A))\cap C^1(\mathbb{R}^+,\mathcal{H})$.*

## 4.3 Exponential and algebraic Stabilities

Here, we introduce our stabilities results for solution of (4.1.1)-(4.1.3). The energy associated with solution is given by

$$\begin{aligned}E(t) &= \frac{1}{2}\int_0^1\Big[\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1 w_t^2+L\psi_x^2+\rho_3\theta^2+\alpha q^2\\&+k(\varphi_x+\psi+lw)^2+k_0(w_x-l\varphi)^2\Big]dx+\frac{\xi}{2\tau}\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx\\&+\frac{1}{2}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx.\end{aligned}\tag{4.3.1}$$

To achieve our goal, we need the following Lemmas.

**Lemma 4.3.** *Let $(\varphi,\psi,w,\theta,q,z,\eta^t)$ be the solution of (4.2.3)-(4.2.4). Then the energy functional, defined by (4.3.1) satisfies*

$$\begin{aligned}E'(t) &\le -\beta\int_0^1 q^2dx-\mu_3\int_0^1\varphi_t^2dx-\mu_4\int_0^1 z^2(x,1,t)\,dx\\&+\frac{1}{2}\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx,\end{aligned}\tag{4.3.2}$$

*where $\mu_3=\left(\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)>0$, $\mu_4=\left(\frac{\xi}{2\tau}-\frac{\mu_2}{2}\right)>0$.*

*Proof.* Multiplying $(4.1.1)_1$, $(4.1.1)_2$, $(4.1.1)_3$, $(4.1.1)_4$, and $(4.1.1)_5$ by $\varphi_t$, $\psi_t$, $w_t$, $\theta$, and $q$, respectively. The integration over $(0,1)$ and using (4.2.5), we get (4.3.2). □

**Lemma 4.4.** *Let $(\varphi,\psi,w,\theta,q,z,\eta^t)$ be the solution of (4.2.3)-(4.2.4). Then the functional*

$$F_1(t):=\alpha\rho_3\int_0^1\theta\int_0^x q(y)\,dydx,\tag{4.3.3}$$

*satisfies, for any* $\varepsilon_1 > 0$, *the estimate*

$$F_1'(t) \le -\frac{\rho_3\kappa}{2}\int_0^1 \theta^2 dx + \varepsilon_1 \int_0^1 \psi_t^2 dx + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 q^2 dx. \tag{4.3.4}$$

*Proof.* Taking the derivative of $F_1$, using (4.2.3)$_5$, (4.2.3)$_6$, we get

$$F_1'(t) = -\rho_3\kappa\int_0^1 \theta^2 dx + \alpha\kappa\int_0^1 q^2 dx + \alpha\gamma\int_0^1 q\psi_t dx - \beta\rho_3\int_0^1 \theta\int_0^x q(y)\,dydx. \tag{4.3.5}$$

Thanks to Cauchy–Schwartz and Young's inequalities with $\varepsilon_1 > 0$ to get (4.3.4). □

**Lemma 4.5.** *Let* $(\varphi,\psi,w,\theta,q,z,\eta^t)$ *be the solution of (4.2.3)-(4.2.4). Then the functional*

$$F_2(t) := \frac{\rho_2\rho_3}{\gamma}\int_0^1 \psi_t \int_0^x \theta(y)\,dydx, \tag{4.3.6}$$

*satisfies, for any* $\varepsilon_1,\varepsilon_2 > 0$, *the estimate*

$$\begin{aligned} F_2'(t) \quad \le \quad & -\frac{\rho_2}{2}\int_0^1 \psi_t^2 dx + \varepsilon_2\int_0^1 (\varphi_x+\psi+lw)^2 dx + \varepsilon_3\int_0^1 \psi_x^2 dx \\ & + c\left(1+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}\right)\int_0^1 \theta^2 dx + c\int_0^1 q^2 dx \\ & + c\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx. \end{aligned} \tag{4.3.7}$$

*Proof.* By differentiating $F_2$, then exploiting the (4.1.1)$_2$, (4.1.1)$_4$ we get

$$\begin{aligned} F_2'(t) \quad = \quad & -\rho_2\int_0^1 \psi_t^2 dx - \frac{\rho_2\kappa}{\gamma}\int_0^1 q\psi_t dx + \rho_3\int_0^1 \theta^2 dx - \frac{L\rho_3}{\gamma}\int_0^1 \theta\psi_x dx \\ & - \frac{k\rho_3}{\gamma}\int_0^1 (\varphi_x+\psi+lw)\int_0^x \theta(y)\,dydx \\ & + \frac{\rho_3}{\gamma}\int_0^1\int_0^\infty g(s)\,\theta\eta_x^t(x,s)\,dsdx. \end{aligned} \tag{4.3.8}$$

We obtain (4.3.7) by applying the Cauchy–Schwarz and Young's inequalities. □

**Lemma 4.6.** *Let* $(\varphi,\psi,w,\theta,q,z,\eta^t)$ *be the solution of (4.2.3)-(4.2.4). Then the functional*

$$F_3(t) := \rho_2\int_0^1 \psi\psi_t dx, \tag{4.3.9}$$

*satisfies*

$$\begin{aligned} F_3'(t) \quad \le \quad & -\frac{L}{2}\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \psi_t^2 dx \\ & + \frac{3k^2}{2cL}\int_0^1 (\varphi_x+\psi+lw)^2 dx + c\int_0^1 \theta^2 dx \\ & + c\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx. \end{aligned} \tag{4.3.10}$$

*Proof.* Taking the derivative of $F_3$ and using (4.2.3)$_3$, it follows that

$$\begin{aligned} F_3'(t) \quad = \quad & -L\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \psi_t^2 dx + \gamma\int_0^1 \psi_x\theta dx - k\int_0^1 \psi(\varphi_x+\psi+lw)\,dx \\ & + \int_0^1 \psi_x(x)\int_0^\infty g(s)\,\eta_x^t(x,s)\,dsdx. \end{aligned} \tag{4.3.11}$$

Thanks to Young and Poincaré's inequalities, to get (4.3.10). □

**Lemma 4.7.** *Let* $(\varphi,\psi,w,\theta,q,z,\eta^t)$ *be the solution of (4.2.3)-(4.2.4). Then the functional*

$$F_4(t) := -\rho_1\int_0^1 \varphi_t(w_x-l\varphi)\,dx-\rho_1\int_0^1 w_t(\varphi_x+\psi+lw)\,dx, \tag{4.3.12}$$

*satisfies the estimate*

$$\begin{aligned} F_4'(t) &\le -\frac{lk_0}{2}\int_0^1 (w_x-l\varphi)^2\,dx-\frac{l\rho_1}{2}\int_0^1 w_t^2dx+c\int_0^1\varphi_t^2dx \\ &\quad +c\int_0^1\psi_t^2dx+lk\int_0^1(\varphi_x+\psi+lw)^2\,dx+c\int_0^1 z^2(x,1,t)\,dx. \end{aligned} \tag{4.3.13}$$

*Proof.* By differentiating $F_4$ and using $(4.2.3)_1$, $(4.2.3)_4$, we obtain

$$\begin{aligned} F_4'(t) &= -lk_0\int_0^1 (w_x-l\varphi)^2\,dx-l\rho_1\int_0^1 w_t^2dx+l\rho_1\int_0^1\varphi_t^2dx \\ &\quad +lk\int_0^1(\varphi_x+\psi+lw)^2\,dx-\rho_1\int_0^1\psi_t w_t dx \\ &\quad +\mu_1\int_0^1\varphi_t(w_x-l\varphi)\,dx+\mu_2\int_0^1 z(x,1,t)(w_x-l\varphi)\,dx. \end{aligned} \tag{4.3.14}$$

We obtain (4.3.13) by applying Young's inequality. □

**Lemma 4.8.** *Let* $(\varphi,\psi,w,\theta,q,z,\eta^t)$ *be the solution of (4.2.3)-(4.2.4) and let (4.1.7) holds. Then the functional*

$$\begin{aligned} F_5(t) &:= \rho_2\int_0^1\psi_t(\varphi_x+\psi+lw)\,dx+\frac{L\rho_1}{k}\int_0^1\varphi_t\psi_x dx \\ &\quad +\frac{L\rho_3}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1\theta\varphi_t dx-\frac{L\kappa}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 q(\varphi_x+\psi+lw)\,dx \\ &\quad -\frac{Ll^2\rho_2}{k_0}\int_0^1\psi\psi_t dx+\frac{Ll\rho_1}{k_0}\int_0^1\psi w_t dx-\frac{\rho_1}{k}\int_0^1\varphi_t\int_0^\infty g(s)\eta_x^t(s)dsdx, \end{aligned} \tag{4.3.15}$$

*satisfies, for any* $\varepsilon_4,\varepsilon_5,\varepsilon_6>0$, *the estimate*

$$\begin{aligned} F_5'(t) &\le -\frac{k}{2}\int_0^1(\varphi_x+\psi+lw)^2\,dx+2\varepsilon_4\int_0^1 w_t^2dx+\left(\frac{L^2l^2}{k}+4\varepsilon_6\right)\int_0^1\psi_x^2dx \\ &\quad +2\varepsilon_5\int_0^1(w_x-l\varphi)^2\,dx+c\left(1+\frac{1}{\varepsilon_4}\right)\int_0^1\psi_t^2dx+c\left(1+\frac{1}{\varepsilon_4}\right)\int_0^1 q^2dx \\ &\quad +c\left(1+\frac{1}{\varepsilon_5}+\frac{1}{\varepsilon_6}\right)\int_0^1\theta^2dx+c\left(1+\frac{1}{\varepsilon_6}\right)\int_0^1 z^2(x,1,t)dx \\ &\quad +c\left(1+\frac{1}{\varepsilon_5}+\frac{1}{\varepsilon_6}\right)\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx+c\left(1+\frac{1}{\varepsilon_6}\right)\int_0^1\varphi_t^2dx \\ &\quad -c\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2dsdx+\frac{L\tilde{\eta}}{\gamma\alpha}\int_0^1\theta_x(\varphi_x+\psi+lw)\,dx, \end{aligned} \tag{4.3.16}$$

*where* $\tilde{\eta}=\left(\kappa^2-\frac{\alpha k\rho_3}{\rho_1}\right)\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)-\frac{\gamma^2\alpha}{L}$.

*Proof.* A simple differentiation of $F_5$ gives

$$\begin{aligned}
F_5'(t) &= \rho_2\int_0^1 \psi_{tt}(\varphi_x+\psi+lw)\,dx+\rho_2\int_0^1 \psi_t(\varphi_x+\psi+lw)_t\,dx \\
&+\frac{L\rho_1}{k}\int_0^1 \varphi_{tt}\psi_x dx-\frac{L\rho_1}{k}\int_0^1 \varphi_t\psi_{xt}dx+\frac{L\rho_3}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 \theta_t\varphi_t dx \\
&+\frac{L\rho_3}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 \theta\varphi_{tt}dx-\frac{L\kappa}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 q_t(\varphi_x+\psi+lw)\,dx \\
&-\frac{L\kappa}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 q(\varphi_x+\psi+lw)_t\,dx-\frac{Ll^2\rho_2}{k_0}\int_0^1 \psi_t^2 dx \\
&-\frac{Ll^2\rho_2}{k_0}\int_0^1 \psi_{tt}\psi dx+\frac{Ll\rho_1}{k_0}\int_0^1 w_{tt}\psi dx+\frac{Ll\rho_1}{k_0}\int_0^1 w_t\psi dx \\
&+\frac{Ll^2\rho_2}{k_0}\int_0^1 \psi_x(x)\int_0^\infty g(s)\,\eta_x^t(x,s)\,dsdx \\
&-\frac{\rho_1}{k}\int_0^1 \varphi_{tt}\int_0^\infty g(s)\eta_x^t(s)dsdx-\frac{\rho_1}{k}\int_0^1 \varphi_t\frac{d}{dt}\left\{\int_0^\infty g(s)\eta_x^t(s)ds\right\}dx. \qquad (4.3.17)
\end{aligned}$$

Using (4.1.1)-(4.1.3), the terms in (4.3.17) take the forme

$$\begin{aligned}
\rho_2\int_0^1 \psi_{tt}(\varphi_x+\psi+lw)\,dx &= -k\int_0^1 (\varphi_x+\psi+lw)^2 dx \\
&-\gamma\int_0^1 \theta_x(\varphi_x+\psi+lw)\,dx \\
&-L\int_0^1 \psi_x(\varphi_x+\psi+lw)_x\,dx \\
&+\int_0^1 (\varphi_x+\psi+lw)\int_0^\infty g(s)\eta_{xx}^t(s)dsdx, \qquad (4.3.18)
\end{aligned}$$

$$\begin{aligned}
\rho_1\int_0^1 \varphi_{tt}\psi_x dx &= k\int_0^1 \psi_x(\varphi_x+\psi+lw)_x\,dx+k_0 l\int_0^1 (w_x-l\varphi)\,\psi_x dx \\
&-\mu_1\int_0^1 \psi_x\varphi_t dx-\mu_2\int_0^1 z(x,1,t)\psi_x dx, \qquad (4.3.19)
\end{aligned}$$

$$\rho_3\int_0^1 \theta_t\varphi_t dx=\kappa\int_0^1 q\varphi_{xt}dx+\gamma\int_0^1 \psi_t\varphi_{xt}dx, \qquad (4.3.20)$$

$$\begin{aligned}
\int_0^1 \theta\varphi_{tt}dx &= -\frac{k}{\rho_1}\int_0^1 \theta_x(\varphi_x+\psi+lw)\,dx+\frac{lk_0}{\rho_1}\int_0^1 \theta(w_x-l\varphi)\,dx \\
&-\frac{\mu_1}{\rho_1}\int_0^1 \theta\varphi_t dx-\frac{\mu_2}{\rho_1}\int_0^1 \theta z(x,1,t)dx, \qquad (4.3.21)
\end{aligned}$$

$$\begin{aligned}
-\int_0^1 q_t(\varphi_x+\psi+lw)\,dx &= \frac{\beta}{\alpha}\int_0^1 q(\varphi_x+\psi+lw)\,dx \\
&+\frac{\kappa}{\alpha}\int_0^1 \theta_x(\varphi_x+\psi+lw)\,dx, \qquad (4.3.22)
\end{aligned}$$

$$\begin{aligned}
-\rho_2\int_0^1 \psi_{tt}\psi dx &= L\int_0^1 \psi_x^2 dx+k\int_0^1 \psi(\varphi_x+\psi+lw)\,dx \\
&-\gamma\int_0^1 \theta\psi_x dx-\int_0^1 \psi\int_0^\infty g(s)\eta_{xx}^t(s)dsdx, \qquad (4.3.23)
\end{aligned}$$

$$\begin{aligned}
-\frac{\rho_1}{k}\int_0^1 \varphi_{tt}\int_0^\infty g(s)\eta_x^t(s)dsdx &= -\int_0^1 (\varphi_x+\psi+lw)_x\int_0^\infty g(s)\eta_x^t(s)dsdx \\
&\quad -\frac{k_0 l}{k}\int_0^1 (\omega_x-l\varphi)\int_0^\infty g(s)\eta_x^t(s)dsdx \\
&\quad +\frac{\mu_1}{k}\int_0^1 \varphi_t\int_0^\infty g(s)\eta_x^t(s)dsdx \\
&\quad +\frac{\mu_2}{k}\int_0^1 z(x,1,t)\int_0^\infty g(s)\eta_x^t(s)dsdx, \qquad (4.3.24)
\end{aligned}$$

$$-\frac{\rho_1}{k}\int_0^1 \varphi_t\frac{d}{dt}\left\{\int_0^\infty g(s)\eta_x^t(s)ds\right\}dx = -\frac{\rho_1}{k}\int_0^1 \varphi_t\left\{g_0\psi_t+\int_0^\infty g'(s)\eta_x^t(s)ds\right\}dx, \qquad (4.3.25)$$

$$\rho_1\int_0^1 w_{tt}\psi dx = -k_0\int_0^1 \psi_x(w_x-l\varphi)dx - kl\int_0^1 \psi(\varphi_x+\psi+lw)dx. \qquad (4.3.26)$$

Substituting (4.3.18)-(4.3.26) into (4.3.17), by (4.1.7), to get

$$\begin{aligned}
F_5'(t) &= -k\int_0^1 (\varphi_x+\psi+lw)^2 dx + \left(\rho_2-\frac{Ll^2\rho_2}{k_0}\right)\int_0^1 \psi_t^2 dx \\
&\quad +\left(l\rho_2+\frac{Ll\rho_1}{k_0}\right)\int_0^1 \psi_t w_t dx + \frac{L\tilde{\eta}}{\alpha\gamma}\int_0^1 \theta_x(\varphi_x+\psi+lw)dx \\
&\quad -\frac{L}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 q\psi_t dx - \frac{bl}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 qw_t dx \\
&\quad -\frac{L\mu_1}{k}\int_0^1 \varphi_t\psi_x dx - \frac{L\mu_2}{k}\int_0^1 \psi_x z(x,1,t)dx \\
&\quad +\frac{Llk_0\rho_3}{\gamma\rho_1}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 \theta(w_x-l\varphi)dx - \frac{\gamma Ll^2}{k_0}\int_0^1 \theta\psi_x dx \\
&\quad -\frac{L\mu_1\rho_3}{\gamma\rho_1}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 \varphi_t\theta dx - \frac{L\mu_2\rho_3}{\gamma\rho_1}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 \theta z(x,1,t)dx \\
&\quad +\frac{L\beta\kappa}{\alpha\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 q(\varphi_x+\psi+lw)dx + \frac{L^2l^2}{k_0}\int_0^1 \psi_x^2 dx \\
&\quad +Ll\left(\frac{k_0}{k}-1\right)\int_0^1 \psi_x(w_x-l\varphi)dx \\
&\quad +\frac{Ll^2\rho_2}{k_0}\int_0^1 \psi_x\int_0^\infty g(s)\eta_x^t(s)dsdx + \frac{\mu_1}{k}\int_0^1 \varphi_t\int_0^\infty g(s)\eta_x^t(s)dsdx \\
&\quad -\frac{lk_0}{k}\int_0^1 (\omega_x-l\varphi)\int_0^\infty g(s)\eta_x^t(s)dsdx \\
&\quad +\frac{\mu_2}{k}\int_0^1 z(x,1,t)\int_0^\infty g(s)\eta_x^t(s)dsdx \\
&\quad +\frac{\rho_1 g_0}{k}\int_0^1 \varphi_t\psi_t dx + \frac{\rho_1}{k}\int_0^1 \varphi_t\int_0^\infty g'(s)\eta_x^t(s)dsdx. \qquad (4.3.27)
\end{aligned}$$

We obtain (4.3.16) by Cauchy–Schwarz and Young's inequalites since $k=k_0$. □

**Lemma 4.9.** *Let $(\varphi,\psi,w,\theta,q,z,\eta^t)$ be the solution of (4.2.3)-(4.2.4). Then, we define the functional*

$$F_6(t) := \int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)d\rho dx. \qquad (4.3.28)$$

*Then the following result holds.*

$$F_6'(t) \le -F_6(t) - \frac{c_1}{2\tau}\int_0^1 z^2(x,1,t)dx + \frac{1}{2\tau}\int_0^1 \varphi_t^2(x,t)dx, \qquad (4.3.29)$$

*where $c>0$.*

*Proof.* Differentiating (4.3.28) with respect to $t$ and using the equation (4.2.2), we have

$$\begin{aligned}\frac{d}{dt}\left(\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho dx\right) &= -\frac{1}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho}zz_\rho(x,\rho,t)\,d\rho dx\\ &= -\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho dx\\ &\quad -\frac{1}{2\tau}\int_0^1\int_0^1\frac{\partial}{\partial\rho}\left(e^{-2\tau\rho}z^2(x,\rho,t)\right)d\rho dx.\end{aligned} \tag{4.3.30}$$

This implies that there exists a positive constant $c_1$ such that (4.3.29) holds. □

The main result is given in the next Theorem

**Theorem 4.2.** *Assume that (4.1.4)–(4.1.6) hold, and (4.1.7). Then the energy functional (4.3.1) satisfies,* $\forall t\geq 0$

$$E(t) \leq \lambda_0 e^{-\lambda_1 t} \quad ,if \quad \tilde{\eta}=0 \tag{4.3.31}$$

$$E(t) \leq C(E_1(0)+E_2(0))t^{-1} \quad ,if \quad \tilde{\eta}\neq 0 \tag{4.3.32}$$

*where the positive constant $\lambda_0$ is directly depending on initial data and the uniform constant $\lambda_1$ is depending only on the coefficients of the system.*

*Proof.* We define a Lyapunov functional

$$\mathcal{L}(t) := NE(t)+N_1F_1(t)+N_2F_2(t)+N_3F_3(t)+F_4(t)+N_5F_5(t)+F_6(t), \tag{4.3.33}$$

where $N, N_1, N_2, N_3, N_5 > 0$.

By differentiating of (4.3.33), and using (4.3.2), (4.3.4), (4.3.7), (4.3.10), (4.3.13), (4.3.16), and (4.3.29), we have

$$
\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\beta N - c_1 N_1\left(1+\frac{1}{\varepsilon_1}\right) - cN_2 - c\left(1+\frac{1}{\varepsilon_4}\right)N_5\right]\int_0^1 q^2 dx \\
& -\left[N\mu_3 - c\left(1+\frac{1}{\varepsilon_6}\right)N_5 - c\right]\int_0^1 \varphi_t^2 dx \\
& -\left[N\mu_4 - c\left(1+\frac{1}{\varepsilon_6}\right)N_5 - c\right]\int_0^1 z^2(x,1,t)\,dx \\
& -\left[\frac{N_1\rho_3\kappa}{2} - cN_2\left(1+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}\right) - cN_3 - c\left(1+\frac{1}{\varepsilon_5}+\frac{1}{\varepsilon_6}\right)N_5\right]\int_0^1 \theta^2 dx \\
& -\left[N_2\frac{\rho_2}{2} - \varepsilon_1 N_1 - \rho_2 N_3 - c\left(1+\frac{1}{\varepsilon_4}\right)N_5 - c\right]\int_0^1 \psi_t^2 dx \\
& -\left[\frac{k}{2}N_5 - \varepsilon_2 N_2 - \frac{3k^2}{2cL}N_3 - c\right]\int_0^1 (\varphi_x + lw + \psi)^2 dx \\
& -\left[\frac{L}{2}N_3 - \varepsilon_3 N_2 - 4\varepsilon_6 N_5 - \frac{L^2 l^2}{k}N_5\right]\int_0^1 \psi_x^2 dx \\
& -\left[\frac{l\rho_1}{2} - 2\varepsilon_4 N_5\right]\int_0^1 w_t^2 dx \\
& -\left[\frac{lk_0}{2} - 2\varepsilon_5 N_5\right]\int_0^1 (w_x - l\varphi)^2 dx \\
& +\left[cN_2 + cN_3 + c\left(1+\frac{1}{\varepsilon_5}+\frac{1}{\varepsilon_6}\right)N_5\right]\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx \\
& +\left[\frac{N}{2} - cN_5\right]\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 ds dx \\
& +\left[\frac{L\tilde{\eta}}{\alpha\gamma}N_5\right]\int_0^1 \theta_x(\varphi_x + lw + \psi)dx - F_6(t). \qquad (4.3.34)
\end{aligned}
$$

By setting

$$
\varepsilon_1 = \frac{\rho_2 N_2}{4N_1}, \varepsilon_2 = \frac{kN_5}{8N_2}, \varepsilon_3 = \frac{LN_3}{8N_2}, \varepsilon_6 = \frac{LN_3}{32N_5}, \varepsilon_4 = \frac{l\rho_1}{8N_5}, \varepsilon_5 = \frac{lk_0}{8N_5}, N_3 = \frac{cL}{6k}N_5,
$$

we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\beta N - c_1 N_1\left(1+\frac{N_1}{N_2}\right) - cN_2 - c(1+N_5)N_5\right]\int_0^1 q^2 dx \\
& -\left[N\mu_3 - c\left(1+\frac{N_5}{N_3}\right)N_5 - c\right]\int_0^1 \varphi_t^2 dx \\
& -\left[N\mu_4 - c\left(1+\frac{N_5}{N_3}\right)N_5 - c\right]\int_0^1 z^2(x,1,t)\,dx \\
& -[\frac{N_1\rho_3\kappa}{2} - cN_2\left(1+\frac{N_2}{N_5}+\frac{N_2}{N_3}\right) - cN_3 \\
& \quad -c\left(1+N_5+\frac{N_5}{N_3}\right)N_5]\int_0^1 \theta^2 dx \\
& -\left[N_2\frac{\rho_2}{4} - \rho_2 N_3 - c(1+N_5)N_5 - c\right]\int_0^1 \psi_t^2 dx \\
& -\left[\frac{k}{8}N_5 - c\right]\int_0^1 (\varphi_x + lw + \psi)^2 dx \\
& -\left[\frac{L}{4}(1-l^2\frac{24}{c})N_3\right]\int_0^1 \psi_x^2 dx - \left[\frac{l\rho_1}{4}\right]\int_0^1 w_t^2 dx \\
& +\left[cN_2 + cN_3 + c\left(1+N_5+\frac{N_5}{N_3}\right)N_5\right]\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx \\
& +\left[\frac{N}{2} - cN_5\right]\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 ds dx - \left[\frac{lk_0}{4}\right]\int_0^1 (w_x - l\varphi)^2 dx \\
& +\left[\frac{L\tilde{\eta}}{\alpha\gamma}N_5\right]\int_0^1 \theta_x(\varphi_x + lw + \psi)dx - F_6(t). \qquad (4.3.35)
\end{aligned}$$

We need now to choose carefully the constants. We start by choosing $N_5$ large enough such that

$$\frac{k}{8}N_5 - c > 0,$$

we fixed $N_5$, and choosing $l$ small enough such that

$$1 - l^2\frac{24}{c} > 0.$$

Moreover, we pick $N_2$ large enough so that

$$N_2\frac{\rho_2}{4} - \rho_2 N_3 - c(1+N_5)N_5 - c > 0,$$

we take $N_1$ large enough such that

$$N_1\frac{\rho_3\kappa}{2} - cN_2\left(1+\frac{N_2}{N_5}+\frac{N_2}{N_3}\right) - cN_3 - c\left(1+N_5+\frac{N_5}{N_3}\right)N_5 > 0.$$

On the other hand, if we let

$$\mathfrak{L}(t) = N_1F_1(t) + N_2F_2(t) + N_3F_3(t) + F_4(t) + N_5F_5(t) + F_6(t),$$

then

$$\begin{aligned}
|\mathfrak{L}(t)| \leq\ & \rho_3\alpha N_1\int_0^1\left|\theta\int_0^x q(y)dy\right|dx+\frac{\rho_2\rho_3}{\gamma}N_2\int_0^1\left|\psi_t\int_0^x\theta(y)dy\right|dx\\
&+\rho_2N_3\int_0^1|\psi_t\psi|\,dx+\rho_1\int_0^1|\varphi_t(w_x-l\varphi)|\,dx\\
&+\rho_2\int_0^1|w_t(\varphi_x+lw+\psi)|\,dx+N_5\rho_2\int_0^1|\psi_t\,(\varphi_x+\psi+lw)\,|dx\\
&+\frac{L\rho_1}{k}N_5\int_0^1|\varphi_t\psi_x|dx+\frac{L\rho_3}{\gamma}|\frac{\rho_1}{k}-\frac{\rho_2}{L}|N_5\int_0^1|\theta\varphi_t|dx\\
&+\frac{L\kappa}{\gamma}|\frac{\rho_1}{k}-\frac{\rho_2}{L}|N_5\int_0^1|q\,(\varphi_x+\psi+lw)\,|dx\\
&+\frac{\rho_1}{k}N_5\int_0^1|\varphi_t\int_0^\infty g(s)\eta_x^t(s)ds|dx+\int_0^1\int_0^1 e^{-2\tau\rho}z^2\,(x,\rho,t)\,d\rho dx\\
&+\frac{Ll^2\rho_2}{k_0}N_5\int_0^1|\psi\psi_t|dx+\frac{Ll\rho_1}{k_0}N_5\int_0^1|\psi w_t|dx. \qquad (4.3.36)
\end{aligned}$$

Thanks to Young's, Cauchy-Schwartz and Poincaré's inequalities, to get

$$\begin{aligned}
|\mathfrak{L}(t)| \leq\ & c\int_0^1\left(\psi_t^2+\psi_x^2+\varphi_t^2+(\varphi_x+lw+\psi)^2+(w_x-l\varphi)^2+\theta^2+q^2\right)dx\\
&+c\int_0^1\int_0^\infty g(s)|\eta_x^t(x,s)|^2ds+c\int_0^1\int_0^1 z^2\,(x,\rho,t)\,d\rho dx\leq cE\,(t)\,.
\end{aligned}$$

Then

$$|\mathfrak{L}(t)|=|\mathcal{L}\,(t)-NE\,(t)|\leq cE\,(t)\,,$$

that is

$$(N-c)\,E\,(t)\leq\mathcal{L}\,(t)\leq(N+c)\,E\,(t)\,. \qquad (4.3.37)$$

Choosing N large enough such that

$$N-c>0,\beta N-c>0,N\mu_3-c>0,N\mu_4-c>0,\frac{N}{2}-c>0,$$

we get

$$c_2E\,(t)\leq\mathcal{L}\,(t)\leq c_3E\,(t)\,,\forall t\geq 0, \qquad (4.3.38)$$

we obtain

$$\begin{aligned}
\mathcal{L}'\,(t) \leq\ & -k_1E\,(t)+\alpha_2\int_0^1\int_0^\infty g(s)|\eta_x^t(x,s)|^2ds\\
&+\alpha_1\int_0^1\theta_x(\varphi_x+lw+\psi)dx,\quad\forall t\geq 0, \qquad (4.3.39)
\end{aligned}$$

for some $k_1,c_2,c_3,\alpha_2>0$, and $\alpha_1=N_5\frac{L\tilde{\eta}}{\gamma\alpha}$.

**Case 1:** If $\tilde{\eta}=0$, in this case, (4.3.39) takes the from

$$\mathcal{L}'\,(t)\leq -k_1E\,(t)+\alpha_2\int_0^1\int_0^\infty g(s)|\eta_x^t(x,s)|^2ds,\quad\forall t\geq 0. \qquad (4.3.40)$$

The last term in (4.3.40) is estimated as following, using (4.1.5), we have

$$\begin{aligned}\alpha_2\int_0^1\int_0^\infty g(s)|\eta_x^t(x,s)|^2ds &= \frac{\alpha_2}{\zeta}\int_0^1\int_0^\infty \zeta g(s)|\eta_x^t(x,s)|^2ds\\ &\leq -\frac{\alpha_2}{\zeta}\int_0^1\int_0^t g'(s)|\eta_x^t(x,s)|^2ds\\ &\leq -\frac{2\alpha_2}{\zeta}E(t).\end{aligned} \tag{4.3.41}$$

Thus, (4.3.40) becomes

$$\mathcal{L}'(t)\leq -k_1E(t)-\frac{2\alpha_2}{\zeta}E'(t),\forall t\geq 0,$$

which can be rewritten as

$$\left(\mathcal{L}(t)+\frac{2\alpha_2}{\zeta}E(t)\right)'\leq -k_1E(t),\forall t\geq 0.$$

By exploiting (4.3.38), we notice that

$$\mathcal{R}(t)=\mathcal{L}(t)+\frac{2\alpha_2}{\zeta}E(t)\sim E(t). \tag{4.3.42}$$

Consequently, for some positive constant $\lambda_1$, we obtain

$$\mathcal{R}'(t)\leq -\lambda_1\mathcal{R}(t),\forall t\geq 0. \tag{4.3.43}$$

where $\lambda_1=\frac{k_1}{c_2}$.

Finally, the integration of (4.3.43) and by (4.3.38) give (4.3.31).

**Case 2:** if, $\tilde{\eta}\neq 0$, and

$$|\tilde{\eta}|<\frac{2kk_1\gamma\alpha}{N_5\eta_2L\rho_1}. \tag{4.3.44}$$

Let

$$E(t)=E(\varphi,\psi,w,\theta,q,z,\eta^t)=E_1(t),$$

and

$$E_2(t)=E(\varphi_t,\psi_t,w_t,\theta_t,q_t,z_t,\eta_t^t).$$

Then

$$\begin{aligned}E_2'(t) &\leq -\beta\int_0^1 q_t^2dx-\mu_3\int_0^1\varphi_{tt}^2dx-\mu_4\int_0^1 z_t^2(x,1,t)\,dx\\ &\quad+\frac{1}{2}\int_0^1\int_0^\infty g'(s)\left|\eta_{tx}^t(x,s)\right|^2dsdx.\end{aligned} \tag{4.3.45}$$

The last term in (4.3.39), by using $(4.2.3)_1$, and Young's inequality, and by setting $K=\frac{\rho_1|\alpha_1|}{k}$ as follows

$$\begin{aligned}\alpha_1\int_0^1\theta_x(\varphi_x+lw+\psi)dx &= -\frac{\alpha_1\rho_1}{k}\int_0^1\theta\varphi_{tt}dx+\frac{k_0l\alpha_1}{k}\int_0^1(w_x-l\varphi)\theta dx\\ &\quad -\frac{\alpha_1\mu_1}{k}\int_0^1\theta\varphi_t dx-\frac{\mu_2\alpha_1}{k}\int_0^1 z(x,1,t)\theta dx\\ &= \frac{K}{2}\int_0^1\varphi_{tt}^2dx+\frac{K}{2}\int_0^1\theta^2dx+\frac{K}{2}\delta_1\int_0^1\theta^2dx\\ &\quad +\frac{K}{2}\delta_2\int_0^1(w_x-l\varphi)^2dx+\frac{K}{2}\delta_3\int_0^1\theta^2dx+\frac{K}{2}\delta_4\int_0^1\varphi_t^2dx\\ &\quad +\frac{K}{2}\delta_5\int_0^1\theta^2dx+\frac{K}{2}\delta_6\int_0^1 z^2(x,1,t)dx, \end{aligned} \tag{4.3.46}$$

then (4.3.39)

$$\begin{aligned}\mathcal{L}'(t) &\le -k_2E_1(t)+\alpha_2\int_0^1\int_0^\infty g(s)|\eta_x^t(x,s)|^2ds\\ &\quad +\frac{K}{2}\int_0^1\varphi_{tt}^2dx+\frac{K}{2}\delta_6\int_0^1 z^2(x,1,t)dx,\end{aligned} \tag{4.3.47}$$

where

$$k_2=k_1-\frac{K}{2}\left(\frac{(1+\delta_1+\delta_3+\delta_5)}{\rho_3}+\frac{\delta_2}{k_0}+\frac{\delta_4}{\rho_1}\right)=k_1-\frac{K}{2}\eta_2.$$

and by (4.3.44) we have $k_2=k_1-\frac{K}{2}\eta_2>0$.

Let

$$G(t)=\mathcal{L}(t)+N_7(E_1(t)+E_2(t)). \tag{4.3.48}$$

With (4.3.38), we obtain

$$G(t) \le c_1E_1(t)+N_7(E_1(t)+E_2(t)).$$

It is note hard to see that

$$m_1(E_1(t)+E_2(t))\le G(t)\le m_2(E_1(t)+E_2(t)), \tag{4.3.49}$$

where $m_1,m_2>0$. By using (4.3.47) and (4.3.48), we obtain

$$\begin{aligned}G'(t) &= \mathcal{L}'(t)+N_7(E_1'(t)+E_2'(t))\\ &\le -k_2E_1(t)+\alpha_2\int_0^1\int_0^\infty g(s)|\eta_x^t(x,s)|^2ds\\ &\quad -(\mu_3N_7-\frac{K}{2})\int_0^1\varphi_{tt}^2dx-(\mu_4N_7-\frac{K}{2}\delta_6)\int_0^1 z^2(x,1,t)dx,\end{aligned} \tag{4.3.50}$$

we choose $N_7$ large enough, such that

$$\begin{cases}\mu_3N_7-\frac{K}{2}>0,\\ \\ \mu_4N_7-\frac{K}{2}\delta_6>0,\end{cases}$$

we have

$$G'(t) \leq -k_2 E_1(t) + \alpha_2 \int_0^1 \int_0^\infty g(s)|\eta_x^t(x,s)|^2 ds, \tag{4.3.51}$$

by using (4.3.41), we get

$$\mathcal{K}'(t) \leq -k_2 E_1(t), \tag{4.3.52}$$

where

$$\mathcal{K}(t) = (G(t) + \frac{\alpha_2}{\zeta} E(t)) \sim E_1(t) + E_2(t).$$

Integrating (4.3.52), we get

$$\int_0^t E_1(y)dy \leq \frac{1}{k_2}(\mathcal{K}(0) - \mathcal{K}(t)) \leq \frac{1}{k_2}\mathcal{K}(0) \leq \frac{m_2}{k_2}(E_1(0) + E_2(0)), \tag{4.3.53}$$

and by

$$(tE_1(t))' = tE_1'(t) + E_1(t) \leq E_1(t), \tag{4.3.54}$$

we have

$$tE_1(t) \leq \frac{m_2}{k_2}(E_1(0) + E_2(0)), \tag{4.3.55}$$

yields (4.3.32). This The proof is complete. □

Chapter 5

# Well Posedness and Stability result for a Thermoelastic Laminated Timoshenko Beam with distributed delay term

## 5.1 introduction

In this chapter, we consider a linear thermoelastic laminated Timoshenko beam with distributed delay, where the heat conduction is given by Cattaneoâs law. We establish the well posedness of the system. For stability results we prove exponential and polynomial stabilities of the system for the cases of equal and nonequal speeds of wave propagation. Our work is a natural extension of Feng's work in [59]. We use $c$ throughout this work to denote a generic positive constant.

In this work, we are concerned with the following system,

$$\begin{cases} \rho\omega_{tt} + G(\psi - \omega_x)_x + \delta\theta_x = 0 \\ I_\rho(3s - \psi)_{tt} - D(3s - \psi)_{xx} - G(\psi - \omega_x) = 0 \\ I_\rho s_{tt} - Ds_{xx} + G(\psi - \omega_x) + \frac{4}{3}\gamma s + \frac{4}{3}\beta s_t + \frac{4}{3}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| s_t(x, t - \varrho) d\varrho = 0 \\ \rho_3\theta_t + q_x + \delta\omega_{tx} = 0 \\ \tau q_t + \alpha q + \theta_x = 0, \end{cases} \tag{5.1.1}$$

where

$$(x, \varrho, t) \in (0,1) \times (\tau_1, \tau_2) \times (0, \infty),$$

with the Neumann-Dirichlet boundary conditions

$$\omega_x(0,t) = \omega_x(1,t) = \psi(0,t) = \psi(1,t) = 0, t \geq 0 \tag{5.1.2}$$

$$s(1,t) = s(0,t) = \theta(0,t) = \theta(1,t) = 0, t \geq 0,$$

and the initial data

$$\begin{aligned} &\omega(x,0)=\omega_0(x), \omega_t(x,0)=\omega_1(x), \psi(x,0)=\psi_0(x) \\ &\psi_t(x,0)=\psi_1(x), s(x,0)=s_0(x), s_t(x,0)=s_1(x), \\ &\theta(x,0)=\theta_0(x), q(x,0)=q_0(x). \end{aligned} \tag{5.1.3}$$

where

$$(x,t)\in(0,1)\times(0,\infty).$$

Here $\rho, G, I_\rho, D, \gamma, \beta, \delta, \rho_3, \alpha$ and $\tau$ are positive constants. $\tau_1, \tau_2$ are tow real numbres with $0 \le \tau_1 \le \tau_2$ , $\mu_2$ is an $L^\infty$function satisfying:

(**H1**) $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho < \beta. \tag{5.1.4}$$

First, as in [133], taking the following new variable

$$y(x,\rho,\varrho,t) = s_t(x, t-\varrho\rho),$$

then we obtain

$$\begin{cases} \varrho y_t(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = 0 \\ y(x,0,\varrho,t) = s_t(x,t). \end{cases}$$

Consequently, the problem is equivalent to

$$\begin{cases} \rho\omega_{tt} + G(\psi-\omega_x)_x + \delta\theta_x = 0 \\ I_\rho(3s-\psi)_{tt} - D(3s-\psi)_{xx} - G(\psi-\omega_x) = 0 \\ I_\rho s_{tt} - Ds_{xx} + G(\psi-\omega_x) + \frac{4}{3}\gamma s + \frac{4}{3}\beta s_t + \frac{4}{3}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t) d\varrho = 0 \\ \rho_3\theta_t + q_x + \delta\omega_{tx} = 0 \\ \tau q_t + \alpha q + \theta_x = 0 \\ \varrho y_t(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = 0, \end{cases} \tag{5.1.5}$$

where

$$(x,\rho,\varrho,t) \in (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty),$$

with the following boundary and initial conditions:

$$\begin{aligned} &\omega_x(0,t) = \omega_x(1,t) = \psi(0,t) = \psi(1,t) = 0, t \ge 0 \\ &\theta(0,t) = \theta(1,t) = s(1,t) = s(0,t) = 0, t \ge 0 \end{aligned}$$

$$\begin{aligned} &\omega(x,0)=\omega_0(x), \omega_t(x,0)=\omega_1(x), \psi(x,0)=\psi_0(x) \\ &\psi_t(x,0)=\psi_1(x), s(x,0)=s_0(x), s_t(x,0)=s_1(x) \\ &q(x,0)=q_0(x), \theta(x,0)=\theta_0(x), \\ &y(x,\rho,\varrho,0) = f_0(x,\rho\varrho)/x \in (0.1), \rho \in (0.1), s \in (0,\tau_2). \end{aligned}$$

Meanwhile, from (5.1.1) and (5.1.3), it follows that

$$\frac{d^2}{dt^2}\int_0^1 \omega(x,t)\,dx = 0. \tag{5.1.6}$$

So, by solving (5.1.6) and using the initial data of $u$, we get

$$\int_0^1 \omega(x,t)\,dx = t\int_0^1 \omega_1(x)\,dx + \int_0^1 \omega_0(x)\,dx.$$

Consequently, if we let

$$\bar{\omega}(x,t) = \omega(x,t) - t\int_0^1 \omega_1(x)\,dx - \int_0^1 \omega_0(x)\,dx, \tag{5.1.7}$$

we get

$$\int_0^1 \bar{\omega}(x,t)\,dx = 0, \forall t \geq 0,$$

and, we have

$$\tau\frac{d}{dt}\int_0^1 q(x,t)\,dx + \alpha\int_0^1 q(x,t)\,dx = 0. \tag{5.1.8}$$

Consequently, if we let

$$\bar{q}(x,t) = q(x,t) - e^{(\frac{\alpha}{\tau})t}\int_0^1 q_0(x)\,dx, \tag{5.1.9}$$

we get

$$\int_0^1 \bar{q}(x,t)\,dx = 0, \forall t \geq 0.$$

Therefore, the use of Poincare's inequality for $\bar{\omega}, \bar{q}$ is justified. In addition, simple substitution shows that $(\bar{\omega}, \psi, s, \theta, \bar{q})$ satisfies system (5.1.1). Henceforth, we work with $\bar{\omega}, \bar{q}$ instead of $\omega, q$ but write $\omega, q$ for simplicity of notation.

## 5.2 Well-posedness

In this section, we give the existence and uniqueness result of the system (5.1.1)-(5.1.3) using the semigroup theory. To achieve our goal,
Introducing the vector function

$$U = (\omega, \omega_t, 3s - \psi, (3s - \psi)_t, s, s_t, \theta, q, y)^T,$$

and the two new dependent variables $v = \omega_t, u = \psi_t, \varphi = s_t$, then the system (5.1.5) can be written as follows:

$$\begin{cases} U_t = \mathcal{A}U \\ U(0) = U_0 = (\omega_0, \omega_1, 3s_0 - \psi_0, 3s_1 - \psi_1, s_0, s_1, \theta_0, q_0, f_0)^T, \end{cases} \tag{5.2.1}$$

where $\mathcal{A}:\mathcal{D}(\mathcal{A})\subset\mathcal{H}:\to\mathcal{H}$ is the linear operator defined by

$$\mathcal{A}U=\begin{pmatrix} v \\ -\frac{1}{\rho}[G(\psi_x-\omega_x)_x+\delta\theta_x] \\ 3\varphi-u \\ \frac{1}{I_\rho}[D(3s-\psi)_{xx}+G(\psi_x-\omega_x)] \\ \varphi \\ \frac{1}{I_\rho}[Ds_{xx}-G(\psi_x-\omega_x)-\frac{4}{3}\gamma s-\frac{4}{3}\beta\varphi-\frac{4}{3}-\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y(x,1,\varrho,t)d\varrho] \\ -\frac{1}{\rho_3}[q_x+\delta v_x] \\ -\frac{1}{\tau}[q+\theta_x] \\ -\frac{1}{\varrho}y_\rho \end{pmatrix}, \tag{5.2.2}$$

and $\mathcal{H}$ is the energy space given by

$$\begin{aligned}\mathcal{H} &= H^1_*\times L^2_*(0,1)\times H^1_0\times L^2(0,1)\times H^1_0\times L^2(0,1)\times L^2(0,1)\times L^2_*(0,1)\\ &\quad\times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)),\end{aligned}$$

where

$$\begin{aligned} L^2_*(0,1) &= \left\{\phi\in L^2(0,1)\ /\ \int_0^1\phi(x)dx=0\right\}\\ H^1_*(0,1) &= H^1(0,1)\cap L^2_*(0,1)\\ H^2_*(0,1) &= \left\{\phi\in H^2(0,1)\ /\ \phi_x(1)=\phi_x(0)=0\right\}.\end{aligned}$$

For any

$$\begin{aligned} U &= (\omega,v,3s-\psi,3\varphi-u,s,\varphi,\theta,q,y)^T\in\mathcal{H},\\ \widehat{U} &= (\widehat{\omega},\widehat{v},3\widehat{s}-\widehat{\psi},3\widehat{\varphi}-\widehat{u},\widehat{s},\widehat{\varphi},\widehat{\theta},\widehat{q},\widehat{y})^T\in\mathcal{H},\end{aligned}$$

we equip $\mathcal{H}$ with the inner product defined by

$$\begin{aligned} <U,\widehat{U}>_{\mathcal{H}} &= \rho\int_0^1 v\widehat{v}dx+I_\rho\int_0^1(3\varphi-u)(3\widehat{\varphi}-\widehat{u})dx+3I_\rho\int_0^1\varphi\widehat{\varphi}dx\\ &\quad+G\int_0^1(\psi-\omega_x)(\widehat{\psi}-\widehat{\omega}_x)dx+\rho_3\int_0^1\theta\widehat{\theta}dx+\tau\int_0^1 q\widehat{q}dx\\ &\quad+D\int_0^1(3s-\psi)_x(3\widehat{s}-\widehat{\psi})_xdx+4\gamma\int_0^1 s\widehat{s}dx+3D\int_0^1 s_x\widehat{s}_xdx\\ &\quad+4\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|y\widehat{y}d\varrho d\rho dx. \end{aligned}\tag{5.2.3}$$

The domain of $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A})=\left\{\begin{array}{l} U\in\mathcal{H}/\omega\in H^2_*\cap H^1_*,3s-\psi,s\in H^2\cap H^1_*,\\ v,q\in H^1_*,3\varphi-u,\varphi,\theta\in H^1_0(0,1),\\ y,y_\rho\in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)),y(x,0,\varrho,t)=\varphi \end{array}\right\}.$$

Clearly, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Now, we can give the following existence result.

**Theorem 5.1.** *Let $U_0 \in \mathcal{H}$ and assume that (5.1.4) holds. Then, there exists a unique solution $U \in \mathcal{C}(\mathbb{R}_+, \mathcal{H})$ of problem (5.2.1).*
*Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then*

$$U \in \mathcal{C}(\mathbb{R}_+, \mathcal{D}(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+, \mathcal{H}).$$

*Proof.* First, we prove that the operator $\mathcal{A}$ is dissipative. For any $U_0 \in \mathcal{D}(\mathcal{A})$ and by using (5.2.3), we have

$$\begin{aligned} < \mathcal{A}U, U >_{\mathcal{H}} &= -4\beta \int_0^1 \varphi^2 dx - 4 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \varphi y(x,1,\varrho,t) d\varrho dx \\ &\quad -4 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| y_\rho y d\varrho d\rho dx - \alpha \int_0^1 q_x^2 dx. \end{aligned} \tag{5.2.4}$$

For the third term of the right-hand side of (5.2.4), we have

$$\begin{aligned} -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y_\rho y d\varrho d\rho dx &= -\frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} \int_0^1 |\mu_2(\varrho)| \frac{d}{d\rho} y^2 d\rho d\varrho dx \\ &= -\frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx \\ &\quad + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,0,\varrho,t) d\varrho dx. \end{aligned} \tag{5.2.5}$$

By using Young's inequality, we get

$$\begin{aligned} -\int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| \varphi y(x,1,\varrho,t) d\varrho dx &\leq \frac{1}{2} (\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho) \int_0^1 \varphi^2 dx \\ &\quad + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx. \end{aligned} \tag{5.2.6}$$

Substituting (5.2.5),(5.2.6) into (5.2.4), using the fact that $y(x,0,\varrho,t) = \varphi(x,t)$ and (5.1.4), we obtained

$$\begin{aligned} < \mathcal{A}U, U >_{\mathcal{H}} &\leq -4(\beta - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho) \int_0^1 \varphi^2 dx - \alpha \int_0^1 q^2 dx \\ &\leq 0. \end{aligned} \tag{5.2.7}$$

Hence, the operator $\mathcal{A}$ is dissipative.
Next, we prove the operator $\mathcal{A}$ is maximal. It is sufficient to show that the operator $(Id - \mathcal{A})$ is surjective.
Indeed, for any $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T \in \mathcal{H}$, we prove that there exists a unique $V = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9) \in \mathcal{D}(\mathcal{A})$ such that

$$(Id - \mathcal{A})V = F. \tag{5.2.8}$$

That is

$$\begin{cases} v_1 - v_2 = f_1 \\ \rho v_2 - G v_{1xx} - G v_{3x} + \delta v_{7x} = \rho f_2 \\ v_3 - v_4 = f_3 \\ I_\rho v_4 - D v_{3xx} - G v_5 + G v_3 + G v_{1x} = I_\rho f_4 \\ v_5 - v_6 = f_5 \\ (I_\rho + \frac{4}{3}\beta) v_6 - D v_{5xx} - G v_3 - G v_{1x} + (3G + \frac{4}{3}\gamma) v_5 \\ \quad + \frac{4}{3}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t) d\varrho = I_\rho f_6 \\ \rho_3 v_7 + v_{8x} + \delta v_{2x} = \rho_3 f_7 \\ (\tau + \alpha) v_8 + v_{7x} = \tau f_8 \\ \varrho y_t(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = \varrho f_9, \end{cases} \tag{5.2.9}$$

We note that the last equation in (5.2.9) with $y(x,0,\varrho,t) = \varphi(x,t)$ has a unique solution given by

$$y(x,\rho,\varrho,t) = e^{-\rho\varrho}\varphi + \varrho e^{\varrho\rho}\int_0^\rho e^{\varrho\sigma} f_9(x,\sigma,\varrho,t) d\sigma, \tag{5.2.10}$$

then

$$y(x,1,\varrho,t) = e^{-\varrho}\varphi + \varrho e^{\varrho}\int_0^1 e^{\varrho\sigma} f_9(x,\sigma,\varrho,t) d\sigma, \tag{5.2.11}$$

and we infer from $(5.2.9)_8$ that

$$v_7 = -(\tau+\alpha)\int_0^x v_8(\sigma) d\sigma + \tau \int_0^x f_8(\sigma) d\sigma, \tag{5.2.12}$$

and we have

$$v_2 = v_1 - f_1, \quad v_4 = v_3 - f_3, \quad v_6 = v_5 - f_5. \tag{5.2.13}$$

Inserting (5.2.11), (5.2.12) and (5.2.13) in $(5.2.9)_2$,$(5.2.9)_4$, $(5.2.9)_6$ and $(5.2.9)_7$, we get

$$\begin{cases} \rho v_1 - G v_{1xx} - G v_{3x} + 3G v_{5x} - \delta(\tau+\alpha) v_8 = h_1 \\ I_\rho v_3 - D v_{3xx} - 3G v_5 + 3G v_{1x} = h_2 \\ \mu_3 v_5 - D v_{5xx} - G v_3 - G v_{1x} = h_3 \\ -\rho_3(\tau+\alpha)\int_0^x v_8(\sigma) d\sigma + v_{8x} + \delta v_{1x} = h_4, \end{cases} \tag{5.2.14}$$

where

$$\begin{cases} \mu_3 = I_\rho + 3G + \frac{4}{3}\beta + \frac{4}{3}\gamma + \frac{4}{3}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| e^{-\varrho} d\varrho \\ h_1 = \rho(f_1 + f_2) - \tau f_8 \\ h_2 = I_\rho (f_3 + f_4) \\ h_3 = I_\rho (f_5 + f_6) + \frac{4}{3}(\beta - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| e^{-\varrho} d\varrho) f_5 \\ \quad -\frac{4}{3}\int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| e^{\varrho} \int_0^1 e^{\varrho\sigma} f_9(x,\sigma,\varrho,t) d\sigma d\varrho. \\ h_4 = \rho_3 f_7 - \rho_3 \tau \int_0^x f_8(\sigma) d\sigma + \delta f_{1x}. \end{cases} \tag{5.2.15}$$

We multiply (5.2.14) by $\widehat{v}_1, \widehat{v}_3, \widehat{v}_5$, and $(\tau+\alpha)\int_0^x \widehat{v}_8(\sigma)d\sigma$ respectively, and integrate their sum over $(0,1)$ to get the following variational formulation:

$$B((v_1,v_3,v_5,v_8),(\widehat{v}_1,\widehat{v}_3,\widehat{v}_5,\widehat{v}_8)) = \Gamma(\widehat{v}_1,\widehat{v}_3,\widehat{v}_5,\widehat{v}_8), \tag{5.2.16}$$

where

$$B : (H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times L_*^2(0,1))^2 \to \mathbb{R},$$

is the bilinear form defined by

$$\begin{aligned}
B((v_1,v_3,v_5,v_8),(\widehat{v}_1,\widehat{v}_3,\widehat{v}_5,\widehat{v}_8)) \quad = \quad & \mu_4\int_0^1 v_5\widehat{v}_5dx + \rho\int_0^1 v_1\widehat{v}_1dx \\
& + I_\rho\int_0^1 v_3\widehat{v}_3dx + (\tau+\alpha)\int_0^1 v_8\widehat{v}_8dx \\
& + D\int_0^1 v_{3x}\widehat{v}_{3x}dx + 3D\int_0^1 v_{5x}\widehat{v}_{5x}dx \\
& + G\int_0^1 (-v_{1x}-v_3+3v_5)(-\widehat{v}_{1x}-\widehat{v}_3+3\widehat{v}_5)dx \\
& + \rho_3(\tau+\alpha)^2\int_0^1 (\int_0^x v_8(\sigma)d\sigma)(\int_0^x \widehat{v}_8(\sigma)d\sigma)dx,
\end{aligned} \tag{5.2.17}$$

where $\mu_4 = 3I_\rho + 4\beta + 4\gamma + 4\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|e^{-\varrho}d\varrho$.

And

$$\Gamma : (H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times L_*^2(0,1)) \to \mathbb{R},$$

is the linear functional given by

$$\begin{aligned}
\Gamma(\widehat{v}_1,\widehat{v}_3,\widehat{v}_5,\widehat{v}_8) \quad = \quad & \int_0^1 h_1\widehat{v}_1dx + \int_0^1 h_2\widehat{v}_3dx + \int_0^1 h_3\widehat{v}_5dx \\
& + \int_0^1 h_4(-(\tau+\alpha)\int_0^x \widehat{v}_8(\sigma)d\sigma)dx.
\end{aligned} \tag{5.2.18}$$

Now, for $V = H_*^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L)\times L_*^2(0,L)$, equipped with the norm

$$\begin{aligned}
\|(v_1,v_3,v_5,v_8)\|_V^2 \quad = \quad & \|(-v_{1x}-v_3+3v_5)\|_2^2 + \|v_1\|_2^2 \\
& + \|v_8\|_2^2 + \|v_{3x}\|_2^2 + \|v_{5x}\|_2^2,
\end{aligned}$$

then, we have

$$\begin{aligned} B((\varphi,\psi,\theta,P),(\varphi,\psi,\theta,P)) &= \mu_4\int_0^1 v_5^2dx+\rho\int_0^1 v_1^2dx \\ &+I_\rho\int_0^1 v_3^2dx+(\tau+\alpha)\int_0^1 v_8^2dx \\ &+D\int_0^1 v_{3x}^2dx+3D\int_0^1 v_{5x}^2dx \\ &+G\int_0^1(-v_{1x}-v_3+3v_5)^2dx \\ &+\rho_3(\tau+\alpha)^2\int_0^1(\int_0^x v_8(\sigma)d\sigma)^2dx \\ &\geq \rho\int_0^1 v_1^2dx+(\tau+\alpha)\int_0^1 v_8^2dx \\ &+D\int_0^1 v_{3x}^2dx+3D\int_0^1 v_{5x}^2dx \\ &+G\int_0^1(-v_{1x}-v_3+3v_5)^2dx, \end{aligned} \tag{5.2.19}$$

then, for some $M_0>0$

$$B((v_1,v_3,v_5,v_8),(v_1,v_3,v_5,v_8))\geq M_0\|(v_1,v_3,v_5,v_8)\|_V^2. \tag{5.2.20}$$

Thus $B$ is coercive. Consequently, using Lax-Milgram theorem, we conclude that (5.1.5) has a unique solution:

$$\begin{aligned} &v_1\in H_*^1(0,1),\\ &v_3,v_5\in H_0^1(0,1),\\ &v_8\in L_*^2(0,1). \end{aligned} \tag{5.2.21}$$

Substituting $v_1,v_3,v_5$, and $v_8$ into (5.2.11), (5.2.12) and (5.2.13), respectively, we have

$$\begin{aligned} &v_2\in H_*^1(0,1),\\ &v_4,v_6\in H_0^1(0,1),\\ &v_7\in H_*^1(0,1),\\ &y,y_\rho\in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)), \end{aligned} \tag{5.2.22}$$

let $\widehat{v}_1\in H_0^1(0,1)$ and denote

$$\widehat{\widehat{v}}_1=\widehat{v}_1(x)-\int_0^1\widehat{v}_1(\xi)d\xi, \tag{5.2.23}$$

which gives us $\widehat{\widehat{v}}_1\in H_*^1(0,1)$. Now we replace $(\widehat{v}_1,\widehat{v}_3,\widehat{v}_5,\widehat{v}_8)$ by $(\widehat{\widehat{v}}_1,0,0,0)$ in (5.2.16) to obtain

$$G\int_0^1(-v_{1x}-v_3+3v_5)(-\widehat{\widehat{v}}_{1x})dx+\rho\int_0^1 v_1\widehat{\widehat{v}}_1dx=\int_0^1 h_1\widehat{\widehat{v}}_1dx, \tag{5.2.24}$$

we get

$$G\int_0^1 v_{1xx}\widehat{\widehat{v}}_{1x}dx = \rho\int_0^1 v_1\widehat{\widehat{v}}_1 dx - G\int_0^1 v_{3x}\widehat{\widehat{v}}_{1x}dx + G\int_0^1 v_{5x}\widehat{\widehat{v}}_{1x}dx - \int_0^1 h_1\widehat{\widehat{v}}_1 dx, \quad \forall \widehat{v}_1 \in H_0^1(0,1), \tag{5.2.25}$$

which yields

$$Gv_{1xx} = \rho v_1 - Gv_{3x} + Gv_{5x} - h_1 \in L^2(0,1). \tag{5.2.26}$$

Thus

$$v_1 \in H^2(0,1). \tag{5.2.27}$$

Moreover, (5.2.26) also holds for any $\Phi \in C^1([0,1])$. Then, by using integration by parts, we obtain

$$Gv_{1x}(1)\Phi(1) - Gv_{1x}(0)\Phi(0) \quad -G\int_0^1 v_{1xx}\Phi dx + \rho\int_0^1 v_1\Phi dx - G\int_0^1 v_{3x}\Phi dx + G\int_0^1 v_{5x}\Phi dx - \int_0^1 h_1\Phi dx = 0. \tag{5.2.28}$$

Then, we get for any $\Phi \in C^1([0,1])$

$$Gv_{1x}(1)\Phi(1) - Gv_{1x}(0)\Phi(0) = 0. \tag{5.2.29}$$

Since $\Phi$ is arbitrary, we get that $v_{1x}(0) = v_{1x}(1) = 0$. Hence, $v_1 \in H_*^2(0,1)$. Using similar arguments as above, we can obtain

$$\begin{aligned} &v_3, v_5 \in H^2(0,1) \cap H_0^1(0,1), \\ &v_7 \in H_0^1(0,1), \\ &v_8 \in H_*^1(0,1). \end{aligned} \tag{5.2.30}$$

Finally, the application of regularity theory for the linear elliptic equations guarantees the existence of unique $U \in \mathcal{D}(\mathcal{A})$ such that (5.2.8) is satisfied.
Consequently, we conclude that $\mathcal{A}$ is a maximal dissipative operator. Hence by Lumer-Philips theorem (see [109],[138]), we have the well-posedness result. This completes the proof. □

## 5.3 Exponential decay

In this section, we state and prove our stability result. We need the following lemmas.

**Lemma 5.1.** *The energy functional $E$, defined by*

$$E(t) = \frac{1}{2}\int_0^1 \{\rho\omega_t^2 + I_\rho(3s-\psi)_t^2 + 3I_\rho s_t^2 + 3Ds_x^2 + 4\gamma s^2 + D(3s-\psi)_x^2 + D(\psi-\omega_x)^2 + \rho_3\theta^2 + \tau q^2 + 4\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)\,d\varrho d\rho\}dx, \tag{5.3.1}$$

*satisfies*

$$E'(t) \leq -\alpha \int_0^1 q^2 dx - 4\eta_0 \int_0^1 s_t^2 dx \leq 0, \tag{5.3.2}$$

*where* $\eta_0 = \beta - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho > 0.$

*Proof.* Multiplying the equations of system (5.1.5) by $\omega_t, (3s-\psi)_t, \theta, q$ respectively, and integrating over $(0,1)$, using integration by parts, and using (5.1.2), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \quad & \int_0^1 \{\rho\omega_t^2 + I_\rho(3s-\psi)_t^2 + 3I_\rho s_t^2 + 3Ds_x^2 + 4\gamma s^2 + D(3s-\psi)_x^2 \\ & + D(\psi-\omega_x)^2 + \rho_3\theta^2 + \tau q^2\} dx + 4\beta \int_0^1 s_t^2 dx + \alpha \int_0^1 q^2 dx \\ & + 4\int_0^1 s_t \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t)\, d\varrho dx = 0. \end{aligned} \tag{5.3.3}$$

Using Young's inequality, we arrive at

$$\begin{aligned}\int_0^1 s_t \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t)\, d\varrho dx \quad \leq \quad & (\frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho) \int_0^1 s_t^2 dx \\ & + \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx. \end{aligned} \tag{5.3.4}$$

Now, multiplying the last equation in (5.1.5) by $y|\mu_2(\varrho)|$, and integrating the result over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$

$$\begin{aligned}\frac{d}{dt} \quad \frac{1}{2} \quad & \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| y^2(x,\rho,\varrho,t) d\varrho d\rho dx \\ = \quad & -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y y_\rho (x,\rho,\varrho,t)\, d\varrho d\rho dx \\ = \quad & -\frac{1}{2}\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \frac{d}{d\rho} y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx \\ = \quad & \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| (y^2(x,0,\varrho,t) - y^2(x,1,\varrho,t)) d\varrho dx \\ = \quad & \frac{1}{2}(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho) \int_0^1 s_t^2 dx \\ & -\frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t)\, d\varrho dx. \end{aligned} \tag{5.3.5}$$

From (5.3.1),(5.3.3),(5.3.4), and(5.3.5), we get (5.3.2).

$$E'(t) \leq -\alpha \int_0^1 q^2 dx - 4(\beta - \int_{\tau_1}^{\tau_2} |\mu_2(s)|) \int_0^1 s_t^2 dx, \tag{5.3.6}$$

then, by (5.1.4), there exists a positive constant $\eta_0$ such that

$$E'(t) \leq -4\eta_0 \int_0^1 s_t^2 dx - \alpha \int_0^1 q^2 dx \leq 0. \tag{5.3.7}$$

Then we obtain $E$ is a non-increasing function. □

**Lemma 5.2.** *The functional*

$$F_1(t) := I_\rho \int_0^1 (3s-\psi)(3s-\psi)_t dx - \rho \int_0^1 \omega_t (\int_0^x (3s-\psi)(y)dy)dx, \tag{5.3.8}$$

*satisfies*

$$\begin{aligned} F_1'(t) &\leq -\frac{D}{2}\int_0^1 (3s-\psi)_x^2 dx + \varepsilon_1 \int_0^1 \omega_t^2 dx \\ &+c(1+\frac{1}{\varepsilon_1})\int_0^1 (3s-\psi)_t^2 dx + c\int_0^1 \theta^2 dx. \end{aligned} \tag{5.3.9}$$

*Proof.* Direct computations using integration by parts, we have

$$\begin{aligned} F_1'(t) &= I_\rho \int_0^1 (3s-\psi)_t^2 dx - D\int_0^1 (3s-\psi)_x^2 dx \\ &+\delta \int_0^1 \theta_x (\int_0^x (3s-\psi)(y)dy)dx \\ &-\rho \int_0^1 \omega_t (\int_0^x (3s-\psi)_t(y)dy)dx. \end{aligned} \tag{5.3.10}$$

Using Young's, Cauchy-Schwarz and poincare's inequalities, we obtain (5.3.9). □

**Lemma 5.3.** *The functional*

$$F_2(t) := \rho \int_0^1 (\psi-\omega_x)(\int_0^x \omega_t(y)dy)dx, \tag{5.3.11}$$

*satisfies*

$$\begin{aligned} F_2'(t) &\leq -\frac{G}{2}\int_0^1 (\psi-\omega_x)^2 dx + \varepsilon_2 \int_0^1 (3s-\psi)_t^2 dx \\ &+c(1+\frac{1}{\varepsilon_2})\int_0^1 \omega_t^2 dx + c\int_0^1 \theta^2 dx + c\int_0^1 s_t^2 dx. \end{aligned} \tag{5.3.12}$$

*Proof.* Differentiating $F_2(t)$ with respect to $t$ and using integration by parts, we have

$$\begin{aligned} F_2'(t) &= \rho \int_0^1 \psi_t (\int_0^x \omega_t(y)dy)dx + \rho\int_0^1 \omega_t^2 dx \\ &-G\int_0^1 (\psi-\omega_x)^2 dx - \delta \int_0^1 (\psi-\omega_x)\theta dx. \end{aligned} \tag{5.3.13}$$

Using the fact that $\psi_t = 3s_t - (3s-\psi)_t$, Young's and Cauchy-Schwarz inequalities, we obtain (5.3.12). □

**Lemma 5.4.** *The functional*

$$F_3(t) := \tau\rho_3 \int_0^1 \theta(\int_0^x q(y)dy)dx, \tag{5.3.14}$$

*satisfies*

$$\begin{aligned} F_3'(t) &\leq -\frac{\rho_3}{2}\int_0^1 \theta^2 dx + \varepsilon_3 \int_0^1 \omega_t^2 dx \\ &+c(1+\frac{1}{\varepsilon_3})\int_0^1 q^2 dx. \end{aligned} \tag{5.3.15}$$

*Proof.* Direct computations and using integration by parts, we have

$$\begin{aligned} F_3'(t) &= \tau\int_0^1 q^2dx+\tau\delta\int_0^1 \omega_t q dx \\ &\quad -\rho_3\alpha\int_0^1\theta(\int_0^x q(y)dy)dx-\rho_3\int_0^1\theta^2dx. \end{aligned} \tag{5.3.16}$$

Using Young's and Cauchy-Schwarz inequalities, we obtain (5.3.15). □

**Lemma 5.5.** *The functional*

$$F_4(t) := -\rho\rho_3\int_0^1\theta(\int_0^x\omega_t(y)dy)dx, \tag{5.3.17}$$

*satisfies*

$$\begin{aligned} F_4'(t) &\leq -\frac{\rho\delta}{2}\int_0^1\omega_t^2dx+\varepsilon_4\int_0^1(\psi-\omega_x)^2dx \\ &\quad +c(1+\frac{1}{\varepsilon_4})\int_0^1\theta^2dx+c\int_0^1 q^2dx. \end{aligned} \tag{5.3.18}$$

*Proof.* We take the derivative of $F_4$ and use (5.1.3) and integrate by parts to obtain

$$\begin{aligned} F_4'(t) &= -\rho\int_0^1 q\omega_t dx-\rho\delta\int_0^1\omega_t^2dx \\ &\quad +\rho_3G\int_0^1\theta(\psi-\omega_x)dx+\rho_3\delta\int_0^1\theta^2dx. \end{aligned} \tag{5.3.19}$$

Using Young's, Cauchy-Schwarz and Poincare's inequalities, we obtain (5.3.18). □

**Lemma 5.6.** *The functional*

$$\begin{aligned} F_5(t) &:= \tau\delta GI_\rho\int_0^1(3s-\psi)_t(\psi-\omega_x)dx-\tau\delta D\rho\int_0^1\omega_t(3s-\psi)_x dx \\ &\quad +\rho_3\tau(D\rho-GI_\rho)\int_0^1\theta(3s-\psi)_t dx \\ &\quad -\tau(D\rho-GI_\rho)\int_0^1 q(3s-\psi)_x dx, \end{aligned} \tag{5.3.20}$$

*satisfies*

$$\begin{aligned} F_5'(t) &\leq -\frac{\tau\delta GI_\rho}{2}\int_0^1(3s-\psi)_t^2dx+\varepsilon_5\int_0^1(3s-\psi)_x^2dx \\ &\quad +\varepsilon_6\int_0^1\theta^2dx+c(1+\frac{1}{\varepsilon_6})\int_0^1(\psi-\omega_x)^2dx+\frac{c}{\varepsilon_5}\int_0^1 q^2dx \\ &\quad +c\int_0^1 s_t^2dx+\chi\int_0^1\theta_x(3s-\psi)_x dx, \end{aligned} \tag{5.3.21}$$

*where*

$$\chi=\tau\delta^2D-(D\rho-GI_\rho)(\frac{\tau\rho_3D}{I_\rho}-1). \tag{5.3.22}$$

*Proof.* By differentiating $F_5$ with respect to $t$, and integrate by parts to get

$$\begin{aligned} F_5'(t) &= \tau G^2\delta\int_0^1(\psi-\omega_x)^2dx+\tau G\delta I_\rho\int_0^1(3s-\psi)_t\psi_t dx \\ &+\delta^2\tau\int_0^1\theta_x(3s-\psi)_x dx+\frac{\tau\rho_3 D}{I_\rho}(D\rho-GI_\rho)\int_0^1\theta_x(3s-\psi)_x dx \\ &+\frac{\tau\rho_3}{I_\rho}(D\rho-GI_\rho)\int_0^1\theta(\psi-\omega_x)dx+\alpha(D\rho-GI_\rho)\int_0^1 q(3s-\psi)_x dx \\ &+(D\rho-GI_\rho)\int_0^1\theta_x(3s-\psi)_x dx. \end{aligned} \tag{5.3.23}$$

Using the fact that $\psi_t=3s_t-(3s-\psi)_t$ and Young's inequality, we obtain (5.3.21). □

**Lemma 5.7.** *The functional*

$$F_6(t) := 3I_\rho\int_0^1 ss_t dx+2\beta\int_0^1 s^2dx, \tag{5.3.24}$$

*satisfies*

$$\begin{aligned} F_6'(t) &\le -3D\int_0^1 s_x^2dx-2\gamma\int_0^1 s^2dx+c\int_0^1(\psi-\omega_x)^2dx \\ &+c\int_0^1 s_t^2dx+c\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)\,d\varrho dx. \end{aligned} \tag{5.3.25}$$

*Proof.* Direct computations using integration by parts, we have

$$\begin{aligned} F_6'(t) &= -3D\int_0^1 s_x^2dx-3G\int_0^1 s(\psi-\omega_x)dx+3I_\rho\int_0^1 s_t^2dx \\ &-4\gamma\int_0^1 s^2dx-4\int_0^1 s\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y(x,1,\varrho,t)\,d\varrho dx. \end{aligned} \tag{5.3.26}$$

Using Young's, Cauchy-Schwarz and Poincare's inequalities, we obtain (5.3.25). □

**Lemma 5.8.** *The functional*

$$F_7(t) := \int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varrho e^{-\varrho\rho}|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)\,d\varrho d\rho dx,$$

*satisfies,*

$$\begin{aligned} F_7'(t) &\le -\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)\,d\varrho d\rho dx+\beta\int_0^1 s_t^2dx \\ &-\eta_1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)\,d\varrho dx, \end{aligned} \tag{5.3.27}$$

*where $\eta_1$ is a positive constant.*

*Proof.* By differentiating $F_7$, with respect to $t$ and using the last equation in (5.1.5), we have

$$\begin{aligned} F_7'(t) &= -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}e^{-\varrho\rho}|\mu_2(\varrho)|yy_\rho(x,\rho,\varrho,t)\,d\varrho d\rho dx \\ &= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varrho e^{-\varrho\rho}|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)\,d\varrho d\rho dx \\ &\quad -\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|[e^{-\varrho}y^2(x,1,\varrho,t)-y^2(x,0,\varrho,t)]d\varrho dx. \end{aligned}$$

Using the fact that $y(x,0,s,t)=s_t(x,t)$, and $e^{-\varrho}\leq e^{-\varrho\rho}\leq 1$,for all $0<\rho<1$, we obtain

$$\begin{aligned}F_7'(t) &= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varrho e^{-\varrho}|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)\,d\varrho d\rho dx\\ &\quad -\int_0^1\int_{\tau_1}^{\tau_2}e^{-\varrho}|\mu_2(\varrho)|y^2(x,1,\varrho,t)\,d\varrho dx+(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\int_0^1 s_t^2dx.\end{aligned}$$

Because $-e^{-\varrho}$ is a increasing function, we have $-e^{-\varrho}\leq -e^{-\tau_2}$, for all $\varrho\in[\tau_1,\tau_2]$.

Finally, setting $\eta_1=e^{-\tau_2}$, and recalling (5.1.4), we obtain (5.3.27). □

We are now ready to prove the following result.

### 5.3.1 Exponential Stability

In this subsection, we study the exponential stability of systems, we consider the case $\chi=0$.

**Theorem 5.2.** *Assume (5.1.4), there exist positive constants $\lambda_1$ and $\lambda_2$ such that the energy functional given by (5.3.1) satisfies*

$$E(t)\leq\lambda_2 e^{-\lambda_1 t},\forall t\geq 0. \tag{5.3.28}$$

*Proof.* We define a Lyapunov functional

$$\mathcal{L}(t):=NE(t)+F_1(t)+\sum_{i=2}^{i=7}N_iF_i(t), \tag{5.3.29}$$

where $N$, and $N_i, i=2...7$, are positive constants to be selected later.

By differentiating (5.3.29) and using (5.3.2), (5.3.9),(5.3.12),(5.3.15),(5.3.18),(5.3.21),(5.3.25) (5.3.27), we have

$$\begin{aligned}\mathcal{L}'(t) &\leq -\left[\frac{D}{2}-\varepsilon_5N_5\right]\int_0^1(3s-\psi)_x^2dx\\ &\quad -[4\eta_0N-cN_2-cN_3-cN_6-\beta N_7]\int_0^1 s_t^2dx\\ &\quad -\left[\frac{\tau G\delta I_\rho}{2}N_5-c(1+\frac{1}{\varepsilon_1})-\varepsilon_2N_2\right]\int_0^1(3s-\psi)_t^2dx\\ &\quad -\left[\frac{G}{2}N_2-\varepsilon_4N_4-c(1+\frac{1}{\varepsilon_6})N_5-cN_6\right]\int_0^1(\psi-\omega_x)^2dx\\ &\quad -\left[\frac{\rho\delta}{2}N_4-\varepsilon_1-c(1+\frac{1}{\varepsilon_2})N_2-\varepsilon_3N_3\right]\int_0^1\omega_t^2dx\\ &\quad -\left[\frac{\rho_3}{2}N_3-\varepsilon_6N_5-c(1+\frac{1}{\varepsilon_4})N_4-cN_2-c\right]\int_0^1\theta^2dx\\ &\quad -\left[\alpha N-cN_4-c(1+\frac{1}{\varepsilon_3})N_3-\frac{c}{\varepsilon_5}N_5\right]\int_0^1 q^2dx\\ &\quad -[2\gamma N_6]\int_0^1 s^2dx-[3DN_6]\int_0^1 s_x^2dx\\ &\quad -[\eta_1N_7-cN_6]\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)\,d\varrho dx\\ &\quad -[N_7\eta_1]\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)\,d\varrho d\rho dx.\end{aligned} \tag{5.3.30}$$

By setting

$$\varepsilon_1 = 1, \varepsilon_2 = \frac{\tau G\delta I_\rho N_5}{4N_2}, \varepsilon_3 = \frac{1}{N_3}, \varepsilon_4 = \frac{GN_2}{4N_4}, \varepsilon_5 = \frac{D}{4N_5}, \varepsilon_6 = \frac{1}{N_5},$$

we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\frac{D}{4}\right]\int_0^1 (3s-\psi)_x^2 dx \\
& -[4\eta_0 N - cN_2 - cN_3 - cN_6 - \beta N_7]\int_0^1 s_t^2 dx \\
& -\left[\frac{\tau G\delta I_\rho}{4}N_5 - 2c\right]\int_0^1 (3s-\psi)_t^2 dx \\
& -\left[\frac{G}{4}N_2 - c(1+N_5)N_5 - cN_6\right]\int_0^1 (\psi-\omega_x)^2 dx \\
& -\left[\frac{\rho\delta}{2}N_4 - c(1+\frac{N_2}{N_5})N_2 - 2\right]\int_0^1 \omega_t^2 dx \\
& -\left[\frac{\rho_3}{2}N_3 - c(1+\frac{N_4}{N_2})N_4 - cN_2 - c - 1\right]\int_0^1 \theta^2 dx \\
& -\left[\alpha N - cN_4 - c(1+N_3)N_3 - cN_5^2\right]\int_0^1 q^2 dx \\
& -[2\gamma N_6]\int_0^1 s^2 dx - [3DN_6]\int_0^1 s_x^2 dx \\
& -[\eta_1 N_7 - cN_6]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t)\, d\varrho dx \\
& -[N_7\eta_1]\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)| y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx.
\end{aligned} \tag{5.3.31}$$

Next, we carefully choose our constants so that the terms inside the brackets are positive.

We fixed $N_6$, and we choose $N_5, N_7$ large enough such that

$$\begin{aligned}
\alpha_1 &= \frac{\tau G\delta I_\rho}{4}N_5 - 2c > 0, \\
\alpha_2 &= \eta_1 N_7 - cN_6 > 0,
\end{aligned}$$

then we choose $N_2$ large enough such that

$$\alpha_3 = \frac{G}{4}N_2 - c(1+N_5)N_5 - cN_6 > 0,$$

then we choose $N_4$ large enough such that

$$\alpha_4 = \frac{\rho\delta}{2}N_4 - c(1+\frac{N_2}{N_5})N_2 - 2 > 0,$$

then we choose $N_3$ large enough such that

$$\alpha_5 = \frac{\rho_3}{2}N_3 - c(1+\frac{N_4}{N_2})N_4 - cN_2 - c - 1 > 0,$$

thus, we arrive at

$$\begin{aligned}
\mathcal{L}'(t) &\leq -\frac{D}{4}\int_0^1 (3s-\psi)_x^2 dx - \alpha_3 \int_0^1 (\psi-\omega_x)^2 dx - [N\eta_0 - c]\int_0^1 s_t^2 dx \\
&-\alpha_2 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)| y^2(x,1,\varrho,t)\, d\varrho dx - \alpha_1 \int_0^1 (3s-\psi)_t^2 dx \\
&-\alpha_5 \int_0^1 \theta^2 dx - \alpha_7 \int_0^1 s_x^2 dx - [\alpha N - c]\int_0^1 q^2 dx - \alpha_4 \int_0^1 \omega_t^2 dx \\
&-\alpha_6 \int_0^1 s^2 dx - \alpha_8 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)| y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx, \qquad (5.3.32)
\end{aligned}$$

where $\alpha_6 = 2\gamma N_6, \alpha_7 = 2DN_6, \alpha_8 = \eta_1 N_7$.

On the other hand, if we let

$$\mathfrak{T}(t) = F_1(t) + \sum_{i=2}^{i=7} N_i F_i(t),$$

then

$$\begin{aligned}
|\mathfrak{T}(t)| &\leq I_\rho \int_0^1 |(3s-\psi)(3s-\psi)_t| dx + \rho \int_0^1 |\omega_t (\int_0^x (3s-\psi)(y)dy)| dx \\
&+N_2\rho \int_0^1 |(\psi-\omega_x)(\int_0^x \omega_t(y)dy)| dx + N_3\tau\rho_3 \int_0^1 |\theta(\int_0^x q(y)dy)| dx \\
&+N_4\rho\rho_3 \int_0^1 |\theta(\int_0^x \omega_t(y)dy)| dx + N_5\tau|(D\rho - GI_\rho)| \int_0^1 |q(3s-\psi)_x| dx \\
&+N_5\tau\delta GI_\rho \int_0^1 |(3s-\psi)_t(\psi-\omega_x)| dx + N_5\tau\delta D\rho \int_0^1 |\omega_t(3s-\psi)_x| dx \\
&+N_5\rho_3\tau|(D\rho - GI_\rho)| \int_0^1 |\theta(3s-\psi)_t| dx + 3I_\rho N_6 \int_0^1 |ss_t| dx \\
&+2\beta \int_0^1 s^2 dx + N_7 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho} |\mu_2(\varrho)| y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx.
\end{aligned}$$

Exploiting Young's, Cauchy-Schwarz, and Poincaré inequalities, we get

$$\begin{aligned}
|\mathfrak{T}(t)| &\leq c\int_0^1 \left(\omega_t^2 + (3s-\psi)_t^2 + (3s-\psi)_x^2 + (\psi-\omega_x)^2 + s^2 + s_x^2 + s_t^2\right) dx \\
&+c\int_0^1 \left(\theta^2 + q^2\right) dx + c\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)| y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx,
\end{aligned}$$

then

$$|\mathfrak{T}(t)| \leq cE(t).$$

Consequently, we obtain

$$|\mathfrak{T}(t)| = |\mathcal{L}(t) - NE(t)| \leq cE(t),$$

that is

$$(N-c)E(t) \leq \mathcal{L}(t) \leq (N+c)E(t). \qquad (5.3.33)$$

Now, by choosing $N$ large enough such that

$$N - c > 0, \alpha N - c > 0, \eta_0 N - c > 0,$$

and exploiting (5.3.1), estimates (5.3.32) and (5.3.33), respectively, give

$$\mathcal{L}'(t) \leq -k_2 E(t), \tag{5.3.34}$$

for some $k_2 > 0$, and

$$c_1 E(t) \leq \mathcal{L}(t) \leq c_2 E(t), \forall t \geq 0, \tag{5.3.35}$$

for some $c_1, c_2 > 0$, we have

$$\mathcal{L}(t) \sim E(t).$$

A combination with (5.3.34), gives

$$\mathcal{L}'(t) \leq -\lambda_1 \mathcal{L}(t), \tag{5.3.36}$$

where $\lambda_1 = \frac{k_2}{c_2}$.

Finally, a simple integration of (5.3.36), we obtain (5.3.28). Then the proof is complete. □

### 5.3.2 Polynomial Stability

In this subsection, we study the polynomial stability of systems, we consider the case $\chi \neq 0$.

**Theorem 5.3.** *Assume (5.1.4), there exist positive constant $C_1$ such that the energy functional given by (5.3.1) satisfies*

$$E(t) \leq \frac{C_1}{t}, \forall t > 0. \tag{5.3.37}$$

*Proof.* First, we introduce second-order energy functional $E_2(t)$ by

$$E_2(t) = E_1(\omega_t, \psi_t, s_t, \theta_t, q_t) = E(\omega_t, \psi_t, s_t, \theta_t, q_t),$$

satisfies

$$\begin{aligned} E_2'(t) &\leq -\alpha \int_0^1 q_t^2 dx - 4\eta_0 \int_0^1 s_{tt}^2 dx \\ &\leq -\alpha \int_0^1 q_t^2 dx. \end{aligned} \tag{5.3.38}$$

An thanks to $(5.1.5)_5$ and Young's inequality, the last term of $F_5'(t)$, gives

$$\begin{aligned} \chi \int_0^1 \theta_x (3s-\psi)_x dx &= -\chi\tau \int_0^1 q_t (3s-\psi)_x dx - \chi \int_0^1 q (3s-\psi)_x dx \\ &\leq \frac{c}{\varepsilon_7} \int_0^1 q_t^2 dx + \frac{c}{\varepsilon_7} \int_0^1 q^2 dx + 2\varepsilon_7 \int_0^1 (3s-\psi)_x^2 dx. \end{aligned}$$

We define a Lyapunov functional

$$\mathcal{G}(t) := N(E(t) + E_1(t)) + F_1(t) + \sum_{i=2}^{i=7} N_i F_i(t), \tag{5.3.39}$$

where $N$, and $N_i, i=2...7$, are positive constants to be selected later.

By differentiating (5.3.39) and using (5.3.2), (5.3.9),(5.3.12),(5.3.15),(5.3.18),(5.3.21),(5.3.25), (5.3.27) and (5.3.38) we have

$$\begin{aligned}
\mathcal{G}'(t) \leq & -\left[\frac{D}{2}-(\varepsilon_5+2\varepsilon_7)N_5\right]\int_0^1(3s-\psi)_x^2dx \\
& -[4\eta_0 N - cN_2 - cN_3 - cN_6 - \beta N_7]\int_0^1 s_t^2 dx \\
& -\left[\frac{\tau G\delta I_\rho}{2}N_5 - c(1+\frac{1}{\varepsilon_1}) - \varepsilon_2 N_2\right]\int_0^1(3s-\psi)_t^2dx \\
& -\left[\frac{G}{2}N_2 - \varepsilon_4 N_4 - c(1+\frac{1}{\varepsilon_6})N_5 - cN_6\right]\int_0^1(\psi-\omega_x)^2dx \\
& -\left[\frac{\rho\delta}{2}N_4 - \varepsilon_1 - c(1+\frac{1}{\varepsilon_2})N_2 - \varepsilon_3 N_3\right]\int_0^1\omega_t^2dx \\
& -\left[\frac{\rho_3}{2}N_3 - \varepsilon_6 N_5 - c(1+\frac{1}{\varepsilon_4})N_4 - cN_2 - c\right]\int_0^1\theta^2dx \\
& -\left[\alpha N - cN_4 - c(1+\frac{1}{\varepsilon_3})N_3 - (\frac{c}{\varepsilon_5}+\frac{c}{\varepsilon_7})N_5\right]\int_0^1 q^2dx \\
& -[2\gamma N_6]\int_0^1 s^2dx - [3DN_6]\int_0^1 s_x^2dx \\
& -[\eta_1 N_7 - cN_6]\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)\,d\varrho dx \\
& -[N_7\eta_1]\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)\,d\varrho d\rho dx \\
& -[N\alpha - \frac{c}{\varepsilon_7}N_5]\int_0^1 q_t^2dx. \qquad (5.3.40)
\end{aligned}$$

By setting

$$\varepsilon_1 = 1, \varepsilon_2 = \frac{\tau G\delta I_\rho N_5}{4N_2}, \varepsilon_3 = \frac{1}{N_3}, \varepsilon_4 = \frac{GN_2}{4N_4}, \varepsilon_5 = \frac{D}{8N_5}, \varepsilon_6 = \frac{1}{N_5}, \varepsilon_7 = \frac{D}{16N_5},$$

we obtain

$$
\begin{aligned}
\mathcal{G}'(t) \leq & -\left[\frac{D}{4}\right]\int_0^1 (3s-\psi)_x^2 dx \\
& -\left[4\eta_0 N - cN_2 - cN_3 - cN_6 - \beta N_7\right]\int_0^1 s_t^2 dx \\
& -\left[\frac{\tau G\delta I_\rho}{4}N_5 - 2c\right]\int_0^1 (3s-\psi)_t^2 dx \\
& -\left[\frac{G}{4}N_2 - c(1+N_5)N_5 - cN_6\right]\int_0^1 (\psi-\omega_x)^2 dx \\
& -\left[\frac{\rho\delta}{2}N_4 - c(1+\frac{N_2}{N_5})N_2 - 2\right]\int_0^1 \omega_t^2 dx \\
& -\left[\frac{\rho_3}{2}N_3 - c(1+\frac{N_4}{N_2})N_4 - cN_2 - c - 1\right]\int_0^1 \theta^2 dx \\
& -\left[\alpha N - cN_4 - c(1+N_3)N_3 - cN_5^2\right]\int_0^1 q^2 dx \\
& -\left[2\gamma N_6\right]\int_0^1 s^2 dx - \left[3DN_6\right]\int_0^1 s_x^2 dx \\
& -\left[\eta_1 N_7 - cN_6\right]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t)\, d\varrho dx \\
& -\left[N_7\eta_1\right]\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)| y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx \\
& -\left[N\alpha - cN_5^2\right]\int_0^1 q_t^2 dx.
\end{aligned} \tag{5.3.41}
$$

Next, we carefully choose our constants so that the terms inside the brackets are positive.

We fixed $N_6$, and we choose $N_5, N_7$ large enough such that

$$
\begin{aligned}
\alpha_1 &= \frac{\tau G\delta I_\rho}{4}N_5 - 2c > 0, \\
\alpha_2 &= \eta_1 N_7 - cN_6 > 0,
\end{aligned}
$$

then we choose $N_2$ large enough such that

$$
\alpha_3 = \frac{G}{4}N_2 - c(1+N_5)N_5 - cN_6 > 0,
$$

then we choose $N_4$ large enough such that

$$
\alpha_4 = \frac{\rho\delta}{2}N_4 - c(1+\frac{N_2}{N_5})N_2 - 2 > 0,
$$

then we choose $N_3$ large enough such that

$$
\alpha_5 = \frac{\rho_3}{2}N_3 - c(1+\frac{N_4}{N_2})N_4 - cN_2 - c - 1 > 0,
$$

thus, we arrive at

$$\begin{aligned}\mathcal{G}'(t) \leq & -\frac{D}{4}\int_0^1 (3s-\psi)_x^2 dx - \alpha_3 \int_0^1 (\psi-\omega_x)^2 dx - [N\eta_0 - c]\int_0^1 s_t^2 dx \\ & -\alpha_2 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)| y^2(x,1,\varrho,t)\, d\varrho dx - \alpha_1 \int_0^1 (3s-\psi)_t^2 dx \\ & -\alpha_5 \int_0^1 \theta^2 dx - \alpha_7 \int_0^1 s_x^2 dx - [\alpha N - c]\int_0^1 q^2 dx - \alpha_4 \int_0^1 \omega_t^2 dx \\ & -\alpha_6 \int_0^1 s^2 dx - \alpha_8 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)| y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx \\ & -[N\alpha - c]\int_0^1 q_t^2 dx, \end{aligned} \tag{5.3.42}$$

where $\alpha_6 = 2\gamma N_6, \alpha_7 = 2DN_6, \alpha_8 = \eta_1 N_7$.

On the other hand, if we let

$$\mathcal{K}(t) = F_1(t) + \sum_{i=2}^{i=7} N_i F_i(t),$$

$$\begin{aligned}|\mathcal{K}(t)| \leq & I_\rho \int_0^1 |(3s-\psi)(3s-\psi)_t| dx + \rho \int_0^1 |\omega_t (\int_0^x (3s-\psi)(y) dy)| dx \\ & + N_2 \rho \int_0^1 |(\psi-\omega_x)(\int_0^x \omega_t(y) dy)| dx + N_3 \tau \rho_3 \int_0^1 |\theta(\int_0^x q(y) dy)| dx \\ & + N_4 \rho \rho_3 \int_0^1 |\theta(\int_0^x \omega_t(y) dy)| dx + N_5 \tau |(D\rho - GI_\rho)| \int_0^1 |q(3s-\psi)_x| dx \\ & + N_5 \tau \delta G I_\rho \int_0^1 |(3s-\psi)_t (\psi-\omega_x)| dx + N_5 \tau \delta D \rho \int_0^1 |\omega_t (3s-\psi)_x| dx \\ & + N_5 \rho_3 \tau |(D\rho - GI_\rho)| \int_0^1 |\theta(3s-\psi)_t| dx + 3 I_\rho N_6 \int_0^1 |s s_t| dx \\ & + 2\beta \int_0^1 s^2 dx + N_7 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho} |\mu_2(\varrho)| y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx. \end{aligned}$$

Exploiting Young's, Cauchy-Schwarz, and Poincaré inequalities, we get

$$\begin{aligned}|\mathcal{K}(t)| \leq & c\int_0^1 \left(\omega_t^2 + (3s-\psi)_t^2 + (3s-\psi)_x^2 + (\psi-\omega_x)^2 + s^2 + s_x^2 + s_t^2\right) dx \\ & + c\int_0^1 \left(\theta^2 + q^2\right) dx + c\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)| y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx, \end{aligned}$$

then

$$|\mathcal{K}(t)| \leq c E_1(t).$$

Consequently, we obtain

$$|\mathcal{K}(t)| = |\mathcal{G}(t) - N(E_1(t) + E_2(t))| \leq cE(t) = cE_1(t),$$

that is

$$(N-c)E_1(t) + NE_2(t)(t) \leq \mathcal{L}(t) \leq (N+c)E_1(t) + NE_2(t). \tag{5.3.43}$$

Now, by choosing $N$ large enough such that

$$N - c > 0, N\alpha - c > 0, N\eta_0 - c > 0,$$

and exploiting (5.3.1), estimates (5.3.43) and (5.3.42), respectively, give

$$m_1(E_1(t)+E_2(t)) \leq \mathcal{G}(t) \leq m_2(E_1(t)+E_2(t)), \forall t \geq 0, \tag{5.3.44}$$

for some $m_1, m_2 > 0$.

we have

$$\mathcal{G}(t) \sim (E_1(t)+E_2(t)),$$

and we have

$$\mathcal{G}'(t) \leq -d_1 E_1(t), \tag{5.3.45}$$

for some $d_1 > 0$.

Integrating (5.3.45), we get

$$\begin{aligned} \int_0^t E_1(y)dy &\leq \frac{1}{d_1}(\mathcal{G}(0)-\mathcal{G}(t)) \leq \frac{1}{d_1}\mathcal{G}(0) \\ &\leq \frac{m_2}{d_1}(E_1(0)+E_2(0)), \end{aligned} \tag{5.3.46}$$

using the fact that

$$(tE_1(t))' = E_1(t)+tE_1'(t) \leq E_1(t), \tag{5.3.47}$$

we get that

$$tE_1(t) \leq \frac{m_2}{d_1}(E_1(0)+E_2(0)), \tag{5.3.48}$$

which gives us

$$E_1(t) \leq \frac{C_1}{t}, \tag{5.3.49}$$

where $C_1 = \frac{m_2}{d_1}(E_1(0)+E_2(0))$, we obtain (5.3.37). Then the proof is complete. □

# Part III

# System of damped Wave equations: General decay, Blow-up and Growth of solutions

Chapter 6

# Blow up of a nonlinear viscoelastic wave equation with distributed delay, strong damping and source termes

## 6.1 introduction

In this chapter, we are concerned with a problem for a nonlinear viscoelastic wave equation with distributed delay, strong damping and source termes, under suitable conditions we proved a blow up result of the solution.

We consider the following problem

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \int_0^t g(t-s)\Delta u(s)ds \\ +\mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u_t(x,t-\varrho)d\varrho = b|u|^{p-2}.u, \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, \ x \in \partial\Omega, \\ u_t(x,-t) = f_0(x,t), \quad (x,t) \in \Omega \times (0,\tau_2), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega, \end{cases} \tag{6.1.1}$$

where $\omega, b, \mu_1 > 0$, $p > 2$ and $\tau_1, \tau_2$ are the time delay with $0 \le \tau_1 < \tau_2$, and $\mu_2$ is an $L^\infty$ function, and $g$ is a differentiable function, and under the assumptions $(A_1), (A_2), and (A_3)$.

In this work, we studied problem (6.1.1), All damping mechanisms were considered at the same time ( i.e. $w > 0$; $g \neq 0$; and $\mu_1 > 0, \mu_2 \in L^\infty$), these assumptions make our problem different from the one previously studied, specially the blow up of solutions.

the our goal is to expand the results of the blow-up results to our strong damping for a viscoelastic problem with distributed delay, we prove the blow-up result under the following suitable assumptions.

(**A1**) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable and decreasing function so that

$$g(t) \geq 0 \quad , \quad 1 - \int_0^\infty g(s)\,ds = l > 0. \tag{6.1.2}$$

(**A2**) There exists a constant $\xi > 0$ such that

$$g'(t) \leq -\xi g(t) \quad , \quad t \geq 0. \tag{6.1.3}$$

(**A3**) $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is an $L^\infty$ function so that

$$(\frac{2\delta - 1}{2}) \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho < \mu_1 \quad , \quad \delta > \frac{1}{2}. \tag{6.1.4}$$

## 6.2 Blow up

In this section, we prove the blow up result of solution of problem (6.1.1).

First, as in [133], we introduce the new varible

$$y(x, \rho, \varrho, t) = u_t(x, t - \varrho\rho),$$

then we obtain

$$\begin{cases} \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0 \\ y(x, 0, \varrho, t) = u_t(x, t). \end{cases} \tag{6.2.1}$$

Let us denote by

$$gou = \int_\Omega \int_0^t g(t-s)|u(t) - u(s)|^2 ds dx. \tag{6.2.2}$$

Therefore, problem (6.1.1) takes the form:

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \int_0^t g(t-s)\Delta u(s) ds \\ +\mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x, 1, \varrho, t) d\varrho = b|u|^{p-2}.u, \qquad x \in \Omega, t > 0, \\ \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0, \end{cases} \tag{6.2.3}$$

with initial and boundary conditions

$$\begin{cases} u(x, t) = 0, \quad x \in \partial\Omega, \\ y(x, \rho, \varrho, 0) = f_0(x, \varrho\rho), \\ u(x, 0) = u_0(x), \; u_t(x, 0) = u_1(x), \end{cases} \tag{6.2.4}$$

where

$$(x, \rho, \varrho, t) \in \Omega \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty).$$

**Theorem 6.1.** *Assume (6.1.2),(6.1.3), and (6.1.4) holds. Let*

$$\begin{cases} 2 < p < \frac{2n-2}{n-2}, \quad n \geq 3; \\ p \geq 2, \qquad n = 1, 2 \end{cases} \tag{6.2.5}$$

*Then for any initial data*

$$(u_0, u_1, f_0) \in \mathcal{H} \ / \ \mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega \times (0,1) \times (\tau_1, \tau_2)),$$

*the problem (6.2.4) has a unique solution*

$$u \in C([0,T];\mathcal{H}),$$

*for some* $T > 0$.

We define the energy functional

**Lemma 6.1.** *Assume (6.1.2),(6.1.3),(6.1.4), and (6.2.5) hold, let* $u(x,t)$ *be a solution of (6.2.3), then* $E(t)$ *is non-increasing, that is*

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u\|_2^2 + \frac{1}{2}(go\nabla u) \\ &+ \frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx - \frac{b}{p}\|u\|_p^p, \end{aligned} \tag{6.2.6}$$

*satisfies*

$$E'(t) \leq -c_1(\|u_t\|_2^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx). \tag{6.2.7}$$

*Proof.* By multiplying the equation $(6.2.3)_1$ by $u_t$ and integrating over $\Omega$, we get

$$\begin{aligned} &\frac{d}{dt}\{\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u\|_2^2 + \frac{1}{2}(go\nabla u) - \frac{b}{p}\|u\|_p^p\} \\ = &-\mu_1\|u_t\|_2^2 - \int_\Omega u_t \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y(x,1,\varrho,t)d\varrho dx \\ &+ \frac{1}{2}(g' o\nabla u) - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \omega\|\nabla u_t\|_2^2, \end{aligned} \tag{6.2.8}$$

and, we have

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx &= -\frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} 2|\mu_2(\varrho)|yy_\rho d\varrho d\rho dx \\ &= +\frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y^2(x,0,\varrho,t)d\varrho dx \\ &\quad -\frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \\ &= \frac{1}{2}(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)d\varrho)\|u_t\|_2^2 \\ &\quad -\frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx, \end{aligned} \tag{6.2.9}$$

then, we get

$$\begin{aligned} \frac{d}{dt}E(t) &= -\mu_1\|u_t\|_2^2 - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|u_t y(x,1,\varrho,t)d\varrho dx + \frac{1}{2}(g' o\nabla u) \\ &- \frac{1}{2}g(t)\|\nabla u\|_2^2 - \omega\|\nabla u_t\|_2^2 + \frac{1}{2}(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)d\varrho)\|u_t\|_2^2 \\ &- \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx. \end{aligned} \tag{6.2.10}$$

By (6.2.8) and (6.2.9), we get (6.2.6).

And by using Young's inequality, (6.1.2),(6.1.3) and (6.1.4) in (6.2.10), we obtain (6.2.7). □

**Lemma 6.2.** *There exists $c > 0$, depending on $\Omega$ only so that*

$$(\int_{\Omega} |u|^p dx)^{s/p} \leq c[\|\nabla u\|_2^2 + \|u\|_p^p], \tag{6.2.11}$$

*for all $u \in L^{p+1}(\Omega)$ and $2 \leq s \leq p$.*

Using the fact that $\|u\|_2^2 \leq c\|u\|_p^2 \leq c(\|u\|_p^p)^{2/p}$, we have

**Corollary 6.1.** *There exists $C > 0$, depending on $\Omega$, such that*

$$\|u\|_2^2 \leq c[\|\nabla u\|_2^{4/p} + (\|u\|_p^p)^{2/p}]. \tag{6.2.12}$$

**Lemma 6.3.** *There exists $C > 0$, depending on $\Omega$, so that*

$$\|u\|_p^s \leq C[\|\nabla u\|_2^2 + \|u\|_p^p], \tag{6.2.13}$$

*for all $u \in L^{p+1}(\Omega)$ and $2 \leq s \leq p$.*

*Proof.* If $\|u\|_p \geq 1$ then

$$\|u\|_p^s \leq \|u\|_p^p.$$

If $\|u\|_p \leq 1$ then, $\|u\|_p^s \leq \|u\|_p^2$. Using Sobolev embedding theorems, we have

$$\|u\|_p^s \leq \|u\|_p^2 \leq c\|\nabla u\|_2^2.$$

□

Now we define the functional

$$\begin{aligned} \mathbb{H}(t) = -E(t) &= \frac{b}{p}\|u\|_p^p - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u\|_2^2 \\ &\quad - \frac{1}{2}(go\nabla u) - \frac{1}{2}\int_{\Omega}\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx. \end{aligned} \tag{6.2.14}$$

**Theorem 6.2.** *Assume (6.1.2)-(6.1.4), and (6.2.5) hold. Assume further that $E(0) < 0$, then the solution of problem (6.2.3) blow up in finite time.*

*Proof.* From (6.2.6), we have

$$E(t) \leq E(0) \leq 0. \tag{6.2.15}$$

Therefore

$$\begin{aligned} \mathbb{H}'(t) = -E'(t) &\geq c_1(\|u_t\|_2^2 + \int_{\Omega}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx) \\ &\geq c_1\int_{\Omega}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \geq 0, \end{aligned} \tag{6.2.16}$$

and

$$0 \leq \mathbb{H}(0) \leq \mathbb{H}(t) \leq \frac{b}{p}\|u\|_p^p. \tag{6.2.17}$$

We set

$$\mathcal{K}(t) = \mathbb{H}^{1-\alpha} + \varepsilon \int_\Omega u u_t dx + \frac{\varepsilon \mu_1}{2}\int_\Omega u^2 dx + \frac{\varepsilon \omega}{2}\int_\Omega (\nabla u)^2 dx, \tag{6.2.18}$$

where $\varepsilon > 0$ to be assigned later and

$$\frac{2(p-2)}{p^2} < \alpha < \frac{p-2}{2p} < 1. \tag{6.2.19}$$

By multiplying $(6.2.3)_1$ by $u$ and with a derivative of (6.2.18), we get

$$\begin{aligned} \mathcal{K}'(t) &= (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon\|u_t\|_2^2 + \varepsilon\int_\Omega \nabla u \int_0^t g(t-s)\nabla u(s) ds dx \\ & -\varepsilon\|\nabla u\|_2^2 + \varepsilon b\int_\Omega |u|^p dx - \varepsilon\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u y(x,1,\varrho,t) d\varrho dx. \end{aligned} \tag{6.2.20}$$

Using

$$\begin{aligned} \varepsilon\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u y(x,1,\varrho,t) d\varrho dx &\leq \varepsilon\Big\{\delta_1(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\|u\|_2^2 \\ & + \frac{1}{4\delta_1}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx\Big\}, \end{aligned} \tag{6.2.21}$$

and

$$\begin{aligned} \varepsilon\int_0^t g(t-s)ds\int_\Omega \nabla u.\nabla u(s) dx ds &= \varepsilon\int_0^t g(t-s)ds\int_\Omega \nabla u.(\nabla u(s) - \nabla u(t)) dx ds \\ & + \varepsilon\int_0^t g(s)ds\|\nabla u\|_2^2 \\ &\geq \frac{\varepsilon}{2}\int_0^t g(s)ds\|\nabla u\|_2^2 - \frac{\varepsilon}{2}(go\nabla u), \end{aligned} \tag{6.2.22}$$

we obtain, from (6.2.20),

$$\begin{aligned} \mathcal{K}'(t) &\geq (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon(1 - \frac{1}{2}\int_0^t g(s)ds)\|\nabla u\|_2^2 + \varepsilon b\|u\|_p^p \\ & -\varepsilon\delta_1(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\|u\|_2^2 - \frac{\varepsilon}{4\delta_1}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx \\ & + \frac{\varepsilon}{2}(go\nabla u). \end{aligned} \tag{6.2.23}$$

Therefore, using (6.2.16) and by setting $\delta_1$ so that, $\frac{1}{4\delta_1 c_1} = \kappa\mathbb{H}^{-\alpha}(t)$, substituting in (6.2.23), we get

$$\begin{aligned} \mathcal{K}'(t) &\geq [(1-\alpha) - \varepsilon\kappa]\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon\|u_t\|_2^2 \\ & -\varepsilon[(1 - \frac{1}{2}\int_0^t g(s)ds)]\|\nabla u\|_2^2 + \varepsilon b\|u\|_p^p \\ & -\varepsilon\frac{\mathbb{H}^\alpha(t)}{4c_1\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\|u\|_2^2 + \frac{\varepsilon}{2}(go\nabla u). \end{aligned} \tag{6.2.24}$$

For $0 < a < 1$, from (6.2.14)

$$\begin{aligned}
\varepsilon b\|u\|_p^p &= \varepsilon p(1-a)\mathbb{H}(t)+\frac{\varepsilon p(1-a)}{2}\|u_t\|_2^2+\varepsilon ba\|u\|_p^p \\
&+\frac{\varepsilon p(1-a)}{2}(1-\int_0^t g(s)ds)\|\nabla u\|_2^2+\frac{\varepsilon}{2}p(1-a)(go\nabla u) \\
&+\frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx. \qquad (6.2.25)
\end{aligned}$$

substituting in (6.2.24), we get

$$\begin{aligned}
\mathcal{K}'(t) &\geq [(1-\alpha)-\varepsilon\kappa]\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\varepsilon[\frac{p(1-a)}{2}+1]\|u_t\|_2^2 \\
&+\varepsilon[(\frac{p(1-a)}{2})(1-\int_0^t g(s)ds)-(1-\frac{1}{2}\int_0^t g(s)ds)]\|\nabla u\|_2^2 \\
&-\varepsilon\frac{H^\alpha(t)}{4c_1\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\|u\|_2^2+\varepsilon p(1-a)\mathbb{H}(t)+\varepsilon ba\|u\|_p^p \\
&+\frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx \\
&+\frac{\varepsilon}{2}(p(1-a)+1)(go\nabla u). \qquad (6.2.26)
\end{aligned}$$

Using (6.2.12), (6.2.17) and Young's inequality, we get

$$\begin{aligned}
\mathbb{H}^\alpha(t)\|u\|_2^2 &\leq (b\int_\Omega|u|^p dx)^\alpha\|u\|_2^2 \\
&\leq c\{(\int_\Omega|u|^p dx)^{\alpha+2/p}+(\int_\Omega|u|^p dx)^\alpha\|\nabla u\|_2^{4/p}\} \\
&\leq c\{(\int_\Omega|u|^p dx)^{(p\alpha+2)/p}+\|\nabla u\|_2^2+(\int_\Omega|u|^p dx)^{p\alpha/(p-2)}\}. \qquad (6.2.27)
\end{aligned}$$

Exploiting (6.2.19), we have

$$2 < p\alpha+2 \leq p, \quad and \quad 2 < \frac{\alpha p^2}{p-2} \leq p.$$

Consequently, by lemma 6.2, we get

$$\mathbb{H}^\alpha(t)\|u\|_2^2 \leq c\{\|u\|_p^p dx+\|\nabla u\|_2^2\}. \qquad (6.2.28)$$

Combining (6.2.26) and (6.2.28), we obtain

$$\begin{aligned}
\mathcal{K}'(t) &\geq [(1-\alpha)-\varepsilon\kappa]\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\varepsilon[\frac{p(1-a)}{2}+1]\|u_t\|_2^2 \\
&+\frac{\varepsilon}{2}(p(1-a)+1)(go\nabla u) \\
&+\varepsilon\{(\frac{p(1-a)}{2}-1)-\int_0^t g(s)ds(\frac{p(1-a)-1}{2}) \\
&-\frac{c}{4c_1\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\}\|\nabla u\|_2^2 \\
&+\varepsilon[ab-\frac{c}{4c_1\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)]\|u\|_p^p+\varepsilon p(1-a)\mathbb{H}(t) \\
&+\frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx. \qquad (6.2.29)
\end{aligned}$$

In this stage, we take $a>0$ small enough so that

$$\alpha_1 = \frac{p(1-a)}{2} - 1 > 0,$$

and we assume

$$\int_0^\infty g(s)ds < \frac{\frac{p(1-a)}{2} - 1}{(\frac{p(1-a)}{2} - \frac{1}{2})} = \frac{2\alpha_1}{2\alpha_1 + 1}, \tag{6.2.30}$$

then we choose $\kappa$ so large that

$$\alpha_2 = (\frac{p(1-a)}{2} - 1) - \int_0^t g(s)ds(\frac{p(1-a)-1}{2}) - \frac{c}{4c_1\kappa}(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|d\varrho) > 0,$$

and

$$\alpha_3 = ab - \frac{c}{4c_1\kappa}(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|d\varrho) > 0,$$

we fixed $\kappa$ and $a$, we appoint $\varepsilon$ small enough so that

$$\alpha_4 = (1-\alpha) - \varepsilon\kappa > 0.$$

Thus, for some $\beta > 0$ , estimate (6.2.29) becomes

$$\begin{aligned} \mathcal{K}'(t) \quad &\geq \quad \beta\{\mathbb{H}(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + (go\nabla u) + \|u\|_p^p \\ &\quad + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx\}, \end{aligned} \tag{6.2.31}$$

and

$$\mathcal{K}(t) \geq \mathcal{K}(0) > 0, \quad t > 0. \tag{6.2.32}$$

Next, using Holder's and Young's inequalities, we have

$$\|u\|_2 = (\int_\Omega u^2 dx)^{\frac{1}{2}} \leq [(\int_\Omega (|u|^2)^{p/2} dx)^{\frac{2}{p}}.(\int_\Omega 1dx)^{1-\frac{2}{p}}]^{\frac{1}{2}} \leq c\|u\|_p, \tag{6.2.33}$$

and

$$|\int_\Omega uu_t dx| \leq \|u_t\|_2.\|u\|_2 \leq c\|u_t\|_2.\|u\|_p,$$

then

$$\begin{aligned} |\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}} \quad &\geq \quad c\|u_t\|_2^{\frac{1}{1-\alpha}}.\|u\|_p^{\frac{1}{1-\alpha}} \\ &\leq \quad c[\|u_t\|_2^{\frac{\theta}{1-\alpha}} + \|u\|_p^{\frac{\mu}{1-\alpha}}], \end{aligned} \tag{6.2.34}$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$, we take $\theta = 2(1-\alpha)$, to get

$$\frac{\mu}{1-\alpha} = \frac{2}{1-2\alpha} \leq p.$$

Subsequently, for $s = \frac{2}{(1-2\alpha)}$, we obtain

$$|\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}} \leq c[\|u_t\|_2^2 + \|u\|_p^s].$$

Therefore, lemma 6.3 gives

$$\begin{aligned} |\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}} &\geq c[\|u_t\|_2^2+\|u\|_p^p+\|\nabla u\|_2^2] \\ &\leq c[\|u_t\|_2^2+\|u\|_p^p+\|\nabla u\|_2^2+(go\nabla u)]. \end{aligned} \tag{6.2.35}$$

Subsequently,

$$\begin{aligned} \mathcal{K}^{\frac{1}{1-\alpha}}(t) &= (\mathbb{H}^{1-\alpha}+\varepsilon\int_\Omega uu_t dx+\frac{\varepsilon\mu_1}{2}\int_\Omega u^2 dx+\frac{\varepsilon\omega}{2}\int_\Omega \nabla u^2 dx)^{\frac{1}{1-\alpha}} \\ &\leq c[\mathbb{H}(t)+|\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}}+\|u\|_2^{\frac{2}{1-\alpha}}+\|\nabla u\|_2^{\frac{2}{1-\alpha}}] \\ &\leq c[\mathbb{H}(t)+\|u_t\|_2^2+\|u\|_p^p+\|\nabla u\|_2^2+(go\nabla u)]. \end{aligned} \tag{6.2.36}$$

From (6.2.31) and (6.2.36), gives

$$\mathcal{K}'(t) \geq \lambda \mathcal{K}^{\frac{1}{1-\alpha}}(t), \tag{6.2.37}$$

where $\lambda > 0$, this depends only on $\beta$ and $c$.

by integration of (6.2.37), we obtain

$$\mathcal{K}^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{\mathcal{K}^{\frac{-\alpha}{1-\alpha}}(0)-\lambda\frac{\alpha}{(1-\alpha)}t}.$$

Hence, $\mathcal{K}(t)$ blows up in time

$$T \leq T^* = \frac{1-\alpha}{\lambda\alpha\mathcal{K}^{\alpha/(1-\alpha)}(0)}.$$

Then the proof is completed. □

Chapter 7

# General decay rate for a viscoelastic wave equation with distributed delay and Balakrishnan-Taylor damping

## 7.1 Introduction and Preliminaries

The objective of this chapter is the study of the general decay of solutions for a viscoelastic wave equation with distributed delay and Balakrishnan-Taylor damping. This type of problem is frequently found in some mathematical models in applied sciences. Especially in the theory of viscoelasticity. What interests us in this current work is the combination of these terms of damping (Memory term, Balakrishnan-Taylor damping and the distributed delay terms), which dictates the emergence of these terms in the problem.

Let $\mathcal{H} = \Omega \times (\tau_1, \tau_2) \times (0, \infty)$, in the present work, we consider the following wave equation

$$\begin{cases} u_{tt} - \Big(\zeta_0 + \zeta_1 \|\nabla u\|_2^2 + \sigma(\nabla u, \nabla u_t)_{L^2(\Omega)}\Big)\Delta u(t) + \alpha(t)\displaystyle\int_0^t h(t-\varrho)\Delta u(\varrho)d\varrho \\ \quad +\beta_1 |u_t(t)|^{m-2}u_t(t) + \displaystyle\int_{\tau_1}^{\tau_2} |\beta_2(s)||u_t(t-s)|^{m-2}u_t(t-s)ds = 0. \\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), \;\; in \;\; \Omega \\ u_t(x,-t) = f_0(x,t), \;\; in \;\; \Omega \times (0,\tau_2) \\ u(x,t) = 0, \;\; in \;\; \partial\Omega \times (0,\infty), \end{cases} \tag{7.1.1}$$

where $\Omega \in \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. $\zeta_0, \zeta_1, \sigma, \beta_1$ are positive constants, $m \geq 1$ for $N = 1, 2$, and $1 < m \leq \frac{N+2}{N-2}$ for $N \geq 3$.
$\tau_1 < \tau_2$ are non-negative constants such that $\beta_2 : [\tau_1, \tau_2] \to \mathbb{R}$ represents distributive time delay, $h, \alpha$ are positive functions.

The combination of these terms of damping (Memory term, Balakrishnan-Taylor damping and the distributed delay ) in one particular problem with the addition of $\alpha(t)$ to the term of memory and the distributed delay term ( $\int_{\tau_1}^{\tau_2} |\beta_2(s)||u_t(t-s)|^{m-2}u_t(t-s)ds$) we believe that it constitutes a new problem worthy of study and research, different from the above that we will try to shed light on.
Our chapter is divided into several sections: in the next section we lay down the hypotheses, concepts and lemmas we need. Finally, we prove our main result.

For studying our problem, in this section we will need some materials.
Firstly, introducing the following hypothesis for $\beta_2$, $h$ and $\alpha$:
**(A1)** $h, \alpha : \mathbb{R}_+ \to \mathbb{R}_+$ are non-increasing $C^1$ functions satisfying

$$h(t) > 0,\ \alpha(t) > 0,\ l_0 = \int_0^{\infty} h(\varrho)d\varrho < \infty,\ \zeta_0 - 2\alpha(t)\int_0^t h(\varrho)d\varrho \geq l > 0, \tag{7.1.2}$$

**(A2)** $\exists \vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing $C^1$ function, satisfying

$$\vartheta(t)h(t) + h'(t) \leq 0,\ \ t \geq 0\ \ and\ \ \lim_{t\to\infty} \frac{-\alpha'(t)}{\vartheta(t)\alpha(t)} = 0. \tag{7.1.3}$$

**(A3)** $\beta_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\beta_2(s)|ds < \beta_1. \tag{7.1.4}$$

Let us indroduce

$$(h \circ \psi)(t) := \int_{\Omega}\int_0^t h(t-\varrho)|\psi(t) - \psi(\varrho)|^2 d\varrho dx.$$

and

$$M(t) := \Big(\zeta_0 + \zeta_1\|\nabla u\|_2^2 + \sigma(\nabla u(t), \nabla u_t(t))_{L^2(\Omega)}\Big).$$

**Lemma 7.1.** *(Sobolev-Poincare inequality [1]). Let* $2 \leq q < \infty (n = 1, 2)$ *or* $2 \leq q < \frac{2n}{n-2} (n \geq 3)$*. Then,* $\exists c_* = c(\Omega, q) > 0$ *such that*

$$\|u\|_q \leq c_*\|\nabla u\|_2,\ \ \forall u \in H_0^1(\Omega).$$

As in [133], taking the following new variables

$$y(x, \rho, s, t) = u_t(x, t - s\rho),$$

which satisfy

$$\begin{cases} sy_t(x, \rho, s, t) + y_\rho(x, \rho, s, t) = 0, \\ y(x, 0, s, t) = u_t(x, t). \end{cases} \tag{7.1.5}$$

So, problem (7.1.1) can be written as

$$\begin{cases} u_{tt}-\Big(\zeta_0+\zeta_1\|\nabla u\|_2^2+\sigma(\nabla u,\nabla u_t)_{L^2(\Omega)}\Big)\Delta u(t)+\alpha(t)\displaystyle\int_0^t h(t-\varrho)\Delta u(\varrho)d\varrho \\ \quad +\beta_1|u_t(t)|^{m-2}u_t(t)+\displaystyle\int_{\tau_1}^{\tau_2}|\beta_2(s)||y(x,1,s,t)|^{m-2}y(x,1,s,t)ds=0. \\ sy_t(x,\rho,s,t)+y_\rho(x,\rho,s,t)=0. \\ u(x,0)=u_0(x),\;\; u_t(x,0)=u_1(x),\;\; in\;\; \Omega \\ y(x,\rho,s,0)=f_0(x,\rho s),\;\; in\;\; \Omega\times(0,1)\times(0,\tau_2) \\ u(x,t)=0,\;\; in\;\; \partial\Omega\times(0,\infty), \end{cases} \tag{7.1.6}$$

where

$$(x,\rho,s,t)\in\Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

Now, we give the energy functional.

**Lemma 7.2.** *The energy functional $E$, defined by*

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\Big(\zeta_0-\alpha(t)\int_0^t h(\varrho)d\varrho\Big)\|\nabla u(t)\|_2^2+\frac{\zeta_1}{4}\|\nabla u(t)\|_2^4 \\ &\quad +\frac{\alpha(t)}{2}(h\circ\nabla u)(t)+\frac{m-1}{m}\int_0^1\int_{\tau_1}^{\tau_2}s|\beta_2(s)|\|y(x,\rho,s,t)\|_m^m dsd\rho, \end{aligned} \tag{7.1.7}$$

*satisfies*

$$\begin{aligned} E'(t) &\leq -\Big(\beta_1-\int_{\tau_1}^{\tau_2}|\beta_2(s)|ds\Big)\|u_t(t)\|_m^m+\frac{\alpha(t)}{2}(h'\circ\nabla u)(t) \\ &\quad -\frac{\alpha'(t)}{2}\Big(\int_0^t h(\varrho)d\varrho\Big)\|\nabla u(t)\|_2^2-\frac{\sigma}{4}\Big(\frac{d}{dt}\Big\{\|\nabla u(t)\|_2^2\Big\}\Big)^2. \end{aligned} \tag{7.1.8}$$

*Proof.* Taking the inner product of $(7.1.6)_1$ with $u_t$, then integrating over $\Omega$, we find

$$\begin{aligned} &(u_{tt}(t),u_t(t))_{L^2(\Omega)}-(M(t)\Delta u(t),u_t(t))_{L^2(\Omega)} \\ &+(\alpha(t)\int_0^t h(t-\varrho)\Delta u(\varrho)d\varrho,u_t(t))_{L^2(\Omega)}+\beta_1(|u_t|^{m-2}u_t,u_t)_{L^2(\Omega)} \\ &+\int_{\tau_1}^{\tau_2}|\beta_2(s)|(|y(x,1,s,t)|^{m-2}y(x,1,s,t),u_t(t))_{L^2(\Omega)}ds=0. \end{aligned} \tag{7.1.9}$$

By computation, integration by parts and the last condition in (1.1.2), we get

$$(u_{tt}(t),u_t(t))_{L^2(\Omega)}=\frac{1}{2}\frac{d}{dt}\Big(\|u_t(t)\|_2^2\Big), \tag{7.1.10}$$

by integration by parts, we find

$$\begin{aligned}
& -(M(t)\Delta u(t),u_t(t))_{L^2(\Omega)} \\
= & -(\Big(\zeta_0+\zeta_1\|\nabla u\|_2^2+\sigma(\nabla u(t),\nabla u_t(t))_{L^2(\Omega)}\Big)\Delta u(t),u_t(t))_{L^2(\Omega)} \\
= & \Big(\zeta_0+\zeta_1\|\nabla u\|_2^2+\sigma(\nabla u(t),\nabla u_t(t))_{L^2(\Omega)}\Big)\int_\Omega \nabla u(t).\nabla u_t(t)dx \\
= & \Big(\zeta_0+\zeta_1\|\nabla u\|_2^2+\sigma(\nabla u(t),\nabla u_t(t))_{L^2(\Omega)}\Big)\frac{d}{dt}\Big\{\int_\Omega |\nabla u(t)|^2dx\Big\} \\
= & \frac{d}{dt}\Big\{\frac{1}{2}\Big(\zeta_0+\frac{\zeta_1}{2}\|\nabla u\|_2^2\Big)\|\nabla u(t)\|_2^2\Big\}+\frac{\sigma}{4}\frac{d}{dt}\Big\{\|\nabla u(t)\|_2^2\Big\}^2,
\end{aligned} \tag{7.1.11}$$

and we have

$$\begin{aligned}
& (\int_0^t h(t-\varrho)\Delta u(\varrho)d\varrho,u_t(t))_{L^2(\Omega)} \\
= & \int_0^t h(t-\varrho)(\Delta u(\varrho),u_t(t))_{L^2(\Omega)}d\varrho \\
= & -\int_0^t h(t-\varrho)\Big[\int_\Omega \nabla u(x,\varrho)\nabla u(x,t)dx\Big]d\varrho,
\end{aligned} \tag{7.1.12}$$

and

$$-\nabla u(x,\varrho).\nabla u(x,t)=\frac{1}{2}\frac{d}{dt}\Big\{|\nabla u(x,\varrho)-\nabla u(x,t)(t)|^2\Big\}-\frac{1}{2}\frac{d}{dt}\Big\{|\nabla u(x,t)|^2\Big\}, \tag{7.1.13}$$

then

$$\begin{aligned}
& -\int_0^t h(t-\varrho)(\nabla u(\varrho),\nabla u_t(t))_{L^2(\Omega)}d\varrho \\
= & -\int_0^t h(t-\varrho)\int_\Omega\Big[\frac{1}{2}\frac{d}{dt}\Big\{|\nabla u(x,\varrho)-\nabla u(x,t)|^2\Big\}\Big]dxd\varrho. \\
& -\int_0^t h(t-\varrho)\int_\Omega\Big[\frac{1}{2}\frac{d}{dt}\Big\{|\nabla u(x,t)|^2\Big\}\Big]dxd\varrho \\
= & \frac{1}{2}\int_0^t h(t-\varrho)\Big[\frac{d}{dt}\Big\{\int_\Omega|\nabla u(x,t)-\nabla u(x,\varrho)|^2dx\Big\}\Big]d\varrho \\
& -\frac{1}{2}\int_0^t h(t-\varrho)\Big[\frac{d}{dt}\Big\{\|\nabla u(x,t)\|_2^2\Big\}\Big]dxd\varrho,
\end{aligned} \tag{7.1.14}$$

by (7.1.2), we find

$$\begin{aligned}
& \frac{1}{2}\int_0^t h(t-\varrho)\Big[\frac{d}{dt}\Big\{\int_\Omega|\nabla u(x,t)-\nabla u(x,\varrho)|^2dx\Big\}\Big]d\varrho \\
= & \frac{1}{2}\frac{d}{dt}\Big\{\int_0^t h(t-\varrho)\Big[\int_\Omega|\nabla u(x,t)-\nabla u(x,\varrho)|^2dx\Big]\Big\}d\varrho \\
& -\frac{1}{2}\int_0^t h'(t-\varrho)\Big[\int_\Omega|\nabla u(x,t)-\nabla u(x,\varrho)|^2dx\Big]d\varrho \\
= & \frac{1}{2}\frac{d}{dt}(h\circ\nabla u)(t)-\frac{1}{2}(h'\circ\nabla u)(t),
\end{aligned} \tag{7.1.15}$$

and

$$\begin{aligned}
&-\frac{1}{2}\int_0^t h(t-\varrho)\left[\frac{d}{dt}\left\{\|\nabla u(t)\|_2^2\right\}\right]dxd\varrho \\
=\;& -\frac{1}{2}\left(\int_0^t h(t-\varrho)d\varrho\right)\left(\frac{d}{dt}\left\{\|\nabla u(t)\|_2^2\right\}\right)dx \\
=\;& -\frac{1}{2}\left(\int_0^t h(\varrho)d\varrho\right)\left(\frac{d}{dt}\left\{\|\nabla u(t)\|_2^2\right\}\right)dx \\
=\;& -\frac{1}{2}\frac{d}{dt}\left\{\left(\int_0^t h(\varrho)d\varrho\right)\|\nabla u(t)\|_2^2\right\}+\frac{1}{2}h(t)\|\nabla u(t)\|_2^2,
\end{aligned} \tag{7.1.16}$$

by inserting (7.1.15) and (7.1.16) into (7.1.14), we find

$$\begin{aligned}
&\left(\alpha(t)\int_0^t h(t-\varrho)\Delta u(\varrho)d\varrho, u_t(t)\right)_{L^2(\Omega)} \\
=\;& \frac{d}{dt}\left\{\frac{\alpha(t)}{2}(h\circ\nabla u)(t)-\frac{\alpha(t)}{2}\left(\int_0^t h(\varrho)d\varrho\right)\|\nabla u(t)\|_2^2\right\} \\
& -\frac{\alpha(t)}{2}(h'\circ\nabla u)(t)+\frac{\alpha(t)}{2}h(t)\|\nabla u(t)\|_2^2 \\
& -\frac{\alpha'(t)}{2}(h\circ\nabla u)(t)+\frac{\alpha'(t)}{2}\left(\int_0^t h(\varrho)d\varrho\right)\|\nabla u(t)\|_2^2.
\end{aligned} \tag{7.1.17}$$

Now, multiplying the equation $(7.1.6)_2$ by $-y|\beta_2(s)|$, and integrating over $\Omega\times(0,1)\times(\tau_1,\tau_2)$, and using $(7.1.5)_2$, we get

$$\begin{aligned}
&\frac{d}{dt}\quad\frac{m-1}{m}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\beta_2(s)|.|y(x,\rho,s,t)|^m dsd\rho dx \\
=\;& -(m-1)\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}|\beta_2(s)|.|y|^{m-1}y_\rho dsd\rho dx \\
=\;& -\frac{m-1}{m}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}|\beta_2(s)|\frac{d}{d\rho}|y(x,\rho,s,t)|^m dsd\rho dx \\
=\;& \frac{m-1}{m}\int_\Omega\int_{\tau_1}^{\tau_2}|\beta_2(s)|\Big(|y(x,0,s,t)|^m-|y(x,1,s,t)|^m\Big)dsdx \\
=\;& \frac{m-1}{m}\left(\int_{\tau_1}^{\tau_2}|\beta_2(s)|ds\right)\int_\Omega|u_t(t)|^m dx \\
& -\frac{m-1}{m}\int_\Omega\int_{\tau_1}^{\tau_2}|\beta_2(s)|.|y(x,1,s,t)|^m dsdx \\
=\;& \frac{m-1}{m}\left(\int_{\tau_1}^{\tau_2}|\beta_2(s)|ds\right)\|u_t(t)\|_m^m \\
& -\frac{m-1}{m}\int_{\tau_1}^{\tau_2}|\beta_2(s)|\|y(x,1,s,t)\|_m^m ds,
\end{aligned} \tag{7.1.18}$$

and by Young's inequalitie, we have

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}|\beta_2(s)|\Big(|y(x,1,s,t)|^{m-2}y(x,1,s,t),u_t(t)\Big)_{L^2(\Omega)}ds \qquad (7.1.19)\\
\leq\;& \frac{1}{m}\left(\int_{\tau_1}^{\tau_2}|\beta_2(s)|ds\right)\|u_t(t)\|_m^m+\frac{m-1}{m}\int_{\tau_1}^{\tau_2}|\beta_2(s)|\|y(x,1,s,t)\|_m^m ds.
\end{aligned}$$

By inserting (7.1.10)–(7.1.11) and (7.1.17)–(7.1.19) into (7.1.9), we obtain (7.1.7) and (7.1.8). Hence, by (7.1.3), we get the function E is a non-increasing $\forall t\geq t_1$. This completes of the proof. □

Now we state the local existence of problem (7.1.6).

**Theorem 7.1.** *Suppose that (7.1.2)-(7.1.4) are satisfied. Then, for any* $u_0, u_1 \in H_0^1(\Omega)\cap L^2(\Omega)$, *and* $f_0 \in L^2(\Omega,(0,1),(\tau_1,\tau_2))$, *there exists a weak solution* $u$ *of problem (1.1.2) such that*

$$u \in C(]0,T[,H_0^1(\Omega))\cap C^1(]0,T[,L^2(\Omega)),$$
$$u_t \in C(]0,T[,H_0^1(\Omega))\cap L^2(]0,T[,L^2(\Omega,(0,1),(\tau_1,\tau_2))).$$

## 7.2 General Decay

In this section, we state and prove the asymptotic behavior of the system (7.1.6). For this goal, we set

$$\Psi(t) := \int_\Omega u(t)u_t(t)dx + \frac{\sigma}{4}\|\nabla u(t)\|_2^4, \tag{7.2.1}$$

$$\Phi(t) := -\int_\Omega u_t \int_0^t h(t-\varrho)(u(t)-u(\varrho))d\varrho dx, \tag{7.2.2}$$

and

$$\Theta(t) := \int_0^1\int_{\tau_1}^{\tau_2} se^{-\rho s}|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m ds d\rho. \tag{7.2.3}$$

**Lemma 7.3.** *The functional* $\Psi(t)$ *defined in (7.2.1) satisfies, for any* $\varepsilon>0$

$$\begin{aligned}\Psi'(t) \le{}& \|u_t\|_2^2-(l-\varepsilon(c_1+c_2))\|\nabla u\|_2^2-\zeta_1\|\nabla u\|_2^4+\frac{\alpha(t)}{4}(h\circ\nabla u)(t)\\ &+c(\varepsilon)\Big(\|u_t\|_m^m+\int_{\tau_1}^{\tau_2}|\beta_2(s)|\|y(x,1,s,t)\|_m^m ds\Big).\end{aligned} \tag{7.2.4}$$

*Proof.* A differentiation of (7.2.1) and using $(7.1.6)_1$, gives

$$\begin{aligned}\Psi'(t) ={}& \|u_t\|_2^2+\int_\Omega u_{tt}udx+\sigma\|\nabla u\|_2^2\int_\Omega \nabla u_t\nabla u dx\\ ={}& \|u_t\|_2^2-\zeta_0\|\nabla u\|_2^2-\zeta_1\|\nabla u\|_2^4\underbrace{-\beta_1\int_\Omega |u_t|^{m-2}u_t u dx}_{I_1}\\ &\underbrace{+\alpha(t)\int_\Omega\nabla u(t)\int_0^t h(t-\varrho)\nabla u(\varrho)d\varrho dx}_{I_2}\\ &\underbrace{-\int_\Omega\int_{\tau_1}^{\tau_2}|\beta_2(s)||y(x,1,s,t)|^{m-2}y(x,1,s,t)udsdx}_{I_3}.\end{aligned} \tag{7.2.5}$$

We estimate the last 3 terms of the RHS of (7.2.5). Applying Hölder's, Sobolev-Poincare and Young's inequalities, (7.1.2) and (7.1.7), we find

$$\begin{aligned} I_1 &\le \varepsilon\beta_1^m\|u\|_m^m + c(\varepsilon)\|u_t\|_m^m \\ &\le \varepsilon\beta_1^m c_p^m\|\nabla u\|_2^m + c(\varepsilon)\|u_t\|_m^m \\ &\le \varepsilon\beta_1^m c_p^m\left(\frac{E(0)}{l}\right)^{(m-2)/2}\|\nabla u\|_2^2 + c(\varepsilon)\|u_t\|_m^m \\ &\le \varepsilon c_1\|\nabla u\|_2^2 + c(\varepsilon)\|u_t\|_m^m, \end{aligned} \tag{7.2.6}$$

and

$$\begin{aligned} I_2 &\le 2\alpha(t)(\int_0^t h(\varrho)d\varrho)\|\nabla u\|_2^2 + \frac{\alpha(t)}{4}(h\circ\nabla u)(t) \\ &\le (\zeta_0 - l)\|\nabla u\|_2^2 + \frac{\alpha(t)}{4}(h\circ\nabla u)(t). \end{aligned} \tag{7.2.7}$$

Similarly to $I_1$, we have

$$I_3 \le \varepsilon c_2\|\nabla u\|_2^2 + c(\varepsilon)\int_{\tau_1}^{\tau_2}|\beta_2(s)|.\|y(x,1,s,t)\|_m^m ds. \tag{7.2.8}$$

Combining (7.2.6)-(7.2.8) and (7.2.5), we get

$$\begin{aligned} \Psi'(t) \le\ & \|u_t\|_2^2 - (l - \varepsilon(c_1+c_2))\|\nabla u\|_2^2 - \zeta_1\|\nabla u\|_2^4 \\ &+ \frac{\alpha(t)}{4}(h\circ\nabla u)(t) + c(\varepsilon)\Big(\|u_t\|_m^m + \int_{\tau_1}^{\tau_2}|\beta_2(s)|\|y(x,1,s,t)\|_m^m ds\Big). \end{aligned}$$

□

**Lemma 7.4.** *The functional* $\Phi(t)$ *defined in (7.2.2) satisfies, for any* $\delta > 0$

$$\begin{aligned} \Phi'(t) \le\ & -\Big(\int_0^t h(\varrho)d\varrho - \delta\Big)\|u_t\|_2^2 + \delta\Big(\zeta_0 + 2(1-l)^2\alpha(t)\Big)\|\nabla u\|_2^2 + \zeta_1\delta\|\nabla u\|_2^4 \\ &+ \delta\frac{\sigma E(0)}{l}\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 + \Big(c(\delta) + (2\delta + \frac{1}{4\delta})(\zeta_0 - l)\alpha(t)\Big)(h\circ\nabla u)(t) \\ &+ c(\delta)\Big(\|u_t\|_m^m + \int_{\tau_1}^{\tau_2}|\beta_2(s)|\|y(x,1,s,t)\|_m^m ds\Big) - \frac{g(0)c_p^2}{4\delta}(h'\circ\nabla u)(t). \end{aligned} \tag{7.2.9}$$

*Proof.* A differentiation of (7.2.2) and using $(7.1.6)_1$, gives

$$
\begin{aligned}
\Phi'(t) &= -\int_\Omega u_{tt}\int_0^t h(t-\varrho)(u(t)-u(\varrho))d\varrho dx \\
&\quad -\int_\Omega u_t\int_0^t h'(t-\varrho)(u(t)-u(\varrho))d\varrho dx-\Big(\int_0^t h(\varrho)d\varrho\Big)\|u_t\|_2^2 \\
&= \underbrace{(\zeta_0+\zeta_1\|\nabla u\|_2^2)\int_\Omega \nabla u\int_0^t h(t-\varrho)(\nabla u(t)-\nabla u(\varrho))d\varrho dx}_{J_1} \\
&\quad \underbrace{+\sigma\int_\Omega \nabla u\nabla u_t dx.\int_\Omega \nabla u\int_0^t h(t-\varrho)(\nabla u(t)-\nabla u(\varrho))d\varrho dx}_{J_2} \\
&\quad \underbrace{-\alpha(t)\int_\Omega\Big(\int_0^t h(t-\varrho)\nabla u(\varrho)d\varrho\Big).\Big(\int_0^t h(t-\varrho)(\nabla u(t)-\nabla u(\varrho))d\varrho\Big)dx}_{J_3} \\
&\quad \underbrace{-\beta_1\int_\Omega |u_t|^{m-2}u_t\Big(\int_0^t h(t-\varrho)(u(t)-u(\varrho))d\varrho\Big)dx}_{J_4} \\
&\quad \underbrace{\begin{aligned}&-\int_\Omega\int_{\tau_1}^{\tau_2}|\beta_2(s)||y(x,1,s,t)|^{m-2}y(x,1,s,t)\\ &\times\Big(\int_0^t h(t-\varrho)(u(t)-u(\varrho))d\varrho\Big)d\varrho dx\end{aligned}}_{J_5} \\
&\quad \underbrace{-\int_\Omega u_t\int_0^t h'(t-\varrho)(u(t)-u(\varrho))d\varrho dx}_{J_6}-\Big(\int_0^t h(\varrho)d\varrho\Big)\|u_t\|_2^2. \qquad (7.2.10)
\end{aligned}
$$

We estimate the terms $J_i, i=1,..,6$ of the RHS of (7.2.10). Applying Hölder's, Sobolev-Poincare and Young's inequalities, (7.1.2) and (7.1.7), we find

$$
\begin{aligned}
|J_1| &\leq (\zeta_0+\zeta_1\|\nabla u\|_2^2)\Big(\delta\|\nabla u\|_2^2+\frac{(\zeta_0-l)}{4\delta}(h\circ\nabla u)(t)\Big) \\
&\leq \delta\zeta_0\|\nabla u\|_2^2+\delta\zeta_1\|\nabla u\|_2^4+\Big(\frac{\zeta_0(\zeta_0-l)}{4\delta}+\frac{\zeta_1(\zeta_0-l)E(0)}{4l\delta}\Big)(h\circ\nabla u)(t),
\end{aligned}
\qquad (7.2.11)
$$

and

$$
\begin{aligned}
J_2 &\leq \delta\sigma\Big(\int_\Omega \nabla u\nabla u_t dx\Big)^2\|\nabla u\|_2^2+\frac{\sigma(\zeta_0-l)}{4\delta}(h\circ\nabla u)(t) \\
&\leq \delta\frac{\sigma E(0)}{l}\Big(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\Big)^2+\frac{\sigma(\zeta_0-l)}{4\delta}(h\circ\nabla u)(t),
\end{aligned}
\qquad (7.2.12)
$$

$$
\begin{aligned}
|J_3| &\leq \delta\alpha(t)\int_\Omega\Big(\int_0^t h(t-\varrho)(|\nabla u(t)-\nabla u(\varrho)|-\nabla|u(t)|)d\varrho\Big)^2 dx \\
&\quad +\frac{1}{4\delta}\alpha(t)\int_\Omega\Big(\int_0^t h(t-\varrho)(\nabla u(t)-\nabla u(\varrho))d\varrho\Big)^2 dx \\
&\leq 2\delta c(\zeta_0-l)^2\alpha(t)\|\nabla u\|_2^2+c\Big(2\delta+\frac{1}{4\delta}\Big)(\zeta_0-l)\alpha(t)(h\circ\nabla u)(t),
\end{aligned}
\qquad (7.2.13)
$$

$$\begin{aligned}
|J_4| &\leq c(\delta)\|\nabla u_t\|_m^m+\delta\beta_1^m\int_\Omega\Big(\int_0^t h(t-\varrho)(u(t)-u(\varrho))d\varrho\Big)^m dx\\
&\leq c(\delta)\|\nabla u_t\|_m^m+\delta\beta_1^m(\zeta_0-l)^{m-1}c_p^m\int_0^t h(t-\varrho)\|\nabla u(t)-\nabla u(\varrho)\|_2^m d\varrho\\
&\leq c(\delta)\|\nabla u_t\|_m^m+\delta\Big(\beta_1^m(\zeta_0-l)^{m-1}c_p^m(\frac{E(0)}{l})^{(m-2)/2}\Big)(h\circ\nabla u)(t)\\
&\leq c(\delta)\|\nabla u_t\|_m^m+\delta c_3(h\circ\nabla u)(t). \qquad (7.2.14)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
J_5 &\leq c(\delta)\|y(x,1,s,t)\|_m^m+\delta c_4(h\circ\nabla u)(t), \qquad (7.2.15)\\
J_6 &\leq \delta\|u_t\|_2^2-\frac{h(0)c_p^2}{4\delta}(h'\circ\nabla u)(t). \qquad (7.2.16)
\end{aligned}$$

A substitution of (7.2.11)-(7.2.16) into (7.2.10), we get

$$\begin{aligned}
\Phi'(t) \leq & -\Big(\int_0^t h(\varrho)d\varrho-\delta\Big)\|u_t\|_2^2+\delta\Big(\zeta_0+2c(\zeta_0-l)^2\alpha(t)\Big)\|\nabla u\|_2^2+\zeta_1\delta\|\nabla u\|_2^4\\
&+\delta\frac{\sigma E(0)}{l}\Big(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\Big)^2+\Big(c(\delta)+(2\delta+\frac{1}{4\delta})(\zeta_0-l)\alpha(t)\Big)(h\circ\nabla u)(t)\\
&+c(\delta)\Big(\|u_t\|_m^m+\int_{\tau_1}^{\tau_2}|\beta_2(s)|\|y(x,1,s,t)\|_m^m ds\Big)-\frac{h(0)c_p^2}{4\delta}(h'\circ\nabla u)(t).
\end{aligned}$$

□

**Lemma 7.5.** *The functional* $\Theta(t)$ *defined in (7.2.3) satisfies*

$$\begin{aligned}
\Theta'(t) \leq & -\eta_1\int_0^1\int_{\tau_1}^{\tau_2}s|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m ds d\rho\\
&-\eta_1\int_{\tau_1}^{\tau_2}|\beta_2(s)|.\|y\,(x,1,s,t)\,\|_m^m ds+\beta_1\|u_t(t)\|_m^m. \qquad (7.2.17)
\end{aligned}$$

*Proof.* By differentiating of $\Theta(t)$, and using $(7.1.6)_2$, we have

$$\begin{aligned}
\Theta'(t) &= -m\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}e^{-s\rho}|\beta_2(s)|.|y|^{m-1}y_\rho\,(x,\rho,s,t)\,ds d\rho dx\\
&= -\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}se^{-s\rho}|\beta_2(s)|.|y(x,\rho,s,t)|^m ds d\rho dx\\
&\quad -\int_\Omega\int_{\tau_1}^{\tau_2}|\beta_2(s)|\Big[e^{-s}|y\,(x,1,s,t)\,|^m-|y\,(x,0,s,t)\,|^m\Big]ds dx.
\end{aligned}$$

Applying $y(x,0,s,t)=u_t(x,t)$, and $e^{-s}\leq e^{-s\rho}\leq 1$, for any $0<\rho<1$, and we set $\eta_1=e^{-\tau_2}$, we obtain

$$\begin{aligned}
\Theta'(t) \leq & -\eta_1\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}s|\beta_2(s)|.|y(x,\rho,s,t)|^m ds d\rho dx\\
&-\eta_1\int_\Omega\int_{\tau_1}^{\tau_2}|\beta_2(s)||y(x,1,s,t)|^m ds dx+\int_{\tau_1}^{\tau_2}|\beta_2(s)|ds\int_\Omega|u_t|^m(t)dx,
\end{aligned}$$

using (7.1.4), we find (7.2.17).

□

Now, we introduce the functional

$$\mathcal{G}(t) \quad := \quad E(t)+\varepsilon_1\alpha(t)\Psi(t)+\varepsilon_2\alpha(t)\Phi(t)+\varepsilon_3\alpha(t)\Theta(t). \tag{7.2.18}$$

for some positive constants $\varepsilon_i, i=1,2,3$ to be determined.

**Lemma 7.6.** *There exist $\mu_1,\mu_2>0$, such that*

$$\mu_1 E(t)\le \mathcal{G}(t)\le \mu_2 E(t). \tag{7.2.19}$$

*Proof.* From (7.2.1)-(7.2.3), by using Hölder, Young's and poincare inequalities, we get

$$\begin{aligned}|\mathcal{G}(t)-E(t)| \quad \le \quad & \varepsilon_1\frac{|\alpha(t)|}{2}\left(\|u_t(t)\|_2^2+c_p\|\nabla u(t)\|_2^2\right)+\varepsilon_1\sigma\frac{|\alpha(t)|}{4}\|\nabla u(t)\|_2^4\\ &+\varepsilon_2\frac{|\alpha(t)|}{2}\|u_t(t)\|_2^2+\varepsilon_2\frac{|\alpha(t)|}{2}c_p(\zeta_0-l)(h\circ\nabla u)(t)\\ &+\varepsilon_3|\alpha(t)|\int_0^1\int_{\tau_1}^{\tau_2}se^{-\rho s}|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m dsd\rho.\end{aligned} \tag{7.2.20}$$

Using the fact that $0<\alpha(t)\le\alpha(0)$ and $e^{-\rho s}<1$, we find

$$\begin{aligned}|\mathcal{G}(t)-E(t)| \quad \le \quad & \varepsilon_1\frac{\alpha(0)}{2}\left(c_p\|\nabla u(t)\|_2^2+\|u_t(t)\|_2^2\right)+\varepsilon_1\sigma\frac{\alpha(0)}{4}\|\nabla u(t)\|_2^4\\ &+\varepsilon_2\frac{\alpha(0)}{2}\|u_t(t)\|_2^2+\varepsilon_2\frac{\alpha(0)}{2}c_p(\zeta_0-l)(h\circ\nabla u)(t)\\ &+\varepsilon_3\alpha(0)\int_0^1\int_{\tau_1}^{\tau_2}se^{-\rho s}|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m dsd\rho\\ \le \quad & C(\varepsilon_1,\varepsilon_2,\varepsilon_3)E(t).\end{aligned} \tag{7.2.21}$$

Choosing $\varepsilon_1,\varepsilon_2$ and $\varepsilon_3$ sufficiently small, then (7.2.19) follows from (7.2.21).

□

**Lemma 7.7.** *There exist $d_5,d_6,t_0>0$ satisfying*

$$\mathcal{G}'(t) \quad \le \quad -d_5\alpha(t)E(t)+d_6\alpha(t)(h\circ\nabla u)(t), \quad t>t_0. \tag{7.2.22}$$

*Proof.* Since the function $h$ is a positive and continuous, for all $t_0>0$, we have

$$\int_0^t h(\varrho)d\varrho\ge\int_0^{t_0}h(\varrho)d\varrho:=h_0,\ \ \forall t\ge t_0.$$

A differentiation of (7.2.18), using 7.1.8, the Lemmas 7.3, 7.4 and 7.5

$$
\begin{aligned}
\mathcal{G}'(t) \quad := \quad & E'(t)+\varepsilon_1\alpha'(t)\Psi(t)+\varepsilon_2\alpha'(t)\Phi(t)+\varepsilon_3\alpha'(t)\Theta(t) \\
& +\varepsilon_1\alpha(t)\Psi'(t)+\varepsilon_2\alpha(t)\Phi'(t)+\varepsilon_3\alpha(t)\Theta'(t) \\
\leq \quad & \alpha(t)\Big(\varepsilon_1-\varepsilon_2(h_0-\delta)\Big)\|u_t\|_2^2 \\
& +\alpha(t)\Big(\varepsilon_2\delta(\zeta_0+2c(\zeta_0-l)^2\alpha(t))-\varepsilon_1(l-\varepsilon(c_1+c_2))\Big)\|\nabla u\|_2^2 \\
& +\alpha(t)\Big(\varepsilon_2\zeta_1\delta-\varepsilon_1\zeta_1\Big)\|\nabla u\|_2^4 \\
& +\alpha(t)\Big(\varepsilon_2\delta\frac{\sigma E(0)}{l}-\frac{\sigma}{\alpha(0)}\Big)\Big(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\Big)^2 \\
& +\alpha(t)\Big(\varepsilon_1\frac{\alpha(t)}{4}+\varepsilon_2(c(\delta)+(2\delta+\frac{1}{4\delta})(\zeta_0-l)\alpha(t))\Big)(h\circ\nabla u)(t) \\
& +\alpha(t)\Big(\frac{1}{2}-\varepsilon_2\frac{h(0)c_p^2}{4\delta}\Big)(h'\circ\nabla u)(t) \\
& +\alpha(t)\Big(\varepsilon_1c(\varepsilon)+\varepsilon_2c(\delta)+\varepsilon_3\beta_1-\frac{\eta_0}{\alpha(0)}\Big)\|u_t\|_m^m \\
& +\alpha(t)\Big(\varepsilon_1c(\varepsilon)+\varepsilon_2c(\delta)-\eta_1\varepsilon_3\Big)\int_{\tau_1}^{\tau_2}|\beta_2(s)\|y(x,1,s,t)\|_m^m ds\Big) \\
& -\alpha(t)\varepsilon_3\eta_1\int_0^1\int_{\tau_1}^{\tau_2}s|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m ds d\rho \\
& -\frac{\alpha'(t)}{2}\Big(\int_0^t h(\varrho)d\varrho\Big)\|\nabla u(t)\|_2^2 \\
& +\varepsilon_1\alpha'(t)\int_\Omega uu_t dx+\varepsilon_2\alpha'(t)\int_\Omega u_t\int_0^t h(t-\varrho)(u(t)-u(\varrho))d\varrho dx \\
& +\varepsilon_3\alpha'(t)\int_0^1\int_{\tau_1}^{\tau_2}se^{-\rho s}|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m ds d\rho. 
\end{aligned} \tag{7.2.23}
$$

By using the fact that $e^{-\rho s}<1$, Young's and Sobolev-Poincare inequalities, we find

$$
\begin{aligned}
& \alpha'(t)\int_\Omega uu_t dx+\alpha'(t)\int_\Omega u_t\int_0^t h(t-\varrho)(u(t)-u(\varrho))d\varrho dx \\
& +\alpha'(t)\int_0^1\int_{\tau_1}^{\tau_2}se^{-\rho s}|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m ds d\rho \\
\leq \quad & -\alpha'(t)\frac{c_p^2}{2}\|\nabla u\|_2^2-\alpha'(t)\|u_t\|_2^2-\alpha'(t)\frac{c_p^2}{2}\Big(\int_0^t h(\varrho)d\varrho\Big)h\circ\nabla u)(t) \\
& -\alpha'(t)\int_0^1\int_{\tau_1}^{\tau_2}s|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m ds d\rho.
\end{aligned}
$$

Hence

$$\begin{aligned}
\mathcal{G}'(t) \leq\ & \alpha(t)\Big(\varepsilon_1-\varepsilon_2(h_0-\delta)-\frac{\alpha'(t)}{\alpha(t)}\Big)\|u_t\|_2^2\\
&+\alpha(t)\Big(\varepsilon_2\delta(\zeta_0+2c(\zeta_0-l)^2\alpha(t))-\varepsilon_1(l-\varepsilon(c_1+c_2))\\
&\qquad -\frac{\alpha'(t)}{2\alpha(t)}(\int_0^t h(\varrho)d\varrho)-\varepsilon_1\frac{\alpha'(t)c_p^2}{2\alpha(t)}\Big)\|\nabla u\|_2^2\\
&+\alpha(t)\Big(\varepsilon_2\zeta_1\delta-\varepsilon_1\zeta_1\Big)\|\nabla u\|_2^4\\
&+\alpha(t)\Big(\varepsilon_2\delta\frac{\sigma E(0)}{l}-\frac{\sigma}{\alpha(0)}\Big)\Big(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\Big)^2\\
&+\alpha(t)\Big(\varepsilon_1\frac{\alpha(t)}{4}+\varepsilon_2(c(\delta)+(2\delta+\frac{1}{4\delta})(\zeta_0-l)\alpha(t))\\
&\qquad -\varepsilon_2\frac{\alpha'(t)c_p^2}{2\alpha(t)}(\int_0^t h(\varrho)d\varrho)\Big)(h\circ\nabla u)(t)\\
&+\alpha(t)\Big(\frac{1}{2}-\varepsilon_2\frac{h(0)c_p^2}{4\delta}\Big)(h'\circ\nabla u)(t)\\
&+\alpha(t)\Big(\varepsilon_1c(\varepsilon)+\varepsilon_2c(\delta)+\varepsilon_3\beta_1-\frac{\eta_0}{\alpha(0)}\Big)\|u_t\|_m^m\\
&+\alpha(t)\Big(\varepsilon_1c(\varepsilon)+\varepsilon_2c(\delta)-\eta_1\varepsilon_3\Big)\int_{\tau_1}^{\tau_2}|\beta_2(s)\|y(x,1,s,t)\|_m^m ds\\
&+\alpha(t)\varepsilon_3\Big(-\eta_1-\frac{\alpha'(t)}{\alpha(t)}\Big)\int_0^1\int_{\tau_1}^{\tau_2}s|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m ds d\rho. \qquad (7.2.24)
\end{aligned}$$

Choosing $\delta,\varepsilon$ so small that

$$h_0-\delta>\frac{1}{2}h_0,\ \ \frac{\delta}{(l-\varepsilon(c_1+c_2))}(\zeta_0+2(1-l)^2)<\frac{1}{4}h_0.$$

For any fixed $\delta>0$, we select $\varepsilon_1,\varepsilon_2$ so small satisfying

$$\frac{h_0}{4}\varepsilon_2<\varepsilon_1<\frac{h_0}{2}\varepsilon_2,$$

and

$$\begin{aligned}
&\varepsilon_2(h_0-\delta)-\varepsilon_1>0,\\
&\varepsilon_1(l-\varepsilon(c_1+c_2)-\varepsilon_2\delta(\zeta_0+2c(\zeta_0-l)^2\alpha(t))>0.
\end{aligned}$$

Then, we select $\varepsilon_1,\varepsilon_2$ and $\varepsilon_3$ so small that (7.2.19) and (7.2.24) to remain valid, and further,

$$\begin{aligned}
&\zeta_1(\varepsilon_1-\varepsilon_2\delta)>0,\ \ \frac{\sigma}{\alpha(0)}-\varepsilon_2\delta\frac{\sigma E(0)}{l}>0,\ \ \frac{1}{2}-\varepsilon_2\frac{h(0)c_p^2}{4\delta}>0,\\
&\frac{\eta_0}{\alpha(0)}-\varepsilon_1c(\varepsilon)-\varepsilon_2c(\delta)-\varepsilon_3\beta_1>0,\ \ \eta_1\varepsilon_3-\varepsilon_1c(\varepsilon)-\varepsilon_2c(\delta)>0,
\end{aligned}$$

where $\eta_0=\beta_1-\int_{\tau_1}^{\tau_2}|\beta_2(s)|ds>0$.

Therefore, (7.2.24) becomes, for positive constants $d_1, d_2, d_3, d_6$

$$\begin{aligned}
\mathcal{G}'(t) &\leq -\alpha(t)\Big(d_1+\frac{\alpha'(t)}{\alpha(t)}\Big)\|u_t\|_2^2-\alpha(t)d_3\|\nabla u\|_2^4 \\
&\quad -\alpha(t)\Big(d_2+\frac{\alpha'(t)}{2\alpha(t)}(\int_0^t h(\varrho)d\varrho)+\varepsilon_1\frac{\alpha'(t)c_p^2}{2\alpha(t)}\Big)\|\nabla u\|_2^2 \\
&\quad +\alpha(t)\Big(d_6-\varepsilon_2\frac{h_0\alpha'(t)c_p^2}{2\alpha(t)}\Big)(h\circ\nabla u)(t) \\
&\quad -\alpha(t)\varepsilon_3\Big(\eta_1-\frac{\alpha'(t)}{\alpha(t)}\Big)\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m ds d\rho. \qquad (7.2.25)
\end{aligned}$$

According (7.1.3), $\lim_{t\to\infty}\frac{\alpha'(t)}{\alpha(t)}=0$, we can choose $t_1>t_0$ so that (7.2.25) can be written as

$$\begin{aligned}
\mathcal{G}'(t) &\leq -\alpha(t)\Big(d_1\|u_t\|_2^2+d_3\|\nabla u\|_2^4+d_2\|\nabla u\|_2^2-d_6(h\circ\nabla u)(t) \\
&\quad +d_4\int_0^1\int_{\tau_1}^{\tau_2}s|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m ds d\rho\Big) \\
&\leq -\alpha(t)d_5E(t)+\alpha(t)d_6(h\circ\nabla u)(t), \quad \forall t\geq t_1. \qquad (7.2.26)
\end{aligned}$$

□

**Theorem 7.2.** *Suppose that (7.1.2)-(7.1.4) are satisfied, for any $(u_0,u_1,f_0)$ satisfy $E(0)>0$. Then, the energy $E(t)$ of (7.1.6) decays to zero exponentially. That is, $\exists\lambda_1,\lambda_2>0$ such that*

$$E(t)\leq\lambda_1 e^{-\lambda_2\int_{t_1}^t\alpha(\varrho)\vartheta(\varrho)d\varrho}, \quad \forall t\geq t_1. \qquad (7.2.27)$$

*Proof.* Multiplying (7.2.22) by $\vartheta(t)$, using (7.1.2) and (7.1.8), we find

$$\begin{aligned}
\vartheta(t)\mathcal{G}'(t) &\leq -d_5\vartheta(t)\alpha(t)E(t)+d_6\alpha(t)\vartheta(t)(h\circ\nabla u)(t) \\
&\leq -d_5\vartheta(t)\alpha(t)E(t)-d_6\alpha(t)(h'\circ\nabla u)(t) \\
&\leq -d_5\vartheta(t)\alpha(t)E(t)-d_6\Big\{2E'(t)-\alpha'(t)\Big(\int_0^t h(\varrho)d\varrho\Big)\|\nabla u(t)\|_2^2\Big\}. \qquad (7.2.28)
\end{aligned}$$

Since $\vartheta(t)$ is non-increasing function, we have

$$\frac{d}{dt}\Big(\vartheta(t)\mathcal{G}(t)+2d_6E(t)\Big)\leq -d_5\vartheta(t)\alpha(t)E(t)-d_6\alpha'(t)\Big(\int_0^t h(\varrho)d\varrho\Big)\|\nabla u(t)\|_2^2. \qquad (7.2.29)$$

From (7.1.7) and (7.1.3) that $l\|\nabla u(t)\|_2^2\leq E(t)$, we find

$$\begin{aligned}
\frac{d}{dt}\Big(\vartheta(t)\mathcal{G}(t)+2d_6E(t)\Big) &\leq -d_5\alpha(t)\vartheta(t)E(t)-d_6\alpha'(t)\Big(\int_0^t h(\varrho)d\varrho\Big)\|\nabla u(t)\|_2^2 \\
&\leq -d_5\alpha(t)\vartheta(t)E(t)-\frac{2d_6\alpha'(t)}{l}E(t) \\
&\leq -\alpha(t)\vartheta(t)\Big(d_5+\frac{2d_6l_0\alpha'(t)}{l\vartheta(t)\alpha(t)}\Big)E(t). \qquad (7.2.30)
\end{aligned}$$

Since $\lim_{t\to\infty}\frac{\alpha'(t)}{\vartheta(t)\alpha(t)}=0$, we can choose $t_1\geq t_0$ such that $d_5+\frac{2d_6l_0\alpha'(t)}{l\alpha(t)\vartheta(t)}>0$ for $t\geq t_1$.

Finally, let

$$\mathcal{K}(t):=\vartheta(t)\mathcal{G}(t)+2d_6E(t)\sim E(t). \qquad (7.2.31)$$

Hence, for some $\lambda_2 > 0$, we obtain

$$\mathcal{K}'(t) \leq -\lambda_2 \vartheta(t)\alpha(t)\mathcal{K}(t), \ \ \forall t \geq t_1. \tag{7.2.32}$$

By integrating (7.2.32) over $(t_1, t)$ yields

$$\mathcal{K}(t) \leq \mathcal{K}(t_1) e^{-\lambda_2 \int_{t_1}^{t} \alpha(\varrho)\vartheta(\varrho)d\varrho}, \ \ \forall t \geq t_1. \tag{7.2.33}$$

Hence, (7.2.27) is established by virtue of (7.2.31) and (7.2.33). The proof is complete. □

Chapter 8

# Exponential growth of solutions with $L_p$-norm of a nonlinear viscoelastic wave equation with strong damping, source and distributed delay termes

## 8.1 introduction

In this chapter, we are concerned with a problem for a nonlinear viscoelastic wave equation with strong damping , source and distributed delay terms. We will show the exponential growth of solutions with $L_p$-norm.

We considered the following problem

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \int_0^t g(t-s)\Delta u(s)ds \\ +\mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(s)| u_t(x,t-s)ds = b|u|^{p-2}.u, \quad x \in \Omega, t > 0, \\ u(x,t) = 0, \ x \in \partial\Omega, \\ u_t(x,-t) = f_0(x,t), \ \ (x,t) \in \Omega \times (0,\tau_2), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega, \end{cases} \tag{8.1.1}$$

where $\omega, b, \mu_1$ are positive constants, $p > 2$ and $\tau_1, \tau_2$ are the time delay with $0 \leq \tau_1 < \tau_2$, and $\mu_2$ is an $L^\infty$ function, and $g$ is a differentiable function.

In this work, we investigated the problem (8.1.1), in which all the damping mechanism have been considered in the same time ( i.e. $w > 0$; $g \neq 0$; and $\mu_1 > 0, \mu_2 \in L^\infty$), these assumptions make our problem different form those studied in the literature, specially the exponential growth of solutions.

We will prove that if the initial energy $E(0)$ of our solutions is negative ( this means that our initial data are large enough), then our local solutions in bounded and

$$\|u_t\|_2^2 + \|\nabla u\|_2^2 \to \infty \tag{8.1.2}$$

as $t$ tends to $+\infty$: In fact it will be proved that the $L_p$-norm of the solution grows as an exponential function. An essential tool of the proof is an idea used in [164], which based on an auxiliary function (which is a small perturbation of the total energy).

Our aim in the present work is to extend the existing exponential growth results to our strong damping for a viscoelastic problem with distributed delay, we prove the exponential growth result under the following suitable assumptions.

(**A1**) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable and decreasing function so that

$$g(t) \geq 0 \quad , \quad 1 - \int_0^\infty g(s)\,ds = l > 0. \tag{8.1.3}$$

(**A2**) There exists a constant $\xi > 0$ such that

$$g'(t) \leq -\xi g(t) \quad , \quad t \geq 0. \tag{8.1.4}$$

(**A3**) $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is an $L^\infty$ function so that

$$(\frac{2\delta - 1}{2}) \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \leq \mu_1 \quad , \quad \delta > \frac{1}{2}. \tag{8.1.5}$$

## 8.2 Main results

In this section, we prove the Exponential Growth result of solution of problem (8.1.1). First, as in [133], we introduce the new varible

$$z(x, \rho, s, t) = u_t(x, t - s\rho),$$

then we obtain

$$\begin{cases} sz_t(x, \rho, s, t) + z_\rho(x, \rho, s, t) = 0 \\ z(x, 0, s, t) = u_t(x, t). \end{cases} \tag{8.2.1}$$

Let us denote by

$$gou = \int_\Omega \int_0^t g(t-s)|u(t) - u(s)|^2 ds. \tag{8.2.2}$$

Therefore, problem (8.1.1) takes the form:

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \int_0^t g(t-s)\Delta u(s) ds \\ +\mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x, 1, s, t) ds = b|u|^{p-2}.u, \qquad x \in \Omega, t > 0, \\ sz_t(x, \rho, s, t) + z_\rho(x, \rho, s, t) = 0, \end{cases} \tag{8.2.3}$$

with initial and boundary conditions

$$\begin{cases} u(x,t) = 0, \quad x \in \partial\Omega, \\ z(x,\rho,s,0) = f_0(x,s\rho), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \end{cases} \tag{8.2.4}$$

where

$$(x,\rho,s,t) \in \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty).$$

We first state a local existence theorem that can be established by combining arguments of Georgiev and Todorova [64].

**Theorem 8.1.** *Assume (8.1.3),(8.1.4), and (8.1.5) holds. Let*

$$\begin{cases} 2 < p < \dfrac{2n-2}{n-2}, \quad n \geq 3; \\ p \geq 2, \quad n = 1,2 \end{cases} \tag{8.2.5}$$

*Then for any initial data*

$$(u_0,u_1,f_0) \in \mathcal{H} \ / \ \mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega \times (0,1) \times (\tau_1,\tau_2)),$$

*with compact support, problem (8.2.4) has a unique solution*

$$u \in C([0,T];\mathcal{H}),$$

*for some $T > 0$.*

In the next theorem we give the global existence result, its proof based on the potential well depth method in which the concept of so-called stable set appears, where we show that if we restrict our initial data in the stable set, then our local solution obtained is global in time, that is to say, the norm

$$\|u_t\|_2 + \|\nabla u\|_2 \tag{8.2.6}$$

in the energy space $L^2(\Omega) \times H_0^1(\Omega)$ of our solution is bounded by a constant independent of the time $t$. We will make use of arguments in [155].

**Theorem 8.2.** *Suppose that (8.1.3),(8.1.4),(8.1.5), and (8.2.5) holds. If $u_0 \in W$, $u_1 \in H_0^1(\Omega)$ and*

$$\frac{bC_*^p}{l}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{8.2.7}$$

*where $C_*$ is the best Poincare's constant. Then the local solution $u(t,x)$ is global in time.*

We introduce the energy functional

**Lemma 8.1.** *Assume (8.1.3),(8.1.4),(8.1.5), and (8.2.5) hold, let $u(t)$ be a solution of (8.2.3), then $E(t)$ is non-increasing, that is*

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_2^2+\frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla u\|_2^2+\frac{1}{2}(go\nabla u) \\ &+\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx-\frac{b}{p}\|u\|_p^p, \end{aligned} \tag{8.2.8}$$

*satisfies*

$$E(t)\leq -c_1(\|u_t\|_2^2+\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx). \tag{8.2.9}$$

*Proof.* By multiplying the equation $(8.2.3)_1$ by $u_t$ and integrating over $\Omega$, we get

$$\begin{aligned} &\frac{d}{dt}\{\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla u\|_2^2+\frac{1}{2}(go\nabla u)-\frac{b}{p}\|u\|_p^p\} \\ =& -\mu_1\|u_t\|_2^2-\int_\Omega u_t\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,s,t)dsdx \\ &+\frac{1}{2}(g'o\nabla u)-\frac{1}{2}g(t)\|\nabla u\|_2^2-\omega\|\nabla u_t\|_2^2, \end{aligned} \tag{8.2.10}$$

and, we have

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx &= -\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}2|\mu_2(s)|zz_\rho dsd\rho dx \\ &= +\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,0,s,t)dsdx \\ &\quad -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx \\ &= \frac{1}{2}(\int_{\tau_1}^{\tau_2}|\mu_2(s)ds)\|u_t\|_2^2 \\ &\quad -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx, \end{aligned} \tag{8.2.11}$$

then, we get

$$\begin{aligned} \frac{d}{dt}E(t) &= -\mu_1\|u_t\|_2^2-\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|u_t z(x,1,s,t)dsdx+\frac{1}{2}(g'o\nabla u) \\ &-\frac{1}{2}g(t)\|\nabla u\|_2^2-\omega\|\nabla u_t\|_2^2+\frac{1}{2}(\int_{\tau_1}^{\tau_2}|\mu_2(s)ds)\|u_t\|_2^2 \\ &-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx. \end{aligned} \tag{8.2.12}$$

By (8.2.10) and (8.2.11), we get (8.2.8).

And by using Young's inequality, (8.1.3),(8.1.4) and (8.1.5) in (8.2.12), we obtain (8.2.9). □

Now we are ready to state and prove our main result. For this purpose, we define

$$\begin{aligned} H(t)=-E(t) &= \frac{b}{p}\|u\|_p^p-\frac{1}{2}\|u_t\|_2^2-\frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla u\|_2^2 \\ &-\frac{1}{2}(go\nabla u)-\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx. \end{aligned} \tag{8.2.13}$$

**Theorem 8.3.** *Suppose that (8.1.3)-(8.1.5), and (8.2.5). Assume further that $E(0) < 0$ holds. Then the unique local solution of problem (8.2.3) grows exponentially.*

*Proof.* From (8.2.8), we have

$$E(t) \leq E(0) \leq 0. \tag{8.2.14}$$

Hence

$$\begin{aligned} H'(t) = -E'(t) &\geq c_1(\|u_t\|_2^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds dx) \\ &\geq c_1 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds dx \geq 0, \end{aligned} \tag{8.2.15}$$

and

$$0 \leq H(0) \leq H(t) \leq \frac{b}{p}\|u\|_p^p. \tag{8.2.16}$$

We set

$$\mathcal{K}(t) = H(t) + \varepsilon \int_\Omega u u_t dx + \frac{\varepsilon \mu_1}{2} \int_\Omega u^2 dx + \frac{\varepsilon \omega}{2} \int_\Omega (\nabla u)^2 dx, \tag{8.2.17}$$

where $\varepsilon > 0$ to be specified later.

By multiplying $(8.2.3)_1$ by $u$ and taking a derivative of (8.2.17), we obtain

$$\begin{aligned} \mathcal{K}'(t) &= H'(t) + \varepsilon \|u_t\|_2^2 + \varepsilon \int_\Omega \nabla u \int_0^t g(t-s) \nabla u(s) ds dx \\ &\quad - \varepsilon \|\nabla u\|_2^2 + \varepsilon b \int_\Omega |u|^p dx - \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| u z(x,1,s,t) ds dx. \end{aligned} \tag{8.2.18}$$

Using

$$\begin{aligned} \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| u z(x,1,s,t) ds dx &\leq \varepsilon \{ \delta_1 ( \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds) \|u\|_2^2 \\ &\quad + \frac{1}{4\delta_1} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds dx \}, \end{aligned} \tag{8.2.19}$$

and

$$\begin{aligned} \varepsilon \int_0^t g(t-s) ds \int_\Omega \nabla u . \nabla u(s) dx ds &= \varepsilon \int_0^t g(t-s) ds \int_\Omega \nabla u . (\nabla u(s) - \nabla u(t)) dx ds \\ &\quad + \varepsilon \int_0^t g(s) ds \|\nabla u\|_2^2 \\ &\geq \frac{\varepsilon}{2} \int_0^t g(s) ds \|\nabla u\|_2^2 - \frac{\varepsilon}{2}(g o \nabla u), \end{aligned} \tag{8.2.20}$$

we obtain, from (8.2.18),

$$\begin{aligned} \mathcal{K}'(t) &\geq H'(t) + \varepsilon \|u_t\|_2^2 - \varepsilon (1 - \frac{1}{2} \int_0^t g(s) ds) \|\nabla u\|_2^2 + \varepsilon b \|u\|_p^p + \frac{\varepsilon}{2}(g o \nabla u) \\ &\quad - \varepsilon \delta_1 ( \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds) \|u\|_2^2 - \frac{\varepsilon}{4\delta_1} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds dx. \end{aligned} \tag{8.2.21}$$

Therefore, using (8.2.15) and by setting $\delta_1$ so that, $\frac{1}{4\delta_1 c_1} = \kappa$, substituting in (8.2.21), we get

$$\begin{aligned} \mathcal{K}'(t) \geq\ & [1-\varepsilon\kappa]H'(t) + \varepsilon\|u_t\|_2^2 \\ & -\varepsilon[(1-\frac{1}{2}\int_0^t g(s)ds)]\|\nabla u\|_2^2 + \varepsilon b\|u\|_p^p \\ & -\frac{\varepsilon}{4c_1\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds)\|u\|_2^2 + \frac{\varepsilon}{2}(go\nabla u). \end{aligned} \tag{8.2.22}$$

For $0 < a < 1$, from (8.2.13)

$$\begin{aligned} \varepsilon b\|u\|_p^p =\ & \varepsilon p(1-a)H(t) + \frac{\varepsilon p(1-a)}{2}\|u_t\|_2^2 + \varepsilon ba\|u\|_p^p \\ & + \frac{\varepsilon p(1-a)}{2}(1-\int_0^t g(s)ds)\|\nabla u\|_2^2 + \frac{\varepsilon}{2}p(1-a)(go\nabla u) \\ & + \frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx. \end{aligned} \tag{8.2.23}$$

Substituting in (8.2.22), we get

$$\begin{aligned} \mathcal{K}'(t) \geq\ & [1-\varepsilon\kappa]H'(t) + \varepsilon[\frac{p(1-a)}{2}+1]\|u_t\|_2^2 \\ & + \varepsilon[(\frac{p(1-a)}{2})(1-\int_0^t g(s)ds) - (1-\frac{1}{2}\int_0^t g(s)ds)]\|\nabla u\|_2^2 \\ & -\frac{\varepsilon}{4c_1\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds)\|u\|_2^2 + \varepsilon p(1-a)H(t) + \varepsilon ba\|u\|_p^p \\ & + \frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx \\ & + \frac{\varepsilon}{2}(p(1-a)+1)(go\nabla u). \end{aligned} \tag{8.2.24}$$

Using Poincare's inequality, we obtain

$$\begin{aligned} \mathcal{K}'(t) \geq\ & [1-\varepsilon\kappa]H'(t) + \varepsilon[\frac{p(1-a)}{2}+1]\|u_t\|_2^2 + \frac{\varepsilon}{2}(p(1-a)+1)(go\nabla u) \\ & + \varepsilon\{(\frac{p(1-a)}{2}-1) - \int_0^t g(s)ds(\frac{p(1-a)-1}{2}) \\ & -\frac{c}{4c_1\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds)\}\|\nabla u\|_2^2 + \varepsilon ab\|u\|_p^p + \varepsilon p(1-a)H(t) \\ & + \frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx. \end{aligned} \tag{8.2.25}$$

At this point, we choose $a > 0$ so small that

$$\alpha_1 = \frac{p(1-a)}{2} - 1 > 0,$$

and assume

$$\int_0^\infty g(s)ds < \frac{\frac{p(1-a)}{2}-1}{(\frac{p(1-a)}{2}-\frac{1}{2})} = \frac{2\alpha_1}{2\alpha_1+1}, \tag{8.2.26}$$

then we choose $\kappa$ so large that

$$\alpha_2 = (\frac{p(1-a)}{2}-1) - \int_0^t g(s)ds(\frac{p(1-a)-1}{2}) - \frac{c}{4c_1\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds) > 0.$$

Once $\kappa$ and $a$ are fixed, we pick $\varepsilon$ so small enough so that

$$\alpha_4 = 1 - \varepsilon\kappa > 0,$$

and

$$\mathcal{K}(t) \leq \frac{b}{p}\|u\|_p^p. \tag{8.2.27}$$

Thus, for some $\beta > 0$ , estimate (8.2.25) becomes

$$\begin{aligned} \mathcal{K}'(t) &\geq \beta\{H(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + (go\nabla u) + \|u\|_p^p \\ &\quad + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx\}, \end{aligned} \tag{8.2.28}$$

and

$$\mathcal{K}(t) \geq \mathcal{K}(0) > 0, \quad t > 0. \tag{8.2.29}$$

Next, using Young's and Poincare's inequalities, from (8.2.17) we have

$$\begin{aligned} \mathcal{K}(t) &= (H + \varepsilon\int_\Omega uu_t dx + \frac{\varepsilon\mu_1}{2}\int_\Omega u^2 dx + \frac{\varepsilon\omega}{2}\int_\Omega \nabla u^2 dx) \\ &\leq c[H(t) + |\int_\Omega uu_t dx| + \|u\|_2^2 + \|\nabla u\|_2^2] \\ &\leq c[H(t) + \|\nabla u\|_2^2 + \|u_t\|_2^2], \end{aligned} \tag{8.2.30}$$

for some $c > 0$: Since, $H(t) > 0$, we have from (8.2.3)

$$\begin{aligned} &-\frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u\|_2^2 - \frac{1}{2}(go\nabla u) \\ &-\frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx + \frac{b}{p}\|u\|_p^p > 0, \end{aligned} \tag{8.2.31}$$

then

$$\begin{aligned} \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u\|_2^2 &< \frac{b}{p}\|u\|_p^p < \frac{b}{p}\|u\|_p^p + (go\nabla u) \\ &\quad + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx. \end{aligned} \tag{8.2.32}$$

In the other hand, using (8.1.3), to get

$$\begin{aligned} \frac{1}{2}(1-l)\|\nabla u\|_2^2 &< \frac{b}{p}\|u\|_p^p < \frac{b}{p}\|u\|_p^p + (go\nabla u) \\ &\quad ++ \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx. \end{aligned} \tag{8.2.33}$$

Consequently,

$$\begin{aligned} \|\nabla u\|_2^2 &< \frac{2b}{p}\|u\|_p^p + 2(go\nabla u) + l\|\nabla u\|_2^2 \\ &\quad + 2\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx. \end{aligned} \tag{8.2.34}$$

Inserting (8.2.34) into (8.2.30), to see that there exists a positive constant $k_1$ such that

$$\begin{aligned}\mathcal{K}(t) \quad \leq \quad & k_1[H(t)+\|\nabla u\|_2^2+\|u_t\|_2^2+\frac{b}{p}\|u\|_p^p+(go\nabla u)(t) \\ & +\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx], \forall t>0. \end{aligned} \tag{8.2.35}$$

From inequalities (8.2.28) and (8.2.35) we obtain the differential inequality

$$\mathcal{K}'(t) \geq \lambda\mathcal{K}(t), \tag{8.2.36}$$

where $\lambda > 0$, depending only on $\beta$ and $k_1$.

a simple integration of (8.2.36), we obtain

$$\mathcal{K}(t) \geq \mathcal{K}(0)e^{(\lambda t)}, \forall t>0. \tag{8.2.37}$$

From (8.2.17) and (8.2.27), we have

$$\mathcal{K}(t) \leq \frac{b}{p}\|u\|_p^p. \tag{8.2.38}$$

By (8.2.37) and (8.2.38), we have

$$\|u\|_p^p \geq Ce^{(\lambda t)}, \forall t>0.$$

Therefore, we conclude that the solution in the $L_p$-norm growths exponentially. This completes the proof. □

# Conclusion

In this thesis, we studied various problems of porous-elastic type, and by adding different dessipations and dampings, we prove the existence of solutions, and the results of stability under suitable conditions. We also expanded the study to other types of hyperbolic systems, by using various methods we have improved many works.
The same thing with wave equations that we allocated for the third part.

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