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Faculty of Economic, Commercial and Management Sciences

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lectures in :

Financial Mathematics

Addressed to second-year bachelor's degree students in all majors.

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Introduction
to Financial Mathematics

Welcome to this series of lectures on Financial Mathematics, specifically tailored for second-year students in Economics, Commercial Sciences, and Management Sciences. This module is designed to equip you with the essential mathematical tools and concepts that form the bedrock of financial analysis and decision-making in the economic and business world.

Financial mathematics is more than just numbers; it's the analytical framework that underpins virtually every aspect of finance, from personal investments to corporate strategic planning. As aspiring economists, business professionals, and managers, understanding these quantitative principles isn't just beneficial—it's crucial for navigating the complexities of modern financial markets and making informed choices.

Throughout these sessions, we'll delve into the core concepts that allow us to evaluate financial instruments and analyze various financial scenarios. We'll explore how the passage of time impacts the value of money, a fundamental principle that influences everything from calculating loan repayments to assessing the profitability of long-term projects. We'll also examine the mechanics of different financial products and how their values are determined through rigorous mathematical models.

By the end of this module, you will have developed the analytical skills necessary to interpret financial data, understand the behavior of financial assets, and apply quantitative reasoning to real-world economic and business challenges. This knowledge will not only enhance your academic performance but also provide a strong practical foundation for your future careers in diverse fields such as banking, finance, accounting, and business administration.

CHAPTER 01: Short-term financial transactions

I.1. Introduction:

Short-term financial transactions refer to a set of financial activities that are focused on the management of funds and assets over a relatively brief period, typically ranging from a few days to a year. These operations are characterized by their temporary nature and their aim to address immediate financial needs and objectives. They involve activities such as cash management, liquidity management, working capital management, short-term borrowing and lending, and the execution of quick financial transactions. Short-term financial operations play a crucial role in maintaining the financial stability and liquidity of businesses, allowing them to meet their short-term obligations, seize investment opportunities, and effectively manage their day-to-day financial activities. In this discussion, we will explore the key aspects of short-term financial operations, their importance, and the strategies employed in managing them effectively."

Feel free to modify and adapt this introduction to suit your specific needs and audience.

I.2. Simple Interest:

I.2.1. Definition of Simple Interest:

Simple interest is the fixed income obtained by the owner of a sum of money borrowed or deposited with a bank during a certain period of time, usually not exceeding a year. The interest is calculated based on the principal amount and does not accumulate or compound over time. It remains constant in each period, provided that the maturity date is updated, and as long as the deposited amount remains unchanged without any increase or decrease.

Simple interest is commonly used in short-term transactions, interbank lending, financial intermediation, current accounts, deposit accounts, discounting commercial papers, and consumer loans offered by financial institutions.

I.2.2. The specific elements of interest are:

I.2.2.1. Current Value:

Also known as the initial amount employed, deposited, invested, or lent, represented by the symbol "CV", the original value has an inverse relationship with the simple interest "I". As the employed amount increases, the simple interest increases by the same proportion, while keeping the rate and duration constant.

I.2.2.2. Time Period:

It refers to the duration of employing the original amount, which can be short-term, such as yearly, monthly, or daily. It represents the period between the deposit day and the withdrawal day, excluding the deposit day (the first day) and including the withdrawal day (the last day).

The relationship between the time period "n" and the simple interest "I" is an Positive relationship. As the employment duration increases, the obtained simple interest increases at the same rate, given a constant rate.

I.2.2.2. Interest Rate:

It represents the rate at which the original amount is employed, representing the return on the employed capital within a specific time unit, often a year, 12 months, or 360 days.

The relationship between the simple interest rate and the interest amount is an inverse relationship. As the interest rate increases, the obtained amount of simple interest increases.

I.2.3. Methods of calculating simple interest:

There are three methods for calculating simple interest:

I.2.3.1. The general formula

Also known as the general formula for calculating simple interest, it is given by the following relationship:

$$\boxed{I = CV \times r \times n}$$

Note:

The above formula cannot be directly applied as is unless there is consistency between the time unit associated with the interest rate and the other time unit associated with the time period.

I.2.3.2. Proportional rates:

We say that rates with different time units are proportional if and only if they produce the same amount of simple interest for the same principal during the same time period.

Therefore, the following condition must be satisfied:

$$\boxed{I = CV \times \boldsymbol{r_a} \times \boldsymbol{n_a} = \boldsymbol{CV} \times \boldsymbol{r_s} \times \frac{\boldsymbol{n_a}}{\boldsymbol{1/2}} = \boldsymbol{CV} \times \boldsymbol{r_t} \times \frac{\boldsymbol{n_a}}{\boldsymbol{1/4}} = \boldsymbol{CV} \times \boldsymbol{r_m} \times \frac{\boldsymbol{n_a}}{\boldsymbol{1/12}}}$$

I.2.3.3.Tiger and denominator method:

This method is used when we have multiple amounts that have been employed for different periods of time but at the same rate of employment. It is applied using the following relationship:

$$\boxed{I = \frac{N}{D}}$$

in which:

N : Tiger

D : denominator

The tiger is calculated, according to the following relationship:

$$N = \sum_{i=1}^{K} CV_i \times n_{ij}$$

in which:

K : The number of amounts employed

As for the denominator, it is calculated as follows:

$$D = \frac{36000}{t_a\%}$$

I.2.4. Future Value

If we add the total interest earned (I) to the initial amount saved or invested

(CV), we obtain the maturity amount, or future value (FV), to which the curren value will grow.

$$FV = CV + I$$
$$FV = CV + CV \times r \times n$$
$$FV = CV(1 + rn)$$

Example:

Using the general formula, proportional rates, and the method of multiplication and division, calculate the total interest generated by investing the following amounts at a semi-annual interest rate of 6%.

Original Value	150000	240000	185000	5440000
Time Period	2 years	8 months	2 semesters	5 semesters

The solution:

- The first method: the general formula

$$I_1 = CV_1 \times r_s \times \frac{n_a}{1/2} = 150000 \times 0.06 \times \frac{2}{1/2} =$$

$$I_2 = a_2 \times t_s \times \frac{n_m}{6} = 150000 \times 0.06 \times \frac{8}{6} =$$

$$I_3 = a_3 \times t_s \times n_s = 150000 \times 0.06 \times 2 =$$

$$I_4 = a_4 \times t_s \times \frac{n_t}{2} = 150000 \times 0.06 \times \frac{5}{2} =$$

$$I = I_1 + I_2 + I_3 + I_4$$

$$I = \quad + \quad + \quad +$$

- Second method: Proportional rates.

$$t_a = 2t_s = 2 \times 0.06 = 0.12$$

$$I_1 = a_1 \times t_a \times n_a = 150000 \times 0.12 \times 2 =$$

$$t_m = \frac{t_s}{6} = \frac{0.06}{6} = 0.01 =$$

$$I_2 = a_2 \times t_m \times n_m = 150000 \times 0.01 \times \frac{8}{6} =$$

$$I_3 = a_3 \times t_s \times n_s = 150000 \times 0.06 \times 2 =$$

$$t_t = \frac{t_s}{2} = \frac{0.06}{2} = 0.03$$

$$I_4 = a_4 \times t_s \times \frac{n_t}{2} = 150000 \times 0.03 \times \frac{5}{2} =$$

$$I = I_1 + I_2 + I_3 + I_4$$

$$I = \quad + \quad + \quad +$$

- Third method: Multiplication and division.

$$t_a = 2 \times t_s = 2 \times 6\% = 12\%$$

$$D = \frac{36000}{t_a\%} = \frac{36000}{12\%} = 2000$$

$$N = \sum_{i=1}^{K} a_i \times n_{ij}$$

$$N = 150000 \times 720j + 240000 \times 240j + 185000 \times 360j + 5440000 \times 450$$

$$N =$$

$$I = \frac{N}{D} = \frac{}{} =$$

I.2. Discount:

I.2.1. Definition of Discount:

Discount is a process that allows the holder of a commercial paper to convert it into liquidity before its maturity date. It is also an amount that represents the value deducted by the banker based on a specific interest rate and the period between the discount date and the maturity date of the paper.

The amount mentioned in the paper is known as the future value: FV. The current value obtained is called the present value: CV. The interest earned by the banker is referred to as the discount.

I.2.2. Types of Discount:

In fact, there are two types of discount:

- Trade Discount
- Real Discount

I.2.2.1. Trade Discount:

Calculated based on the nominal value mentioned on the paper.

$$\boxed{TD = \frac{FV \times n}{D}}$$

Where:

$TD:$ Trade Discount
$FV:$ The future value of the discounted commercial paper
D : The divisor calculated at an annual rate of discount
n :The period between the discount date and the due date

- **Current Value:**

The current value is the value remaining after subtracting the discount value from the futre value.

It can be calculated as follows:

$$\boxed{Current\ value\ =\ futre\ value\ -\ trade\ discount\ value}$$

I.2.2.2. Real discount:

in this case, the rate is applied to the current value, making the real discount more accurate than the trade discount. This is because the future value and the current value may differ. Therefore, the calculation of the real discount is as follows:

$$\boxed{RD = \frac{CV \times n}{D} \quad \dots\dots\dots\dots (1)}$$

Since the:

$$CV = FV - TD \implies FV = CV + \frac{CV \times n}{D} \Rightarrow FV = CV + \frac{CV \times n}{D} = \frac{CV \times D + CV \times n}{D}$$

$$FV = \frac{CV(D+n)}{D} \Rightarrow FV = \frac{CV \times D}{(D+n)} \ldots\ldots\ldots\ldots.(2)$$

Substituting Equation 2 into Equation 1, we get:

$$RD = \frac{\frac{FV \times D}{(D+n)} \times n}{D}$$

After simplification, we have the formula for the real discount as follows:

$$\boxed{RD = \frac{FV \times n}{(D+N)}}$$

I.2.3. The relationship between trade discount and real discount is as follows:

- **Comparing the two discounts**:

$$\boxed{TD > RD}$$

- **The difference between the two discounts:**

$$\boxed{\mathbf{TD - RD = \frac{\boldsymbol{FV \times n^2}}{\boldsymbol{D(D+n)}}}}$$

- **The ratio between the two discounts:**

$$\boxed{\frac{\textbf{TD}}{\textbf{RD}} = \frac{(\boldsymbol{D}+\boldsymbol{n})}{\boldsymbol{D}}}$$

- **The difference between the reciprocals of the two discounts:**

$$\boxed{\frac{\mathbf{1}}{\textbf{RD}} - \frac{\mathbf{1}}{\textbf{TD}} = \frac{\mathbf{1}}{\boldsymbol{FV}} \Rightarrow \boldsymbol{FV} = \frac{\textbf{TD} \times \textbf{RD}}{\textbf{TD} - \textbf{RD}}}$$

I.2.4. Discount Elements of Commercial Papers

In the event that the bank discounts the commercial paper before its due date, the bank deducts an amount from the future value of the debt (commercial paper). The deducted amount is determined based on the future value of the commercial paper and the discount period. The longer the discount period, the larger the deducted amount, and vice versa. The deducted amount is also determined by the discount rate, increasing with its increase and decreasing with its decrease. Moreover, the bank does not only deduct the discount amount from the future value of the commercial paper, but there are other elements that the bank deducts during the discounting process.

- Bank Commission

This is the fee charged by the bank for acting as an intermediary between the issuer of the commercial paper and the beneficiary, and for providing liquidity to the beneficiary. Often, this commission is calculated as a percentage or per thousand of the future value of the commercial paper. Therefore, the bank commission is not affected by the discount period or the discount rate.

- Collection Expenses:

These are the expenses charged by the bank for collecting the value of the commercial paper from the debtor without the intervention of the beneficiary. These expenses are calculated as a percentage or per thousand of the nominal value of the commercial paper. Usually, there is a minimum fixed amount required as collection expenses for each commercial paper. This element is also not affected by the discount period of the commercial paper.

- Endorsement Commission:

It is applied when the ownership of the commercial paper is transferred from the holder to the bank before its maturity date. The purpose of this commission is to cover potential rediscount with the central bank. This commission is tied to the time interval between the discount date and the maturity date because during this period, the risk remains as the bank has not yet recovered the principal of the Debt.

Based on the above:

$$\boxed{Agio = TD + C}$$

Where: C: Sum of deducted commissions

I.2.5. Commercial Paper Net Worth and Real Rate of Discount:

The net value of a commercial paper is the current value collected by the beneficiary of the discount, and is calculated as follows:

$$\boxed{NCV = FV - Agio}$$

As for the real discount rate, it is the rate that takes all the real costs, and we denote it with the symbol Rr, and it is calculated according to the following relationship:

$$\boxed{Rr = \frac{Agio \times 360}{NCV \times n}}$$

- **Example:**

A commercial paper with a current value of 70 000 M.U deducted 42 days before its maturity date under the following conditions :

Discount rate : 7.5 %

Discount commission : 1‰

Endorsement commission: 0.5%

Other fixed commissions 10 M.U

Required:

- Calculate trade discount
- Calculate real discount
- Calculate total burden

Solution:

- Calculation trade discount:

$$D = \frac{36000}{7.5} = 4800$$

$$TD = \frac{FV \times n}{D} = \frac{70000 \times 42}{4800} = 612.5 \text{ M.U}$$

- Calculation real discount:

$$RD = \frac{FV \times n}{(D + N)} = \frac{70000 \times 42}{(4800 + 42)} = 607.18 \text{ M.U}$$

- Calculation total burden:

$$Agio = TD + C$$

$$C = 70000 \times 0.001 + \frac{70000 \times 0.005 \times 42}{360} + 10 = 120.83$$

$$Agio = 612.5 + 120.83$$

$$Agio = 733.33$$

Unsolved exercises

Exercise 01:

- Calculate the simple interest accrued from investing a monetary amount in a bank amounting to 10000 M.U. at an interest rate of 5%, for the following durations: 28 days, then 5 months, then 3 years.
- Calculate the total amount resulting from investing another amount of 15000 M.U. at a simple interest rate of 3% during the following durations:

n: from March 10th to May 20th (same year).

n: from June 16th to August 31st (same year).

Exercise 02:

Complete the following table (Simple Interest):

Case	Current	Rate	Duration	Interest	Future value
1	240000	Semi-annually 8 %	....	....	259200
2	…….	3.75% Quarterly	....	12625	265125
3	480000	12 % year	9m	....	………….
4	320000	.... year	1t et 15j	....	336800
5	………	Semi-annually 5 %	2a	40000	………….

Exercise 03:

A person was employed in a bank for a period of one year at an annual interest rate of 15%. After the employment period, an additional amount of 720000 M.U was added to the accrued balance. Then, the person was employed in another bank for a duration of four months at a semiannual interest rate of 12%, resulting in an interest of 80000 M.U from the second bank.

Required: Calculate the initial principal employed in the first bank and the interest earned from the employment in it.

Exercise 04:

A person employed an amount of 150,000 M.U. at an annual interest rate of 12%. Calculate the interest earned from the employment in each case: n=3a, n=4s, n=3t, n=6m, n=9m et 1t et 15j, n=5m et 10j.

Exercise 05:

Employ a person with an amount of 800000 M.U. for a duration of 1 year and 6 months. Calculate the simple interest and the accrued balance in the following cases: $r_a = 6\%$, $r_s = 9\%$, $r_t = 6\%$.

Exercise 06:

If the total of three amounts is 48000 M.U. and they are proportionate to the numbers 3, 4, 5 respectively, we need to calculate the probability of each amount.

Then, we deposited these amounts in a bank for different durations at an interest rate of 5% to achieve a total interest of 7200 M.U. If the interest on the first amount is half the interest on the second amount and the interest on the third amount is equal to the sum of the interest on the first and second amounts.

The required calculations are:

- Calculate the interest for each amount.
- Calculate the duration of deposit for each amount.

Exercise 07:

If the difference between the trade discount and the true discount is 25 M.U. for a debt due after 10 months with an annual interest rate of 8%, we need to calculate the future value of the debt, the trade discount, and the true discount.

Exercise 08:

A person presented two commercial papers for discounting at a commercial bank with a total value of 2400 M.U. The first paper is due after 90 days, and the second paper is due after 45 days. The total commercial discount for both papers amounted to 22.5 N.

The requested information:

- Determine the nominal value of each paper.
- Calculate the commercial discount for each paper if the discount rate is known to be 4.5%.

Exercise 09:

On April 18, 2022, a trader discounted two commercial papers. The first paper has a future value of 80000 M.U and is due on April 28, 2022. The second paper has a future value of 120000 M.U and is due on August 3, 2022.

Given that the semi-annual interest rate is 3%, the discount commission is 1% of the future value of the paper, the endorsement commission is 2‰ of the future value of the paper, and the tax is 5 M.U for each discount operation.

calculate:

- ER_2 ; ER_1 ; EC_1 ; EC_2
- The commercial current value and the real current value.
- The agio value for each commercial paper (L'AGIO).

Exercise 09:

A debt is due after 9 months from now, knowing that 3.6 = ER - EC and 8% = ta. Calculate the future value of the debt .

Exercise 10:

Two commercial papers were discounted at the same rate. If the commercial discount for the first paper is 1625 dinars, and the real discount for the second commercial paper is 32000 dinars, with the discount period of the second paper being 150 days longer than the discount period of the first paper, and the differences between the discounts are given as follows:

$$EC_1 - ER_1 = 25$$

$$EC_2 - ER_2 = 2000$$

The requested:

Calculate the real discount for the first commercial paper and the commercial discount for the second commercial paper.

Calculate the nominal value of each commercial paper.

Exercise 11:

The nominal value of a debt is 4731000 M.U due after 3 months.

Calculate the applied semi-annual discount rate, knowing that

$EC - ER = 6412.50$

Chapter 02
Long-term financial transactions

II.1. Introduction:

Long-term financial transactions refer to financial activities and strategies that involve a significant time horizon, typically extending beyond one year. These operations are crucial for organizations and individuals looking to secure funding for substantial investments, such as acquiring fixed assets, expanding business transactions, or funding major projects.

Unlike short-term financial transactions that focus on managing day-to-day cash flows and immediate financial obligations, long-term financial operations revolve around planning, acquiring, and allocating financial resources over an extended period. These operations require careful analysis, strategic decision-making, and a thorough understanding of the long-term financial goals and objectives of the entity involved.

Long-term financial transactions involve various key aspects, including capital budgeting, long-term financing, and investment management. Capital budgeting entails evaluating potential investment opportunities and determining which projects align with the organization's long-term objectives and offer the most favorable returns. Long-term financing involves securing funds through various sources, such as issuing bonds, obtaining loans, or attracting equity investors, to support the organization's long-term growth and development. Investment management focuses on effectively allocating financial resources and optimizing the organization's investment portfolio to generate long-term returns while managing risks.

Furthermore, long-term financial operations require considering factors such as interest rates, market conditions, industry trends, and regulatory requirements. Rigorous financial analysis, forecasting, and risk assessment play a critical role in ensuring the success and sustainability of long-term financial transactions.

In summary, long-term financial transactions encompass a range of activities aimed at strategically managing financial resources and investments over an extended period. By carefully planning and executing these transactions, organizations and individuals can pursue their long-term goals, maximize financial returns, and navigate the complexities of the financial landscape.

II.2. Compound interest:

II.2.1. Definition of compound interest:

Compound interest is the return earned from a capital employed for a specific period of time at a predetermined rate. The interest is added to the principal at the end of each cycle, and the total amount is reinvested during the next cycle. Adding interest to the principal is called compounding. Compound interest can be calculated annually, semi-annually, monthly, or quarterly.

II.2.2. The general formula for calculating compound interest:

If an individual invests an amount CV at an annual compound interest rate of r for a certain number of years, the following can be inferred:

- **The future value at the end of the investment period:**

$$\boxed{FV = CV(1+r)^n}$$

- **Compound interest at the end of the investment period.**

$$\boxed{I = CV[(1+r)^n - 1]}$$

It can also be calculated as follows:

$$\boxed{I = CV \times r[1 + (1+r) + (1+r)^2 + \quad \ldots\ldots\ldots\ldots\ldots\ldots + (1+r)^{n-1}\ ]}$$

- **Annual compound interest**:

$$\boxed{I_n = [CV \times (1+r)^{n-1}] \times r}$$

It should be noted that all the previous mathematical formulas cannot be applied directly unless consistency is achieved between the time unit associated with the employment period with the time unit associated with the employment rate.

II.2.3. Calculating the components of compound interest:

Using the general formula for calculating the future value, the following can be inferred:

- **Calculation of the current value:**

$$\boxed{CV = \frac{FC}{(1+r)^n}}$$

- **Calculation of the compounding term:**

$$\boxed{n = \frac{\ln\left(\frac{FV}{CV}\right)}{\ln(1+r)}}$$

- **Calculation of the rate of compound interest:**

$$\boxed{r = \sqrt[n]{\frac{FV}{CV}} \; - 1}$$

Example:

Capital with a current value of 20000 M.U was deposited for a duration of 8 years at a rate of 9% .

The required :

calculations are as follows:

- Calculate of the future value afteru8 years.
- Calculate of the total interest generated during this period.
- Calculate of the interest for year 8.
- Calculate of the time required to double the current value of the capital.

Solution:

- Calculation of the future value afteru8 years.

$$FV = CV(1+r)^n = 20000(1.09)^8 = 39851.25$$

- Calculation of the total interest generated during this period.

$$I = CV[(1+r)^n - 1] = 20000[(1.09)^8 - 1] = 19851.25$$

- Calculation of the interest for year 8.

$$I_8 = [CV(1+r)^{n-1}] \times r = [20000(1.09)^7] \times 0.09 = 3290.47$$

- Calculation of the time required to double the current value of the capital.

$$FV = 2 \times 20000 = 40000$$

$$CV = 40000$$

$$n = \frac{\ln\left(\frac{FV}{CV}\right)}{\ln(1+r)} = \frac{\ln\left(\frac{40000}{20000}\right)}{\ln(1.09)} = 8.04$$

II.3. The CURRENT VALUE:

I.2.1. Definition of Current value:

Current value is the value of an amount before its date of maturity, meaning it represents the determination of a certain amount at the present time, knowing that it will be due or paid in the future.

The current value formula can be derived from the general formula for calculating the future value of compound interest. It allows us to calculate the current value using the following general formula:

$$CV = FV(1+r)^{-n}$$

I.2.1. Calculation of current value elements:

From the general formula for calculating the acquired sentence, the following can be concluded:

- **Calculation of the future value :**

$$FV = \frac{CV}{(1+t)^n}$$

- **Calculation of the compounding term :**

The term (n) can also be obtained in relation to all other compounding factors:

$$n = \frac{\ln\left(\frac{FV}{CV}\right)}{\ln(1+r)}$$

- **Calculation of the rate of compound interest:**

The compound interest rate can be found in terms of all other remaining factors

in the compounding formula: FV, CV, and *n*.

$$r = \sqrt[n]{\frac{FV}{CV}} - 1$$

Example:

An industrialist purchased a machine for 500 000 M.U, which is due for payment after 10 years, at a compound interest rate of 9% annually. He proposed three methods of payment to the debtor:

- The first method: Payment at the time of purchase.
- The second method: Payment 4 years before the specified payment date.
- The third method: Payment 3 years after the specified payment date.

The required:

1. Determine the amounts to be paid according to the three methods.
2. What is the value of the machine if it is purchased 2 years before the specified purchase date?

Solution:

1. Determining the payment amounts according to the three methods:

- Payment at the time of purchase:

$$CV = FV(1+r)^{-n} = 500000(1.09)^{-10} = 211205.40 \text{ M.U}$$

- Payment four years before the specified payment date:

$$CV = FV(1+r)^{-n} = 500000(1.09)^{-4} = 35412.60 \text{ M.U}$$

Or in another way

$$FV = CV(1+r)^n = 211205.40(1.09)^6 = 354212.60 \text{ M.U}$$

- Repayment after 3 years from the specified repayment date:

$$FV = CV(1+r)^n = 500000(1.09)^3 = 647514.5 \text{ M.U}$$

Or in another way

$$FV = CV(1+r)^n = 211205.40(1.09)^{13} = 647514.5 \text{ M.U}$$

2. The value of the machine shall be determined if it was purchased two years prior to the specified purchase date:

$$CV = FV(1+r)^{-n} = 211205.40(1.09)^{-2} = 1777767.023 \text{ M.U}$$

Or in another way

$$FV = CV(1+r)^n = 500000(1.09)^6 = 211205.40 \text{ M.U}$$

II.4. Debt replacement "parity theory":

II.4.1. Definition of debt replacement:

When dealing with commercial papers, there may come a time when their maturity date arrives and the debtor is unable to fulfill their value. In such a case, the debtor is required to settle their debts and renegotiate with their creditors to establish new terms and conditions for resolving the delinquent debts. This is done through the creation of new commercial papers or debts with different maturity dates and conditions from the previous ones.

The changes in the repayment terms of the old commercial papers result in the creation of new debts that are equivalent to them. Therefore, the nominal value of the new debts is different from the old ones. The fundamental principle in debt replacement is that the sum of the present values of the old debts at the settlement date (replacement date) must be equal to the sum of the present values of the new debts. This ensures that neither party is disadvantaged.

The general rule for debt replacement is that the new debts should be equivalent in their present value at the settlement date. In other words, the commercial papers are only considered equivalent at the settlement date .

It is also known as debt replacement process, and equivalence is a financial process through which the parties involved in commercial papers can replace them fairly, provided that the replaced papers are equivalent, whether it is two papers or several commercial papers on the compensation day (replacement day).

The equivalence equation is as follows:

II.4.2. Equivalent capital situation of two capitals:

We say that two capitals are equivalent if their present values are equal at a specific date, which is the date of equivalence, at a discount rate of one. This can be expressed as follows:

$$\boxed{\begin{gathered} CV_1 = CV_2 \\ CV_1(1-t)^{-n_1} = \quad CV_2(1-t)^{-n_2} \end{gathered}}$$

- **Example:**

A merchant debtor wanted to replace his debt, which is valued at 200000 M.U and is due in four years, with another debt due in six years. If the compound interest rate is 8%

Required:

Find the nominal value of the new debt.

$$CV_1 = CV_2$$

$$FV_1(1+t)^{-n_1} = FV_2(1+t)^{-n_2}$$

$$200000(1.08)^{-4} = FV_2(1.08)^{-6}$$

$$200000(1.08)^{-4} = FV_2(1.08)^{-6}$$

$$FV_2 = \frac{200000(1.08)^{-4}}{(1.08)^{-6}}$$

$$FV_2 = \quad 233280$$

- **Example:**

A person wants to replace a debt valued at 700,000, due in 6 years, with another debt valued at 850,000 due for payment after n2 years, with an equivalence rate of 9%.

Required:

Calculate n_2.

$$CV_1 = CV_2$$

$$FV_1(1+t)^{-n_1} = FV_2(1+t)^{-n_2}$$

$$700000(1.09)^{-6} = 850000(1.09)^{-n_2}$$

$$(1.09)^{-n_2} = \frac{700000(1.09)^{-6}}{850000} = 0.491043681$$

$$ln(1.09)^{-n_2} = \ln 0.491043681$$

$$-n_2 \ln(1.09) = \ln 0.491043681$$

$$-n_2 = \frac{\ln 0.491043681}{\ln(1.09)}$$

$$n_2 = 8.25 \Rightarrow n_2 = 8\ years\ and\ 3\ months$$

II.4.3. Equivalence of capital with a set of capital heads:

We say that a capital is equivalent to a group of capitals, or vice versa, if its current value is equal to the sum of the current values of the group of capitals at a specific date called the reconciliation date, at a discount rate of one.

$$\boxed{CV = CV_1 + CV_2 + \ \dots\dots\dots\dots + CV_n}$$

- **Example:**

Person A is indebted with two amounts: First amount:

- 500 000 M.U payable after 5 years.
- Second amount: 100 000 M.U payable after 10 years.

The merchant wants to replace them with a debt that matures after 7 years, knowing that the prevailing compound interest rate is 9% annually.

- **Required:**

Find the nominal value of this debt.

- **Solution:**

$$FV(1+t)^{-n} = FV_1(1+t)^{-n_1} + FV_2(1+t)^{-n_2}$$

$$FV(1.09)^{-7} = 500000(1.09)^{-5} + 100000(1.09)^{-10}$$

$$FV = \frac{500000(1.09)^{-5} + 100000(1.09)^{-10}}{(1.09)^{-7}}$$

$$FV = 136623.34$$

II.4.4. Equivalence of two sets of capital heads:

We say that a set of capital heads is equivalent to another set of capital heads if the sum of the present values of the first set of capital heads is equal to the sum of the present values of the second set of capital heads at the date of equivalence with a discount rate of one.

$$\boxed{CV_1 + \mathrm{CV}_2 + CV_3 + \ \dots\dots\dots\dots CV_n \ = \grave{C}V_1 + \grave{C}V_2 + \grave{C}V_3 + \ \dots\dots\dots\dots \grave{C}V_n}$$

- **Example:**

Person is indebted with the following amounts:

- 80000 M.U due in 3 years.
- 120000 M.U due in 5 years.

He wants to replace these two debts with two new equal debts. The first debt is due in 4 years, and the second debt is due in 7 years.

Find the nominal value of the two new debts if the interest rate is 9% per annum.

- **Solution:**

$$CV_1 + CV_2 = \grave{C}V_1 + \grave{C}V_2$$

$$800000(1.09)^{-3} + 120000(1.09)^{-5} = FV(1.09)^{-4} + FV(1.09)^{-7}$$

$$FV = \frac{800000(1.09)^{-3} + 120000(1.09)^{-5}}{(1.09)^{-11}}$$

$$FV = 111326.92$$

II.4.4. Using the principle of equivalence:

The principle of equivalence is used to determine the future value of any debt or the applicable discount rate. It is especially used to determine the due date or maturity period of specific debts in the case of replacement, or a common maturity period for the payment of multiple debts at once. In the context of replacement or equivalence, we encounter three important and well-known dates: the date of equivalence, the date of common due, and the date of average due.

II.4.4.1. Date of Equivalence:

The date of equivalence is the date chosen by both the debtor and the creditor to replace or substitute old debts with new ones. It is always before the nearest first due date because a debt can only be replaced before its due date. The date of equivalence can be at the time of maturity or at any other due date.

To determine the date of equivalence, we calculate the maturity period of a debt and then go back in time by the same calculated period.

- **Example:**

Four years ago, a merchant borrowed an amount of 60000 M.U due on 5/03/2023. The merchant wants to replace it with a new debt with a nominal value of 300000 M.U and a compound annual interest rate of 15%.

- Required:
  - Determining the due date.
  - Finding the due date of the new debt.
- **Solution**:
  - Determining the due date:

The due date is located two years before the maturity date: 5/03/2023, which means it is on 5/03/2021.

- Finding the due date of the new debt:

To find the due date of the new debt, we need to calculate the maturity period based on the interest rate.

$$CV_1 = CV_2$$

$$FV_1(1+t)^{-n_1} = FV_2\,(1+t)^{-n_2}$$

$$60000(1.15)^{-4} = 300000\,(1.15)^{-n_2}$$

$$60000(1.15)^{-4} \times (1.15)^{n_2} = 300000$$

$$(1.15)^{n_2} = \frac{150000}{95000(1.15)^{-4}} = 2.76158881579$$

$$(1.15)^{n_2} = 2.76158881579$$

$$\Rightarrow n_2 \ln(1.15) = \ln\ 2.76158881579$$

$$n_2 = \frac{\ln\ 2.76158881579}{\ln(1.15)}$$

$$n_2 = 7.26$$

So, the maturity period of the debt is 7 years, 3 months, and 4 days.

II.4.4.2. Date of Common Due:

The date of common due is the due date of a single commercial paper that replaces a group of commercial papers, where the future value of this paper differs from the sum of the future values of the papers being replaced.

$$FC \neq FC_1 + FC_2 + \ \ldots\ldots\ldots FC_K$$

$$FV(1+t)^{-n} = FV_1(1+t)^{-n_1} + FV_2(1+t)^{-n_2} + \ \ldots\ldots + FV_K(1+t)^{-n_k}$$

$$(1+t)^n = \frac{FV}{FV_1(1+t)^{-n_1} + FV_2(1+t)^{-n_2} + \ \ldots\ldots + FV_K(1+t)^{-n_k}}$$

$$n\ln(1+t) = \ln FV \ - \ln[\, FV_1(1+t)^{-n_1} + FV_2(1+t)^{-n_2} + \ \ldots\ldots + FV_K(1+t)^{-n_k}]$$

$$n = \frac{\ln FV \ - \ln[\, FV_1(1+t)^{-n_1} + FV_2(1+t)^{-n_2} + \ \ldots\ldots + FV_K(1+t)^{-n_k}]}{\ln(1+t)}$$

- **Example:**

On a certain date, a company found itself indebted to the bank with the following amounts:

- 4000 M.U due in 2 years
- 6000 M.U due in 2 years
- 9000 M.U due in 7 years

The company decided, in agreement with the bank, to repay these debts with a single amount of 200000 M.U with a compound interest rate of 7% per year.

- **Required:**

Calculate the common maturity period. Determine the common maturity date.

- **Solution:**
  - Calculation of the common maturity period.

$$FC \neq FC_1 + FC_2 + \ \dots\dots\dots FC_K$$
$$FV(1+t)^{-n} = \ FV_1(1+t)^{-n_1} + \ FV_2(1+t)^{-n_2} + \ \dots\dots + FV_K(1+t)^{-n_k}$$
$$200000(1.07)^{-n} = 40000(1.07)^{-2} + \ 60000(1.07)^{-4} + 90000(1.07)^{-7}$$
$$200000(1.07)^{-n} = 0.683793693$$
$$(1.07)^n = 1.4624294$$

By applying the natural logarithm to both sides, we find:

$$n = \frac{\ln(1.46242939974)}{\ln(1.07)}$$
$$n = 5.61$$

The duration of maturity is 5 years and 220 days.

- Determining the common due date:

The common due date will be after 5 years and 220 days, which is on 10/07/2023. It is the date on which the three debts will be paid off simultaneously.

II.4.4.3Average Maturity Date:

The date of average maturity is the due date of a single paper that replaces a group of papers, where the future value of this single paper equals the sum of the future values of the papers being replaced.

The date of average maturity is achieved by calculating the due date for the total future values of various debts, considering them as the future value of a single paper or debt, where:

$$FC = \ FC_1 + FC_2 + FC_3 + \cdots \dots\dots\dots. + FC_K$$

$$FV(1+t)^{-n} = \ FV_1(1+t)^{-n_1} + \ FV_2(1+t)^{-n_2} + \ \dots\dots + FV_K(1+t)^{-n_k}$$

$$(1+t)^n = \frac{FV}{FV_1(1+t)^{-n_1} + \ FV_2(1+t)^{-n_2} + \ \dots\dots + FV_K(1+t)^{-n_k}}$$

$$n\ln(1+t) = \ln FV \ - \ln[\ FV_1(1+t)^{-n_1} + \ FV_2(1+t)^{-n_2} + \ \dots\dots + FV_K(1+t)^{-n_k}]$$

$$n = \frac{\ln FV \; - \ln[\, FV_1(1+t)^{-n_1} + \; FV_2(1+t)^{-n_2} + \; \ldots\ldots + FV_K(1+t)^{-n_k}]}{\ln(1+t)}$$

- **Example:**

On a certain date, a company found itself indebted to the bank with the following amounts:

- 4000 M.U due after 2 years.
- 6000 M.U due after 2 years.
- 9000 M.U due after 7 years.

The company decided, in agreement with the bank, to settle these debts with a single amount of 190000 M.U at a compound interest rate of 7% per year.

- **Required:**
  - Calculate the average maturity period.
  - Determine the average maturity date.
- **Solution:**
  - Calculating the average maturity period:

$$FV(1+t)^{-n} = \; FV_1(1+t)^{-n_1} + \; FV_2(1+t)^{-n_2} + \; FV_3(1+t)^{-n_3}$$

$$190000(1.07)^{-n} = 40000(1.07)^{-2} + \; 60000(1.07)^{-4} + 90000(1.07)^{-7}$$

$$(1.07)^{-n} = 0.719782834$$

$$(1.07)^{n} = 1.38930793118$$

By applying the natural logarithm to both sides, we find:

$$n \; = \frac{\ln(1.38930793118)}{\ln(1.07)}$$

$$n = 4.85$$

The duration of maturity is 4 years, 10 months, and 6 days.

- Average Maturity Date:

The duration of maturity is 4 years, 10 months, and 6 days. Therefore, the average maturity date would be on 09/11/2022.

II.3. Annuities:

II.3.1. Definition of Annuities:

Although the word *annuity* was meant originally to refer to an annual payment, the concept has been expanded to represent more than simply an annual payment. An **annuity** is any set of equal payments made at equal intervals of time. In this sense, all periodic saving or investment deposits, purchases on installment, mortgages, auto loans, life insurance premiums, and even Social Security deductions are types of annuities.

II.3.2. Types of annuities

Annuities are classified according to three criteria: time of payments, term of annuity, and the timing of the compounding process. The following annuities are based on time of payments:

- *Ordinary annuity:* an annuity for which payments are made at the end of the interval period. If the annuity is monthly, payments are due at the end of each month. This annuity is also called an **annuity immediate**.
- *Annuity due:* an annuity for which payments are made at the beginning of each interval period. So if it is a monthly annuity, payments are due at the beginning of each month.
- *Deferred annuity:* an ordinary annuity but with a deferred term. In this case, the entire payment process would not begin until after a certain designated time has passed. An example is a debt paid off by a certain set of equal payments made at the end of equal intervals, but with the first payment due 3 years from now. So it is an ordinary annuity with a 3-year deferment.

The **term of annuity** is the time between the beginning of the first interval and the end of the last interval. For example, a 3-year term would be between January 1, 2009 and December 31, 2011.

The following annuities are based on the term of the annuity:

- *Annuity certain:* an annuity for which the beginning and end of the term are designated and recognized in advance. A mortgage and an auto loan are examples of an annuity certain.

- *Contingent annuity:* an annuity for which the beginning of the term is known but the end is contingent upon a certain event. The best example is life insurance. Payment of the premium starts at the time of purchase and continues while the insured is alive, and the end occurs at the time of the insured's death.
- *Perpetuity:* an annuity for which the beginning of the term is known but the end is infinite. An example is a certain principal fund that is kept invested indefinitely and thus continues to generate interest income indefinitely.

The following annuities are based on the timing of the compounding process:

- *Simple annuity:* an annuity for which the compounding occurs at a time matching the time of payment. For example, the interest on an annuity that is due at the end of each quarter would be compounded quarterly at the end of each quarter.
- *General annuity:* an annuity for which the compounding process does not match the payment intervals, so interest is assessed either more or less often than the payments. An example is a quarterly annuity where the interest is compounded monthly, called a **complex annuity**.

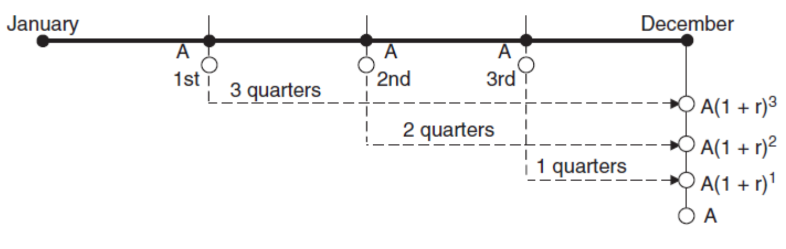
The most common annuity in the financial world, an ordinary and certain annuity, is our focus hereafter.

II.3.3. Ordinary annuity

II.3.3.1. Future value of an ordinary annuity

Let a payment of a quarterly annuity be *A* and assume a term of annuity of 1 year. The first payment would be due at the end of the first quarter and the last would be due at the end of the last quarter (see Figure). We can arrange the compounded FVs of these payments as follows:

The to



$$FV = A + A(1+r) + A(1+r)^2 + \cdots + A(1+r)^{n-2} + A(1+r)^{n-1}$$

By applying the formula for the sum of an increasing geometric series, we find:

$$FV = \frac{A[(1+r)^n - 1]}{r}$$

This is the formula for a future value of an ordinary annuity.

Example:

The person has been repaying a debt in the form of equal monthly payments for a duration of 20 months. They agreed with the lender that each payment would be 5000 M.U , with a compound interest rate of 5%.

What is the total amount that the debtor will repay at the end of the agreed-upon duration?

Solution:

$$FV = \frac{A[(1+r)^n - 1]}{r} = \frac{5000\,[(1+0{,}05)^{20} - 1]}{0.05}$$

$$= 165329{,}7705\ DA$$

II.3.3. 2. Current value of an ordinary annuity

If we consider that a person is obligated under a contract to pay an annual amount due at the end of each year included in the contract, and given that the applicable annual interest rate is fixed throughout the term of the contract, the current value of the total payments made at the end of the contract can be found through the following graphic representation:

Through the previous graphic representation, we find the general formula for calculating the The current value of an ordinary annuity as follows:

$$CV4 = A(1 + r)^{-4} + A(1 + r)^{-3} + A(1 + r)^{-2} + A(1 + r)^{-1}$$

So the formula for the sum is as follows:

$$\boxed{CV = \frac{A[1 - (1 + r)^{-n}]}{r}}$$

This is the formula for a future value of an ordinary annuity, which is the evaluation of the payments for a period before the date of the date of the first payment.

Example:

A man wants to cash in his trust fund, which pays him 500 M.U a month for the next 10 years. The interest on the fund is 6.5 % compounded.

How much would he receive?

Solution:

The monthly interest rate is 0.065/12 =0 .0054; the monthly term is 10 × 12 = 120.

$$CV = \frac{A[1 - (1 + r)^{-n}]}{r} = \frac{500\ [1 - (1 + .0054)^{-120}]}{0.0054}$$

$$CV = 48{,}518.85$$

II.3.3. 3. Finding the determinants of an ordinary annuity

- **Finding the payment of an ordinary annuity**

We can rearrange the formulas of both the future value and the current value of the annuity and rewrite them in terms of *A*.

• Annuity payment (*A*) when FV is given:

$$FV = \frac{A[(1 + r)^n - 1]}{r}$$

$$\Downarrow$$

• Annuity payment (A) when the current value is given: $A = \frac{FV \cdot r}{(1 + r)^n - 1}$

$$A = \frac{CV \cdot r}{1 - (1 + r)^{-n}}$$

- **Finding the term of an ordinary annuity**

The term of annuity (n) can be found by rearranging the future value of an annuity formula:

$$\mathrm{FV} = \frac{\mathrm{A}[(1 + \mathrm{r})^{\mathrm{n}} - 1]}{r}$$

$$\Downarrow$$

$$\boxed{\mathrm{n} = \frac{\ln\left(\frac{\mathrm{FV} \times \mathrm{r}}{A} + 1\right)}{\ln(1 + r)}}$$

II.3.4. Annuity Due:

II.3.4.1. Future value of an annuity due

Annuity due is just like an ordinary annuity except that the payments occur at the beginning of each payment term instead of at the end. Therefore, the entire term of an annuity due starts at the date of the first payment and ends at the end of the interval of the last payment. In other words, it would end one payment interval after the last payment has been paid. Insurance premiums and property rentals are typical examples of annuities due. As a result of the change in the timing of payments, the future and current values would also change. The future value of an annuity due formula would reflect that change:

$$\boxed{FV_d = \left[A \frac{(1 + r)^n - 1}{r}\right](1 + r)}$$

Example :

Jack and his wife will deposit 550 M.U in their savings account at the first day of each month for the next 5 years in order to have a large enough down payment to buy a larger house. If their savings account pays 7% interest compounded.

how much down payment will they collect?

Solution:

The monthly rate =0 .07/12 = 0.00583; n = 5 × 12 = 60.

$$FV_d = \left[A\frac{(1 + r)^n - 1}{r}\right](1 + r) = \left[550\frac{(1.00583)^{60} - 1}{0.00583}\right](1 .00583)$$

$$FV_d = 39{,}606$$

II.3.4.2. Current value of an annuity due:

The current value of an annuity due can be obtained using the following

formula:

$$\boxed{CV_d = \left[A\frac{1-(1 + r)^{-n}}{r}\right](1 + r)}$$

Example:

What is the current value of an annuity due for an investment

fund paying 5.75% interest compounded if 750 M.U is deposited at the

beginning of each month for 3 years?

The monthly rate =0 .0575/12 =0 .00479; n = 3 × 12 = 36.

$$CV_d = \left[A\frac{1-(1 + r)^{-n}}{r}\right](1 + r) = \left[750\frac{1-(1 .00479)^{-n}}{0.00479}\right](1 .00479)$$

$$CV_d = 24{,}864$$

II.3.4.3. Finding the determinants of an annuity due

- **Finding the payment of an annuity due**

Just like the ordinary annuity payment, the annuity due payment can be obtained

by a rearrangement of either the future value, or the current value formula, depending on which one is available. If the future value is given, the payment

formula would be:

$$\boxed{A_d = \frac{FV \times r}{(1+r)^{n+1} - (1+r)}}$$

If the current value is given, the payment formula would be

$$\boxed{A_d = \frac{CV \times r}{(1+r) - (1+r)^{1-n}}}$$

Example :

Sam purchased a stereo set on his appliance store card. He was told that with the store rate of 18% compounded, he would end up paying 1744 M.U on a 2-year payment plan.

How much would he pay at the start of each month?

Solution:

Since we know the future value 1744 M.U , we apply the A_d with the future

value formula:

$$A_d = \frac{FV \times r}{(1+r)^{n+1} - (1+r)}$$

The monthly rate = .18/12 =0 .015; n = 2 × 12 = 24.

$$A_d = \frac{1744 \times 0.015}{(1.015)^{24+1} - (1.015)} = 60$$

- **Finding the term of an annuity due**

The term (n) can also be found depending on whether the future value or the current value is given. If the future value is given, the n formula would be

$$\boxed{n = \frac{\ln\left[1 + \frac{FV \times r}{A(1+r)}\right]}{\ln(1+r)}}$$

and if the current value is given, the n formula would be

$$\boxed{n = -\frac{\ln\left[1 - \frac{CV \times r}{A(1+r)}\right]}{\ln(1+r)}}$$

Example:

Tim and Cathy will need a down payment of 15000 M.U for their
first home. If they can save 600 M.U at the beginning of each month in an account bearing 8.5 % compounded.
determine how long it would take them to buy a home.

Solution:

The monthly rate = 0.085/12 = 0.007083.

$$n = \frac{\ln\left[1 + \frac{FV \times r}{A(1+r)}\right]}{\ln(1+r)} = n = \frac{ln\left[1 + \frac{15000 \times 0.007083}{600(1.007083)}\right]}{ln\ (1.007083)}$$

$$n = 22.95 \quad \approx 23\ month$$

II.3.5. Deferred annuity

A deferred annuity is an ordinary annuity except that the payoff starts at a later time. In this case, the first payment is made after a certain time has been passed according to the financial contract. This time is called the deferment period. The important note to make here is that because of the fact that the payments are made at the end of each payment term as an ordinary annuity, calculation of the deferment period needs attention. For example, if the first payment is made at the fifth payment period, the annuity would be deferred four payment periods, and by the same logic, we conclude that if an annuity is deferred for seven payment periods, its first payment has to be made at the end of the eighth payment.

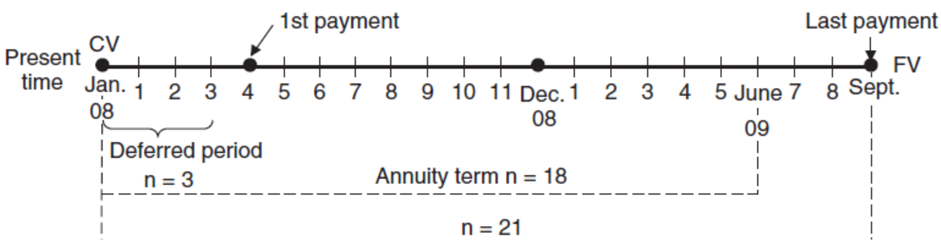
Example :

Determine the current value of a deferred annuity of 250 M.U a month for 18 months from January 2008 to June 2009 if the first payment is due at the end of April 2008, given that the interest rate is 7% compounded. (see Figure).

Solution:

Since the first payment is made at the end of April 2008, there would be three months of deferment: January, February, and March. The future value would occur (or the annuity will be paid off) on September 2009, three months after the original date of June 2009, because of the three months' skip in advance. As for the current value calculation, it can be done by bringing back the future value from September 2009 to January 2008; but that is 21 periods, and that is why we also need to calculate the CV of the deferment period and subtract it from the entire CV. Therefore, we calculate two current values; the first where *n* is equal to the regular annuity term of 18 months plus the deferred period of 3 months ($n = 18 + 3 = 21$). The second current value is for the deferment only where *n* is equal to 3. Then we subtract the second CV from the first to get the correct CV, where *n* is 18.

The monthly rate = 0.07/12 =0 .00583.



$$= \frac{250[1-(1.00583)^{-21}]}{0.00583}$$

= 4927.85

$$CV_2 = \frac{250[1-(1.00583)^{-3}]}{0.00583}$$

$$= 741.34$$

$$CV = CV_1 - CV_2$$

$$CV = 4927.85 - 741.34$$

$$= 4186.51$$

Unsolved exercises

Exercise01 :

A person deposited an amount of 72800.00 with a bank for a duration of 9 years at a compound interest rate of 9.5% per annum.

The required calculations are as follows:

- Calculate the interest for the first year.
- Calculate the interest for the fourth year.
- Calculate the interest for the sixth year.
- Calculate the total amount accumulated at the end of the sixth year.

- **Exercise 02 :**

A person employed an amount of 215000 M.U in a bank for a period of 6 years at a compound interest rate of 9.5% per annum.

The required:

- Calculate the interest for the first year and the fourth year.
- Calculate the total amount acquired at the end of the investment.

The person borrowed this amount, and it will be recovered after four years for a total of 583170.1224 M.U .

- Calculate the annual interest rate applied to this amount.
- What is the future value for a capital of 100000 M.U invested for 10 years at a compound interest of 6% annually?

•Exercise 03 :

A company employed an amount of 725000 M.U. in a bank for a period of 8 years at a compound interest rate of 6.5% per annum.

The required :

- Calculate the interest for the first year and the third year.

- Calculate the total amount acquired at the end of the investment.

The company deposited this amount at a rate of 9% to achieve 1846151.589 .

- Determine the duration of the deposit**.**

- **Exercise 04 :**

A person deposited an amount of 98,200 M.U in a commercial bank at a compound annual interest rate of 8.5% for a duration of 7 years.

The required calculations are as follows:

- Calculate the interest earned at the end of the first year.
- Calculate the interest earned at the end of the fifth year only.
- Calculate the total amount received at the end of the duration.

- **Exercise 05** :

A company employed an amount of 430000 M.U in a bank for a duration of three years at a semi-annual interest rate of 5%.

- Calculate the total amount at the end of the duration.
- Calculate the interest earned from the investment.

The accumulated amount was lent and is expected to be recovered after three years with a value of 736026.523 M.U.

- Calculate the annual interest rate applied to the amount.

The accumulated amount was deposited to yield 843,675 dinars at an interest rate of 10%.

- What was the duration of the deposit?

- **Exercise 06 :**

The employee has the following options for paying off a home:

- Option 1: Pay the purchase price of the house on the purchase date.

- Option 2: Pay 1,250,000 dinars on the purchase date and 332750 M.U after 3 years from the purchase date.
- Option 3: Pay the purchase price with a nominal value of 1996500 M.U dinars.
- Option 4: Pay the purchase price with a nominal value of 2623509.375 M.U after 4 years from the purchase date.

The required:

assuming an annual interest rate of 10%:

- Calculate the purchase price of the house on the purchase date.
- Determine the due date for the purchase price according to Option 3.
- Find the applied discount rate according to Option 4.

- **Exercise 07 :**

The merchant bought a car, and the seller proposed to him to pay an amount of 2420000 M.U after one year from the date of purchase, and an amount of 2662000 M.U after two years from the date of payment of the first amount.

Calculate the purchase price of the car on the date of purchase. (ta = 10%)

- **Exercise 08 :**

A person deposited an amount in a bank at an interest rate of 10%, and after 4 years, he found that the balance in the bank had reached 48000 M.U.

Calculate the amount that this person deposited.

- **Exercise 09 :**

A person has two debts:

The first debt: 50000 M.U due after 5 years.

The second debt: 100000 M.U due after 10 years.

He wants to replace them with a single amount due after 7 years in order to completely get rid of these debts. If we know that the prevailing compound interest rate is 9% annually.

The required :

find the future value of this debt.

- **Exercise 10 :**

A client is indebted to a company with the following amounts:

- 2450 M.U due after 2 years.
- 5000 M.U due after 5 years.
- 7500 M.U due after 7 years.

The client requested to replace these debts with a single debt due after 4 years. If the interest rate is 10% annually.

The required:

- Calculate the value of the new debt.
- Calculate the average due date for these debts.

- **Exercise 11** :

A person is indebted to a bank with the following amounts:

- First debt: 10000 M.U due after 1 year.
- Second debt: 20000 M.U due after 2 years.
- Third debt: 40000 M.U due after 4 years.

The person requested to replace these three debts with a single debt, and the bank proposed the following options:

- Replace them with a single debt of nominal value 72000 M.U.
- Replace them with a single debt of nominal value 70000 M.U.
- Pay them off with a single payment within a period of 3 years.

The required:

- Determine the common due date for the three debts according to the first proposal.
- Calculate the average due date for the three debts according to the second proposal.
- Calculate the nominal value of the single payment according to the third proposal.

Chapter 03:
Loan Amortization

III.1. Introduction:

A loan is a financial transaction where a lender provides a certain amount of money or assets to a borrower, who agrees to repay the loan with interest over a specified period. Loans are commonly used by individuals, businesses, and organizations to finance various activities, such as purchasing property, funding projects, or covering expenses.

III.2. Definition of Ordinary Loan

The Ordinary loan is a loan that is made between natural or legal entities, and this type of loan is called indivisible loans (Emprunts indivis), where the loan agreement involves a single lender."

In an ordinary loan, the borrower usually adheres to the following conditions:

- Paying interest on a regular basis on the borrowed capital (the principal of the loan) and the unpaid ones.
- Repayment of the borrowed capital, and this repayment is called: “amortization of loans”, and the loan may be repaid at once or in many installments in most cases.

The debtor or borrower periodically makes a payment that contains interest on the remaining capital plus depreciation. as follows:

$$\boldsymbol{The\ payment} =$$
$$\boldsymbol{The\ interest\ on\ the\ remaining\ capital} + \boldsymbol{a\ portion\ of\ the\ principal}\ (\boldsymbol{Amortization})$$

The process of loan amortization with equal or fixed installments is similar to the process of repaying loans with end-of-term payments. At the end of the loan term, the sum of the payments is equal to the total amount of the loan paid. The loan principal or its present value at the beginning of the first repayment year is equal to the present value of the payments. If:

P_0 :The value of the loan principal at time zero, which is the beginning of the first year of repayment.

PYT : The periodic payment.

PP_i : The principal portion or

amortization in year i.

I_i :The interest in year i.

n : The loan term (number of years)

r: The loan rate.

III.3. Methods of Loan Amortization:

Loan amortization refers to the process of repaying a loan over time through a series of scheduled payments. When you borrow money, whether it's for a mortgage, car loan, or personal loan, the lender typically requires you to make regular payments that include both principal (the amount borrowed) and interest (the cost of borrowing).

Loan amortization methods are mathematical formulas used to determine the allocation of each payment towards the principal and interest components of the loan. These methods help borrowers understand how their loan balance decreases over time and how much interest they will pay throughout the loan term.

There are several commonly used loan amortization methods, including the straight-line method, the declining balance method, and the annuity method. Each method has its own calculations and benefits, and the choice of method depends on factors such as the loan terms, interest rates, and borrower's preferences.

The straight-line method is the simplest amortization method, where equal payments are made throughout the loan term, with the same amount allocated towards both principal and interest. This method provides a clear and consistent payment schedule but may result in higher interest payments in the earlier years of the loan.

The declining balance method, also known as the reducing balance method, allocates more of each payment towards the principal as the loan balance decreases. This method helps borrowers pay off the loan faster and reduces the overall interest paid over the loan term.

The annuity method, also called the equal installment method, involves fixed payments that remain the same throughout the loan term. These payments are structured to ensure that the principal and interest components are appropriately allocated, with the interest portion decreasing over time as the principal is paid down.

Choosing the right loan amortization method depends on factors such as the borrower's financial situation, goals, and preferences. It's important to carefully consider the implications of each method, including the total interest paid, the duration of the loan, and the impact on monthly cash flow.

In conclusion, loan amortization methods play a crucial role in determining the repayment structure of a loan. Understanding these methods and their implications can help borrowers make informed decisions, manage their debt effectively, and plan for a successful loan repayment journey.

III.3.1 Ordinary loans paid in fixed or equal payments

According to this method, the loan is repaid through fixed payments that are paid at the end of each agreed-upon time period. Based on this, if we have an ordinary loan repaid through n fixed payments and an interest rate of t, we can infer the following relationships:

- **The mathematical formula for the loan principal in terms of the fixed payment:**

$$\boxed{P_0 = PYT \ \frac{1-(1+r)^{-n}}{r}}$$

- **The mathematical formula for the loan principal in terms of the consumptions:**

- **The mathematical formula for the loan principal in terms of the initial consumption:**

$$P_0 = \sum_{i=0}^{n} PP_i$$

$$\boxed{P_0 = PP_1 \ \frac{(1+r)^n - 1}{r}}$$

- **The mathematical formula for the fixed payment in terms of the loan principal:**

The general formula for calculating the periodic payment or fixed installment in terms of the loan principal is given by the following mathematical equation:

$$PYT = P_0 \ \frac{r}{1-(1+r)^{-n}}$$

$$PYT = PP_i + I_i$$

III.3.1.1 Amortization Schedule Loan:

A loan amortization schedule is a table or chart that outlines the repayment plan for a loan. It breaks down the loan into periodic payments, Monthly, quarterly or yearly payments..., and provides details on the principal amount, interest payment, and remaining balance for each payment period. The schedule helps borrowers understand how their payments are applied towards reducing the loan balance and how much interest they will pay over the loan term.

The loan amortization schedule is as follows:

i	P_{i-1}	I_i	PP_i	TPY_i	P_i
1	P_0	$I_1 = P_o \times r$	PP_1	TPY	$P_1 = P_0 - PP_1$
2	P_1	$I_2 = P_1 \times r$	PP_2	TPY	$P_2 = P_1 - PP_2$
3	P_2	$I_3 = P_2 \times r$	PP_3	TPY	$P_3 = P_2 - PP_3$
⋮	⋮	⋮	⋮	⋮	⋮
n	P_{n-1}	$I_n = P_{n-1} \times r$	PP_n	TPY	$P_n = P_{n-1} - PP_n$

III.3.1.2. The relationship between the elements of loan amortization:

The elements of loan amortization, including the principal amount, interest rate, loan term, and payment frequency, are interconnected and influence each other. The principal amount represents the initial loan balance, which is gradually reduced with each payment. The interest rate determines the cost of borrowing and affects the amount of interest paid over time. The loan term defines the duration of the loan, and the payment frequency determines the number of installments required for repayment. By adjusting these elements, borrowers can modify their loan amortization schedule to meet their financial goals and manage their loan effectively.

At this point, we will try to show the relationships that link the various elements of rodent consumption as follows:

- **The relationship between consecutive installments:**

$$\boxed{PP_n = PP_{n-1}(1+t)}$$

- **The relationship between two amortizations:**

$$\boxed{PP_n = PP_p(1+t)^{n-p}}$$

- **The relationship between the amortizations and the first amortization:**

$$\boxed{PP_n = PP_1(1+t)^{n-1}}$$

- **The relationship between the payments and the amortization.**

  ✓ The relationship between the final payment and the last amortization:

$$\boxed{\text{TPY} = PP_n(1+t)}$$

  ✓ The relationship between the initial payment and the first amortization:

$$\boxed{\text{TPY} = PP_1(1+t)^1}$$

  ✓ The relationship between the payment and the amortization:

$$\boxed{\text{TPY} = PP_p(1+r)^{n-p+1}}$$

- **The relationship between the loan and the first amortization:**

$$\boxed{P_0 = PP_1 \quad \frac{(1+r)^n - 1}{r}}$$

- **The relationship between the loan and the fixed payments**:

$$\boxed{\text{TPY} = \text{P}_0 \frac{r}{1-(1+r)}}$$

- **The relationship between interest and amortization**

Since the fixed payment value equals the principal reduction amount plus the annual interest value, the payments are equal:

$$\boxed{I_{n-1} - I_n = M_{n-1} \times r}$$

- **The loan principal:**

We denote the amount paid from the principal of the loan until the payment k as: PP_{k} which is equal to the sum of amortizations paid up to that period.

✓ The principal of the loan paid until the payment "K" is made:

$$\boxed{\mathrm{PP}_K = PP_1 \times \frac{(1+r)^K - 1}{r}}$$

✓ Principal amount remaining after payment "P" has been made:

If we denote the remaining capital as P_p , then:

$$\boxed{P_p = P_0 - PP_K}$$

Example:

The institution obtained a loan of 20000000 to be repaid over a period of 4 years with annual fixed installments, at an interest rate of 12%, starting from the end of the contract year.

Required:

Calculate the value of the fixed payment and prepare the loan amortization schedule**.**

Solution:

- Calculation the value of the payment:

$$\mathrm{TPY} = \mathrm{P}_0 \frac{r}{1-(1+r)} = 20000000 \ \frac{0.12}{1-(1+0.12)} = 65846.88$$

- Preparing a loan amortization schedule:

i	P_{i-1}	I_i	PP_i	TPY_i	P_i
1	**200000**	**24000**	**41846.88**	**65846.88**	**158153.12**
2	**158153.12**	**18978.3744**	**46868.5056**	**65846.88**	**111284.6144**
3	**111284.6144**	**13354.15373**	**52492.72627**	**65846.88**	**58791.88813**
4	**58791.88813**	**7055.026576**	**58791.88813**	**65846.88**	**0**
	/	**200.8622480**	**200000**	**263387.52**	/

III.3.2. Ordinary loans repaid at the end of the contract and periodic interest

According to this method, the borrower is obligated to pay the periodic interest at the end of each borrowing period, with an additional payment of the loan principal at the end of the final period.

III.3.2.1.The relationship between the elements of loan amortization is as follows:

From the above, we can conclude that the periodic interest remains constant throughout the contract period. The annual amortization is equal to zero throughout the contract period, except for the last year where it is equal to the loan amount. Therefore, the payments are also equal throughout the contract period and equal to the periodic interest, except for the last year where the payment is equal to the interest plus the loan amount. To summarize the above:

- The periodic interest remains constant throughout the contract.
- The annual amortization is zero except for the last year, where it is equal to the loan amount.
- The payments are equal throughout the contract, equal to the periodic interest, except for the last year where the payment is equal to the interest plus the loan amount.

In summary, we can summarize the above as follows:

- **The relationship between interest:**

$$\boxed{I_1 = I_2 = I_3 = \ \ldots\ldots\ldots. = I_n}$$

- **The relationship between amortization:**

$$\boxed{PP_1 = PP_2 = PP_3 = \ \ldots\ldots\ldots. = PP_{n-1}}$$

- **The relationship between the final amortization and the principal of the loan:**

$$\boxed{PP_n = P_0}$$

- **The relationship between payments and interest:**

$$\boxed{TPY_1 = TPY_2 = TPY_3 = \ \ldots\ldots\ldots. = TPY_{n-1} = I}$$

- **The relationship between the final payment and the principal of the loan:**

$$\boxed{TPY_n = PP_n + I}$$

Or

$$\boxed{TPY_n = P_0 + I}$$

$$\boxed{TPY_n = PP_n + I}$$

$$\boxed{TPY_n = P_0 + I}$$

III.3.2.2. Loan amortization schedule:

According to this method, the loan amortization schedule is as follows:

i	P_{i-1}	I_i	PP_i	TPY_i	P_i
1	P_0	$I = P_o \times r$	0	$TPY_1 = I$	P_0
2	P_0	$I = P_0 \times r$	0	$TPY_2 = I$	P_0
3	P_0	$I = P_0 \times r$	0	$TPY_3 = I$	P_0
⋮	⋮	⋮	⋮	⋮	⋮
n	P_0	$I = P_0 \times r$	P_0	$TPY_n = I + P_0$	0

Example:

A trader borrowed a ordinary loan of 3000000 with an annual interest rate of 10% for a duration of 5 years. The loan is to be repaid at the end of the contract, and annual interest is accrued at the end of each year throughout the contract period.

Required: Prepare the loan amortization schedule for this loan.

Solution:

$$I = P_0 \times r$$

$$I = 3000000 \times 0.1 = 300000$$

$$TPY_1 = TPY_2 = \cdots = TPY_4 = I = 300000$$

$$TPY_5 = I + P_0 = 300000+3000000=3300000$$

i	P_{i-1}	I_i	PP_i	TPY_i	P_i

1	**3000000**	**300000**	**0**	**300000**	**3000000**
2	**3000000**	**300000**	**0**	**300000**	**3000000**
3	**3000000**	**300000**	**0**	**300000**	**3000000**
4	**3000000**	**300000**	**0**	**300000**	**3000000**
5	**3000000**	**300000**	**3000000**	**3300000**	**0**

III.3.3. Ordinary loans repaid with a fixed amortization:

The debt is repaid periodically (annually, semi-annually, or monthly) according to this method, with payments that include a fixed portion of the loan principal and interest on the remaining loan balance each period.

III.3.3.1. The relationship between the elements of loan amortization is as follows:

- **Determining the value of the fixed amortization:**

It is calculated directly by dividing the original loan principal by the number of amortization.

$$\boxed{PP = \frac{P_0}{n}}$$

- **Determine the interest rate:**

$$\boxed{I_i = P_{i-1} \times r}$$

- **Determine the payment amount:**

$$\boxed{TPY_i = I_i + PP}$$

III.3.3.2. Loan amortization schedule:

According to this method, the loan amortization schedule is as follows:

i	P_{i-1}	I_i	PP	TPY_i	P_i
1	P_0	$I_1 = P_o \times r$	PP	$TPY_1 = I_1 + PP$	$P_1 = P_0 - PP$
2	P_1	$I_2 = P_1 \times r$	PP	$TPY_2 = I_2 + PP$	$P_2 = P_1 - PP$
3	P_2	$I_3 = P_2 \times r$	PP	$TPY_3 = I_3 + PP$	$P_3 = P_2 - PP$
⋮	⋮	⋮	⋮	⋮	⋮
n	P_{n-1}	$I_n = P_{n-1} \times r$	PP	$TPY_n = I_n + PP$	$P_n = P_{n-1} - PP$

Example:

A merchant borrowed 30000 M.U from a bank and committed to repay it in five annual installments using the fixed installment method, with the remaining interest paid at the end of each year at a rate of 6% .

required:

Prepare the loan amortization schedule.

Solution:

$$PP_i = \frac{P_0}{n} = \frac{30000}{5} = 6000\ M.U$$

i	P_{i-1}	I_i	PP_i	TPY_i	P_i
1	300000	1800	6000	7800	24000
2	24000	1440	6000	7440	18000
3	18000	1080	6000	7080	12000

4	**12000**	**720**	**6000**	**6720**	**6000**
5	**6000**	**360**	**6000**	**6360**	**0**

Unsolved exercises

Exercise 01 :

Borrowed a person from the bank an amount of 100,000 M.U and committed to repay it in 5 equal annual Payments. Knowing that the interest rate is 5% per year. Prepare an amortization table for this loan.

Exercise 02 :

A person borrowed an amount of 25324.5 M.U from the bank, with the condition to repay the principal and interest in equal Payments every six months for a period of 3 years. If the annual interest rate is 12% and the interest is compounded semi-annually, prepare an amortization schedule for this loan.

Exercise 03 :

Present the first five lines of an amortization schedule for a loan with fixed payments over a period of 10 years, given that the first payment is 43426.5 M.U and the third payment is 49719.000 M.U.

Exercise 04 :

A person invested an amount (x) at the beginning of each year for a period of 15 years, with a compound interest rate of 4.5%. At the end of the 15th year, the accumulated amount was borrowed by another person, who committed to repay it over 20 years in fixed annual payments, with the first payment due after 3 years from the loan date, at an interest rate of 5.5%. After repaying the 12th installment, the debtor and the creditor agreed to settle the remaining amount immediately and in one payment, which amounted to 28570.47. The required information is as follows:

- Calculate the fixed payment installment.
- Determine the borrowed principal amount.
- Find the investment payment value (x).

Exercise 05 :

From a regular loan amortization table with a repayment period of 5 years and annual interest rates, you have provided the following information:

- Second payment: 101250 M.U
- Fifth payment: 1226250 M.U

Required:

- Calculate the following:
- Loan principal

- The applied interest rate.
- Provide the loan amortization schedule.

Exercise 06 :

I have the following information about a regular loan that is repaid with annual interest:

Loan principal: 900000 M.U

Total interest: 432000 M.U

First payment: 72000 M.U

The required calculations are:

- Calculate the loan duration.
- Calculate the applied interest rate
- provide the loan amortization schedule.

Exercise 07 :

From a regular loan amortization schedule with a fixed installment, I have the following information:

Annual interest rate: 7.5 %

First-year installment: 250000 M.U

Total payments: 2275000 M.U

The required calculations are:

- Calculate the number of installments.
- Calculate the loan principal.
- Provide the loan amortization schedule.

Exercise 08 :

From a regular loan amortization schedule with a fixed installment, I have the following information:

Interest rate: 6 %

Sum of the first and third interest payments: 43200 M.U

Value of the installment: 120000 M.U

The required calculations are:

- Calculate the loan principal.
- Calculate the number of installments.

Exercise 09 :

The institution borrowed an amount of 1000000 to be repaid in fixed payments over 15 years, at a compound interest rate of 7% annually.

Required:

- Calculate the tenth consumption amount؛
- Calculate the repaid amount of the loan after paying the tenth payment؛
- Calculate the amount remaining to be repaid from the loan after paying the twelfth payment؛
- Calculate the interest involved in the final payment.

Exercise 10:

From an amortization table of a Ordinary loan repaid through 8 annual fixed payments, the following information has been extracted:

Year 2 interest: 18251.1

Year 3 interest: 16327.3

Remaining amount to be repaid in Year 1: 182511

Required:

- Loan rate
- First payment

- Fixed payment
- Loan principal
- Last row of the amortization table

Exercise 11:

A loan value of P_0 is repaid through 10 fixed annual payments. You have the following information extracted from the amortization table of this loan: M3 = 23460.22 M.U

M6 = 30381.61 M.U

Required: Calculate in order:

- Loan rate
- Borrowed capital
- Value of payment
- Remaining amount after the 7th payment
- Provide a portion of the amortization table related to the last three years.

Chapter 04

Loan Amortization

4.1.Basic Definitions

In practice, financial operations, as a rule, stipulate payments and cash inflows in- terspaced in time.

Let us call the sequence (range) of payments and inflows arranged for different time moments a *payment stream.*

The payment stream whose elements are positive values and time intervals be- tween two consequent payments are constant are called a *financial contract* or *annuity* ir- respectively of the purpose, destination and origin of these payments.

The annuity is described through the following parameters:

1. Annuity component – the value of each separate payment.
2. Annuity interval – time interval between payments.
3. Annuity term – time from the beginning of the annuity till the end of its last period (in- terval).
4. Interest rate – the rate used for accumulation or discounting of payments of which the annuity is composed.

Besides, there may be additional parameters: the number of annual payments, the number of annual interest calculations, payment moments (at the beginning or the end of annuity period) etc.

Annuity Types

An annuity is called annual if its period equals one year.

An annuity is called p-due, if its period is less than a year and the number of annual payments is p.

These annuities are discrete since their payments are coordinated with discrete time points. There are continuous annuities when the payment stream is characterized with the continuous function.

Annuities may be constant and variable. An annuity is constant if all its payments are equal and do not change in time. If the amounts of payment depend on time, the annuity is variable.

The annuity is certain (terminating) if the number of payments is finite; otherwise the annuity is called indefinite or perpetual. For example, a long-term commitment when the financial transaction term is continuous and is not agreed on in advance makes a per- petual annuity.

Annuities may be immediate or postponed (deferred). The term of immediate annui- ties starts from the moment of contract conclusion. If the annuity is postponed, then the starting date of payments is moved aside for a certain time.

If payments are made at the end of a period, then such an annuity is called ordinary or annuity-immediate. If payments are made at the beginning of a period, then such an annuity is called annuity-due.The annuity is *certain* (*terminating*) if the number of payments is finite; otherwise the annuity is called *indefinite* or *perpetual*. For example, a long-term commitment when the financial transaction term is continuous and is not agreed on in advance makes a per- petual annuity.

Annuities may be *immediate* or *postponed (deferred)*. The term of immediate annui- ties starts from the moment of contract conclusion. If the annuity is postponed, then the starting date of payments is moved aside for a certain time.

If payments are made at the end of a period, then such an annuity is called *ordinary* or *annuity-immediate*. If payments are made at the beginning of a period, then such an annuity is called *annuity-due.*

Generalized Characteristics of Payment Streams

To analyze payments streams, it is necessary to know how to calculate their gener- alized characteristics. There are two such characteristics: the accumulated sum and the present value of the annuity.

The *accumulated sum* of a payment stream is the sum of all the successive payments with their calculated interest by the end of the annuity.

The *present value* of a payment stream is the sum of all the payments discounted for a certain time point that coincides with the beginning of the payment stream or that predicts it.

The accumulated sum is determined, for instance, in order to be aware of the to- tal amount of indebtedness at a certain time point, the total volume of investments, money reserve accumulated by the evaluation moment etc.

The present value is a most important index when evaluating financial effectiveness of commercial transactions, real and financial investments, etc.

Let us consider the problems for determining the accumulated sums for various types of annuity-immediate.

4.2.The Accumulated Sum of the Annual Annuity

When calculating the formula for accumulated annuity sums it is important to cor- rectly indicate the moment of the consequent payment's coming and determine the term (in years) during which this payment will be charged with interest. After this, in order to de- termine the amount of payment with the interest calculated for this term, it is sufficient to use the formula for compound interest accumulation. This problem is solved most simply for the annual annuity with interest calculation once a year. Such an annuity is called ordinary.

Let

S – accumulated sum of annuity؛

R – amount of individual payment؛

i – interest rate in form of a decimal fraction؛

n – annuity term in years.

Let us consider the process of payment accumulation for the annuity-immediate.

The first payment of the R amount comes at the end of the first period, it will be charged with interest at the rate i for years (that much money is left from the end of the first period to the term end); as a result, the amount $R(1 \square i)^{n \square 1}$ will be obtained at the end of the term. The second payment will come in at the end of the second year and the amount $R(1+i)^{n-2}$ will be obtained at the end of the term. With this process continued, at the term end we will obtain the sum $R(1+i)$ for the payment coming at the end of the $(n\text{-}1)$ year. The last payment will come in at the end of the n year, it will not be charged with interest.

To determine the accumulated value of the annuity S, it is necessary to sum the payment series obtained, i.e. $S = R(1+i)^{n-1} + R(1+I)^{n-2} + ... + R(1+I) + R$ or

$$S = R\sum_{t=0}^{n-1}(1+I)^{t} . \qquad (2.1)$$

We should note that this series makes a geometric sequence with the first member R and denominator that equals $(1+i)$, and the number of members n.

Let us remind that the geometric sequence is a series whose each member comes out of the preceding one by multiplication by one and the same number.

The formula for the sum n of the geometric sequence members is

$$S_n = a_1(q^n - 1)/(q-1), \tag{2.2}$$

where a_1 – the first sequence member, q – the sequence denominator.

Using this formula from (1.1) we will obtain a relation for calculating the accumulated sum of the annual annuity:

$$S = R\frac{(1+i)^n-1}{(1+i)-1} = R\frac{(1+I)^n-1}{i}.$$

$$= \frac{(1+i)^n-1}{i}$$

The value

$$s(n,i) = \sum_{t=0}^{n-1}(1+i)^t$$

is called the annual annuity accumulation factor.

Using the value introduced, we may write the expression for the accumulated sum of the annual annuity the following way:

$$S = Rs(n,i).$$

If payments come in at the beginning of the term, the annuity accumulation factor equals

$$s(n,i) = \sum_{t=1}^{n}(1+i)^t = (1+i)\cdot\frac{(1+i)^n-1}{i}.$$

In the case with simple interest, the accumulated sum of payments equals:

$$R\cdot[1+(n-1)\cdot i]+R\cdot[1+(n-2)\cdot i]+...+R\cdot[1+i]+R$$

$$=R\left[n+i\cdot(1+2+...+(n-1))\right]=R\left[n+i\frac{n\cdot(n-1)}{2}\right]=R\cdot s(n,i),$$

where

$$s(n,i)=\left[n+i\frac{n\cdot(n-1)}{2}\right]$$

We have a geometric sequence with the first member R and sequence denominator that Equals $(1+\frac{j}{m}).^{m}$ there are n series members. Using Formula (2.2) for the sum of the geomet- ric sequence members, we will obtain

$$S=R\frac{\left(1+\frac{J}{m}\right)-1^{m,n}}{(1+\frac{j}{m})^{m-1}}$$

Let us indicate the following value as the accumulation factor

$$s(n,j/m)=\sum_{t=0}^{n-1}(1+\frac{j}{m})^{mt}=\frac{(1+\frac{j}{m})^{m\cdot n}-1}{(1+\frac{j}{m})^{m}-1}.$$

Then the accumulated sum is

$$S=Rs(n,j/m).$$

For the annuity-due the annuity accumulation factor equals

$$s(n,j/m)=\sum_{t=1}^{n}(1+\frac{j}{m})^{mt}=(1+\frac{j}{m})^{m}\frac{(1+\frac{j}{m})^{m\cdot n}-1}{(1+\frac{j}{m})^{m}-1}.$$

4.3.Accumulated Sum of – due Annuity p

Let the payments come in p times a year with equal amounts, and the interest be calculated once at the end of the year $(m=1)$.

If R is the annual sum, then a single payment equals R/p. Since there are p payments per year, the interval between them will equal $1/p$ years. The first payment will come in at the time moment $1/p$. Arguing analogously, we will obtain for the accumulated annuity value

$$\boldsymbol{S = \frac{R}{P}\sum_{t=1}^{np}(1+i)^{t1lp)}}$$

We have a geometric sequence whose first member equals R/ p , and the denominator is

$(1+i)^{1\ p}$, with the total number of members n p.

Using the formula for the sum of the geometric sequence members (2.2), we will

obtain

$$S = \frac{R}{p}\frac{((1+i)^{1/p})^{np}-1}{(1+i)^{1/p}-1} = R\cdot s(p,n,i),$$

where $s^{(p)}$ the accumulation factor of the p -due annuity with interest calculation once a year, which equals

$$s(p,n,i) = \frac{1}{p}\sum_{t=0}^{np-1}(1+i)^{t(1/p)} = \frac{(1+i)^{n}-1}{((1+i)^{1/p}-1)}.$$

If payments come in at the beginning of the period, then for the accumulation factor of the p -due annuity-due we will obtain:

$$s(p,n,i) = \frac{1}{p}\sum_{t=1}^{np}(1+i)^{t(1/p)} = (1+i)^{1/p}\frac{(1+i)^{n}-1}{p\cdot((1+i)^{1/p}-1)}.$$

4.4. Accumula

$m \neq 1$

The parameters of such an annuity are: p payments per year and interest calculation m times a year at the nominal rate j. The annuity with such conditions is called *total*. The principle of obtaining the formulas for the accumulated sum is analogous to the cases men- tioned above.

The first payment equals R / p , the moment of this payment's receipt (the time point from the annuity term beginning) equals 1/ p and at the end of the annuity term this payment will give a sum with interest

$$\frac{R}{p}(1+\frac{j}{m})^{(n-\frac{1}{p})m}$$

The second payment with calculated interest will equal

$$\frac{R}{P}(1+\frac{j}{m})^{(n-\frac{2}{p})m}$$

The last but one payment will give a sum of

$$\frac{R}{p}(1+\frac{j}{m})^{(n-\frac{np-1}{p})m}=\frac{R}{p}(1+\frac{j}{m})^{\frac{m}{p}}.$$

The last payment will not be charged with interest and will equal R / p . Let us sum the members of the series obtained with the use of (2.21) for the sum of the geometric sequence members with the following parameters:

the first sequence member equals $\frac{r}{p}$ the denominator is $(1+\frac{j}{m})^{\frac{m}{p}}$ the number of sequence members being np . We will obtain

$$S=\frac{R}{P}\sum_{t=0}^{np-1}(1+\frac{j}{m})^{t(m\,p)}=R\frac{(1+\frac{j}{m})^{mn}-1}{p(1+\frac{j}{m})^{p}-1}$$

Let us indicate

$$s(p,n,j/m)=\frac{1}{p}\sum_{t=0}^{np-1}(1+\frac{j}{m})^{t(m/p)}=\frac{(1+\frac{j}{m})^{mn}-1}{p((1+\frac{j}{m})^{\frac{m}{p}}-1)}$$

the accumulation factor of the p – due annuity with interest calculation mtimes a year. Then the formula for the accumulated sum will be

S = Rs(p,n, j /m) .

Making analogous arguments, for the annuity-due accumulation factor we will have an expression

$$s(p,n,j/m)=\frac{1}{p}\sum_{t=1}^{np}(1+\frac{j}{m})^{t(m/p)}=(1+\frac{j}{m})^{\frac{m}{p}}\frac{(1+\frac{j}{m})^{mn}-1}{p((1+\frac{j}{m})^{\frac{m}{p}}-1)}.$$

4.5.The Present Value of the Ordinary Annuity

The present value of the annuity is a most important property of the payment stream that determines the value of the future money stream at the present time point. This prop- erty serves the base for many methods of financial analysis. By definition, the present val- ue is the sum of all the discounted members of the payment stream at the initial or its preceding time point. Sometimes, instead of the term the present value, the terms reduced or capitalized sum of payments. When determining the present value of the payment stream, it is important to correctly determine the time period from the beginning of the stream (time point for which evaluation is made) to the moment of the payment’s receipt (in years). Af- ter this, discount formulas may be applied. Let us indicate v – the discount multiplier,

$v=\frac{1}{1+i}$, where i is the annual rate.

The discounting process is as follows. The first payment of an amount of R comes in at the end of the first year, its present value at the term beginning equals Rv. The second payment comes in at the end of the second year, its present value equals Rv^2 . Continuing this process, we will obtain the present value Rv^n of the last payment that must come in at the term end.

To obtain the present value of the stream, let us sum all the members of the result- ing series. It is obvious that we have a geometric sequence with the features: the first member of the sequence equals Rv ; the sequence denominator – v ; the number of se- quence members – n .

As a result we obtain

$$A = R\sum_{t=1}^{n} v^t = R\sum_{t=1}^{n} \frac{1}{(1+i)^t}.$$

The value

$$a(n,i) = \sum_{t=1}^{n} \frac{1}{(1+i)^t} = \frac{1-(1+i)^{-n}}{i}$$

is called the *annual annuity reduction factor*. Expression (2.20) is obtained with the consid- eration of Formula (2.2).

The diagram on Figure 2.4 illustrates the process of discounted payment stream formation at the beginning of the annuity term. The indices of the degrees — time peri- ods in years from the beginning of the annuity to the moment of the payment's receipt.

If payments are received at the beginning of the period, the annuity reduction factor equals

or

$$A = R \cdot a(n,i),$$

where

$$a(n,i) = \sum_{k=1}^{n} \frac{1}{1+i \cdot k}$$

by the annual annuity reduction factor at the simple interest rate.

4.6.The Present Value of the Annual Annuity with Interest Calculation m Times a Year

Let us insert the multiplier$(1+\frac{j}{m})^{-m}$ into the obtained formula for the present value of the annual annuity instead of the discount multiplier $(1+i)^{-1}$ We will obtain.

Let us indicate $$A = R\frac{1-(1+j/m)^{-mn}}{(1+j/m)^m - 1}.$$

$$a(n, j/m) = \sum_{t=1}^{n} \frac{1}{(1+j/m)^{mt}} = \frac{1-(1+j/m)^{-mn}}{(1+j/m)^m - 1},$$

Then A = Ra(n, j / m) .

For the annuity-due the reduction factor equals

$$a(n, j/m) = \sum_{t=0}^{n-1} \frac{1}{(1+j/m)^{mt}} = (1+j/m)^m \frac{1-(1+j/m)^{-mn}}{(1+j/m)^m - 1}.$$

4.7.The Present Value of the p – Due Annuity $(m=1)$

The interval between payments of such an annuity equals $1/p$, the amount of payment is R/p. Let us consider the order of forming discounting payments. The present value of the first payment equals $\frac{R}{p}v^{1/p}$, of the second payment - $\frac{R}{p}v^{2/p}$ etc., until the last payment $\frac{R}{p}v^{n/p}$.

We have a geometric sequence with the number of members that equals np, the first sequence member is $\frac{R}{P}V^{1P}$ the sequence denominator - V^{1P} Using Formula

for the sum of the geometric sequence members, we will obtain

$$A=\frac{R}{p}\frac{v^{\frac{1}{p}}(v^{\frac{1}{p}np}-1)}{v^{\frac{1}{p}}-1}=\frac{R}{p}\frac{(v^{n}-1)}{(1-\frac{1}{v^{1/p}})}=R\frac{1-(1+i)^{-n}}{p((1+i)^{1/p}-1)}.$$

Let us indicate

$$a(p,n,i)=\frac{1}{p}\sum_{t=1}^{np-1}\frac{1}{(1+i)^{t/p}}=\frac{1-(1+i)^{-n}}{p((1+i)^{1/p}-1)}$$

– the reduction factor of the p – due annuity.

Then the formula for calculating the present value of the p – due annuity will be

$$A=Ra(p,n,i)$$

For the ‹

$$a(p,n,i)=\frac{1}{p}\sum_{t=0}^{np}\frac{1}{(1+i)^{t/p}}=(1+i)^{1/p}\frac{1-(1+i)^{-n}}{p((1+i)^{1/p}-1)}.$$

2.8. The Present Value of the p – Due Annuity with m≠ 1, p ≠ m

In this case, the discounting factor is $v=\frac{1}{1+j\,m}$. Then the first discounted payment equals$\frac{R}{P}V^{m\,p}$ the second discounted payment equals$\frac{R}{P}V^{2mp}$ the last but on *discounted payment is* $\frac{R}{P}V\,\frac{(np-1).^{m}}{p}$

and, finally, the last payment being$\frac{R}{P}$ V^{mn} We should remind here that the exponent of the discount multiplier is the time interval (measured in years) from the beginning of the annuity to the moment of the payment's receipt with adjustments for m–interest calculation once a year. We have a geometric sequence with the number of members np , whose first member is$\frac{R}{P}V^{mp}$ and the sequence denominator is V $^{m\ p}$ The sum of the members of this sequence is

$$A=\frac{R}{p}\frac{v^{m/p}(v^{mn}-1)}{v^{m/p}-1}=\frac{R}{p}\cdot\frac{1-(1+j/m)^{-mn}}{(1+j/m)^{m/p}-1}.$$

Let us indicate

$$a(p,n,j/m)=\frac{1}{p}\sum_{t=1}^{np}\frac{1}{(1+j/m)^{t(m/p)}}=\frac{1-(1+j/m)^{-mn}}{p((1+j/m)^{m/p}-1)}$$

– the ordinary annuity reduction factor.

Then it is definitely that the formula for the present value of this annuity is A = Ra(p, n, j / m)

In practice, the number of interest calculations and payments on them (or on the receipts) often coincides, i.e. p = m. Then

$$a(p,n,j/m) = \frac{1}{p}\frac{1-(1+j/m)^{-mn}}{j/m}.$$

For the annuity-due, instead of (2.30) we will obtain:

$$a(p,n,j/m) = \frac{1}{p}\sum_{t=0}^{np-1}\frac{1}{(1+j/m)^{t(m/p)}} = (1+j/m)^{m/p}\frac{1-(1+j/m)^{-mn}}{p((1+j/m)^{m/p}-1)}.$$

4.10.The Relation between the Accumulated and Present Values of Annuity

Let A be the present value of the annual annuity for the beginning of the term with interest calculation once a year, S – the accumulated sum of this annuity. Then it is possible to show that $A(1+i)^n = S$, i.e. interest calculation on the sum A for n periods gives an accumulated

sum of the annuity. Indeed,

$$A(1+i)^n = R\frac{1-(1+i)^{-n}}{i}(1+i)^n = R\frac{(1+i)^{+n}-1}{i} = Rs(n,i) = S.$$

Besides, it is obvious that we have another range of relations:

$$Sv^n = A\text{, where } v = \frac{1}{1+i},$$

$$a_{n,i}(1+i)^n = s(n,i)\text{, } s(n,i)v^n = a(n,i)\text{ .}$$

4.11.Determining Annuity Parameters

In this section, let us consider several problems that arise in connection with the notion of annuity.

1. Determining the amount of payment of the annuity (member).
2. Determining the term of the annuity.
3. Determining the interest rate.

In order not to overload this topic's presentation by lengthy computations and formulas, let us consider the most simple annuity type, in particular, the annual annuity with interest calcula- tion once a year. For other annuity types the solutions will be analogous.

Determining the Amount of Payment

It is obvious that if the accumulated sum is given, then $R = \frac{S}{S(n\ i)}$

If the present value of the annuity is given, then $R = \frac{A}{a(n\ i)}$

Determining the Term of the Annuity

If the accumulated sum S is known, then the term n is determined from the solution of
this equation:

$$R = \frac{(1+i)^n - 1}{i} = s$$

After simple transformations we will have

$$(1+i)^n = \frac{s}{R}i + 1$$

Taking the logarithm of the right and left parts and expressing out of the obtained relation n , we will have

If the reduced value of anr we obtain analogously

$$n = \frac{\ln(1+\frac{S}{R}i)}{\ln(1+i)}$$

$$n = -\frac{1n(1 - \frac{A}{R}\,i)}{1n\,(1 + i)}$$

Determining the Term of the Annuity

The estimated value of the rate is of importance in the financial and economic analysis when determining the financial and commercial transaction yield.

If the accumulated sum is known, to determine the rate, it is necessary to solve the equation in relation to the unknown value I

$$S=R\frac{(1+i)^{n}-1}{i}$$

If the present value of the annuity is known, then we solve the equation

$$A=R\frac{(1+i)^{-n}}{i}$$

We have nonlinear equations, whose accurate solution (as a calculation formula for i) is not possible. The data of the equation allow only an approximate solution with the use of special numerical techniques, for instance, Newton's method that is easily realized with the help of such tabular processing units as Microsoft Excel.

4.12.Annuity Conversion

Converting the annuity means changing the conditions of the financial contract that stipu- lates payment of this annuity.

Simple types of conversion

To simple types of conversion belong:

1. Annuity buyout – a replacement of the annuity with a lump sum payment. It follows from the principle of financial equivalency that in this case, instead of the annuity, its present value is paid.
2. Payment by instalments – a replacement of a lump sum payment with an annuity.

In this case, the present value of the annuity must be equated with the value of the substi- tutable payment and either the amount of payment of such an annuity at the given term or the annuity term (if the amount of payment is known) must be determined. The data of the problem reduce themselves to the problems of determining the annuity parameters that were considered earlier (see Subsection 2.11).

Compound types of conversion

Compound types of conversion include the replacement of one annuity with another that means changing annuity parameters. It follows from the conditions of financial equiva- lency that at such a replacement, the present values of these annuities must be equal. In other words, if $A1$ is the present value of the annuity to be replaced,, $A2$ *is the present value of the replacing annuity at the same time point, then the following condition must be observed:* $A1 = A2$.

Ex ample s of conve r sion

Replacing the immediate annuity with the deferred one.

Let there be an annual immediate annuity with the parameters of R_1, n_1, i . This annuity is replaced with the one deferred for t years with the term n_2 and rate i . The present problem is reduced to determining the amount of payment of R_2 of the second annuity. From the condition of financial equivalency we have an equation

$$R_1 a(n_1, i) = R_2 a(n_2, i) v^t ,$$

where v – the discount multiplier at the rate i , from which we obtain

$$R_2 = R_1 \left(\frac{a(n_1, i)}{a(n_2, i)} \right) (1+i)^t .$$

If interest is calculated m times a year, then the problem is solved by analogy. Let us suppose further that $R_1 = R_2$, and since the annuity is deferred, its term also changes that must also be determined. Solving the equation of equivalency (2.44) in relation to n_2 , we will obtain

$$n_2 = \frac{-\ln\left\{1 - \left[1 - (1+i)^{-n_1}\right](1+i)^t\right\}}{\ln(1+i)} .$$

Changing the duration and the term structure of annuity. Let us consider an ordinary annual annuity with the parameters R_1,n_1,i . This annuity is replaced with another one whose parameters are R_2,n_2

I Let us suppose that the payment R_2 is known, it is necessary to determine the term n_2 . Making the present values of annuities equal, we will obtain the equation

$$R_1a(n_1,i) = R_2a(n_1,i)$$

from which we find the term of the second annuity n_2 . This problem is solved analogously with the problem of determining the annual annuity term (see Subsection 2.11). If the term is known, then it is possible to estimate R_2 from the same equation

Let us consider this problem in the case when the term structure changes. Let the p – due annuity with the parameters $p1,R1,n1,i1$ be replaced with the p -due annuity with the parameters p_2,R_2,n_2,i_2 . It is necessary to determine the parameter of the second annuity the amount of payment R_2 . From the equation condition of the payment stream present values we have

$$R_1a(p_1,n_1,i) = R_2a(p_2,n_2,i)$$

from which R2 is easily determined. If R2 is known, then by solving this equation it is also
possible to determine n2 .

3. Annuity integration.

Let us suppose that k annual annuities with interest calculation once a year, whose parameters are known, are replaced with a single annual annuity. Let A be the present value of the replacing annuity; Aq, q =1,2,...k – the present values of the substitutable annuities. Then the condition of financial equivalency is written as: $A=\sum_{q=1}^{k} Aq$

If the moments of annuity beginning do not coincide, the present values of these annuities are determined for the starting moment of the earliest annuity. When integrating annuities two tasks arise:

1) determining the payment amount of the replacing annuity if its term is

given;

2) determining the term of the replacing annuity with other parameters given.

Since the present value of the replacing annuity is known, both these problems are reduced to the corresponding problems of determining the annual annuity parameters

If the accumulated sum S is known, then the term n is determined from the solution of this equation:

$$R\frac{(1+i)^n-1}{i}=S$$

After simple transformations we will have

$$(1+i)^n=\frac{S}{R}i+1$$

Taking the logarithm of the right and left parts and expressing out of the obtained relation n, we will have

$$n=\frac{\ln(1+\frac{S}{R}i)}{\ln(1+i)}. \qquad (2.34)$$

If the reduced value of annuity is known, we obtain analogously

$$n=-\frac{\ln(1-\frac{A}{R}i)}{\ln(1+i)}. \qquad (2.35)$$

Determining the interest rate

The estimated value of the rate is of importance in the financial and economic analysis when determining the financial and commercial transaction yield.

If the accumulated sum is known, to determine the rate, it is necessary to solve the equation in relation to the unknown value i

$$S=R\frac{(1+i)^n-1}{i}. \qquad (2.36)$$

If the present value of the annuity is known, then we solve the equation

$$A=R\frac{1-(1+i)^{-n}}{i}. \qquad (2.37)$$

We have nonlinear equations, whose accurate solution (as a calculation formula for i) is not possible. The data of the equation allow only an approximate solution with the use of special numerical techniques, for instance, Newton's method that is easily realized with the help of such tabular processing units as Microsoft Excel.

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In this case, the present value of the annuity must be equated with the value of the substitutable payment and either the amount of payment of such an annuity at the given term or the annuity term (if the amount of payment is known) must be determined. The data of the problem reduce themselves to the problems of determining the annuity parameters that were considered earlier (see Subsection 2.11).

Compound types of conversion

Compound types of conversion include the replacement of one annuity with another that means changing annuity parameters. It follows from the conditions of financial equivalency that at such a replacement, the present values of these annuities must be equal. In other words, if $A1$ is the present value of the annuity to be replaced,, $A2$ *is the present value of the replacing annuity at the same time point, then the following condition must be observed:* $A1 = A2$.

Examples of conversion

1. Replacing the immediate annuity with the deferred one.

Let there be an annual immediate annuity with the parameters of R_1, n_1, i. This annuity is replaced with the one deferred for t years with the term n_2 and rate i. The present problem is reduced to determining the amount of payment of R_2 of the second annuity. From the condition of financial equivalency we have an equation

$$R_1 a(n_1, i) = R_2 a(n_2, i) v^t,$$

where v – the discount multiplier at the rate i, from which we obtain

$$R_2 = R_1 \left(\frac{a(n_1, i)}{a(n_2, i)} \right) (1+i)^t.$$

If interest is calculated m times a year, then the problem is solved by analogy.

Let us suppose further that $R_1 = R_2$, and since the annuity is deferred, its term also changes that must also be determined. Solving the equation of equivalency (2.44) in relation to n_2, we will obtain

$$n_2 = \frac{-\ln\left\{1 - \left[1-(1+i)^{-n_1}\right](1+i)^t\right\}}{\ln(1+i)}.$$

2. Changing the duration and the term structure of annuity.

Let us consider an ordinary annual annuity with the parameters R_1, n_1, i. This annuity is replaced with another one whose parameters are R_2, n_2, i

Let us suppose that the payment R_2 is known, it is necessary to determine the term n_2. Making the present values of annuities equal, we will obtain the equation

$$R_1 a(n_1, i) = R_2 a(n_1, i),$$

from which we find the term of the second annuity n_2. This problem is solved analogously with the problem of determining the annual annuity term (see Subsection 2.11). If the term is known, then it is possible to estimate R_2 from the same equation.

Let us consider this problem in the case when the term structure changes. Let the p–due annuity with the parameters p_1, R_1, n_1, i_1 be replaced with the p-due annuity with the parameters p_2, R_2, n_2, i_2. It is necessary to determine the parameter of the second annuity – the amount of payment R_2. From the equation condition of the payment stream present values we have

$$R_1 a(p_1, n_1, i) = R_2 a(p_2, n_2, i),$$

from which R_2 is easily determined. If R_2 is known, then by solving this equation it is also possible to determine n_2.

3. Annuity integration.

Let us suppose that k annual annuities with interest calculation once a year, whose parameters are known, are replaced with a single annual annuity. Let A be the present value of the replacing annuity; $A_q,\ q = 1, 2, ...k$ – the present values of the substitutable annuities. Then the condition of financial equivalency is written as:

$$A = \sum_{q=1}^{k} A_q .$$

If the moments of annuity beginning do not coincide, the present values of these annuities are determined for the starting moment of the earliest annuity. When integrating annuities two tasks arise:

1) determining the payment amount of the replacing annuity if its term is given;

2) determining the term of the replacing annuity with other parameters given.

Since the present value of the replacing annuity is known, both these problems are reduced to the corresponding problems of determining the annual annuity parameters.

Questions for Self-test

1. What is the financial contract? What parameters is it described with?
2. List financial contract types.
3. Give the definition of accumulated sum and present value of payment stream.
4. Determine the formula for accumulated sum of payment stream.
5. Determine the formula for accumulated sum of payment stream with interest calcu- lations m times a year.
6. Determine the formula for accumulated sum of p - due annuity.
7. Determine the formula for accumulated sum of p - due annuity with interest calcula- tions m times a year.
8. Determine the formula for the present value of payment stream.
9. Determine the formula for the present value of payment stream with interest calcula- tions m times a year.
10. Determine the formula for the present value of p - due annuity.
11. Determine the formula for the present value of p - due annuity with interest calcula- tions m times a year.
12. Write the connection between the factors of annuity accumulation and annuity dis- counting factor.
13. How to determine the amount of payment, annuity period and amount of interest rate?
14. Write the formula for accumulated value of discrete annuity with continuous interest calculation.

15. Write the formula for the present value of discrete annuity with continuous interest calculation.
16. Write the formula for accumulated value of continuous annuity with discrete interest calculation.
17. How should the immediate annuity be replaced with the deferred one?
18. How should one annual annuity be replaced with another?

Chapter 05 Financial Transaction Yield

A transaction is called financial if its beginning and ending have a money valuation — $P(0)$ and $S(T)$ correspondingly and the purpose of its performance is in maximizing the difference $S(T) - P(0)$ or another similar index. A most important property of the trans- action is its yield.

In the definition, $P(0)$ is the really invested funds at the moment $t = 0$, $S(T)$ — monetary funds really returned as the result of a transaction whose term is T time units. It is natural to measure the effect from investing in the form of the interest accumulation rate, which in this case is called the yield.

5.1.The Absolute and Average Annual Transaction Yield

Two types of financial transaction yield are distinguished - absolute and average annual.

Absolute yield d *(*the yield for all the term) of the transaction is determined from the equation $P(0)(1+d)=S(T)$ or $d=(S(T)-P(0)/P(0)=S(T)/P(0)-1.$ The value $S(T)/P(0)$ is called *accumulationfactor* or *multiplier.* It is obvious that $S(T)/P(0)=1+d$.

Average annual yield r of the financial transaction is the simple or compound inter- est rate that is used to measure the effectiveness of a financial transaction.

By definition, the average annual financial transaction yield is the positive value r that satisfies the equality:

$$P(0)(1+r\cdot T)=S(T) \quad (3.1)$$

or

$$P(0)(1+r)^T = S(T). \tag{3.2}$$

From (3.1) and (3.2) we find the connection between d and r:

$1+r\cdot T = 1+d$ (for the simple rate) и

$(1+r)^T = 1+d$ (for the compound rate).

Thus, the financial transaction is associated with an equivalent transaction of accumulating the sum $P(0)$ at the rate r within the time T. Such an approach enables comparing the obtained yield value with the yields on alternative invested funds.

5.2.Tax and Inflation Accounting

Taxes and inflation remarkably influence the effectiveness of the financial transaction. Let us analyze tax accounting. The tax is charged, as a rule, on the interest acquired when placing an amount of money into growth. Let us assume that the sum P_0 within the time n was charged with interest at the rate i, g - is the tax rate on interest. Then the value of interest is

$$I(n) = S_n - P_0,$$

and the amount of tax is $G_n = g\cdot I(n)$. The accumulated sum after paying the tax makes

$$S(n) = S_n - G_n.$$

Since $S(n) < S_n$, then tax accounting actually reduces the accumulation rate. Hence,

$$S(n) = S_n - G_n = S_n - g\cdot I(n) = S_n - g\cdot(S_n - P_0) = S_n(1-g) + gP_0$$

If i is the simple interest rate, then $S_n = P_0(1+i\cdot n)$. Thus,

$$S(n) = P_0(1+i\cdot(1-g)n).$$

We see that in the reality, accumulation is made according to the rate $i(1-g) < i$.

If i is the compound interest rate, then $S_n = P_0(1+i)^n$. Thus,

$$S(n) = P_0\left((1+i)^n(1-g)+g\right).$$

Example 3.1. When granting a credit for 2 years with an annual compound interest rate of 0.08, the creditor retains the commission fee at the rate of 0.5% of the credit amount. The tax amount on interest is 10%. What is the transaction yield for the creditor?

If P_0 is the credit amount, and S_n is the sum of the dischargeable debt, then $S_n = P_0(1+i)^n$, where i = 0.08 , n = 2. The commission fee cP_0, where c = 0.005. Then the sum actually issued as a loan will make $P(0) = P_0(1-c)$. After paying the tax, the creditor will have $S(n) = P_0\left((1+i)^n(1-g)+g\right)$, where g = 0.1 is the tax rate. The equation for the yield is $S(n) = P(0)(1+r)^n$. When solving this equation in accordance with r, we will obtain

$$r = \left(\frac{S(n)}{P(0)}\right)^{\frac{1}{n}} - 1 = \left(\frac{(1+i)^n(1-g)+g}{1-c}\right)^{\frac{1}{n}} - 1 = 0.07496.$$

It must be noted that without tax accounting (g = 0) the transaction yield would be 0.08271.

Inflation – depreciation of money that is revealed in the growth of the prices for goods and services that is followed by the reductions of purchasing capacity of money.

Inflation is characterized by two quantitative indices – price index and inflation rate. Let us suppose that a time unit is chosen. Let us consider a time segment $[0, n]$, whose length is t time units from the starting moment t = 0.

The price index for the time $[0,n]$ is the number

$$J(n) = \frac{K(n)}{K(0)},$$

that indicates the number of times by which the cost of the consumer goods basket has grown for the time period $[0,n]$.

The inflation rate for the time $[0,n]$ is the number

$$H(n) = \frac{K(n)-K(0)}{K(0)},$$

indicating by how much percent the cost of the consumer goods basket has grown in the time period $[0,n]$. Since $H(n) = \frac{K(n)}{K(0)} - 1$, then the relations between the inflation rate and the price index are:

$$H(n) = J(n) - 1$$

And

$$j_k = 1 + h_k, \quad k = 1,2,...,m,$$

and the price index for $[t_{k-1}, t_k]$ equals

$$j(t_{k-1}, t_k) = j^{(t_k - t_{k-1})} = (1+h_k)^{(t_k - t_{k-1})}, \quad k = 1,2,...,m.$$

By the price index definition we have

$$J(n) = j_1^{t_1} \cdot j_2^{(t_2 - t_1)} \cdots j_m^{(t_m - t_{m-1})}.$$

Then

$$J(n) = (1+h_1)^{t_1} \cdot (1+h_2)^{(t_2 - t_1)} \cdots (1+h_m)^{(t_m - t_{m-1})},$$

$$1+H(n)=(1+h_1)^{t_1}\cdot(1+h_2)^{(t_2-t_1)}\cdots(1+h_m)^{(t_m-t_{m-1})}.$$

If $h_1=h_2=...=h_n=h$, то

$$J(n)=(1+h)^n$$

$$1+H(n)=(1+h)^n .$$

Here h is the inflation rate for 1 time unit at the time segment $[0,n]$, $J(n)$ и $H(n)$ - the price index and inflation rate for the time period $[0,n]$.

Let us suppose that an accumulated sum of the deposit S_n is acquired for n time units. The price index for the period $[0,n]$ has grown to the value $J(n)$. Then the real amount of the deposit as the result of the decrease in the purchasing capacity of money will be

$$S(n)=\frac{S_n}{J(n)} .$$

The price index $J(n)$ is calculated by one of the formulas given above: (3.5) or (3.7) depending on the initial data. Since $J(n)>1$, then $S(n)<S_n$, which means the actual decrease in the accumulation rate.

Example 2.2. The expected annual inflation rate of the first two years of the deposit is 3%, and of the next three – 4%. What minimum annual compound interest rate should the bank offer the client in order that the real annual yield of the deposit be no less than 8% ?

Here $t=0$ - is the moment of placing the deposit, 1 year – is the unit of measuring time, the deposit term n = 5 years. h_1 = 0.03 и h_2 = 0.04 – the average annual inflation rates on the time segments [0,2], [2,5]. For the yield on the deposit r, the condition $r \geq 0.08$ must be fulfilled. Let i be the annual compound interest rate, at which the sum P_0 is deposited. Then the accumulated sum of the deposit after n years $S_n = P_0(1+i)^n$. Adjusted for inflation, the real sum of the deposit will equal $S(n) = \frac{S_n}{J(n)}$, where the price index, according to (3.8), equals $J(t) = (1+h_1)^2 \cdot (1+h_2)^3$. The yield equation is: $S(n) = P(0)(1+r)^n$. Solving this equation in accordance with r and taking the required condition for the yield into consideration, we will obtain:

$$r = \frac{1+i}{(1+h_1)^{\frac{2}{5}}(1+h_2)^{\frac{3}{5}}} - 1 \geq 0.08.$$

Hence $i \geq$0.11887. It means that the minimum interest rate of placing the deposit makes 0.11887 versus 0.08 without inflation accounting.

5.3.Payment Stream and its Yield

Let $\{R_k, t_k\}$ be the payment stream with t_k – time points, and R_k – payments. Let us say that the stream under consideration has the present value A at the yield level j, if $\sum_k R_k/(1+j)^{t_k} = A.$ If the stream is the annual annuity with an annual payment R and the duration n, then the annuity has the present value A at the yield level j, if $R \cdot a_{n,j} = A$. Let us fix $A,$ then with the increase in R the yield of the annuity increases. It is also possible to say it in another way: in order to increase the annuity yield, the annual payment must be increased.

All these considerations are especially well seen with the example of the perpetual annuity since for this $A = R/j$, or, otherwise: the perpetual annuity yield is $j = R/A$. It is important to note that the yield of the payment stream defined this way does not depend on the interest rate, but on the value and moments of payments themselves, due to which it is often called the domestic return of the payment stream.

More precisely, the domestic return of the payment stream is such its yield in the recently defined sense at which the present value of this stream equals zero (such is a characteristic is not of every payment stream). The equality $A=0$ is possible only then when there are negative values in the payment stream.

Example 3.1. A promissory note was discounted at rate of i =10% 160 days be- fore its maturity (the base annual number is 360 days). At the execution of the transaction a commission fee was retained from the note owner that equaled 0.5% of the principal amount of the note. Find the yield of the transaction

Solution. When calculating the yield of the promissory note, its face value does not often play a role. The absolute yield of the transaction without accounting the commission:

$$d = \frac{\quad}{P} - 1 = \frac{\quad}{N(1 - i \cdot m)} - 1,$$

where S, P – the final and initial cost of the promissory note; N - the face value of the promissory note; $m = 160/360$. Let us insert the given data, we will obtain: d = 0.046, i.e. d = 4.6%.

With consideration of the commission, the absolute yield equals:

$$d = \frac{S}{P} - 1 = \frac{N}{N(1 - i \cdot m - 0.005)} - 1 = 5.2\%$$

The transaction effectiveness (the relative yield), i.e. the yield in the interest per annum,

$(1.046)^{360/160} - 1 = 0.106$, i.e. 10,6% —, with commission,

$(1.052)^{360/160} - 1 = 0.1208$, i.e. 12,08% – without commission.

5.4.Instant Profit

Let at the moment t the capital is $K(t)$, and after some time Δt the capital equals $K(t + \Delta t)$, then the average yield d on the segment $[t, t + \Delta t]$ in interest per annum (in amounts) equals

$$K(t + \Delta t) / K(t) = (1 + d)^{\Delta t},$$

at the small Δt the value $(1 + d)^{\Delta t}$ with an accuracy to infinitely small of the second order equals $1 + d \cdot \Delta t$. Directing Δt to zero, we obtain

$$d = \lim_{\Delta t \to 0} [K(t + \Delta t) - K(t)]/[K(t) \cdot \Delta t] = K'(t) / K(t) = [\ln K(t)]'.$$

Thus, instant profit is a derivative with respect to time of the capital's base logarithm or, as they say, logarithmic derivative.

In particular, at the constant instant profit *d,* the capital grows in time on the exponent: $K(t) = K(0) \cdot e^{d \cdot t}$.

Example 3.2. The capital grows in time with a constant speed v, i.e. $K(t) = K_0(1 + vt)$. Find the instant profit at an arbitrary time point.

Solution. Let us indicate the unknown instant profit $d(t)$, then $d(t) = K'(t) / K(t) = K_0 v / K_0(1 + vt) = v / (1 + vt)$.

Thus, the yield changes with time. It is also clear – the capital gain per time unit is constant and equals $K_0 v$, and the capital itself grows.

Questions for Self-test

1. What is a financial transaction? What types of financial transaction yields are there and how are they determined?
2. How are the inflation rate and price index determined? How do taxes and inflation influence the yield?
3. How is the yield of the payment stream determined?

A borrowing, credit, loan are the earliest financial transactions. Creditum means ‘loan’ in Latin; the Russian word credit has a stress on the second syllable (credit with the first syllable stressed is the right member of accounting entries).

All three words —— borrowing, credit, loan —— mean one and the same — giving money or goods as a loan on conditions of recurrence and, as a rule, with interest payment. The one who sells money or goods on credit is called creditor, the one who takes — loan debtor (or debtor). The conditions for credit (borrowing, loan) issue and re- payment are quite versatile. Here, only the simplest and most widespread ways of repay- ing borrowings are considered.

6.1.Total Yield Index of a Financial and Credit Transaction

The income from financial and credit transactions and different commercial transactions may have a different form, in particular, it may be:

- interest on the loans issued,
- commission fees,
- return on bonds and other securities etc.

As a rule, one and the same transaction implies a few source of income: the loan brings the creditor interest and commission payments, besides the interest on the bond, the bond holder receives the difference between the buy-back price of the bond and its purchase price.

In connection with this, there arises a problem of measuring the effectiveness (yield) of the transaction with regard to all the income sources. The generalized characteristic of yield must be universal and applicable to all the financial transaction types.

The degree of financial effectiveness of such transactions is usually measured in the form of the annual compound interest rate. The given rate as an index of effectiveness (total yield, i.e. yield with regard to all the income sources stipulated in the transaction) is received on account of the general principle, in particular: all the income types discounted at the unknown rate (capitalized value of income), stipulated by the given transaction, are equated to the expenses reduced at the same rate and at the same time point. The unknown rate is determined from the

equation obtained. For the loan transaction it means the equivalence of the debit amount of the credit, i.e. the credit with the deduction of commission fees, to the sum of discounted inflows. The higher the rate, the higher the effectiveness of the transaction.

This interest rate does not appear in contracts directly and depending on the transaction type bears different titles: in loan operations – the term effective interest rate, in analysis of trans- action yield – yield at the repayment moment, in the analysis of industrial investments an analogous index is called the internal rate of return (intrinsic interest norm).

The base for calculating the total yield of the financial transaction is the relation called the balance of a financial and credit transaction.

6.2.The Balance of a Financial and Credit Transaction

The necessary condition for any financial and credit transaction is an equilibrium of in- vestments and return. Let us consider the notion of financial and balance transaction balance.

Let a credit of a K_0 amount be issued for a term T, the interest on the credit equals i. Let two payments whose amounts are R_1 and R_2 be made for all this term on account of dis- charging the debt, and at the term end the final amount R_3 is paid. All the term is divided into three periods with the duration t_1, t_2, t_3. For the time period t_1, the debt increases to the amount D_1, (since interest is charged on the credit amount). With termination of this time, thesum R_1 is deposited on account of repaying the credit and the amount of debt becomes equal with K_1. Then, for the time t_2, the sum K_1 will increase to the value D_2. With termination of the time interval t_2 the subsequent sum R_2 is paid, the amount of debt decreases and becomes equal with the sum K_2. Finally, for the time t_3 the amount of debt increases to the value D_3 and with the termination of this time interval, at the time point *T,* the sum R_3 is paid. For a balanced transaction, the amount of payment R_3 must be so that the debt be discharged. Figure 4.1 illustrates the described process as a graph that is called *financial and credit transaction con- tour*. A balanced transaction must have a closed contour.

Figure. 4.1. Financial and credit transaction contour

Mathematically, the process of repaying the debt may be described with an equation:

$$K_1 = K_0 q^{t_1} - R_1,\ K_2 = K_1 q^{t_2} - R_2,$$

$$K_2 q^{t_3} - R_3 = 0, \tag{4.1}$$

where $q = (1+i)$ – the accumulation factor at the compound rate.

The latter equation is called accounting equation and describes the condition of trans- action equilibrium.

Let us define K_2 through K_0 and insert the result into the accounting equation (4.1). We will obtain

$$K_2 = (K_0 q^{t_1} - R_1) q^{t_2} - R_2,$$

$$[(K_0 q^{t_1} - R_1) q^{t_2} - R_2] q^{t_3} - R_3 = 0,$$

$$K_0 q^T - (R_1 q^{t_2+t_3} + R_2 q^{t_3} + R_3) = 0, \tag{4.2}$$

Where $T = \sum_i t_i$.

From the latter equation it is obvious that the financial and credit transaction may be conventionally divided into two counter processes:

1) accumulation of the initial debt for all the time period;
2) accumulation of discharging payments for the period from the payment moment to the end of the transaction.

This equation may be transformed through multiplication by a discount factor

$$v^T = \left(\frac{1}{1+i} \right)^T .$$

As a result, we will obtain

$$K_0 - (R_1 v^{t_1} + R_2 v^{t_1+t_2} + R_3 v^T) = 0. \tag{4.3}$$

Thus, the sum of the present values of discharging payments at the moment of credit issue with the total equilibrium equals the credit amount.

For a general case n of discharging payments, the accounting equation will be analogous and as follows

$$K_0 q^{T\,n} \qquad \sum^{n} R_j q$$

$j=1$

$$_j = 0, \qquad (4.4)$$

where $T_j = \sum_{r=j+1} t_r$ - time from the payment moment R_j to the term end.

On the basis of accounting equations, it is possible to measure the financial and credit transaction yield. An accounting equation must be formed for this where accumulation or discounting is made at an unknown rate that characterizes the total yield; then this equation must be solved with regard to the rate sought for.

6.3.Determining the Total Yield of Loan Operations With Commission

The yield index of a loan operation without considering commission fees is the annual compound interest rate that is equivalent to the interest rate used in thi operation. The creditor often takes a commission fee for opening a credit and other services and this obviously increases the yield of the transaction for them because the sum actually issued decreases. Let the loan of the sum D be issued for the term n. At its issue, a commission of the G amount is retained. This means, the actually issued loan equals $(D - G)$. The deal stipulates charging simple interest at the rate i. The total yield rate will be defined by i_e. When determining the yield of this transaction as the annual compound interest rate, we proceed from the premise that the accumulation of the value $(D - G)$ at this rate must give the same result as the accumulation of the value D at the simple interest rate i. The equilibrium equation for this transaction is (see Figure 4.2)

$$(D-G)(1+i_e)^n = D(1+n \cdot i) . \qquad (4.5)$$

Рис 4.2. A graphic illustration of accounting equation

The higher the amount of the commission, the more effective the rate (total yield rate). Figure 3.2 graphically illustrates this equation.

Let $G = D \cdot g$, where g – commission interest. Then, when solving Equation (4.1) with regard i_e we will obtain

If the deal stipulates compound interest calculation, then the accounting equation will be

$$(D-G)(1+i_e)^n = D(1+i)^n \text{ ,} \tag{4.7}$$

from which an expression for the effective rate follows:

$$i_e = \frac{1+i}{(1-g)^{1/n}} - 1. \tag{4.8}$$

The effective rate i_e does not appear in the conditions for the transaction directly, it is fully determined by the interest rate on the credit and the value of commission.

Loans with Regular Interest Payments

If the commission is not paid, then the yield of such a transaction equals the compound interest rate that is equivalent to any rate applicable in this deal.

Let us suppose that commission is stipulated. Let the loan D be discharged after n years, and the interest on the simple rate i be paid regularly once at the end of the year. Then the amount of the interest paid equals $D \cdot i$. On account of the commission, the loan amount equals $D(1-g)$. Let us write the accounting equation:

$$D(1-g) - (D \cdot i \cdot a(n,i_e) + Dv^n) = 0 \text{ ,} \tag{4.10}$$

where $v = \frac{1}{1+i_e}$; $a(n,i_e) = \frac{1-(1+i_e)^{-n}}{i_e}$ - the annual annuity reduction factor.

This equation equals the following:

$$f(i_e) = v^n + i \cdot a(n,i_e) - (1-g) = 0 \text{ .} \tag{4.11}$$

We have a nonlinear equation that must be solved with regard to the variable i_e - the total yield rate. The accurate solution for this equation in the form of a calculation formula for i_e is impossible to obtain. The rate sought for may be determined relatively with the use of some numerical method.

If the interest on the credit is paid p times a year, then the equation to determine the transaction yield will be

$$f(i_e) = v^n + i \cdot a(p,n,i_e) - (1-g) = 0 \text{ ,} \tag{4.11a}$$

$$1-(1+i_e)^{-n}$$

Loans with Recurrent Expenses

Let interest be paid on the loan regularly and the principal balance of the debt be discharged, moreover, the amount of expenses be constant. We suppose that the payments will be made once at the end of the year. Let R be the annual amount on the debt service.
If the debt equals D, then we will obtain from the condition for transaction equilibrium

$$D = Ra(n,i),$$

where i – compound interest rate on the credit. Hence

$$R = \frac{D}{a(n,i)}.$$

If the commission is not paid, then the effectiveness of such a transaction as a compound annual interest rate coincides with the rate on the credit i. If the commission is paid, then the loan debtor does in fact receive the sum equivalent to $D(1-g)$. Then the accounting equation will be

$$D(1-g)-Ra(n,i_e)=0, \tag{4.13}$$

or

$$f(i_e)=a(n,i_e)-a(n,i)(1-g)=0.$$

The rate i_e that is sought for may be determined with the use of numerical methods.

If the maturing payments are made p times a year, then in this equation, the annual annuity reduction factor $a(n,i)$ must be replaced with the reduction factor of the p - due annuity $a(p,n,i)$.

Repaying a loan with an irregular payment stream

So far we have supposed that the debt is discharged with equal payments. Let us consider the case when the debt is discharged by repaying an irregular payment stream: $R_1, R_2, ..., R_n$.

In this case, the accounting equation is written as:

$$f(i_e)=D(1-g)-\sum_{j=1}^{n} R_j v^j=0, \tag{4.14}$$

where $v=\frac{1}{1+i_e}$ - the discount multiplier at the total yield rate of the transaction. The amount of the final payment R_n here depends on the amount of the preceding payments and must be determined from the condition for equilibrium of the financial transaction (the final payment must discharge the debt), i.e.

$$R_n=Dq^n-\sum_{j=1}^{n-1} R_j q^{T_j}=0, \tag{4.15}$$

where T_j is the term of paying the j payment till the end of the deal, $q=1+i$ – accumulation factor with the use of the rate on the credit i.

6.4.Method of Comparing and Analyzing Commercial Contracts

There are often cases in the commercial practice when one and the same type of goods may be bought from different suppliers each of whom offers their conditions for sales (price, credit conditions, payment of the deal etc.). Since both the moments of pay- ing sums of money and their amounts do not, as a rule, coincide for different variants, these conditions can often mismatch directly. For a reasonable choice of the most profit- able variant, the consumer must have a certain analytical procedure that allows to deter- mine the comparable indices of contract costs that account all the conditions stipulated in the contracts.

We will consider the classical approach that is based on comparing the present values of all the payments stipulated by the contracts. Payments are reduced to one time point, as a rule, to the beginning of the agreement validity. The variant with the lowest present value from the financial viewpoint is considered preferable to the consumer along with the acceptance of all other conditions.

When calculating the present values for contract comparison, the central moment is choosing the level of the interest rate on which discounting is made. This rate is called the *effective percentage rate*. The higher the rate, the less impact the outer payments have on the present value of expenditures. Therefore, the increase in the effective percentage rate makes contracts with longer terms for debt repayment preferable for investors. Indices cal- culated at the accepted effective percentage rate are conventional, however, the contract rating adjusted in this way turns out to be stable.

Let us first consider a problem with competitive conditions of debt repayment. Let th price of the goods be constant in all the variants. In the general case, the rates on the credit may be different. The longer the life of the credit, the higher the rate, since it is necessary to compensate the decrease in the cost of more outer payments. Each variant of the contract states the following basic conditions for debt repayment:

1) advance payments, their amount and moments of repayment;
2) duration and conditions for interest payments in the grace period, if it is stipulated (in the grace period, the principal balance of the debt is not repaid, it is only the interest on the principal balance of the debt that is paid);
3) term for debt repayment;
4) method of debt repayment.

The problem involves writing out the equation for calculating the present values of stream payments stipulated by the competing contracts. Since in the real practice, a set of different variants may appear, it is not possible to write a single formula applicable for all the cases. That is why, let us consider several most characteristic variants.

Example 4.1. A shipbuilding company offers to pay 8 million dollars for its order. Two payment options are offered:

1) 5% at the conclusion of the contract, 5% at the ship launch after 6 months and fur- ther on within 5 years equal expenses for the debt service, i.e. the remainder of the debt is to be discharged within 5 with equal sums. This option does not stipulate any grace period.

2) 5% at the conclusion of the contract, 10% at the ship launch after 6 month; a grace pe- riod is stipulated that starts right after the repayment of the second advance payment and equals six months. Payment of interest calculated with the compound rate is stipulated at the end of the grace period. After the end of the grace period the debt repayment occurs within 8 years with equal payments.

The interest on the credit in both variants is i = 10%.

Let us accept the effective percentage rate q = 15%. It is the rate with which discount- ing of all payments stipulated by the contract occurs.

Let us write an equation for the present value of payments on the first contract. Two ad- vance payments are stipulated in it: Q_1 and Q_2, the latter to be paid at the time point t = 0.5 years from the beginning of the contract validity. Consequently, it must be discounted through multiplying it by the factor v^t, where $v = 1/(1+q)$ - is the discount multiplier at the rate q. The process of discharging the balance of the debt $D_1 = P-(Q_1+Q_2)$ starts from the moment of paying the second advance payment. Taking into account that pay- ments are received once at the end of the year, the stream of payments discharging the debt may be considered as a constant annual annuity postponed for t years with parame- ters R_1, n_1, whose present value is defined with the use of the rate equal to the effective

percentage rate q. Here n_1 is the time period during which the principal balance of the debt for the first variant is discharged (n_1=5). It is necessary to define the reduced value of this an- nuity for the initial time point. The present value of the annuity at the moment of paying
the second advance payment equals $R_1a(n_1,q)$, and at the initial time point it is $R_1a(n_1,q)v^t$. The amount of a single discharging payment is calculated by the formula $R_1 = D_1 / a(n_1,i)$. Note that for calculating R_1, the rate on the credit i is used.

Thus, mathematically the process of discharging the debt (the present value of expenses) for the first variant is described by the equation:

$$A_1 = Q_1 + Q_2v^t + R_1a(n_1,q)v^t .$$

Let us consider the second variant. A grace period is stipulated by this variant. Consequently, in the equation for the present value of payments it is necessary to account the interest paid at the end of this period. Let L be the duration of the grace period; $D_2 = P - (Q_3 + Q_{4)}$, here Q_3,Q_4 are advance payments. Then the sum of the interest in the grace period equals $D_2[(1+i)^L - 1]$. In order to reduce this payment to the zero moment, it must be multiplied by the discount multiplier v^{t+L}. Discharge of the principal balance of the debt starts after the end of the grace period. The stream of payments discharging the debt is an annuity postponed for $t+L$ years. The present value of the discounted at the initial moment discharging payments equals $R_2a(n_2,q)v^{t+L}$, where $R_2 = D_2 / a(n_2,i)$. The equation to determine the present value of payments in the second variant is formed as in the first and has the following form

$$A_2 = Q_3 + Q_4v^t + D[(1+i)^L - 1]v^{t+L} + R_2a(n_2,q)v^{t+L}, \qquad (4.17)$$

here n_2 is the time period during which the principal balance of the debt is discharged for the second variant (n_2 =8).

Making calculations, we will obtain: A_1 = \$ 6 710 000, A_2 = \$ 6 382 000.

Therefore, in accordance with the method described above, the second variant has turned more profitable because $A_2 < A_1$.

The sums obtained are conventional and are used only in order to determine the contract rating. It is possible to show that if the present value of payments on one of the contracts compared is higher than that on the other, then a similar relation is preserved for other values of the effective percentage rate, in the case when they exceed the highest of the rates of the contracts under comparison or when the effective percentage rates are smaller than the smallest of these rates. In other words, if for any two contracts $A_2 < A_1$, besides, in the first contract, the rate on the credit is i_1, and in the second – i_2, $i_1 > i_2$ at a certain effective percentage rate q, then the same relation between A_2 and A_1 will be preserved for all the other values of q, such that $q > i_1$ or for all the values $q < i_2$.

Let us consider one more typical variant of the contract. Let the contract conditions are as follows: an advance payment in the sum Q is paid once at the beginning of the deal, while a single supply of the goods after the time period t from the moment of contract conclusion is stipulated, a grace period of the duration of L is stipulated that starts from the moment of supplying the goods; the discharge of the debt occurs by equal payments, what is more, the process of discharging the debt starts from the moment of the grace period's termination, the interest in the grace period is paid regularly once at the end of the year, the total contract term is $n + t + L$ years.

Considering all these conditions, let us write the equation that determines the present value of payments on the contract:

$$A = Q + (P - Q)\left(\frac{a(n,q)}{a(n,i)} v^{t+L} + i \cdot a(L,q)v^t\right). \qquad (4.18)$$

In this equation, the sum of annual interest payments in the grace period equals $(P - Q)i$. Since the interest is paid regularly from the beginning of supplying the goods, the stream of these payments may be considered as an annual annuity postponed for t years with the parameters R, i, L, q. The size of the discharging payments equals $R = (P - Q)/a(n,i)$. The stream of these payments is the annual annuity postponed for $t + L$ years.

6.5.Planning Long-Term Debt Repayment

Expenses on the Debt Service

The notion of regular expenses connected with the borrowing is called *debt service*. A single sum for the debt service is called *a due payment*. Due payments include:

- the current interest payments;
- the means intended for repayment (amortization) of the principal balance of the debt (the

principal).

The methods of determining the amount of due payments depends on the conditions of the borrowing. These conditions stipulate the term, the duration of the grace period (if it exists), the level of the interest rate, the method of repayment and payment of the interest and the principal balance of the debt.

Usually the interest is paid at the duration of the whole borrowing regularly. Sometimes it is charged and added to the principal balance of the debt. The main amount is usually repaid in parts, besides, the amounts of discharging payments (due payment) may be equal or change.

The principal balance of the debt is not discharged in the grace period, the interest may be paid by regular payments or at the end of the period by one payment, or not paid at all but added to the principal balance of the debt.

Let D be the amount of the debt, I - the interest on the borrowing, L - the duration of the grace period, R - annual expenses on discharging the principal balance of the debt, g - the interest rate on the conditions of the borrowing, Y - due payment.

In the period when the principal balance of the debt is discharged, the due payment contains two elements: $Y = R + I$. In the grace period it is $Y = I$.

The calculation methods substantially depend on the type of the borrowing. The main feature of borrowing classification is the method of loan repayment. The following types of borrowings are distinguished:

1. Borrowings without mandatory redemption (*perpetual bonds*) – the loan debtor is committed to pay the creditor a profit in a form of fixed interest in certain terms, the borrowed sum is not to be returned. The loan debtor keeps the right to buy out or discharge all the certificates of indebtedness at any time.

2. Borrowings with mandatory redemption at one term. The loan debtor gives the borrowed amount back at the specified time limits and pays interest regularly or at the term end.

3. Borrowings with mandatory redemption in several terms. The loan debtor gives the borrowed sum back in parts and regularly pays a profit on the borrowing in form for interest.

Forming a Fund

To secure the repayment of debt, a repayment fund is usually created. Sometimes it is stated in the agreement on the borrowing issue. The fund is formed from subsequent payments that are charged with interest (for example, a bank account). It clearly has a meaning if the loan debtor has an opportunity of receiving larger interest on the repayment fund money than that on which they have taken the loan. Different variants of forming the fund and repaying the debt are possible. We will consider three models of forming the repayment fund:

1. The principal balance of the debt is discharged from the fund at the term end by a lump sum payment. The sum of payments with interest in the fund must be identical with the debt at the moment of its payment. The interest on the debt is paid not from the fund.

2. The conditions for financial commitment instead of regular interest payment stipulate their addition to the principal balance of the debt.

3. The fund is formed in such a way that regular payment of interest on the debt (from the fund) and the repayment of the principal balance of the debt at the term end are secured.

Let us consider the first model. Let the accumulation of finances in the fund be made by regular annual payments whose amount equals R, and let interest at the rate i be charged on these payments. Payment of interest charged on the debt at the rate g occurs simultaneously (the interest is paid not from the fund). Then the due payment is

$Y = Dg + R,$ where Dg - is the interest on the debt, R - payments in the fun.

The fund must be accumulated for n years. Payments in the fund are an annuity with the parameters R,n,i. *Knowing* the amount of the debt, it is easy to determine a single payment in the fund:

$$R = \frac{}{s(n,i)},$$

where $s(n,i)$ - annual annuity accumulation factor that equals $s(n,i) = \frac{(1+i)^n - 1}{i}$.

On account of this, the due payment is $Y = D(g +$

In the second model, the conditions for the financial commitment stipulate adding the interest to the principal balance of the debt, which is why the payments in the fund at the term end must secure the accumulation of the sum $D(1+g)^n$. In this case, the due payment equals

$$Y = \frac{D(1+g)^n}{s(n,i)}, \tag{4.20}$$

The rate i characterizes the speed of the repayment fund growth, and the rate g — the amount of interest paid for the borrowing.

Let us consider the third model of forming the fund. Let the credit be discharged the fol- lowing way:

- at the beginning of each year, equal amounts R are deposited to the repayment fund;
- the interest on the credit starts to be discharged after the first calculation of interest on payments in the fund;
- the creditor receives payments on interest from the fund;

at the term end the principal balance of the debt will be paid plus the interest on the last year of the credit.

The peculiarity of this problem is that: a) the payment in the fund is made not at the end of the year, but as we have supposed before, at its beginning; b) the interest on the credit is paid regularly from the fund.

Let us consider the process of fund formation (see Figure 4.3). The first payment into the fund of the sum R is made at the moment of taking the credit. For the sum R interest for n years (n - the total credit term) at the compound rate i is calculated. At the term end this sum will equal $R(1+i)^n$. At the beginning of the second year, the sum R is deposited into the fund and

Figure 4.3. The process of forming the repayment fund on Model 3.

and interest on the debt G is paid from the fund. Therefore, in fact it is the sum $(R-G)$ that is deposited into the fund. It will be charged with the interest during $(n-1)$ years, by the end of the term we will obtain an amount $(R-G)(1+i)^{n-1}$. At the beginning of the third year we deposit the sum R into the fund and simultaneously we take the sum G from the fund. This deposit (after the deduction of interest on the debt) will be charged with the interest within $(n-2)$ years. As the result, by the end of the term we will obtain the sum $(R-G)(1+i)^{n-2}$. The deposit into the fund is made at the time point $(n-1)$ and with the calculated interest (with consideration of withdrawing the amount of interest on the debt) it will give the sum $(R-G)(1+i)$. At the term end it is necessary to secure payment of the sum whose amount equals $(D+G)$ (the principal balance of the debt plus the interest on the debt for the previous year).

Let us derive an accounting equation in which all the payments into the fund after the deduction of the interest on the debt, with consideration of their interest calculation in the fund will be equated to the debt amount plus the interest for the preceding year. We have

$$R(1+i)^n + (R-G)(1+i)^{n-1} + \ldots + (R-G)(1+i)^2 + (R-G)(1+i) = D+G.$$

It is necessary to determine the amount of deposits into the fund R from this equation

After quite simple transformations the accounting equation may be written as

$$R\sum_{k=1}^{n}(1+i)^k - G\sum_{k=0}^{n-1}(1+i)^k = D.$$

Note that $\sum_{k=0}^{n-1}(1+i)^k$ is the accumulation factor of the annual annuity that may be writ ten as:

$= s(n,i)$. Besides, we have

$$\sum_{k=1}^{n}(1+i)^k = (1+i)\sum_{k=0}^{n-1}(1+i)^k .$$

Considering these relations, the accounting equation will be written as follows:

$$R\frac{(1+i)((1+i)^n-1)}{i} - G\frac{(1+i)^n-1}{i} = D.$$

When solving it, let us determine the value of the deposit into the fund:

$$R = \frac{D}{(1+i)s(n,i)} + \frac{G}{1+i}. \tag{4.21}$$

As it is seen from the obtained relation, the deposit into the fund is formed from two constituents: the first item is the sum that secures repayment of the principal balance of the debt (note, there is the factor $(1+i)$ reflecting the occurrence of deposits into the fund at the beginning of the year); the second item is the sum that secures regular discharge of the interest on the debt by payments from the fund.

Repaying the Debt by Equal Instalments

One of the ways to repay the debt by instalments is to pay it by equal amounts.

Let the borrowing D be discharged within n years by equal payments. Then the sum spent on discharge of the debt annually equals d, where $d = D/n$. Apart from paying the principal balance of the debt, the debtor pays the interest on the *remainder* of the debt. Let us assume that the interest is paid once at the end of the year at the rate g. Then the first interest payment at the end of the first year equals Dg. At the end of the second year the interest amount equals $(D-d)g$. At the end of the third year it is $(D-2d)g$ etc.

Due payments are formed the following way:

The first payment equals $Y_1 = d + Dg$, the second payment – $Y_2 = d + (D-d)g$, the third payment - $Y_3 = d + (D-2d)g$ etc.

The due payment at an arbitrary time point t equals:

$$Y_t = d + D_t g, \tag{4.22}$$

where D_t is the remainder of the debt at the beginning of the year t $(t = 1,2,...,n)$ that is defined through the following recursion relation:

$$D_{t+1} = D_t - d = D - td,\ D_1 = D.$$

Repaying the Debt by Instalments with Equal Due Payments

Along all the discharging term a constant due payment is paid regularly, its one part counts for the repayment of the principal balance of the debt, and the other part – for the repayment of the interest on the borrowing. The value of the principal balance of the debt decreases after each payment. However, in connection with the decrease in payments on the interest, the amounts going for discharging the principal balance of the debt increase in time. Such a method is called *progressive repayment***.** The due payment

$$Y = D_t g + d_t = const\,. \tag{4.24}$$

The regularly paid amounts Y may be viewed as a constant annual annuity whose member is determined by the formula:

$$Y = \frac{D_1}{a(n,g)}\,. \tag{4.25}$$

Let us determine the structure of the due payment, i.e. its part that goes for discharging the principal balance of the debt and the part that goes for discharging the interest.

The amount of the first maturing payment is $d_1 = Y - D_1 g$.

From Formula (3.25) it follows that $D_1 = Ya(n,g)$, consequently,

$$d_1 = Y - D_1 g = Y(1 - g \cdot a(n,g)) = Yv^n\,, \tag{4.26}$$

where $v = \dfrac{1}{1+g}$.

Then

$d_2 = Y - D_2 g = Y - (D_1 - d_1)g$ (here it is taken into account that $D_2 = D_1 - d_1$) or

$$d_2 = Yv^n + Yv^n g = Yv^{n-1} = d_1(1+g). \quad (4.27)$$

For the payment at the time point t we have $d_t = Yv^{n-(t-1)}$, or

$$d_t = d_1(1+g)^{t-1}. \quad (4.28)$$

Therefore, it is possible to write the following system of relations.

The amount of the maturing payment in the year t:

$$d_t = Y - D_t g = d_{t-1}(1+g), \; t = 1,2,...,n, \quad (4.29)$$

$$d_1 = Y - D_1 g = Yv^n = D_1 v^n / a(n,g) = D_1 / s(n,g). \quad (4.30)$$

The remainder of the debt for the beginning of the year $t+1$:

$$D_{t+1} = D_t - d_t = D_t(1+g) - Y. \quad (4.31)$$

The amount of the repaid debt for the beginning of the year $t+1$:

$$W_t = \sum_{k=0}^{t-1} d_1(1+g)^k = d_1 s(t,g), \quad (4.32)$$

where $s(t,g)$ - is the annual annuity accumulation factor whose term equals t years.

If the maturing payments and interest is paid p times a year, then the given problem is solved analogously, only with the use of the corresponding formulas for the p–due annuities.

Questions for Self-test

1. What characteristic is used to evaluate the yield of the financial and credit transac- tion?
2. What is the balance of the credit transaction? Write the balance equation.
3. How to determine the total yield of the financial and credit transaction without com- mission?
4. How to determine the total yield of a loan operation with commission?
5. How to determine the total yield of a loan operation with regular interest payments?
6. How to determine the total yield of a loan operation with recurrent expenses?
7. How to determine the total yield of a loan operation with irregular payment stream?
8. What is the essence of the method of comparing different contracts based on pay- ment capitalization?
9. How are the loans classified in accordance with the method of their repayment?
10. Name the main methods of forming a repayment fund.
11. How is the repayment fund formed to discharge the principal balance of the debt from the

fund at the term end by a lump sum payment , and the interest on debt is paid not from the fund?

12. How is the repayment fund formed to discharge the principal balance of the debt and the interest at the term end by a lump sum payment?
13. How is the repayment fund formed to discharge the principal balance of the debt at the term end by a lump sum payment and secure regular interest payments on the debt (from the fund)?
14. How is the debt discharged through repayment by equal instalments?
15. How is the debt discharged through repayment by instalments with equal due pay- ments?

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