DEMOCRATIC AND POPULAR ALGERIAN REPUBLIC

Ministry of Higher Education and Scientific Research

ECHAHID HAMMA LAKHDAR UNIVERSITY

Faculty of Exact Sciences

Department of mathematics

Thesis For End of Study

For obtention Diplome of **MASTER**

Domain : Mathematics and informatics

Field : Mathematics

Specialty : Fundamental Mathematics and Applied

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Theme

Finite Element Methods of a Parabolic Integro-Differential Equations

Sustained: June,2020

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University Year : 2019/2020

APPRECIATION

This thesis is wholeheartedly appreciated and consecrated

Firstly, I consecrated it to the Almighty God **"Allah"**, thank you for the guidance, strength, power of mind, protection and skills and for giving us a healthy life.

Especially to my director teacher **"Doudi Nadjet"**, huge thanks for giving me this opportunity, without her support, and advice this thesis would not have been done.

To my beloved parents, Who have been my source of inspiration and gave me strength when I thought of giving up.

To my sister, brothers, relatives, friends, and classmates who shear their words of advice and encouragement to finish this work.

Finally, all our professors and more particularly the members of the jury who agreed to judge our work.

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PREFACE

Finite element method (FEM) is a numerical technique for solving a differential or integral equations. It has been applied to a number of physical problems, where the governing differential equations are available. The method essentially consists of assuming the piecewise continuous function for the solution and obtaining the parameters of the functions in a manner that reduces the error in the solution.

The FEM has been in existence for more than 70 years. One of the authors, John Argyris, invented this technique in World War II in the course of the check on the analysis of the swept back wing of the twin engined Meteor Jet Fighter. In this work, he also consistently applied matrix calculus and introduced triangular membrane elements in conjunction with two new definitions of triangular stresses and strains which are now known as the component and total measures. In fact, he was responsible for the original formulation of the matrix force and displacement methods.

In this work we will present the numerical solution using the finite element method for a parabolic integro-differential equations with boundary conditions. This type of equations arise quite frequently as mathematical models in diverse disciplines. The origins of the study of integral and integro-differential equations may be traced to the work of Abel, Lotka, Fredholm, Malthus, Verhulst and Volterra on problems in mechanics, mathematical biology and economics. The work of Volterra on the problem of competing species is of fundamental importance for the development of mathematical modeling of real world problems. From those beginnings, the theory and applications of Volterra integro-differential equations with bounded and unbounded delays have emerged as new areas of investigation. This continuous process of development during the past few decades is reflected in the large number of research papers and books covering some of these areas.

Many numerical methods have been developed for solving integral and integro-differential equations such as finite-difference methods, finite-element methods, boundary-element methods, etc. Each of these numerical methods has its inherent advantages and disadvantages and the search for alternative, more general, easier and more accurate numerical methods is a continuous and ongoing process. In the past several decades, many effective methods for obtaining solutions of nonlinear Volterra integral and integro-differential equations have been presented.

The contents of each chapter are summarized as follows.
In the first chapter, we will present some preliminary concepts in the functional analysis required by the method of finite elements and some special concepts that we need to study the parabolic integro-differential equations.

Chapter two is concerned with giving a simple theoretical introduction to the method of finite elements applied to a partial differential equation. Where we gave the method of finding the linear system and how to study error estimation. Then we performed a numerical application on a ordinary differential equation.

In third chapter, finite element approximation is established for parabolic integro-differential equations. First, we obtain variational form of the problem, and then, finite element method and basic functions will be used. Also, the error analysis of the method is considered. Finally, optimal error estimate is deduced for backward Euler discrete scheme. Furthermore, the efficiency of the proposed method will be considered through numerical example.

NOTATIONS

A	positive self adjoint elliptic second order operator.
$A(.,.)$	bilinear form associated with A.
B	second order partial differential operator.
B^*	adjoint operator to B.
$B(.,.)$	bilinear form associated with B.
C, c	constants.
$\mathcal{C}^r(\overline{\Omega})$	space of continuously differentiable functions up to r^{th} order.
D_t, D_x	derivatives with respect to t and x.
$D(\Omega)$	vector space of $\mathcal{C}^\infty(\Omega)$ functions, with compact support.
$e = u - u_h$	the error.
h	step size in space variable x.
$H^m(\Omega) = W_2^m(\Omega)$	Hilbert space.
I_h	interpolation operator.
k	step length in time variable t.
$K(t)$	kernel function under Volterra integral.
$L^p(\Omega)$	space of integrable functions.
PDE	partial differential equation.
$PIDE$	parabolic integro-differential equation.
P_h	L^2- projection operator.
q	conjugated exponent of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$.
R_h	Ritz-projection operator associated with A.
S_h	piecewise polynomial subspace.
t	time variable.
$u = u(t, x)$	unknown.
$u_0 = u(0, x)$	initial value.
u_t, u_x	derivatives with respect to t and x respectively.
u_h	finite element solution to space discretization.
$u_{h,t}$	$D_t u_h$.
U^n	fully discrete finite element solution at time t_n.
V_h	Ritz-Volterra projection operator associated with A and B.
$W_p^m(\Omega) = W^{m,p}(\Omega)$	Sobolev space.
$x = (x_1, \ldots, x_n)$	space variable.

$\theta = u_h - V_h u$	the error.
$\rho = V_h u - u_h$	the error.
Ω	domain in space, with boundary $\partial\Omega$.
Ω_h	mesh domain with stepsize h.
$\overline{\partial}_t U^n = (U^n - U^{n-1})/k$	backward difference quotient.
$\lvert . \rvert_{\mathbb{R}^N}$	norm in $\mathbb{R}^N$.

CHAPTER 1

PRELIMINARY IN FUNCTIONAL ANALYSIS

In functional analysis many different fields of mathematics come together. The purpose of this chapter is to give an introduction of functional analysis that the finite elements methods required, and the basic tools necessary for the effective use of the analysis of the method with integro-differential problem.

1.1 Normed Spaces. Banach Spaces

Throughout $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 1.1.1 (**Normed space**).
Let E be a vector space. A map $E \longrightarrow \mathbb{R}$, $x \longmapsto \|x\|$ is called a "norm" on E if :

i) $\|x\| \geq 0, \quad \forall x \in E$ *and* $\|x\| = 0 \Leftrightarrow x = 0$.

ii) $\|\alpha x\| = |\alpha|\|x\|, \quad \forall x \in E$ *and* $\alpha \in \mathbb{K}$.

iii) $\|x+y\| \leq \|x\| + \|y\|, \quad \forall x, y \in E$. *(triangle inequality).*

We call $(E, \|.\|)$ or simply E a "normed space" [7].

Example 1.1. *For $E = \mathbb{R}^N$ and $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, we define the norms:*

$$\|x\|_1 = \sum_{i=1}^{N} |x_i|, \qquad \|x\|_2 = \left(\sum_{i=1}^{N} x_i^2 \right)^{1/2} \text{ (Euclidean norm)}, \qquad \|x\|_\infty = \sup_i |x_i|.$$

Lemma 1.1.
Let $(E, \|.\|)$ be a normed space and define

$$d(x, y) = \|x - y\| \tag{1.1}$$

for all $x, y \in E$. Then (E, d) is a "metric space".

Definition 1.1.2 (**Banach space**).
A Banach space is a complete normed space (complete in the metric defined by the norm (1.1)) [7].

Example 1.2. *The simplest example of a Banach space is $\mathbb{R}^N$ or $\mathbb{C}^N$ with the Euclidean norm.*

1.2 Hilbert Spaces

1.2.1 Inner Product Spaces

Throughout we let E denote a vector space over $\mathbb{K}$.

Definition 1.2.1 (Inner product, inner product space).
A mapping $\langle .,. \rangle : E \times E \longrightarrow \mathbb{K}$ is called an "inner product" or "scalar product" for $x, y, z \in E$ and $\alpha \in \mathbb{K}$, if

(i) $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

(ii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

(iii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

(iv) $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

We say that E equipped with $\langle .,. \rangle$ is an "inner product space" or "pre-Hilbert space" [6].

Remark 1.1.
An inner product on E defines a norm on E given by

$$\|x\| = \sqrt{\langle x, x \rangle}. \tag{1.2}$$

and a metric on E given by

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle}. \tag{1.3}$$

Definition 1.2.2 (Hilbert space).
A Hilbert space is a complete inner product space (complete in the metric defined by the inner product (1.3)).

Remark 1.2.
Inner product spaces are normed spaces, and Hilbert spaces are Banach spaces.

Orthogonality:

Definition 1.2.3. *An element x of an inner product space E is said to be orthogonal to an element $y \in E$ if*

$$\langle x, y \rangle = 0.$$

We also say that x and y are orthogonal, and we write $x \perp y$. Similarly, for subsets $A, B \subset E$ we write $x \perp A$ if $x \perp a$ for all $a \in A$, and $A \perp B$ if $a \perp b$ for all $a \in A$ and all $b \in B$ [7].

Projection:

Theorem 1.1 (Projection Theorem).
Let H be a Hilbert space, and let C be non-empty convex and closed part of H. Thus for all $x \in H$, there is a unique $y \in C$ such that:

$$\|x - y\| = dist(x, C).$$

We say $y = P_C(x)$ is a projections of x onto C. It is characterized by the property:

$$y \in C \text{ and } Re\langle x - y, z - y \rangle \leq 0, \quad \forall z \in C.$$

If the set is not convex, $P_C(x)$ can consist of several points, if it is convex, it is precisely one.

Definition 1.2.4 (**Cauchy-Schwarz inequality, triangle inequality**).
An inner product and the corresponding norm satisfy the Cauchy-Schwarz inequality and the triangle inequality as follows.

a) ***Cauchy-Schwarz inequality:***
$$|\langle x,y\rangle| \leq \|x\|\|y\| \tag{1.4}$$
where the equality sign holds if and only if $\{x,y\}$ is a linearly dependent set.

b) ***Triangle inequality:***
$$\|x+y\| \leq \|x\| + \|y\| \tag{1.5}$$
where the equality sign holds if and only if $y=0$ or $x=cy$ (c real and ≥ 0).

[7].

1.3 Functional Spaces

1.3.1 Notations

- Let Ω be an open not empty set of $\mathbb{R}^N$, $\quad (N \in \mathbb{N}^*)$.
- A vector $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ is called a multi-index, and $|\alpha| = \alpha_1 + \ldots + \alpha_N$ is the length of the multi-index.
- Let u be a function of Ω into $\mathbb{R}$, α an element of $\mathbb{N}^N$. We note $D^\alpha u$ the α order derivative of u, we write:
$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \ldots . \partial^{\alpha_N} x_N}$$

1.3.2 Continuous Function Spaces

- We note by $\mathcal{C}^r(\Omega)$ the space of derivable functions till order r, whose the derivatives of order r are continuous.
We write :
$$u \in \mathcal{C}^r(\Omega) \Leftrightarrow \forall \alpha \in \mathbb{N}^N, \quad |\alpha| \leq r, \quad D^\alpha u \text{ exist and continuous.}$$
- Also we note by $\mathcal{C}^r(\overline{\Omega})$ the space of derivable functions till order r on Ω, whose the partials derivatives till order r are continuous, and extendable by continuity to $\overline{\Omega}$.
We write:
$$\mathcal{C}^r(\overline{\Omega}) = \Big\{ u \in \mathcal{C}^r(\Omega), \text{ such that } \forall \alpha = (\alpha_1, \dots, \alpha_N) \text{ with } |\alpha| \leq r, \quad D^\alpha u \text{ continually extends to } \overline{\Omega} \Big\}.$$
With norm:
$$\|u\|_{\mathcal{C}^r(\overline{\Omega})} = \sup_{x \in \overline{\Omega}, |\alpha| \leq r} |D^\alpha u(x)|.$$

Remark 1.3. *The functional space $\mathcal{C}^r(\overline{\Omega})$ equipped with $\|.\|_{\mathcal{C}^r(\overline{\Omega})}$ is Banach space. (Ω is bounded) [1].*

Definition 1.3.1 (**Support of a function**).
*Let Ω be a open of $\mathbb{R}^N$. We call a **support** of $u \in \mathcal{C}^r(\Omega)$ the set:*

$$supp(u) := \overline{\{x \in \Omega, \quad u(x) \neq 0\}}.$$

Definition 1.3.2.
We note by $\mathcal{D}^r(\Omega)$ the space of functions of class $\mathcal{C}^r$ with compact support.

$$\mathcal{D}^r(\Omega) := \{u \in \mathcal{C}^r(\Omega), \text{ with compact support }\}.$$

Remark 1.4. *$r = +\infty$, $\mathcal{D}^\infty(\Omega)$ noted by $\mathcal{D}(\Omega)$ is called "test function" space.*
We write

$$\mathcal{D}(\Omega) = \{u \in \mathcal{C}^\infty(\Omega) | \exists K \text{ compact}, K \subset \Omega, u \in \mathcal{D}_K(\Omega)\}$$

where $\mathcal{D}_K(\Omega) = \{u \in \mathcal{C}^\infty(\Omega), \quad supp(u) \subset K\}$.

Definition 1.3.3.
The distribution space with support in Ω is the topological dual of $\mathcal{D}(\Omega)$, noted by $\mathcal{D}'(\Omega)$ [1].

1.3.3 Lebesgue Spaces

Definition 1.3.4 (**Measurable function**).
A function $u : \Omega \longrightarrow \mathbb{R}$, it said to be "measurable" if for all $\alpha \in \mathbb{R}$, the set

$$E_\alpha = \{x \in \Omega \mid u(x) \geq \alpha\}$$

is measurable in the sense of Lebesgue.

Definition 1.3.5 (**Integrable function**).
It is said that a measurable function $u : \Omega \longrightarrow \mathbb{R}$ can be integrated in the Lebesgue sense if

$$\int_\Omega |u| < \infty.$$

[11].

Definition 1.3.6 ($L^p(\Omega)$ **spaces,** $1 \leq p \leq +\infty$).

- *Let $p \in \mathbb{R}$ with $1 \leq p < +\infty$, we pose:*

  $$L^p(\Omega) = \{u : \Omega \longrightarrow \mathbb{R}, u \quad \text{measurable, and} \quad \int_\Omega |u(x)|^p dx < +\infty\}.$$

  We define and note the norm in $L^p(\Omega)$ by

  $$\|u\|_{L^p} := \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}}$$

- *For $p = +\infty$, we pose:*

  $$L^\infty(\Omega) := \{u : \Omega \longrightarrow \mathbb{R}; u \quad \text{measurable, and} \quad \exists c > 0 \quad \text{such that} \quad |u(x)| \leq c \quad \text{almost all} \quad \Omega\}$$

  We define and note the norm in $L^\infty(\Omega)$ by

  $$\|u\|_{L^\infty} := \inf\{c > 0 \quad \text{such that} \quad |u(x)| \leq c \quad \text{almost all} \quad \Omega\}$$

Remark 1.5.
We call $L^\infty(\Omega)$ the space of essentially bounded functions, and we say that $\|u\|_{L^\infty}$ is the essential sup of u:

$$\|u\|_{L^\infty} := supess(u).$$

Remark 1.6.
$\|.\|_{L^p}$ and $\|.\|_{L^\infty}$ defines a norm on $L^p(\Omega)$, with $1 \leq p \leq +\infty$, for which $L^p(\Omega)$ is a Banach space [5].

Hölder Inequality:

Let (p,q) a couple of $[1,+\infty]^2$ with $\frac{1}{p}+\frac{1}{q}=1$. The application :

$$L^p(\Omega)\times L^q(\Omega)\longrightarrow \mathbb{C}, \quad (u,v)\longmapsto \int_\Omega u(x)v(x)dx,$$

is continuous bilinear with values in $L^1(\Omega)$ and

$$\|uv\|_1 \leq \|u\|_{L^p}\|v\|_{L^q}.$$

Minkowski Inequality:

Let $u,v \in L^p(\Omega)$ with $1 \leq p < +\infty$. Thus, $u+v \in L^p(\Omega)$, and:

$$\|u+v\|_{L^p} \leq \|u\|_{L^p} + \|v\|_{L^p},$$

[12].

1.4 Sobolev Spaces

Weak Derivative:

Let $u \in \mathcal{C}^1(\Omega)$ and $\varphi \in \mathcal{C}_0^1(\Omega)$, then, by integrating by parts we will have:

$$\int_\Omega u\frac{\partial\varphi}{\partial x_i}dx = -\int_\Omega \frac{\partial u}{\partial x_i}\varphi dx, \quad \forall\varphi\in\mathcal{C}_0^1(\Omega), \quad i=1,2,\ldots,N$$

note that the term on the left has a meaning if $u \in L^1_{loc}(\Omega)$, when the term on the right only has meaning if $u \in \mathcal{C}^1(\Omega)$.

Definition 1.4.1.
It is said that the function $u \in L^p(\Omega)$, admits a first weak partial derivative with respect to x_i if there exists a function $v_i \in L^r(\Omega)$ for $1 \leq r < \infty$ such that

$$\int_\Omega u\frac{\partial\varphi}{\partial x_i}dx = -\int_\Omega v_i\varphi dx, \quad \forall\varphi\in L^q(\Omega)$$

v_i is noted $\dfrac{\partial u}{\partial x_i}$.

More general: Let $m \geq 1$ and $\alpha \in \mathbb{N}^N$, it is said that u admits a weak derivative of the order α, if there exists v_α such that:

$$\int_\Omega u\frac{\partial^{|\alpha|}\varphi}{\partial^{\alpha_1}x_1\partial^{\alpha_2}x_2\ldots\partial^{\alpha_N}x_N}dx = (-1)^{|\alpha|}\int_\Omega v_\alpha\varphi dx, \quad \forall\varphi\in\mathcal{C}_0^\infty(\Omega),$$

v_α is noted by $\dfrac{\partial^{|\alpha|}u}{\partial^{\alpha_1}x_1\partial^{\alpha_2}x_2\ldots\partial^{\alpha_N}x_N}$ or $D^\alpha u$ [10].

1.4.1 The H^1 Spaces

Definition 1.4.2 (**The** $H^1(\Omega)$ **space**).
Let p be an element of $[1,+\infty[$, Ω *an open set from* $\mathbb{R}^N$. *We call a "Sobolev space" of the first order, and we note* $W^{1,p}(\Omega)$ *(resp* $H^1(\Omega)$ *if* $p=2$*), the set:*

$$W^{1,p}(\Omega)=\left\{u\in L^p(\Omega)\mid \forall i,\quad 1\le i\le n,\quad \frac{\partial u}{\partial x_i}\in L^p(\Omega)\right\}$$

where the derivative $\dfrac{\partial u}{\partial x_i}, i=1,2,\dots,N$ *is taken in the sense of the distributions [12].*

Proposition 1.1.
For $p\in[1,+\infty]$, *and for all* Ω *an open from* $\mathbb{R}^N$. *We have the following properties:*

1. *The Sobolev space* $W^{1,p}(\Omega)$ *is a Banach space for the norm:*

$$u\longrightarrow \|u\|\mapsto\begin{cases}\left(\|u\|^p_{L^p}+\displaystyle\sum_{i=1}^{N}\|\frac{\partial u}{\partial x_i}\|^p_{L^p}\right)^{\frac{1}{p}}, & if\quad 1\le p<+\infty\\ \max\left(\|u\|_{L^\infty},\|\dfrac{\partial u}{\partial x_i}\|_{L^\infty}\quad 1\le i\le N\right), & if\quad p=\infty\end{cases}$$

2. *For* $p=2$, $H^1(\Omega)$ *is a Hilbert space for the inner product:*

$$(u,v)\mapsto\langle u,v\rangle=\int_\Omega u\overline{v}dx+\sum_{i=1}^{N}\int_\Omega\frac{\partial u}{\partial x_i}\frac{\partial\overline{v}}{\partial x_i}dx.$$

3. *For* $1<p<+\infty$, *the space* $W^{1,p}(\Omega)$ *is reflexive.*

4. *For* $1\le p<+\infty$, *the space* $W^{1,p}(\Omega)$ *is separable.*

Remark 1.7.
For all p,q from $[1,+\infty[$, *the bidual of* $L^p(\Omega)$, *also the dual of its dual* $L^q(\Omega)$ *with* $\dfrac{1}{p}+\dfrac{1}{q}=1$, *identifies algebraically and topologically with* $L^q(\Omega)$. *It is said that space* $L^p(\Omega)$ *is reflexive.*

The $H^1_0(\Omega)$ **Space:**

Definition 1.4.3.
We denote by $H^1_0(\Omega)$ *the closure of* $\mathcal{D}(\Omega)$ *in* $H^1(\Omega)$.
We write:

$$H^1_0(\Omega):=\left\{v\in H^1(\Omega):\forall\varepsilon>0,\exists\varphi\in\mathcal{D}(\Omega)\ such\ that\ \|v-\varphi\|_{H^1(\Omega)}<\varepsilon\right\}=\overline{\mathcal{D}(\Omega)}.$$

Remark 1.8. *The semi-norm of* $H^1(\Omega)$ *given by:*

$$|v|^2_{H^1(\Omega)}:=\int|\nabla v(x)|^2_{\mathbb{R}^N}dx,\quad\forall v\in H^1(\Omega),$$

defined a norm on $H^1_0(\Omega)$ *equivalent to that of* $H^1(\Omega)$.

Proposition 1.2. *The space* $(H^1_0(\Omega),|.|_{H^1(\Omega)})$ *is a Hilbert space [5].*

Poincaré Inequality:

Let Ω be a bounded domain of $\mathbb{R}^N$. There is a constant $C > 0$ such that:

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

Green's Formula:

Proposition 1.3.
Let $\Omega \in \mathbb{R}^N$ be a bounded open and $\partial\Omega$ is Lipschitzian, then for all functions $u, v \in H^1(\Omega)$

$$\int_\Omega \Delta u v dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} v ds - \int_\Omega \nabla u \nabla v dx,$$

where, $n = (n_i)_{1\leq i\leq N}$ is the unit vector normal to $\partial\Omega$ directed outwards [10].

1.4.2 The $H^m, m \in \mathbb{N}^*$ Spaces

Definition 1.4.4 (**The** $H^m(\Omega)$ **space**).
Let Ω be a non empty open of $\mathbb{R}^N$, we define $H^m(\Omega)$ by:

$$H^m(\Omega) = \left\{v \in L^2(\Omega) : \quad \forall \alpha \in \mathbb{N}^N, \quad |\alpha| \leq m, \quad D^\alpha v \in L^2(\Omega)\right\}.$$

Where D^α is the derivative of the order α taken in the sense of the distributions [12].

Proposition 1.4.
$H^m(\Omega)$ is a Hilbert space if we define the following scalar product, norm and semi-norm $\forall u, v \in H^m(\Omega)$, there:

$$\langle u, v\rangle_{H^m} := \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u(x).D^\alpha v(x) dx,$$

$$\|u\|_{H^m} := \left(\sum_{|\alpha|\leq m} \int_\Omega |D^\alpha u(x)|^2 dx\right)^{1/2},$$

$$|u|_{H^m} := \left(\sum_{|\alpha|= m} \int_\Omega |D^\alpha u(x)|^2 dx\right)^{1/2}.$$

[10].

The $H_0^m(\Omega)$ **Spaces:**

Definition 1.4.5.
For $m \in \mathbb{N}$, we denote by $H_0^m(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^m(\Omega)$. We note

$$H_0^m(\Omega) = \overline{\mathcal{D}(\Omega)} = \overline{\mathcal{C}_0^\infty(\Omega)}.$$

Proposition 1.5. *The space $H_0^m(\Omega)$ is a Hilbert space.*

Remark 1.9 (**Poincaré Inequality in** $H^m(\Omega)$).
Let Ω be a bounded domain of $\mathbb{R}^N$. There is a constant $C > 0$ such that:

$$\|u\|_{H^m(\Omega)} \leq C|u|_{H^m(\Omega)}, \quad \forall u \in H_0^m(\Omega).$$

The $H^{-m}(\Omega), m \in \mathbb{N}^*$ Spaces:

Definition 1.4.6.
We denote by $H^{-m}(\Omega)$ the dual topological of the space $H_0^m(\Omega)$. We equipped $H^{-m}(\Omega)$ with the norm:

$$\|v\|_{H^{-m}(\Omega)} := \sup\Big\{|v(u)| : u \in H_0^m(\Omega) \text{ such that } \|u\|_{H^m(\Omega)} \leq 1\Big\}.$$

Proposition 1.6. *$(H^{-m}(\Omega), \|.\|_{H^{-m}(\Omega)})$ is a Hilbert space.*

1.5 Some Important Operators

1.5.1 The Interpolation Operator I_h

It is straight forward to extend the concept of interpolation to continuous piecewise linear functions. Given a continuous function f we define the continuous piecewise linear interpolate $\pi f \in V_h$ to f by

$$\pi f(x) = \sum_{i=1}^{N} f(x_i)\varphi_i.$$

Where V_h is the space of continuous piecewise linear functions, and $\{\varphi_i\}_{i=0}^{N}$ is a basis of V_h. (Will be introduce in the next chapter).

We have the following estimates for continuous piecewise linear interpolation.

Proposition 1.7. *The following interpolation estimates hold.*

$$\|f - \pi f\|^2_{L^2(\Omega)} \leq C \sum_{i=1}^{N} h_i^4 \|f''\|^2_{L^2(\Omega_i)}.$$

$$\|(f - \pi f)'\|^2_{L^2(\Omega)} \leq C \sum_{i=1}^{N} h_i^2 \|f''\|^2_{L^2(\Omega_i)}.$$

1.5.2 The L^2-projection Operator P_h

Interpolation is a simple way of approximating a continuous function, but there are, of course, other ways. In this section we shall study orthogonal, or L^2-projection as a technique for approximating functions. L^2-projection gives a good on average approximation, as opposed to interpolation which is exact at the nodes. Moreover, in contrast to interpolation L^2-projection does not require the function we seek to approximate to be continuous or have well-defined node values.

Definition 1.5.1.
Given a function $f \in L^2(\Omega)$ the L^2-projection $P_h f \in V_h$ of f is defined by

$$\int_\Omega (f - P_h f) v\, dx = 0, \quad \forall v \in V_h. \tag{1.6}$$

In analogy with projection onto subspaces of $\mathbb{R}^N$, (1.6) defines a projection of f onto V_h, since the difference $f - P_h f$ is required to be orthogonal to all functions $v \in V_h$.

Definition 1.5.2 (Derivation of a Linear System of Equations).
To compute the L^2-projection $P_h f$ we first note that the definition (1.6) is equivalent to

$$\int_{\Omega} (f - P_h f)\varphi_i dx = 0, \quad i = 1, 2, \ldots, N. \tag{1.7}$$

Where $\varphi_i, i = 1, 2, \ldots, N$ are the hat basis functions (V_h basis). This is a consequence of the fact that if (1.6) is satisfied for any choice of v as a hat function, then it is also satisfied for a linear combination of hat functions, and, conversely, since any function $v \in V_h$ is a linear combination of hat functions (1.6) implies (1.7).
Now, since $P_h f$ belongs to V_h it can be written as the linear combination

$$P_h f = \sum_{i=0}^{N} \xi_i \varphi_i.$$

with $n+1$ unknown coefficients $\xi_i, i = 0, 1, \ldots, N$ to be determined [13].

1.5.3 The Ritz-projection Operator R_h

Ritz projection is a technique for approximating a given function u, and is very similar to L^2-projection. Both L^2 - and Ritz projection compute the orthogonal projection of u onto a finite dimensional subspace with respect to a certain scalar product. For L^2-projection the subspace is V_h and the scalar product the usual L^2-product $\int uv dx$. However, for Ritz projection the subspace is $V_{h,0} \subset V_h$ and the scalar product $\int au'v'dx$, where $a \geq a_0 > 0$ is a positive weight function. The practical consequence of this is that the mass matrix should be replaced by the stiffness matrix when switching from computing L^2 - to Ritz projections. We shall not study Ritz projection in depth, but only state its definition and approximation properties.

Definition 1.5.3.
The Ritz projection $R_h u \in V_{h,0}$ to a given function $u \in V_h$ is defined by

$$\int_{\Omega} a(u - R_h u)'v'dx = 0, \quad \forall v \in V_{h,0}.$$

With this definition we have the following approximation result [13].

Proposition 1.8. *The following estimate holds.*

$$\|u - R_h u\| \leq Ch^2 \|D^2 u\|.$$

1.5.4 The Ritz-Volterra Projection Operator V_h

If we consider the following initial boundary value problem of a parabolic integro-differential equation:

$$\begin{cases} u_t + Au = \int_0^t Bu(s)ds + f, & (x,t) \in Q_T = \Omega \times J, J = (0,T], \\ u = 0 \quad \text{on} \quad \partial\Omega \times J, \\ u(0) = u_0 \quad \text{in } \Omega. \end{cases} \tag{1.8}$$

where A is a uniformly elliptic operator of second order, and B is an arbitrary partial differential operator of second order.

In order to analyse the semidiscrete finite element approximation to the parabolic problem, we introduce the Ritz-Volterra projection (or quasielliptic projection) $V_h u(t) \in S_h$ which satisfies:

$$A(V_h u(t) - u(t), v) = \int_0^t B(V_h u(s) - u(s), v)\, ds, \quad v \in S_h. \tag{1.9}$$

Where S_h is a piecewise subspace [2].

Also it is shown that using $V_h u$ as a comparison function, the error analysis of the finite element for PIDE can be simplified (see chapter 3).

CHAPTER 2

PRESENTATION OF FINITE ELEMENT METHODS

In this chapter we shall introduce the finite element method (FEM) as a numerical technique for solving problems which are described by partial differential equations or can be formulated as functional minimization. A domain of interest is represented as an assembly of finite elements. Approximating functions in finite elements are determined in terms of nodal values of a physical field which is sought. A continuous physical problem is transformed into a discretized finite element problem with unknown nodal values. For a linear problem a system of linear algebraic equations should be solved. Values inside finite elements can be recovered using nodal values.

2.1 General Principle of The Methods

2.1.1 General formulation

The general approach of the finite element method is as follows. We have an PDE to solve on a domain $\Omega \in \mathbb{R}^n, (n = 1, 2, 3)$. We write the variational formulation of this PDE, and then we obtain a problem of the type:

$$\text{Find } u \in V \text{ such that } a(u, v) = L(v), \quad \forall v \in V. \tag{2.1}$$

Where assuming that V is a Hilbert space [8].

We next try to approximate u by a continuous piecewise linear function. To this end we introduce a mesh on a domain $\Omega \in \mathbb{R}^n, (n = 1, 2, 3)$, consisting of N subintervals, and the corresponding space V_h of finite dimension is a subspace of V (which is usually of infinite dimension), of all continuous piecewise linears. Since we are dealing with functions vanishing at the end-points of Ω, then the approximate problem is:

$$\text{Find } u_h \in V_h \text{ such that } a(u_h, v_h) = L(v_h), \quad \forall v_h \in V_h. \tag{2.2}$$

The existence and uniqueness of a solution for this problem is proved using the Lax-Milgram theorem [13].

Theorem 2.1 (Lax-Milgram).
Let V be a Hilbert space with norm $\|.\|_V$ and scalar product $(.,.)_V$, and assume a is a bilinear form on $V \times V$, and L is a linear form on V, that satisfy:

- *a is is symmetric, i.e. $a(u,v) = a(v,u), \quad \forall u,v \in V$.*
- *a is continuous, i.e. $\exists M \in \mathbb{R}$ such that*

$$|a(u,v)| \leq M\|u\|_V\|v\|_V, \quad \forall u,v \in V. \tag{2.3}$$

- *a is V-elliptic (coercive), i.e. $\exists \eta > 0$ such that*

$$a(u,u) \geq \eta\|u\|_V^2, \quad \forall v \in V. \tag{2.4}$$

- *L is continuous, i.e. $\exists K \in \mathbb{R}$ such that*

$$|L(u)| \leq K\|u\|_V, \quad \forall v \in V. \tag{2.5}$$

Then there is a unique function $u \in V$ such that $a(u,v) = L(v)$ for all $v \in V$.

2.1.2 Error Estimation

We wish to bound from above the error $e = u - u_h$ in the ambient norm $\|.\|_V$. We first observe an important property:

Lemma 2.1 (Galerkin orthogonality).
The error $e = u - u_h$ satisfies

$$\forall v_h \in V_h, \quad a(e, v_h) = 0. \tag{2.6}$$

Proof.
The exact solution u satisfies $a(u,v) = L(v)$ for any $v \in V$. Since $V_h \subset V$, we can take a test function in V_h and thus deduce that $a(u,v_h) = L(v_h)$ for any $v_h \in V_h$.
Using that $a(u_h,v_h) = L(v_h)$ for any $v_h \in V_h$, we immediately get the result. □

We are now in position to establish the celebrated Céa lemma, which states that the error $u - u_h$ is bounded from above (up to an explicit multiplicative constant) by the best approximation error.

Lemma 2.2 (Céa lemma).
We suppose that assumptions (2.3),(2.4) and (2.5) holds. Let u be the solution to (2.1) and u_h be the solution to (2.2). Then

$$\|u - u_h\|_V \leq \frac{M}{\eta} \inf_{v_h \in V_h} \|u - v_h\|_V. \tag{2.7}$$

Proof.
For any $v_h \in V_h$ we deduce from the Galerkin orthogonality (2.6) that $a(u - u_h, u_h - v_h) = 0$.
Therefore, we have

$$\eta\|u - u_h\|_V^2 \leq a(u - u_h, u - u_h) = a(u - u_h, u - v_h) \leq M\|u - u_h\|_V\|u - v_h\|_V.$$

We can now divide by $\|u - u_h\|_V$ to obtain

$$\|u - u_h\|_V \leq \frac{M}{\eta}\|u - v_h\|_V$$

As the function v_h is arbitrary in V_h, we deduce (2.7). □

The quantity $\inf_{v_h \in V_h} \|u - v_h\|_V$ is usually called the best approximation error. It is an upper bound (up to a multiplicative constant) of the error $u - u_h$. Obviously, it is also a lower bound of the error, since $\inf_{v_h \in V_h} \|u - v_h\|_V \leq \|u - u_h\|_V$.
If the best approximation error converges to 0, then so does the error $u - u_h$, at the same rate [8].

2.1.3 The linear system

The space V_h is of finite dimension, therefore the problem (2.2) can actually be recasted as a linear system, as we now show. Denote N the dimension of V_h and let $(\varphi_1, \ldots, \varphi_N)$ be a basis of V_h. We expand $u_h \in V_h$ on the basis of V_h as:

$$u_h = \sum_{i=1}^{N} u_i \varphi_i$$

with $u_i = u_h(x_i)$ is the approximate value of the exact solution at the point x_i. And by considering φ_j as test function, we see that the problem (2.2) is equivalent to:

$$\text{Find } u_1, \ldots, u_N \text{ such that } \sum_{i=1} u_i a(\varphi_i, \varphi_j) = L(\varphi_j), \quad \forall j = 1, \ldots, N. \tag{2.8}$$

We thus introduce the matrix $A \in \mathbb{R}^{N \times N}$ defined by:

$$A_{ij} = a(\varphi_i, \varphi_j)$$

and the vector $F \in \mathbb{R}^N$ defined by:

$$F_j = L(\varphi_j)$$

and recast the above problem as the linear system:

$$\begin{pmatrix} a(\varphi_1, \varphi_1) & \cdots & a(\varphi_n, \varphi_1) \\ \vdots & & \vdots \\ a(\varphi_1, \varphi_n) & \cdots & a(\varphi_n, \varphi_n) \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} l(\varphi_1) \\ \vdots \\ l(\varphi_n) \end{pmatrix} \tag{2.9}$$

In matrix form we write this :

$$AU = F.$$

The matrix A is called the stiffness matrix, by reference to problems in mechanics where it has first been introduced. This matrix has properties that directly come from the properties satisfied by the bilinear form a, and F as the load vector [6], [8].

2.1.4 Lagrange Finite Elements

Definition 2.1.1.
A finite element of Lagrange is a triplet (K, Σ, P) such that:

- *K is a geometric element of $\mathbb{R}^n$, $(n = 1, 2,$ or $3)$, compact, connexe, and interior not empty.*
- *$\Sigma = \{a_1, \ldots, a_N\}$ is a finite set of N distinct points of K.*
- *P is a finite dimensional vector space of real functions defined on K.*

Definition 2.1.2.
Let (K, Σ, P) be a finite element of Lagrange. We call "the local base function" of the element the N functions $p_i, i = 1, \ldots, N$ of P such that:

$$p_i(a_j) = \delta_{ij}, \quad 1 \leq i, j \leq N.$$

We can easily verify that $(p_1, \ldots, p_N)$ defined a basis of P.

Definition 2.1.3.
We call operator of P-interpolation on Σ the operator π_K, which at any function v defined on K, combines the function $\pi_K v$ of P defined by:

$$\pi_K v = \sum_{i=1}^{N} v(a_i) p_i.$$

Theorem 2.2.
$\pi_K v$ is therefore the only element of P which takes the same values as v on the points of Σ.

Example 2.1 (Lagrange Finite Elements)**.**
We denote by P_k the vector space of polynomials of total degree less than or equal to k.

- *On $\mathbb{R}$, $P_k = Vect\{1, X, \ldots, X^k\}$ and dim $P_k = k + 1$.*
- *On $\mathbb{R}^2$, $P_k = Vect\{X^i Y^j; 0 \leq i + j \leq k\}$ and dim $P_k = \dfrac{(k+1)(k+2)}{2}$.*
- *On $\mathbb{R}^3$, $P_k = Vect\{X^i Y^j Z^l; 0 \leq i + j + l \leq k\}$ and dim $P_k = \dfrac{(k+1)(k+2)(k+3)}{6}$*

We denote by Q_k the vector space of polynomials of degree less than or equal to k with respect to each variable.

- *On $\mathbb{R}$, $Q_k = P_k$.*
- *On $\mathbb{R}^2$, $Q_k = Vect\{X^i Y^j; 0 \leq i, j \leq k\}$ and dim $Q_k = (k+1)^2$.*
- *On $\mathbb{R}^3$, $Q_k = Vect\{X^i Y^j Z^l; 0 \leq i, j, l \leq k\}$ and dim $Q_k = (k+1)^3$.*

[10].

2.2 Model Problem

2.2.1 Variational Formulation

Let us consider the following homogeneous Dirichlet problem in dimension $n = 1$:

$$\begin{cases} -\alpha u''(x) + \beta u(x) = f(x), \quad on \quad \Omega =]a, b[\\ u(a) = u(b) = 0 \end{cases} \tag{2.10}$$

Where α, β are constants and f is a given continuous function [14].

The derivation of a finite element method always starts by rewriting the differential equation under consideration as variational equation. In our case this so-called variational formulation is obtained by multiplying $-\alpha u'' + \beta u = f$ by a test function v, which is assumed to vanish at the end-points of the interval Ω, and integrate by parts. This leads to the following variational formulation of the problem (2.10):

$$\begin{cases} \text{Find } u \in H_0^1([a,b]) \text{ where:} \\ \int_a^b \alpha u'(x)v'(x)dx + \int_a^b \beta u(x)v(x)dx = \int_a^b f(x)v(x)dx, \quad \forall v \in H_0^1([a,b]). \end{cases} \tag{2.11}$$

Thus the following form of the problem (2.11):

$$\begin{cases} \text{Find } u \in H_0^1([a,b]) \\ a(u,v) = L(v), \quad \forall v \in H_0^1([a,b]). \end{cases} \tag{2.12}$$

Where

$$a(u,v) = \int_a^b \alpha u'(x)v'(x)dx + \int_a^b \beta u(x)v(x)dx \tag{2.13}$$

and

$$L(v) = \int_a^b f(x)v(x)dx.$$

The existence and uniqueness of a solution for this problem is proved using the Lax-Milgram theorem.
Where, a is a bilinear form, and from (2.13) we have

$$|a(u,v)| = \left| \int_a^b \alpha u'(x)v'(x)dx + \int_a^b \beta u(x)v(x)dx \right|.$$

Using the Cauchy–Schwarz, Poincaré inequalities and Sobolev norm, we have

$$\begin{aligned} |a(u,v)| &\leq \alpha \|u'\|_{L^2}\|v'\|_{L^2} + \beta\|u\|_{L^2}\|v\|_{L^2} \\ &\leq \alpha\|u\|_{H_0^1}\|v\|_{H_0^1} + c^2\beta\|u\|_{H_0^1}\|v\|_{H_0^1} = M\|u\|_{H_0^1}\|v\|_{H_0^1}. \end{aligned}$$

Where $M = \alpha + c^2\beta$. Thus a is continuous.
Furthermore, we prove the V-ellipticity of a. For this purpose, we have for $v \in H_0^1([a,b])$

$$a(v,v) = \alpha \int_a^b v'(x)v'(x)dx + \beta \int_a^b v(x)v(x)dx \geq \alpha \int_a^b (v'(x))^2 dx = \alpha\|v\|_{H_0^1}^2. \tag{2.14}$$

Thus

$$a(v,v) \geq \eta\|v\|_{H_0^1}^2, \tag{2.15}$$

where $\eta = \alpha$. Thus a is a V-elliptic if $\eta > 0$. Therefore, by Lax-Milgram theorem the equation (2.10) has a unique solution.

2.2.2 Domain Discretization

Admissible Meshing on 1D

Let's introduce a discretization of the interval $[a,b]$ in N subintervals or elements $T_i = [x_{i-1}, x_i]$. The elements T_i dose not necessarily have the same length. V_h is then the space of piecewise affine continuous functions (affines on T_i segments) and zero at $x = a$ and $x = b$.

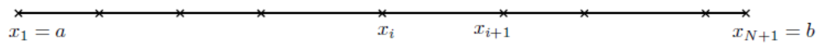


Figure 2.1: Discretization (mesh) of the segment $[a, b]$ in finite element P_1

with a local discretization step $h_i = x_{i+1} - x_i, \quad 0 \leq i \leq N$.

We pose that $h = max_{0 \leq i \leq N} h_i$. The points $\{x_i\}_{0 \leq i \leq N+1}$ constitute the vertices of the mesh and the $\{x_i\}_{0 \leq i \leq N}$ are the inner vertices. Les segments $\{[x_i, x_{i+1}]\}_{0 \leq i \leq N}$ are called the elements of the mesh or more simply the meshes. In a dimension of space, the relationship between the number of vertices N_v, the number of interior vertices $N_{i,v}$, and the number of meshes N_m is very simple:

$$N_v = N_{i,v} + 2 = N_m + 1 = N + 2$$

For reasons of simplicity, we would often assume that the vertices of the mesh are regularly spaced so that for all $0 \leq i \leq N + 1$, we have $x_i = a + ih$ with $h = \dfrac{b-a}{N+1}$.

Remember that each function $v_h \in V_h$ is uniquely determined by the data of its point values x_i for $i = 2, \dots, N$. Space V_h is of dimension $N - 1$ and it is generated by the Lagrange base which is formed of $N - 1$ functions $\varphi_i \in V_h$ defined by:

$$\varphi_i\left(x_j\right) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if not} \end{cases} \quad \forall i = 2, \dots, N \quad \text{and} \quad \forall j = 2, \dots, N$$

Hence:

$$\varphi_i(x) = \begin{cases} \frac{x_i - x}{x_i - x_{i-1}} & \text{if } x \in [x_{i-1}, x_i] \\ \frac{x_i - x}{x_i - x_{i+1}} & \text{if } x \in [x_i, x_{i+1}] \\ 0 & \text{if } x \notin [x_{i-1}, x_{i+1}], \end{cases}$$

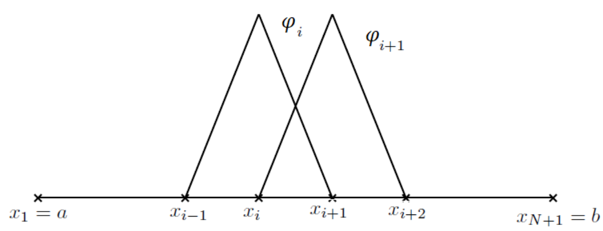


Figure 2.2: Basic functions of P_1.

Any function v_h is written in this base:

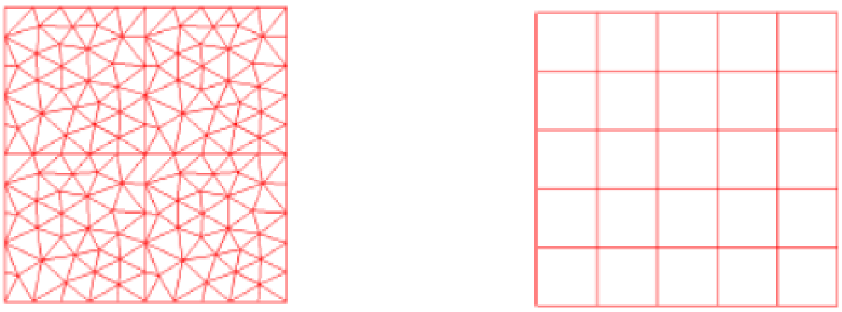
$$v_h(x) = \sum_{i=2}^{N} v_i \varphi_i(x).$$

With $v_i = v_h(x_i)$. The coefficients v_i are therefore the values of v_h at x_i points [6].

Remark 2.1 (**Admissible Meshing on 2D**).
Let τ_h a rectangle on a domain Ω using convex quadrilaterals K_i of diameter $\leq h$, so that means: if K_i and K_j are any two but distinct elements of τ_h, we would have:

$$K_i \cap K_j = \begin{cases} \text{Let } \phi. \\ \text{Let be a common summit.} \\ \text{Let a common side.} \end{cases}$$

In the latter eventuality, the volumes K_i and K_j will be said to be adjacent.
One can also use an admissible triangular mesh which fulfills the preceding conditions [10].



(a) Triangular mesh. (b) Rectangular mesh.

Figure 2.3: Admissible mesh on 2D.

2.2.3 The approximate problem

In that case, the approximation method is said to be conformal. In the sequel, we approximate the continuous problem (2.12) by the following problem:

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a(u_h, v_h) = L(v_h), \quad \forall v_h \in V_h. \end{cases} \tag{2.16}$$

Where

$$a(u_h, v_h) = \int_a^b \alpha u_h'(x) v_h'(x) dx + \int_a^b \beta u_h(x) v_h(x) dx \tag{2.17}$$

and

$$L(v_h) = \int_a^b f(x) v_h(x) dx.$$

On the other hand, if u_h is solution of the problem approached in V_h, we can express it in the base of φ_i as follows:

$$u_h(x) = \sum_{i=2}^{N} u_j \varphi_j(x),$$

with $u_j = u_h(x_j)$ is the approximate value of the exact solution at the point x_j. Thus the approached problem is written as:

$$\begin{cases} \text{Find } u_1, u_2, \ldots, u_N \text{ such that} \\ \sum_{j=2}^{N} \left(\int_a^b \alpha \varphi_j'(x) \varphi_i'(x) dx + \int_a^b \beta \varphi_j(x) \varphi_i(x) dx \right) u_j = \int_a^b f(x) \varphi_i(x) dx. \end{cases} \tag{2.18}$$

Where:

$$F_i := \int_a^b f(x)\varphi_i(x)dx$$

and

$$A_{ij} := \int_a^b \alpha\varphi_j'(x)\varphi_i'(x)dx + \int_a^b \beta\varphi_j(x)\varphi_i(x)dx.$$

We notice that the approximate problem takes the form of a linear system of $N-1$ equations with $N-1$ unknowns, which can be written in the following matrix form:

$$AU = F.$$

2.2.4 Error Estimation

We have u_h is an approximate solution of (2.16), and u is the exact solution of (2.12), we get:

$$a(u, v_h) = L(v_h), \quad \forall v_h \in V_h \tag{2.19}$$

and also

$$a(u_h, v_h) = L(v_h), \quad \forall v_h \in V_h. \tag{2.20}$$

If $e = u - u_h$, Then from Lemma 2.1

$$a(e, v_h) = 0, \quad \forall v_h \in V_h.$$

We define the inner product energy as follows:

$$\begin{aligned}&(.,.) : V \times V \longrightarrow R\\ &(u, v)_a = a(u, v)\end{aligned}$$

and also the energy norm $\|u\|_E^2 = (u, u)_a$.
By the relation between the inner product and energy norm, using the Schwarz inequality, we have

$$|a(v, w)| \le \|v\|\|w\|, \quad \forall v, w \in V.$$

Since $(e, v_h)_a = a(e, v_h) = 0$, Lemma 2.1 yields that e is orthogonal to any $v_h \in V_h$.

Also, for each particular $\widehat{v}_h = \sum_{i=1}^{n} \xi_i\varphi_i(x) \in V_h$, and $\|u - u_h\| = \min\{\|u - v_h\|; v_h \in V_h\}$, and using Cea's Lemma 2.2, we have

$$\inf \|u - v_h\|_V \le \|u - \widehat{v}_h\|_V$$

if $\widehat{v}_h$ is equal to $\widehat{u}_h$, and we get an upper bound for the interpolation error. Also,

$$\|u - u_h\|_V \le C\|u - \widehat{u}_h\|_V \le Ch^{\zeta}, \quad \zeta > 0$$

where the constant $C = \dfrac{M}{\eta}$ is independent of h. Therefore

$$\|u - u_h\|_V \le Ch^{\zeta}.$$

Hence the norm of error tends to zero as $h \longrightarrow 0$, and the order of the method is $O(h^{\zeta})$ [19].

2.2.5 Numerical Application

We propose to seek an approximate solution to the following problem:
Find $u(x)$ in the domain $\Omega = [0,1]$ satisfying the differential equation

$$\frac{d^2u(x)}{dx^2} + 2\frac{du(x)}{dx} - 3x = 0 \tag{2.21}$$

with the border conditions of Ω

$$u(0) = 0 \text{ and } \left[\frac{du(x)}{dx}\right]_{x=1} = 0.$$

In this problem, Ω is a dimension 1 domain and time does not appear.

- Variational Formulation:

  We have $V = \left\{u \in H^1(\Omega)/u(0) = 0, \dfrac{\partial u}{\partial x}(1) = 0\right\}$.
  Solving the differential equation is equivalent to Find $u(x) \in V$ such that

  $$\int_\Omega v(x)\left(\frac{d^2u(x)}{dx^2} + 2\frac{du(x)}{dx} - 3x\right)dx = 0, \quad \forall v \in V, \tag{2.22}$$

  with

  $$u(0) = 0 \text{ and } \left[\frac{du(x)}{dx}\right]_{x=1} = 0.$$

  This is the variational formulation of the problem (2.21).

  We obtain, after using the integration by parts and the boundary conditions, $u(0) = 0$ and $\left[\dfrac{du(x)}{dx}\right]_{x=1} = 0$, the following variational formulation of the problem (2.21), so we have:
  Find $u(x)$ such that

  $$\begin{aligned} &-\int_0^1 \frac{dv(x)}{dx}\frac{du(x)}{dx}dx + 0 - v(0)\left[\frac{du(x)}{dx}\right]_{x=0} \\ &-2\int_0^1 \frac{dv(x)}{dx}u(x)dx + 2v(1)u(1) - 0 \\ &-3\int_0^1 v(x)u(x)dx = 0, \quad \forall v(x) \in V. \end{aligned} \tag{2.23}$$

- Choice of mesh:

  We arbitrarily divide Ω into three meshes of the same size (see figure 2.4)

  $$\Omega_1 = [0, \frac{1}{3}], \quad \Omega_2 = [\frac{1}{3}, \frac{2}{3}], \quad \Omega_3 = [\frac{2}{3}, 1].$$

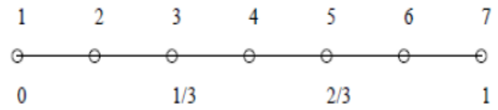


Figure 2.4: Meshing of the problem.

- Choice of nodes and local fields:

We decide to take elements with 3 nodes, and for the family of local fields of the polynomials of degree 2. The nodes are chosen at the ends and in the middle of each mesh.

Each local field can therefore be expressed as a function of the values at the nodes. Indeed, there exists only one polynomial of the second degree which satisfies the following conditions:

$$\begin{cases} a_1 x_1^2 + b_1 x_1 + c_1 = u_1 \\ a_1 x_2^2 + b_1 x_2 + c_1 = u_2 \\ a_1 x_3^2 + b_1 x_3 + c_1 = u_3. \end{cases}$$

Where x_1, x_2, and x_3 are the coordinates of the 3 nodes of the elements, and u_1, u_2, and u_3 are values of u at the 3 nodes of the element.
We obtain

$$a_1 = \frac{x_2 u_1 - x_2 u_3 + x_1 u_3 - u_1 x_3 - u_2 x_1 + u_2 x_3}{(x_2 - x_3)(x_1 - x_3)(x_1 - x_2)}, \tag{2.24}$$

$$b_1 = \frac{x_1^2 u_3 - x_1^2 u_2 + x_3^2 u_2 - u_1 x_3^2 - u_3 x_2^2 + u_1 x_2^2}{(x_2 - x_3)(x_1 - x_3)(x_1 - x_2)}, \tag{2.25}$$

$$c_1 = \frac{x_1^2 x_2 u_3 - x_1^2 u_2 x_3 - u_3 x_2^2 x_1 + x_3^2 u_2 x_1 + u_1 x_2^2 x_3 - x_3^2 x_2 x_1}{(x_2 - x_3)(x_1 - x_3)(x_1 - x_2)}. \tag{2.26}$$

The polynomial with the coefficients a_1, b_1, and c_1 above is called "the interpolation function" of element 1.

The interpolation polynomials of the other 2 are of the same form: For the second, we replace the indices $1;2;3$ by indices $3;4;5$ and for the third, we replace the indices $1;2;3$ by indices $5;6;7$.

Hence, we obtain the three interpolation polynomials of the 3 elements, as follows:

$$\tilde{u}_1 = (18u_3 - 36u_2 + 18u_1)x^2 + (-3u_3 + 12u_2 - 9u_1)x + u_1. \tag{2.27}$$

$$\tilde{u}_2 = (18u_5 - 36u_4 + 18u_3)x^2 + (-15u_5 + 36u_4 - 21u_3)x + 3u_5 - 8u_4 + 6u_3. \tag{2.28}$$

$$\tilde{u}_3 = (18u_7 - 36u_6 + 18u_5)x^2 + (-27u_7 + 60u_6 - 33u_5)x + 10u_7 - 24u_6 + 15u_5. \tag{2.29}$$

In this example, we have 7 nodes. If an arbitrary value is given to each node, the interpolation functions are determined and define by piecewise a c_0 continuous function $\tilde{u}$ on the domain Ω. The space of functions $\tilde{U}$, defined in 3 pieces, is of dimension 7.

- Discretization of the problem:

In our example, if we choose the formulation on the (2.23) , the functions $v_i(x)$ must have a nonzero derivative everywhere, so that the formulation is not trivial. On the other hand, in this formulation, the second derivative of the function $u(x)$, which will be replaced by the approximation $\tilde{u}(x)$; no longer appears. We could therefore choose a simpler approximation family $\tilde{u}(x)$.

We choose for example the 7 test functions $v_1, v_2, \ldots, v_7$ defined by the following figure:

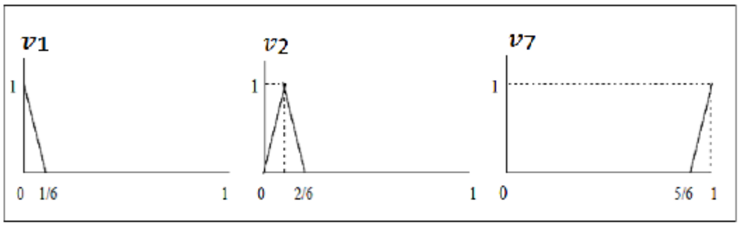


Figure 2.5: The seven $v_i(x)$ functions.

Their derivatives are constant (+6, or -6), or zero and we have $v_i(x) = \delta_{ij}$

For each test function v_i, we write a scalar equation:

$$-\int_0^1 \frac{dv_i(x)}{dx}\frac{d\tilde{u}(x)}{dx}dx - v_i(0)\left[\frac{d\tilde{u}(x)}{dx}\right]_{x=0} - 2\int_0^1 \frac{dv_i(x)}{dx}\tilde{u}(x)dx + 2v_i(1)\tilde{u}(1)$$
$$-3\int_0^1 v_i(x)xd(x) = 0 \text{ pour } i = \overline{1,7}.$$

Where $\tilde{u}(x)$ are the function of x, and values at nodes u_1, u_2, and u_3.

The integrals of these 7 equations are easy to calculate: they are reduced to integrals on the segment where the derivatives of v_i are non-zero.
For example, the first equation (with v_i) is:

$$-\int_0^{\frac{1}{6}} \frac{dv_1(x)}{dx}\frac{d\tilde{u}_1(x)}{dx}dx - v_1(0)\left[\frac{d\tilde{u}_1(x)}{dx}\right]_{x=0}$$
$$-2\int_0^{\frac{1}{6}} \frac{dv_1(x)}{dx}\tilde{u}_1(x)dx + 2v_1(1)\tilde{u}_1(1) - 3\int_1^{\frac{1}{6}} v_1(x)xdx = 0.$$

Either still

$$-\int_0^{\frac{1}{6}}(-6)\frac{d\tilde{u}_1(x)}{dx}dx - \left[\frac{d\tilde{u}_1(x)}{dx}\right]_{x=0} - 2\int_0^{\frac{1}{6}}(-6)\tilde{u}_1(x)dx - 3\int_0^{\frac{1}{6}}(1-6x)xdx = 0. \quad (2.30)$$

Where $\tilde{u}(x) = ax^2 + bx + c$ on the segment $[0, \frac{1}{6}]$, with a, b, and c defined on (2.24), (2.25), (2.26). So the same is done for the other 6 equations.

We thus obtain a system of 7 equations with 7 unknowns (the values at nodes $u_1, u_2, \ldots, u_7$):

$$\begin{cases} \frac{23}{6}u_1 - \frac{14}{6}u_2 + \frac{17}{6}u_3 - \frac{1}{72} & = 0 \\ 5u_1 - 12u_2 + 7u_3 - \frac{1}{12} & = 0 \\ \frac{1}{6}u_1 + \frac{14}{3}u_2 - 12u_3 + \frac{22}{3}u_4 - \frac{1}{6}u_5 - \frac{1}{6} & = 0 \\ \frac{1}{6}u_3 + \frac{14}{3}u_4 - 12u_5 + \frac{22}{3}u_6 - \frac{1}{6}u_7 - \frac{1}{3} & = 0 \\ 5u_5 - 12u_6 + 7u_7 - \frac{5}{12} & = 0 \\ \frac{1}{6}u_5 + \frac{14}{3}u_6 - \frac{29}{6}u_7 - \frac{17}{72} & = 0 \end{cases} \tag{2.31}$$

This system can be written in the following matrix form

$$AU = F$$

Hence

$$\begin{bmatrix} \frac{23}{16} & -\frac{14}{6} & \frac{17}{6} & 0 & 0 & 0 & 0 \\ 5 & -12 & 7 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{14}{6} & -12 & \frac{22}{3} & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & 5 & -12 & 7 & 0 & 0 \\ 0 & 0 & \frac{1}{6} & \frac{14}{3} & -12 & \frac{22}{3} & -\frac{1}{6} \\ 0 & 0 & 0 & 0 & 5 & -12 & 7 \\ 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{14}{3} & -\frac{29}{6} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} \frac{1}{72} \\ \frac{1}{12} \\ \frac{1}{6} \\ \frac{1}{4} \\ \frac{1}{3} \\ \frac{5}{12} \\ \frac{17}{72} \end{bmatrix}$$

The matrix A is invertible therefore $u = A^{-1}F$.

The solution of this system gives the values to the nodes of the approximation $\tilde{u}(x)$ sought

$$u_1 = \frac{4107}{54872}, \quad u_2 = -\frac{7063}{8664}, \quad u_3 = -\frac{236893}{164616}, \quad u_4 = -\frac{25563}{13718}$$
$$u_5 = -\frac{350779}{164616}, \quad u_6 = -\frac{374605}{164616}, \quad u_7 = -\frac{127275}{54872}.$$

CHAPTER 3

FINITE ELEMENT APPROXIMATION OF A PARABOLIC INTEGRO-DIFFERENTIAL EQUATION

This chapter is divided in two section, the first one discusses the error estimates for the numerical solution by means of the Galerkin finite element method of an integro-differential equation of parabolic type with the case of weakly singular kernel. Optimal-order estimates are shown for spatially semidiscrete and completely discrete methods. And the second section discusses a numerical model of the method.

3.1 General Principle

3.1.1 General Problem

We shall consider the initial value problem (with $u_t = \partial u/\partial t$)

$$\begin{cases} u_t(x,t) + Au(x,t) = \int_0^t K(x,t-s)Bu(x,s)ds + f(x,t), \quad x \in \Omega, \quad \text{for } t > 0 \\ u = 0 \quad \text{on } \partial\Omega, \quad t > 0 \\ u(x,0) = u_0(x) \quad x \text{ in } \Omega. \end{cases} \tag{3.1}$$

Where A is a linear positive self adjoint elliptic and B a general partial differential operator of second order with smooth, time-independent coefficients, where K is a weakly singular kernel $K(t)$ such that

$$|K(t)| \leq Ct^{-\alpha} \text{ with } 0 \leq \alpha < 1, \text{ for } t > 0, \tag{3.2}$$

and Ω is a sufficiently smooth domain in $\mathbb{R}^n, n > 1$.

3.1.2 Discretization of the Problem

For the numerical solution we assume that we are given a family $\{S_h\}$ of finite dimensional sub-spaces of $H_0^1 = H_0^1(\Omega)$ such that

$$\inf_{\chi \in S_h} \{\|v - \chi\| + h\|v - \chi\|_1\} \leq Ch^2\|v\|^2, \quad \forall v \in H^2 \cap H_0^1, \tag{3.3}$$

where $\|.\|$ is the norme in $L^2 = L^2(\Omega)$ and $\|.\|_s$ that in $H^s = H^s(\Omega)$.

- **Semidiscretization:**

We consider first the semidiscrete problem of finding $u_h : [0,\infty) \longrightarrow S_h$ such that

$$\begin{aligned} (u_{h,t},\chi) + A(u_h,\chi) &= \int_0^t K(t-s)B(u_h(s),\chi)ds + (f(t),\chi), \quad \forall \chi \in S_h, \quad t>0 \\ u_h(0) &= u_{0h}. \end{aligned} \tag{3.4}$$

Where $(.,.)$ is the inner product in L^2 and $A(.,.)$ and $B(.,.)$ are the bilinear forms on H_0^1 associated with the differential operators A and B, and where u_{0h} is an appropriate approximation in S_h of the initial data in (3.1). We shall show that, for each $T>0$, we then have the error estimate

$$\|u_h(t) - u(t)\| \leq C_T h^2 \left\{ \|u_0\|_2 + \int_0^t \|u_t\|_2 ds \right\}, \text{ for } t \leq T. \tag{3.5}$$

- **Full Discretization:**

We shall also consider the discretization in time of (3.4). Thus, let k be a time step such that $k = \frac{T}{N}$, and let $U^n \in S_h$ be the approximation of the exact solution of (3.1) at time $t_n = nk$.
The time discretization considered will be based on the backward difference quotient

$$\overline{\partial}_t U^n = \frac{U^n - U^{n-1}}{k}.$$

The integral term then has to be evaluated by numerical quadrature from the values of the U^n, but since the integrand is singular, even when the solution is smooth, we shall use product integration:
We shall approximate ϕ in $J_n(\phi) = \int_0^{t_n} K(t_n - s)\phi(s)ds$ by the piecewise constant function taking the value $\phi(t_j)$ in (t_j, t_{j+1}), and thus use

$$J_n(\phi) \approx Q_n(\phi) = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} K(t_n - s)\phi(t_j)ds = \sum_{j=0}^{n-1} \kappa_{n-j}\phi(t_j),$$

where

$$\kappa_j = \int_{t_{j-1}}^{t_j} K(s)ds. \tag{3.6}$$

Our completely discrete scheme is therefore

$$\begin{aligned} (\overline{\partial}_t U^n,\chi) + A(U^n,\chi) &= \sum_{j=0}^{n-1} \kappa_{n-j} B(U^j,\chi) + (f(t_n),\chi), \quad \forall \chi \in S_h, \quad n \geq 1, \\ U^0 &= u_{0h}. \end{aligned} \tag{3.7}$$

For this completely discrete method we shall show

$$\|U^n - u(t_n)\| \leq C_T(h^2 + k) \left\{ \|u_0\|_2 + \int_0^{t_n} (\|u_{tt}\| + \|u_t\|_2) ds \right\} \text{ for } t \leq T. \tag{3.8}$$

Before the analysis of these discrete methods, we shall mention the existence of the solution of (3.1).
And the following is the main existence result.

Theorem 3.1.
Assume that $u_0 \in H^{\beta} \cap H_0^1$, $f \in \mathcal{C}([0,T], H^{\beta-2})$ and $t^{\gamma} f_t \in L^{\infty}([0,T], H^{\beta-2})$ with $\beta > 2$, $0 < \gamma < 1$. Then there exist a unique solution of (3.1) in $\mathcal{C}([0,T], L^2)$.
Furthermore,

$$u \in \mathcal{C}([0,T], H^2 \cap H_0^1), \quad u_t \in \mathcal{C}([0,T], L^2) \cap L^1([0,T], H^2 \cap H_0^1), \quad \text{and } u_{tt} \in L^1([0,T], L^2).$$

Proof. see [2],[3]. □

Then, we shall need the following version of Gronwall's lemma [3].

Lemma 3.1.
Assume that y is a non-negative function in $L^1(0,T)$ which satisfies

$$y(t) \leq b(t) + \beta \int_0^t (t-s)^{-\alpha} y(s) ds, \text{ for } 0 < t \leq T. \tag{3.9}$$

Where $b(t) \geq 0$, $\beta \geq 0$. Then there is a constant C_T such that

$$y(t) \leq b(t) + C_T \int_0^t (t-s)^{-\alpha} b(s) ds, \text{ for } t \leq T.$$

We shall also need the following lemma [3].

Lemma 3.2.
Let $K \in L^1(0,T)$. Then for each $\varepsilon > 0$ there is a constant $C_{\varepsilon} = C_{\varepsilon}(\|K\|_{L^1(0,T)})$ such that

$$\left| \int_0^T \int_0^t K(t-s) f(s) f(t) ds dt \right| \leq \varepsilon \int_0^T f(t)^2 dt + C_{\varepsilon} \int_0^T |K(T-t)| \int_0^t f(s)^2 ds dt. \tag{3.10}$$

3.1.3 Error Estimation

- **Discretization in Space:**

In this section we shall derive the error estimate (3.5) stated above for the semidiscrete method (3.4).
Where $u_{0h} \in S_h$ is an approximation to u_0 satisfying

$$\|u_{0h} - u_0\| \leq C h^2 \|u_0\|_2.$$

The main result is the following [2].

Theorem 3.2.
Assume that $u(t)$ and $u_h(t) \in S_h$ are the solutions of (3.1) and (3.4) respectively, then for each $T > 0$ there is a constant C_T, therefore

$$\|u_h(t) - u(t)\| \leq C_T h^2 \left\{ \|u_0\|_2 + \int_0^t \|u_t\|_2 ds \right\}, \text{ for } t \leq T.$$

Before proving the theorem, we need to mention the following two lemmas.

For the analysis we introduce the Ritz-Volterra projection $V_h u \in S_h$ satisfying

$$A(V_h u(t) - u(t), \chi) = \int_0^t K(t-s)B(V_h u(s) - u(s), \chi)ds, \quad \forall \chi \in S_h, \quad t \geq 0. \tag{3.11}$$

Lemma 3.3.
For the Ritz-Volterra projection, we have:

$$\|(V_h u - u)(t)\| + h\|(V_h u - u)(t)\|_1 \leq Ch^2 \sup_{s\leq t} \|u(s)\|_2 \leq Ch^2 \left\{ \|u_0\|_2 + \int_0^t \|u_t\|_2 ds \right\}.$$

Proof. Let $W = V_h u$ and $\rho = W - u$. We begin with an H^1-estimate, and introduce also the standard Ritz projection R_h defined by

$$A(R_h u - u, \chi) = 0, \quad \forall \chi \in S_h.$$

Recalling under the assumption (3.3) that

$$\|R_h u - u\| + h\|R_h u - u\|_1 \leq Ch^2 \|u\|_2.$$

We have, using the definition of W and notation $\theta = W - R_h u \in S_h$, that with $c > 0$

$$\begin{aligned} c\|\theta(t)\|_1^2 \leq A(\theta, \theta) = A(\rho, \theta) &= \int_0^t K(t-s)B(\rho(s), \theta(t))ds \\ &\leq C\|\theta(t)\|_1 \int_0^t (t-s)^{-\alpha} \|\rho(s)\|_1 ds \end{aligned}$$

and hence

$$\|\rho(t)\|_1 \leq C \int_0^t (t-s)^{-\alpha} \|\rho(s)\|_1 ds + \|(R_h u - u)(t)\|_1.$$

Where $\rho(t) = \theta + (R_h u - u)$.
Lemma 3.1 now implies

$$\|\rho(t)\|_1 \leq C_T \sup_{s\leq t} \|(R_h u - u)(s)\|_1 \leq C_T h \sup_{s\leq t} \|u(s)\|_2.$$

Consider now the L^2-estimate, which will be derived from a duality argument. Using

$$\|\rho(t)\| = \sup_{\|\phi\|=1} (\rho(t), \phi).$$

For each such ϕ, let ψ be the solution of

$$A\psi = \phi \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega,$$

and recall that

$$\|\psi\|_2 \leq C\|\phi\| = C. \tag{3.12}$$

Then, for $\chi \in S_h$,

$$(\rho(t), \phi) = A(\rho, \psi) = A(\rho, \psi - \chi) + A(\rho, \chi),$$

where

$$\begin{aligned} A(\rho,\chi) &= \int_0^t k(t-s)B(\rho(s),\chi)ds \\ &= \int_0^t k(t-s)B(\rho(s),\chi-\psi)ds + \int_0^t k(t-s)(\rho(s),B^*\psi)ds. \end{aligned}$$

Hence, with $\chi = R_h\psi$, using (3.12),

$$\begin{aligned} (\rho(t),\phi) &\le C\sup_{s\le t}\|\rho(s)\|_1\|R_h\psi-\psi\|_1 + C\int_0^t (t-s)^{-\alpha}\|\rho(s)\|ds\|\psi\|_2 \\ &\le C\left\{h^2\sup_{s\le t}\|u(s)\|_2 + \int_0^t (t-s)^{-\alpha}\|\rho(s)\|ds\right\}. \end{aligned}$$

Thus,

$$\|\rho(t)\| \le Ch^2\sup_{s\le t}\|u(s)\|_2 + C\int_0^t (t-s)^{-\alpha}\|\rho(s)\|ds.$$

By using Lemma 3.1, the proof of Lemma 3.3 is complete. □

For the time derivatives of ρ in the Ritz-Volterra projection there is the following estimate.

Lemma 3.4.
Under the assumptions of Lemma 3.3 we have for $\rho = V_hu - u$,

$$\int_0^t (\|\rho_t\| + h\|\rho_t\|_1)ds \le Ch^2\left\{\|u_0\|_2 + \int_0^t \|u_t\|_2 ds\right\}.$$

For the proof of Lemma 3.4 see [3].

Proof of Theorem 3.2:

In a standard fashion we write

$$u_h - u = (u_h - V_hu) + (V_hu - u) = \theta + \rho.$$

Lemma 3.3 immediately gives the desired estimate for ρ, so it remains to get a bound for θ. We have the following directly from our definition,

$$(\theta_t,\chi) + A(\theta,\chi) = \int_0^t K(t-s)B(\theta(s),\chi)ds + (\rho_t,\chi), \quad \forall\chi\in S_h,$$

setting $\chi = \theta$, it leads to

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + A(\theta,\theta) \le C\int_0^t (t-s)^{-\alpha}\|\theta(s)\|_1\|\theta(t)\|_1 ds + \|\rho_t\|.\|\theta\|.$$

By integration in t and ellipticity of A, this yields

$$\begin{aligned} &\|\theta(T)\|^2 + \int_0^T \|\theta\|_1^2 dt \\ &\le C\left\{\|\theta(0)\|^2 + \int_0^T\int_0^t (t-s)^{-\alpha}\|\theta(s)\|_1\|\theta(t)\|_1 dsdt + C\int_0^T \|\rho_t\|.\|\theta\|dt\right\}. \end{aligned}$$

Using Lemma 3.2 with a suitable choice of ε for the double integral, we thus have

$$\begin{aligned}
&\|\theta(T)\|^2 + \int_0^T \|\theta\|_1^2 dt \\
&\qquad \le C\left\{\|\theta(0)\|^2 + \int_0^T \|\rho_t\|.\|\theta\| dt + \int_0^T (T-t)^{-\alpha}\int_0^t \|\theta(s)\|_1^2 ds dt\right\}.
\end{aligned}$$

and obtain by Lemma 3.1 the bound

$$\|\theta(T)\|^2 + \int_0^T \|\theta\|_1^2 dt \le C_T\left\{\|\theta(0)\|^2 + \int_0^T \|\rho_t\|.\|\theta\| dt\right\},$$

Further, using Lemma 3.4 and noting that $V_h(0) = R_h$, it becomes

$$\begin{aligned}
\|\theta(T)\| &\le C_T\left\{\|\theta(0)\| + \int_0^T \|\rho_t\| dt\right\} \\
&\le C_T\left\{\|\theta(0)\| + \int_0^T (\|\rho_t\| + h\|\rho_t\|_1) ds\right\} \\
&\le C_T\left\{\|u_{0h} - R_h u_0\| + h^2\left(\|u_0\|_2 + \int_0^T \|u_t\|_2 ds\right)\right\} \\
&\le Ch^2\left\{\|u_0\|_2 + \int_0^T \|u_t\|_2 ds\right\}.
\end{aligned}$$

This completes the proof of the desired estimate for θ, and thus of the theorem. □

- **The Completely Discrete Scheme:**

In this section we shall consider the completely discrete method (3.7).
The following error estimate is the main result of this section.

Theorem 3.3.
For each $T > 0$, there is a constant C_T such that for the solutions of (3.7) and (3.1),

$$\|U^n - u(t_n)\| \le C_T(h^2 + k)\left\{\|u_0\|_2 + \int_0^{t_n} (\|u_{tt}\| + \|u_t\|_2) ds\right\} \ \textit{for } t \le T.$$

To prove Theorem 3.3 we need some lemmas.
In the next lemma we estimate a time-discrete $L^1([0,T];L^2(\Omega))$ type norm of the quadrature error

$$\varepsilon_n(\phi) = \sum_{j=0}^{n-1} \kappa_{n-j}\phi(t_j) - \int_0^{t_n} K(t_n - s)\phi(s) ds,$$

where κ_j is defined by (3.6).

Lemma 3.5.
For each $T > 0$, there is a constant C_T such that, if $\phi_t \in L^1([0,T],L^2)$, then

$$k\sum_{n=1}^{N} \|\varepsilon_n(\phi)\| \le C_T k\int_0^{t_N} \|\phi_t(s)\| ds \quad \textit{for } Nk \le T.$$

Proof. By the definition of the κ_j we have

$$\varepsilon_n(\phi) = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} K(t_n - s)(\phi(t_j) - \phi(s))ds,$$

and for each $x \in \Omega$, since $|K(t)| \leq Ct^{-\alpha}$,

$$\begin{aligned} |\varepsilon_n(\phi)| &\leq \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} |K(t_n - s)| \int_{t_j}^{t_{j+1}} |\phi_t(\sigma)| d\sigma ds \\ &\leq C \sum_{j=0}^{n-1} \mu_{\alpha,n-j} \int_{t_j}^{t_{j+1}} |\phi_t(\sigma)| d\sigma, \end{aligned}$$

where

$$\mu_{\alpha,j} = \int_{t_{j-1}}^{t_j} s^{-\alpha} ds = \frac{t_j^{1-\alpha} - t_{j-1}^{1-\alpha}}{1-\alpha} \tag{3.13}$$

and

$$\sum_{n=j+1}^{N} \mu_{\alpha,n-j} = \int_0^{t_{N-j}} s^{-\alpha} ds \leq C_T = \frac{T^{1-\alpha}}{1-\alpha}.$$

Using integration in x and Minkowski inequality, this yields

$$\|\varepsilon_n(\phi)\| \leq C \sum_{j=0}^{n-1} \mu_{\alpha,n-j} \int_{t_j}^{t_{j+1}} \|\phi_t\| ds,$$

and, by interchanging the order of summation, we find

$$\sum_{n=1}^{N} \|\varepsilon_n(\phi)\| \leq C \sum_{j=0}^{N-1} \sum_{n=j+1}^{N} \mu_{\alpha,n-j} \int_{t_j}^{t_{j+1}} \|\phi_t\| ds \leq C_T \int_0^{T_N} \|\phi_t\| ds.$$

This completes the proof. □

The following two Lemmas are discrete analogues of Lemmas 3.1 and 3.2 respectively, and can be proved similarly [2],[3].

Lemma 3.6.
let $\mu_{\alpha,j}$ be defined by (3.13) and assume that $y_n \geq 0$ and satisfies

$$y_n \leq b_n + \beta \sum_{j=0}^{n-1} \mu_{\alpha,n-j} y_j, \quad \text{for } n \geq 0,$$

Where $b_n \geq 0$, $\beta \geq 0$. Then for each $T > 0$ there is a constant C_T such that

$$y_n \leq b_n + C_T \sum_{j=0}^{n-1} \mu_{\alpha,n-j} b_j, \quad \text{for } nk \leq T.$$

Lemma 3.7.
Let $K \in L^1(0,T)$, and let κ_j be defined by (3.6). Then for each $\varepsilon > 0$ there is a constant $C_\varepsilon = C_\varepsilon(\|K\|_{L^1(0,T)})$ such that

$$\left|\sum_{n=1}^{N}\sum_{j=0}^{n-1}\kappa_{n-j}f_jf_n\right| \leq \varepsilon\sum_{n=1}^{N}f_n^2 + C_\varepsilon\sum_{n=0}^{N-1}|\kappa_{N-n}|\sum_{j=0}^{n-1}f_j^2$$

Proof of Theorem 3.3:

With the Ritz-Volterra projection $W = V_h u$ introduced in (3.11), we write

$$U^n - u(t_n) = (U^n - W^n) + (W^n - u(t_n)) = \theta^n + \rho^n$$

The term ρ^n has been estimated in Lemma 3.3. For the term θ^n, by our definition, we have

$$(\bar{\partial}_t\theta^n,\chi) + A(\theta^n,\chi) = \sum_{j=0}^{n-1}\kappa_{n-j}B(\theta^j,\chi) + (\tau_n,\chi), \tag{3.14}$$

where

$$\tau_n = u_t^n - \bar{\partial}_t W^n + \varepsilon_n(B_h W).$$

We show by an energy argument that

$$\|\theta^N\| \leq C_T\left(\|\theta^0\| + k\sum_{n=1}^{N}\|\tau_n\|\right), \qquad \text{for } Nk \leq T. \tag{3.15}$$

In fact, choosing $\chi = \theta^n$ in (3.14), which yields

$$\frac{1}{2}\bar{\partial}_t\|\theta^n\|^2 + \frac{1}{2}k\|\bar{\partial}_t\theta^n\|^2 + A(\theta^n,\theta^n) = \sum_{j=0}^{n-1}\kappa_{n-j}B(\theta^j,\theta^n) + (\tau_n,\theta^n),$$

where

$$\bar{\partial}_t\|\theta^n\|^2 + \|\theta^n\|_1^2 \leq C\sum_{j=0}^{n-1}\mu_{\alpha,n-j}\|\theta^j\|_1\|\theta^n\|_1 + C\|\tau_n\|.\|\theta^n\|,$$

and, after summation,

$$\|\theta^N\|^2 + k\sum_{n=1}^{N}\|\theta^n\|_1^2 \leq \|\theta^0\|^2 + Ck\sum_{n=1}^{N}\sum_{j=0}^{n-1}\mu_{\alpha,n-j}\|\theta^j\|_1\|\theta^n\|_1 + Ck\sum_{n=1}^{N}\|\tau_n\|.\|\theta^n\|.$$

Using Lemma 3.7 with $K(t) = Ct^{-\alpha}$, we conclude that

$$\|\theta^N\|^2 + k\sum_{n=1}^{N}\|\theta^n\|_1^2 \leq \|\theta^0\|^2 + Ck\sum_{n=1}^{N}\|\tau_n\|.\|\theta^n\| + C\sum_{n=0}^{N-1}\mu_{\alpha,N-n}\left(k\sum_{j=0}^{n-1}\|\theta^j\|_1^2\right).$$

Combining with Lemma 3.6, in which y_n is replaced by $y_N = k\sum_{n=1}^{N}\|\theta^j\|_1^2$, this shows

$$\begin{aligned}
\|\theta^N\|^2 \le \|\theta^N\|^2 + k\sum_{n=1}^{N}\|\theta^n\|_1^2 \\
&\le \|\theta^0\|^2 + Ck\sum_{n=1}^{N}\|\tau_n\|.\|\theta^n\| + C\sum_{n=0}^{N-1}\mu_{\alpha,N-n}\left(k\sum_{j=0}^{n-1}\|\theta^j\|_1^2\right) \\
&\le C_T\left(\|\theta^0\|^2 + k\sum_{n=1}^{N}\|\tau_n\|.\|\theta^n\|\right), \quad \text{for } Nk \le T,
\end{aligned}$$

and (3.15) follows.
Going back to estimate τ_n, we then write

$$\tau_n = \sum_{l=1}^{4}\tau_n^l = \tau_n^1 + \tau_n^2 + \tau_n^3 + \tau_n^4,$$

where

$$\begin{aligned}
\tau_n^1 &= u_t^n - \overline{\partial}_t u^n, \\
\tau_n^2 &= \overline{\partial}_t(u^n - W^n) = -\overline{\partial}_t\rho^n, \\
\tau_n^3 &= \varepsilon_n(B_h u), \\
\tau_n^4 &= \varepsilon_n(B_h\rho).
\end{aligned}$$

We have at once

$$k\sum_{n=1}^{N}\|\tau_n^1\| \le Ck\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\|u_{tt}\|ds = Ck\int_0^{t_N}\|u_{tt}\|ds,$$

and, by Lemma 3.4,

$$\begin{aligned}
k\sum_{n=1}^{N}\|\tau_n^2\| &\le \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\|\rho_t\|ds = \int_0^{t_N}\|\rho_t\|ds \\
&\le Ch^2\left\{\|u_0\|_2 + \int_0^{t_N}\|u_t\|_2 ds\right\}.
\end{aligned}$$

To estimate τ_n^3, we note that when u is smooth, $B_h u = P_h Bu$ and hence, by Lemma 3.5,

$$k\sum_{n=1}^{N}\|\tau_n^3\| \le Ck\int_0^{t_N}\|P_h Bu_t\|ds \le Ck\int_0^{t_N}\|u_t\|_2 ds.$$

Using the inverse assumption $\|\chi\|_1 \le Ch^{-1}\|\chi\|, \quad \forall \chi \in S_h$, we have

$$(B_h\rho, \chi) = B(\rho, \chi) \le C\|\rho\|_1\|\chi\|_1 \le Ch^{-1}\|\rho\|_1\|\chi\|,$$

so that

$$\|B_h\rho\| \le Ch^{-1}\|\rho\|_1.$$

Hence, for τ_n^4, by Lemmas 3.4 it follows

$$\begin{aligned}
k\sum_{n=1}^{N}\|\tau_n^4\| &\le C_T k\int_0^{t_N}\|B_h\rho_t\|ds \le C_T kh^{-1}\int_0^{t_N}\|\rho_t\|_1 ds \\
&\le C_T k\left\{\|u_0\|_2 + \int_0^{t_N}\|u_t\|_2 ds\right\}.
\end{aligned}$$

Substituting these estimates into (3.15), it leads to

$$\|\theta^N\| \le C_T\|u_{0h} - R_h u_0\| + C_T(h^2+k)\left\{\|u_0\|_2 + \int_0^{t_N}(\|u_{tt}\| + \|u_t\|_2)ds\right\}.$$

This completes the proof. □

3.2 Numerical Example

Consider the following parabolic partial Volterra integro-differential equation:

$$u_t(x,t) - u_{xx}(x,t) = f(x,t) + \int_0^t K(x,t,s)u(x,s)ds, \quad -1 \le x \le 1, \quad 0 \le t \le 1 \tag{3.16}$$

with the following initial and boundary conditions

$$u(x,0) = 1 - x^2, \quad x \in [-1,1], \tag{3.17}$$

$$u(-1,t) = u(1,t) = 0, \quad t \in [0,1]. \tag{3.18}$$

Where $K(x,t,s) = -\sin(x(t-s))$,
and $f(x,t) = (1-x^2)e^t + \dfrac{(x^2-1)(x\cos(xt) + \sin(xt) - xe^t)}{x^2+1} + 2e^t$,
with the exact solution $u(x,t) = (1-x^2)\exp(t)$.

3.2.1 Finite Element Method

Now, we explain how to solve the problem with finite element method using the radial basis functions (RBFs) for discretization of both time and space variables. Let

$$\Omega = \left\{(x_i,t_j), \quad -1 \le x_i \le 1, \quad 0 \le t_j \le 1, \quad i,j = 1,2,\ldots,n\right\}$$

be a set of scattered nodes. Then the approximated solution of the problem (3.16)-(3.17)-(3.18) is considered as follows:

$$\widehat{u}(x,t) = \sum_{i=1}^{N} \alpha_i \varphi_i(x,t) \tag{3.19}$$

where $\varphi_i(x,t) = \varphi(\|(x,t) - (x_k,t_l)\|), k,l = 1,2,\ldots,n, \quad i = 1,2,\ldots,N = n^2$, for a radial function and $\alpha_i, i = 1,2,\ldots,N$ are unknown coefficients that must be found.

The collocation technique is used for finding unknowns $\alpha_i, i = 1,2,\ldots,N$. Let

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4, \tag{3.20}$$

where

$$\begin{aligned}
1th \text{ side}: &\quad \Omega_1 = \left\{\left(x_i,t_j\right), -1 \le x_i \le 1, t_j = 0, i,j = 1,2,\ldots,n\right\}\\
2th \text{ side}: &\quad \Omega_2 = \left\{\left(x_i,t_j\right), x_i = -1, 0 < t_j \le 1, i,j = 1,2,\ldots,n\right\}\\
3th \text{ side}: &\quad \Omega_3 = \left\{\left(x_i,t_j\right), x_i = 1, 0 < t_j \le 1, i,j = 1,2,\ldots,n\right\}\\
\text{Inner area}: &\quad \Omega_4 = \left\{\left(x_i,t_j\right), -1 < x_i < 1, 0 < t_j \le 1, i,j = 1,2,\ldots,n\right\}.
\end{aligned}$$

Also assuming that $\Omega_i \neq \emptyset$, for $i = 1, 2, 3, 4$.
Now (3.16)-(3.17)-(3.18) are approximated by using (3.19). Thus we have

$$\sum_{i=1}^{N} \alpha_i \varphi_i(x_k, t_l) = 1 - x_k^2, \quad (x_k, t_l) \in \Omega_1. \tag{3.21}$$

$$\sum_{i=1}^{N} \alpha_i \varphi_i(x_k, t_l) = 0, \quad (x_k, t_l) \in \Omega_2. \tag{3.22}$$

$$\sum_{i=1}^{N} \alpha_i \varphi_i(x_k, t_l) = 0, \quad (x_k, t_l) \in \Omega_3. \tag{3.23}$$

$$\sum_{i=1}^{N} \alpha_i \left[\frac{\partial}{\partial t} \varphi_i(x_k, t_l) - \frac{\partial^2}{\partial x^2} \varphi_i(x_k, t_l) \right] - A(x_k, t_l) = f(x_k, t_l), \quad (x_k, t_l) \in \Omega_4, \tag{3.24}$$

where

$$\begin{aligned} A(x_k, t_l) &= \int_0^{t_l} K(x_k, t_l, s) \sum_{i=1}^{N} \alpha_i \varphi_i(x_k, s) ds \\ &= -\int_0^{t_l} \sin(x_k(t_l - s)) \sum_{i=1}^{N} \alpha_i \varphi_i(x_k, s) ds \\ &= -\sum_{i=1}^{N} \alpha_i \int_0^{t_l} \sin(x_k(t_l - s)) \varphi_i(x_k, s) ds, \quad (x_k, t_l) \in \Omega_4. \end{aligned} \tag{3.25}$$

and

$$f(x_k, t_l) = (1 - x_k^2) e^{t_l} + \frac{(x_k^2 - 1)(x_k \cos(x_k t_l) + \sin(x_k t_l) - x_k e^{t_l})}{x_k^2 + 1} + 2e^{t_l}. \tag{3.26}$$

For obtaining $A(x_k, t_l), (x_k, t_l) \in \Omega_4$, by change of variable $s = \frac{t_l}{2}(\xi + 1)$ where $\{\xi\}_{i=1}^{M}$ are Legendre-Guass nodes [18], so (3.25) can be written as:

$$\begin{aligned} A(x_k, t_l) &= \frac{t_l}{2} \int_{-1}^{1} K(x_k, t_l, \frac{t_l}{2}(\xi + 1)) \sum_{i=1}^{N} \alpha_i \varphi_i(x_k, \frac{t_l}{2}(\xi + 1)) d\xi \\ &= -\sum_{i=1}^{N} \alpha_i \int_{-1}^{1} \sin\left(x_k(t_l - \frac{t_l}{2}(\xi + 1)) \right) \varphi_i(x_k, \frac{t_l}{2}(\xi + 1)) d\xi, \quad (x_k, t_l) \in \Omega_4. \end{aligned} \tag{3.27}$$

By applying numerical integration method given in [18], we can approximate the integral in (3.27) and we can get:

$$\begin{aligned} A(x_k, t_l) &\simeq \frac{t_l}{2} \sum_{j=0}^{M} w_j K(x_k, t_l, \frac{t_l}{2}(\xi_j + 1)) \sum_{i=1}^{N} \alpha_i \varphi_i(x_k, \frac{t_l}{2}(\xi_j + 1)), \quad (x_k, t_l) \in \Omega_4. \\ &= -\frac{t_l}{2} \sum_{j=0}^{M} w_j \sum_{i=1}^{N} \alpha_i \sin\left(x_k(t_l - \frac{t_l}{2}(\xi_j + 1)) \right) \varphi_i(x_k, \frac{t_l}{2}(\xi_j + 1)), \end{aligned} \tag{3.28}$$

where $w_j, j = 0, 1, , \dots, M$ are the weights [18]. We can have a system of equations that can be solved to obtain the coefficient $\alpha_i, i = 1, 2, \dots, N$.

There are many different RBFs can be used [20]. Here to obtain the following results we have been used the multiquadratics (MQ) $\varphi(r) = (r^2 + c^2)^{\frac{\beta}{2}}, \beta \neq 0, \beta \neq 2\mathbb{N}$, and Gaussians (GA) $\varphi(r) = e^{-c^2 r^2}$.

In figure 3.1, the exact solution, numerical solution and the corresponding absolute error function $|u - \widehat{u}|$ are plotted by using GA-RBF with $c = 0.1$ and the uniform set of collocation points $x_i = -1 + (i-1)/5$ and $t_i = (i-1)/10$, $i = 1, 2, \ldots, 11$.

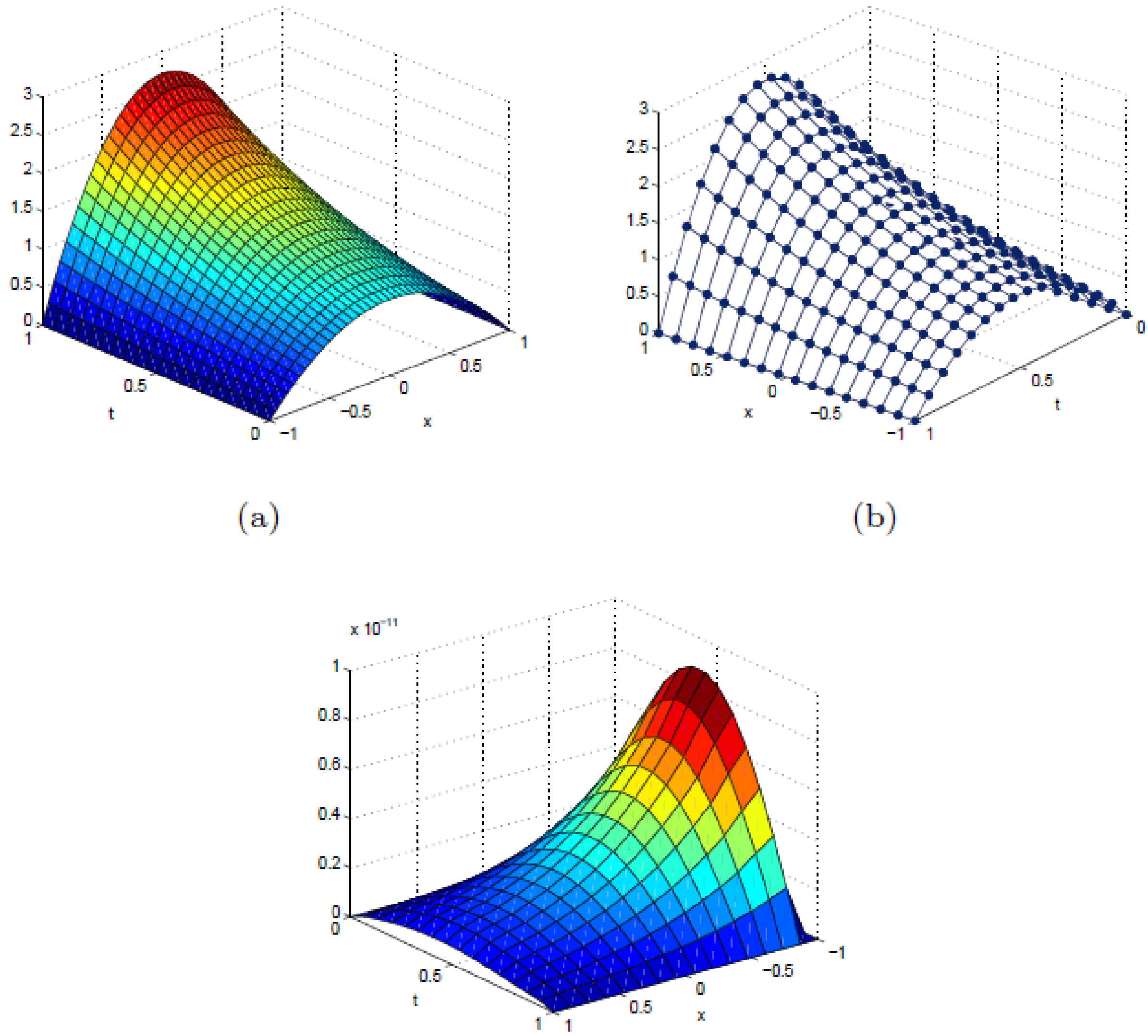


Figure 3.1: Exact solution (a), approximated solution (b) and absolute error function $|u - \widehat{u}|$ for $N = 121$ using GA-RBF with $c = 0.1$.

3.2.2 Error Analysis

Madych have proven exponential convergence property of multiquadratic approximation [17]. He has shown that under certain conditions, the interpolation error is $\epsilon = O(\lambda^{\frac{c}{h}})$ where c is the shape parameter, h is the mesh size and $0 < \lambda < 1$ is a constant. It implies we can improve the approximated solution either by reducing the size of h or by increasing the magnitude of c. It means that if $c \longrightarrow \infty$ then $\epsilon \longrightarrow 0$. Since increasing of c can improve the accuracy exponentially without extra computation, it is preferred to decrease error rather than reducing h.

However, according to "uncertainty principle" of Schaback [15], as the error becomes smaller, the matrix becomes more ill-conditioned, hence the solution will break down as c becomes too large. The experimental results confirm such behavior of the error values as c becomes larger.

The numerical results for the example are demonstrated in Tables 3.1, 3.2 and Figures 3.2, 3.3 which show according to the findings of Madych, the error functions decrease exponentially as c becomes larger in bounded interval. After that according to the research of Schaback the error values decline as c becomes too large. The best c is different for various problems and not the same RBFs.

c	MQ-RBF
0.1	1.31×10^{-1}
1	3.16×10^{-3}
5	1.82×10^{-5}
19	3.75×10^{-11}
25	3.31×10^{-11}
30	2.54×10^{-11}
35	8.01×10^{-11}

Table 3.1: The maximum absolute error by using MQ-RBF with different c.

c	GA-RBF
0.001	3.49×10^{-4}
0.01	8.51×10^{-12}
0.1	9.25×10^{-12}
0.5	8.38×10^{-7}
1	3.63×10^{-4}
1.2	1.59×10^{-3}
2	2.82×10^{-2}
3	8.55×10^{-2}

Table 3.2: The maximum absolute error by using GA-RBF with different c.

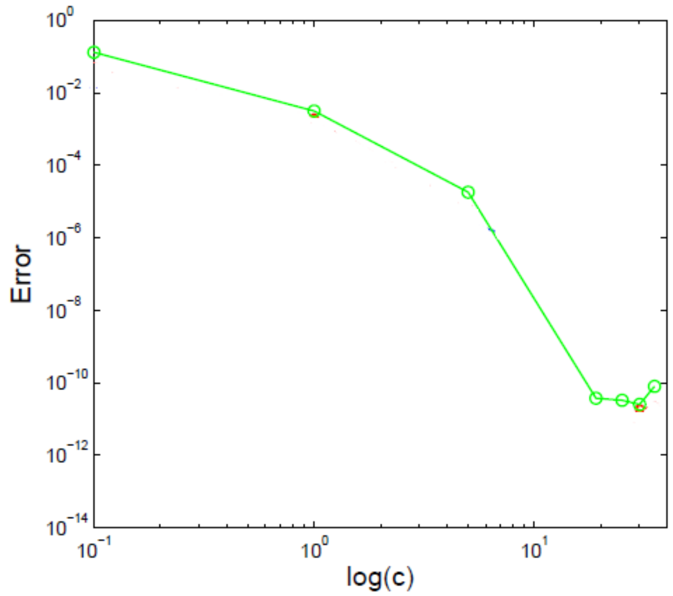


Figure 3.2: Horizontal axis is related to shape parameter (c) with log mode and vertical axis shows absolute error values with log mode when the solutions are approximated by using MQ-RBF.

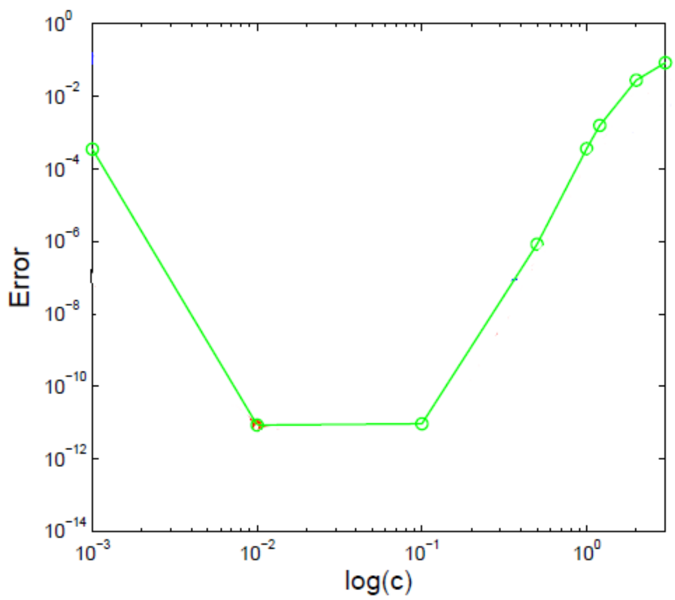


Figure 3.3: Horizontal axis is related to shape parameter (c) with log mode and vertical axis shows absolute error values with log mode when the solutions are approximated by using GA-RBF.

BIBLIOGRAPHY

[1] A. Munnier, Espaces de Sobolev et introduction aux équations aux dérivées partielles, 2007-2008.

[2] Chen Chuanmiao, Shih Tsimin, FINITE ELEMENT METHODS FOR INTEGRODIFFERENTIAL EQUATIONS, 1998.

[3] C.Chen, V.Thomée, and L.B.Wahlbin, FINITE ELEMENT APPROXIMATION OF A PARABOLIC INTEGRO-DIFFERENTIAL EQUATION WITH A WEAKLY SINGULAR KERNEL, April 1992.

[4] Daniel Daners, Introduction to Functional Analysis, 2006.

[5] Daniel Li, Cours d'analyse fonctionnelle, 2013.

[6] Eric Blayo, Notes de cours sur la méthode des éléments finis, 2010.

[7] Erwin Kreyszig, INTRODUCTORY FUNCTIONAL ANALYSIS WITH APPLICATIONS, 1978.

[8] Frédéric LEGOLL, Partial Differential Equations and the Finite Element Method, 2019.

[9] G.P.Nikishkov, INTRODUCTION TO THE FINITE ELEMENT METHOD, 2004.

[10] Guediri Fouzia, Rezzoug Aicha, Introduction à la méthode des éléments finis, Mémoire, 2014.

[11] Jonathan ROCHAT, Les espaces de Sobolev, décembre 2009.

[12] Marie-Thérèse, Lacroix-Sonrier, Distributions Espaces de Sobolev Applications, 1998.

[13] Mats G. Larson, Fredrik Bengzon, The Finite Element Method: Theory, Implementation, and Practice, 2010.

[14] NGUYEN Thi Thuy Trang, TRINH Thi Huyen, ELEMENTS FINIS : DU CLASSIQUE AU ISOGEOMETRIQUE, MÉMOIRE DE MASTÈRE 2011-2012.

[15] R.Schaback, Error estimate and condition numbers for radial basis function interpolation, 1995.

[16] V.Lakshmikantham and M.Rama Mahana Rao, Theory of lntegro-Differential Equations, 1995.

[17] W.R.Madych, Miscellaeous error bounds for multiquadratic and related interpolators, 1992.

[18] Z.Avazzadeh, M.Heydari, W.Chen and G.B.Loghmani, SMOOTH SOLUTION OF PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS USING RADIAL BASIS FUNCTIONS, May 2014.

[19] Zeidabad Hamedi, Mortaza Gachpazan, Asghar Kerayechian, Finite Element Method for solving Linear Volterra Integro-Differential Equations of the second kind, 2014.

[20] R.L.Hardy, Multiquadratic equation of topology and other irregular surface, 1905-1915.

Abstract

In this work, we gave a simple idea how to apply the Galerkin finite element method to an integro-differential equation of parabolic type with a memory term containing a weakly singular kernel. Optimal-order estimates are shown for spatially semidiscrete and completely discrete methods. Furthermore, the efficiency of the proposed method will be considered through numerical example.

ملخص

في هذا العمل قدمنا فكرة بسيطة عن كيفية تطبيق طريقة العناصر المنتهية لجاليركين على معادلة تفاضلية تكاملية من نوع مكافئ مع عبارة ذاكرة تحتوي على نواة مفردة ضعيفة. يتم تقديم التقدير الأمثل للخطأ بالنسبة لنصف التقسيم على الفضاء والتقسيم الكامل على الفضاء والزمن. بالإضافة إلى ذلك، سيتم إثبات كفاءة الطريقة المقترحة من خلال مثال عددي.

Résumé

Dans ce travail, nous avons donné une idée simple comment appliquer la méthode des éléments finis de Galerkin à une équation intégro-différentielle de type parabolique avec un terme de mémoire contenant un noyau faiblement singulier. Des estimations d'ordre optimal sont présentées pour les méthodes spatialement semi-discrètes et complètement discrètes. De plus, l'efficacité de la méthode proposée sera prise en compte à travers un exemple numérique.