People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

ECHAHID HAMMA LAKHDAR UNIVERSITY OF EL-OUED

Faculty of Exact Sciences

Department of Mathematics

THESIS IN VIEW OF OBTAINING THE DIPLOMA OF DOCTORATE OF LMD IN MATHEMATICS

Speciality: FUNCTIONAL ANALYSIS AND APPLICATIONS

WEAK MULTIVALUED CONTRACTIONS AND FIXED POINTS RESULTS WITH APPLICATIONS TO DIFFERENTIAL INCLUSIONS

Presented by: MAROUA MENECEUR

Defended on: 22-12-2022

THE JURY COMPOSED OF:

Abdelouahab Mansour	Prof	El-Oued.Univ	President
Said Beloul	Prof	El-Oued.Univ	Supervisor
Khadidja Nisse	M.C.A	El-Oued.Univ	Examiner
Khaled Berrah	M.C.A	Tebessa.Univ	Examiner
Mostafa Djedidi	M.C.A	El-Oued.Univ	Examiner
Taieb Hamaizia	M.C.A	Oum El-Bouaghi.Univ	Examiner

University year: 2022-2023

Abstract

In this thesis, we combine the notion of α-admissible mappings with θ-contraction and weak contraction concepts to introduce a new type of contractions with some related fixed point results in complete metric spaces and b-metric spaces. We also prove the existence of fixed points in complete partially ordered metric spaces and in complete metric spaces endowed with a graph by using our main results.

Finally, we provide two applications to prove the existence of the solutions for a Volterra integral inclusion and a boundary value problem of fractional differential inclusions.

Keywords:
Metric spaces, fixed point theorem, θ-contraction, Berinde contraction, α-admissible, fractional differential inclusions.

Résumé

Dans cette thèse, nous combinons la notion des applications α-admissibles avec les concepts de θ-contraction et de contraction faible pour introduire un nouveau type de contractions et quelques résultats de points fixes associés dans des espaces métriques complets et dans espaces b-métriques complets. Nous prouvons également l'existence de points fixes dans des espaces métriques partillement ordonnés et dans des espaces métriques complets muni d'un graphe en utilisant nos principaux résultats.

Enfin, nous donnons des applications pour prouver l'existence des solutions d'une inclusion intégrale de tupe Volterra et d'un problème aux limites d'inclusions différentielles fractionnaires.

Mots clés:
Espaces métriques, point fixe, θ-contraction, contraction de Berinde, α-admissible, inclusion différentielle fractionnaire.

ملخص

في هذه الأطروحة قمنا بالجمع بين مفهوم α-مقبول مع مفاهيم θ-تقلص و تقلص ضعيف لإدخال نوع جديد من التقلصات للحصول على بعض النتائج المرتبطة بالنقطة الثابتة في فضاءات مترية تامة وفضاءات مترية من نوع(b)، نستنتج أيضا وجود نقط ثابتة في فضاءات مترية مرتبة جزئيا وفضاءات مترية تامة مزودة ببيان بإستخدام نتائجنا الرئيسية.

في الأخير طبقنا النتائج المحصل عليها لإثبات وجود حلول لمسألة القيمة الحدية لاحتواءات تفاضلية كسرية مع شروط حدية تكاملية.

الكلمات المفتاحية:

الفضاءات المترية، نظرية النقطة الثابتة، θ-تقلص ،تقلص من نوع بريند، α-مقبول، احتواءات تفاضلية كسرية.

Dedication

I dedicate this thesis

To my mother

To my husband

To my daughters Fatima and Siwar

To my sisters my brothers

To my friends

for encouraging me.

Acknowledgement

First of all, I would like to thank **ALLAH** for his blessing and success that made me complete this work.

In these lines, I would like to extend a special thanks to the people who helped me enter the doctoral phase first and then formulate my thesis, and the first of them is the professor **Said Beloul** who suggested this topic to me and helped me a lot in completing the thesis.

I would also like to thank him for his kindness, availability and time spent on my work. I would also like to thank Professors: Abdelouahab Mansour, Khadidja Nisse, Khaled Berrah, Taieb Hamaizia and Mostafa Djedidi.

I also thank the members of the Department of Mathematics for allowing me to work in good conditions during the performance of my work.

Thanks also to all the teachers who helped me during my studies, not forgetting their valuable advice.

I also thank everyone from near and far contributed to the finalization of this work.

A big thank you to all of my family who were always there when I needed them.

Contents

Notations

The most frequently used notations, symbols and abbreviations are listed below:

$\mathbb{N}$	:	The set of natural numbers.
$\mathbb{R}$	:	The set of real numbers.
$\mathbb{R}^+$	:	The set of positive real numbers.
$C([a,b],\mathbb{R})$	:	The space continuous functions from $[a,b]$ to $\mathbb{R}$.
$H(A,B)$	:	The Hausdorff distance between A and B.
$\mathcal{P}(X)$	:	The family of non-empty subsets of X.
$CL(X)$	:	The set of all closed subsets of X.
$CB(X)$	:	The set of all closed and bounded subsets of X.
$K(X)$	:	The set of all compact subsets of X.
$T : X \longrightarrow \mathcal{P}(Y)$	:	T is a multivalued map from X to $\mathcal{P}(Y)$.
$Dom(T)$	:	The domain of T.
$Graph(T)$	:	The graph of T.
$Im(T)$	:	The image of T.
$V(G)$	:	The set of vertices of a graph G.
$E(G)$	:	The set of edges (arcs) of a graph G.
$(X,\preceq,d)$	:	be a complete ordered metric space.
$(X,\widetilde{d},s)$	:	be a b-metric space.

Introduction

Fixed point theory has had major influences on the development of many mathematical disciplines associated with the fundamental problems of functional analysis. This theory plays a major role in both linear and non linear analysis and because of its important applications in game theory, economics, and numerical analysis in biology,..

Multivalued maps theory plays an important role in various branches of pure and applied mathematics because it has many applications, for example, in real and complex areas of analysis as well as in problems of optimal control. Over the years, this theory has increased in importance and thus in the literature there are many documents that focus on discussing abstract problems and practices that have multivalued applications. In fact, among the various methods used to develop this theory, one of the most interesting is based on the fixed point theory, often given the constructive nature of fixed point theorems especially in its metric branch.

Multivalued fixed point theory was known some development, starting from the results of Nadler [50], where he has established the existence of multivalued fixed point by using the Hausdorff metric some generalizations were given in this way, for instance, see [4, 10, 18, 32, 34] and references therein.

Berinde [14] introduced the concept of almost as a generalization to weak contraction in the context of single valued mappings, which has been extended to multivalued case in [16], later some results have been obtained using this concept.

Samet and al. [56] furnished a new concept called as α-admissible and obtained some fixed point results for $\alpha-\psi$-contraction maps, later some results were established in this direction, see [6, 33, 34]. Recently, Jleli and Samet [35] introduced a new contractions type called θ-contraction to prove the existence of fixed points for such contractions. It is worth mentioning here, that a contraction in the sense of Banach is a particular of θ contraction, while there are some θ-contractions which are not Banach contraction. After that, several authors studied on different variations of θ-contraction for single-valued and multivalued mappings, for example, see [2, 21, 24, 61].

This thesis is divided into five chapters divided as follows:

Chapter 1: We touched on some concepts of metric space, the Pompeiu -Hausdorff metric and multivalued analysis with some examples, and then we defined α-admissible and α-continue.

Chapter 2: We have given some types of contractions, whether in single-valued or multivalued maps.

Chapter 3: We will present the fixed point theorems for generalized almost (α, ψ, θ) contraction and their proofs, in addition to practical examples of theorems, theorems fixed point on partially ordered metric spaces and theorems fixed point on metric spaces endowed with graph.

Chapter 4: we present some fixed point theorems in b-metric space for almost (α, θ, κ) contractions of Hardy-Rogers type, in addition to practical examples of theorems, theorems fixed point on partially ordered b-metric spaces and theorems fixed point on b-metric spaces endowed with graph.

Chapter 5: This chapter will be devoted to some applications concerning the study of some problems such as:

1. The existence of solutions for a Volterra integral inclusions.

2. The existence of solutions for a boundary problem of fractional differential inclusions with integral boundary conditions.

Chapter 1

Preliminaries

In this chapter, we present the notations and definitions and some preliminary properties that are used in this thesis.

1.1 The Pompeiu -Hausdorff metric

The objective of this section is to present the definition of the Pompeiu-Hausdorff. This notion will allow us to define the notion of a multivalued contraction.

Firstly, let us first introduce the distance from a point to a set.

Definition 1.1.1. [1] *Let (X, d) be a metric space, $x \in X$ and A a subset of X. The distance from point x to A noted $d(x, A)$ is defined by:*

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

By convention $d(x, \emptyset) = \infty$. If on the contrary, A is not empty, then for all $\varepsilon > 0$, it exists an element $y \in A$ such that $d(x, y) \leq d(x, A) + \varepsilon$.

Theorem 1.1.1. *Let $x \in X, A \subset K(X)$, so it exists $x_0 \in A$ such that $d(x, A) = d(x, x_0)$.*

Proof. A function $x \mapsto d(x, y)$ is continuous, with real values. It reaches its lower limit on any compact. □

Lemma 1.1.2. *Let (X, d) a metric space. For each $B \in CL(X)$and $x \in X$, we have*

$$d(x, B) = 0 \Leftrightarrow x \in B.$$

Proof. By the definition of $d(a, B)$, for each $n_0 \in \mathbb{N}$, it exists a sequence $\{b_n\} \in B$, such that $d(a, b_{n_0}) < \frac{1}{n_0}$.

Now, we have a sequence $\{b_n\}$ in B such that $d(a, b_n) < \frac{1}{n}$, for all $n \in \mathbb{N}$. Next, we show that $\lim\limits_{n \longrightarrow \infty} d(a, b_n) = 0$. Let $\varepsilon > 0$ by property Archimede it exist $N \in \mathbb{N}$, such that $\frac{1}{N} < \varepsilon$, where for all $n \in \mathbb{N}$ we find $\frac{1}{n} \leq \frac{1}{N}$.

So $d(a, b_n) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$, then $\lim\limits_{n \longrightarrow \infty} d(a, b_n) = 0$, so $\lim\limits_{n \longrightarrow \infty} b_n = a$. As B is closed we have $a \in B$. $\square$

Since we are working in the field of multivalued analysis, we need to define a distance between two sets. A notion of this type was proposed by Pompeiu [53].

In this thesis, we will also use the Hausdorff metric [30] constructed from this function.

Definition 1.1.2. [53] *Let (X, d) be a metric space and let $e : CB(X) \times CB(X) \to \mathbb{R}_+$ be a function defined by:*

$$e(A, B) = \sup\{d(a, B) : a \in A\}.$$

For $A = \emptyset$ and $B \neq \emptyset$, we have: $e(\emptyset, B) = 0$.
For $A = B = \emptyset$, we have: $e(\emptyset, \emptyset) = +\infty$
and $e(A, \emptyset) = +\infty$ for each $A \subset X$ with $A \neq \emptyset$.

Example 1.1.1. *Let $X = \mathbb{R}$, $A = [0, 1]$, $B = [2, 3]$*
$e(A, B) = \sup\{d(a, B), \quad a \in A\} = 2$
$e(B, A) = \sup\{d(b, A), \quad b \in B\} = 2$

Now, we present some properties:

Proposition 1.1.1. [1] *Let (X, d) a metric space and $A, B, C \in CB(X)$. Then we have:*

(a) $e(A, B) = 0 \Leftrightarrow A \subset B$.

(b) $B \subset C \Rightarrow e(A, C) \leq e(A, B)$.

(c) $e(A \cup B, C) = \max\{e(A, C), e(B, C)\}$.

(d) $e(A, B) \leq e(A, C) + e(C, B)$.

Proof. (a) By the definition of the function e, we have:

$$\begin{aligned} e(A,B) &= 0 \Leftrightarrow \sup_{x\in A} d(x,B) = 0 \\ &\Leftrightarrow d(x,B) = 0 \text{ for each } \; x \in A. \end{aligned}$$

Since B is closed in X,

$$d(x,B) = 0 \Leftrightarrow x \in B.$$

So,

$$e(A,B) = 0 \Leftrightarrow A \subset B.$$

(b) We have

$$B \subset C \Rightarrow d(x,C) \leq d(x,B) \text{ for each } x \in X.$$

$$B \subset C \Rightarrow \sup_{x\in A} d(x,C) \leq \sup_{x\in A} d(x,B).$$

So

$$B \subset C \Rightarrow e(A,C) \leq e(A,B).$$

(c) We have

$$e(A\cup B, C) = \sup_{x\in A\cup B} d(x,C) = \max\{\sup_{x\in A} d(x,C), \sup_{x\in B} d(x,C)\}.$$

So

$$e(A\cup B, C) = \max\{e(A,C), e(B,C)\}.$$

(d) Let $a \in A, b \in B$ and $c \in C$. So

$$d(a,B) \leq d(a,b) \leq d(a,c) + d(c,b),$$

$$d(a,B) \leq d(a,c) + \inf_{b\in B} d(c,b).$$

Then

$$d(a,B) \leq d(a,c) + d(c,B) \leq d(a,c) + e(C,B).$$

This implies

$$d(a,B) \leq d(a,c) + e(C,B),$$

We have

$$d(a,B) \leq \inf_{c\in C} d(a,c) + e(C,B).$$

Then

$$d(a,B) \leq d(a,C) + e(C,B).$$

We have

$$\sup_{x\in A} d(a,B) \leq \sup_{x\in A} d(a,C) + e(C,B).$$

So

$$e(A,B) \leq e(A,C) + e(C,B).$$

$\square$

Definition 1.1.3. [30]

Let (X,d) be a metric space, and let $CB(X)$ be a non empty, closed and bounded subsets of X, the Pompeiu -Hausdorff metric is defined as:

$$H(A,B) = \max\{e(A,B), e(B,A)\} = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\},$$

for all $A, B \in CB(X)$.

Remark 1.1.1. *if $A = \{a\}$ (singleton) and $B = \{b\}$, then $H(A,B) = d(a,b)$.*

Example 1.1.2. *Let $X = \mathbb{R}$, $A = [0,2]$,$B = [4,5]$*
So, $\sup_{a\in A} d(a,B) = 4$ *and* $\sup_{b\in B} d(b,A) = 3$
Then, $H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\} = 4.$

Proposition 1.1.2.

1. *For $A, B \in CB(X)$, $H(A,B) = 0 \Longleftrightarrow A = B$.*

2. *$H(A,B) = H(B,A)$, for each $A, B \in CB(X)$.*

3. *$H(A,B) \leq H(A,C) + H(C,B)$, for each $A, B, C \in CB(X)$.*

Proposition 1.1.3. *If (X,d) is a metric space, then H is a metric on $CB(X)$.*

Proof. (1) By the definition of H, we have $H(A,B) \geqslant 0$, and

$$\begin{aligned} H(A,B) &= 0 \Leftrightarrow \max\{e(A,B), e(B,A)\} = 0 \\ &\Leftrightarrow e(A,B) = 0 \text{ and } e(B,A) = 0 \\ &\Leftrightarrow A \subset B \text{ and } B \subset A \\ &\Leftrightarrow A = B. \end{aligned}$$

(2) From the properties of max, we find that

$$\begin{aligned} H(A,B) &= \max\{e(A,B), e(B,A)\} \\ &= \max\{e(B,A), e(A,B)\} \\ &= H(B,A). \end{aligned}$$

(3) Using the proposition (1.1.1), we get

$$\begin{aligned} H(A,B) &= \max\{e(A,B), e(B,A)\} \\ &\leqslant \max\{e(A,C)+e(C,B), e(B,C)+e(C,A)\} \\ &\leq \max\{e(A,C), (C,A)\} + \max\{e(B,C), e(C,B)\} \\ &= H(A,C) + H(C,B). \end{aligned}$$

□

Remark 1.1.2. *It is easy to ensure that the completeness of (X,d) implies the completeness of $(CB(X),H)$ and $(K(X),H)$.*

Proposition 1.1.4. *If (X,d) is a complete metric space, then $K(X)$ is a closed set in metric space $(CB(X),H)$.*

Lemma 1.1.3. [50] *Let $A, B \in CB(X)$ and $a \in A$. Then for $\varepsilon > 0$. There exists $b \in B$ such that:*

$$d(a,b) \leq H(A,B) + \varepsilon$$

Lemma 1.1.4. [50] *Let (X,d) be a metric space and $A, B \in CB(X)$ with $H(A,B) > 0$. Then, for each $h > 1$ and for each $a \in A$, there exists $b = b(a) \in B$ such that*

$$d(a,b) < hH(A,B). \tag{1.1}$$

Proof. Let $H(A,B) = 0$ then $a \in B$ and the above inequality holds for $b = a$.
Now, suppose that $H(A,B) > 0$ and we choose

$$\varepsilon = (h-1)H(A,B) > 0. \tag{1.2}$$

Then, using the definition of $d(a, B)$ and $H(A, B)$ it follows that, for all $\varepsilon > 0$, it exists $b \in B$ such as

$$d(a, b) \leq d(a, B) + \varepsilon \leq H(A, B) + \varepsilon. \tag{1.3}$$

Finally, replacing (1.2) in (1.3), we get the inequality (1.1).

□

Definition 1.1.4. [31] *Let (X, d) be a metric space, and the sequence $\{A_n\} \in CB(X)$. The sequence $\{A_n\}$ is said*

(1) Convergent if there exists $A \in CB(X)$ such that:

$$H(A_n, A) \longrightarrow 0, when \quad n \longrightarrow \infty,$$

(2) Cauchy if for all $m, n > 0$,

$$H(A_n, A_m) \longrightarrow 0, when \quad n, m \longrightarrow \infty,$$

1.2 Multivalued maps

As we mentioned earlier, in this thesis we will focus on the concept of multivalued maps. These maps are not necessarily points, but rather groups.

In this part we give somes definitions of the multivalued maps and their properties examples.

Definition 1.2.1. [7] *Let X, Y be two non-empty sets. a multivalued mapping T defines of X in Y is a map which associates to each $x \in X$ with a non-empty subset Tx of Y, which called image or the value of T in X. We denote it by $T : X \longrightarrow \mathcal{P}(Y)$, or $T : X \longrightarrow 2^Y$ where $\mathcal{P}(Y) = 2^Y$ denotes the set of non-empty subsets of Y.*

Remark 1.2.1. *The single-valued maps is also a multivalued maps since for any $x \in X$ singleton $T(x) = y$ is a subset of Y .*

Example 1.2.1. *We present some examples:*

1. *Let $T : \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ defines by*

$$Tx = \begin{cases} \{-1\} & \text{if } x < 0, \\ [-1,1] & \text{if } x = 0, \\ \{1\} & \text{if } x > 0. \end{cases}$$

Then T is multivalued mapping.

2. *The mapping $T : [0,1] \longrightarrow \mathcal{P}([0,1])$ defined by*

$$Tx = [0, x^2], \text{ for each } x \in [0,1],$$

is a multivalued mapping.

3. *Let $T : \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$,and $n \in \mathbb{Z}$ defines by*

$$Tx = \begin{cases} [n-1, n], & \text{if } x = n, \\ n; & \text{if } x \in]n, n+1[. \end{cases}$$

So T is a multivalued mapping.

4. *Let $T : \mathbb{R}_+ \longrightarrow \mathcal{P}(\mathbb{R})$,defines by*

$$Tx = [e^{-x}, e^x], \text{ for each } x \in \mathbb{R}_+.$$

T is a multivalued mapping.

5. *The mapping $T : \mathbb{R} \to K(\mathbb{R})$ defined by $Tx = [a, b]$, with $a, b \in \mathbb{R}$ such that $a < b$, is a constant multivalued mapping.*

In order to work well within the framework of the multivalued analysis, let us now state some definitions and characteristics specific to multivalued mappings. Notions of this part come from [7].

Definition 1.2.2. *Let X, Y tow sets, $T : X \longrightarrow \mathcal{P}(Y)$ is a multivalued mapping*

1. *We call domain of T the set*

$$Dom(T) = \{x \in X : Tx \neq \emptyset\}$$

When Dom(T) $\neq \emptyset$ (resp. Dom(T) $= X$), the multivalued mapping T is said to be proper (resp. strict).

2. *We call image of T the set*

$$Im(T) = \{y \in \mathcal{P}(Y), \exists x \in X, y \in Tx\}.$$

3. *The graph of T is the subset of $X \times Y$ defined by*

$$Graph(T) = \{(x, y) \in X \times Y, y \in Tx\}.$$

Definition 1.2.3. *The inverse of T is a multivalued mapping $T^{-1} : Y \to \mathcal{P}(X)$ such that*

$$x \in T^{-1}y \Leftrightarrow y \in Tx \Leftrightarrow (x, y) \in Graph(T).$$

Thus Dom(T^{-1}) = Im(T) and Im(T^{-1}) = Dom(T).

Definition 1.2.4. *Let $T : X \to \mathcal{P}(Y)$ is a multivalued mapping, A a subset of X. The image of A by T is defined by:*

$$T(A) := \bigcup_{x \in A} Tx.$$

Definition 1.2.5. *Let $T : X \to \mathcal{P}(Y)$ be a multivalued mapping, and B a subset of Y. The reciprocal image of B by T denoted by: $T^{-1}(B)$ is defined by:*

$$T^{-1}(B) := \{x \in X : Tx \cap B \neq \emptyset\}.$$

Example 1.2.2. *Let $T : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ such that $Tx = [x, x + 1]$, for all $x \in \mathbb{R}$ we have:*

$$T([0, 1]) = \bigcup_{x \in [0,1]} Tx = [0, 2], \quad T(\mathbb{R}_+) = \mathbb{R}_+, \quad T(\mathbb{R}_-) =]-\infty, 1].$$

$$T^{-1}(\mathbb{R}_+) \; = \mathbb{R}_+, \quad T^{-1}(\mathbb{R}_-) = \mathbb{R}_-.$$

Example 1.2.3. *Consider the multivalued mapping $T : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ defined by:*

$$Tx = \begin{cases} \emptyset, & \text{if } x < 0, \\ \{-\sqrt{x}, \sqrt{x}\}, & \text{if } x \geq 0. \end{cases}$$

It is clear that T is proper and not strict because Dom(T) $= \mathbb{R}_+$.
Here

$$Im\ T = \bigcup_{x \in \mathbb{R}_+} \{-\sqrt{x}, \sqrt{x}\} = \mathbb{R},$$

and

$$Graph(T) = \{(x, \sqrt{x}) : x \geq 0\} \cup \{(x, -\sqrt{x}) : x \geq 0\}.$$

Moreover, the inverse $T^{-1} : \mathbb{R} \to \mathbb{R}$ is given by: $T^{-1}x = \{x^2\}$.

Definition 1.2.6. *Let A be a subset of X, we denote by $T_{|A}$ the restriction of T on A, defined by:*

$$T_{|A}x = \begin{cases} Tx, & \text{if } x \in A, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We have Dom(T)$_{|A}$ = Dom(T) $\cap A$.

1.3 Semi-continuous maps

Definition 1.3.1. [23] *Let X, Y be a metric spaces. Let $T : X \to \mathcal{P}(Y)$ be a multivalued mapping and a point $x_0 \in X$. The mapping T is called*

- *upper semi-continuous on point x_0 if*

$$\lim_{n\to\infty} e(T(x_n), T(x_0)) = 0,$$

  for each sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_0.

- *lower semi-continuous on point x_0 if*

$$\lim_{n\to\infty} e(T(x_0), T(x_n)) = 0,$$

  for each sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_0.

Example 1.3.1.

1. *The multivalued mapping* $T_1 : \mathbb{R} \to \mathbb{R}$ *defined by*

$$T_1 x = \begin{cases} [-1,1] & \text{if} \quad x \neq 0, \\ \{0\} & \text{if} \quad x = 0. \end{cases}$$

   is lower semi continuous in 0 *but is not upper semi continuous in* 0.

2. *The multivalued mapping* $T_2 : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *defined by*

$$T_2 x = \begin{cases} \{0\} & \text{if} \quad x \neq 0, \\ [-1,1] & \text{if} \quad x = 0. \end{cases}$$

   is upper semi continuous in 0 *but is not lower semi continuous in* 0.

Lemma 1.3.1. [1] *Let* $T : X \to \mathcal{P}(Y)$ *be a multivalued mapping, then the following statements are equivalent.*

1. T *is upper semi continuous.*

2. $V \subset Y \Rightarrow T^{-1}(\overline{V})$ *is closed in* X.

Lemma 1.3.2. [1] *Let* $T : X \to \mathcal{P}(Y)$ *be a multivalued mapping, then the following statements are equivalent.*

1. T *is lower semi continuous.*

2. $V \subset Y \Rightarrow T^{-1}(int(V))$ *is open in* X.

Definition 1.3.2. [23] *We say that* T *is* ***continuous*** *at* x *if* T *is lower semi continuous and upper semi continuous at* x*, and* T *is continuous if and only if it is continuous at every point* $x \in Dom(T)$.

Now, we will define continuity in the sense of Hausdorff:

Definition 1.3.3. [23] *A multivalued mapping $T : X \to CB(X)$ is continuous on x_0 with respect to H if*

$$\lim_{n \longrightarrow +\infty} H(T(x_n), T(x_0)) = 0$$

for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converging to x_0.

Definition 1.3.4. [40] *Let (X, d) be a metric space. A multivalued mapping $T : X \to CB(X)$ is α-**continuous**, if for $x \in X$ and a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} d(x_n, x) = 0$ and $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, implies*

$$\lim_{n\to\infty} H(Tx_n, Tx) = 0.$$

Remark 1.3.1. *Note that the continuity of T implies the α-continuity of T, for all a function α . In general, the converse is not true, (see Example 1.3.2).*

Example 1.3.2. *Let $X = \mathbb{R}_+, \lambda \in [10, 20]$ and $d(x, y) =| x - y |$, for all $x, y \in X$. We define $T : X \to CB(X)$ and $\alpha(x, y) : X \times X \to \mathbb{R}_+$ by*

$$Tx = \begin{cases} \{\lambda x^2\} & \text{if} \quad x \in [0, 1], \\ \{x\} & \text{if} \quad x > 1, \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} \cosh(x^2 + y^2) & \text{if} \quad x, y \in [0, 1], \\ \tanh(x + y) & \text{else}. \end{cases}$$

It is clear that T is not continuous on $(CB(X), H)$. Indeed, for a sequence $x_n = \{1 + \frac{1}{n}\}$ in X, we see that $x_n = 1 + \frac{1}{n} \to 1$, but
$Tx_n = \{1 + \frac{1}{n}\} \xrightarrow{H} \{1\} \neq \{\lambda\} = T(1)$.

Now, we are going to show that T is α-continuous on $(CB(X), H)$. Let x_n be a sequence in X such that $x_n \xrightarrow{d} x \in X$ as $n \to \infty$, and $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$. Then we have $x, x_n \in [0, 1]$ for all $n \in \mathbb{N}$. So $Tx_n = \{\lambda x_n^2\} \xrightarrow{H} \{\lambda x^2\} = Tx$. This shows that T α-continues on $(CB(X), H)$.

Now, we are going to give the concept of α-lower semi-continuous and α-upper semi-continuous multivalued mappings.

Definition 1.3.5. [33] *Let (X,d) be a metric space. A mapping $T\colon X \to CL(X)$ is*

(1) α-lower semi-continuous, if for $x \in X$ and a sequence $\{x_n\}$ in X with $\lim\limits_{n\to\infty} d(x_n,x)=0$ and $\alpha(x_n,x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, implies

$$\liminf_{n\to\infty} d(x_n,Tx_n) \geq d(x,Tx).$$

(2) α-upper semi-continuous, if for $x \in X$ and a sequence $\{x_n\}$ in X with $\lim\limits_{n\to\infty} d(x_n,x)=0$ and $\alpha(x_n,x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, implies

$$\limsup_{n\to\infty} d(x_n,Tx_n) \leq d(x,Tx).$$

Remark 1.3.2. *A multivalued mapping T is α- continuous if it is α-lower semi continuous and α-upper semi continuous.*

Example 1.3.3. *Let $X=\mathbb{R}$ with usual metric d. Then (X,d) is a metric space. We define $T_1\colon X \to 2^X$, $\alpha : X \times X \to \mathbb{R}_+$ by*

$$T_1x = \begin{cases} [-1,1] & \text{if } x \neq 0, \\ \{0\} & \text{if } x = 0, \end{cases}$$

$$\alpha(x,y) = \begin{cases} 0 & \text{if } x,y \neq 0, \\ 2 & \text{if } x=y=0, \end{cases}$$

First, we show that T_1 is not a multivalued mapping upper semi-continuous . For this, either $V=[-1,1] \ \subset 2^X$, then $T_1^{-1}(\overline{V}) = T_1^{-1}\big([-1,1]\big) = \mathbb{R}-\{0\}$ which is not closed in $\mathbb{R}$, so by Lemma (1.3.2), T_1 is not upper semi continuous. But T_1 is a multivalued mapping α upper semi-continuous . Indeed, $\alpha(x_n,x_{n+1}) \geq 1$ for a sequence $x_n=0$, for all $n \in \mathbb{N}$. Then, x_n approximates to $x=0$ only. Therefore, if $x_n \to 0$, then $T_1x_n=\{0\}$ and $T_1x=\{0\}$. This implies that

$$\limsup_{n\to\infty} d(x_n,T_1x_n) = 0 = d(x,T_1x).$$

On the other hand

$$\lim_{n\to\infty} H(T_1x_n,T_1x) = 1.$$

Therefore, T_1 is not an α-continue multivalued mapping.

Example 1.3.4. *Consider X as in Example (1.3.3). We define $T_2\colon X \to 2^X$, $\alpha : X \times X \to \mathbb{R}_+$ by*

$$T_2x = \begin{cases} \{0\} & \text{if } x \neq 0, \\ [-1,1] & \text{if } x = 0, \end{cases}$$

$$\alpha(y,x) = \begin{cases} 1 & \text{if } \ x, y \neq 0, \\ 0 & \text{if } \ x = y = 0. \end{cases}$$

First, we will show that T_2 is not a lower semi-continuous multivalued mapping. For this, either $V = [-1,1] \ \subset 2^X$, then $T_2^{-1}(int(V)) = T_2^{-1}\big(]-1,1[\big) = \{0\}$ which is not open in $\mathbb{R}$, so by Lemma (1.3.1) , T_2 is not lower semi continuous. But T_2 is a multivalued mapping α lower-semi-continuous. Indeed, $\alpha(x_n, x_{n+1}) \geq 1$ for a sequence x_n of non zero real numbers. Here are two cases:

Case 1. $x_n \to x = 0$.

If $x_n \to 0$, then $T_2x_n = \{0\}$ and $T_2x = [-1,1]$ such that $d(x_n, T_2x_n) = d(x_n, \{0\}) = x_n$ and $d(x, T_2x) = d(0, [-1,1]) = 0$. This implies that

$$\liminf_{n\to\infty} d(x_n, T_2x_n) = \liminf_{n\to\infty} x_n = x = 0 = d(x, T_2x).$$

Case 2. $x_n \to x \neq 0$.

If $x_n \to x$, then $T_2x_n = \{0\}$ and $T_2x = \{0\}$ such that $d(x_n, T_2x_n) = d(x_n, \{0\}) = x_n$ and $d(x, T_2x) = x$. This implies that

$$\liminf_{n\to\infty} d(x_n, T_2x_n) = \liminf_{n\to\infty} x_n = x = d(x, T_2x).$$

*On the other hand, in **Case 1.** we have*

$$\lim_{n\to\infty} H(T_2x_n, T_2x) = 1.$$

Therefore, T_2 is not an α-continue multivalued mapping.

1.4 Admissible maps

1.4.1 Single-valued case

Definition 1.4.1. [56] *Let (X, d) be a metric space, $T : X \to X$ and $\alpha \colon X \times X \to \mathbb{R}_+$ two applications. We say that T is α-**admissible** if for $x \in X$,*

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Example 1.4.1. *Let $X = \mathbb{R}_+$. We define $T : X \to X$ and $\alpha \colon X \times X \to \mathbb{R}_+^*$ by $Tx = \ln x$, for all $x \in X$, and*

$$\alpha(x, y) = \begin{cases} 2 & \text{if} \quad x \geq y, \\ 0 & \text{if} \quad x < y, \end{cases}$$

then T is α-admissible.

Example 1.4.2. *Let $X = [0, +\infty)$. We define $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ by $Tx = \sqrt{x}$ for all $x \in X$ and*

$$\alpha(x, y) = \begin{cases} e^{x-y} & \textit{if } x \geq y, \\ 0 & \textit{if } x < y. \end{cases}$$

Then T is α-admissible.

Remark 1.4.1. *It is clear that every definite map of X into itself non-decreasing T is α-admissible.*

1.4.2 Multivalued case

First, Asl et al [56] have adapted the notion of α-admissible to multivalued maps as α_*-admissible. Subsequently, Mohammadi et al [49] introduced the concept of α-admissible for multivalued maps.

Definition 1.4.2. [49] *Let (X, d) be a metric space and $\alpha \colon X \times X \to \mathbb{R}_+$ a given function, and let $T \colon X \to CL(X)$ a multivalued mapping. T is said to be α-**admissible**, if for all $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $\alpha(y, z) \geq 1$ for all $z \in Ty$.*

Definition 1.4.3. [56] *Let (X,d) be a metric space and $\alpha\colon X\times X\to\mathbb{R}_+$ a given function, and let $T\colon X\to CL(X)$ a multivalued maps. We say that T is α_***-admissible**, if $\alpha(x,y)\geq 1$ implies $\alpha_*(Tx,Ty)\geq 1$, where*

$$\alpha_*(Tx,Ty)=\inf\left\{\alpha(a,b):\ a\in Tx, b\in Ty\right\}.$$

Example 1.4.3. *Let $X=\mathbb{R}_+$, and $\delta\in\,]0,1[$ be a fixed number. We define $T\colon X\to CL(X)$ by $Tx=[0,\delta x]$ for all $x\in X$ and $\alpha\colon X\times X\to\mathbb{R}_+$ by*

$$\alpha(x,y)=\begin{cases}1 & \text{if}\quad x,y\in[0,1],\\ 0 & \text{if}\quad x\notin[0,1]\text{ or } y\notin[0,1].\end{cases}$$

Now we show that T is α_*-admissible.

If $\alpha(x,y)\geqslant 1$, then $x,y\in[0,1]$ and therefore Tx and Ty are subsets of $[0,1]$. Thereby, $a,b\in[0,1]$ for all $a\in Tx$ and $b\in Ty$. Therefore $\alpha(a,b)=1$ for all $a\in Tx$ and $b\in Ty$. This implies that

$$\alpha_*(Tx,Ty)=\inf\left\{\alpha(a,b):\ a\in Tx, b\in Ty\right\}=1.$$

So T is α_*-admissible.

Remark 1.4.2. *It is clear that any α_*-admissible map is also α-admissible, but the converse is not true in general as shown in the following example.*

Example 1.4.4. *Let $X=[-1,1]$ and $\alpha\colon X\times X\to\mathbb{R}_+$ defined by $\alpha(x,x)=0$ and $\alpha(x,y)=1$ for $x\neq y$. We define $T\colon X\to CB(X)$ by*

$$Tx=\begin{cases}\{-x\} & \text{if}\quad x\notin\{-1,0\},\\ \{0,1\} & \text{if}\quad x=-1,\\ \{1\} & \text{if}\quad x=0.\end{cases}$$

Let $x=-1$ and $y=0\in Tx=\{0,1\}$, then $\alpha(x,y)\geqslant 1$, but $\alpha_*(Tx,Ty)=\alpha_*(\{0,1\},\{1\})=0$. So T is not α_*-admissible.

Let us now show that T is α-admissible with the following cases:

***Case 1.** If $x=0$, then $y=1$ and $\alpha(x,y)\geqslant 1$. Also, $\alpha(y,z)\geqslant 1$ since $z=-1\in Ty=\{-1\}$.*

Case 2. *If* $x = -1$, *then* $y \in \{0, 1\}$ *and* $\alpha(x, y) \geqslant 1$. *Also,* $\alpha(y, z) \geqslant 1$ *for all* $z \in Ty$.

Case 3. *If* $x \notin \{-1, 0\}$, *then* $y = -x$ *and* $\alpha(x, y) \geqslant 1$. *Also,* $\alpha(y, z) \geqslant 1$ *since* $z = x \in Ty = \{x\}$.

Chapter 2

Weak contractions in Berinde sense

In this chapter, we will give definitions and basic notions concerning some classes of weak contractions in both cases: single-valued and multivalued. Additionally, the fixed point theorem for each type.

2.1 Banach contraction principle and its generalizations

In 1922, Stefan Banach [9] established a remarkable fixed point theorem known as "Banach's Contraction Principle" which is one of the most important results of analysis and considered to be the source principle of the metric fixed point theory.

Definition 2.1.1. [1] *Let (X, d) be a metric space. A $T : X \to X$ mapping is said to be:*

1. *Lipschitz (or k-Lipschitz) if and only if there exists a constant $k > 0$ such that for all $x, y \in X$ we have: $d(Tx, Ty) \leq kd(x, y)$;*

2. *contraction if $k < 1$;*

3. *no-expansive if and only if it is 1-Lipschitz.*

Theorem 2.1.1 (Banach contraction principle). [9] *Let (X, d) be a complete metric space and $T : X \to X$ be a contraction. So*

(i) T has a unique fixed point $x^* \in X$.

(*ii*) *For all* $x \in X$*, the Picard sequence* $\{x_n\}$ *defined by* $Tx_n = x_{n+1}$*,* $n \in \mathbb{N}$ *converges to* x^**.*

(*iii*) *For all* $x \in X$*,*

$$d(T^n x, x^*) \leq \frac{k^n}{1-k} d(x, Tx), n \in \mathbb{N}.$$

Example 2.1.1. *Let* $X = \mathbb{R}$ *and* $T : \mathbb{R} \to \mathbb{R}$ *a mapping defined by* $Tx = \frac{2}{3}x + 1, x \in \mathbb{R}$*. then* T *is a contraction and* $x^* = 3$ *a fixed point of* T*.*

Example 2.1.2. $X = \mathbb{R}$ *and*

$$Tx = \begin{cases} 1 & , \quad x \in \mathbb{Q} \\ 0 & , \quad x \notin \mathbb{Q} \end{cases}$$

T *is not continuous, so is not a contraction.*

$$T^2 x = \begin{cases} 1 & , \quad x \in \mathbb{Q} \\ 1 & , \quad x \notin \mathbb{Q} \end{cases}$$

T^2 *is a contraction, while* T^2 *and* T *has the same fixed point* 1*.*

Edelstein's contraction

In the year 1962, Edelstein [25] proved the following generalization of Banach's principle.

Theorem 2.1.2. [25] *Let* (X, d) *be a compact metric space and* $T : X \to X$ *be a mapping. Suppose that*

$$d(Tx, Ty) < d(x, y) \ \ \textit{for all } x, y \in X \textit{ with } x \neq y. \tag{2.1}$$

Then T *has a unique fixed point in* X*.*

Kannan contraction

Theorem 2.1.3. [38] *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a map such that there exists a real number* $\alpha \in \left[0, \frac{1}{2}\right[$ *and for all* x, y

$$d(Tx, Ty) \leq \alpha[d(Tx, x) + d(Ty, y)]. \tag{2.2}$$

So

(i) *T has a unique fixed point $x^* \in X$.*

(ii) $\lim\limits_{n\to\infty} T^n x = x^*$.

(iii) *For all $x \in X$*

$$d(T^n x, x^*) \leq \frac{\beta^n}{1-\beta} d(x, Tx), \quad \beta \in [0,1[\,.$$

Chatterjea's contraction

Theorem 2.1.4. [19] *Let (X, d) be a complete metric space and $T : X \to X$ be a map such that there exists a real number $\alpha \in \left[0, \frac{1}{2}\right[$ and for all $x, y \in X$*

$$d(Tx, Ty) \leq \alpha[d(Tx, y) + d(Ty, x)]. \tag{2.3}$$

Then T has a unique fixed point in X.

Reich contraction

Theorem 2.1.5. [54] *Let (X, d) be a complete metric space and $T : X \to X$ a map such that there exist positive numbers $a, b, c \in [0, 1[$ matching $a + b + c < 1$ and:*

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty) \text{ for all } x, y \in X. \tag{2.4}$$

Then T has a unique fixed point in X.

Hardy and Rogers contraction

Theorem 2.1.6. [29] *Let (X, d) be a complete metric space and $T : X \to X$ be a map such that there exist positive numbers $a_1, a_2, a_3, a_4, a_5 \in [0, 1[$ with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ and :*

$$d(Tx, Ty) \leq a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, Ty) + a_4 d(y, Tx) + a_5 d(x, y), \tag{2.5}$$

for all $x, y \in X$.
Then, T has a unique fixed point in X.

Non-linear contraction

Theorem 2.1.7. [17] *Let (X,d) be a metric space complete and $T : X \to X$ be a mapping. Suppose there exists an upper semi-continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for each $t > 0$ and*

$$d(Tx, Ty) \leq \phi(d(x,y)), \text{ for each } x,y \in X. \tag{2.6}$$

Then T has a unique fixed point x^*. Also, for each $x \in X$, we have $\lim\limits_{n\to\infty} T^n x = x^*$.

2.2 Nadler contraction and its generalizations

Definition 2.2.1. [50] *A multivalued mapping $T : X \to CB(X)$ is called*

(1) Lipschitz if and only if there exists $k > 0$ such that

$$H(T(x), T(y)) < kd(x,y) \quad \forall x, y \in X.$$

(2) a contraction if and only if it is Lipschitz with $k < 1$.

Definition 2.2.2. [7] *Let X be a non-empty set and let $T : X \longrightarrow \mathcal{P}(Y)$. A point $x \in X$ is a fixed point of T if $x \in Tx$.*

Theorem 2.2.1 (Nadler). [50] *Let (X,d) be a complete metric space. If $T : X \to CB(X)$ is a multivalued contraction, then T has at least one fixed point, i.e. there exists $x \in X$ such that $x \in Tx$.*

Proof. Let $k \in [0,1[$ be the Lipschitz constant of T and $x_0 \in X$. We choose $x_1 \in Tx_0$. Since $Tx_0, Tx_1 \in CB(X)$ and $x_1 \in Tx_0$, by lemma 1.1.3, there exists a point $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq H(Tx_0, Tx_1) + k.$$

Now, when $Tx_1, Tx_2 \in CB(X)$, there is a point $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq H(Tx_1, Tx_2) + k^2.$$

Continuing in this manner, we obtain a sequence $\{x_i\}_{i=1}^{\infty}$ of points of X such that $x_{i+1} \in Tx_i$ and

$$d(x_i, x_{i+1}) \leq H(Tx_{i-1}, Tx_i) + k^i, \text{ for all } i \geq 1.$$

We note that

$$\begin{aligned} d(x_i, x_{i+1}) &\leq H(Tx_{i-1}, Tx_i) + k^i \\ &\leq kd(x_{i-1}, x_i) + k^i \\ &\leq kH(Tx_{i-2}, Tx_{i-1}) + k^{i-1} + k^i \\ &\leq k^2 d(x_{i-2}, x_{i-1}) + 2k^i \\ &\leq \dots \\ &\leq k^i d(x_0, x_1) + ik^i, \end{aligned}$$

for all $i \geq 1$. So

$$\begin{aligned} d(x_i, x_{i+j}) &\leq d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + \cdots + d(x_{i+j-1}, x_{i+j}) \\ &\leq k^i d(x_0, x_1) + ik^i + k^{i+1} d(x_0, x_1) + (i+1)k^{i+1} \\ &\quad + k^{i+j-1} d(x_0, x_1) + (i+j-1)k^{i+j-1} \\ &\leq \left(\sum_{n=1}^{i+j-1} k^n\right) d(x_0, x_1) + \sum_{n=i}^{i+j-1} nk^n, \end{aligned}$$

for all $i, j \geq 1$.

Therefore $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence and since the completeness of (X, d) then $\{x_i\}_{i=1}^{\infty}$ converges to a point $z \in X$. In this case

$$d(x_{n+1}, Tz) \leq H(Tx_n, Tz) \leq kd(x_n, z)$$

Passing to the limit when $n \to \infty$ we get

$$d(z, Tz) = 0.$$

So $z \in \overline{Tz} = Tz$, because Tz is closed. Then z is a fixed point of T. □

Example 2.2.1. *Let $I = [0, 1]$ be the unitary interval of real numbers (with the usual metric) and let $f : I \to I$ given by*

$$f(x) = \begin{cases} \dfrac{-2}{3}x + 1 & \text{if } x \in \left[0, \dfrac{2}{3}\right], \\[2ex] \dfrac{2}{3}x + \dfrac{1}{3} & \text{if } x \in \left]\dfrac{2}{3}, 1\right]. \end{cases}$$

We define $T : I \to 2^I$ by $Tx = \{0\} \cup \{f(x)\}$ for all $x \in I$.
Then, T is a multivalued contraction (in Nadler sense) with fixed points of T is $\left\{0, \dfrac{3}{5}, 1\right\}$.

Remark 2.2.1. *Example (2.2.1) shows that the fixed point of a multivalued contraction mapping is not necessarily unique.*

2.3 Weak single-valued contractions

In [13], Berinde introduced the concept of weak contractions (or (δ, L)-weak contraction) and studied the existence of fixed points for this class of maps. Moreover, he proved that several contraction-like applications (Kannan contraction, Chatterjee contraction, Zamfirescu contraction, Hardy-Rogers contraction and many others) are weak contractions.

2.3.1 Berinde contraction

Definition 2.3.1. [13] *Let (X, d) be a complete metric space and $T : X \to X$ be a single-valued mapping. Then T is called a Berinde contraction*

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx), \tag{2.7}$$

for all $x, y \in X$.

Remark 2.3.1. *Due to the distance is symmetric, the condition 2.7 implicitly includes the following dual condition*

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(x, Ty)$$

Berinde's theorem

Theorem 2.3.1. [12] *Let (X, d) be a complete metric space and $T : X \to X$ be a weak contraction such that there exists $\delta \in [0, 1[$ and $L \geq 0$ such that, for all $x, y \in X$,*

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx).$$

Then T has a unique fixed point in X.

Definition 2.3.2. [13] *Let (X, d) be a complete metric space and $T : X \to X$ be a single-valued mapping. Then T is called a generalized Berinde contraction, if there exists $\delta \in [0, 1)$ and a function $\varphi : X \to [0, \infty)$ such that*

$$d(Tx, Ty) \leq \delta d(x, y) + \varphi(y)d(y, Tx), \text{ for all } x, y \in X.$$

Comparison function and c-comparison function

Definition 2.3.3. [15] *A function $\psi : [0, +\infty) \to [0, +\infty)$ is called a comparison function if the following conditions are satisfied:*

(1) : *ψ is non-decreasing,*

(2) : $\lim_{n\to\infty} \psi^n(t) = 0$, *for all $t \in [0, +\infty)$.*

Throughout this section, we will denote by Ψ the set of a comparison functions

Example 2.3.1. *1) The function $\psi(t) = at$ where $0 < a < 1$ is a comparison function.*

2) $\psi(t) = \frac{t}{t+1}$ for all $t > 0$.

Definition 2.3.4. [15] *A function p $\psi : [0, +\infty) \to [0, +\infty)$ is called a c-comparison function if the following conditions are satisfied:*

(1) : *ψ is non-decreasing,*

(2) : $\sum_{n=0}^{\infty} \psi^n(t) < \infty$, *for all $t \in [0, +\infty)$.*

Lemma 2.3.2. *For each function $\psi \in \Psi$ and it is increasing, then for all $t \in [0, +\infty)$, $\lim_{n\to+\infty} \psi^n(t) = 0$ implies $\psi(t) < t$.*

Proof. Suppose ψ is increasing, $\lim_{n\to\infty} \psi^n(t) = 0$ and there exists $t_0 > 0$ such that $\psi(t_0) \geq t_0$. Since ψ is increasing, then

$$\psi^2(t_0) \geq \psi(t_0) \geq t_0.$$

Continually by induction, we conclude $\psi(t_0) \geq t_0$ and therefore $\psi^n(t) \nrightarrow 0$ when $n \to \infty$. It is a contradiction. $\square$

Berinde [13] in 2004 defined a new type of non linear contraction using a comparison function as follows.

Definition 2.3.5. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function of c-comparison. Then T is called contraction if there exists $\delta \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \psi(d(x, y)) + Ld(x, Ty),$$

for all $x, y \in X$.

Theorem 2.3.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a map and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function of c-comparison such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq \psi(d(x, y)) + Ld(x, Ty).$$

Then T has a fixed point.

Definition 2.3.6. *A function $\varphi : (0, \infty) \to [0, 1)$ is called $\mathcal{MT}$-function if it satisfies the Mizoguchi-Takahashi condition, i.e.*

$$\limsup_{t \to s^+} \varphi(t) < 1 \text{ for all } s \in [0, +\infty)$$

Remark 2.3.2.

(1) *If the function $\varphi : (0, \infty) \to [0, 1)$ is non-increasing and non-decreasing, then it is $\mathcal{MT}$-function.*

(2) *The constant function $\varphi : (0, \infty) \to [0, 1)$ such that $\varphi(t) = c \in [0, 1)$ is a $\mathcal{MT}$-function.*

Example 2.3.2. *The function $\varphi : (0, \infty) \to [0, 1)$ such that*

$$\varphi(t) = \begin{cases} \frac{\sin t}{t} & \text{if } t \in (0, \frac{\pi}{2}] \\ 0 & \text{if } t \notin (0, \frac{\pi}{2}] \end{cases},$$

is a $\mathcal{MT}$-function.

Example 2.3.3. *The function $\varphi : (0, \infty) \to [0, 1)$ such that*

$$\varphi(t) = \begin{cases} 2t & \text{if } 0 \leq t < \frac{1}{2} \\ 0 & \text{if } t \geq \frac{1}{2} \end{cases},$$

is a $\mathcal{MT}$-function.

2.3.2 Wordowski contraction

In 2012, Wardowski [62] introduced a new type of contractions called F-contraction.

Definition 2.3.7. [62] *Let $\mathcal{F}$ be the set of all functions $F\ :]0,+\infty[\to R$ satisfying the following conditions:*

(F1): F is strictly increasing, that is, for all $t_1,t_2\in]0,+\infty[$ such that $t_1<t_2, F(t_1)<F(t_2)$.

(F2): For any sequence $\{t_n\}_{n\in N}$ $\lim\limits_{n\to+\infty} t_n=0$ if and only if $\lim\limits_{n\to+\infty} F(t_n)=-\infty$.

(F3): There exists $k\in]0,1[$ such that $\lim\limits_{t\to 0^+} t^kF(t)=0$.

Definition 2.3.8. [62] *Let (X,d) be a metric space. A map $T:X\to X$ is said to be an F-contraction or an F-contracting map if $F\in\mathcal{F}$ and there exists $\tau>0$ such that*

$$\forall x,y\in X,[d(Tx,Ty)>0\Longrightarrow \tau+F(d(Tx,Ty))\leqslant F(d(x,y))]. \tag{2.8}$$

If we replace Wardowski's contraction condition in (2.8) by different examples of the functions F, we obtain various types of contraction, some known types of which have been found in the literature.

Remark 2.3.3. *From (F_1) and 2.8, we get $d(Tx,Ty)<d(x,y)$, for all $x,y\in X, Tx\neq Ty$, that is, every F-contractual mapping is contractive. Moreover, it is well known that if T is contractive, then it is non-expansive, hence Lipschitz, and then it is continuous. Therefore, any F-contracting mapping is continuous.*

The following theorem, which was given by Wardowski, is a generalization of Banach's principle of contraction of a way different from the principle well known in the literature.

Theorem 2.3.4. [62] *Let (X,d) be a complete metric space and $T:X\to X$ a F-contraction. Then T has a unique fixed point $z\in X$. Moreover, for all $x_0\in X$, we have $\lim\limits_{n\to+\infty} T^nx_0=z$.*

In the sequel, we will give another version of the theorem by Wardowski. We will first need the following lemma. For the proof we refer to [57].

Lemma 2.3.5. *Let $F\ :]0,+\infty)\to\mathbb{R}$ be an increasing function and $\{t_n\}_{n\in\mathbb{N}}$ a sequence. Then, the following statements are satisfied:*

1. *If* $\lim_{n\to+\infty} F(t_n) = -\infty$, *then* $\lim_{n\to-\infty} t_n = 0$.

2. *If* $inf F = -\infty$ *and* $\lim_{n\to+\infty} t_n = 0$, *then* $\lim_{n\to+\infty} F(t_n) = -\infty$.

Remark 2.3.4. *According to the previous lemma, Secelean* [57] *showed that the condition* (F_2) *in the definition 2.3.9 can be replaced by an equivalent and very simple condition as follows:*
$(F_2^{'})$: $inf F = -\infty$.
Also, Piri and Kumam [52] *used the following condition instead of (F3):*
$(F_3^{'})$: F *is continuous on* $]0, +\infty[$.
Denoting by $\mathcal{F}'$ *the set of all functions* F *satisfying the conditions* $(F_1), (F_2^{'})$ *and* $(F_3^{'})$.

Example 2.3.4. *Let* $F_1(t) = \frac{-1}{t}$, $F_2(t) = \dfrac{1}{1-e^t}$, *then we easily see that* $F_1, F_2 \in \mathcal{F}$.

Theorem 2.3.6. [52] *Let (X,d) be a complete metric space and* $T : X \to X$ *a mapping. Suppose there is* $F \in \mathcal{F}$ *and* $\tau > 0$ *such that:*

$$\forall x, y \in X, [d(Tx, Ty) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \leqslant F(d(x, y))].$$

Then T *has a unique fixed point* $z \in X$. *Moreover, for all* $x_0 \in X$, *we have*
$\lim_{n\to+\infty} T^n x_0 = z$.

Definition 2.3.9. [46] *Let* (X, d) *be a metric space. A map* $T : X \to X$ *is said to be an weak F-contraction or an F-contraction of Berinde type map if* $F \in \mathcal{F}$ *and there exists* $\tau > 0$ *and* $L \geq 0$ *such that*

$$\forall x, y \in X, [d(Tx, Ty) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \leqslant F(d(x, y)) + Ld(y, Tx)], \tag{2.9}$$

and

$$\forall x, y \in X, [d(Tx, Ty) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \leqslant F(d(x, y)) + Ld(x, Ty)]. \tag{2.10}$$

Theorem 2.3.7. [46] *Let* (X, d) *be a metric complete space and* $T : X \to X$ *is Berinde type F-contraction mapping, then* T *has a fixed point in* X.

2.3.3 Theta type contractions

In 2014, Jleli and Samet [35] introduced a new type of single-valued contraction called θ-contraction and established new fixed point theorems for such a contraction in the context of generalized metric spaces.

Definition 2.3.10. *Let Θ be the set of all functions $\theta : (0, +\infty) \to (1, +\infty)$ be a function satisfying:*

(θ_1) : *θ is non decreasing,*

(θ_2) : *for each sequence $\{t_n\}$ in $(0, +\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$,*

(θ_3) : *there exists $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \dfrac{\theta(t) - 1}{t^r} = l$.*

Example 2.3.5.
1. $\theta_1(t) = e^t$.
2. $\theta_2(t) = e^{te^t}$.
3. $\theta_3(t) = e^{\sqrt{t}}$.
4. $\theta_4(t) = e^{\sqrt{t}e^t}$.

Definition 2.3.11. [35] *Let (X, d) be a metric space. a map $T : X \to X$ is called θ-contraction if there exists $\theta \in \Theta$ and $k \in (0, 1)$ such that*

$$\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k . \tag{2.11}$$

for all $x, y \in X, d(Tx, Ty) > 0$.

Theorem 2.3.8. [35] *Let (X, d) be a complete metric space, and $T : X \to X$ is a θ-contraction. Then T has a unique fixed point in X.*

Remark 2.3.5. *The theorem 2.3.8 is a modified version of Banach's principle contraction. Indeed, if $T : X \to X$ is a Banach contraction map with a constant $\lambda \in (0, 1)$, i.e.,*

$$d(Tx, Ty) \leq \lambda d(x, y), \forall x, y \in X.$$

We have $\theta(t) = e^{\sqrt{t}} \in \Theta, t > 0$, passing to the above inequality, we find

$$e^{\sqrt{d(Tx,Ty)}} \leq e^{\sqrt{\lambda d(x,y)}} = \left[e^{\sqrt{d(x,y)}}\right]^{\sqrt{\lambda}} = \left[e^{\sqrt{d(x,y)}}\right]^k, \forall x, y\ in X,$$

where $k = \sqrt{\lambda}$. We follow from theorem 2.3.8 that T has a unique fixed point in X.

2.4 Weak multivalued contractions

We now introduce the class of multivalued contraction maps and obtain a fixed point theorem for this class of maps.

2.4.1 Weak contractions

In 2007, M. Berinde and V. Berinde [16] proved the following fixed point theorem.

Theorem 2.4.1 (Berinde and Berinde)**.** [16] *Let (X, d) be a complete metric space and $T : X \to CB(X)$ a multivalued mapping. If there exists $L \geq 0$ and a function $\varphi : [0, \infty) \to [0, 1)$ satisfying*

$$\limsup_{t \to s^+} \varphi(t) < 1 \ \textit{ for all } \ s \geq 0,$$

and for all $x, y \in X$

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + Ld(y, Tx).$$

Then T has at least one fixed point.

Note that if we take $L = 0$ in Berinde and Berinde's theorem, then we can obtain the Mizoguchi-Takahashi fixed point theorem [48] which is a partial answer to Reich's problem 9 in [55]. Indeed, S. Reich established the following.

Theorem 2.4.2 (Reich)**.** [55] *Let (X, d) be a complete metric space and $T : X \to K(X)$ a multivalued mapping. If there exists $L \geq 0$ and a function $\varphi : [0, \infty) \to [0, 1)$ satisfying*

$$\limsup_{t \to s^+} \varphi(t) < 1 \ \textit{ for all } \ s \geq 0,$$

and for all $x, y \in X,\ x \neq y$

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y).$$

Then T has at least one fixed point.

Theorem 2.4.3 (Mizoguchi and Takahashi)**.** [48] *Let (X, d) be a complete metric space and $T : X \to CB(X)$ a multivalued mapping. If there exists $L \geq 0$ and a function $\varphi : [0, \infty) \to [0, 1)$ satisfying*

$$\limsup_{t \to s^+} \varphi(t) < 1 \ \textit{ for all } \ s \geq 0,$$

and for all $x, y \in X,\ x \neq y$

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y).$$

Then T *has at least one fixed point.*

Definition 2.4.1. [16] *Let* (X, d) *be a metric space,* $T : X \to CB(X)$ *is a multivalued map,* T *is said to be a weak multivalued contraction or* (δ, L)*-multivalued contraction if and only if there exist two constants* $\delta \in (0, 1)$ *and* $L \geq 0$ *such that*

$$H(Tx, Ty) \leq \delta(d(x, y)) + Ld(y, Tx),$$

for all $x, y \in X$.

Theorem 2.4.4. [16] *Let* (X, d) *be a metric space and* $T : X \to CB(X)$ *a map* (δ, L)*-multivalued contraction. So* T *has at least one fixed point.*

2.4.2 Theta type weak contractions

In 2016, Minak and Altun [45] introduced the notion of non-linear multivalued θ-contraction in this way.

Definition 2.4.2. [45] *Let* (X, d) *be a metric space,* $T : X \to CB(X)$ *and* $\theta \in \Theta$*. Then* T *is said to be* θ*-non-linear multivalued contraction if there exists a function* $k : (0, \infty) \to [0, 1)$ *such that*

$$\theta(H(Tx, Ty)) \leq [\theta(d(x, y))]^{k(d(x,y))}, \tag{2.12}$$

for all $x, y \in X$*, with* $H(Tx, Ty) > 0$.

Theorem 2.4.5. [45] *Let* (X, d) *be a complete metric space,* $T : X \to K(X)$ *is a multivalued non linear* θ*-contraction multivalued. If there is* $k : (0, \infty) \to [0, 1)$ *such that*

$$\limsup_{t \to s^+} k(t) < 1, \quad \textit{for all} \ \ s \in [0, \infty).$$

Then T *has a fixed point.*

The concept of θ-contraction from the case of single-valued maps extended to multivalued maps in 2017 by Hançer and al [28] and they introduced the concept of multivalued θ-contraction.

Definition 2.4.3. [28] *Let (X,d) be a metric space, $T : X \to CB(X)$ be an application and let $\theta \in \Theta$. We say that T is a multivalued θ-contraction if there exists $k \in (0,1)$ such that*

$$\theta(H(Tx,Ty)) \leq [\theta(d(x,y))]^k . \tag{2.13}$$

for all $x, y \in X$, with $H(Tx,Ty) > 0$.

Remark 2.4.1. *We can easily get that any multivalued contraction is also a θ-multivalued contraction with $\theta(t) = e^{\sqrt{t}}$.*

Theorem 2.4.6. [28] *Let (X,d) be a complete metric space and $T : X \to K(X)$ be a multivalued map. If T is a multivalued θ-contraction, then T has a fixed point $z \in X$, and for all $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to z.*

In this theorem 2.4.6 we notice that, Tx is compact for all $x \in X$, Thus, they presented the following problem: Can we replace $CB(X)$ instead of $K(X)$. Unfortunately, the answer is negative with the same conditions shown in the following example.

Example 2.4.1. *Let $X = [0,2]$. We define the distance d on X and the map $T : X \to CB(X)$, by*

$$d(x,y) = \begin{cases} 0 & , if \quad x = y \\ 1 + |x-y| & , if \quad x \neq y \end{cases}, \qquad Tx = \begin{cases} \mathbb{Q}_X & , if \quad x \in X \backslash \mathbb{Q}_X \\ X \backslash \mathbb{Q}_X & , if \quad x \in \mathbb{Q}_X \end{cases},$$

where $\mathbb{Q}_X$ is the set of all rational numbers in X. Then (X,d) is also a complete and bounded metric space, hence all subsets of X are closed and bounded.
Let $\theta \in \Theta$ which is defined by

$$\theta(t) = \begin{cases} e^{\sqrt{t}} & , if \quad t \leq 1 \\ 9 & , if \quad t > 1 \end{cases}.$$

Now we show that T is a multivalued θ-contraction. Note that for all $x, y \in X$: $H(Tx,Ty) > 0 \implies \{x,y\} \cap \mathbb{Q}$ is singleton. Therefore, we have

$$\begin{aligned} H(Tx,Ty) > 0 \quad &\implies \quad H(Tx,Ty) = 1 \text{ and } d(x,y) = 1 + |x-y| > 1 \\ &\implies \quad \theta(H(Tx,Ty)) = e \text{ and } [\theta(d(x,y))]^{\frac{1}{2}} = \sqrt{9} = 3 \\ &\implies \quad \theta(H(Tx,Ty)) \leq [\theta(d(x,y))]^{\frac{1}{2}}. \end{aligned}$$

Therefore, T is a multivalued θ-contraction, But T has not fixed point.

To assert that for any map that satisfies the condition 2.13 of theorem 2.4.6 taking $CB(X)$ instead of $K(X)$ to have a fixed point, we must add a weak condition on $\theta : (0,\infty) \to (1,\infty)$

(θ_4) : $\quad \theta(\inf A) = \inf(\theta(A))$ for all $A \subset (0,\infty)$ with $\inf A > 0$.

We notice that, if θ satisfies (θ_1), then it satisfies (θ_4) if and only if it is right continuous. Let Θ_* be the family of all functions θ satisfying $(\theta_1) - (\theta_4)$.

Example 2.4.2. *The function $\theta : (0,+\infty) \to (1,+\infty)$ defined by $\theta(t) = e^{\sqrt{t}}$ for $t \leq 1$, and $\theta(t) = 3$ for $t > 1$, belongs to $\Theta \backslash \Theta_*$.*

Therefore, by considering the set Θ_*, we can provide the following theorem, which is a real generalization of Nadler's result.

Theorem 2.4.7. [28] *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multivalued map. If T is a multivalued θ-contraction such as $\theta \in \Theta_*$, then T admits a fixed point $z \in X$, and for all $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to z.*

Example 2.4.3. [28] *Consider the complete metric space (X,d), where $X = \mathbb{N}$ and $d : X \times X \to [0,1)$ is given by*

$$d(x,y) = \begin{cases} 0 & if \quad x = y \\ x+y & if \quad x \neq y \end{cases},$$

Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \{0,1\} & if \quad x \in \{0,1,2\} \\ \{0, x-1\} & if \quad x \geq 3 \end{cases},$$

Let $x \geq 3$ and $y = 0$, then we get

$$\lim_{x\to\infty} \frac{H(Tx,Ty)}{d(x,y)} = \lim_{x\to+\infty} \frac{x-1}{x} = 1.$$

Thus, T is not a multivalued contraction. Now, we can assert that, T is multivalued θ-contraction with $\theta(t) = e^{\sqrt{te^t}}$ with $k = e^{-\frac{1}{2}}$.

To find 2.13, we have to show that

$$\frac{H(Tx,Ty)}{d(x,y)} e^{H(Tx,Ty)-d(x,y)} \leq e^{-1}.$$

First of all, we notice that

$$H(Tx,Ty)>0 \Leftrightarrow (x\neq y \text{ and } \{x,y\}\cap\{0,1,2\} \text{ is empty or singleton }).$$

Now, without loss of generality, we can assume $x>y$ in the following cases.

Case 1. *If $\{x,y\}\cap\{0,1,2\}$ is singleton, then*

$$H(Tx,Ty)=x-1 \text{ and } d(x,y)\in\{x,x+1,x+2\}.$$

Thus, we have

$$\frac{H(Tx,Ty)}{d(x,y)}e^{H(Tx,Ty)-d(x,y)}\leq\frac{x-1}{x}e^{-1}\leq e^{-1}.$$

Case 2. *If $\{x,y\}\cap\{0,1,2\}$ is empty, then*

$$H(Tx,Ty)=x-1 \text{ and } d(x,y)=x+y.$$

Thus, we have

$$\frac{H(Tx,Ty)}{d(x,y)}e^{H(Tx,Ty)-d(x,y)}\leq\frac{x-1}{x+y}e^{-1-y}\leq e^{-1}.$$

This shows that all the conditions of the theorem 2.4.6 (or of theorem 2.4.7) are satisfied and therefore T admits a fixed point.

Theorem 2.4.8. [45] *Let (X,d) be a complete metric space, $T:X\to CB(X)$ is a multivalued non-linear θ-contraction map with $\theta\in\Theta_*$. If there is $k:(0,\infty)\to[0,1)$ such that*

$$\limsup_{t\to s^+}k(t)<1, \quad \text{for all} \quad s\in[0,\infty).$$

Then T has a fixed point.

In the same year, Minak and Altun [47] introduced the notion of (α,θ)-multivalued contraction and gave some fixed point results for maps of this type on a complete metric space.

Definition 2.4.4. [47] *Let (X,d) be a metric space, $T:X\to CB(X)$ a multivalued application and $\alpha:X\times X\to[0,\infty)$ a function. We then say that T is a multivalued (α,θ)-contraction if there exists $\theta\in\Theta$ and $k\in(0,1)$ such that*

$$\theta\big(H(Tx,Ty)\big)\leq\big[\theta(d(x,y))\big]^k,$$

for all $(x,y)\in S$, where $S=\big\{(x,y):\alpha(x,y)\geq 1 \text{ and } H(Tx,Ty)>0\big\}$.

Theorem 2.4.9. [47] *Let (X,d) be a complete metric space and $T : X \to K(X)$ a multi-valued (α,θ)-contraction, with $\theta \in \Theta$ and $alpha : X \times X \to [0,\infty)$ a function. Suppose the following conditions are satisfied:*

1. *T is α-admissible.*
2. *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.*
3. *T is upper semi-continuous or X is α-regular, i.e., for any sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, therefore $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

We can take $CB(X)$ instead of $K(X)$ by adding the condition (θ_4) on Θ and we obtain the theorem

Theorem 2.4.10. [47] *Let (X,d) be a complete metric space and $T : X \to CB(X)$ a (α,θ)- contraction, with $\theta \in \Theta_*$ and $\alpha : X \times X \to [0,\infty)$ a function. Suppose the following conditions are satisfied:*

1. *T is α-admissible.*
2. *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.*
3. *T is upper semi-continuous or X is α-regular, i.e., for any sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Theorem 2.4.11. [47] *Let (X,d) be a complete metric space and $T : X \to CB(X)$ (resp. $K(X)$) a multivalued map and $\alpha : X \times X \to [0,\infty)$ a function. Suppose the following conditions are satisfied:*

1. *T is α_*-admissible.*

2. *There exists* $\theta \in \Theta_*$ *(resp.* $\theta \in \Theta$*) and* $k \in (0,1)$ *such that*

$$\theta\big(H(Tx,Ty)\big) \leq \big[\theta(d(x,y))\big]^k,$$

for all $(x,y) \in S_*$*, where*

$$S_* = \big\{(x,y) : \alpha_*(x,y) \geq 1 \text{ and } H(Tx,Ty) > 0\big\} \subset X \times X.$$

3. *There exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$.

4. T *is upper semi-continuous or* X *is* α*-regular, i.e., for any sequence* $\{x_n\}$ *in* X *such that* $x_n \to x \in X$ *and* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$*, we have* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}$.

Then T *has a fixed point.*

Chapter 3

Fixed point theorems for generalized almost θ-contractions

In this chapter, we will present our results published in [42] which concern some fixed point theorems by combining generalized almost contraction with the concept of α-admissibility in the context of multivalued maps. Moreover, we will establish new results of fixed points on partially ordered metric spaces and in metric spaces equipped with a graph.

3.1 Fixed point theorems for generalized almost (α, ψ, θ) contractions

Definition 3.1.1. *Let (X, d) be a metric space and $\alpha : X \times X \to \mathbb{R}$. A mapping $T : X \to CB(X)$ is called a generalized almost $(\alpha, \psi, \theta, k)$ contraction, if there exist a function $\theta \in \Theta$, $\psi \in \Psi$, $L \geq 0$ and $k : (0, \infty) \to [0, 1)$ satisfies $\lim_{t \to s^+} \sup k(t) < 1$ for all $s \in (0, \infty)$ such that*

$$\theta(H(Tx, Ty)) \leq \Big[\theta(\psi(M(x, y))\Big]^{k(M(x,y)))} + LN(x, y), \tag{3.1}$$

for all $x, y \in X$ with $\alpha(x, y) \geq 1$ and $H(Tx, Ty) > 0$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}$$

and $N(x, y) = min\{d(x, Ty), d(y, Tx)\}$.

Theorem 3.1.1. *Let (X,d) be a complete metric space and $T : X \to K(X)$ be a generalized almost (α,ψ,θ,k) contraction, with $\theta \in \Theta$. Assume that the following conditions are satisfied:*

1. *T is α-admissible.*
2. *There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$.*
3. *T is α-lower semi-continuous, or X is α-regular, that is, for every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

***Proof*.** From (2) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$, then

$$H(Tx_0,Tx_1) \geq d(x_1,Tx_1) > 0,$$

otherwise $x_1 \in Tx_1$, or, $x_0 = x_1$, which implies x_1 is a fixed point and the proof completes. For $H(Tx_0,Tx_1) > 0$ using (3.1) we get:

$$\begin{aligned}\theta(d(x_1,Tx_1)) &\leq \theta(H(Tx_0,Tx_1))\\ &\leq \Big[\theta(\psi(M(x_0,x_1)))\Big]^{k(M(x_0,x_1))} + LN(x_1,x_0) < [\theta(M(x_0,x_1))]^{k(M(x_0,x_1))}.\end{aligned}$$

If $d(x_0,x_1) \leq d(x_1,Tx_1)$, we get

$$\theta(d(x_1,Tx_1)) \leq \Big[\theta(\psi(d(x_1,Tx_1)))\Big]^{k(d(x_1,Tx_1))} + LN(x_0,x_1) < \theta(d(x_1,Tx_1),$$

which is a contradiction. Then we have

$$\theta(d(x_1,Tx_1)) \leq \theta(H(Tx_0,Tx_1)) \leq \Big[\theta(\psi(d(x_0,x_1)))\Big]^{k(d(x_0,x_1))}.$$

Since Tx_1 is compact, then there exists $x_2 \in Tx_1$ such that

$$\begin{aligned}\theta(d(x_1,x_2)) = \theta(d(x_1,Tx_1)) &\leq \theta(H(Tx_0,Tx_1))\\ &\leq \Big[\theta(d(x_0,x_1))\Big]^{k(d(x_0,x_1))} < \theta(d(x_0,x_1)).\end{aligned}$$

If $x_1 = x_2$, or $x_2 \in Tx_2$, then x_2 is a fixed point. Suppose $x_1 \neq x_2$ and $x_2 \notin Tx_2$, so $H(Tx_2, Tx_1) > 0$ and since T is α-admissible we have $\alpha(x_1, x_2) \geq 1$. Using (3.1) we get:

$$\theta(d(x_2, Tx_2)) \leq \theta(H(Tx_1, Tx_2)) \leq \Big[\theta(\psi(M(x_1, x_2)))\Big]^{k(M(x_1,x_2))} + LN(x_1, x_2)$$
$$< \Big[\theta(M(x_1, x_2))\Big]^{k(M(x_1,x_2))}.$$

If $d(x_1, x_2) \leq d(x_2, Tx_2)$, we get

$$\theta(d(x_2, Tx_2)) \leq \Big[\theta(\psi(d(x_2, Tx_2)))\Big]^{k(d(x_2,Tx_2))} + LN(x_1, x_2) < \theta(d(x_2, Tx_2)),$$

which is a contradiction. Then we have

$$\theta(d(x_2, Tx_2)) \leq \theta(H(Tx_1, Tx_2)) \leq \Big[\theta(\psi(d(x_1, x_2)))\Big]^{k(d(x_1,x_2))}.$$

The compactness of Tx_2 implies that there exists $x_3 \in Tx_2$ such that

$$\theta(d(x_2, x_3)) = \theta(d(x_2, Tx_2)) \leq \theta(H(Tx_1, Tx_2))$$
$$\leq \Big[\theta(\psi(d(x_1, x_2)))\Big]^{k(d(x_1,x_2))} < \theta(d(x_1, x_2)).$$

Continuing in this manner we can construct a sequence (x_n) in X, if $x_n = x_{n+1}$ or $x_{n+1} \in Tx_{n+1}$, then x_{n+1} is a fixed point, otherwise we get

$$\theta(d(x_n, Tx_n)) \leq \Big[\theta(\psi(M(x_n, x_{n-1})))\Big]^{k(M(x_n,x_{n-1}))} + LN(x_n, x_{n-1}).$$

As the same arguments in previous steps, we get

$$d(x_{n+1}, Tx_{n+1}) \leq d(x_n, x_{n+1}),$$

so we obtain

$$\theta(d(x_n, x_{n+1})) \leq \theta(H(Tx_n, Tx_{n-1})) \leq \Big[\theta(\psi(d(x_n, x_{n-1})))\Big]^{k(d(x_n,x_{n+1}))}$$
$$= \Big[\theta(\psi(d(x_n, x_{n-1})))\Big]^{k(d(x_n,x_{n-1}))} < \theta(d(x_n, x_{n-1})).$$

Since θ is increasing, then the sequence $(d(x_n, x_{n+1}))_n$ is decreasing, further it is bounded at below so it is convergent. On the other hand, $\limsup_{t\to s^+} k(t) < 1$, then there exist $\delta \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that $k(d(x_n, x_{n+1})) < \delta$, for all $n \geq n_0$. Thus we have

$$1 < \theta(d(x_n, x_{n+1})) \leq \Big[\theta(d(x_{n_0}, x_{n_0+1}))\Big]^{\delta^{n-n_0}}, \tag{3.2}$$

for all $n \geq n_0$.

Letting $n \to \infty$ in (3.2), we get

$$\lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = 1,$$

By (θ_2), we infer that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now, we prove $\{x_n\}$ is a Cauchy sequence, from (θ_3) there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r} = l.$$

If $l < \infty$, let $2\varepsilon = l$, so from the definition of limit there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, we have

$$\varepsilon = l - \varepsilon < \frac{\theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r}$$

$$(d(x_n, x_{n+1}))^r < \frac{\theta(d(x_n, x_{n+1})) - 1}{\varepsilon}.$$

Then (3.2) gives

$$n(d(x_n, x_{n+1}))^r < \frac{n(\theta(d(x_{n_0}, x_{n_0+1}))^{\delta^{n-n_0}} - 1)}{\varepsilon}. \tag{3.3}$$

In the case where $l = \infty$, let A be an arbitrary positive real number, so from the definition of the limit there exists $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$, we have

$$\frac{\theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r} > A,$$

which implies that

$$n(d(x_n, x_{n+1}))^r \leq \frac{n(\theta(d(x_{n_0}, x_{n_0+1}))^{\delta^{n-n_0}} - 1)}{A}. \tag{3.4}$$

Letting $n \to \infty$ in (3.3), or in (3.4), we obtain

$$\lim_{n\to\infty} n(d(x_n, x_{n+1}))^r = 0.$$

From the definition of the limit, there exists $n_3 \geq \max\{n_0, n_1, n_2\}$ such that for all $n \geq n_3$, we have

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{r}}},$$

This implies
$$\sum_{n=n_3}^{\infty} d(x_n, x_{n+1}) \leq \sum_{1}^{\infty} \frac{1}{n^{\frac{1}{r}}} < \infty.$$
For all $m, n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$, $m, n \geq n_0$
$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{r}}} \leq \sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{r}}} \longrightarrow 0.$$

Then $\{x_n\}$ is a Cauchy sequence.

The completeness of (X, d) implies that $\{x_n\}$ converges to a some $x \in X$.

Now, we show that x is a fixed point of T. In fact, if T is α-lower continuous, then for all $n \in \mathbb{N}$ we have
$$0 \leq d(x_n, Tx_n) \leq d(x_n, x_{n+1}).$$

Letting $n \to +\infty$, we get
$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

The α-lower semi continuity of T implies
$$0 \leq d(x, Tx) < \lim_{n\to\infty} \inf d(x_n, Tx_n) = 0.$$

Hence $d(x, Tx) = 0$ and x is a fixed point of T.

If X is regular, so $\alpha(x_n, x) \geq 1$ and $H(Tx_n, Tx) > 0$, by using (3.1) we get
$$1 < \theta(d(x_{n+1}, Tx)) \leq \theta(H(Tx_n, Tx)) < \Big[\theta(d(x_{n_0}, x))\Big]^{\delta^{n-n_0}}.$$

Letting $n \to +\infty$, we get
$$\lim_{n\to\infty} \theta(d(x_n, Tx)) = 1,$$

so (θ_2) gives
$$\lim_{n\to\infty} d(x_n, Tx) = d(x, Tx) = 0,$$

which implies that $x \in Tx$. □

Theorem 3.1.2. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a generalized almost (α, ψ, θ) contraction, with θ is right continuous. Assume that the following conditions are satisfied:*

(H_1) : *T is α-admissible,*

(H_2) : *there exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0, x_1) \geq 1$,

(H_3) : *for every sequence* $\{x_n\}$ *in* X *converging to* $x \in X$ *with* $\alpha(x_n, x_{n+1}) \geq 1$, *for all* $n \in \mathbb{N}$, *then* $\alpha(x_n, x) \geq 1$, *for all* $n \in \mathbb{N}$.

Then T *has a fixed point.*

Proof. From (H_2) there are $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$, if $x_0 = x_1$, or, $x_1 \in Tx_1$, so x_1 is a fixed point. Suppose the contrary, then $H(Tx_0, Tx_1) \geq d(x_1, Tx_1) > 0$ and by using (3.1) we get

$$\begin{aligned}\theta(d(x_1, Tx_1)) &\leq \theta(H(Tx_0, Tx_1)) \\ &\leq \Big[\theta(\psi(M(x_0, x_1)))\Big]^{k(M(x_0,x_1))} + LN(x_0, x_1) \\ &< \Big[\theta(M(x_0, x_1))\Big]^{k(M(x_0,x_1))} + LN(x_0, x_1).\end{aligned}$$

By right continuity of θ, there exists $h_1 > 1$ such that

$$\theta(h_1 H(Tx_0, Tx_1)) \leq \Big[\theta(\psi(M(x_0, x_1)))\Big]^{k(M(x_0,x_1))} + LN(x_0, x_1).$$

As in proof of Theorem 3.1.1 we get $M(x_0, x_1) = d(x_0, x_1)$ and $N(x_0, x_1) = 0$, then by using Lemma 1.1.4, there exist $x_2 \in Tx_1$ such that

$$\begin{aligned}\theta(d(x_1, x_2)) &\leq \theta(h_1 H(Tx_0, Tx_1)) \\ &\leq \Big[\theta(\psi(d(x_0, x_1)))\Big]^{k(d(x_0,x_1))} \\ &< \Big[\theta(\psi(d(x_0, x_1)))\Big]^{k(d(x_0,x_1))} \\ &< \theta(d(x_0, x_1)).\end{aligned}$$

Since T is α-admissible, then $\alpha(x_1, x_2) \geq 1$. Assume that $x_1 \neq x_2$ and $x_2 \notin Tx_2$, so $H(Tx_1, Tx_2) \geq d(x_2, Tx_2) > 0$ and using (3.1), we obtain

$$\begin{aligned}1 < \theta(d(x_2, Tx_2)) \leq \theta(H(Tx_1, Tx_2)) &\leq \Big[\theta(\psi(M(x_1, x_2)))\Big]^{k(M(x_1,x_2))} + LN(x_1, x_2) \\ &< \Big[\theta(d(x_1, x_2))\Big]^{k(d(x_1,x_2))}.\end{aligned}$$

As in previous step, we have $M(x_1, x_2) = d(x_1, x_2)$, so we get

$$\theta(d(x_2, Tx_2)) \leq \theta(H(Tx_1, Tx_2)) \leq \Big[\theta(\psi(d(x_1, x_2)))\Big]^{k(d(x_1,x_2))}$$

$$< \Big[\theta(d(x_1,x_2))\Big]^{k(d(x_1,x_2))}.$$

Since θ is right continuous and from Lemma 1.1.4, there exist $h_2 > 1$ and $x_3 \in Tx_2$ such that

$$\begin{aligned}
\theta(d(x_2,x_3)) &\leq \theta(h_2 H(Tx_1,Tx_2)) \\
&\leq \Big[\theta(\psi(d(x_1,x_2)\Big]^{k(d(x_1,x_2))} \\
&< \Big[\theta(d(x_1,x_2))\Big]^{k(d(x_1,x_2))} \\
&< \theta(d(x_1,x_2)).
\end{aligned}$$

Continuing in this manner, we can construct two sequences $\{x_n\} \subset X$ and $(h_n) \subset (1,\infty)$ such that $x_n \neq x_{n+1}$, $x_{n+1} \in Tx_n$, $\alpha(x_n,x_{n+1}) \geq 1$ and

$$\begin{aligned}
1 < \theta(d(x_n,x_{n+1})) &\leq \theta(h_n H(Tx_{n-1},Tx_n)) \\
&\leq \Big[\theta(d(x_n,x_{n-1}))\Big]^{k(d(x_n,x_{n-1}))} + LN(x_n,Tx_{n-1}) \\
&< \theta(d(x_n,x_{n-1})),
\end{aligned}$$

which implies that $(d(x_n,x_{n+1}))_n$ is a decreasing sequence and bounded at below, so there exist $\delta \in (0,1)$ and $n_0 \in \mathbb{N}$ such that $k(d(x_n,x_{n+1})) < \delta$, for all $n \geq n_0$. Thus we have

$$1 < \theta(d(x_n,x_{n+1})) < \Big[\theta(d(x_{n_0},x_{n_0+1}))\Big]^{\delta^{n-n_0}}, \tag{3.5}$$

for all $n \geq n_0$.

On taking the limit as $n \to \infty$, we get $\lim\limits_{n\to\infty} \theta(d(x_n,x_{n+1})) = 1$, so (θ_2) gives

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

The rest of the proof is like in the proof of Theorem 3.1.1. □

Corollary 3.1.1. *Let (X,d) be a complete metric space, $\alpha\colon X\times X \to [0,+\infty)$ be a function and $T\colon X \to K(X)$ (resp $CB(X)$ with θ is right continuous) be an α-admissible multivalued mapping and the following assertions hold:*

(i) T is α-admissible.

(*ii*) *There exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0, x_1) \geq 1$.

(*iii*) T *is* α*-lower semi-continuous, or, for every sequence* $\{x_n\}$ *in* X *such that* $x_n \to x \in X$ *and* $\alpha(x_n, x_{n+1}) \geq 1$, *for all* $n \in \mathbb{N}$, *we have* $\alpha(x_n, x) \geq 1$, *for all* $n \in \mathbb{N}$.

(*iv*) *There exist* $\theta \in \Theta$, $\psi \in \Psi$ *and a function* $k : (0, \infty) \to [0, 1)$ *satisfying* $\lim\limits_{t \to s^+} \sup k(t) < 1$ *such for* $x, y \in X$ $H(Tx, Ty) > 0$ *implies*

$$\alpha(x, y)\theta\big(H(Tx, Ty)\big) \leq \theta\Big[(\psi(M(x, y)))\Big]^{k(M(x,y))} + LN(x, y), \tag{3.6}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\}$$

and $N(x, y) = \min\{d(x, Ty), d(y, Tx)\}$.

Then T *has a fixed point.*

Proof. For $x, y \in X$ with $\alpha(x, y) \geq 1$ and $H(Tx, Ty) > 0$, we have

$$\begin{aligned}\theta\big(H(Tx, Ty)\big) &\leq \alpha(x, y)\theta\big(H(Tx, Ty)\big)\\ &\leq \theta\Big[(\psi(M(x, y)))\Big]^{k(M(x,y))} + LN(x, y),\end{aligned}$$

which implies that the inequality (3.1) holds. Thus, the rest of proof is like in the proof of Theorem 3.1.2 (resp. Theorem 3.1.1). □

If $\alpha(x, y) = 1$, for all $x, y \in X$, we get the following corollary.

Corollary 3.1.2. *Let* (X, d) *be a complete metric space and* $T : X \to K(X)$ *(resp.* $CB(X)$ *with* θ *is right continuous) be a multivalued mapping such that there exist* $\theta \in \Theta$, $\psi \in \Psi$ *and a function* $k : (0, \infty) \to [0, 1)$ *satisfying* $\lim\limits_{t \to s^+} \sup k(t) < 1$ *for all* $s \in (0, \infty)$ *such that*

$$\theta\big(H(Tx, Ty)\big) \leq \theta\Big[(\psi(M(x, y)))\Big]^{k(M(x,y))} + LN(x, y), \tag{3.7}$$

for $x, y \in X$ *with* $H(Tx, Ty) > 0$ *where*

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\}$$

and $N(x, y) = \min\{d(x, Ty), d(y, Tx)\}$. *Then* T *has a fixed point in* X.

Example 3.1.1. *Let $X = \{1,2,3\}$ and $d(x,y) = |x-y|$. Define $T\colon X \to CB(X)$ and $\alpha : X \times X \to [0,\infty)$ by*

$$Tx = \begin{cases} \{1\}, & x \in \{1,3\} \\ \{2\}, & x = 2 \end{cases}$$

and $\alpha(x,y) = e^{|x-y|}$. Taking $\theta(t) = e^t$, $\psi(t) = \frac{4}{5}t$, $L = 2$ and $k(t) = \frac{1}{2}$.
Now, we show that the contraction condition holds.
For $x, y \in X$, we have $|x-y| \geq 0$, which implies $e^{|x-y|} \geq 1$. Then T is α-admissible.
On other hand, $H(Tx,Ty) > 0$ and $\alpha(x,y) \geq 1$ for all $(x,y) \in \{(1,2),(2,1),(2,3),(3,2)\}$.
Then we have the following cases:

1. *for $x = 2$ and $y = 3$, we have*

$$H(T2,T3) = 1, \quad d(2,3) = 1, \quad \psi(d(2,3)) = \frac{4}{5} \quad \text{and} \quad d(2,T3) = d(3,T2) = 1,$$

then

$$e = e^{H(T1,T3)} < (e^{\psi(d(1,3))})^{\frac{1}{2}} + LN(2,3)$$
$$e < e^{\frac{2}{5}} + 2.$$

2. *For $x = 2$ and $y = 1$, we have*

$$H(T2,T1) = 1, \quad d(2,1) = 1, \quad \psi(d(2,1)) = \frac{4}{5} \quad \text{and} \quad d(2,T1) = d(1,T2) = 1,$$

then

$$e = e^{H(T2,T3)} < (e^{\psi(d(2,1))})^{\frac{1}{2}} + LN(2,1)$$
$$e < e^{\frac{2}{5}} + 2.$$

There exist $x_0 = 3$ and $x_1 = 1 \in Tx_0$ such that $\alpha\,(3,1) \geq 1$.

It is clear that T is α- lower semi continuous. Consequently, all conditions of Theorem 3.1.1 are satisfied. Then T has fixed points which are$\{1,3\}$.

3.2 Fixed point on partially ordered metric spaces

Now, we give an existence theorem of fixed point in a partially order metric space, by using the results provided in previous section.

Theorem 3.2.1. *Let $(X, \preceq, d)$ be a complete ordered metric space and $T\colon X \to CB(X)$ be a multivalued mapping. Assume that the following assertions hold:*

1. *For each $x \in X$ and $y \in Tx$ with $x \preceq y$, we have $y \preceq z$ for all $z \in Ty$;*
2. *There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$.*
3. *For every non decreasing sequence $\{x_n\}$ in X such that $x_n \to x \in X$, we have $x_n \preceq x$, for all $n \in \mathbb{N}$.*
4. *There exist a right continuous function $\theta \in \Theta$, $\psi \in \Psi$ and $k : (0, \infty) \to [0, 1)$ satisfies $\lim\limits_{t \to s^+} \sup k(t) < 1$ for all $s \in (0, \infty)$ such that*

$$\theta(H(Tx, Ty)) \leq \Big[\theta(\psi(M(x, y)))\Big]^{k(M(x,y))} + LN(x, y), \tag{3.8}$$

for all $x, y \in X$ with $x \preceq y$ and $H(Tx, Ty) > 0$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\}$$

and $N(x, y) = \min\{d(x, Ty), d(y, Tx)\}$.

Then T has a fixed point.

Proof. Define $\alpha\colon X \times X \to [0, +\infty)$ as follows:

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

From (1), for each $x \in X$ and $y \in Tx$ with $x \preceq y$, i.e., $\alpha(x, y) = 1 \geq 1$, we have $z \preceq y$, for all $z \in Ty$, i.e., $\alpha(x, y) = 1 \geq 1$. Thus T is α-admissible.

From (2), there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$, i.e., $\alpha(x_0, x_1) = 1 \geq 1$.

Condition (3) implies α- lower semi continuity of T, or regularity of X.

From (4), for $x \preceq y$, we have $\alpha(x, y) = 1 \geq 1$ then the inequality (3.1) holds, which implies that T is a generalized almost $(\alpha, \psi, \theta, k)$ contraction. $\square$

3.3 Fixed point on metric spaces endowed with a graph

In this section, as a consequence of our main results, we present an existence theorem of fixed point for a multivalued mapping in a metric space X, endowed with a graph, into the space of non-empty closed and bounded subsets of the metric space.

Definition 3.3.1. *Let (X, d) be a metric space. A directed graph G is a pair formed by two sets, a set $V(G)$ whose elements are called vertices, and a set $E(G)$ contained in the product Cartesian $X \times X$, whose elements are called arcs or edges. We denote $G = (V(G), E(G))$.*

When $a = (x, y) \in E(G)$ we say that a is the edge of G with extremities x and y, or that a joins x and y, or that a passes through x and y. The vertices x and y are said to be adjacent in G.

Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$, where $\Delta = \{(x, x) : x \in X\}$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$.

Theorem 3.3.1. *Let (X, d) be a complete metric space endowed with a graph G and $T : X \to CB(X)$ be a multivalued mapping. Assume that the following conditions are satisfied:*

1. *For each $x \in X$ and $y \in Tx$ with $(x, y) \in E(G)$, we have $(y, z) \in E(G)$ for all $z \in Ty$,*

2. *There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$,*

3. *T is G-lower semi-continuous, that is, for $x \in X$ and a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} d(x_n, x) = 0$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, implies*

$$\liminf_{n\to\infty} d(x_n, Tx_n) \geq d(x, Tx)$$

   or, for every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, we have $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$,

4. *There exist a right continuous function $\theta \in \Theta$, $\psi \in \Psi$ and $k : (0, \infty) \to [0, 1)$ satisfying $\limsup_{t\to s^+} k(t) < 1$ for all $s \in (0, \infty)$ such that*

$$\theta(H(Tx, Ty)) \leq \Big[\theta(\psi(M(x, y)))\Big]^{k(M(x,y))} + LN(x, y), \tag{3.9}$$

for all $x, y \in X$ with $(x, y) \in E(G)$ and $H(Tx, Ty) > 0$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\}$$

and $N(x, y) = \min\{d(x, Ty), d(y, Tx)\}$.

Then T has a fixed point.

Proof. This result is a direct consequence of results of Theorem 3.1.2 by taking the function $\alpha\colon X \times X \to [0, +\infty)$ defined by:

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

$\square$

Chapter 4

Fixed point theorems for θ-contractions in complete b-metric space

The aim of this chapter is to give results which concern the fixed point theorem by combining some types of contractions with the concept of α_s-admissibility in b-metric spaces (also called metric spaces of type (b)[43]. Some consequences are established on partially ordered b-metric spaces and b-metric spaces endowed with a graph.

4.1 b-Metric spaces

It is well known that in some functional spaces, the natural "norm" cannot satisfy the triangle inequality for example, on $L^p(\mathbb{R}^n)$ when $p \in]0, 1[$.

In view of this observation, Bakhtin [8] proposed a generalization of metric spaces called b-metric spaces in a way that allows the extension of fixed point theory to cover these spaces of functions.

As a result, Czerwik [22] used the notion of b-metric to generalize Banach's principle of contraction. Later, several results were published for various classes of single-valued and multivalued maps in this type of spaces.

In this section, we will see some preliminaries, definitions and properties on b-metric spaces.

Definition 4.1.1. [5] *For a non empty set X and a given real number with $s \geq 1$. A* *function* $\widetilde{d} : X \times X \to [0, \infty)$ *is a b-metric on* X *if for all* $x_1, x_2, x_3 \in X$, *the following conditions hold:*

(b1) $\widetilde{d}(x_1, x_2) = 0$ *if and only if* $x_1 = x_2$,

(b2) $\widetilde{d}(x_1, x_2) = \widetilde{d}(x_2, x_1)$,

(b3) $\widetilde{d}(x_1, x_3) \leq s[\widetilde{d}(x_1, x_2) + \widetilde{d}(x_2, x_3)]$.

A triplet $(X, \widetilde{d}, s)$ *is called a b-metric space.*

Remark 4.1.1. *Also, every metric space is a b-metric space but the converse is not necessarily true.*

Example 4.1.1. *Let* $X = [0; 1]$ *and* $\widetilde{d} : X \times X \to \mathbb{R}^+$ *such that* $d(x, y) = |x - y|^2$ *for all* $x, y \in X$. *Clearly,* $(X, \widetilde{d}, 2)$ *is a b-metric space, but it is not metric.*

Now, we will give two very famous examples of b-metric spaces which are not metric.

Example 4.1.2. [8] *Space* $l^p(\mathbb{R}) = \left\{ \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$ *(where* $p \in]0, 1[$*), such as the function* $\widetilde{d} : l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to \mathbb{R}$ *is given by:*

$$\widetilde{d}(\{x_n\}, \{y_n\}) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}},$$

for all $\{x_n\}, \{y_n\} \in l^p(\mathbb{R})$ *is a b-metric space.*

Indeed, by an elementary calculation we get:

$$\widetilde{d}(x, z) \leq 2^{\frac{1}{p}}[\widetilde{d}(x, y) + \widetilde{d}(y, z)].$$

Therefore, $s = 2^{\frac{1}{p}} > 1$ *in this case.*

Definition 4.1.2. [20] *Let* $(X, \widetilde{d}, s)$ *be a b-metric space. The following concepts are natural deductions from the corresponding metric versions:*

1. *A sequence* $\{x_n\}$ *in* X *converges to a point* $x \in X$ *if* $\lim_{n \longrightarrow \infty} \widetilde{d}(x_n, x) = 0$.

2. *A sequence* $\{x_n\}$ *in* X *is Cauchy if for all* $\epsilon > 0$, *there exists* N *such that*

$$\widetilde{d}(x_n, x_m) < \epsilon,$$

for all $n, m \geq N$.

3. *We say that $(X, \widetilde{d}, s)$ is a complete b-metric space if every Cauchy sequence is convergent in X.*

Definition 4.1.3. [22] *Let $(X, \widetilde{d}, s)$ be a metric space, and let $CB(X)$ be a set of non empty, closed and bounded subsets of X, the Pompeiu -Hausdorff metric is defined as:*

$$H(A,B) = \max\{e(A,B), e(B,A)\} = \max\{\sup_{a\in A} \widetilde{d}(a,B), \sup_{b\in B} \widetilde{d}(b,A)\},$$

for all $A, B \in CB(X)$. and $\widetilde{d}(a,B) = inf\{\widetilde{d}(a,b), b \in B\}$.
H is called Hausdorff's b-metric.

Lemma 4.1.1. [5] *Let $(X, \widetilde{d}, s)$ be a b-metric space, for all $A, B, C \in CB(X)$. The following properties are satisfied.*

1) $\widetilde{d}(x,B) \leq \widetilde{d}(x,b)$ for all $x \in X$, $b \in B$.
2) $\widetilde{d}(a,B) \leq H(A,B)$ for all $a \in A$.
3) $\widetilde{d}(x,A) \leq s(\widetilde{d}(x,y) + \widetilde{d}(y,A))$ for all $x, y \in X$.
4) $H(A,A) = 0$.
5) $H(A,B) = H(B,A)$.
6) $H(A,B) \leq s[H(A,B) + H(B,C)]$.

Lemma 4.1.2. [22] *Let $(X, \widetilde{d}, s)$ be a b-metric space and $A, B \in CL(X)$ with $H(A,B) > 0$. Then, for each $b \in B$, there exists $a = a(b) \in A$ such that*

$$\widetilde{d}(a,b) \leq sH(A,B).$$

Definition 4.1.4. [59] *For a non-empty set X and a self mapping $T : X \to X$, $\alpha : X \times X \to [0,\infty)$ be two maps. For a given real number $s \geq 1$, T is weak α-admissible of type S if for $x \in X$ and $\alpha(x,Tx) \geq s$, we have $\alpha(Tx,TTx) \geq s$.*

Definition 4.1.5. [5] *Let $(X, \widetilde{d}, s)$ be a b-metric space. For a given function $\alpha\colon X \times X \to [0,+\infty)$ a multivalued mapping $T\colon X \to CL(X)$ is*
(1) α_s-admissible, if for each $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geq s^2$, we have $\alpha(y,z) \geq s^2$ for each $z \in Ty$.
(2) α_s^*-admissible, if for $x, y \in X$ with $\alpha(x,y) \geq s$ we have $\alpha^*(Tx,Ty) \geq s^2$, where $\alpha^*(Tx,Ty) = \inf\{\alpha(a,b) : a \in Tx, b \in Ty\}$.

Definition 4.1.6. [33] *Let $(X, \widetilde{d}, s)$ be a metric space, multivalued map $T : X \to CL(X)$ and a function $\alpha : X \times X \to [0, \infty)$. T is called to be an α_s-lower semi continuous map if for $x \in X$ and a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \widetilde{d}(x_n, x) = 0$ and $\alpha(x_n, x_{n+1}) \geq s^2$ for all $n \geq 0$ implies*

$$\lim_{n\to\infty} \inf \widetilde{d}(x_n, Tx_n) \geq \widetilde{d}(x, Tx).$$

In [22], Czerwik presented a generalization of Banach contraction in b-metric spaces according to the following theorem:

Theorem 4.1.3. *Let $(X, \widetilde{d}, s)$ be a complete b-metric space. suppose that $\psi \in \Psi$ and $T : X \to X$ map satisfying :*

$$\widetilde{d}(Tx, Ty) \leq \psi\widetilde{d}(x, y), \quad for \quad each \quad x, y \in T.$$

Then T has a unique fixed point z in X and $\lim_{n\to\infty} T^n x = z$, for each $x \in T$.

4.2 Fixed point theorems for almost (α, θ, κ) contractions of Hardy-Rogers type

Definition 4.2.1. *Let Θ_s be the set of all functions $\theta : (0, +\infty) \to (1, +\infty)$ such that:*

(θ_1) : *θ is a non decreasing function.*

(θ_2) : *For each sequence of real positive numbers $\{\omega_n\}$ $\lim_{n\to\infty} \theta(\omega_n) = 1$ if and only if $\lim_{n\to\infty} \omega_n = 0$.*

(θ_3) : *There exist $\rho \in (0, 1)$ and $\chi \in (0, \infty]$ such that $\lim_{\omega\to 0^+} \frac{\theta(\omega) - 1}{\omega^\rho} = \chi$.*

(θ_4) : *For each sequence (u_n) in $\mathbb{R}_+$ such that $\theta(su_n) \leq \Big[\theta(u_{n-1})\Big]^k$, where $k \in (0, 1)$, then $\theta(s^n u_n) \leq \Big[\theta(s^{n-1} u_n)\Big]^k$.*

Definition 4.2.2. *Let $(X, \widetilde{d}, s)$ be a b-metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. A mapping $T : X \to CB(X)$ is called a multivalued almost (α, θ, κ) contraction of Hardy-Rogers type if there exist $\theta \in \Theta_s$, $L \geq 0$ and $\kappa : (0, \infty) \to [0, 1)$ satisfies $\lim_{\omega\to s^+} \sup \kappa(\omega) < 1$*

for all $s \in (0, \infty)$ and non-negative real numbers a_1, a_2, a_3, a_4, a_5 with $a_1+a_2+a_3+2sa_4 = 1$ and $a_3 \neq 1$ such that

$$\theta(s^3 H(Tx,Ty)) \leq \Big[\theta(N_s(x,y))\Big]^{\kappa(\widetilde{d}(x,y))} + L\min\{\widetilde{d}(x,Ty), \widetilde{d}(y,Tx)\}, \tag{4.1}$$

for all $x, y \in X$ with $\alpha(x,y) \geq s^2$ and $H(Tx,Ty) > 0$ where

$$N_s(x,y) = a_1\widetilde{d}(x,y) + a_2\widetilde{d}(x,Tx) + a_3\widetilde{d}(y,Ty) + a_4\widetilde{d}(x,Ty) + a_5\widetilde{d}(y,Tx).$$

Theorem 4.2.1. *Let $(X, \widetilde{d}, s)$ be a complete b-metric space and $T : X \to CB(X)$ be a multivalued almost (α, θ, κ)- contraction of Hardy-Rogers type. Assume that the following conditions hold:*

1. *T is an α_s-admissible,*
2. *There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq s^2$,*
3. *T is α_s-lower semi continuous, or X is α_s-regular, that is, for every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $\alpha(x_n, x_{n+1}) \geq s^2$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq s^2$, for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. From the hypothesis (2), there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq s^2$. If $x_0 = x_1$ or $x_1 \in Tx_1$, then x_1 is a fixed point of T and so the proof is completed. Because of this, assume that $x_0 \neq x_1$ and $x_1 \notin Tx_1$, then $\widetilde{d}(x_1,Tx_1) > 0$ and hence $H(Tx_0,Tx_1) > 0$. From Lemma 4.1.2 there exists $x_2 \in Tx_1$ such that

$$\widetilde{d}(x_1,x_2) \leq sH(Tx_0,Tx_1) \leq s^2H(Tx_0,Tx_1),$$

which implies

$$s\widetilde{d}(x_1,x_2) \leq s^3H(Tx_0,Tx_1).$$

Since θ is strictly increasing, then by using (3.1) we get

$$\theta(s\widetilde{d}(x_1,x_2)) \leq \theta(s^3H(Tx_0,Tx_1)) \leq [\theta(N_s(x_0,x_1))]^{\kappa(\widetilde{d}(x_0,x_1))} < \theta(N_s(x_0,x_1)),$$

which gives

$$\theta(s\widetilde{d}(x_1,x_2)) < \theta(N_s(x_0,x_1)),$$

Since θ is increasing, then we have

$$s\widetilde{d}(x_1, x_2) < N_s(x_0, x_1)$$

where

$$N_s(x_0, x_1) = a_1\widetilde{d}(x_0, x_1) + a_2\widetilde{d}(x_0, Tx_0) + a_3\widetilde{d}(x_1, Tx_1) + a_4\widetilde{d}(x_0, Tx_1) + a_5\widetilde{d}(x_1, Tx_0).$$

Then

$$\begin{aligned} N_s(x_0, x_1) &= a_1\widetilde{d}(x_0, x_1) + a_2\widetilde{d}(x_0, x_1) + a_3\widetilde{d}(x_1, x_2) + a_4\widetilde{d}(x_0, x_2). \\ &\leq a_1\widetilde{d}(x_0, x_1) + a_2\widetilde{d}(x_0, x_1) + a_3\widetilde{d}(x_1, x_2) + sa_4[\widetilde{d}(x_0, x_1) + \widetilde{d}(x_1, x_2)]. \\ &\leq (a_1 + a_2 + sa_4)\widetilde{d}(x_0, x_1) + (a_3 + sa_4)\widetilde{d}(x_1, x_2), \end{aligned}$$

which implies

$$s\widetilde{d}(x_1, x_2) \leq (a_1 + a_2 + sa_4)\widetilde{d}(x_0, x_1) + (a_3 + sa_4)\widetilde{d}(x_1, x_2).$$

Then

$$\widetilde{d}(x_1, x_2) \leq \frac{a_1 + a_2 + sa_4}{s - a_3 - sa_4}\widetilde{d}(x_0, x_1),$$

since $a_1 + a_2 + a_3 + 2sa_4 = 1$, we get

$$\widetilde{d}(x_1, x_2) < \widetilde{d}(x_0, x_1).$$

Thus,

$$s\widetilde{d}(x_1, x_2) \leq (a_1 + a_2 + sa_4)\widetilde{d}(x_0, x_1) + (a_3 + sa_4)\widetilde{d}(x_0, x_1) = \widetilde{d}(x_0, x_1),$$

so

$$s\widetilde{d}(x_1, x_2) \leq \widetilde{d}(x_0, x_1).$$

Then we have

$$\theta(s\widetilde{d}(x_1, x_2)) \leq \left[\theta(\widetilde{d}(x_0, x_1))\right]^{\kappa(\widetilde{d}(x_0,x_1))}.$$

Assume that $x_1 \neq x_2$, then $x_2 \notin Tx_2$ and $\widetilde{d}(x_2, Tx_2) > 0$,so $H(Tx_1, Tx_2) > 0$. From Lemma 4.1.2 there exists $x_3 \in Tx_2$ such that

$$\theta(s\widetilde{d}(x_2, x_3)) \leq \theta(s^3H(Tx_1, Tx_2)) \leq [\theta(N_s(x_1, x_2))]^{\kappa(\widetilde{d}(x_1,x_2))} < \theta(N_s(x_1, x_2)).$$

Then

$$\theta(s\widetilde{d}(x_2,x_3)) < \theta(N_s(x_1,x_2)),$$

which gives

$$s\widetilde{d}(x_2,x_3) < N_s(x_1,x_2)$$

where

$$\begin{aligned} N_s(x_1,x_2) &= a_1\widetilde{d}(x_1,x_2) + a_2\widetilde{d}(x_1,Tx_1) + a_3\widetilde{d}(x_2,Tx_2) + a_4\widetilde{d}(x_1,Tx_2) + a_5\widetilde{d}(x_2,Tx_1) \\ &\leq a_1\widetilde{d}(x_1,x_2) + a_2\widetilde{d}(x_1,x_2) + a_3\widetilde{d}(x_2,x_3) + sa_4(\widetilde{d}(x_1,x_2) + \widetilde{d}(x_2,x_3)) \\ &\leq (a_1 + a_2 + sa_4)\widetilde{d}(x_1,x_2) + (a_3 + sa_4)\widetilde{d}(x_2,x_3). \end{aligned}$$

Then

$$s\widetilde{d}(x_2,x_3) \leq (a_1 + a_2 + sa_4)\widetilde{d}(x_1,x_2) + (a_3 + sa_4)\widetilde{d}(x_2,x_3),$$

which implies

$$(s - a_3 - sa_4)\widetilde{d}(x_2,x_3) \leq (a_1 + a_2 + sa_4)\widetilde{d}(x_1,x_2)$$

since $a_1 + a_2 + a_3 + 2sa_4 = 1$, we get

$$\widetilde{d}(x_2,x_3) < \widetilde{d}(x_1,x_2).$$

Then we have

$$\theta(s\widetilde{d}(x_2,x_3) \leq \left[\theta(\widetilde{d}(x_1,x_2))\right]^{\kappa(\widetilde{d}(x_1,x_2))}.$$

By continuing in this manner, we construct a sequence (x_n) in X, if there exists n_0 such that $x_{n_0} = x_{n_0+1}$, or $x_{n_0+1} \in Tx_{n_0+1}$ then x_{n_0+1} is fixed point. If $x_n \neq x_{n+1}$ and $x_{n+1} \notin Tx_{n+1}$, then $H(Tx_n, Tx_{n+1}) > 0$. From Lemma 4.1.2, there exists $x_{n+1} \in Tx_n$ such that

$$\theta(s\widetilde{d}(x_n,x_{n+1})) \leq \left[\theta(\widetilde{d}(x_{n-1},x_n))\right]^{\kappa(\widetilde{d}(x_{n-1},x_n))}, \quad \text{for all } n \in \mathbb{N}. \tag{4.2}$$

It follows by (4.2) and (θ_4) that

$$\theta(s^n\widetilde{d}(x_n,x_{n+1})) \leq \left[\theta(s^{n-1}\widetilde{d}(x_{n-1},x_n))\right]^{\kappa(\widetilde{d}(x_{n-1},x_n))}, \quad \text{for all } n \in \mathbb{N}. \tag{4.3}$$

Since θ is increasing, then the sequence $(\widetilde{d}(x_n,x_{n+1}))_n$ is decreasing, further it is bounded at below so it is convergent, then there exist $\delta \in (0,1)$ and $n_0 \in \mathbb{N}$ such that $k(\widetilde{d}(x_n,x_{n+1})) < \delta$, for all $n \geq n_0$. Thus, from (4.3), we deduce

$$\begin{aligned} 1 &< \theta(s^n \widetilde{d}(x_n, x_{n+1})) \\ &\leq \left[\theta(s^{n-1}\widetilde{d}(x_{n-1}, x_n))\right]^{\kappa(\widetilde{d}(x_{n-1},x_n))} \\ &\leq \left[\theta(s^{n-2}\widetilde{d}(x_{n-2}, x_{n-1}))\right]^{\kappa(\widetilde{d}(x_{n-2},x_{n-1}))\kappa(\widetilde{d}(x_{n-1},x_n))} \\ &\vdots \\ &\leq \left[\theta(\widetilde{d}(x_0, x_1))\right]^{\delta^{n-n_0}}, \end{aligned} \tag{4.4}$$

On taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \theta(s^n \widetilde{d}(x_n, x_{n+1})) = 1.$$

(θ_2) gives

$$\lim_{n\to\infty} s^n \widetilde{d}(x_n, x_{n+1}) = 0.$$

Now, we prove $\{x_n\}$ is a Cauchy sequence, by (θ_3) there exist $\rho \in [0, 1)$ and $\chi \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(s^n \widetilde{d}(x_n, x_{n+1})) - 1}{(s^n \widetilde{d}(x_n, x_{n+1}))^\rho} = \chi.$$

If $\chi < \infty$, let $2\varepsilon = \chi$. By the definition of limit, there exists $n_1 \in \mathbb{N}$ such that

$$(s^n \widetilde{d}(x_n, x_{n+1}))^\rho \leq \varepsilon^{-1}[\theta(s^n \widetilde{d}(x_n, x_{n+1})) - 1], \quad \text{for all } n \geq n_1.$$

Using (4.4) and the above inequality, we deduce

$$n(s^n \widetilde{d}(x_n, x_{n+1}))^\rho \leq \varepsilon^{-1} n([\theta(\widetilde{d}(x_0, x_1))]^{\delta^{n-n_0}} - 1), \quad \text{for all } n \geq n_1.$$

This implies that

$$\lim_{n\to\infty} n(s^n \widetilde{d}(x_n, x_{n+1}))^\rho = 0.$$

In the case where $\chi = \infty$, let A be an arbitrary positive real number, so from the definition of the limit there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$\frac{\theta(s^n \widetilde{d}(x_n, x_{n+1})) - 1}{(s^n \widetilde{d}(x_n, x_{n+1}))^\rho} = \chi. > A,$$

which implies that

$$n(s^n\widetilde{d}(x_n, x_{n+1}))^\rho < \frac{n([\theta(\widetilde{d}(x_0, x_1))]^{\delta^{n-n_0}} - 1)}{A}. \tag{4.5}$$

Letting $n \to \infty$ in (4.5), we obtain

$$\lim_{n\to\infty} n(s^n\widetilde{d}(x_n, x_{n+1}))^\rho = 0.$$

Thence, there exists $n_2 \in \mathbb{N}$ such that

$$s^n\widetilde{d}(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{\rho}}}, \quad \text{for all } n \geq n_2. \tag{4.6}$$

Let $m > n \geq \max\{n_0, n_1, n_2\}$. Then, using the triangular inequality and (4.6), we have

$$\widetilde{d}(x_n, x_m) \leq \sum_{j=n}^{m-1} \widetilde{d}(x_j, x_{j+1}) \leq \sum_{j=n}^{m-1} s^n\widetilde{d}(x_j, x_{j+1}) \leq \sum_{j=n}^{m-1} \frac{1}{j^{\frac{1}{p}}} \leq \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{p}}} < \infty,$$

Hence $\{x_n\}$ is a Cauchy sequence.

Since $(X, \widetilde{d}, s)$ is complete, so $\{x_n\}$ converges to some $x^* \in X$.

If T is α_s-lower semi continuous, then for all $n \in \mathbb{N}$, we have

$$\widetilde{d}(x_n, Tx_n) \leq \widetilde{d}(x_n, x_{n+1}).$$

Passing to the limit, we get

$$\lim_{n\to\infty} \widetilde{d}(x_n, \widetilde{T}x_n) = 0.$$

Then taken in the account T is α_s-lower semi continuous, we obtain

$$0 < \widetilde{d}(x^*, Tx^*) \leq \lim_{n\to\infty} \inf \widetilde{d}(x_n, Tx_n) = 0,$$

which gives $\widetilde{d}(x^*, Tx^*)$. Hence x^* is a fixed point.

If X is α_s-regular, so for every sequence $\{x_n\}$ converges to x^* with $\alpha(x_n, x_{n+1}) \geq s^2$, then $\alpha(x_n, x^*) \geq s^2$ so $\widetilde{d}(x_{n+1}, Tx^*) > 0$, which implies $H(Tx_n, Tx^*) > 0$, then by (4.1), we get

$$\begin{aligned} 1 < \theta(s\widetilde{d}(x_{n+1}, Tx^*)) &\leq \theta(s^3H(Tx_n, Tx^*) \\ &\leq [\theta(N_s(\widetilde{d}(x_0, x_1)))]^{\delta^{n-n_0}} \\ &< [\theta(\widetilde{d}(x_0, x_1))]^{\delta^{n-n_0}}. \end{aligned}$$

Passing to the limit, we have

$$\lim_{n\to\infty} \theta(\widetilde{d}(x_{n+1}, Tx^*)) = 1,$$

then (θ_2) gives

$$\lim_{n\to\infty} \widetilde{d}(x_{n+1}, Tx^*) = 0,$$

which implies $\widetilde{d}(x^*, Tx^*) = 0$. Hence x^* is a fixed point of T. $\square$

Since each α_s^*-admissible mapping is also α_s-admissible, we obtain following result.

Corollary 4.2.1. *Let $(X, \widetilde{d}, s)$ be a complete b-metric space, $\alpha\colon X \times X \to [0, +\infty)$ be a function and $T\colon X \to CB(X)$ be a multivalued mapping. Assume that the following conditions hold:*

(i) T is an α_s^*-admissible.

(ii) There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq s^2$.

(iii) For every sequence $\{x_n\} \subset X$ converges to some x in X and $\alpha^*(x_n, x_{n+1}) \geq s^2$, for all $n \in \mathbb{N}$. Then $\alpha^*(x_n, x) \geq s^2$, for all $n \in \mathbb{N}$.

(iv) There exist $\theta \in \Theta$, $L \geq 0$ and $\kappa : (0, \infty) \to [0, 1)$ satisfies $\lim\limits_{t\to s^+} \sup k(t) < 1$ for all $s \in (0, \infty)$ such that

$$\theta(s^3 H(Tx_1, Tx_2)) \leq \left[\theta(N_s(x_1, x_2))\right]^{\kappa(\widetilde{d}(x_1,x_2))} + L \min\{\widetilde{d}(x_1, Tx_2), \widetilde{d}(x_2, Tx_1)\},$$

where

$$N_s(x_1, x_2) = a_1\widetilde{d}(x_1, x_2) + a_2\widetilde{d}(x_1, Tx_1) + a_3\widetilde{d}(x_2, Tx_2) + a_4\widetilde{d}(x_1, Tx_2) + a_5\widetilde{d}(x_2, Tx_1).$$

Then T has a fixed point.

Corollary 4.2.2. *Let $(X, \widetilde{d}, s)$ be a complete b-metric space, $\alpha\colon X \times X \to [0, +\infty)$ be a function and $T\colon X \to CB(X)$ be a multivalued mapping. Assume that the following conditions are satisfied:*

(i) T is an α_s-admissible;

(ii) *there exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0, x_1) \geq s^2$;

(iii) T *is* α_s*-lower semi-continuous, or* X *is* α_s*-regular;*

(iv) *there exist* $\theta \in \Theta_s$, $L \geq 0$ *and* $\kappa : (0, +\infty) \to [0, 1)$ *satisfies* $\lim\limits_{\omega \to z^+} \sup \kappa(\omega) < 1$ *for all* $z \in (0, +\infty)$ *and non negative real numbers* a_1, a_2, a_3, a_4, a_5 *with* $a_1 + a_2 + a_3 + 2sa_4 = 1$, *and* $a_3 \neq 1$ *such that*

$$\theta(s^3\alpha(x,y)H(Tx,Ty)) \leq \left[\theta(N_s(x,y))\right]^{\kappa(\widetilde{d}(x,y))} + L\min\{\widetilde{d}(x,Ty), \widetilde{d}(y,Tx)\},$$

for all $x, y \in X$ *with* $H(Tx, Ty) > 0$, *where*

$$N_s(x,y) = a_1\widetilde{d}(x,y) + a_2\widetilde{d}(x,Tx) + a_3\widetilde{d}(y,Ty) + a_4\widetilde{d}(x,Ty) + a_5\widetilde{d}(y,Tx).$$

Then T *has a fixed point.*

Proof. For all $x, y \in X$, we have

$$H(Tx,Ty) \leq \alpha(x,y)H(Tx,Ty),$$

since θ is increasing function, we get

$$\begin{aligned}\theta(s^3H(Tx,Ty)) &\leq \theta(s^3\alpha(x,y)H(Tx,Ty)) \\ &\leq \left[\theta(N_s(x,y))\right]^{\kappa(\widetilde{d}(x,y))} + L\min\{\widetilde{d}(x,Ty), \widetilde{d}(y,Tx)\}.\end{aligned}$$

So this result is a consequence of Theorem 4.2.1. □

Corollary 4.2.3. *Let* $(X, \widetilde{d}, s)$ *be a complete b-metric space,* $\alpha \colon X \times X \to [0, +\infty)$ *be a function and* $T \colon X \to CB(X)$ *be a multivalued mapping. Assume that the following conditions hold:*

(i) T *is almost* (θ, κ)*-contraction of Hardy Rogers type.*

(ii) T *is lower semi continuous.*

Then T *has a fixed point.*

Proof. It suffices to take $\alpha(x, y) = s^2$ for all $x, y \in X$ in Theorem 4.2.1. □

Example 4.2.1. *Let $X = [0, 2]$ be a set endowed with a b-metric $\widetilde{d}(x, y) = |x - y|^2$. Define $T : X \to CB(X)$ and $\alpha : X \times X \to [0, \infty)$ by*

$$Tx = \begin{cases} [0, \frac{x}{4}], & x \in [0, 2) \\ \{2\}, & x = 2 \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 4, & (x, y) \in [0, 2) \\ 0, & otherwise. \end{cases}$$

Taking $\theta(t) = e^t$, $\kappa = 3/4$, $s = 2, L = 0, a_1 = \frac{2}{3}$, $a_2 = a_4 = a_5 = 0$ and $a_3 = 1/3$.
For all $x, y \in (0, 2)$, we have $\alpha(x, y) = 4$, $H(Tx_1, Tx_2) > |\frac{x_1 - x_2}{4}|^2 > 0$ and $\widetilde{d}(x, y) = |x_1 - x_2|^2$. Then

$$8H(Tx, Ty) = \frac{1}{2}|x - y|^2 = \frac{3}{4}(\frac{2}{3}|x - y|^2) \leq \frac{3}{4}N_s(x, y),$$

which implies that

$$e^{8H(Tx,Ty)} \leq e^{\frac{3}{4}\widetilde{d}(x,y)} \leq e^{\frac{3}{4}N_s(x,y)}.$$

$$e^{8H(Tx,Ty)} \leq \left[e^{N_s(x,y)}\right]^{\frac{3}{4}}.$$

T is α-continuous. since if (x_n) is a sequence in X converges to x^* with $\alpha(x_n, x_{n+1}) \geq 4$, then $(x_n) \subset [0, 2)$ which implies $Tx_n = [0, \frac{x_n}{4}]$ and $\lim_{n\to\infty} Tx_n = [0, \frac{x^*}{4}] = Tx^*$.
Consequently, all conditions of Theorem 4.2.1 satisfied. Then T has a fixed point which is 2.

Now, we give new fixed point results on a b-metric space endowed with a partial ordering/graph, by using the results provided in previous section.

4.3 Fixed point on partially ordered b-metric space

Define

$$\alpha\colon X\times X\to[0,+\infty),\quad \alpha(x,y)=\begin{cases} s^2, & \text{if } x\preceq y,\\ 0, & \text{otherwise.}\end{cases}$$

Then the following result is a direct consequence of our results.

Theorem 4.3.1. *Let $(X,\preceq,\widetilde{d})$ be a complete ordered b-metric space and $T\colon X\to CB(X)$ be a multivalued mapping. Assume that the following assertions hold.*

1. *For each $x\in X$ and $y\in Tx$ with $x\preceq y$, we have $y\preceq z$ for all $z\in Ty$.*

2. *There exist $x_0\in X$ and $x_1\in Tx_0$ such that $x_0\preceq x_1$;*

3. *For $x^*\in X$ and a sequence $\{x_n\}$ in X with $\lim_{n\to\infty}\widetilde{d}(x_n,x^*)=0$ and $x_n\preceq x_{n+1}$ for all $n\in\mathbb{N}$, implies*

   $$\lim\inf_{n\to\infty}\widetilde{d}(x_n,Tx_n)\geq\widetilde{d}(x^*,Tx^*)$$

   or, for every sequence $\{x_n\}$ in X such that $x_n\to x^*\in X$ and $x_n\preceq x_{n+1}$ for all $n\in\mathbb{N}$, we have $x_n\preceq x$ for all $n\in\mathbb{N}$.

4. *There exist $\theta\in\Theta_s$, $L\geq 0$ and $\kappa:(0,\infty)\to[0,1)$ satisfies $\lim\sup_{t\to s^+}\kappa(t)<1$ for all $s\in(0,\infty)$ such that*

   $$\theta(s^3H(Tx,Ty))\leq\Big[\theta(N_s(x,y))\Big]^{\kappa(\widetilde{d}(x,y))}+L\min\{\widetilde{d}(x,Ty),\widetilde{d}(y,Tx)\},$$

   where

   $$N_s(x,y)=a_1\widetilde{d}(x,y)+a_2\widetilde{d}(x,Tx)+a_3\lceil(y,Ty)+a_4\widetilde{d}(x,Ty)+a_5\widetilde{d}(y,Tx).$$

Then T has a fixed point.

4.4 Fixed point on b-metric space endowed with a graph

Now, we present the existence of fixed point for multivalued mappings from a b-metric space X, endowed with a graph, into the space of non empty closed and bounded subsets of the metric space.
We define the function

$$\alpha\colon X \times X \to [0, +\infty), \quad \alpha(x, y) = \begin{cases} s^2, & \text{if } (x, y) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.4.1. *Let $\left(X, \widetilde{d}, s\right)$ be a complete b-metric space endowed with a graph G and $T\colon X \to CB(X)$ be a multivalued mapping. Assume that the following conditions hold:*

1. *For each $x \in X$ and $y \in Tx$ with $(x, y) \in E(G)$, we have $(y, z) \in E(G)$ for all $z \in Ty$;*
2. *There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$;*
3. *For every sequence $\{x_n\}$ in X such that $x_n \to x^* \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, we have $(x_n, x^*) \in E(G)$ for all $n \in \mathbb{N}$;*
4. *There exist $\theta \in \Theta_s$ and $\kappa : (0, \infty) \to [0, 1)$ satisfies $\limsup_{t \to s^+} \kappa(t) < 1$ such that*

$$\theta(s^3 H(Tx, Ty)) \leq \left[\theta(N_s(x, y))\right]^{\kappa(\widetilde{d}(x,y))} + L \min\{\widetilde{d}(x, Ty), \widetilde{d}(y, Tx)\}, \tag{4.7}$$

where

$$N_s(x, y) = a_1\widetilde{d}(x, y) + a_2\widetilde{d}(x, Ty) + a_3\widetilde{d}(y, Ty) + a_4\widetilde{d}(x, Ty) + a_5\widetilde{d}(y, Tx).$$

Then T has a fixed point.

Chapter 5

Applications

researchers try to highlight the quantitative approximation and the solution nature of a problem. One of the techniques, among all the many strategies, is the theory of the fixed point. The fixed point theory can be used as an essential tool to discuss the existence and uniqueness of the solution of several types integral, differential, integro-differential, fractional integrals and integral inclusions or differential equations.

In this chapter, we will apply theorems (3.1.2) and (4.2.1) to study the problem of existence of the solution of fractional differential inclusions, and integral inclusions of the Volterra type.

5.1 Application to integral inclusions

5.1.1 Notions on integral inclusions

Integral of multivalued mapping

Let $F \subset X$, we note by ω the set of integrable functions of $K : X \to \mathcal{P}(Y)$.

Definition 5.1.1. [25] *The integral of the multivalued mapping K over F is the set of integrals of integrable functions of K :*

$$\int_F K ds := \{ \int_F f ds \mid f \in \omega \}.$$

Integral inclusions

Definition 5.1.2. [25] *Let E be a closed, bounded set of a vector space and X a space of definite functions from E to $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$, assumed to be a Banach space. An integral inclusion is a functional problem of the following form:*

$$To\ find\ x \in X,\ \forall t \in E,\ g(t)x(t) \in f(t) + \int_F K(t,s,x(s))ds \tag{5.1}$$

where f a function in $X, g : E \to \mathbb{K}$ which is not necessarily in X and K a function such that $K : E^2 \times \mathbb{K} \to \mathcal{P}(\mathbb{K})$.

Linearity

Depending on the form of function K, the integral can be linear or non-linear. In the particular case where the function K is written in the form:

$$K(t,s,x) = k(t,s)x$$

where $k : E^2 \to \mathcal{P}(\mathbb{K})$; then our inclusion becomes linear.
In the case where we cannot separate the third variable, our inclusion is said to be non-linear.
Between linearity and non-linearity, the analytical and numerical studies of our inclusion are totally different.
In the linear case, the analytical study is based on the theory of bounded operators, especially the study of the spectrum and the notions of convergence of operator sequences, the latter is used in numerical studies and projection techniques.
On the other hand, for the non-linear case, then one applies the theorems of fixed points.

Types of integral inclusions:

In the mathematical and physical literature, there are two main classes of integral inclusions :

Integral Fredholm-type inclusions

The first class concerns integral Fredholm inclusions, these inclusions are characterized by the condition $E = F$. They are then written in the following form:

$$x \in X,\ \forall t \in E,\ g(t)x(t) \in f(t) + \int_E K(t, s, x(s))ds \tag{5.2}$$

Integral Volterra-type inclusions

The second class defines the integral inclusions of Volterra, it is characterized by the fact that $F \subset E$, where the domain of integration depends on the variable t, so we have the following form:

$$x \in X,\ \forall t \in E,\ g(t)x(t) \in f(t) + \int_{F(t)} K(t, s, x(s))ds \tag{5.3}$$

Remark 5.1.1. *If we take $E = [a, b]$ an interval of $\mathbb{R}$, the Fredholm inclusions are given in the form:*

$$x \in [a, b],\ \forall t \in E,\ g(t)x(t) \in f(t) + \int_a^b K(t, s, x(s))ds \tag{5.4}$$

and those of Volterra are written:

$$x \in [a, b],\ \forall t \in E,\ g(t)x(t) \in f(t) + \int_a^t K(t, s, x(s))ds \tag{5.5}$$

Classes of integral inclusions

The function $g(t)$ makes it possible to specify the classes of integral inclusion, we then distinguish three classes of integral.

1. If $g = 0$, the integral inclusion is said to be of the first class. The most studied case is the linear integral inclusion of Fredholm.
2. For the non-linear case, the studies were not exhaustive if $g(t) = \lambda$(where λ is a constant), the inclusion is said to be of the second class. This kind of inclusion is the most treated, there are a large number of studies that treat the types of Fredholm and Volterra in the linear or non-linear case.
3. If $g(t)$ is any function, we say of the third class. This kind of inclusion is recent, but it is very interesting from a mathematical and physical point of view.

5.1.2 Selection of multivalued maps

Definition 5.1.3. [7] *We call selection of a multivalued mapping $K : X \to \mathcal{P}(Y)$, any function $k : X \to Y$:*

$$k(x) \in Kx, \quad \forall x \in X.$$

A selection theorem ensured the existence of a single-valued function (selection) belongs to a given multivalued mapping. There are various selection theorems, and they are important in differential inclusion theory and economics. Among these theorems, the selection theorem of Michael which was proved in 1956 by Ernest Michael. In what follows, we consider X, Y two Banach spaces, then Michael's selection theorem reads as follows:

Theorem 5.1.1. [44] *Let $K : X \to \mathcal{P}(Y)$ a lower semi-continuous multivalued mapping non-empty closed convex. Then K has continuous selection, that is, there exists a continuous function $k : X \to Y$ such that for all $x \in X, k(x)$ belongs to Kx.*

Example 5.1.1. *Let $K : [0,1] \to [0,1]$ a multivalued mapping defined by:*

$$K(x) = \begin{cases} \dfrac{3}{4} & \text{if} \quad x \in [0, 0.5[, \\ [0,1] & \text{if} \quad x = 0.5, \\ \dfrac{1}{4} & \text{if} \quad x \in]0.5, 1]. \end{cases}$$

Clearly, K is a non-empty closed convex-valued multivalued mapping, but it is not lower semi continuous at 0.5. Then, we cannot apply Michael's theorem in this case, i.e. K does not admit a continuous selection.

5.1.3 Existence of solutions for an integral inclusion

In this section, we apply our obtained results to prove an existence theorem of solution for an integral inclusion of Volterra-type . For this purpose, let $X := C([a,b], \mathbb{R})$ be the space of all continuous real valued functions on $J = [a,b]$. Note that X is complete b- metric space by considering

$$d(x,y) = \sup_{t \in [a,b]} |x(t) - y(t)|^2,$$

with $s = 2$ and define a function:

$$\alpha : X \times X \to \mathbb{R}_+ \text{ by } \alpha(x,y) = 4,$$

for all $x, y \in X$.

Consider now the following problem

$$x(t) \in p(t) + \int_a^t F(t,r,x(r))dr, \qquad t \in J. \tag{5.6}$$

where $p \in X$ and $F : J \times J \times \mathbb{R} \to K(\mathbb{R})$.

Consider the set-valued operator $T : X \to CL(X)$ as follows

$$Tx(t) = \left\{ y \in X : y \in p(t) + \int_a^t F(t,r,x(r))dr, \quad t \in J \right\}.$$

We consider the following hypotheses:

$(\mathcal{A}_1)$: For each $x \in X$, the multivalued operator $F_x : (t,r) \mapsto F(t,r,x(r))$, is lower semi continuous.

$(\mathcal{A}_2)$: There exists a continuous function $\xi : J \times J \to [0,+\infty)$ such that

$$|q_x(t,r) - q_y(t,r)| \leq \xi(t,r)|x(r) - y(r)|,$$

for all $x, y \in X$, all $q_x \in F_x, q_y \in F_y$ and for each $(t,r) \in J \times J$.

$(\mathcal{A}_3)$: There exists $\gamma > 0$ such that

$$\sup_{t \in J} \int_a^t |\xi(t,r)|dr \leq (\frac{e^{-\gamma}}{8})^{\frac{1}{2}}.$$

Theorem 5.1.2. *The integral inclusion (5.6) has a solution in X provided the assumptions $(\mathcal{A}_1) - (\mathcal{A}_3)$ hold.*

Proof. The set-valued operator $F_x(t,r) : J \times J \to K(\mathbb{R})$ is lower semi continuous, then from Michael's selection theorem, for $x \in X$ there exists a continuous function $q_x : J \times J \to \mathbb{R}$ such that $q_x(t,r) \in F_x(t,r)$ for all $t, r \in J$.

It follows that $p(t) + \int_a^t q_x(t,r)dr \in Tx$, so Tx is non-empty for all $x \in X$.

Since p and q_x are continuous on J, resp. J^2, their ranges are bounded and closed and hence

Tx is bounded, i.e., $T : X \to K(X)$.

Let $x, y \in X$ and let $v \in Tx$. Then

$$v(t) \in p(t) + \int_a^t F(t, r, x(r))dr, \quad t \in J.$$

It follows that there exists $q_x \in F_x(t, r)$ such that

$$v(t) = p(t) + \int_a^t q_x(t, r)dr, \quad (t, r) \in J \times J,$$

From $(\mathcal{A}_2)$, there exists $q_y(t, r) \in F_y(t, r)$ such that

$$|q_x(t, r) - q_y(t, r)| \leq \ \xi(t, r) \cdot |x(r) - y(r)|,$$

for all $(t, r) \in J \times J$. Let P be a multi valued operator defined by

$$P(t, r) = F_y(t, r) \cap \{z \in \mathbb{R}\colon |q_x(t, r) - z| \leq \xi(t, r) \cdot |x(r) - y(r)|\},$$

for all $(t, r) \in J \times J$. Since, by $(\mathcal{A}_1)$, P is lower semi-continuous, there exists a continuous function $q_y(t, r) \in P(t, r)$. Then we have

$$u(t) = p(t) + \int_a^t q_y(t, r)dr \in p(t) + \int_a^t F(t, r, y(r))dr, \quad t \in J$$

and

$$\begin{aligned}
\widetilde{d}(v, Ty) &\leq |v(t, r) - u(t, r)|^2 \leq \Big(\int_a^t |q_x(t, r) - q_y(t, r)|dr\Big)^2 \\
&\leq \Big(\int_a^t \xi(t, r)|x(r) - y(r)|dr\Big)^2 \\
&\leq \sup_{r \in [a,b]} |x(r) - y(r)|^2 \Big(\int_a^t \xi(t, r)dr\Big)^2 \\
&= \widetilde{d}(x, y)(\int_a^t \xi(t, r)dr)^2 \\
&\leq \frac{e^{-\gamma}}{8}\widetilde{d}(x, y).
\end{aligned}$$

Consequently, we have

$$8\widetilde{d}(v, Ty) \leq e^{-\gamma}\widetilde{d}(x, y),$$

interchanging the role of x and y, we get

$$8H(Tx, Ty) \leq e^{-\gamma}\widetilde{d}(x, y).$$

Taking exponents we get

$$e^{8H(Tx,Ty)} \leq \left[e^{\widetilde{d}(x,y)}\right]^{e^{-\gamma}}$$

Then, the mapping T satisfies all the conditions of Corollary 4.2.1 with $\theta(t) = e^t$, $a_1 = 1$, $a_i = L = 0, i = 2,3,4,5$ and $\kappa = e^{-\gamma}$. So, T has a fixed point, which implies that the integral inclusion (5.6) has a solution in X. □

5.2 Application to fractional differential inclusions

5.2.1 Integral and Derivative Fractional

Integral Fractional

Definition 5.2.1. [39] *The fractional integral of the function $f \in C([a,b])$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(t)dt.$$

Proposition 5.2.1. *We have the following properties:*

i) $I^0 f(t) = f(t)$,

ii) $I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$,

iii) the integral operator I^α is linear,

$$I^\alpha(\lambda f(t) + g(t)) = \lambda I^\alpha f(t) + I^\alpha g(t), \quad \alpha \in \mathbb{R}_+, \quad \lambda \in \mathbb{C},$$

iv) $\frac{d}{dt}(I^\alpha f)(t) = I^{\alpha-1} f(t)$.

Fractional derivative

There are several approaches for fractional differentiation, in this part we will give some definitions and proprieties of fractional derivative.

Definition 5.2.2. [39] *Let f be an integrable function on $[a,b]$, then the derivative fractional order α (with $n-1 \leq \alpha < n$) in the sense of* ***Riemann-Liouville*** *is defined by:*

$$\begin{aligned} {}^{RL}D^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s) ds \\ &= \frac{d^n}{dt^n} (I^{n-\alpha} f(t)) = D^n I^{n-\alpha} f(t). \end{aligned}$$

here $n = [\alpha] + 1$ and $[\alpha]$ denoting the integer part of α.

Proposition 5.2.2. *We have the following properties:*

1. *The fractional differentiation operator in the sense of Riemann-Liouville is a left inverse*

$$^{RL}D^{r} \left(I^{r} f(t)\right) = f(t),$$

in general we have

$$^{RL}D^{r_1} \left(I^{r_2} f(t)\right) =^{RL} D^{r_1 - r_2} f(t)$$

and if $r_1 - r_2 < 0, ^{RL} D^{r_1-r_2} f(t) = I^{r_2 - r_1} f(t)$.

2. *In general derivation and fractional integration do not commute*

$$^{RL}D^{-r_1} \left({}_a^{RL}D_t^{r_2} f(t)\right) =^{RL} D^{r_2 - r_1} f(t) - \sum_{k=1}^{m} \left[{}^{RL}D_t^{r_2-k} f(t)\right]_{t=a} \frac{(t-a)^{r_1-k}}{\Gamma\left(r_1 - k + 1\right)}$$

with $m - 1 \leq r_2 < m$

3. *if :$f^{(k)}(a) = 0$ for all $k = 0, 1, 2, \ldots, n-1$*

$$\frac{d^n}{dt^n} \left({}^{RL}D^{r} f(t)\right) =^{RL} D^{n+r} f(t),$$

but

$$^{RL}D^{r} \left(\frac{d^n}{dt^n} f(t)\right) =^{RL} D^{n+r} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-r-n}}{\Gamma(k-r-n+1)}.$$

4. *For the Riemann-Liouville fractional differential operator, the corresponding interpolation property reads*

$$\lim_{\alpha \to n} {}^{RL}D^{\alpha} f(t) = f^{(n)}(t).$$

$$\lim_{\alpha \to n} {}^{RL}D^{\alpha} f(t) = f^{(n-1)}(t)$$

5. *is a linear operator* $\alpha \in \mathbb{R}_+, \lambda \in \mathbb{C}$

$$^{RL}D^{\alpha}(\lambda f(t) + g(t)) = \lambda^{RL}D^{\alpha}f(t) +^{RL} D^{r}g(t).$$

Definition 5.2.3. [39] *Let f be an integrable function on $[a, b]$, then the derivative fractional order α (with $n-1 \leq \alpha < n$) in the sense of Caputo is defined by:*

$$^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-s)^{n-\alpha-1}f^{(n)}(s)ds,$$

here $n = [\alpha] + 1$ and $[\alpha]$ denoting the integer part of α.

Proposition 5.2.3. *We have the following proprieties:*

1. *The relationship with the Riemann-Liouville derivative.*

   Suppose that $n-1 < \alpha < n$, $m, n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and let the function $f(t)$ be such that $^{c}D^{\alpha}f(t)$ and $^{RL}D^{\alpha}f(t)$ exist, then:

$$^{c}D^{\alpha}f(t) =^{RL} D^{\alpha}f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}.$$

   We deduce that if $f^{(k)}(a) = 0$, for $k = 0, 1, 2, ..., n-1$, we will have:

$$^{c}D^{\alpha}f(t) =^{RL} D^{\alpha}f(t).$$

2. *Suppose that $n-1 < \alpha < n$, $m, n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and let the function f(t) be such that $^{c}D^{\alpha}f(t)$ exists, then:*

$$^{c}D^{\alpha}D^{m}f(t) = {}^{c}D^{\alpha+m}f(t) \neq D^{m c}D^{\alpha}f(t).$$

3. *For the Caputo fractional differential operator, the corresponding interpolation property reads*

$$\lim_{\alpha \to n} {}^{c}D^{\alpha}f(t) = f(t).$$
$$\lim_{\alpha \to n} {}^{c}D^{\alpha}f(t) = f^{(n-1)}(t) - f^{n-1}(0)$$

4. *it is a linear operator* $\alpha \in \mathbb{R}_+, \lambda \in \mathbb{C}$

$$^{c}D^{\alpha}(\lambda f(t) + g(t)) = \lambda^{c}D^{\alpha}f(t) +^{c} D^{r}g(t).$$

Lemma 5.2.1. [39] *Let $\alpha > 0$, then*

$$I^{\alpha c}D^{\alpha}f(t) = f(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}$$

for $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1, n = [\alpha] + 1,$

5.2.2 Existence of solutions for fractional differential inclusions

Consider the following boundary value problem of fractional order differential inclusion with boundary integral conditions:

$$\begin{cases} {}^cD^q x(t) \in F(t,x(t)),\ 0 \leq t \leq 1,\ 1 < q \leq 2 \\ ax(0) - bx'(0) = 0 \\ x(1) = \int_0^1 h(s)g(s,x(s))ds \end{cases} \tag{5.7}$$

where ${}^cD^q$, $1 < q \leq 2$ is the Caputo fractional derivative, F, g, and h are given continuous functions, where

$F : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow K(\mathbb{R})$, $g : [0,1] \times \mathbb{R} \to \mathbb{R}$, $h \in L^1([0,1])$, $a + b > 0$, $\frac{a}{a+b} < q - 1$ and $h_0 = \|h\|_{L^1}$.

Denote by $X = C([0,1],\mathbb{R})$ the Banach space of continuous functions $x : [0,1] \longrightarrow \mathbb{R}$, with the uniform convergence norm

$$\| x \|_\infty = \sup\{|x(t)|,\ \ t \in I = [0,1]\}.$$

X can be endowed with the partial order relationship $\preceq$, that is, for all $x, y \in X$ $x \preceq y$ if and only if $x(t) \leq y(t)$, so $(X, d_\infty, \preceq)$ is a complete order metric space.

x is a solution of problem (5.7) if there exists $v(t) \in F(t,x(t)))$, for all $t \in I$ such that

$$\begin{cases} {}^cD^q x(t) = v(t),\ 0 \leq t \leq 1,\ 1 < q \leq 2 \\ ax(0) - bx'(0) = 0 \\ x(1) = \int_0^1 h(s)g(s))ds \end{cases} \tag{5.8}$$

Lemma 5.2.2. *Let $1 < q \leq 2$ and $v \in \mathcal{AC}(I,\mathbb{R}) = \{v : I \to \mathbb{R}, f \text{ is absolutely continuous}\}$. A function x is a solution of (5.8) if and only if it is a solution of the integral equation:*

$$x(t) = \int_0^1 G(t,s)v(s)ds + \frac{at+b}{a+b}\int_0^1 h(s)g(s)ds,$$

where G is the Green function given by

$$G(t,s) = \begin{cases} \frac{(at+b)(1-s)^{q-1}}{(a+b)\Gamma(q)} - \frac{(t-s)^{q-1}}{\Gamma(q)}, & s \leq t \\ \frac{(at+b)(1-s)^{q-1}}{(a+b)\Gamma(q)}, & t \leq s. \end{cases} \tag{5.9}$$

Proof. For some constants $c_0, c_1 \in \mathbb{R}$, we have

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s)ds + c_0 + c_1 t.$$

Using the boundary conditions on (5.8), we get

$$ac_0 - bc_1 = 0,$$

$$\frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds + c_0 + c_1 = \int_0^1 h(s)g(s)ds.$$

Therefore

$$c_0 = \frac{b}{a+b} \left[\frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} g(s,x(s))ds + \int_0^1 h(s)g(s,x(s))ds \right].$$

$$c_1 = \frac{a}{a+b} \left[\frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds + \int_0^1 h(s)g(s,x(s))ds \right].$$

It means that

$$\begin{aligned} x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s)ds &+ \frac{b}{a+b} \left[\frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds + \int_0^1 h(s)g(s,x(s))ds \right] \\ &+ \frac{at}{a+b} \left[\frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds + \int_0^1 h(s)g(s,x(s))ds \right] \\ = \int_0^t \left[\frac{(at+b)(1-s)^{q-1}}{(a+b)\Gamma(q)} - \frac{(t-s)^{q-1}}{\Gamma(q)} \right] v(s)ds &+ \int_t^1 \frac{(at+b)(1-s)^{q-1}}{(a+b)\Gamma(q)} v(s)ds \\ + \frac{at+b}{a+b} \int_0^1 h(s)g(s,x(s))ds = \int_0^1 G(t,s)v(s)ds &+ \frac{at+b}{a+b} \int_0^1 h(s)g(s)ds. \end{aligned}$$

$\square$

Moreover, we have

$$\begin{aligned} \int_0^1 G(t,s)ds &= \frac{(1}{\Gamma(q)} \Big[\int_0^t (t-s)^{q-1} ds + \frac{at+b}{a+b} \int_0^1 (1-s)^{q-1} ds \Big] \\ &\leq \frac{1}{\Gamma(q+1)} t^q + \frac{1}{\Gamma(q+1)} \leq \frac{2}{\Gamma(q+1)}. \end{aligned}$$

Define a set valued mapping

$$Tx_1(t) = \{ z \in X, z(t) = \int_0^1 G(t,s)v(s)ds + \frac{at+b}{a+b} \int_0^1 h(s)g(s,x_1(s)ds \}.$$

The problem (5.7) has a solution if and only if T has a fixed point. Assume that the following assumptions hold:

- (A_1) : For each $x_1 \in X$ and $x_2 \in Tx_1$ with $x_1 \preceq x_2$ we have $x_2 \preceq x_3$ for all $x_3 \in Tx_2$.

- (A_2) : There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$.

- (A_3) : There exist $K > 0$ and $L > 0$ such that for all $x_1, x_2 \in \mathbb{R}$, we have

$$H(F(t, x_1(t)), F(t, x_2(t))) \leq K|x_1 - x_2|$$

and

$$|g(t, x_1(t)) - g(t, x_2(t))| \leq L|x_1 - x_2|,$$

with $k_0 = \frac{2K}{\Gamma(q+1)} + h_0 L < \frac{1}{2}$.

Theorem 5.2.3. *Under the assumptions $(A_1) - (A_3)$ the problem (5.7) has a solution in X.*

Proof. Since F is continuous, it has a selection, i,e., there exists a continuous function $v_1 \in F(t, x_1(t))$ such that Tx_1 is non empty and has compact values.
Let $x_1, x_2 \in X$ and $z_1 \in Tx_1$, then there exists $v_1 \in F(t, x_1(t))$ such that

$$z_1(t) = \int_0^1 G(t, s)v_1(s)ds + \frac{at + b}{a + b} \int_0^1 h(s)g(s, x_1(s))ds.$$

Then by using (A_2), we get

$$\begin{aligned} d(v_1, Fx_2) = \inf_{u \in Fx_2} |v_1 - u| &\leq H(F(t, x_1(t)), F(t, x_2(t))) \\ &\leq K|x_1 - x_2|, \end{aligned}$$

the compactness of $F(t, x_2(t))$ implies that there exists $u^* \in F(t, x_2(t))$ such that

$$d(v_1, Fx_2) = |v_1 - u^*| \leq K|x_1 - x_2|.$$

Define an operator $P(t) = \{u^* \in \mathbb{R}, |v_1(t) - u^*| \leq K|x_1(t) - x_2(t)|\}$. Clearly $P \cap F(t, x_2(t))$ is continuous, so it has a selection v_2 such that

$$|v_1 - v_2| \leq K|x_1 - x_2|.$$

Define

$$z_2 = \int_0^1 G(t, s)v_2(s)ds + \frac{at + b}{a + b} \int_0^1 h(s)g(s, x_2(s)ds.$$

For all $t \in I$, we have

$$\begin{aligned}
|z_1 - z_2| &\leq \int_0^1 |G(t,s)||v_1 - v_2|ds + \frac{at+b}{a+b}\int_0^1 |h(s)||g(s,x_1(s)) - g(s,x_2(s))|ds \\
&\leq K|x_1 - x_2|\int_0^1 |G(t,s)|ds + \frac{at+b}{a+b}h_0 L|x_1(s) - x_2(s)| \\
&\leq (\frac{2K}{\Gamma(q+1)} + h_0 L)|x_1 - x_2| = k_0|x_1 - x_2|.
\end{aligned}$$

Then, we have

$$\sup_{z_1 \in Tx_1}\Big[\inf_{z_2 \in Tx_2}|z_1 - z_2|\Big] \leq k_0|x_1 - x_2|.$$

Hence, by interchanging the role of x_1 and x_2 we obtain

$$H(Tx_1, Tx_2) \leq k_0|x_1 - x_2|.$$

On taking the exponential of two sides, we get

$$\begin{aligned}
e^{H(Tx_1,Tx_2)} &\leq (e^{2k_0|x_1-x_2|})^{\frac{1}{2}} \\
&\leq e^{k_0|x_1-x_2|} + d(x_2, Tx_1).
\end{aligned}$$

If $\{x_n\}$ is a non decreasing sequence in X which is convergent to $x \in X$, so for all $t \in I$ and $n \in \mathbb{N}$ we have $x_n(t) \leq x(t)$, which implies that x is an upper bound for all terms x_n (see [51]), then $x_n \preceq x$.

Consequently, all the conditions of Theorem 3.2.1 are satisfied, with $\theta(t) = e^t$, $\psi(t) = 2k_0 t$ and $k(t) = \frac{1}{2}$.

Hence, T has a fixed point which is a solution of the problem (5.7). □

Conclusion and perspectives

Conclusion

This thesis is devoted to the study of existence of multivalued fixed point for a new type of contractions in metric spaces. Generalized almost θ-contractions type in metric spaces, as well as certain consequences which concern the existence of a fixed point in partially ordered metric spaces and in metric spaces equipped with a graph. As an application, we have studied the existence of the solution of an integral inclusion. In addition, results have been established for contraction of Hardy-Rogers type in b-metric spaces.

Perspectives

In the future, we will look at the following issues as examples:

(1) We will try to apply the obtained results to study the existence problem of the solution of a Fredholm-type integral inclusions.

(2) We will look for other combinations of the various contractions to improve our results and find new results.

Bibliography

[1] R. P. Agarwal, D. O'Regan, D. R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications, Topological Fixed Point Theory and Its Applications*, Springer, New York, 6(2009).

[2] J. Ahmad, A. E. Al-Mazrooei, Y. J. Cho and Y.O. Yang, emph Fixed point re- sults for generalized θ-contractions, J. Non-linear Sci. Appl., 10 (2017), 2350-2358.

[3] A. Ali, S. Mahideb and S. Beloul, *On multivalued fixed point for* $(\alpha_*, \eta_*, \theta)$ *contractions with an application*, J. Appl. Pure Math.Vol3 (5-6), 2021, 215-229.

[4] M. U. Ali and T. Kamran, *Multivalued F-Contractions and Related Fixed Point Theorems with an Application*, Filomat 30:14, (2016), 3779-3793.

[5] M. U. Ali, T. Kamran and M. Postolache, *Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem*, Non-linear Anal. Model. Control **22**(1) (2017), 17-30.

[6] H. Asl, J, Rezapour and S, Shahzad, *On fixed points of* $\alpha-\psi$*-contractive multifunctions*, Fixed Point Theory Appl. 2012, Article ID 212 (2012).

[7] J-P. Aubin, H. Frankowska, *Set-valued analysis*, Topological, Université de Paris-Dauphine and CNRS, (1990).

[8] I.A. Bakhtin, The contraction mapping principle in quasi metric spaces, Funct. Anal. Ulianowsk Gos. Ped. Inst, 30(1989), 26-37.

[9] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., 3 (1922), 133-181.

[10] S. Beloul, *Common fixed point theorems for multi-valued contractions satisfying generalized condition(B) on partial metric spaces*, Facta Univ Nis Ser. Math. Inform., vol. 30 (5) (2015), 555-566.

[11] S. Beloul, *A common fixed point theorem For generalized almost contractions in metric-like spaces*. Appl. Maths. E - Notes.18(2018), 27-139.

[12] V. Berinde, *Approximating fixed points of weak φ-contractions using the Picard iteration*, Fixed Point Theory, (2003), 131-142.

[13] V. Berinde, *Iterative Approximation of Fixed Points*, Editura Efemeride, Baia Mare, 2002.

[14] V. Berinde, *Iterative Approximation of Fixed Points*, Springer-Verlag, Berlin Heidelberg, 2007.

[15] Y. Berinde, *Generalized contractions in quasimetric spaces*, Seminar on Fixed Point Theory, (1993), No. 3, 3-9.

[16] M. Berinde, V. Berinde, *On a general class of multivalued weakly Picard mappings*, J. Math. Anal. Appl, 326(2007), 772-782.

[17] D. W. Boyd and S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc., 20 (1969), 458-464.

[18] L. B. Ćirić, *Multi-valued non linear contraction mappings*, Non linear Anal. (2009), 2716-2723.

[19] S.K. Chaterjea, *Fixed point theorems*. C. R. Acad. Bulgare Sci. 25, (1972), 727-730.

[20] M. Cosentino, M. Jleli, B. Samet, C. Vetro, *Solvability of integrodifferential problems via fixed point theory in b-metric spaces*, Fixed Point Theory and Appl, 70(2015).

[21] M. Cosentino and P. Vetro, *Fixed point results for F-contractive mappings of Hardy-Rogers-type*, Filomat 28:4 (2014), 715-722.

[22] S. Czerwik, Non-linear set-valued contraction mappings in b-metric spaces, Atti Sem. Mat. Fis. Univ. Modena 46(1998), 263-276.

[23] K. Deimling, Multivalued differential equations. Walter de Gruyter, Berlin, New York, (1992).

[24] G. Durmaz, *Some theorems for a new type of multivalued contractive maps on metric space*, Turkish J. Math., 41 (2017), 1092-1100.

[25] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc, 37(1962), 74-79.

[26] J. Fahd, T. Abdeljawad, and D. Baleanu,*Caputo-type modification of the Hadamard fractional derivatives*, Advances in Difference Equations 2012.1 (2012), 1-8.

[27] L. Fornari, V. Pata, *A note on weak contractions in compact metric spaces*, preprint (2019)

[28] H. A. Hançer, G. Mınak and I. Altun, On a broad category of multivalued weakly Picard operators, Fixed Point Theory, 18(2017), No. 1, 229-236.

[29] G. E. Hardy, T. D. Rogers, *A generalization of a fixed point theorem of Riech.* Bull.Cand.Math. Soc.16, (1963); 201-206.

[30] F. Hausdorff. Beweis eines Satzes von Arzelà. Math. Z, 26(1927), 135–137.

[31] S. Hu and N. S. Papageorgiou, Handbook of multivatued anatysis, Vol. I. Theory, Kluwer, Dordrecht, 1997.

[32] N. Hussain, P. Salimi and A. Latif, *Fixed point results for single and set-valued $\alpha-\eta-\psi$-contractive mappings*, Fixed Point Theory Appl. 2013, Article ID 212 (2013).

[33] I. Iqbal and N.Hussain, *Fixed point results for generalized multivalued non linear F-contractions*, J. Non linear Sci. Appl. 9 (2016), 5870-5893.

[34] H. Isik and C. Ionescu, *New type of multivalued contractions with related results and applications* U.P.B. Sci. Bull., Series A, 80(2) (2018), 13-22.

[35] M. Jleli and B. Samet, *A new generalization of the Banach contraction principle*, J. Inequal. Appl.38,(2014), 8 pp.

[36] H. Kaddouri, H. Isik and S. Beloul, *On new extensions of F-contraction with an application to integral inclusions*, U.P.B. Sci. Bull., Series A, Vol.81(3) (2019), 31-42.

[37] H. Kaddouri and S.Beloul, *Fixed point theorems for multivalued wordowski type contractions in b-metric spaces with an application to integral inclusions*, TWMS J. App. and Eng. Math. 11(4) (2021), 1061- 1071.

[38] R.Kannan, *Some results on fixed points.* Bull. Cal. Math. Soc. 60, (1968), 71-76.

[39] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V, Amsterdam (2006).

[40] M. A. Kutbi, W. Sintunavarat, *On new fixed point results for* (α, ψ, ξ)*-contractive multivalued mappings on* α*-complete metric spaces and their consequences*, Fixed Point Theory Appl., 2015 (2015), 15 pages.

[41] S. Mahideb, A. Ali and S. Beloul, *On generalized almost* θ*-contractions with an application to fractional differential equations*, U.P.B. Sci. Bull., Series A, 83(3) (2021), 35-44.

[42] M. Meneceur, S. Beloul, *On multivalued theta-contractions of Berinde type with an application to fractional differential inclusions.*, Facta Univ. Series Mathematics and Informatics. Vol. 36, No 5 (2021), 1047-1063.

[43] M. Meneceur, S. Beloul and H. Isik, *Multivalued almost* θ *contractions, related fixed point results in complete b-metric spaces and applications*, submitted.

[44] E. Michael, *Continuous selections*, I, Ann. of Math, 63(1956), No. 2, 361-382.

[45] G. Minak, I. Altun, *Overall approach to Mizoguchi-Takahashi type fixed point results*, Turk. J. Math, 40 (2016), 895-904.

[46] G. Minak, A. Helvaci, I. Altun, *Ciric Type Generalized F-contraction On complete Metric Spaces and Fixed Point Results*, Filomat, 28(2014), No. 6, 1143-1151.

[47] G. Minak, I. Altun, *On the effect of α-admissibility and θ-contractivity to the existence of fixed points of multivalued mappings*, Non linear Analysis: Modelling and Control, Vol. 21, No. 5, 673–686.

[48] N. Mizoguchi, W. Takahashi, *Fixed point theorems for multi-valued mappings on complete metric spaces*, J. Math. Anal. Appl. 141, 8 (1989), 177–18.

[49] B. Mohammadi, S. Rezapour, N. Shahzad, Some results on fixed points of α-ψ-Ciric generalized multifunctions, Fixed Point Theory and Appl, 2013(2013), Art. No. 24.

[50] S. B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math. 30 (1969), 475-488.

[51] J. J. Nieto, and R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order. 22(2005), 223-239.

[52] H. Piri, P. Kumam, *Some fixed point theorems concerning F-contraction in complete metric spaces*, Fixed Point Theory and Appl, 2014(2014), Art. No. 210.

[53] D. Pompeiu. Sur la continuité des fonctions de variables complexes. Ann. Fac. Sci. Toulouse, (1905), No. 3, pp. 265–315.

[54] S. Reich, *Some remarks concerning contraction mapping*. Canad. Math. Bull. 14, (1971), 121-124.

[55] S. Reich, *Fixed points of contractive functions*. Boll. Unione Mat. Ital. 5, 26–42 (1972).

[56] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Non linear Anal, 75(2012), No. 4, 2154-2165.

[57] N-A. Secelean, *Iterated function systems consisting of F-contractions*. Fixed Point Theory and Appl, 2013.

[58] C. Shiau, K. K. Tan and C.S. Wong, *Quasi-non expansive multi-valued maps and selections*, Fund. Math. 87 (1975), 109-119.

[59] W. Sintunavarat, *Non linear integral equations with new admissibility types in b-metric spaces*, J. Fixed Point Theory Appl. 18, (2016), 397-416.

[60] A. Tomar, S. Beloul, R. Sharma and Sh. Upadhyay, *Common fixed point theorems via generalized condition (B) in quasi-partial metric space and applications*, Demonst.maths journal 50 (2017),278-298.

[61] F. Vetro, *A generalization of Nadler fixed point theorem*, Carpathian J. Math.31(3) (2015), 403-410.

[62] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory and Appl, (2012), Art, No. 94.